To the Graduate Council:

I am submitting herewith a dissertation written by Richard Eugene Woods, Jr., entitled "Bounds, Sampling, and Amplitude Uncertainty of Band-Limited Functions." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Electrical Engineering.

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BOUNDS, SAMPLING, AND AMPLITUDE UNCERTAINTY
OF BAND-LIMITED FUNCTIONS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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ABSTRACT

This work examines the amplitude fluctuations of band-limited functions and the representation of these functions by piecewise linearly interpolated samples. Such representations are implicitly assumed in many digital signal processing computations, including the estimation of level crossing profiles and the location of extrema in a discrete sequence of samples.

A comprehensive analysis of the average rate of change of band-limited functions is presented, including a derivation of the least upper bound on functions for which either the zeroth absolute moments (spectral areas) or Fourier amplitude spectra of the functions are bounded. These results are extended to the case of functions that are limited in both frequency and amplitude. The average rate of change of a function over an interval in which one end point of the interval is an extremum of the function is similarly bounded and used to establish a sampling rate which guarantees that, between successive samples of a band-limited function, the function itself does not deviate from either sample by more than some predefined amplitude change. Based on these results, several problems of practical importance, including slope overload error in delta modulation, aperture time and amplitude uncertainty in analog-to-digital conversion, the estimation of level crossing statistics, and the detection and elimination of high frequency pulses from a low frequency signal, are examined.

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CHAPTER 1

INTRODUCTION

While the amplitude fluctuations of an arbitrary realizable signal may appear to be quite complicated and random, there are definite limits to the signal's variability. A close examination of the signal would reveal that its slopes and curvatures are finite. Thus, although random in appearance, the signal is subject to several important constraints. These constraints impose bounds on the lower and upper frequencies of the signal, i.e., the signal has a finite bandwidth.

One of the important consequences of finite signal bandwidth is that continuous signals can be represented by discrete sequences of samples taken at isolated instants in time. It is well known that a function (or signal) $x(t)$ which is band-limited to W rad/sec is completely determined by giving its ordinates at a series of points spaced π/W seconds apart [E. T. Whittaker (1915), J. M. Whittaker (1929), Kotel'nikov (1933), Shannon (1949)]. The function $x(t)$ and its samples $x(nT)$ are related by the cardinal series expansion

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin W(t - nT)}{W(t - nT)} \tag{1.1}$$

where $T = \pi/W$ is called the sampling period. If $x(t)$ is deterministic, this series converges to $x(t)$ for almost all t[a] [Jagerman and Fogel

[a]Two functions which differ at most on a set of measure zero have the same representation unless they differ at a sampling instant.

(1956), Helms and Thomas (1962)]; if x(t) is stochastic, the equality in Equation 1.1 must be interpreted as convergence in the mean square sense [Balakrishnan (1957), Peterson (1963)].

From a practical point of view, Equation 1.1 permits the replacement of continuous band-limited signals by discrete sequences of samples without loss of information, and specifies the lowest rate (the Nyquist rate) necessary to perfectly reconstruct the original signals. This permits, for instance, the time sharing of transmission facilities by time-division multiplexing [Stark and Tuteur (1979)]. More importantly, these samples can be quantized for subsequent processing by digital computers. If this processing should require additional information concerning the behavior of the continuous function between samples, Equation 1.1 may be used to generate (digitally) the required number of intermediate values.[a]

While the cardinal series expansion of Equation 1.1 provides an important theoretical basis for the exact reconstruction of a band-limited function x(t) between sampling instants, approximate interpolation procedures are often employed in practice. One such method, which is referred to as linear or straight-line interpolation, is defined by the simple expression

$$\hat{x}(t) = mt + b$$

$$m = \frac{1}{T}\left[x(nT) - x((n-1)T)\right]$$

$$b = x((n-1)T) - m(n-1)T \;, \quad (n-1)T \leqslant t \leqslant nT \;. \qquad (1.2)$$

---

[a]In practice, it is impossible to perfectly recreate a continuous signal from its samples; this would require the summation of an infinite number of terms.

Linear interpolation is an implicit assumption in many digital signal processing computations, including the estimation of level crossing profiles, amplitude distributions, and moments [Suthasinekal et al. (1976), Mitchell and Gonzalez (1978), Hummel (1975)]. When computing level crossing profiles, for example, linearly interpolated data is assumed so that $x(t)$ may be considered to be completely contained between $x((n-1)T)$ and $x(nT)$ for $(n-1)T \leqslant t \leqslant T$. Then, utilizing the intermediate value theorem [Tierney (1969)], all levels between $x((n-1)T)$ and $x(nT)$ are considered to be crossed.

Figure 1.1 shows a typical Nyquist sampling of the band-limited function $x(t) = 2\cos\omega_0 t + \sin 3\omega_0 t$ and the corresponding piecewise linearly interpolated function $\hat{x}(t)$. As can be seen, $\hat{x}(t)$ does not in general capture all the "significant" detail of the original signal $x(t)$. Note, in particular, that $x(t)$ is not confined to the amplitude interval between successive samples. In addition, $\hat{x}(t)$ does not reflect the correct location or number of extrema of $x(t)$.

The above example raises several important questions. For instance, is there a sampling rate which will guarantee that $x(t)$ remains within the closed interval $[x((n-1)T), x(nT)]$ for $(n-1)T \leqslant t \leqslant nT$? If so, what is this rate and what are its implications on such problems as the estimation of level crossing profiles, amplitude distributions, and the detection of extrema? If not, what can be guaranteed about $|x(t) - x((n-1)T)|$ and $|x(t) - x(nT)|$ as a function of the sampling rate? To answer these questions, bounds on the amplitude fluctuations of band-limited signals (their slope and curvature) are essential. Moreover, these bounds should be easily related to the signal's frequency spectrum.

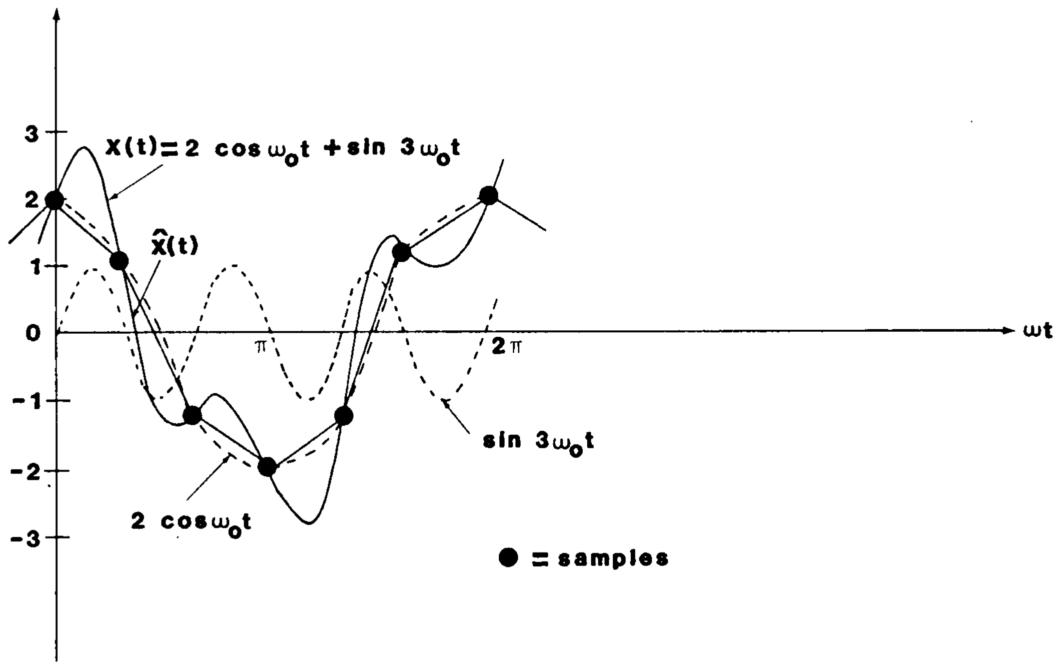


Figure 1.1. Linear interpolation with samples taken at the Nyquist rate.

This work examines the amplitude fluctuations of band-limited functions and the representation of these functions by piecewise linearly interpolated samples. Known results on the sampling of band-limited signals are reviewed and various bounds on their amplitude fluctuations are discussed. In particular, a comprehensive analysis of the average rate of change of band-limited functions is presented, including the first derivation of the least upper bound on functions for which either the zeroth absolute moments (spectral areas) or Fourier amplitude spectra of the functions are bounded. These results are then extended to the case of bounded band-limited functions. The average rate of change of a function referenced to an extremum, a new concept that measures the average rate of change of a function over an interval in which at least one end point of the interval is an extremum of the function, is also defined and similarly bounded. These bounds are subsequently used to establish a new sampling theorem which guarantees that, between successive samples of a band-limited function, the function itself does not deviate from either sample by more than some predefined amplitude change. Based on these results, several problems of practical importance, including slope overload error in delta modulation, aperture time and amplitude uncertainty in analog-to-digital conversion, the estimation of level crossing statistics, and the detection and elimination of high frequency pulses from a low frequency signal, are examined. Finally, an assessment of the work done and recommendations for future research are provided.

CHAPTER 2

BACKGROUND

2.1 Introduction

The analytical relations between time and frequency are established by the Fourier series and Fourier integral. These relations play a central role in signal theory, since signals can be characterized by both time-domain and frequency-domain parameters. The purpose of this chapter is to provide some basic definitions and known results concerning these relations, and establish a unified notation for examining the problems associated with approximating band-limited functions by linearly interpolated samples.

Some elementary concepts of Fourier analysis are first reviewed. Several analytical relations between time and frequency are then discussed. In particular, bounds on the magnitude of a signal's frequency spectrum are derived and related to the signal and its derivatives. The signal and its derivatives are then shown to be bounded by the magnitude of the frequency spectrum. The chapter ends with a survey of the literature on the sampling of band-limited signals.

2.2 Basic Concepts

The following definitions and results can be found in most textbooks on Fourier analysis [Carslaw (1930), Papoulis (1962), Arfgan (1970)] or the analysis of linear systems [Cheng (1961), Schwarz and Friedland (1965)].

2.2.1 Signals and their Fourier series

A *signal* can be defined as a function that conveys information about the state or behavior of a physical system. Signals are represented mathematically as functions of one or more independent variables, which may be either continuous or discrete. *Continuous-time signals* are defined at a continuum of times and represented by functions of continuous variables, while *discrete-time signals* are defined at discrete times and represented by sequences of numbers. The amplitude of a signal can also be either continuous or discrete. *Digital signals* are discrete in both time and amplitude; *analog* or *continuous signals* are continuous in both time and amplitude.

A continuous signal x(t), which is either periodic with period T or of interest only over a finite open interval (-T/2, T/2), can be resolved into a series of complex exponentials

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{jn\omega_0 t} \tag{2.1}$$

where the complex coefficients X_n are given by

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{-jn\omega_0 t} \, dt \tag{2.2}$$

and the *fundamental frequency* ω_0 is related to T by

$$\omega_0 = \frac{2\pi}{T} . \tag{2.3}$$

This resolution, called the *complex* or *exponential Fourier series*, can be made under the *Dirichlet conditions* [Carslaw (1930)]:

1. $x(t)$ has at most a finite number of extrema in the interval
2. $x(t)$ has at most a finite number of discontinuities in the interval
3. The integral $\int_{-T/2}^{T/2} |x(t)|dt$ is finite

These conditions guarantee uniform convergence to $x(t)$ at a point of continuity, but to $\frac{1}{2}[x(t_0^-) + x(t_0^+)]$ at a point $t = t_0$ of discontinuity.

The set of complex coefficients $\{X_n\}$ is referred to as the (discrete) *Fourier spectrum* of $x(t)$. These coefficients may be written

$$X_n = |X_n|e^{j\theta_n} . \tag{2.4}$$

The set of real amplitudes $\{|X_n|\}$ is called the *amplitude spectrum* of $x(t)$, while the set of phase values $\{\theta_n\}$ is known as the *phase spectrum*. If $x(t)$ is a real signal, $X_n = X^*_{-n}$ and the amplitude and phase spectra are even and odd functions of n, respectively.[a]

It is sometimes convenient to express $x(t)$ as a sum of sinusoids. If $x(t)$ is a real function of time, it is easily established that

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos(n\omega_0 t + \theta_n) \tag{2.5}$$

---

[a]The symbol "*" denotes the complex conjugate.

where

$$X_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt \tag{2.6}$$

is the average or d-c value of x(t). This equation may be alternately written

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t \; dt + b_n \sin n\omega_0 t) \tag{2.7}$$

with

$$a_n = 2|X_n| \cos \theta_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos n\omega_0 t \; dt \tag{2.8}$$

$$b_n = -2|X_n| \sin \theta_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \sin n\omega_0 t \; dt \tag{2.9}$$

$$\frac{a_0}{2} = X_0 \; . \tag{2.10}$$

Equation 2.7, which is called the *trigonometric Fourier series,* is related to the exponential Fourier series by

$$|X_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \tag{2.11}$$

and

$$\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \; . \tag{2.12}$$

EXAMPLE 2.1 Consider the pulse train shown in Figure 2.1(a). The complex coefficients X_n of its exponential Fourier series are

$$X_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{T}\ \frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{jn\omega_0}$$

$$= A\left(\frac{\tau}{T}\right)\frac{\sin\,(n\omega_0\tau/2)}{n\omega_0\tau/2}\ ,\ \tau < T\ . \tag{2.13}$$

The discrete Fourier spectrum of x(t) is depicted in Figure 2.1(b). □

2.2.2 Fourier transforms

An aperiodic function or signal x(t) that is defined for all time cannot be represented by a single Fourier series expansion. It is possible, however, to resolve x(t) into complex exponentials of the form $e^{j\omega t}$, where the continuous variable ω replaces the discrete variable $n\omega_0$ of the Fourier series. The resulting resolution, called the *Fourier transform* of x(t), is defined by

$$X(j\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\, dt \tag{2.14}$$

where $j = \sqrt{-1}$. If $X(j\omega)$ is integrable, the *inverse Fourier transform*

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\ e^{j\omega t}\, d\omega \tag{2.15}$$

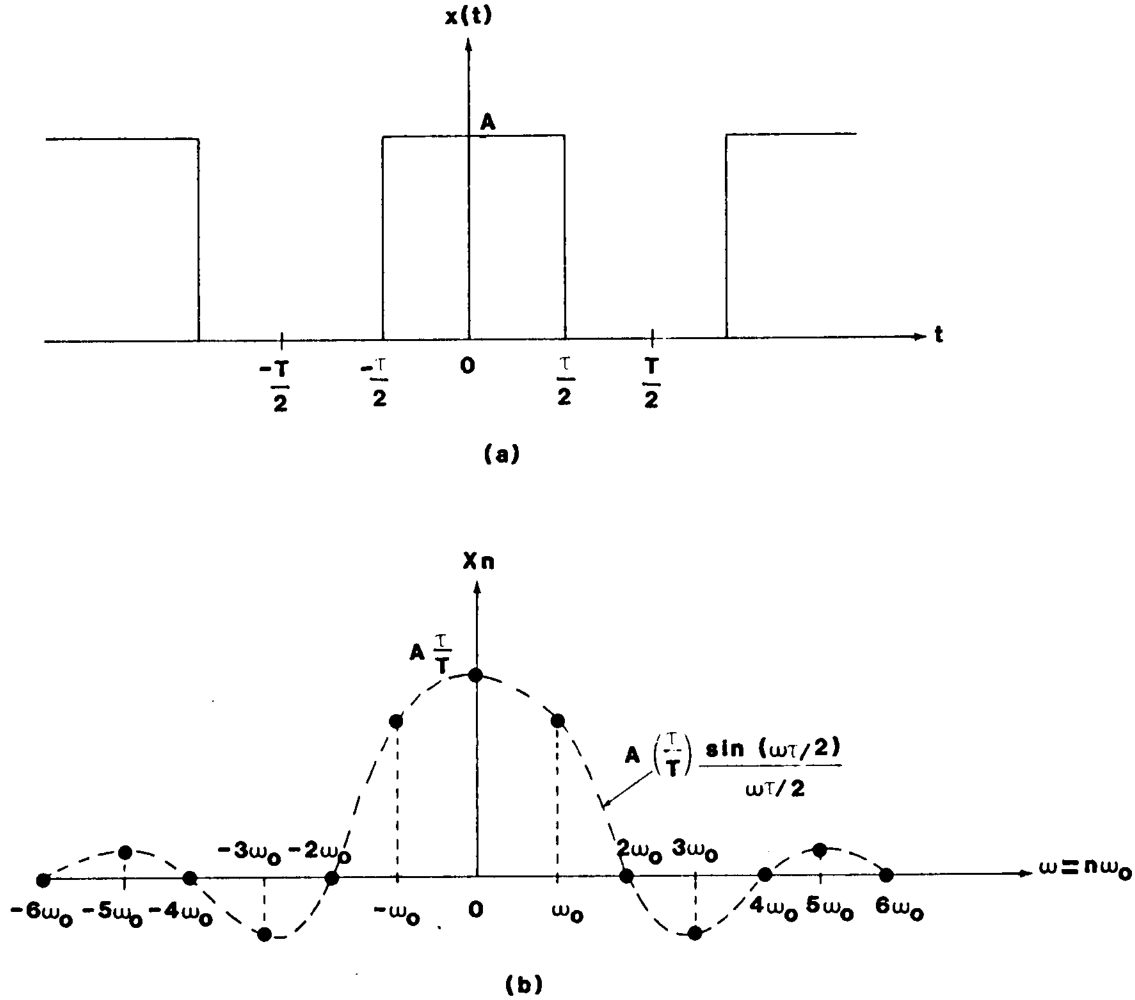


Figure 2.1. Pulse train (a) and its discrete Fourier spectrum (b).

exists and expresses $x(t)$ as the integral transform of $X(j\omega)$. Sufficient conditions for the existence of $X(j\omega)$ are that the signal $x(t)$ satisfies the Dirichlet conditions in every finite interval and

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty \, . \tag{2.16}$$

Equations 2.14 and 2.15 are called the *Fourier transform pair.* The variable ω is referred to as the *frequency* variable – a name which arises from the fact that by replacing $e^{j\omega t}$ with $\cos \omega t + j \sin \omega t$, the integrand in Equation 2.15 can be interpreted as an infinite summation of sine and cosine terms, where each value of ω determines the frequency of its corresponding sine-cosine pair. If the Fourier transform of Equation 2.14 vanishes for all values of ω (frequency) outside some closed interval [-W, W], the inverse Fourier transform of Equation 2.15 becomes

$$x(t) = \int_{-W}^{W} X(j\omega) \, e^{j\omega t} \, d\omega \tag{2.17}$$

and the function $x(t)$ is said to be *band-limited* to W rad/sec.

The magnitude of the Fourier transform $|X(j\omega)|$ is called the *Fourier spectrum* or *amplitude spectrum* of the signal $x(t)$, while the phase of $X(j\omega)$ is referred to as the *phase spectrum* of $x(t)$. The quantity $|X(j\omega)|^2$ is called the *energy density spectrum.* The mean square value of $x(t)$ is related to the energy density spectrum by *Parseval's theorem*

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \ . \tag{2.18}$$

If $x(t)$ represents the real voltage across a resistor, $x^2(t)$ is proportional to the energy dissipated in the resistor. Thus, the total energy of a real signal may be obtained by integrating the square of the magnitude of its Fourier transform over all frequency. The energy contained in a band of frequencies $\omega_1 \leqslant \omega \leqslant \omega_2$ is

$$\frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |X(j\omega)|^2 d\omega \ . \tag{2.19}$$

<u>EXAMPLE 2.2</u> The Fourier transform of a pulse of amplitude A and width τ [see Figure 2.2(a)] is

$$\begin{aligned} X(j\omega) &= \int_{-\tau/2}^{\tau/2} A\, e^{-j\omega t} dt \\ &= A \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-\tau/2}^{\tau/2} \\ &= A\tau \frac{\sin \omega\tau/2}{\omega\tau/2} \ . \end{aligned} \tag{2.20}$$

The corresponding amplitude spectrum is shown in Figure 2.2(b). Note that the spectrum is a continuous function of ω. □

2.3 Analytical Relations between Time and Frequency

Consider a signal $x(t)$ whose amplitude spectrum is shown in Figure 2.3. The *effective bandwidth* W of this spectrum may be defined [Gabor (1946)]

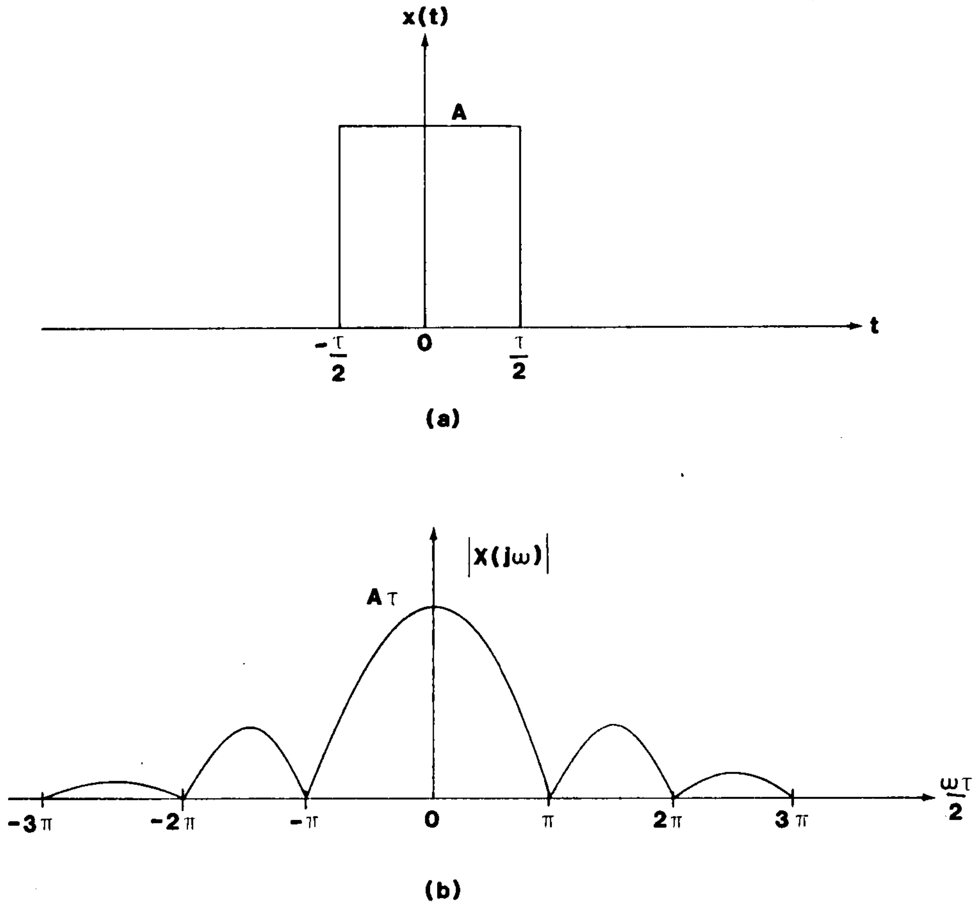


Figure 2.2. Fourier amplitude spectrum (b) of a single rectangular pulse (a).

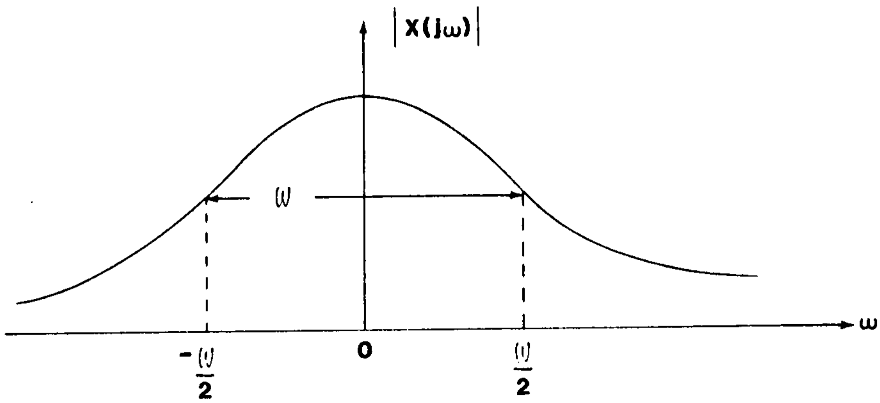


Figure 2.3. Typical amplitude spectrum.

$$\mathcal{W}^2 = \frac{\int_0^\infty \omega^2 |X(j\omega)|^2 d\omega}{\int_0^\infty |X(j\omega)|^2 d\omega} . \tag{2.21}$$

Only positive frequencies are included in this definition since the behavior of $X(j\omega)$ at positive frequencies completely determines any real $x(t)$. A qualitative relation between this measure of signal bandwidth and the "shape" of $x(t)$ can be obtained by observing that

$$F\left[\frac{d^n x}{dt^n}\right] = (j\omega)^n \, X(j\omega) \tag{2.22}$$

so that [Schwartz and Friedland (1965)]

$$\mathcal{W}^2 = \frac{\int_{-\infty}^\infty \left(\frac{dx}{dt}\right)^2 dt}{\int_{-\infty}^\infty x^2(t)\, dt} . \tag{2.23}$$

Thus, the effective bandwidth of $x(t)$ is a measure of its "roughness." A signal which is discontinuous has infinite bandwidth, since its derivative has an impulse whose square integral is infinite.

The effective bandwidth of $x(t)$ may be alternately defined as [Ziemer and Tranter (1976)]

$$\mathcal{W} = \frac{1}{X(jo)} \int_{-\infty}^\infty |X(j\omega)| d\omega . \tag{2.24}$$

This definition, which is based on the approximation of $|X(j\omega)|$ by a rectangular spectrum of equal area and height $X(jo)$ [see Figure 2.4(b)], is particularly useful in problems involving pulse transmission, where the detection and resolution of pulses at a filter output is of interest. In Figure 2.4(a), a pulse with a single maximum at $t = 0$ is shown with a rectangular approximation of height $x(o)$ and duration T. The approximating pulse and $|x(t)|$ have equal areas so that

$$T\,x(o) = \int_{-\infty}^{\infty} |x(t)|dt \geqslant \int_{-\infty}^{\infty} x(t)dt = X(jo) \tag{2.25}$$

where the relationship

$$X(jo) = F[x(t)]\Big|_{t=0} = \int_{-\infty}^{\infty} x(t)e^{-j\omega(o)}dt \tag{2.26}$$

has been used. Similarly, from Figure 2.4(b),

$$\mathcal{W}\,X(jo) = \int_{-\infty}^{\infty} |X(j\omega)|d\omega \geqslant \int_{-\infty}^{\infty} X(j\omega)d\omega = x(o)\ . \tag{2.27}$$

Equations 2.25 and 2.27 result in the pair of inequalities

$$\frac{x(o)}{X(jo)} \geqslant \frac{1}{T} \tag{2.28}$$

and

$$\mathcal{W} \geqslant \frac{x(o)}{X(jo)} \tag{2.29}$$

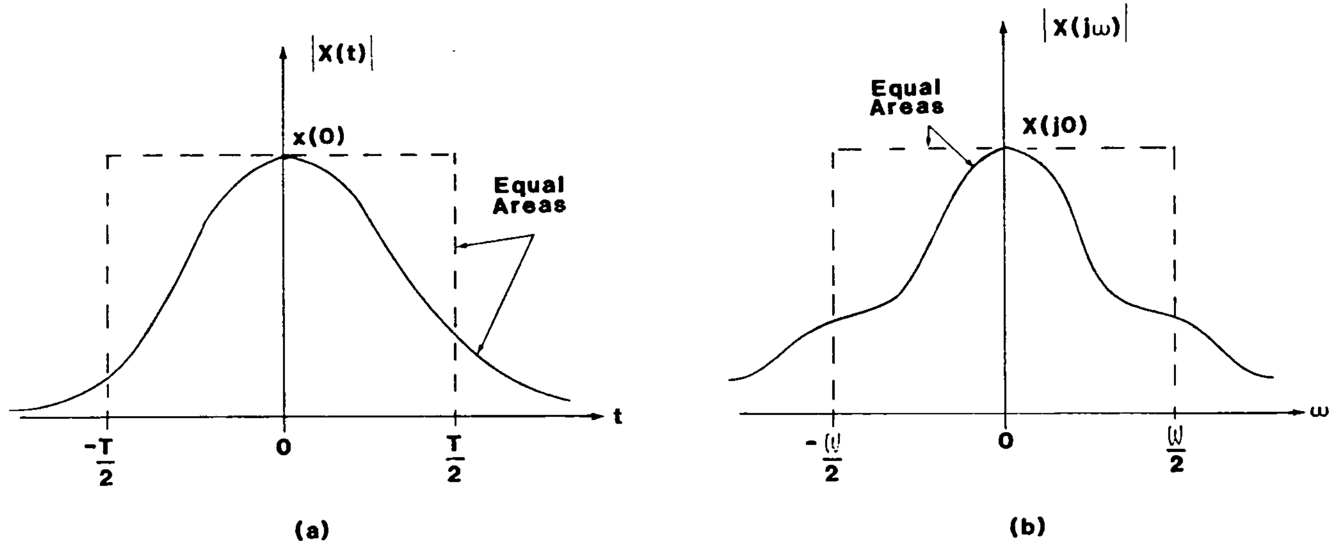


Figure 2.4. Arbitrary pulse signal (a) and spectrum (b).

which, when combined, result in the pulse duration bandwidth relationship

$$W \geqslant \frac{1}{T} . \quad (2.30)$$

This result agrees with the commonly used rule of thumb that the shape of a pulse will be preserved by a system whose 3-dB frequency is approximately equal to the reciprocal of the pulse width [Millman and Taub (1965)].

Equations 2.23 and 2.30 are indicative of the known analytical relations between time and frequency. These relations are usually based on the *differentiation property* (Equation 2.22) of the Fourier transform. Using this property, bounds on $|X(j\omega)|$ can be related to x(t) and its derivatives, or x(t) and its derivatives can be bounded by $|X(j\omega)|$.

2.3.1 Bounds on the Fourier spectrum $|X(j\omega)|$

The following definitions and results are based on Mix (1969). They are useful for determining the approximate spectrum of a signal x(t).

The *content* C_x, *variation* V_x, and *wiggliness* W_x of x(t) are defined as

$$C_x = \int_{-\infty}^{\infty} |x(t)|dt \quad (2.31)$$

$$V_x = \int_{-\infty}^{\infty} \left|\frac{dx}{dt}\right|dt \quad (2.32)$$

$$W_x = \int_{-\infty}^{\infty} \left|\frac{d^2x}{dt^2}\right|dt . \quad (2.33)$$

The content C_x is the total area under x(t), while the variation V_x represents the total change in amplitude of the function. The wiggliness W_x is a measure of the total change in the slope of x(t).

In accordance with the differentiation property, Equation 2.22, the frequency spectrum of x(t) can be related to its derivatives by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.34}$$

$$j\omega\ X(j\omega) = \int_{-\infty}^{\infty} \frac{dx}{dt}\ e^{-j\omega t}dt \tag{2.35}$$

$$-\omega^2\ X(j\omega) = \int_{-\infty}^{\infty} \frac{d^2x}{dt^2}\ e^{-j\omega t}dt\ . \tag{2.36}$$

Taking the absolute value of both sides of these equations and using the relationship

$$\left| \int_a^b f(t)\ g(t)\ dt \right| \leqslant \int_a^b \left|f(t)\right|\ \left|g(t)\right|dt\ , \tag{2.37}$$

they become

$$\left|X(j\omega)\right| \leqslant \int_{-\infty}^{\infty} \left|x(t)\right|dt = C_x \tag{2.38}$$

$$|\omega|\ \left|X(j\omega)\right| \leqslant \int_{-\infty}^{\infty} \left|\frac{dx}{dt}\right|dt = V_x \tag{2.39}$$

$$\omega^2 \left|X(j\omega)\right| \leqslant \int_{-\infty}^{\infty} \left|\frac{d^2x}{dt^2}\right| dt = W_x \tag{2.40}$$

or

$$\left|X(j\omega)\right| \leqslant \begin{cases} C_x \\ V_x/|\omega| \\ W_x/\omega^2 \end{cases} . \tag{2.41}$$

EXAMPLE 2.3 A signal x(t) and its derivatives are shown in Figure 2.5. According to Equation 2.41, the spectrum of x(t) is bounded by

$$\left|X(j\omega)\right| \leqslant \begin{cases} 3 \\ 2/|\omega| \\ 4/\omega^2 \end{cases} . \tag{2.42}$$

Figure 2.6 illustrates these bounds with the magnitude of $X(j\omega)$ superimposed. □

2.3.2 Bounds on a signal x(t)

The definitions and results of this section are based on the book by Papoulis (1962). It is shown that a signal x(t) and its derivatives are bounded by the absolute moments of its Fourier spectrum.

The *nth absolute moment* M_n of $X(j\omega)$ is defined by

$$M_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^n \left|X(j\omega)\right| d\omega . \tag{2.43}$$

Using this definition and Equation 2.22, it is easily verified that

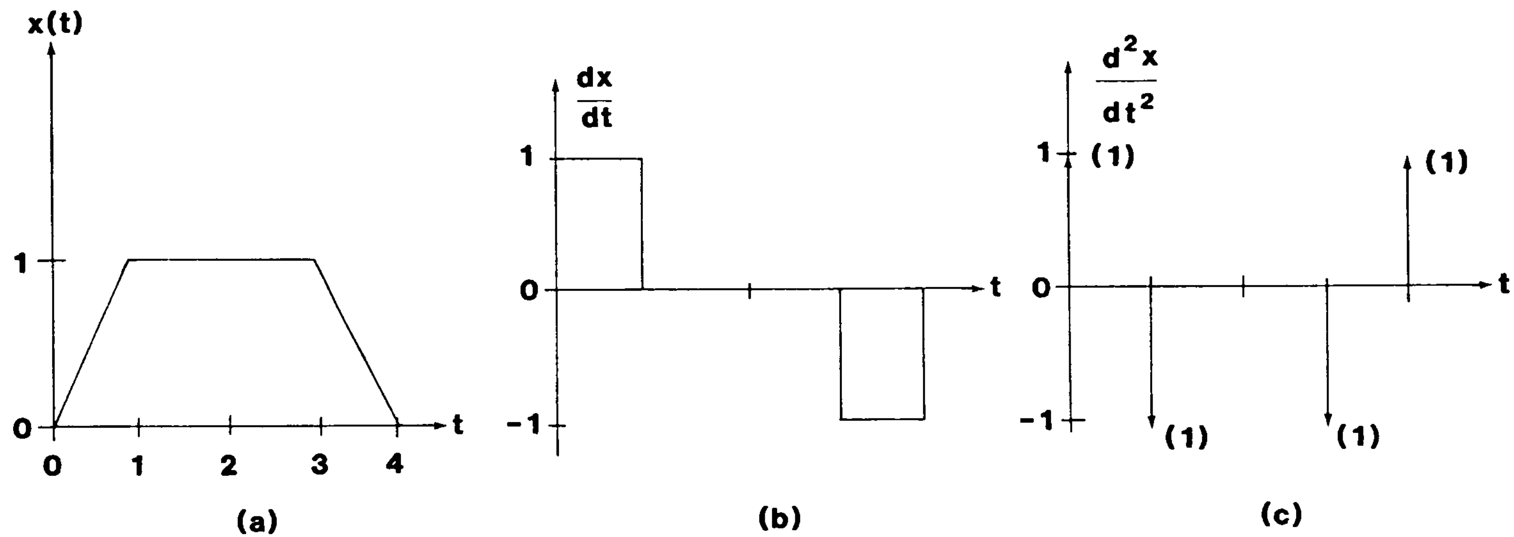


Figure 2.5. A signal x(t) and its derivatives.

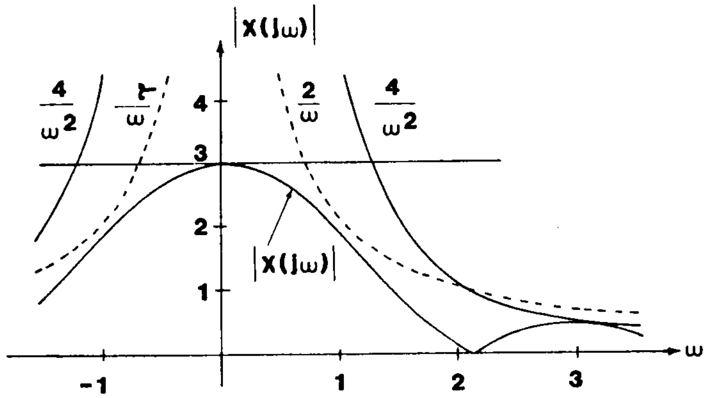


Figure 2.6. The spectrum of the signal shown in Figure 2.5.

$$\left|\frac{d^n x}{dt^n}\right| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega)^n X(j\omega)e^{j\omega t}d\omega\right| \leqslant M_n \ . \tag{2.44}$$

The following estimate of the variation of x(t) is a direct consequence of this result:

$$\left|x(t_2) - x(t_1)\right| \leqslant M_1 \left|t_2 - t_1\right| \ . \tag{2.45}$$

This can be shown in integrating dx/dt from t_1 to t_2 and using Equation 2.44 or, more directly, from

$$x(t_2) - x(t_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)[e^{j\omega t_2} - e^{j\omega t_1}]d\omega \tag{2.46}$$

and the inequality (see Figure 2.7)

$$\left|e^{j\omega t_2} - e^{j\omega t_1}\right| = 2 \sin \left|\frac{\omega(t_2 - t_1)}{2}\right| \leqslant \left|\omega(t_2 - t_1)\right| \ . \tag{2.47}$$

If x(t) is band-limited to W rad/sec,

$$M_1 = \int_{-W}^{W} |\omega| \left|X(j\omega)\right| d\omega \leqslant W \int_{-W}^{W} \left|X(j\omega)\right| d\omega = W\ M_0 \tag{2.48}$$

and

$$\left|x(t_2) - x(t_1)\right| \leqslant W\ M_0 \left|t_2 - t_1\right| \ . \tag{2.49}$$

Equation 2.44 reveals that the nth derivative of x(t) is bounded by the nth absolute moment of $X(j\omega)$; Equation 2.49 simply indicates that

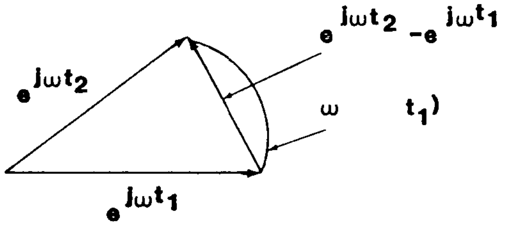


Figure 2.7. An approximation of $[e^{j\omega t_2} - e^{j\omega t_1}]$.

the average rate of change of $x(t)$ must be less than its instantaneous rate of change, which is bounded by $M_1 \leqslant W\, M_0$. A similar bound on the instantaneous derivative of $x(t)$ is given in Stark and Tuteur (1979).

EXAMPLE 2.4 Consider the modulated signal $x(t)\cos\omega_0 t$. If the spectrum of its envelope is band-limited to W rad/sec and $W \ll \omega_0$, the variation of $x(t)$ within the period $T = 2\pi/\omega_0$ of the carrier is small. A quantitative estimate of this variation can be obtained from Equation 2.49:

$$\left| x(t + T) - x(t) \right| \leqslant WM_0T = 2\pi M_0\left(\frac{W}{\omega_0}\right) \tag{2.50}$$

□

2.4 Sampling of Continuous-Time Signals

As mentioned in Chapter 1, one of the most important consequences of finite signal bandwidth is that continuous signals can be sampled and quantized for subsequent processing by digital systems. The problem of effective conversion of continuous information to digital data has received considerable attention in the literature and is currently a field of active research. Interest in this field may be traced back to Nyquist (1928).

2.4.1 Sampling for optimal signal reconstruction

The following survey of sampling theorems for optimal signal reconstruction is based on a tutorial review by Jerri (1977) of the various extensions and applications of the Shannon sampling theorem. Conventional approaches to the selection of sampling rates and quantization intervals are based on minimizing the mean square error between the sampled function and an estimate which is reconstructed from the samples. Since Shannon's theorem guarantees perfect reconstruction via Equation 1.1, most sampling theorems for optimal signal reconstruction are extensions of Shannon's basic result, which may be stated as follows:

THEOREM 2.1 (Shannon Sampling Theorem) If a function $x(t)$ contains no frequencies higher than W rad/sec it is completely determined by giving its ordinates at a series of points spaced π/W sec apart. □

A sequence $x(n)$ with values $x(n) = x(nT)$ is said to be derived from $x(t)$ by periodic sampling, and $T = \pi/W$ is called the *sampling period*. The reciprocal of T is referred to as the *sampling frequency* or *sampling rate*.

Shannon's sampling theorem was extended by Parzen (1956) to include the sampling of band-limited functions of n variables. This result was later extended by Miyakawa (1959) to include stationary stochastic variables in n dimensions. In Peterson and Middleton (1962), a detailed treatment of the sampling of both the amplitude and gradient of n-dimensional stochastic fields is given. The following theorem from Reza (1961) is indicative of these results:

THEOREM 2.2 Let $x(t_1, t_2, \ldots, t_n)$ be a function of n real variables, whose n-dimensional Fourier integral exists and is identically zero outside an n-dimensional rectangle which is symmetrical about the origin; that is,

$$g(y_1, y_2, \ldots, y_n) = 0, \ |y_k| > |W_k|, \ k = 1, 2, \ldots, n \ . \tag{2.51}$$

Then

$$x(t_1, t_2, \ldots, t_n) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} x\left(\frac{\pi m_1}{W_1}, \ldots, \frac{\pi m_n}{W_n}\right)$$

$$\frac{\sin(\omega_1 t_1 - m_1\pi)}{\omega_1 t_1 - m_1\pi} \cdots \frac{\sin(\omega_n t_n - m_n\pi)}{\omega_n t_n - m_n\pi} \ . \tag{2.52}$$

□

When Shannon introduced the sampling theorem, he remarked that the value of x(t) could also be reconstructed from a knowledge of x(t) and its derivative $x'(t)$ at every other sample point, and then extended his remarks to higher derivatives. This was later considered by Fogel (1955), who stated and proved the following theorem:

THEOREM 2.3 If a function x(t) contains no frequencies higher than W rad/sec it is determined by giving M function derivative values at each of a series of points extending throughout the time domain, the sampling interval $T = M\pi/W$ being the time interval between instantaneous observations. □

A generalization of the above result and an explicit procedure for reconstructing x(t) when the value of the function and its first R derivatives are given at equidistant sampling points $(R + 1)\pi/W$ sec apart was presented by Linden (1959). Sampling with derivatives increases the sample spacing required, allowing the reconstruction of a band-limited signal with a sampling rate less than the Nyquist rate. The estimated velocity and position of an aircraft, for example, can be used to determine a continuous course plot of the path with half the Nyquist rate.

Another extension of the Shannon sampling theorem was considered by Balakrishnan (1957), who showed that the sampling theorem can be used to represent a random process of a continuous-time parameter. The following theorem is indicative of his results:

<u>THEOREM 2.4</u> Let $x(t)$, $-\infty < t < \infty$, be a real or complex valued stochastic process, stationary in the "wide sense" (or second-order stationary), possessing a spectral density which vanishes outside the interval $[-W, W]$. Then $x(t)$ has the representation

$$x(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{-N}^{N} x\left(\frac{n\pi}{W}\right) \frac{\sin(Wt - n\pi)}{Wt - n\pi} \tag{2.53}$$

for every t, where l.i.m. stands for limit in the mean square. □

The proof of this theorem consists of using Shannon's sampling theorem for the covariance function of the process, since it is assumed to have a band-limited Fourier transform. Then the optimal estimate of $x(t)$ is constructed (by using the sampling series) to show that the mean square error is zero. The optimal reconstruction of multidimensional random fields has been examined by Petersen (1963). Sampling theorems for nonstationary random processes have also been discussed in the literature [Zakai (1965), Piranashvili (1967), Gardner (1972)].

Sampling expansions for bandpass functions which lie in a frequency band $(W_0, W_0 + W)$, instead of the usual low-pass function with band limits $(-W, W)$, were first considered by Kohlenberg (1953). He introduced "second order sampling," which involves two interleaved sequences of equispaced sampling points. In general, *pth-order sampling* is defined as

$$g(t) = \sum_{i=1}^{p} g_i(t) = \sum_{i=1}^{p} \sum_{n} x(a_i n + k_i)\ S_i(t - a_i n - k_i) \tag{2.54}$$

in which the ith sampling series $g_i(t)$ has *sample spacing* a_i, *phase* k_i, and *sampling function* $S_i(t)$.

<u>THEOREM 2.5</u> For a function $x(t)$ in a band of frequencies $(W_0, W_0 + W)$, the exact interpolation formula is

$$x(t) = \sum_{n} \left[x\left(\frac{2\pi n}{W}\right) S\left(t - \frac{2\pi n}{W}\right) + x\left(\frac{2\pi n}{W}\right) + x\left(\frac{2\pi N}{W} + k\right) S\left(\frac{2\pi n}{W} + k - t\right)\right] \tag{2.55}$$

where

$$S(t) = \frac{\cos[(W_0 + W)t - (r + 1)Wk/2] - \cos[(rW - W_0)t - (r + 1)Wk/2]}{Wt \sin(r + 1)Wk/2}$$

$$+ \frac{\cos[(rW - W_0)t - rWk/2] - \cos[W_0 t - rWk/2]}{Wt \sin rWk/2} . \tag{2.56}$$

□

In Equation 2.55, there are two groups of samples, each taken at a rate of W samples per second. The two sample groups are shifted by a phase k from each other, where k is a constant such that kWr/2, kW(r + 1)/2

$\neq 0, 1, \ldots$, and r is an integer such that $2W_0/W < r < 2W_0/W + 2\pi$. The sampling of bandpass functions is also discussed by Middleton (1960), Linden (1959), and Parzen (1956).

The above extensions of the Shannon sampling theorem may be termed *explicit* samplings in the sense that a band-limited function $x(t)$ is represented in terms of its samples $x(t_n)$ at preselected instances $\{t_n\}$ which are independent of $x(t)$. *Implicit sampling* refers to those cases in which a function $x(t)$ is represented in terms of the instants $\{t_n\}$ at which $x(t)$ assumes a predetermined value — for example, its zero crossings $\{t_n: x(t_n) = 0\}$ or its crossings with a cosine function $\{t_n: x(t_n) = C \cos 2\pi\omega t_n\}$. The first implicit sampling expansion is attributed to Bond and Cahn (1958), who considered the expansion of $x(t)$ from its zero crossings. The important case of implicit sampling in terms of real variables only was considered by Bar-David (1974). He considered bounded functions which are band-limited in the usual sense or as extended by Zakai (1965) to provide the following implicit sampling theorem:

THEOREM 2.6 Let $x(t)$ be bounded, $|x(t)| < A$, and of bandwidth W. Let $\{t_k\}$ denote the set of instants for which $x(t) = C \cos 2\pi\omega t$ where $C > A$ and $\omega > W$. Then $x(o)$ and the set $\{t_k\}$ determine $x(t)$ uniquely. □

The extension of the sampling theorem to the prediction of band-limited processes from past samples was examined by Brown (1972), who considered a signal $x(t)$ (deterministic or stochastic) which is band-limited to the frequency interval $|\omega| \leqslant \pi$ with $|X(j\omega)|^2$ integrable on

$[-\pi, \pi]$. He showed that for any constant sampling interval $0 < T < 1/2$, $x(t)$ can be approximated arbitrarily well by a linear combination of past samples $x(t - kT)$; that is

$$\lim \left| x(t) - \sum_{k=1}^{n} a_{kn}\, x(t - kT) \right| = 0 \tag{2.57}$$

for $-\infty < t < \infty$. The coefficients $a_{kn} = (-1)^{k+1}(\cos \pi T)^k \binom{n}{k}$ are independent of the detailed structure of $x(t)$.

A sampling expansion for $x^2(t)$, rather than $x(t)$, is discussed in Papoulis (1968), where it is shown that

$$x^2(t) = \sum_{n=-\infty}^{\infty} x^2(nT) \frac{\sin W_0(t - nT)}{W_2(t - nT)} \tag{2.58}$$

where $W_2 = \pi/T \geqslant 2W$ and W_0 is such that $2W < W_0 < 2W_2 - 2W$. Note that with $T \leqslant \pi/2W$, Equation 2.58 requires more than double the usual sampling rate.

Other extensions of the Shannon sampling theorem include sampling with nonuniformly spaced sampling points [Black (1953), Yen (1956)] and the sampling of generalized functions or distributions [Campbell (1968), Pfaffelhuber (1971)]. Shannon's theorem has also been extended for the analysis of continuous signals specified by time-varying spectra with time-varying bands [Horiuchi (1968)]. Several other extensions of this theorem are included in a paper by Jerri (1977), which has a rather exhaustive bibliography.

2.4.2 Other criteria for sampling rate selection

While considerable interest has been devoted to determining the sampling rates under which a continuous function may be perfectly reconstructed from its samples, several other criteria for sampling rate selection have been proposed. Some of these criteria will now be reviewed.

The selection of optimal sampling and quantization intervals for detecting impulse waveforms was considered by Knight (1967), who based this selection on the impulse response of the system producing the signal to be sampled. The essence of his approach is illustrated in Figure 2.8. A digitizer, which consists of a sampler with period T and quantizer that divides the signal into $1/q$ intervals between 0 and 1, produces $N = -\log_2 q$ bit samples of the impulse waveform $s(t)$ at a bit rate $R = N/T$. The impulse waveform intersects level q at T_1 and T_2. If k samples of the pulse are to be required for pulse detection, the sampling period T becomes $T = (T_2 - T_1)/k$. Calling q the threshold of the lowest quantizing level establishes the number of bits per sample N, and N/T gives the required bit rate R. The essential question is how does R vary, and is there a q which minimizes R for a given $s(t)$ and k? As was indicated by Knight (1967), who solved the problem for the impulse response $s(t) = n\, e^{-t}u(t)$, this question can be answered analytically for certain simple $s(t)$, since N and T can then be represented by explicit functions of q. Thus, R becomes a closed form function of one variable (q for a given k) which can be adjusted to minimize R. The important concept here is that sampling rates need not be chosen independently of quantization intervals, as is usually the case.

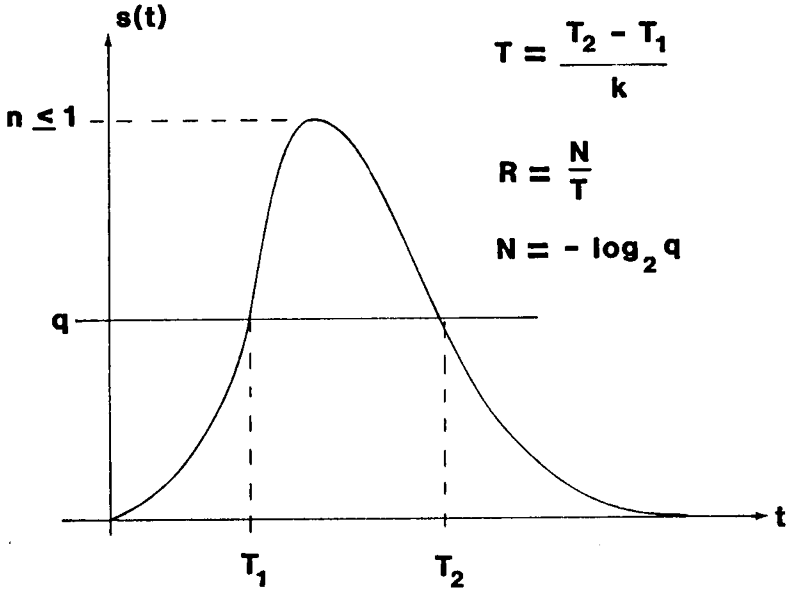


Figure 2.8. Choosing q to minimize R.

Along these same lines, Curry (1976) examined the problem of locating the "center" of band-limited pulses by sampled template matching. Given a band-limited pulse of known shape occurring randomly in time, he experimentally determined the number of samples that must be taken to accurately locate the center of the pulse. To locate the center, a signal sampled at the experimentally determined rate is matched with a previously stored replica of the pulse.

Another criterion for sampling rate selection was considered by Shiva (1965), who examined sampling from the point of view of the piecewise approximation of functions. In particular, he indicated that the choice of sampling interval T is dependent on the degree n of the polynomial assumed for piecewise approximation. To show this, only those functions x(t) that could be piecewise approximated by an nth degree polynomial

$$x(t + T) = x(t) + \frac{dx}{dt} T + \frac{d^2x}{dt^2} \frac{T^2}{2!} + \ldots + \frac{d^nx}{dt^n} \frac{T^n}{n!} \qquad (2.59)$$

and represented by a truncated trigonometric Fourier series[a] (Equation 2.7) were considered. Substituting Equation 2.59 into 2.7, he showed that the resulting series is valid for

$$\left(\frac{T}{T_N}\right)^{n+1} = C \frac{(n + 2)!}{\pi^{n+1}} \text{, n odd, } C \ll 1 \qquad (2.60)$$

---

[a]All band-limited signals cannot be represented by a truncated Fourier series for all t.

$$\left(\frac{T}{T_N}\right)^{n+2} = C\,\frac{(n + 2)!}{\pi^{n+2}}\ , \ n \text{ even}, \ C \ll 1 \tag{2.61}$$

where T_N represents the Nyquist sampling interval $\pi/N\omega_0$. For $n = 1$ (which corresponds to approximating $x(t)$ with a series of ramps) and $C = 0.001 \ll 1$, $T/T_N = 0.0245$ and a sampling rate about 40 times greater than the Nyquist rate is required. No indication of the accuracy of the piecewise approximation was provided.

Other criteria for sampling rate selection have been proposed for deconvolution in image restoration [McDonnel (1975)] and the discrete-time modeling of continuous-time systems [Liff and Wolf (1966)]. In Liff and Wolf (1966), the proposed sampling rates are shown to be a function of the parameters of the system to be modeled. For a system with a simple real pole α in the s-plane, for example, the optimum sampling interval T is shown to be $-1/\alpha$ for $\alpha < 0$.

2.5 Summary

In this chapter, the analytical relations between time and frequency have been reviewed. Since these relations are established by the Fourier series and transform, the fundamental concepts of Fourier analysis were first presented. Several properties arising from the fact that signals are generally band-limited were also discussed. In particular, a survey of the literature on sampling, one of the most important consequences of finite signal bandwidth, was presented. Some of these results were seen to be based on sampling criteria other than optimal signal reconstruction, the fundamental criterion behind the Shannon sampling theorem and its extensions. While signals cannot be strictly confined to a finite time and frequency interval, one can define reasonable confinement intervals in practice.

CHAPTER 3

BOUNDS ON THE RATE OF CHANGE OF FREQUENCY AND AMPLITUDE LIMITED FUNCTIONS

3.1 Introduction

Consider again the band-limited function $x(t) = 2 \cos \omega_0 t + \sin 3\omega_0 t$ and its piecewise linearly interpolated estimate $\hat{x}(t)$ shown in Figure 1.1, page 4. In Chapter 1, it was noted that $\hat{x}(t)$ does not in general capture all the detail of $x(t)$. Recall, for instance, that $x(t)$ is not confined to the amplitude interval between successive samples, and that $\hat{x}(t)$ does not reflect the correct location or number of extrema of $x(t)$. Thus, as previously mentioned, it is of interest to determine what can be guaranteed about the absolute differences $|x(t) - x((n - 1)T)|$ and $|x(nT) - x(t)|$ for an arbitrary T and $(n - 1)T \leqslant t \leqslant nT$. Bounds on these absolute differences provide useful estimates of the accuracy with which a band-limited function can be approximated by its piecewise linearly interpolated samples.

Figure 3.1 shows both $|x(t) - x((n - 1)T)|$ and $|x(nT) - x(t)|$ for two important cases of interest. In Figure 3.1(a), $x(t)$ is depicted as a monotonically increasing function of t in the interval $[(n - 1)T, nT]$, while in Figure 3.1(b), $x(t)$ has an extremum at some $t = t_0$ in this interval. The purpose of this chapter is to develop analytical bounds on $|x(t) - x((n - 1)T)|$ and $|x(nT) - x(t)|$ for each of these cases.

Least upper bounds on the average rate of change of $x(t)$ are first derived for two sets of band-limited functions — $\{x(t)$: M_0

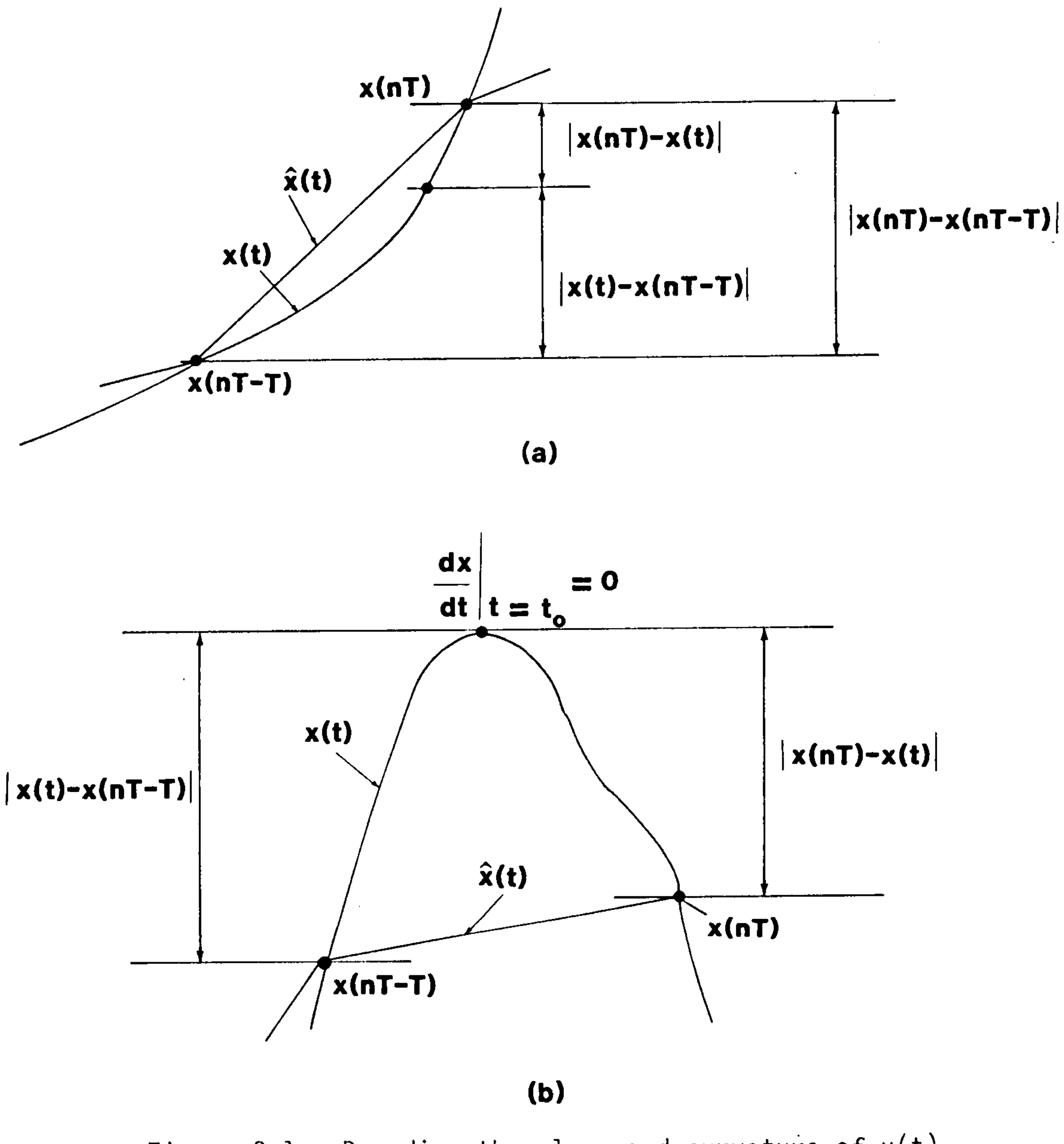


Figure 3.1. Bounding the slope and curvature of x(t).

$\leqslant A\}$ and $\{x(t)\colon |X(j\omega)| \leqslant f(\omega),\ f(\omega) \geqslant 0\}$. The average rate of change of x(t) referenced to an extremum, a new concept that measures the average rate of change of a function over an interval in which at least one end point of the interval is an extremum of the function, is then introduced and shown to be similarly bounded. Together, these two bounds provide limits on both the slope and curvature of x(t) — and on the absolute differences $|x(t) - x((n-1)T)|$ and $|x(nT) - x(t)|$ shown in Figure 3.1. Such bounds are also developed for bounded band-limited functions.

3.2 Bounds on the Rate of Change of Band-Limited Functions with Bounded Spectral Area

3.2.1 A bound on the instantaneous and average rate of change

Consider a band-limited function x(t) (as defined by Equation 2.17) in some open interval I containing the point $t = t_0$. Since it is customary to denote the difference $t_2 - t_1$ by Δt (where Δ is regarded as a *difference operator*), let $t_0 + \Delta t$ denote a second point of I, distinct from t_0 for $\Delta t \neq 0$. The difference $x(t_0 + \Delta t) - x(t_0)$ between the corresponding values of x(t) is denoted by Δx and the quotient $\Delta x/\Delta t$, called a *difference quotient,* is defined by [Tierney (1969)]

$$\frac{\Delta x}{\Delta t} = \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}\,. \tag{3.1}$$

The difference quotient of Equation 3.1 denotes the *average rate of change* of the function x(t), with respect to t, corresponding to or over the interval Δt. Thus, $\Delta x/\Delta t$ is a measure of the rate at which x is changing with respect to t for the interval Δt; it is an average in the sense that

$$\frac{\Delta x}{\Delta t} = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \left(\frac{dx}{dt}\right) dt \ . \tag{3.2}$$

In other words, $\Delta x/\Delta t$ is the average value of the derivative of x(t) over the interval Δt.

The following expression for the average rate of change of a band-limited function x(t) is a direct consequence of this definition:

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \frac{1}{\pi|\Delta t|}\int_{-W}^{W} \left|X(j\omega)\right| \ \left|\sin\frac{\omega\Delta t}{2}\right| d\omega \ , \ \left|\Delta t\right| > 0 \ . \tag{3.3}$$

This can be shown by substituting Equation 2.17 into 3.1 to obtain

$$\frac{\Delta x}{\Delta t} = \frac{1}{2\pi\Delta t}\int_{-W}^{W} X(j\omega) \ \left[e^{j\omega(t_0+\Delta t)} - e^{j\omega t_0}\right] d\omega \tag{3.4}$$

and observing that

$$\begin{aligned} e^{j\omega(t_0+\Delta t)} - e^{j\omega t_0} &= e^{j\omega(t_0+\Delta t/2)} \left[e^{j\omega\ \Delta t/2} - e^{-j\omega\ \Delta t/2}\right] \\ &= 2j\ e^{j\omega(t_0+\Delta t/2)} \sin\frac{\omega\Delta t}{2} \end{aligned} \tag{3.5}$$

so that

$$\frac{\Delta x}{\Delta t} = \frac{j}{\pi \Delta t} \int_{-W}^{W} X(j\omega)\ e^{j\omega(t_0+\Delta t/2)}\ \sin \frac{\omega \Delta t}{2}\, d\omega\ . \tag{3.6}$$

Equation 3.3 then follows by taking the absolute value of both sides of this expression and using Equation 2.37.

If $|\omega\Delta t/2|$ in Equation 3.3 is less than or equal to $\pi/2$ for some Δt and all ω in the interval $[-W, W]$, $\sin \omega\Delta t/2$ is a monotonically increasing function of ω for $0 \leqslant \omega \leqslant W$. Since $|\sin \omega\Delta t/2|$ is also an even function of ω, the maximum value of $|\sin \omega\Delta t/2|$ occurs when $\omega = \pm W$ and

$$\left|\sin \frac{\omega \Delta t}{2}\right| \leqslant \sin \frac{W|\Delta t|}{2} \tag{3.7}$$

for all ω. Thus, Equation 3.3 reduces to

$$\begin{aligned} \left|\frac{\Delta x}{\Delta t}\right| &\leqslant \frac{1}{\pi|\Delta t|} \int_{-W}^{W} \left|X(j\omega)\right|\ \sin \frac{W|\Delta t|}{2}\, d\omega \\ &\leqslant \frac{2}{|\Delta t|} \left[\frac{1}{2\pi} \int_{-W}^{W} \left|X(j\omega)\right| d\omega\right] \sin \frac{W|\Delta t|}{2} \end{aligned} \tag{3.8}$$

where the inequality

$$\int_{a}^{b} |A(t)|\ |f(t)|dt \leqslant \int_{a}^{b} |B(t)||f(t)|dt\ ,\ |A(t)| \leqslant |B(t)| \tag{3.9}$$

has been used. The quantity $1/2\pi \int_{-W}^{W} |X(j\omega)|d\omega$ in Equation 3.8 is proportional to the total area of the Fourier spectrum of x(t). Letting M_0 represent this quantity, which is actually the zeroth absolute moment of $X(j\omega)$ as defined by Equation 2.43,

$$M_0 = \frac{1}{2\pi} \int_{-W}^{W} |X(j\omega)|d\omega \ , \tag{3.10}$$

it follows that

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant WM_0 \left[\frac{\sin \frac{W|\Delta t|}{2}}{\frac{W|\Delta t|}{2}}\right] \tag{3.11}$$

for all $W|\Delta t|/2 \leqslant \pi/2$ or $|\Delta t| \leqslant \pi/W$. When $|\omega\Delta t/2| \geqslant \pi/2$ (in Equation 3.3) for some Δt and ω in the interval $[-W, W]$, $|\sin \omega\Delta t/2| = 1$, its maximum value, for more than one value of ω and Equation 3.3 becomes

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \frac{2M_0}{|\Delta t|} \ , \quad |\Delta t| \geqslant \frac{\pi}{W} \tag{3.12}$$

While Equation 3.1 can be used to compute the average rate of change of x with respect to t for various values of $\Delta t \neq 0$, it cannot be used to compute the rate at which x is changing with respect to t at $t = t_0$, regardless of how small Δt becomes. Consequently, the average rate of change of Equation 3.11 was not defined for $\Delta t = 0$. The difficulty in defining the instantaneous rate of change of x with respect to t at $t = t_0$ is that there is no interval Δt over which to take the average.

If, however, the quotient $\Delta x/\Delta t$ approaches a limit as Δt approaches zero, it is only natural to define the *instantaneous rate of change* of x with respect to t at $t = t_0$ as this limit, which is in fact the derivative of $x(t)$ with respect to t at $t = t_0$. Thus, for a band-limited function $x(t)$ whose average rate of change is bounded by Equation 3.11, it is evident that

$$\left|\frac{dx}{dt}\right| = \lim_{|\Delta t|\to 0} \left|\frac{\Delta x}{\Delta t}\right| \leqslant WM_0 \tag{3.13}$$

where the relationship $\lim\limits_{t\to 0} (\sin t)/t = 1$ has been used. Note that this bound is equivalent to the one established by Equations 2.44 and 2.48 in Section 2.3.2.

Equations 3.11, 3.12, and 3.13 bound the magnitude of the instantaneous and average rate of change of a band-limited function $x(t)$ with Fourier transform $X(j\omega)$ for any interval Δt. All three results are incorporated into the following theorem:

<u>THEOREM 3.1</u> Let the function $x(t)$ with Fourier transform $X(j\omega)$ be such that (i) $X(j\omega) = 0$ for $|\omega| > W$ and (ii) $M_0 = 1/2\pi \int_{-W}^{W} |X(j\omega)|d\omega \leqslant A$ for some constant $A > 0$. Then the instantaneous and average rate of change of $x(t)$ over any interval Δt satisfies the inequality

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \begin{cases} WA\ , \ \Delta t = 0 \\ WA\left[\dfrac{\sin \frac{W|\Delta t|}{2}}{\frac{W|\Delta t|}{2}}\right],\ 0 < |\Delta t| \leqslant \frac{\pi}{W} \\ \dfrac{2A}{|\Delta t|}\ ,\ |\Delta t| \geqslant \frac{\pi}{W} \end{cases} \tag{3.14}$$

□

This theorem establishes an important connection between the Fourier spectrum of a function and its maximum possible instantaneous and average rate of change. By considering the functions $A \sin Wt$ and $A \sin (\pi/\Delta t)t$ for $0 \leqslant \Delta t \leqslant \pi/W$ and $\Delta t \geqslant \pi/W$, respectively, it is easily verified that Equation 3.14 provides a least upper bound on $|\Delta x/\Delta t|$ for the set of functions $\{x(t)\colon M_0 \leqslant A\}$; that is, the set of functions for which the zeroth absolute moments of the functions' Fourier spectra are bounded. Since M_0 is proportional to the area of a function's Fourier spectrum, Theorem 3.1 may be alternately viewed as providing a bound on the set of functions with bounded spectral area $2\pi M_0 \leqslant 2\pi A$. When used to bound a specific function $x(t)$ with zeroth absolute moment M_0, rather than a set of functions $\{x(t)\colon M_0 \leqslant A\}$, Equation 3.14 of Theorem 3.1 may be reduced to Equations 3.11-3.13 by observing that A may be replaced by M_0.

EXAMPLE 3.1 Consider the constant function

$$x(t) = K \; , \; -\infty \leqslant t \leqslant \infty \; . \tag{3.15}$$

This function is certainly band-limited. In fact, $W = 0$. The Fourier transform of $x(t)$ is

$$X(j\omega) = 2\pi K \; \delta(\omega) \tag{3.16}$$

where $\delta(\omega)$ is the unit impulse function. Thus, in accordance with Theorem 3.1,

$$A = M_0 = \frac{1}{2\pi} \int_{-0}^{+0} |2\pi K \; \delta(\omega)| d\omega = K \tag{3.17}$$

and

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant WM_0 \frac{\sin \frac{W}{2}|\Delta t|}{\frac{W}{2}|\Delta t|} = (0)(K)(\frac{\sin 0}{0}) = 0 \ , \ 0 \leqslant \Delta t \leqslant \infty \qquad (3.18)$$

where it has been noted that $\pi/W = \pi/0 = \infty$. Thus, the maximum instantaneous and average rate of change of the constant function $x(t) = K$ is zero for all Δt; the bounds established by Theorem 3.1 are in complete agreement with the known behavior of the function. □

<u>EXAMPLE 3.2</u> The following estimate of the amplitude variation of a band-limited modulated signal $x(t) \cos \omega_0 t$ within the period $T = 2\pi/\omega_0$ of the carrier was provided by Equation 2.50 (Example 2.4):

$$\left|x(t + T) - x(t)\right| \leqslant 2\pi M_0 (\frac{W}{\omega_0}) \ , \ W << \omega_0 \ . \qquad (3.19)$$

Theorem 3.1 provides the following bound on this amplitude variation:

$$\left|x(t + T) - x(t)\right| \leqslant \begin{cases} 2M_0 \sin \pi \left(\frac{W}{\omega_0}\right) \ , & \frac{W}{\omega_0} \leqslant \frac{1}{2} \\ \\ 2M_0 \ , & \frac{W}{\omega_0} \geqslant \frac{1}{2} \end{cases} \qquad (3.20)$$

This result can be employed to evaluate the error $E(\frac{W}{\omega_0})$, defined to be the absolute difference between Equations 3.19 and 3.20, incurred by using Equation 3.19. Both Equations 3.19 and 3.20 are plotted in Figure 3.2. When $W/\omega_0 = 0.5$, the estimate provided by Equation 3.19 is off by a factor of π. As $W/\omega_0 \to 0$, $E(\frac{W}{\omega_0}) \to 0$. □

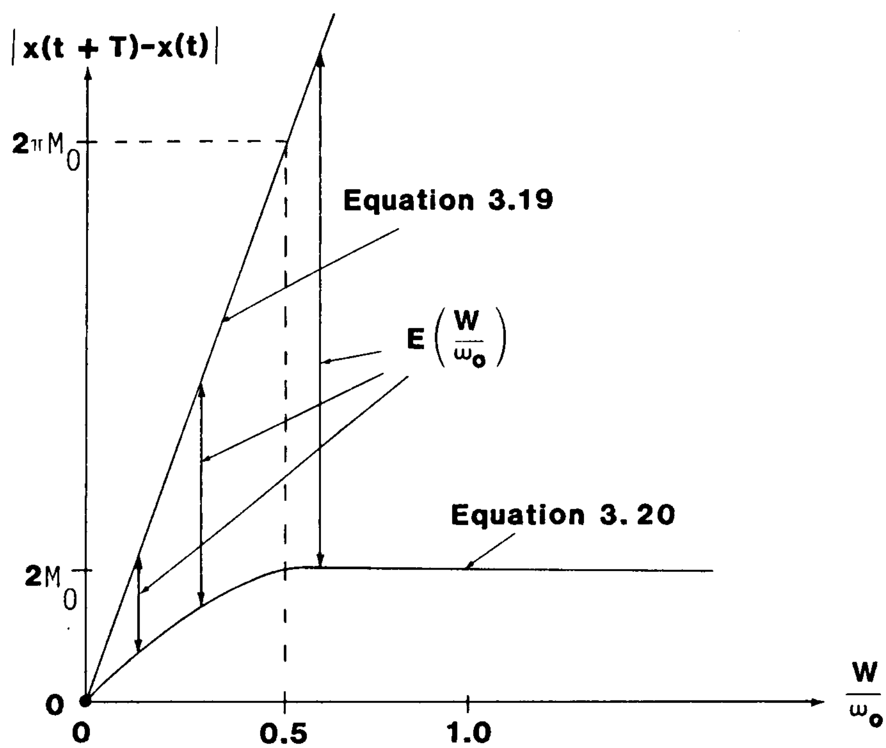


Figure 3.2. A comparison of Equations 3.19 and 3.20.

EXAMPLE 3.3 Consider the sinusoidal function $x(t) = K \sin \omega_0 t$ over the interval Δt from $t = -\Delta t/2$ to $\Delta t/2$. The instantaneous and average rate of change of $K \sin \omega_0 t$ is maximum in this interval. Since the amplitude change Δx (corresponding to Δt) is

$$\Delta x = x(\tfrac{\Delta t}{2}) - x(\tfrac{-\Delta t}{2}) = 2K \sin \frac{\omega_0 \Delta t}{2} , \tag{3.21}$$

it follows that

$$\frac{\Delta x}{\Delta t} = \omega_0 \ K \left[\frac{\sin \frac{\omega_0 \Delta t}{2}}{\frac{\omega_0 \Delta t}{2}} \right] \tag{3.22}$$

and

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \omega_0 \ K . \tag{3.23}$$

When $\Delta t \geqslant \pi/\omega_0$, however, $x(t)$ oscillates between $\pm K$ in the inverval Δt — obtaining the maximum average rate of change $2K/\Delta t$. Thus,

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \begin{cases} \omega_0 |K| \ , \ \Delta t = 0 \\ \omega_0 |K| \left[\dfrac{\sin \omega_0 |\Delta t|/2}{\omega_0 |\Delta t|/2} \right] , \ 0 < |\Delta t| \leqslant \dfrac{\pi}{\omega_0} \\ 2|K|/|\Delta t| \ , \ |\Delta t| \geqslant \dfrac{\pi}{\omega_0} . \end{cases} \tag{3.24}$$

This result is readily verified by observing that

$$X(j\omega) = -j\pi K[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \tag{3.25}$$

and utilizing Theorem 3.1. □

EXAMPLE 3.4 As a final illustration of the use of Theorem 3.1, let $x(t)$ be the finite sum of sinusoids $x(t) = \sum_{n=1}^{N} K_n \sin n\omega_0 t$. This function is band-limited with maximum frequency $W = N\omega_0$. Its Fourier transform is $X(j\omega) = \sum_{n=1}^{N} -j\pi K_n[\delta(\omega - n\omega_0) - \delta(\omega + n\omega_0)]$. In accordance with Theorem 3.1,

$$\left|\frac{dx}{dt}\right| = \lim_{|\Delta t|\to 0} \left|\frac{\Delta x}{\Delta t}\right| \leqslant N\omega_0 \sum_{n=1}^{N} |K_n| \ . \tag{3.26}$$

For the trivial case of $N = 1$, Equation 3.26 reduces to Equation 3.23 of Example 3.3. When $N = 2$, however, $x(t) = K_1 \sin \omega_0 t + K_2 \sin 2\omega_0 t$ and Equation 3.26 reveals that $|dx/dt| \leqslant 2\omega_0[|K_1| + |K_2|]$. Noting that $dx/dt = K_1\omega_0 \cos \omega_0 t + K_2\omega_0 \cos 2\omega_0 t \leqslant |K_1|\omega_0 + 2|K_2|\omega_0$, it is evident that $x(t)$ never achieves the predicted maximum instantaneous rate $2|K_1|\omega_0 + 2|K_2|\omega_0$. □

3.2.2 A bound on the average rate of change referenced to an extremum

Since Theorem 3.1 provides a least upper bound on both the instantaneous and average rate of change of band-limited functions for which $M_0 \leqslant A$, it can be used to establish least upper bounds on the absolute differences $|x(t) - x((n - 1)T)|$ and $|x(nT) - x(t)|$ of Figure 3.1(a), page 39. In this figure, $x(t)$ is a monotonic function of t in the interval $[(n - 1)T, nT]$. Theorem 3.1 does not, however, provide a least upper bound on the absolute differences depicted in Figure 3.1(b), where $x(t)$ has an extremum for some $(n - 1)T \leqslant t_0 \leqslant nT$. In this section, such a bound is developed.

The following expression for the maximum change in the derivative $x'(t)$ of a band-limited function $x(t)$ is a direct consequence of the differentiation property of the Fourier transform (Equation 2.22):

$$\left|x'(t_0 + \Delta t) - x'(t_0)\right| \leqslant \begin{cases} 2WM_0 \sin \dfrac{W|\Delta t|}{2}, & t \leqslant \dfrac{\pi}{W} \\ 2WM_0, & |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.27}$$

This can be shown by forming the difference

$$x'(t_0 + \Delta t) - x'(t_0) = \frac{j}{2\pi} \int_{-W}^{W} \omega X(j\omega)[e^{j\omega(t_0+\Delta t)} - e^{j\omega t_0}]d\omega \tag{3.28}$$

and utilizing Equation 3.5 to obtain

$$x'(t_0 + \Delta t) - x'(t_0) = \frac{1}{\pi} \int_{-W}^{W} \omega X(j\omega)e^{j\omega(t_0+\frac{\Delta t}{2})} \sin \frac{\omega\Delta t}{2}\, d\omega \tag{3.29}$$

Equation 3.27 then follows by taking the absolute value of both sides of this expression, noting that

$$\left|\omega \sin \frac{\omega\Delta t}{2}\right| \leqslant \begin{cases} W \sin \dfrac{W|\Delta t|}{2}, & |\Delta t| \leqslant \dfrac{\pi}{W} \\ W, & |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.30}$$

and using Equations 2.37, 3.9, and 3.10. (The procedure for deriving Equation 3.27 is much the same as that used to obtain Equation 3.11).

The function $x'(t) = WM_0 \sin Wt$ obtains the bound given in Equation 3.27 in the interval Δt from $-\Delta t/2$ to $\Delta t/2$ when $|\Delta t| \leqslant \pi/W$. This function is depicted in Figure 3.3(a). The corresponding function $x(t) = \int x'(t)dt = -M_0 \cos Wt$ is shown in Figure 3.3(b). Note that this function, $x(t)$, has an extremum at $t = 0$. Thus, for any $|\Delta t| \leqslant \pi/W$, the change in the derivative of a band-limited function $x(t)$ for a given M_0 is bounded by the change in derivative about an extremum of $-M_0 \cos Wt$ from $-\Delta t/2$ to $\Delta t/2$. Since the change in derivative of $x(t) = -M_0 \cos Wt$ is maximum for all $|\Delta t| \leqslant \pi/W$ taken symmetrically about the extremum, the amplitude change Δx of any band-limited function $x(t)$ is subject to the inequality

$$|\Delta x| \leqslant M_0[1 - \cos W|\Delta t|] \ , \ |\Delta t| \leqslant \frac{\pi}{2W} \tag{3.31}$$

where one end point of the interval Δt must be an extremum of $x(t)$. Dividing this expression by $|\Delta t|$, noting that $\lim_{t \to 0} (1 - \cos Wt)/Wt = 0$, and using Theorem 3.1, it follows that

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0 \ , \ \Delta t = 0 \\ WM_0 \left[\dfrac{1 - \cos W|\Delta t|}{W|\Delta t|} \right] \ , \ 0 < |\Delta t| \leqslant \dfrac{\pi}{W} \\ \dfrac{2M_0}{|\Delta t|} \ , \ |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.32}$$

where the notation $|\Delta x/\Delta t|_{x'(t_0)=0}$ has been used to refer to the average

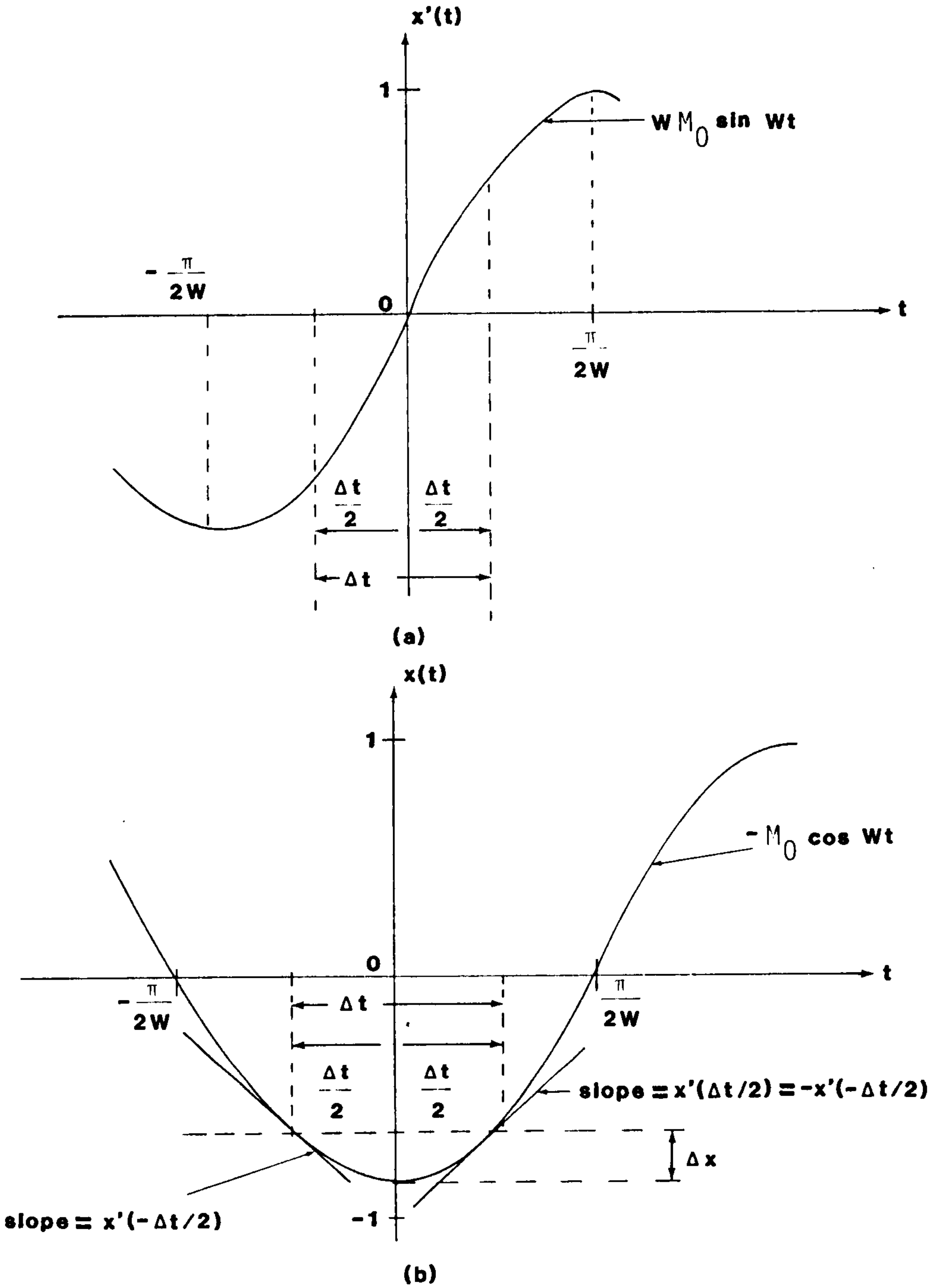


Figure 3.3. A function which obtains the bounds of Equation 3.27.

rate of change of $x(t)$ over an interval Δt in which at least one end point of Δt is an extremum of $x(t)$; that is,

$$\left.\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} = \left.\frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}\right|_{x'(t_0)=0} \tag{3.33}$$

The quantity $|\Delta x/\Delta t|_{x'(t_0)=0}$ will be referred to as the *average rate of change referenced to an extremum.* The following theorem is a direct consequence of Equation 3.32.

THEOREM 3.2 Let the function $x(t)$ with Fourier transform $X(j\omega)$ satisfy the conditions (i) $X(j\omega) = 0$ for $|\omega| > W$, (ii) $x'(t_0) = 0$ for some t_0, and (iii) $M_0 = 1/2\pi \int_{-W}^{W} |X(j\omega)| d\omega \leqslant A$ for some constant $A > 0$. Then the average rate of change of $x(t)$ over the interval Δt from $t = t_0$ to $t_0 + \Delta t$ satisfies the inequality

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0 \text{ , } \Delta t = 0 \\ WA \left[\dfrac{1 - \cos W|\Delta t|}{W|\Delta t|} \right] \text{ , } 0 < |\Delta t| \leqslant \dfrac{\pi}{W} \\ \dfrac{2A}{|\Delta t|} \text{ , } |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.34}$$

□

Thus, the maximum average rate of change referenced to an extremum (of a band-limited function), like both the maximum instantaneous and average rate of change in Theorem 3.1, can be related to the area of the Fourier spectrum of $x(t)$. By considering the function $A \cos Wt$ for $0 \leqslant \Delta t \leqslant \pi/W$ and employing Theorem 3.1, it is easily established that Theorem 3.2 provides a least upper bound on $|\Delta x/\Delta t|_{x'(t_0)=0}$ for the set of functions $\{x(t): M_0 \leqslant A\}$. When used to bound a specific function x(t) with zeroth absolute moment M_0, Equation 3.34 of Theorem 3.2 may be reduced to Equation 3.32 by noting that A may be replaced by M_0.

EXAMPLE 3.5 Consider the Fourier spectrum shown in Figure 3.4(a):

$$|X(j\omega)| = \begin{cases} \left(\frac{K}{\omega_0}\right)\omega + K \; , & -\omega_0 \leqslant \omega \leqslant 0 \\ -\left(\frac{K}{\omega_0}\right)\omega + K \; , & 0 \leqslant \omega \leqslant \omega_0 \\ 0 \; , & |\omega| > \omega_0 \end{cases} \tag{3.35}$$

In accordance with Theorem 3.2,

$$A = M_0 = \frac{1}{2\pi}\left[\int_{-\omega_0}^{0}\left|\frac{K}{\omega_0}\omega + K\right|d\omega + \int_{0}^{\omega}\left|\frac{K}{\omega_0}\omega + K\right|d\omega\right] = \frac{\omega_0 K}{2\pi} \tag{3.36}$$

and

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0 \; , & \Delta t = 0 \\ \frac{\omega_0{}^2 K}{2\pi}\left[\frac{1 - \cos \omega_0 \Delta t}{\omega_0 |\Delta t|}\right] \; , & 0 < |\Delta t| \leqslant \frac{\pi}{\omega_0} \\ \frac{\omega_0 K}{\pi|\Delta t|} \; , & |\Delta t| \geqslant \frac{\pi}{\omega_0} \end{cases} \tag{3.37}$$

The corresponding amplitude variation $|\Delta x|_{x'(t_0)=0}$, obtained by multiplying both sides of this result by $|\Delta t|$, is plotted in Figure 3.4(b). The minimum possible pulse duration T_p of a band-limited pulse with spectrum $|X(j\omega)|$ and amplitude $|\Delta x|$ is

$$T_p = \frac{2}{\omega_0}\cos^{-1}\left[1 - \frac{2\pi}{\omega_0 K}|\Delta x|\right] \; , \quad 0 \leqslant |\Delta x| \leqslant \frac{\omega_0 K}{2\pi} \; . \tag{3.38}$$

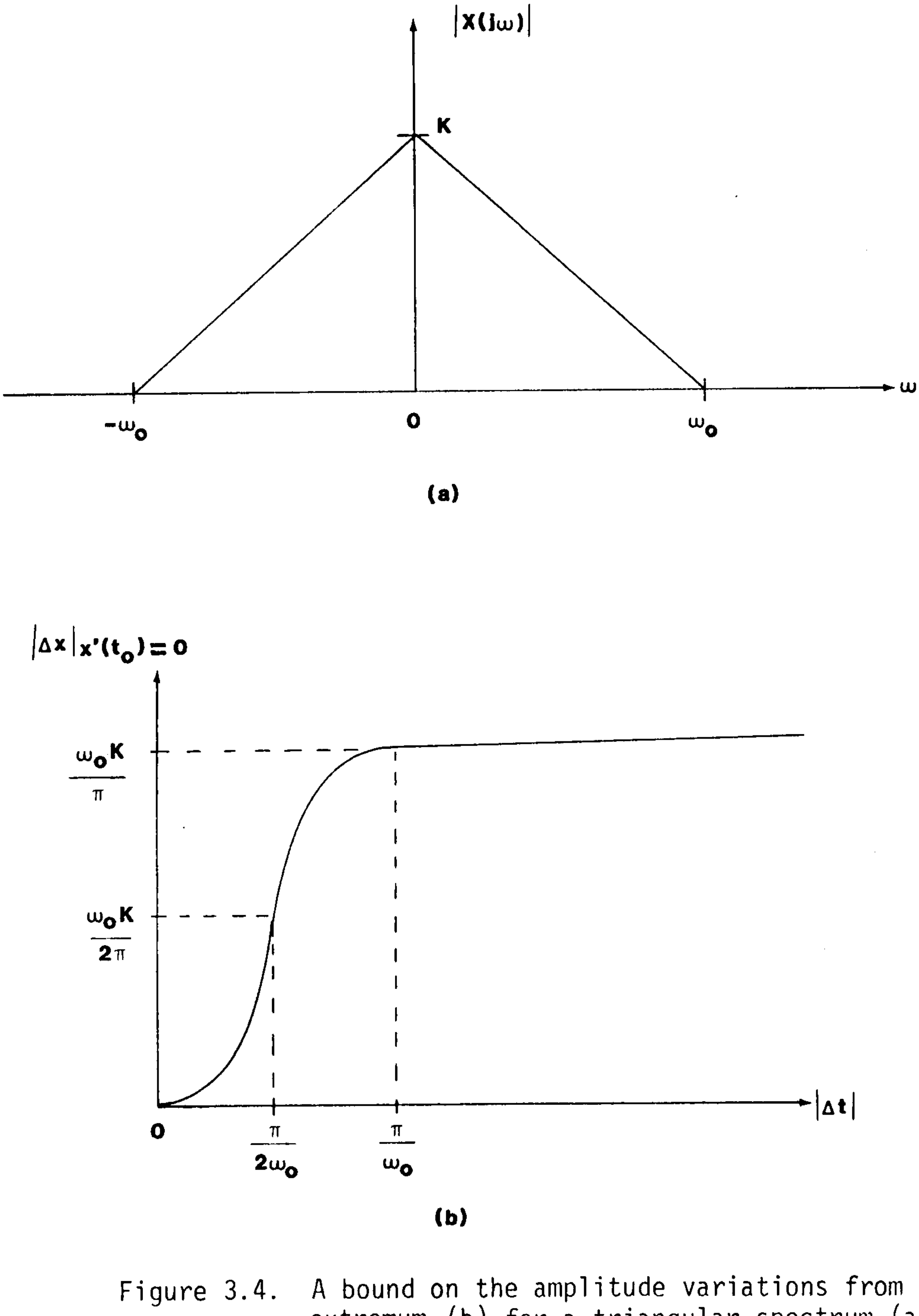


Figure 3.4. A bound on the amplitude variations from an extremum (b) for a triangular spectrum (a).

For a pulse of amplitude $|\Delta x| = \omega_0 K/2\pi$, for example, $T_p = \pi/\omega_0$. □

3.3 Bounds on the Rate of Change of Band-Limited Functions with Bounded Spectra

Theorems 3.1 and 3.2 of the previous section provide least upper bounds on the slope and curvature of the set of band-limited functions $\{x(t): M_0 \leqslant A\}$; that is, those functions whose zeroth absolute moments are bounded by A or whose spectral areas are bounded by $2\pi A$. Similar bounds can be developed for the set of functions whose Fourier amplitude spectra $|X(j\omega)|$ are bounded by some positive function $f(\omega)$ for all ω; that is, on the set of functions $\{x(t): |X(j\omega)| \leqslant f(\omega),\ f(\omega) \geqslant 0\}$.

EXAMPLE 3.6 Consider again the finite sum of sinusoids $x(t) = \sum_{n=1}^{N} K_n \sin n\omega_0 t$ of Example 3.4 for N = 2. Using Theorem 3.1, it was shown that $|dx/dt| \leqslant 2\omega_0[|K_1| + |K_2|]$. An even lower bound can be obtained by noting that

$$\left|\frac{dx}{dt}\right| \leqslant \frac{1}{2\pi}\int_{-W}^{W} |\omega|\ |X(j\omega)|d\omega\ . \tag{3.39}$$

This result follows from the differentiation property (Equation 2.22) and Equation 2.37. In accordance with this result,

$$\begin{aligned}\left|\frac{dx}{dt}\right| &< \frac{1}{2\pi}\int_{-W}^{W} |\omega|\ \Big|-j\pi\Big[K_1[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \\ &\qquad + K_2[\delta(\omega - 2\omega_0) - \delta(\omega + 2\omega_0)]\Big]\Big|d\omega \\ &\leqslant |K_1|\omega_0 + 2|K_2|\omega_0\ .\end{aligned} \tag{3.40}$$

While Equation 3.39 can be integrated for this simple function, integration is not always possible for an arbitrary x(t). If integration is not possible, a numerical estimate based on Equation 3.39 may be employed. □

Just as Equation 3.39 may be used to obtain a bound on the instantaneous rate of change of x(t) which is lower than that established by Theorem 3.1, Equation 3.3 can be employed to determine a bound on the average rate of change of x(t) which is lower than the one given in Theorem 3.1. Similarly, the following inequality often provides a bound on $|\Delta x/\Delta t|_{x'(t_0)=0}$ which is lower than the bound established by Theorem 3.2:

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \frac{1}{2\pi|\Delta t|}\int_{-W}^{W} |X(j\omega)|(1 - \cos \omega\Delta t)d\omega \ , \ |\Delta t| > 0 \qquad (3.41)$$

This result can be obtained from Equation 3.4 by observing that

$$e^{j\omega(t_0+\Delta t)} - e^{j\omega t_0} = e^{j\omega t_0}\,[(\cos \omega\Delta t - 1) + j \sin \omega\Delta t] \qquad (3.42)$$

and considering only those x(t) such that $x(t - \Delta t) = x(t + \Delta t)$ so that[a]

$$\int_{-W}^{W} X(j\omega)\, e^{j\omega t_0} \sin \omega\Delta t \, d\omega = 0 \ . \qquad (3.43)$$

---

[a]This assumption restricts attention to those functions which obtain the bound on $|\Delta x/\Delta t|_{x'(t_0)=0}$ on both sides of the extremum at $t = t_0$. This is not a serious restriction, since it was shown by Theorem 3.2 that such functions exist for all $|\Delta t|$.

Substitution of Equation 3.42 and 3.43 into 3.4 and the use of Equations 2.37 and 3.9 then lead to the desired result, Equation 3.41. The following two theorems are direct consequences of this result and Equations 3.3 and 3.39:

<u>THEOREM 3.3</u> Let the function $x(t)$ with Fourier transform $X(j\omega)$ be such that (i) $X(j\omega) = 0$ for $|\omega| > W$ and (ii) $|X(j\omega)| \leqslant f(\omega)$ for some positive function $f(\omega)$ and all ω. Then the instantaneous and average rate of change of $x(t)$ over any interval Δt satisfies the inequality

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-W}^{W} |\omega|\ f(\omega)\ d\omega\ , & \Delta t = 0 \\[2ex] \dfrac{1}{\pi|\Delta t|}\displaystyle\int_{-W}^{W} f(\omega)\ \left|\sin\frac{\omega\Delta t}{2}\right|\ d\omega\ , & |\Delta t| > 0 \end{cases} \tag{3.44}$$

□

<u>THEOREM 3.4</u> Let the function $x(t)$ with Fourier transform $X(j\omega)$ satisfy the conditions (i) $X(j\omega) = 0$ for $|\omega| > W$, (ii) $x'(t_0) = 0$ for some t_0, and (iii) $|X(j\omega)| \leqslant f(\omega)$ for some positive function $f(\omega)$ and all ω. Then, the average rate of change of $x(t)$ over the interval Δt from $t = t_0$ to $t_0 + \Delta t$ satisfies the inequality

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0\ , & \Delta t = 0 \\[2ex] \dfrac{1}{2\pi|\Delta t|}\displaystyle\int_{-W}^{W} f(\omega)\ (1 - \cos\omega\Delta t)\ d\omega\ , & |\Delta t| > 0 \end{cases} \tag{3.45}$$

□

Theorems 3.3 and 3.4 bound the slope and curvature of the set of band-limited functions $\{x(t): |X(j\omega)| \leqslant f(\omega),\ f(\omega) \geqslant 0\}$. By considering the real functions

$$x(t) = \frac{1}{\pi} \int_0^W f(\omega) \sin \omega t \, d\omega \tag{3.46}$$

and

$$x(t) = -\frac{1}{\pi} \int_0^W f(\omega) \cos \omega t \, d\omega \tag{3.47}$$

over the intervals from $t = -\Delta t/2$ to $\Delta t/2$ and $t = 0$ to Δt, respectively, it is easily established that Theorems 3.3 and 3.4 provide least upper bounds on the set of real functions whose Fourier spectra are bounded by $f(\omega)$. These theorems may be used (instead of Theorems 3.1 and 3.2) to bound the slope and curvature of a specific band-limited function $x(t)$, rather than the set of functions $\{x(t): |X(j\omega)| \leqslant f(\omega),\ f(\omega) \geqslant 0\}$, by noting that $f(\omega)$ (in Equation 3.44 and 3.45) may be replaced by $|X(j\omega)|$. The bounds so obtained will usually be lower than the bounds established by Theorems 3.1 and 3.2. If the integrations in Theorems 3.3 and 3.4 cannot be expressed in a closed form, numerical integration may be employed to obtain an approximation of the desired bounds. In either case, the resulting bounds will not exceed the bounds established by Theorems 3.1 and 3.2.

<u>EXAMPLE 3.7</u> Noise possessing a nonzero and constant power density spectrum over a finite frequency band and zero elsewhere is called band-limited white noise. The general form of the power spectrum of band-limited white noise is depicted in Figure 3.5(a); that is,

$$|X(j\omega)| = \begin{cases} K \ , & -W \leqslant \omega \leqslant W \\ 0 \ , & \text{elsewhere} \ . \end{cases} \tag{3.48}$$

This spectrum, which is also the transfer function of the ideal low pass filter, is frequently encountered in practice. The following bounds on the rate of change of those functions possessing such a spectrum are a direct consequence of Theorems 3.1 and 3.2:

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \begin{cases} \dfrac{2WK}{\pi|\Delta t|} \sin \dfrac{W|\Delta t|}{2} \ , & 0 < |\Delta t| \leqslant \dfrac{\pi}{W} \\ \dfrac{2WK}{\pi|\Delta t|} \ , & |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.49}$$

$$\left|\frac{dx}{dt}\right| \leqslant \frac{W^2 K}{\pi} \tag{3.50}$$

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0 \ , & \Delta t = 0 \\ \dfrac{WK}{\pi|\Delta t|} (1 - \cos W \Delta t) \ , & 0 < |\Delta t| \leqslant \dfrac{\pi}{W} \\ \dfrac{W^2 K}{\pi} \ , & |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.51}$$

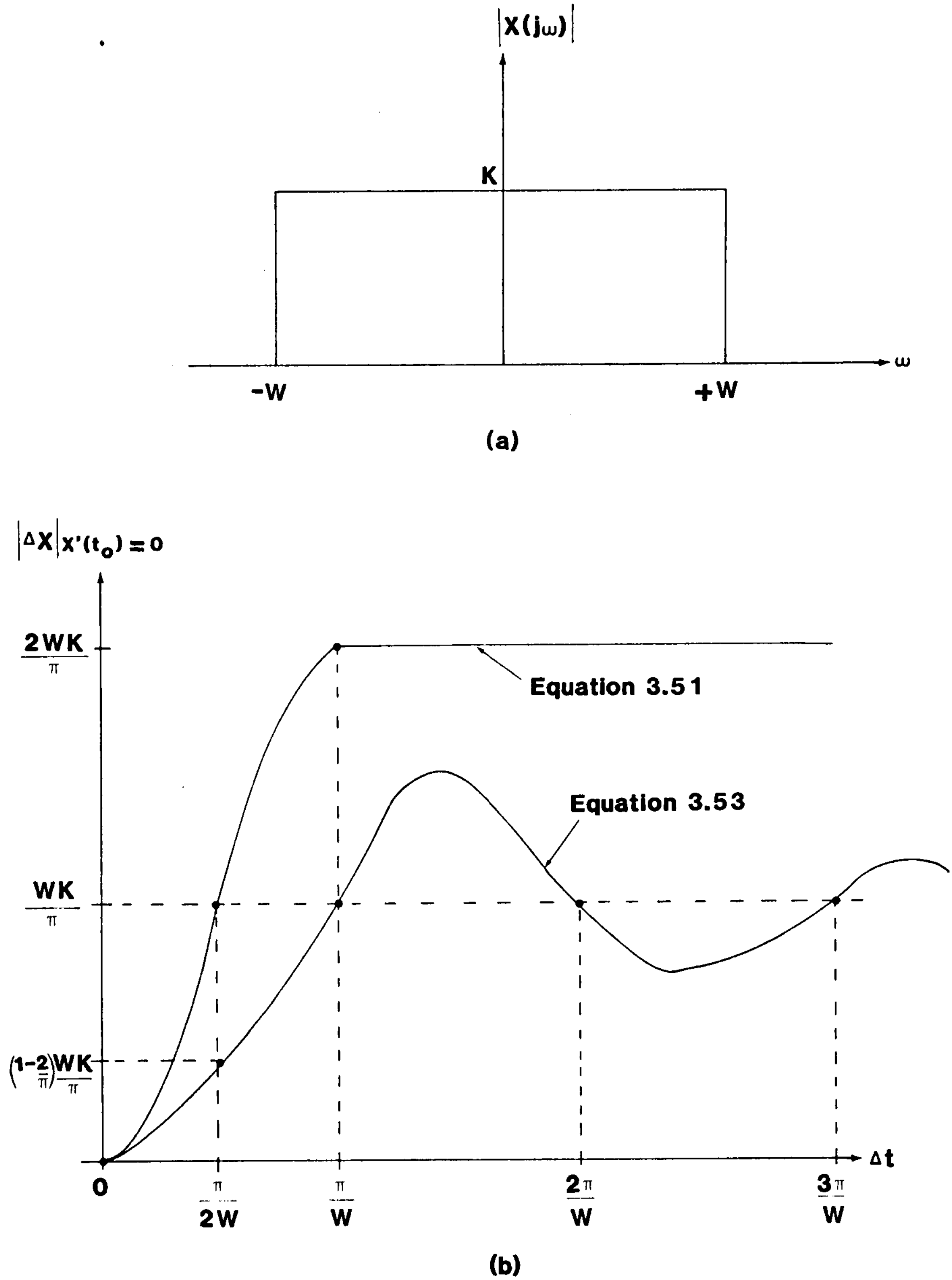
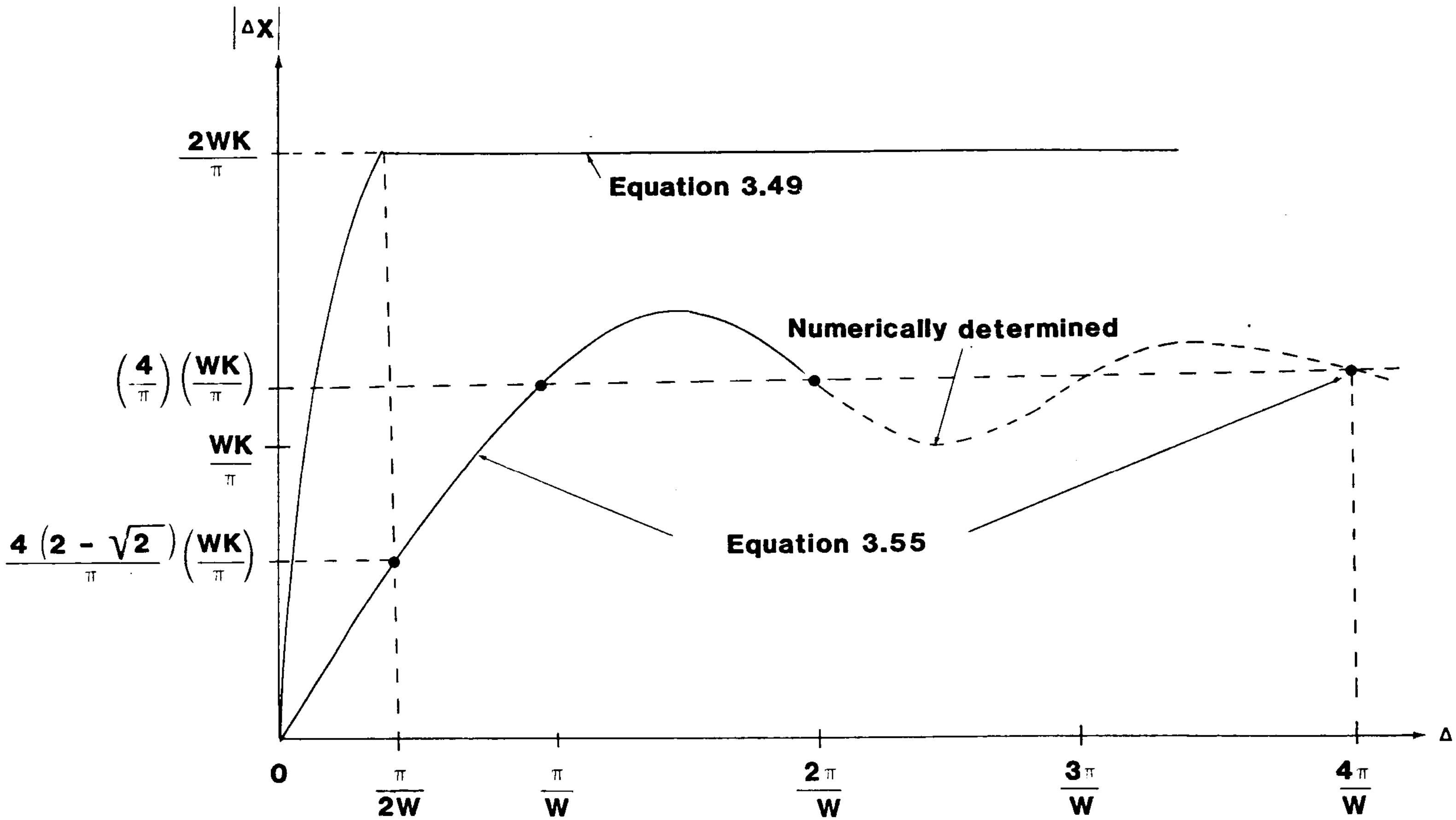


Figure 3.5. Bounds on the rate of change for a rectangular spectrum.



(c)

Figure 3.5. (Continued)

These bounds would be obtained for any spectrum of bandwidth W and area 2WK, the area of the rectangular spectrum in Figure 3.5(a). In view of the importance of this spectrum and the simplicity of its mathematical representation (Equation 3.48), it is worthwhile to employ Theorems 3.3 and 3.4 for the determination of even lower bounds on the rate of change of its corresponding functions.

Consider the instantaneous rate of change of a function whose spectrum has the form of Equation 3.48. In accordance with Theorem 3.3,

$$\left|\frac{dx}{dt}\right| \leqslant \frac{1}{2\pi}\int_{-W}^{W} |\omega| \;\; |K| d\omega = \frac{KW^2}{2\pi} \tag{3.52}$$

Thus, for this particular spectrum, the bound established by Equation 3.50, which is the least upper bound on the class of functions of bandwidth W and $M_0 \leqslant WK/\pi$, is high by a factor of two. While the spectrum of Figure 3.5(a) and Equation 3.48 is a member of this class, Equation 3.52 reveals that functions having this spectrum do not obtain the bounds of the class. The set of functions bounded by Equation 3.52, which includes all functions whose Fourier amplitude spectra are bounded by the rectangular spectrum of Figure 3.5(a), represents a subset of those functions bounded by Equation 3.50. In other words, the set of functions whose amplitude spectra are completely contained within the rectangular spectrum of Figure 3.5(a) is a subset of the set of functions whose spectral areas are bounded by the area of this rectangular spectrum.

The following bound on the average rate of change from an extremum of a function whose spectrum is completely contained within that of

Figure 3.5(a) follows from Theorem 3.4:

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \frac{1}{2\pi|\Delta t|}\int_{-W}^{W} K\ (1 - \cos\omega\Delta t)d\omega$$

$$\leqslant \frac{KW}{\pi|\Delta t|}\left(1 - \frac{\sin W|\Delta t|}{W|\Delta t|}\right),\ |\Delta t| > 0\ . \tag{3.53}$$

This result and the corresponding bound established by Theorem 3.2, i.e., Equation 3.51, are plotted in Figure 3.5(b). As can be seen, the bound resulting from the use of Theorem 3.4 is significantly lower than the one which resulted from Theorem 3.2. In fact, when $|\Delta t| = \pi/2W$, the bound of Theorem 3.4 is only 36% of the one corresponding to Theorem 3.2.

Finally, Theorem 3.3 can be used to bound the average rate of change of those functions whose spectra are contained within the rectangular spectrum of Figure 3.5(a). In accordance with this theorem, the following integration must be performed:

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \frac{1}{\pi|\Delta t|}\int_{-W}^{W} |K|\ \left|\sin\frac{\omega\Delta t}{2}\right|d\omega$$

$$\leqslant \frac{2K}{\pi|\Delta t|}\int_{0}^{W} \left|\sin\frac{\omega\Delta t}{2}\right|d\omega\ ,\ |\Delta t| > 0 \tag{3.54}$$

This integral does not have a single closed form for all $|\Delta t|$. It can, however, be evaluated for $|\Delta t| \leqslant 2\pi/W$ and $|\Delta t| = k(2\pi/W)$ for $K = 1, 2, 3, \ldots$:

$$\left|\frac{\Delta x}{\Delta t}\right| < \begin{cases} \dfrac{4K}{\pi \Delta t^2}\left(1 - \cos\dfrac{W\Delta t}{2}\right), & |\Delta t| \leq \dfrac{2\pi}{W} \\[2ex] \dfrac{4WK}{\pi^2 |\Delta t|}, & |\Delta t| = k\left(\dfrac{2\pi}{W}\right), \ k = 1, 2, 3, \ldots \end{cases} \tag{3.55}$$

The values of $|\Delta x/\Delta t|$ in the open intervals $\{(2\pi(m - 1)/W, 2\pi m/W): m = 2, 3, \ldots\}$ may be estimated by a numerical integration of Equation 3.54. Figure 3.5(c) depicts the bounds resulting from both Theorems 3.1 and 3.3 (Equations 3.49 and 3.55). Note that even with such a mathematically simple spectrum (Equation 3.48), the integrations required by Theorems 3.3 and 3.4 are not always possible for all Δt. □

3.4 Bounds on the Rate of Change of Bounded Band-Limited Functions

Theorems 3.1 and 3.2 of Section 3.2 completely bound the slope and curvature of the set of band-limited functions $\{x(t): M_0 \leq A\}$, while Theorems 3.3 and 3.4 of Section 3.3 provide similar bounds on the set of band-limited functions $\{x(t): |X(j\omega)| \leq f(\omega), f(\omega) \geq 0\}$. As will be seen in Chapter 4, these theorems can be utilized to establish sampling theorems which will bound the allowable amplitude fluctuations of a function between successive samples. In addition, a rather straightforward manipulation of Theorems 3.1 and 3.2 leads to several interesting results on the behavior of those functions that are limited in both amplitude and frequency.

The following bound on the absolute value of $x(t)$ is a consequence of Equations 2.17, 2.37, and 3.10:

$$|x(t)| = \frac{1}{2\pi}\left|\int_{-W}^{W} X(j\omega)e^{j\omega t}d\omega\right| \leqslant \frac{1}{2\pi}\int_{-W}^{W} |X(j\omega)|d\omega = M_0 \tag{3.56}$$

The absolute value of $x(t)$ is bounded by M_0, a quantity which is proportional to the area under the Fourier spectrum of $x(t)$. Since a function $x(t)$ is bounded if $|x(t)| \leqslant A$ for some A and all t, requiring $M_0 \leqslant A$ is sufficient to guarantee that $|x(t)|$ will not exceed A. In other words, if

$$|x(t)| \leqslant M_0 \leqslant A , \tag{3.57}$$

$x(t)$ will remain within the amplitude interval $[-A, A]$ for all t. Thus by substituting A for A in Equations 2.14 and 3.34 of Theorems 3.1 and 3.2, respectively, the following two corollaries result:

<u>COROLLARY 3.1</u> Let the function $x(t)$ with Fourier transform $X(j\omega)$ satisfy the conditions (i) $|x(t)| \leqslant A$ for all t and (ii) $X(j\omega) = 0$ for $|\omega| > W$. Then the instantaneous and average rate of change of $x(t)$ over any interval Δt satisfies the inequality

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \begin{cases} WA , & \Delta t = 0 \\ WA\left[\dfrac{\sin \frac{W|\Delta t|}{2}}{\frac{W|\Delta t|}{2}}\right] , & 0 < |\Delta t| \leqslant \frac{\pi}{W} \\ \dfrac{2A}{|\Delta t|} , & |\Delta t| \geqslant \frac{\pi}{W} \end{cases} \tag{3.58}$$

□

<u>COROLLARY 3.2</u> Let the function $x(t)$ with Fourier transform $X(j\omega)$ satisfy conditions (i) and (ii) of Corollary 3.1 and let $x'(t_0) = 0$ for some t_0. Then, the average rate of change of $x(t)$ over the interval Δt from $t = t_0$ to $t_0 + \Delta t$ satisfies the inequality

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0 \ , \ \Delta t = 0 \\ WA \left[\dfrac{1 - \cos W|\Delta t|}{W|\Delta t|} \right] , \ 0 < |\Delta t| \leqslant \dfrac{\pi}{W} \\ \dfrac{2A}{|\Delta t|} \ , \ |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{3.59}$$

□

In contrast to Theorems 3.1-3.4, the above bounds on the slope and curvature of bounded band-limited functions are independent of the Fourier spectrum of these functions (other than the fact that the spectrum is band-limited). Since most realizable signals are bounded in both amplitude and frequency, these results provide significant computational advantages over Theorems 3.1-3.4, i.e., they require no knowledge of the specific shape of the Fourier spectrum of the functions of interest. It is easily shown that Corollaries 3.1 and 3.2 provide least upper bounds on the rate of change of bounded band-limited functions for a given A, since the functions $A \sin Wt$ and $A \sin (\pi/\Delta t)t$ for $0 \leqslant |\Delta t| \leqslant \pi/W$ and $|\Delta t| \geqslant \pi/W$, respectively, obtain these bounds.

<u>EXAMPLE 3.8</u> An amplifier is said to be slew rate limited to S V/sec if the instantaneous rate of change of its output voltage is bounded by S. Since the information portion of a standard NTSC composite television signal is 1V peak to peak and limited in frequency to approximately

4.3 MHz, video amplifiers should possess a slew rate in excess of WA $= 2\pi(0.5)(4.3 \times 10^6)$ V/sec $= 4.3\pi$ V/μsec to avoid distortion due to slewing. This result follows directly from Corollary 3.1. □

3.5 Extensions and Observations

In Section 3.2, it was noted that Theorems 3.1 and 3.2 may be employed to bound the slope and curvature of a single band-limited function $x(t)$, as well as a set of such functions $\{x(t)\colon M_0 \leqslant A\}$. To do so, A (in Equations 3.14 and 3.34 of Theorems 3.1 and 3.2) is replaced by M_0. Similarly, Theorems 3.3 and 3.4 of Section 3.3 may be used to bound a specific function $x(t)$, rather than a set of functions $\{x(t)\colon |X(j\omega)| \leqslant f(\omega),\ f(\omega) \geqslant 0\}$, by replacing $f(\omega)$ in Equations 3.44 and 3.45 with $|X(j\omega)|$. As previously mentioned, the bounds so obtained will never exceed the bounds provided by Theorems 3.1 and 3.2. In this section, the use of Theorems 3.1-3.4 for the determination of bounds on the rate of change of several functions of practical importance is examined.

Table 3.1 lists several commonly encountered functions $x(t)$ and the corresponding quantities that may be used to replace A in Equations 3.14 and 3.34 of Theorems 3.1 and 3.2. The first entry in this table defines the general procedure for bounding a function $x(t)$ with Theorems 3.1 and 3.2; that is, the replacement of A by M_0. If $x(t)$ is bounded by A, the second entry in Table 3.1 reveals that, as was noted in Section 3.4, A may be replaced by A. All other entries in this table are discussed separately below.

TABLE 3.1

SIMPLIFICATIONS OF THEOREMS 3.1 AND 3.2

Condition on x(t)	Quantity which replaces A
x(t)	$M_0 = \frac{1}{2\pi}\int_{-W}^{W} \lvert X(j\omega)\rvert d\omega$
x(t): $\lvert x(t)\rvert \leqslant A$	A
x(t): REAL	$\frac{1}{\pi}\int_{0}^{W} \lvert X(j\omega)\rvert d\omega$
x(t): REAL BANDPASS	$\frac{1}{\pi}\int_{W_1}^{W_2} \lvert X(j\omega)\rvert d\omega$
x(t): PERIODIC	$\sum_{n=-W/\omega_0}^{W/\omega_0} \lvert X_n\rvert$
x(t): REAL PERIODIC	$\sum_{n=0}^{W/\omega_0} \lvert X_n\rvert$

When x(t) is a real function of t, the Fourier transform of x(t) is conjugate symmetric; that is, $X(j\omega) = X^*(-j\omega)$. As a result, $|X(j\omega)|$ is an even function of ω and

$$A = M_0 = \frac{1}{\pi}\int_0^W |X(j\omega)|d\omega \; . \tag{3.60}$$

If, in addition, x(t) is a bandpass function (rather than a band-limited function) so that $|X(j\omega)| = 0$ for $|\omega| < W_1$, and $|\omega| > W_2$,

$$A = M_0 = \frac{1}{\pi}\int_{W_1}^{W_2} |X(j\omega)|d\omega \; . \tag{3.61}$$

An additional simplification in the computation of A can be realized when x(t) is a periodic function of period T. The Fourier transform of a periodic function x(t) is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} 2\pi X_n \, \delta(\omega - n\omega_0) \tag{3.62}$$

where the set $\{X_n\}$ are the coefficients of the complex Fourier series of x(t). Substituting this expression into Equation 3.10, it follows that

$$A = M_0 = \frac{1}{2\pi}\int_{-W}^{W} \left| \sum_{n=-W/\omega_0}^{W/\omega_0} 2\pi X_n \, \delta(\omega - n\omega_0)\right| d\omega = \sum_{n=-W/\omega_0}^{W/\omega_0} |X_n| \tag{3.63}$$

where the series is assumed to be computed over the period T of x(t). If x(t) is a real function of t,

$$A = M_0 = 2 \sum_{n=0}^{W/\omega_0} |X_n| \ . \tag{3.64}$$

Theorems 3.3 and 3.4 can also be simplified when bounding the rate of change of a real, bandpass, and/or periodic function x(t). When x(t) is a real function of t, Equations 3.44 and 3.45 of Theorems 3.3 and 3.4 become

$$\left|\frac{dx}{dt}\right| \leqslant \frac{1}{\pi} \int_0^W |\omega| \ |X(j\omega)| d\omega \tag{3.65}$$

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \frac{2}{\pi|\Delta t|} \int_0^W |X(j\omega)| \ \left|\sin \frac{\omega \Delta t}{2}\right| d\omega \ , \ |\Delta t| > 0 \tag{3.66}$$

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \frac{1}{\pi|\Delta t|} \int_0^W |X(j\omega)|(1 - \cos \omega\Delta t) d\omega \ , \ |\Delta t| > 0 \tag{3.67}$$

If x(t) is also a bandpass function, the limits on these integrations become $[W_1, W_2]$. For real periodic functions, Equations 3.65-3.67 reduce to the summations

$$\left|\frac{dx}{dt}\right| \leqslant 2 \sum_{n=0}^{W/\omega_0} n\omega_0 \ |X_n| \tag{3.68}$$

$$\left|\frac{\Delta x}{\Delta t}\right| \leqslant \frac{4}{|\Delta t|} \sum_{n=0}^{W/\omega_0} \left|\sin \frac{n\omega_0 \Delta t}{2}\right| \ |X_n| \ , \ |\Delta t| > 0 \tag{3.69}$$

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \frac{2}{|\Delta t|} \sum_{n=0}^{W/\omega_0} |X_n|(1 - \cos n\omega_0 \Delta t) \ , \ |\Delta t| > 0 \qquad (3.70)$$

where the set $\{X_n\}$ are the coefficients of the complex Fourier series of $x(t)$ computed over its period.

3.6 Summary and Discussions

Table 3.2 summarizes the results of Sections 3.2-3.4. Results for both band-limited functions and those functions that are limited in frequency and amplitude are included. For clarity, the bounds provided by Theorems 3.1, 3.2, and Corollary 3.1 on the instantaneous rate of change of a function have been separated from the corresponding bounds on its average rate of change.

The bounds provided by Theorems 3.1 and 3.2 of Section 3.2 are located in column two of Table 3.2. As mentioned in Section 3.2, these theorems provide least upper bounds on the slope and curvature of the set of band-limited functions whose zeroth absolute moments are bounded by A. Figure 3.6 depicts the spectra of several band-limited functions whose slope and curvature may be bounded in accordance with Theorems 3.1 and 3.2 by letting $A = M_0$. The functions corresponding to the spectra of Figure 3.6(a)-(d) would be identically bounded by these theorems, since $M_0 = K/2\pi$ and $W = W_1$ in each case. The spectra in Figure 3.6(e) and (f) would be assigned lower bounds than those of (a)-(d). Although the spectrum of Figure 3.6(e), for instance, has the same bandwidth W_1 as the spectra in (a)-(d), $M_0 < K/2\pi$. Similarly, while the spectrum in Figure 3.6(f) has the same value of M_0 as those

TABLE 3.2. BOUNDS ON THE RATE OF CHANGE OF A FUNCTION $x(t)$.

	BAND-LIMITED FUNCTIONS $X(j\omega)=0,\ \lvert\omega\rvert > W$		
	BOUNDED SPECTRUM $\lvert X(j\omega)\rvert \le f(\omega)$	BOUNDED SPECTRAL AREA $M_0 = \frac{1}{2\pi}\int_{-W}^{W} \lvert X(j\omega)\rvert d\omega \le A$	AMPLITUDE LIMITED $\lvert x(t)\rvert \le A$
AVERAGE RATE OF CHANGE $\left\lvert\frac{\Delta x}{\Delta t}\right\rvert$	$\frac{1}{\pi\lvert\Delta t\rvert}\int_{-W}^{W} f(\omega)\left\lvert \sin\frac{\omega\Delta t}{2}\right\rvert d\omega$ for $\lvert\Delta t\rvert > 0$ (Eq. 3.44)	$WA\left[\frac{\sin W\lvert\Delta t\rvert/2}{W\lvert\Delta t\rvert/2}\right]$ for $0 < \lvert\Delta t\rvert \le \frac{\pi}{W}$; $\frac{2A}{\lvert\Delta t\rvert}$, $\lvert\Delta t\rvert \ge \frac{\pi}{W}$ (Eq. 3.14)	$WA\left[\frac{\sin W\lvert\Delta t\rvert/2}{W\lvert\Delta t\rvert/2}\right]$ for $0 < \lvert\Delta t\rvert \le \frac{\pi}{W}$; $\frac{2A}{\lvert\Delta t\rvert}$, $\lvert\Delta t\rvert \ge \frac{\pi}{W}$ (Eq. 3.58)
INSTANTANEOUS RATE OF CHANGE $\left\lvert\frac{dx}{dt}\right\rvert = \left\lvert\frac{\Delta x}{\Delta t}\right\rvert_{\Delta t\to o}$	$\frac{1}{2\pi}\int_{-W}^{W}\lvert\omega\rvert\ f(\omega)d\omega$ (Eq. 3.44)	WA (Eq. 3.14)	WA (Eq. 3.58)
AVERAGE RATE OF CHANGE REFERENCED TO AN EXTREMUM $\left\lvert\frac{\Delta x}{\Delta t}\right\rvert_{x'(t_o)=0}$	$\frac{1}{2\pi\lvert\Delta t\rvert}\int_{-W}^{W} f(\omega)(1-\cos\omega\Delta t)d\omega$ for $\lvert\Delta t\rvert > 0$ (Eq. 3.45)	$WA\left[\frac{1-\cos W\Delta t}{W\lvert\Delta t\rvert}\right]$ for $0 < \lvert\Delta t\rvert \le \frac{\pi}{W}$; $\frac{2A}{\lvert\Delta t\rvert}$, $\lvert\Delta t\rvert \ge \frac{\pi}{W}$ (Eq. 3.34)	$WA\left[\frac{1-\cos W\Delta t}{W\lvert\Delta t\rvert}\right]$ for $0 < \lvert\Delta t\rvert \le \frac{\pi}{W}$; $\frac{2A}{\lvert\Delta t\rvert}$, $\lvert\Delta t\rvert \ge \frac{\pi}{W}$ (Eq. 3.59)

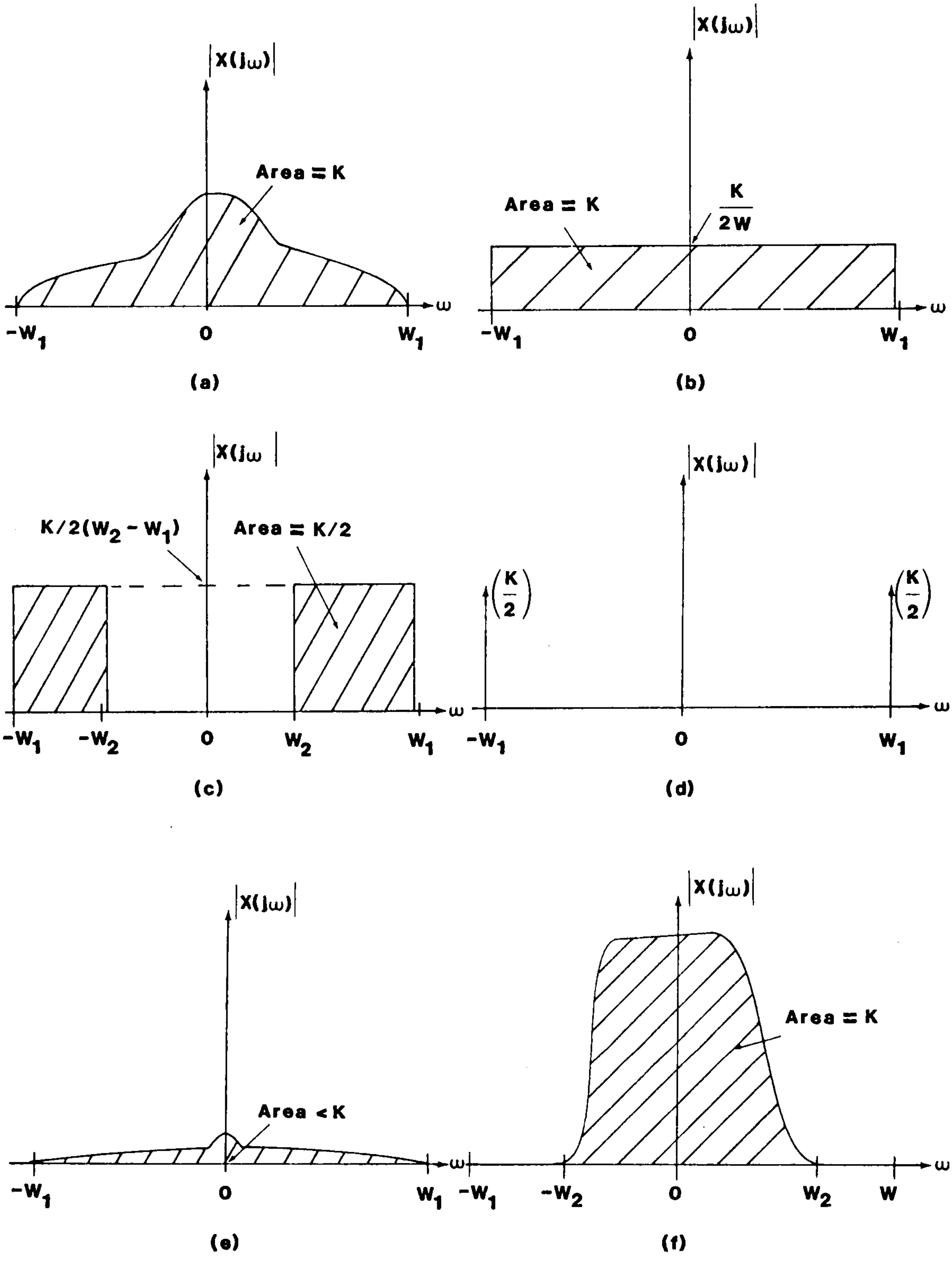


Figure 3.6. Typical spectra which may be bounded by Theorems 3.1 and 3.2.

in (a)-(d), it has a smaller bandwidth $W_2 < W_1$. It should be noted that although Theorems 3.1 and 3.2 provide identical bounds on each of the spectra in Figure 3.5(a)-(d), lower bounds may be individually computed for each spectrum by employing Theorems 3.3 and 3.4 with $f(\omega) = |X(j\omega)|$. The bounds provided by these theorems are located in column one of Table 3.2.

CHAPTER 4

SAMPLING AND AMPLITUDE UNCERTAINTY

4.1 Introduction

As mentioned in Chapter 2, conventional sampling theory is an outgrowth of the well-known Shannon sampling theorem, which is based on optimal signal reconstruction. While several other criteria for sampling rate selection have been proposed (see Section 2.4.2), there is a noticeable lack of results that bound the amplitude fluctuations of a sampled signal between samples. As is evidenced by the literature [Knight (1967), Curry (1976), Mitchell and Gonzalez (1978), Rix and Malengé (1980)], this lack of results does not indicate an absence of problems in which such sampling would be useful or is essential. Having developed analytical bounds on the slope and curvature of band-limited functions in Chapter 3, it is possible to establish sampling rates that will guarantee the detection of specific amplitude variations or bound the magnitude of such variations between consecutive samples. In this chapter, such sampling rates are examined in conjunction with several problems of practical importance. Moreover, the amplitude uncertainty inherent in all sampling processes is examined. This amplitude uncertainty can be defined as and attributed to the allowable amplitude variations of a signal during the sampling process itself or between consecutive samples.

4.2 Amplitude Uncertainty in Sampling and Modulation

4.2.1 Delta modulation and slope overload

Delta modulation (DM), the simplest form of differential pulse code modulation (DPCM), is one of the most valuable modern modulation methods. The block diagram of a DM system is shown in Figure 4.1(a). As can be seen, the difference between the incoming signal and a predicted value, the weighted sum of all past outputs, is quantized to one bit for transmission. The fundamental parameters of a DM system are the sampling rate f_s and quantization step size k. In terms of these parameters, the maximum possible average slope of the predicted signal $x_p(t)$ is kf_s.

A DM system is said to be slope overloaded [see Figure 4.1(b)] if the predicted signal[a] $x_p(t)$ cannot follow x(t) within an error magnitude of k. In order that a system not be overloaded, the following condition must be satisfied:

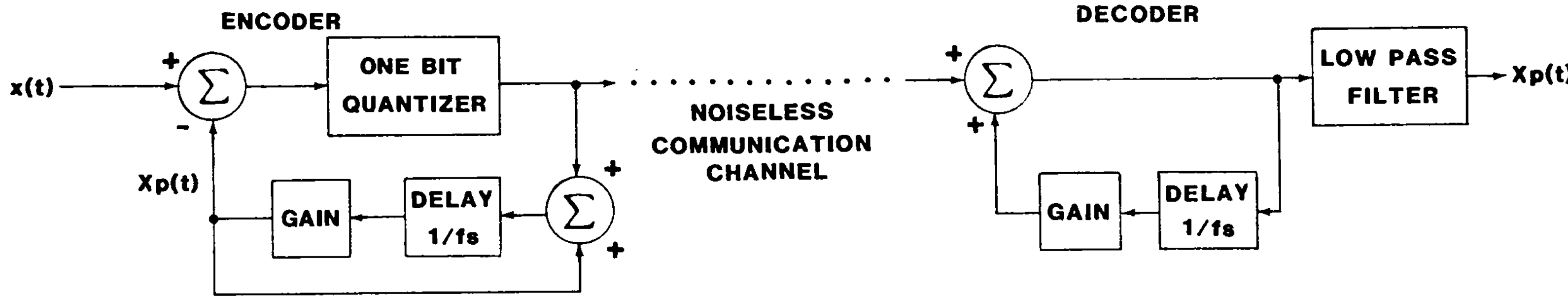
$$kf_s \geqslant |x'(t)| \tag{4.1}$$

While kf_s is usually chosen to be greater than $|x'(t)|_{max}$ [Peebles (1976)], it is sufficient to let

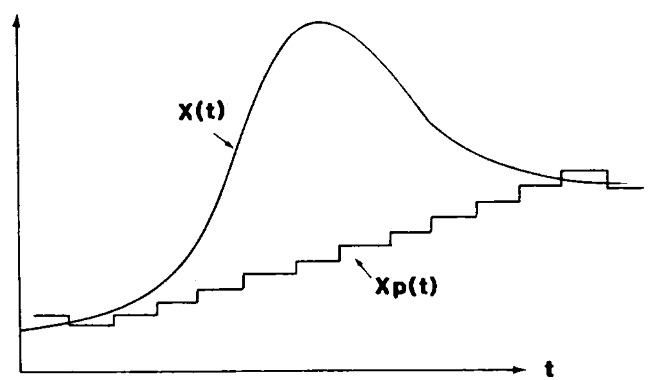
$$kf_s \geqslant \left|\frac{\Delta x}{\Delta t}\right|_{\Delta t=1/f_s} = 2Af_s \sin \frac{1}{2Af_s} \ . \tag{4.2}$$

---

[a]In the absence of noise, the decoded output of a DM system is equal to $x_p(t)$.

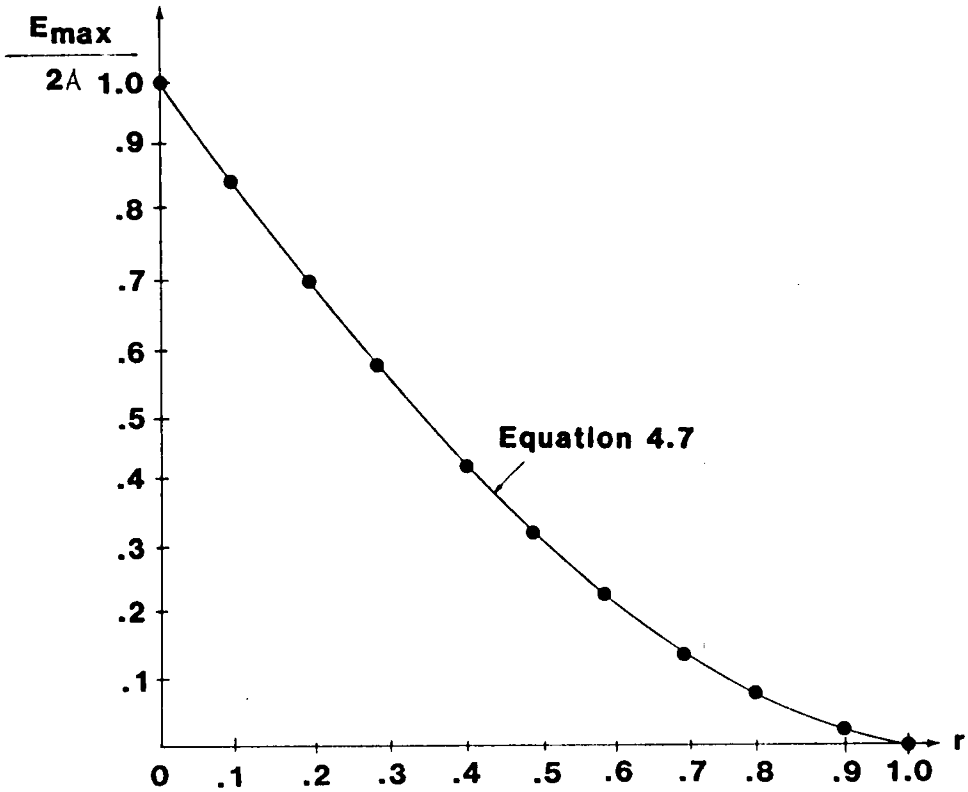


(a)



(b)

Figure 4.1. Delta modulation system (a), slope overload (b), and maximum error (c).



(c)

Figure 4.1. (Continued)

This result, which is defined for the set of functions $\{x(t): M_0 \leqslant A\}$, follows from Theorem 3.1. For a given input signal x(t) and $kf_s < 2Af_s \sin(1/2Af_s)$, slope overload can be reduced by increasing the step size k and/or sampling rate f_s.

Since slope overload reduces the accuracy with which the transmitted and reconstructed signal approximates x(t), it is desirable to
(1) measure the degree by which x(t) loads a given DM system, and
(2) bound the maximum error between $x_p(t)$ and x(t) for a given kf_s.
The problem of characterizing the degree by which x(t) loads a given DM system was considered by Abate (1967), who defined the slope loading factor s as the ratio of the slope capability of the DM system (kf_s) to the effective value of $x'(t)$. For band-limited white noise, $s = \sqrt{3}\, kf_s/2\pi f_{hi}$. The maximum error between $x_p(t)$ and x(t) can be bounded as follows.

Consider a set of functions $\{x(t): M_0 \leqslant A\}$ which are band-limited to $W = 2\pi f_{hi}$ rad/sec. The average rate of change of any member of this set is bounded by Equation 3.14 of Theorem 3.1. Since the maximum average slope of a DM system is kf_s, the difference $E = x(\Delta t) - x_p(\Delta t)$ may be expressed as

$$E = 2A \sin \frac{W|\Delta t|}{2} - kf_s\ |\Delta t|\ ,\ |\Delta t| \leqslant \frac{\pi}{W}\ . \qquad (4.3)$$

Thus, the error E is a function of Δt. Taking the derivative of E with respect to $|\Delta t|$ and setting it equal to zero, it follows that

$$|\Delta t| = \frac{2}{W}\cos^{-1}\frac{kf_s}{AW}\ , \ |\Delta t| \leqslant \frac{\pi}{W} \tag{4.4}$$

so that

$$E_{max} = 2A \sin \cos^{-1}\frac{kf_s}{AW} - \frac{2kf_s}{W}\cos^{-1}\frac{kf_s}{AW}\ , \ 0 \leqslant \frac{kf_s}{AW} \leqslant 1 \tag{4.5}$$

Defining the *rate loading factor* r as

$$r = \frac{kf_s}{AW} = \frac{kf_s}{2\pi A f_{hi}}\ , \tag{4.6}$$

Equation 4.5 becomes

$$\frac{E_{max}}{2A} = \sin \cos^{-1} r - r \cos^{-1} r\ , \ 0 \leqslant r \leqslant 1\ . \tag{4.7}$$

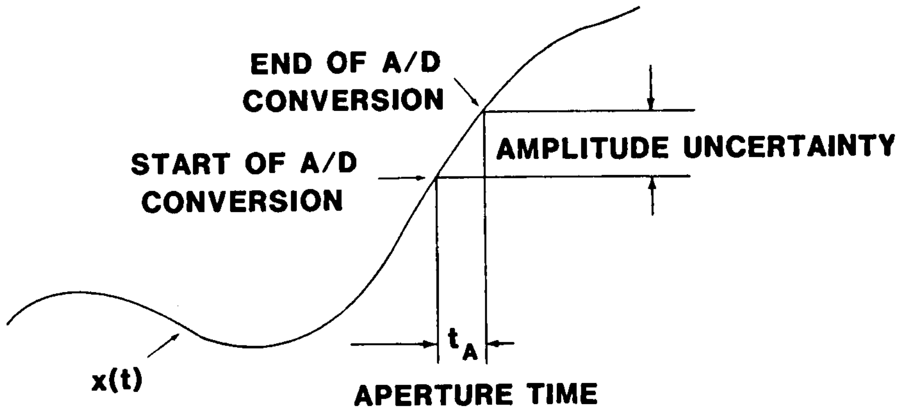
Since $|x(t)| \leqslant A$ (by Equation 3.56), $E_{max}/2A$ can be interpreted as a normalized or fractional error. The rate loading factor r represents the ratio of the slope capability of the DM system to the maximum instantaneous slope of the set of functions $\{x(t): M_0 \leqslant A\}$, $WA = 2\pi f_{hi}A$ of Equation 3.14. Figure 4.1(c) shows the normalized errors $E_{max}/2A$ associated with various values of rate loading factor r. Note that when $r = 1$, the slope capability of the DM system is equivalent to the maximum slope of x(t) and there is no error due to slope overload. When $r = 0$, the system cannot follow any signal since $kf_s = 0$, and the error is maximum.

EXAMPLE 4.1 Let x(t) be band-limited to $W = 2\pi \times 10^{-3}$ rad/sec and confined to the amplitude interval [-5, 5]. If a DM system with rate loading factor r = 0.5 is used to quantize x(t), the maximum error which could occur is $E_{max}/2(5) = 0.342$ or $E_{max} = 3.42$. The corresponding slope capability of the DM system is $ks = 5\pi \times 10^{-3}$. If r were increased to 0.9, the maximum possible error would decrease to 0.3. This error is considerably less than the 34.2% error which can result if r = 0.5. □

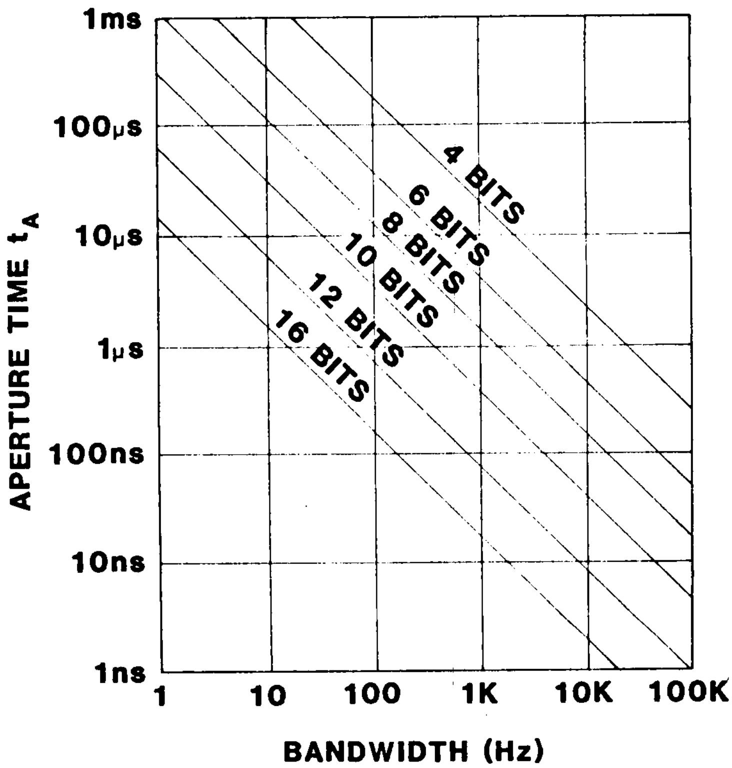
In a DM system, slope overload occurs if the signal changes by more than the step size during any sample period. A single bit is used to represent the difference between x(t) and the predicted value $x_p(t)$. If this difference is quantized and coded into an N bit code for each sample interval, the system is called a delta or differential pulse code modulation (DPCM) system. The results of this section can be extended to such systems.

4.2.2 Aperture time and amplitude uncertainty

An analog-to-digital (A/D) converter performs the operations of sampling, quantization, and coding in some finite amount of time. The amount of time required, called the aperture time, depends both on the resolution of the converter (i.e., its number of bits) and the particular conversion method used. As shown in Figure 4.2(a), any given aperture time t_A is associated with some amplitude uncertainty. The aperture time and amplitude uncertainty are related by the time rate of change of the signal to be sampled. Thus, the speed of conversion or aperture time required in a particular situation



(a)



(b)

Figure 4.2. Aperture time and amplitude uncertainty of bounded band-limited signals.

depends on the allowable rate of change of the signal to be sampled and the amount of resolution desired.

Consider the N bit A/D conversion of a band-limited signal x(t) that is confined to the amplitude interval [-A, A]. Let $f_h = W/2\pi$ represent the highest frequency present in the Fourier transform of x(t). The maximum amplitude uncertainty that can be tolerated in an A/D conversion requiring one bit resolution is $2A/(2^N - 1)$, where the N bits of the A/D converter are assumed to represent the entire amplitude range of x(t). Letting $\Delta x = 2A/(2^N - 1)$ and $t_A = \Delta t$ in Equation 3.58 of Corollary 3.1, it follows that

$$t_A \leqslant \frac{1}{\pi f_{hi}} \sin^{-1} \left[\frac{1}{2^N - 1}\right] . \tag{4.8}$$

This result, which defines the aperture time required to guarantee a conversion accuracy of one part in 2^N, is easily extended to other resolutions.

<u>EXAMPLE 4.2</u> The aperture time required to digitize a 4.3 MHz video signal to 8 bits resolution is

$$t_A \leqslant \frac{1}{\pi(4.3 \times 10^6)} \sin^{-1} \left(\frac{1}{255}\right) = 290 \text{ psec} \tag{4.9}$$

The digitization of a 1 KHz signal with the same resolution would require an aperture time of 1.25 μsec. For 10 bits resolution, the same 1 KHz signal requires an aperture time of 311 nsec. □

Amplitude uncertainty is inherent in all methods of A/D conversion. As can be seen in Example 4.2, even slowly varying signals at moderate resolution levels require an extremely fast conversion time to allow no more than one bit amplitude uncertainty. Fortunately, there is a simple and inexpensive solution to this problem – the use of a sample and hold to take a rapid sample of the signal and hold its value for the required conversion time. For a given signal bandwidth and desired resolution, Equation 4.8 specifies the minimum acceptable aperture time of the required A/D converter or sample and hold. Aperture times for various bandwidths and resolutions are shown in Figure 4.2(b).

4.3 Amplitude Uncertainty between Samples

In the previous section, two commonly encountered problems involving the detection of amplitude variations were considered. Each of these problems is peculiar to some aspect of a sampling or modulation process. In this section, the general problem of detecting the amplitude variations of a band-limited signal from equally spaced samples is examined. The sampling process itself is considered to be ideal.[a]

4.3.1 A sampling theorem

Let x(t) be a band-limited function that is confined to some frequency interval [-W, W], and let $M_0 \leqslant A$ for some constant $A > 0$. As illustrated in Figure 4.3(a), it is in general impossible to guarantee

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[a]All samples take on the instantaneous values of the signal with no amplitude uncertainty.

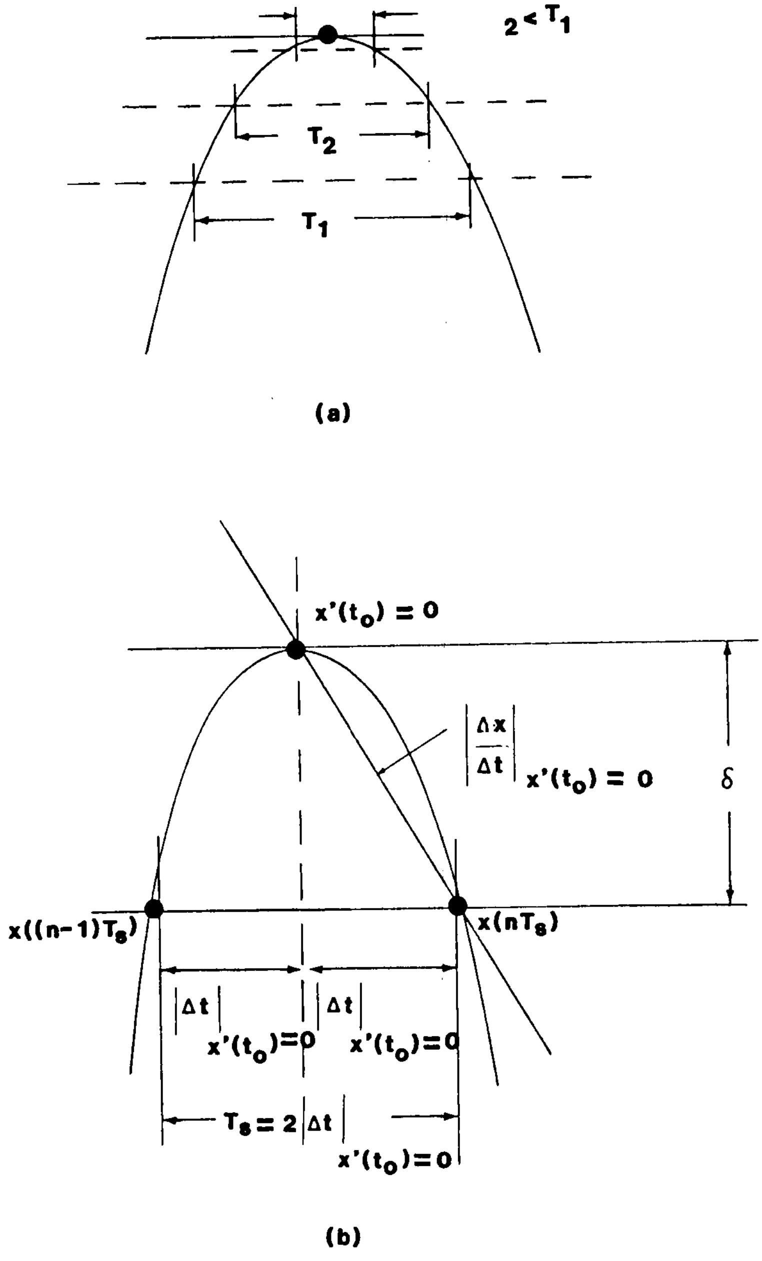

Figure 4.3. The amplitude variations between samples of $x(t)$.

the detection of an extremum of $x(t)$ with any finite sampling interval T_s. The best that can be guaranteed is that

$$|x(t) - x((n-1)T_s)| \leqslant \delta$$

and

$$|x(nT_s) - x(t)| \leqslant \delta \tag{4.10}$$

for some finite nonzero amplitude variation δ and $(n - 1)T_s \leqslant t \leqslant nT_s$. As long as $x(t)$ is strictly increasing or decreasing between samples, $x(t)$ is confined between $x((n - 1)T_s)$ and $x(nT_s)$. Thus, the maximum sample spacing T_s that will guarantee that Equation 4.10 is satisfied for all n can be determined from Figure 4.3(b), where $|\Delta x/\Delta t|_{x'(t_0)=0}$ is defined by Theorem 3.2. In accordance with this theorem,

$$\left|\frac{\Delta x}{\Delta t}\right|_{x'(t_0)=0} \leqslant \begin{cases} 0 \; , \; \Delta t = 0 \\ WA \left[\dfrac{1 - \cos W|\Delta t|}{W|\Delta t|} \right] \; , \; 0 < |\Delta t| \leqslant \dfrac{\pi}{W} \\ \dfrac{2A}{|\Delta t|} \; , \; |\Delta t| \geqslant \dfrac{\pi}{W} \end{cases} \tag{4.11}$$

Letting $|\Delta x| = \delta$ and solving for $|\Delta t|_{x'(t_0)=0}$, it follows that

$$|\Delta t|_{x'(t_0)=0} \leqslant \frac{1}{W} \cos^{-1} \left(1 - \frac{\delta}{A}\right) \; , \; |\delta| \leqslant 2A \tag{4.12}$$

or

$$T_s \leqslant \frac{1}{\pi f_{hi}} \cos^{-1} \left(\frac{A - \delta}{A}\right) \; , \; |\delta| \leqslant 2A \tag{4.13}$$

where it has been noted that $T_s = 2|\Delta t|_{x'(t_0)=0}$ [see Figure 4.3(b)] and $W = 2\pi f_{hi}$. The following sampling theorem is a direct consequence of this result and the relationship $f_s = 1/T_s$:

THEOREM 4.1 Let the function $x(t)$ with Fourier transform $X(j\omega)$ be such that $X(j\omega) = 0$ for $|\omega| > W = 2\pi f_{hi}$, and let $M_0 = 1/2\pi \int_{-W}^{W} |X(j\omega)|d\omega \leqslant A$ for some constant $A > 0$. If

$$f_s \geqslant \frac{\pi f_{hi}}{\cos^{-1}\left[\frac{A-\delta}{A}\right]}, \quad |\delta| \leqslant 2A \tag{4.14}$$

then

$$\begin{gathered} |x(t) - x((n-1)T_s)| \leqslant \delta \\ \text{and} \\ |x(nT_s) - x(t)| \leqslant \delta \end{gathered} \tag{4.15}$$

for all $(n-1)T_s \leqslant t \leqslant nT_s$ and n an integer. □

This theorem defines the lowest sampling rate f_s that will guarantee that, between successive samples of a function in the set $\{x(t): M_0 \leqslant A\}$, $|x(t)|$ does not deviate from either sample by more than δ. Figure 4.4 depicts this rate f_s (normalized by f_{hi}) for various values of δ. Since the Nyquist rate $f_N = 2f_{hi}$, Equation 4.14 of Theorem 4.1 may be written

$$\frac{f_s}{f_N} \geqslant \frac{\pi}{2\cos^{-1}\left[\frac{A-\delta}{A}\right]}, \quad |\delta| \leqslant 2A \tag{4.16}$$

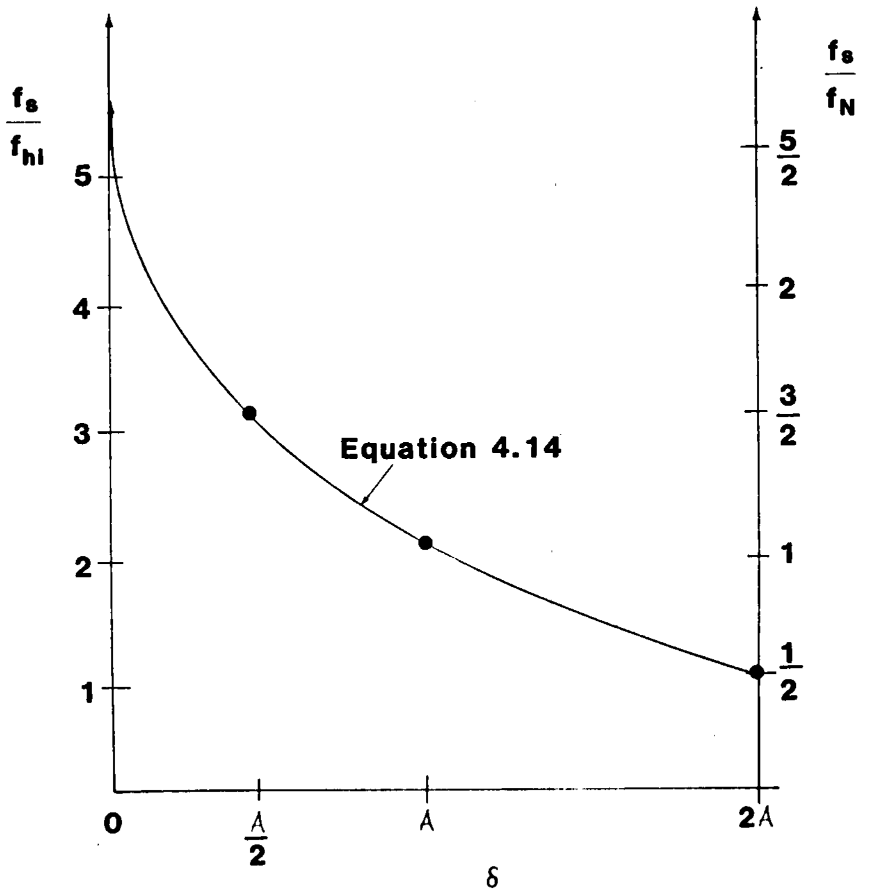

Figure 4.4. Normalized sampling rates f_s/f_{hi} and f_s/f_N from Theorem 4.1 for various values of δ.

and it is evident that when x(t) is sampled at the Nyquist rate, $f_s/f_N = 1$ and $\delta = A$. This maximum deviation $\delta = A$ can occur, for instance, when sampling a sinusoid $A \sin 2\pi f_{hi} t$ at its zero crossings. If x(t) is bounded in amplitude, $|x(t)| \leq A$, Equation 4.14 becomes (see Table 3.1, page 69)

$$f_s \geq \frac{\pi f_{hi}}{\cos^{-1}\left[\frac{A - \delta}{A}\right]}, \quad |\delta| < 2A . \tag{4.17}$$

It should be noted that Theorem 4.1 does not guarantee the detection of every extremum of x(t); this would require an infinite sampling rate.

<u>EXAMPLE 4.3</u> The electrocardiogram (ECG or EKG) is a graphic recording or display of the time-variant voltages produced by the myocardium during the cardiac cycle. It is used clinically in diagnosing various diseases and conditions associated with the heart and serves as a time reference for other measurements. A cardiologist looks critically at the various time intervals, polarities, and amplitudes in an EKG to arrive at his diagnosis. Since the voltages produced by a normal myocardium are bounded in frequency to 100 Hz, an EKG must be sampled at a rate

$$f_s \geq \frac{\pi(100)}{\cos^{-1}\left(\frac{A - .1A}{A}\right)} = 697 \text{ Hz} \tag{4.18}$$

to guarantee that the EKG does not deviate from its samples by more than 10% of its maximum amplitude. It must be sampled at 2.22 KHz to ensure that there are no deviations greater than 1%. □

<u>EXAMPLE 4.4</u> Equation 3.53 of Example 3.7 bounds the average rate of change referenced to an extremum of those functions x(t) whose spectra are contained within the rectangular spectrum of Figure 3.5(a), page 61. The following expression for the sampling rate required to guarantee that these x(t) do not deviate from their nearest samples by more than δ is a direct consequence of this result:

$$f_s \geqslant \frac{\pi f_{hi}}{\mathrm{Sinc}^{-1}\left[1 - \frac{\delta}{2Kf_{hi}}\right]}, \ |\delta| \leqslant 2Kf_{hi} \tag{4.19}$$

where Sinc x = (sin x)/x. This result follows from Equation 3.53 by the same procedure used to derive Theorem 4.1 from Equation 3.34 of Theorem 3.2. □

4.3.2 <u>Estimating level crossing profiles</u>

While statistical characterizations of random signals often employ various moments of the signals' probability density, autocorrelation, or power spectral density functions, neither the moment nor the probability density function provide any information concerning the dynamic behavior of a signal. In this context, the term dynamic behavior is used to refer to such signal characteristics as oscillating activities and excursion rates at a given amplitude level. Level crossing statistics, on the other hand, offer an approach to the interpretation and characterization of time signals that provide both frequency and amplitude information. In particular, the measurement of a signal's *level crossing rate,* defined to be the number of crossings per unit time of an amplitude level, yields information as to the "apparent frequency" associated with the level.

The expected crossing rate of level u by a continuous, stationary, and ergodic random signal x(t) has been shown to be [Rice (1944, 1945)],

$$\overline{N}_u(x\ ;\ 1) = \int_{-\infty}^{\infty} |v|\ p_{x,x'}(u,v)dv \qquad (4.20)$$

where $p_{x,x'}(u,v)$ is the joint probability density function of x(t) and its derivative x′(t). By assuming stationarity, the number of crossings during time T is obtained from

$$\overline{N}_u(x\ ;\ T) = T\ \overline{N}_u(x\ ;\ 1)\ . \qquad (4.21)$$

Thus, an observation of the number of times that x(t) crosses level u during time T provides an estimate of the crossing rate. Although the theoretical expression for the level crossing rate (Equation 4.20) appears awesome (since it involves the joint probability density function of the signal and its derivative), the computation of a signal's level crossing rates is quite simple.

A commonly used method [Suthasinekul (1976), Mitchell and Gonzalez (1978)] for generating level crossing rates is illustrated in Figure 4.5. A piecewise linearly interpolated approximation to x(t), denoted $\hat{x}(t)$, is first obtained by sampling x(t) every T_s seconds and quantizing the sampled values to integers in the interval $[0, 2^N - 1]$, where N represents the number of bits used for quantization. Assuming that x(t) is completely contained between $\hat{x}((n-1)T)$ and $\hat{x}(nT)$ for $(n-1)T \leqslant t \leqslant nT$ and utilizing the intermediate value theorem [Tierney (1969)], all

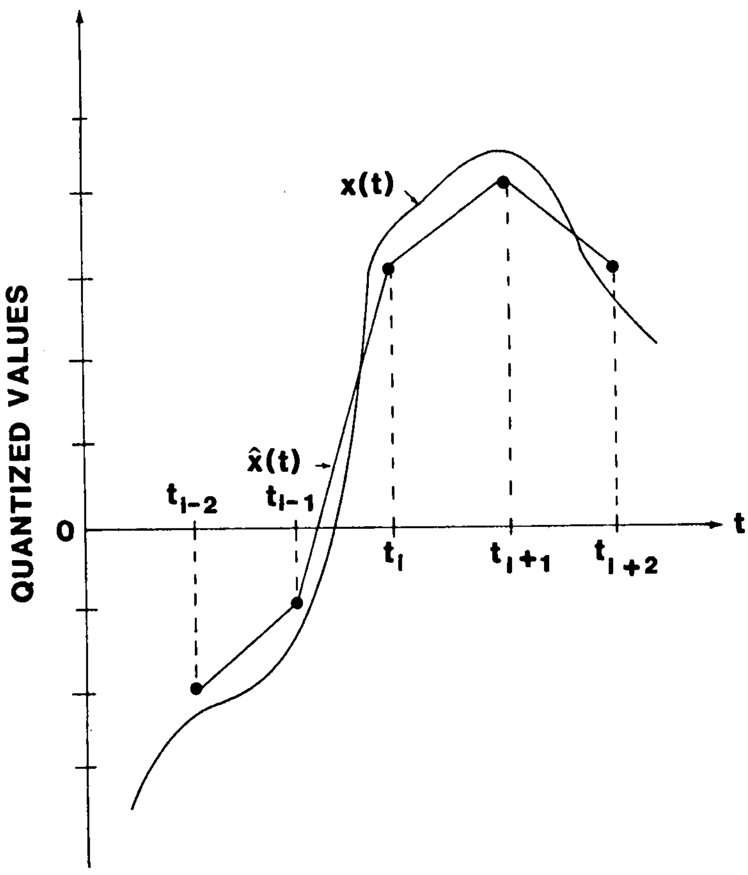

Figure 4.5. A quantized piecewise linearly interpolated signal.

levels between $\hat{x}((n - 1)T)$ and $\hat{x}(nT)$ are considered to be crossed. Thus, a level crossing counter for each level of interest is incremented when its value lies between the amplitudes of consecutive sample points. At the end of some observation interval, the counter for each level indicates the number of times that the level was crossed. The estimated crossing rates can then be established by dividing the total number of counts for each level by the time of observation. The curve obtained by plotting the crossing rates versus their respective levels is called a *level crossing profile* LCP. Figure 4.6 depicts a typical LCP.

The accuracy with which an estimated LCP represents the true behavior of a random signal is a function of the amount of data on which it is based. The estimation of level crossing rates (as well as probability density functions) generally requires a considerably large amount of data, since events with very small and very large probabilities are of equal importance. These large amounts of data, which correspond to long observation intervals, are needed to collect sufficient high amplitude level crossings to form statistically viable LCP estimates. In general, statistical reliability decreases as the observation interval (for a given sampling rate) becomes shorter. While the problem of specifying optimum observation intervals remains unsolved, some general guidelines to the selection of observation intervals have been proposed [Steinberg et al. (1955), Bendat and Piersol (1971), Beuchamp and Yuen (1979)].

EXAMPLE 4.5 Consider again the band-limited function $x(t) = 2 \cos \omega_0 t + \sin 3\omega_0 t$ and its piecewise linearly interpolated approximation $\hat{x}(t)$

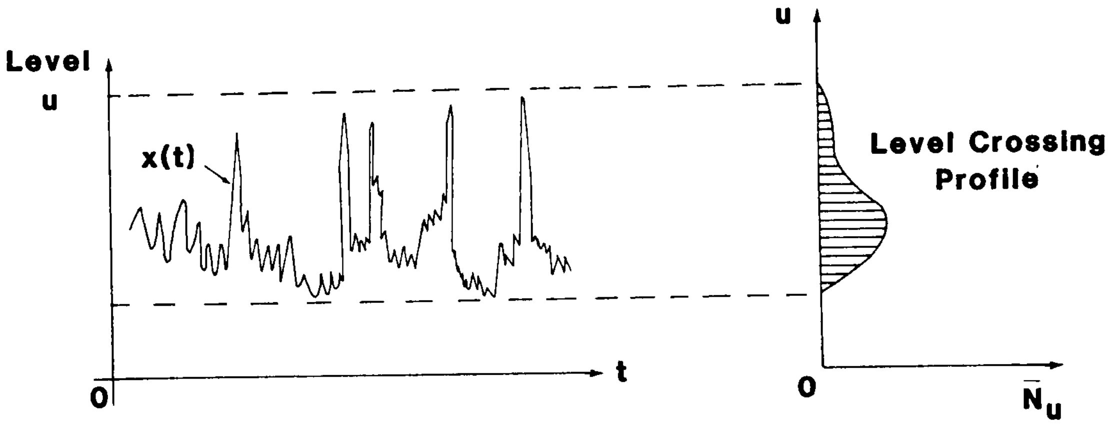

Figure 4.6. A typical level crossing profile.

shown in Figure 1.1, page 4. As can be seen, $x(t)$ crosses the level $x = 1$ four times in every period; the crossing rate for this level is $2\omega_0/\pi$ crossings/sec. Observing $\hat{x}(t)$ (which was obtained from samples of $x(t)$ taken at the Nyquist rate) over one period of $x(t)$, an estimated crossing rate of only ω_0/π crossings/sec is obtained. Moreover, this estimated rate, which is a mere 50% of the actual crossing rate, does not increase as the time of observation increases. Thus, the estimated rate can only be improved by increasing the sampling rate. □

As illustrated in the above example, crossing rates which are obtained from a linearly interpolated approximation of a signal are always less than the actual crossing rates of the signal. This results from the fact that level crossings are missed when an extremum of a signal occurs above (or below) a level while the preceding and following sample points are below (or above) the level. In fact, as can be seen in Figure 4.7, no finite sampling rate can guarantee the detection of every level crossing of a signal. As an extremum of a signal becomes tangent to some level L_n, the sampling rate T_s^{-1} required to detect the resulting pair of level crossings approaches infinity. Thus, for any finite sampling rate, it is possible to miss at least two level crossings (the "up" and "down" crossing of a single level) for every extremum of the signal.

While no sampling rate can guarantee the detection of every level crossing of a function or signal, Theorem 4.1 can be used to obtain the following expression for those sampling rates f_s that will ensure that

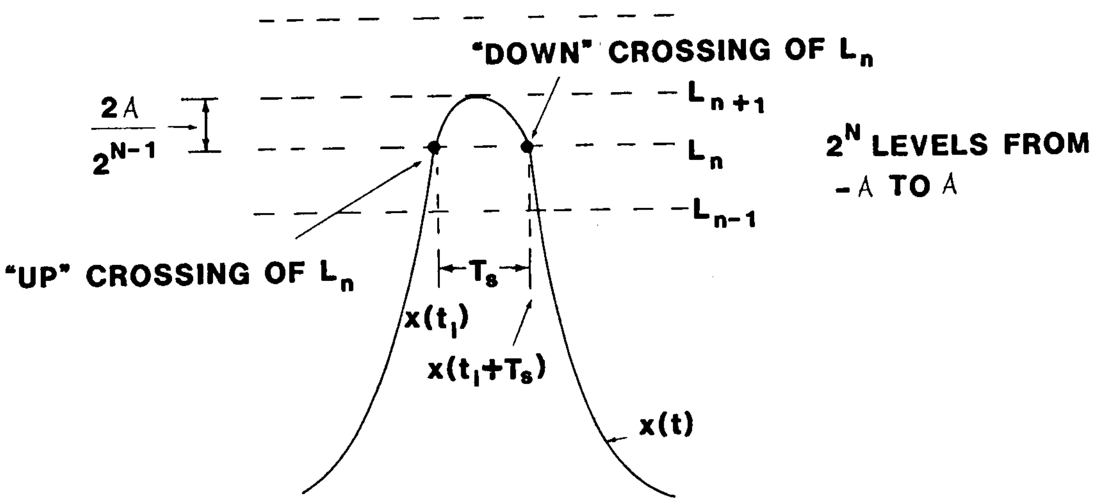

Figure 4.7. Sampling to ensure that no more than one level crossing is missed.

no more than two crossings of one level will be missed per extremum of a band-limited function:

$$f_s \geqslant \frac{\pi f_{hi}}{\cos^{-1}\left[\frac{2^N - 3}{2^N - 1}\right]} . \tag{4.22}$$

This result is a direct consequence of the fact that (see Figure 4.7) those sampling rates which allow the crossing of more than one level to be missed will not guarantee that, between successive samples, the function will not deviate from either sample by more than the amplitude difference between levels. Thus, Equation 4.22 follows from Theorem 4.1 by noting that $|x(t)| \leqslant A$ (Equation 3.57) and letting $\delta = 2A/(2^N - 1)$, where N is the number of bits used to represent the amplitude interval [-A, A]. (Note that δ is the quantization interval). To miss no more than 2m level crossings of m levels per extremum,

$$f_s \geqslant \frac{\pi f_{hi}}{\cos^{-1}\left[1 - \frac{2m}{2^N - 1}\right]} , \; m \geqslant 1 \tag{4.23}$$

or

$$\frac{f_s}{f_{hi}} \geqslant \frac{\pi}{\cos^{-1}\left[1 - \frac{2m}{2^N - 1}\right]} , \; m \geqslant 1 . \tag{4.24}$$

This equation is evaluated for several values of m and N in Table 4.1, and plotted as a function of m for various N in Figure 4.8. As can be seen, f_s/f_{hi} decreases as m + 1/N increases.

TABLE 4.1

NORMALIZED SAMPLING RATE f_s/f_{hi}

Number of Bits N	Maximum Number of Levels Missed m per Extremum														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	2.6	1.6	-	-	-	-	-	-	-	-	-	-	-	-	-
3	4.1	2.8	2.2	1.8	1.6	1.3	1	-	-	-	-	-	-	-	-
4	6	4.2	3.4	2.9	2.6	2.3	2.1	1.9	1.8	1.6	1.5	1.4	1.3	1.2	1
5	8.7	6.1	5	4.3	3.8	2.4	3.2	2.9	2.8	2.6	2.5	2.3	2.2	2.1	2
6	12.4	8.8	7.1	6.2	5.5	5	4.6	4.3	4.1	3.8	3.6	3.5	3.3	3.2	3.1
7	17.7	12.5	10.2	8.8	7.9	7.2	6.6	6.2	5.8	5.5	5.3	5	4.8	4.6	4.5
8	25.1	17.7	14.5	12.5	11.2	10.2	9.4	8.8	8.3	7.9	7.5	7.2	6.9	6.6	6.4
9	35.5	25.1	20.5	17.7	15.9	14.5	13.4	12.5	11.8	11.2	10.7	10.2	9.8	9.4	9.1
10	50.2	35.5	29	25.1	22.5	20.5	19	17.7	16.7	15.9	15.1	14.5	13.9	13.4	12.9
11	71.7	50.2	41	35.5	31.8	29	26.8	25.1	23.7	22.5	21.4	20.5	19.7	19	18.3
12	100	71.7	58	50.3	44.9	41	38	35.5	33.5	31.8	30.3	29	27.9	26.8	25.9

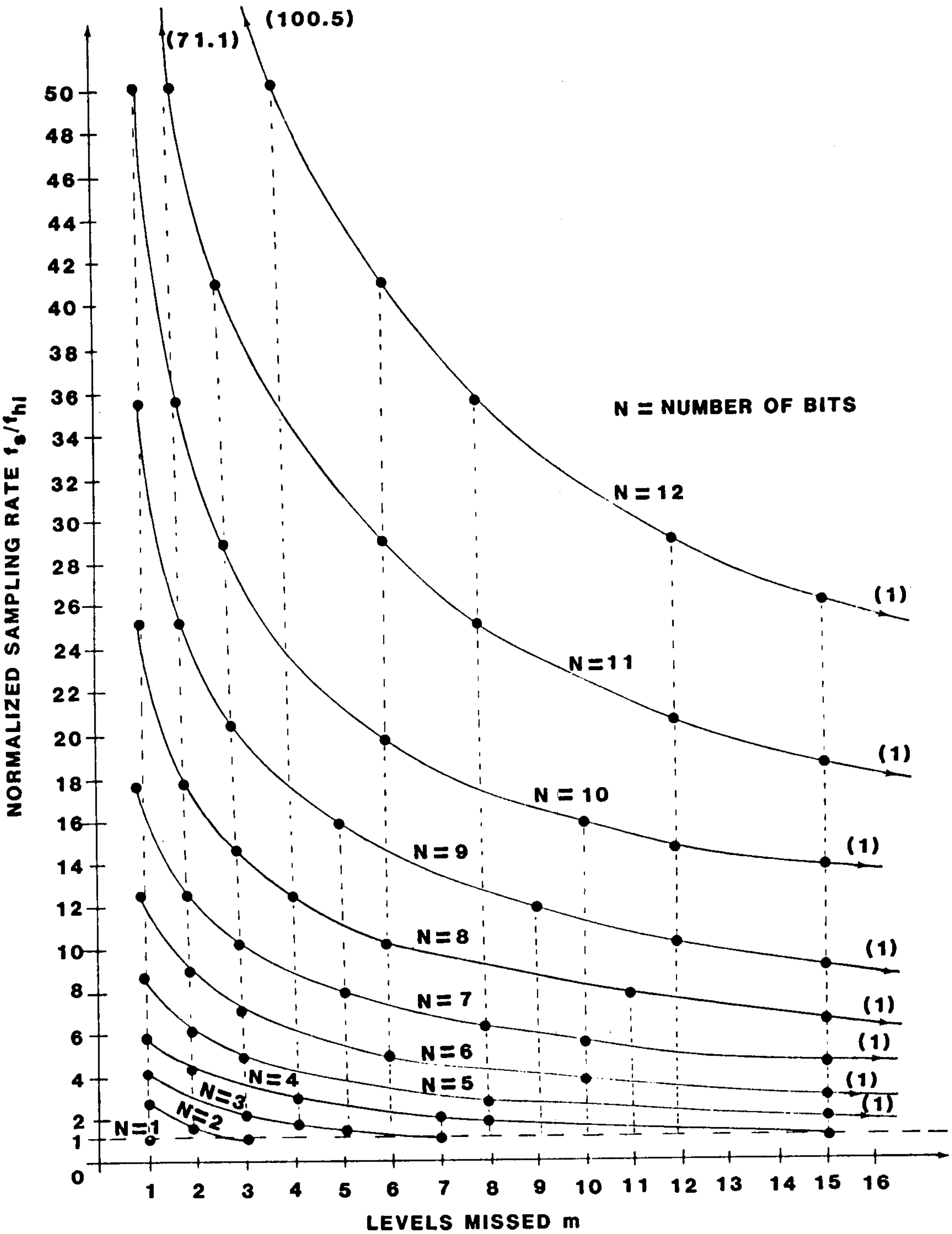

Figure 4.8. The normalized sampling rate f_s/f_{hi} versus maximum levels missed m per extremum for various sampling resolutions N.

<u>EXAMPLE 4.6</u> Consider a band-limited signal that is quantized with 8 bit resolution, N = 8. To form a LCP using 256 levels, the signal should be sampled at a rate $f_s = 25.1\ f_{hi}$ to miss no more than one level crossing per extremum (m = 1) and at $f_s = 6.4\ f_{hi}$ to miss no more than 15 (m = 15). If the signal is sampled at the Nyquist rate, it is possible to miss 128 level crossings for every extremum. □

<u>EXAMPLE 4.7</u> Equation 4.19 of Example 4.4 provides an expression for the sampling rate required to guarantee that a function x(t) whose Fourier spectrum is bounded by the rectangular spectrum of Figure 3.5(a) (page 61) does not deviate from its nearest samples by more than δ. The following expression for the sampling rate required to ensure that the LCP of x(t) will not miss more than 2m level crossings of m levels per extremum of x(t) is a direct consequence of this result:

$$f_s \geqslant \frac{\pi f_{hi}}{\mathrm{Sinc}^{-1}\left[1 - \frac{m}{2^N - 1}\right]}, \quad m \geqslant 1 \ . \tag{4.25}$$

□

Unlike the histogram approach to approximating a probability density function (PDF), where only the amplitude bin in which a sample point occurs is incremented, the above methods for computing level crossing profiles linearly interpolates crossings between sample points. This is due to the fact that the PDF of x(t), denoted $p_x(u)$, is a descriptor of the relative amount of time x(t) spends in any interval from u to u + du, whereas the level crossing rate $\overline{N}_u$ is a measure of the number of times that x(t) crosses level u. The result is a relatively smooth level crossing profile as opposed to an often jagged histogram. It has been shown that a smoother approximation of $p_x(u)$ can be obtained from the LCP of x(t) [Suthasinekul (1976)]. Such an approximation is shown for the case of a Gaussian PDF in Figure 4.9.

4.3.3 Pulse detection and elimination

The sampling rate developed in the Section 4.3.1 can also be employed in the detection and elimination of high frequency pulses from a low frequency signal. Such pulses [see Figure 4.10(a)] cannot be eliminated by conventional filtering techniques, since they may possess significant spectral components within the frequency band of the signal. As illustrated in Figure 4.10(b), filtering would merely spread the pulses into the signal. A digital solution to this problem, which is commonly encountered in the monitoring of nuclear power plants, is depicted in Figure 4.11.

As can be seen in Figure 4.11, two A/D converters are employed to continually digitize the incoming signal x(t). One converter is used to sample x(t), while the other is used to sample an amplified and limited version of x(t). (The amplification and limiting facilitates efficient use of the A/D converter's amplitude range.) Successive samples of the limited signal are clocked into a shift register, which outputs to a digital-to-analog (D/A) converter. As long as there are no pulses present in x(t), the limited signal is simply delayed by the shift register and converted back to analog form. If, however, a pulse is detected, the D/A converter is inhibited (causing it to hold its

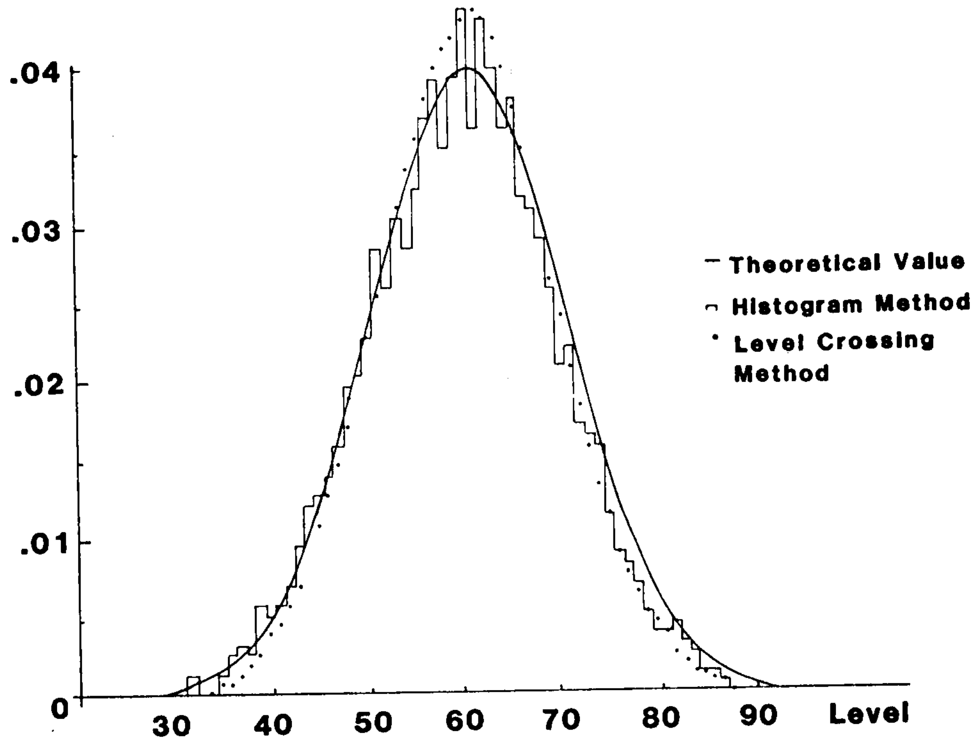

Figure 4.9. Approximating a Gaussian probability density function from a histogram and level crossing profile.

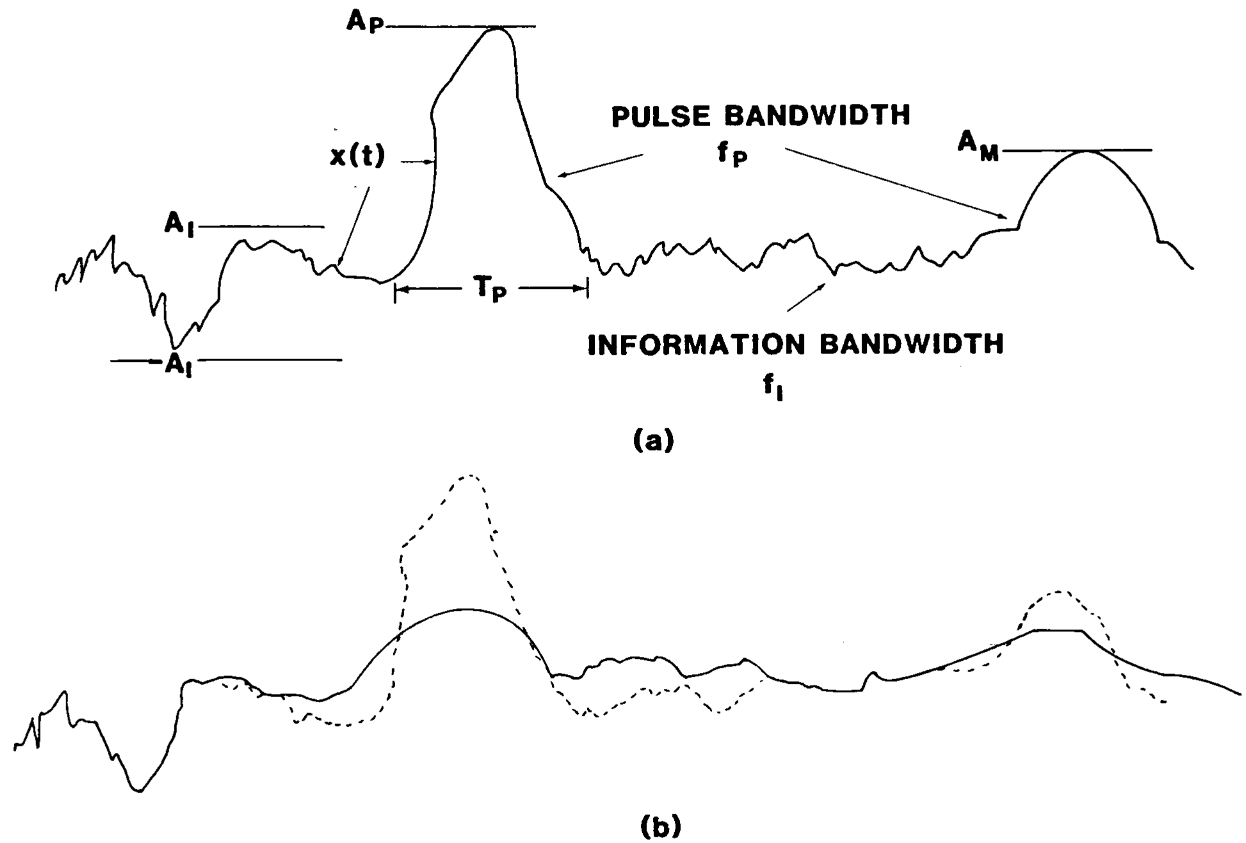

Figure 4.10. The pulse elimination problem.

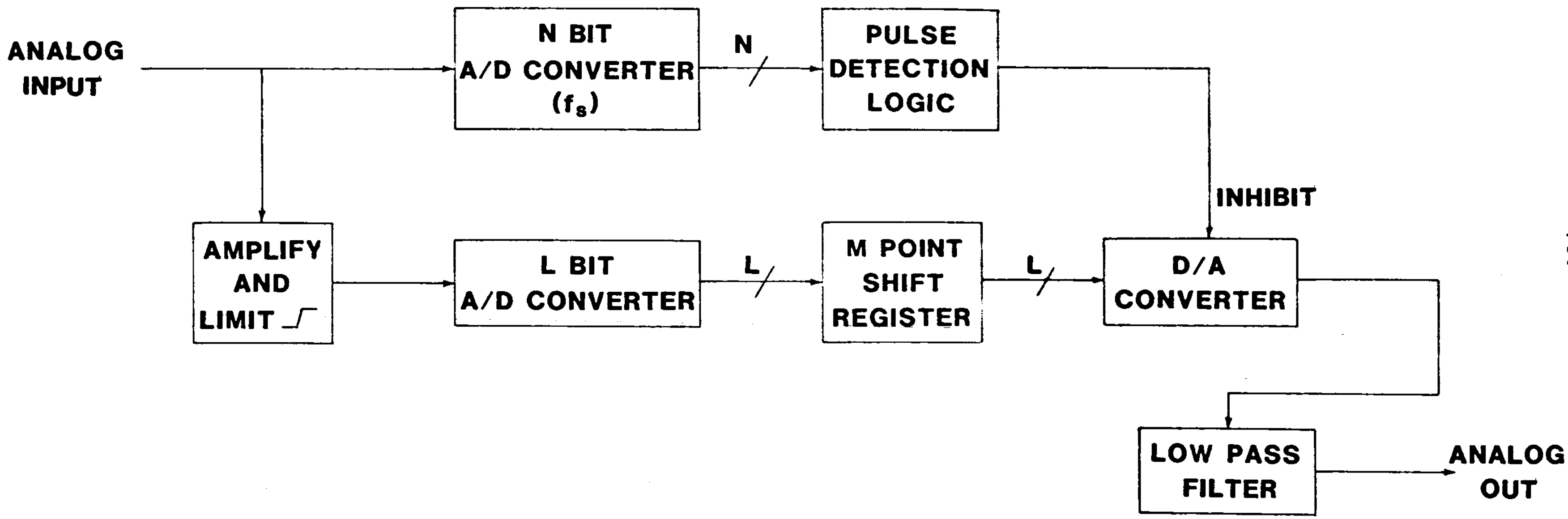

Figure 4.11. A digital solution to the pulse elimination problem.

last value) until the pulse has been clocked out of the shift register. The performance of the system is limited by the quantization interval and sampling rate associated with the A/D converter used for pulse digitization, and the length of the shift register. The quantization interval and sampling rate associated with the converter used for sampling the limited signal should be selected for optimal signal reconstruction via low pass filtering.

Consider a signal x(t) as shown in Figure 4.10(a), and let the information portion of x(t), denoted $x_I(t)$, be limited in frequency to $W_I = 2\pi f_I$ rad/sec and confined to the amplitude interval $[-A_I, A_I]$. Suppose also that the pulses to be eliminated are characterized by bandwidth $W_p = 2\pi f_p$ rad/sec, maximum amplitude A_p, minimum amplitude A_M, and maximum pulse width T_p sec, where $A_p \geqslant A_M > A_I$. Since $A_M > A_I$, the maximum acceptable quantization interval q of the A/D converter employed for pulse digitization is

$$q \leqslant \frac{A_M - A_I}{2} \tag{4.26}$$

This quantization interval divides the amplitude range of the pulses into $4A_p/(A_M - A_I)$ bins, and separates A_M from A_I by two bins. Thus, the sampling rate required for pulse detection follows from Theorem 4.1 by letting $A = A_p$, $f_{hi} = f_p$, and $\delta = (A_M - A_I)/2$:

$$f_s \geqslant \frac{\pi f_p}{\cos^{-1}\left[\dfrac{2A_p + A_M + A_I}{2A_p}\right]} \tag{4.27}$$

Note that this sampling rate f_s, defined for the A/D converter used in pulse digitization, is independent of the bandwidth f_I of the information signal. In practice, q would be chosen so that

$$q = \left\langle \frac{2A_p}{2^N - 1} \right\rangle \leqslant \frac{A_M - A_I}{2} \tag{4.28}$$

where the brackets ⟨ ⟩ are used to indicate "the least integer N such that," and the required sampling rate would be determined from Equation 4.22 of the Section 4.3.2.

The minimum length shift register for the system of Figure 4.11 is determined by noting that every pulse originates below level A_I and must cross level A_M before it can be detected. The minimum possible time Δt_p of this transition follows from Corollary 3.1:

$$\Delta t_p \geqslant \frac{1}{\pi f_p} \sin^{-1} \left(\frac{A_M - A_I}{2A_p}\right) . \tag{4.29}$$

Thus, the shift register must contain at least

$$\langle M \rangle \geqslant 2f_s(T_p - \Delta t_p) \tag{4.30}$$

locations. Substituting Equations 4.27 and 4.29 into this result and letting

$$\gamma = \cos^{-1} \left(\frac{2A_p - A_M + A_I}{2A_p}\right) , \tag{4.31}$$

it follows that

$$\langle M \rangle \geq \frac{2\pi f_p T_p}{\gamma} - \frac{2}{\gamma} \sin^{-1} \left(\frac{A_M - A_I}{2A_p}\right) . \tag{4.32}$$

The first term in this expression is representative of the number of samples occurring during a pulse, while the second term may be associated with the samples during a pulse transition from A_I to A_M.

When a pulse is detected and M samples of x(t) are eliminated, the last value of x(t) before the pulse is held by the D/A converter for N/f_s sec. The integrity of the resulting signal, which is an approximation of $x_I(t)$ with various constants replacing the pulses, can be evaluated by noting that the maximum possible change of $x_I(t)$ in N/f_s sec is

$$\Delta x_I \leq \begin{cases} 2A_I \sin \dfrac{\pi M f_I}{f_s} , & f_s \geq 2Mf_I \\ 2A_I , & f_s \leq 2Mf_I \end{cases} \tag{4.33}$$

This result follows from Corollary 3.1. Using Equations 4.27 and 4.31, it is evident that

$$\frac{\Delta x_I}{2A_I} \leq \begin{cases} \sin M\gamma \left(\dfrac{f_I}{f_p}\right) , & f_s \geq 2Mf_I \\ 1 , & f_s \leq 2Mf_I \end{cases} \tag{4.34}$$

Since $\Delta x_I/2A_I$ is the normalized amplitude change of $x_I(t)$, it is clear that the integrity of the signal whose pulses have been eliminated is a function of the ratio of the bandwidths of the information and pulses f_I/f_p and the number of samples eliminated. The bandwidth ratio required to allow no more than the fractional amplitude change $\Delta x_I/2A_I$ is

$$\frac{f_I}{f_p} \geqslant \frac{1}{M\gamma} \sin^{-1} \frac{\Delta x_I}{2A_I} , \; f_s \geqslant 2Mf_I \; . \tag{4.35}$$

It should be noted that when $f_s \geqslant 2Mf_I$, $x_I(t)$ may be perfectly reconstructed by low pass filterings.[a]

EXAMPLE 4.8 Consider a signal as depicted in Figure 4.10(a) with parameters:

$$\begin{aligned} A_p &= 3 \\ A_M &= 2 \\ A_I &= 1 \\ f_p &= 1 \text{ KHz} \\ f_I &= 1 \text{ Hz} \\ T_p &= 1 \text{ msec} \end{aligned} \tag{4.36}$$

Using Equations 4.26, 4.27, 4.32, and 4.34, it follows that

$$\begin{aligned} q &= 0.5 \\ f_s &= 5.36 \text{ KHz} \\ M &= 11 \\ \frac{\Delta x_I}{2A_I} &= 0.5\% \; . \end{aligned} \tag{4.37}$$

---

[a]In practice, it is impossible to perfectly recreate a continuous signal from its quantized samples.

Since $q = 0.5$, six bins are established in the interval $[0, 3]$, so that a level $x = 1.5$ falls between $A_I = 1$ and $A_M = 2$. By sampling at 5.36 KHz, at least one sample on or above $x = 1.5$ is guaranteed per pulse. When a pulse is detected, the current sample and the five preceding and following samples must be eliminated. Since the bandwidth ratio f_I/f_p is 0.001, the maximum possible amplitude change of the information signal during the pulse is 0.01, which is only 0.5% of its total possible change. If f_I were 10 Hz or 100 Hz, the information signal would be able to change 6.5% and 60% of its maximum possible deviation, respectively. □

4.4 Summary

In this chapter, the results of Chapter 3 have been used to examine the amplitude uncertainty inherent in all sampling processes. This amplitude uncertainty can be attributed to signal amplitude fluctuations during the sampling process itself or between consecutive samples. The slope overload phenomenon associated with delta modulation was first examined, and a bound on the maximum error due to slope overload was derived for the set of band-limited signals $\{x(t): M_0 \leqslant A\}$. The amplitude uncertainty resulting from finite conversion time was then discussed. Finally, a sampling rate which guarantees that, between successive samples of a band-limited function in the set $\{x(t): M_0 \leqslant A\}$, the function itself does not deviate from either sample by more than some predefined amplitude change was established. This sampling rate was shown to be of valuable use in the estimation of level crossing profiles and the detection and elimination of high frequency pulses from a low frequency signal.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

This work has examined the amplitude fluctuations of band-limited functions and the representation of these functions by piecewise linearly interpolated samples. Such representations are implicitly assumed in many digital signal processing computations, including the estimation of level crossing profiles and the location of extrema in a discrete sequence of samples.

In Chapter 2, the known analytical relations between time and frequency were reviewed. Since these relations are established by the Fourier series and transform, the fundamental concepts of Fourier analysis were presented. Several properties arising from the fact that signals are generally band-limited were also discussed. In particular, a survey of the literature on sampling, one of the most important consequences of finite signal bandwidth, was presented. Some of the results surveyed were seen to be based on sampling criteria other than optimal signal reconstruction, the fundamental criterion behind the Shannon sampling theorem and its extensions. There was a noticeable lack of results on the sampling of signals to bound the amplitude fluctuations of the sampled signal between samples.

A comprehensive analysis of the allowable amplitude fluctuations of band-limited functions was presented in Chapter 3. New bounds on the average rate of change of band-limited functions were developed, and the average rate of change referenced to an extremum, a new concept that measures the average rate of change of a function over an interval in which at least one end point of the interval is an extremum of the function, was defined and similarly bounded. These bounds were derived for the two sets of band-limited functions $\{x(t): M_0 \leqslant A\}$ and $\{x(t): |X(j\omega)| \leqslant f(\omega),\ f(\omega) \geqslant 0\}$, and shown to be least upper bounds. It was noted that M_0 is a measure of the area of a function's Fourier transform, and that numerical procedures must sometimes be employed to evaluate bounds on the set of functions $\{x(t): |X(j\omega)| \leqslant f(\omega),\ f(\omega) > 0\}$. The slope and curvature of band-limited white noise was bounded for both sets of functions. Least upper bounds on the rate of change of those functions which are limited in both amplitude and frequency were also developed.

In Chapter 4, the results of Chapter 3 were used to examine several problems of practical importance. These problems were seen to be related to the amplitude fluctuations of a signal (1) during a sampling or modulation process or (2) between consecutive samples. The slope overload phenomenon associated with delta modulation was examined, and a bound on the maximum error due to slope overload was derived for the set of band-limited signals $\{x(t): M_0 \leqslant A\}$. The amplitude uncertainty resulting from finite conversion time was also discussed, and a sampling theorem which guarantees that, between successive samples of a band-limited function in the set $\{x(t): M_0 \leqslant A\}$, the function itself does not deviate from either sample by more than some predefined amplitude change was established. This sampling theorem was shown to be useful in the estimation of level crossing profiles and the detection and elimination of high frequency pulses from a low frequency signal. In regard to the estimation of level crossing profiles, this theorem led to the development of a sampling rate which guarantees that no more than 2m ($m \geqslant 1$) level crossings of m levels are missed per extremum of the sampled function.

Although several problems associated with detecting the amplitude variations of band-limited signals and their representation by piecewise linearly interpolated samples have been identified and clarified by this study, there are still many interesting and unsolved problems that require further investigation. The results of Chapter 3 can be easily extended, for example, to n dimensions, and an n-dimensional sampling theorem similar to Theorem 4.1 established. Such an extension could have some interesting implications in the processing of pictorial data — e.g., on the ability to detect and accurately locate the edges of an $N \times N$ image. In addition, further research is needed to determine the number of missed level crossings m (of Equation 4.23) which can be tolerated for a particular observation interval without appreciable error in the resulting LCP. While all the implications of the results in Chapters 3 and 4 are not immediately apparent, it is clear that these results are useful in the analysis of several current problems of practical significance.

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