

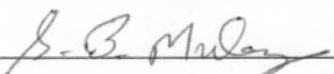
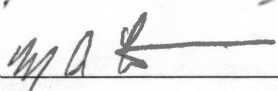
To the Graduate Council:

I am submitting herewith a dissertation written by Neal Oliver Smith entitled "Graph-Theoretic Properties of the Zero-Divisor Graph of a Ring". I have examined the final paper copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a Major in Mathematics.

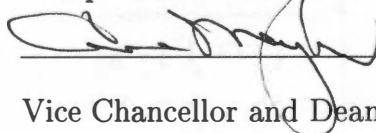
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We have read this dissertation
and recommend its acceptance:

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Accepted for the Council:



Vice Chancellor and Dean of
Graduate Studies

Graph-Theoretic Properties of the Zero-Divisor Graph of a Ring

A Dissertation
Presented For the
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Chapter 1

Introduction

The notion of the zero-divisor graph of a commutative ring was introduced by I. Beck in [6], who was primarily interested in colorings of the graph. The zero-divisor graph has subsequently been studied in [1], [2], [3], [4], [5], [9], [13], [14] [15], and [17]. In [2], the finite commutative rings whose zero-divisor graph has *clique number* 3 are classified, among other things. In [3], [4], [15], and [17], many important properties of the zero-divisor graph are proved. It is shown, for example, that the zero-divisor graph of a commutative ring is *connected*, the *diameter* of the zero-divisor graph is less than or equal to 3, and those zero-divisor graphs that contain a cycle must have *girth* less than or equal to 4. In [9] and [15], those finite commutative rings whose zero-divisor graph contains no cycle are classified.

The zero-divisor graph makes sense in the context of semigroups as well. This is studied in [8]. In [17] and [18], S. Redmond discusses the zero-divisor graph of a noncommutative ring. In those two works, other possible zero-divisor graphs of commutative rings are discussed, and in [20], a zero-divisor graph is studied where the vertices of the graph are the (proper) ideals of the ring.

In this work, R will denote a ring with $1 \neq 0$. R may or may not be commutative depending on the setting. $Z(R)$ will denote the set of zero-divisors of R , and $U(R)$ will denote the group of units of R . Some facts that we will cite about $Z(R)$ are due to N. Ganesan and can be found in [10] and [11]. One such fact that will be used is that if R is an infinite commutative ring which is not a domain, then $Z(R)$ is an infinite set. The integers (mod n) and the field with p^k elements will be denoted, as usual, by $\mathbb{Z}_n$ and $\mathbb{F}_{p^k}$. We will often speak of annihilators. Given $x \in R$ we denote $\{r \in R \mid rx = 0\}$ by $(0 :_R x)$. If the ring is clearly understood, we may sometimes simply write $(0 : x)$. Given a set S , we will denote the cardinality of S by $|S|$. If S is an infinite set, and we are not interested in its cardinality beyond this fact, we will often write $|S| = \infty$. We will also tacitly assume facts about arithmetic in the extended real numbers. In particular, we will often use the fact that if n is a positive real number, then $n + \infty = \infty$ and $n\infty = \infty$.

We first begin by fixing some notation. When we speak of the zero-divisor graph of a commutative ring R , denoted $\Gamma(R)$, we mean the graph whose vertices are the nonzero zero-divisors of R . Two vertices x and y are joined by an edge in $\Gamma(R)$ if and only if $xy = 0$. One should note that in [2] and [6], a different definition of the zero-divisor graph is used, as every element of the ring (including 0) is allowed to be a vertex in the graph. Figures 1.1-1.3 give examples of $\Gamma(R)$. Note that non-isomorphic rings may have the same zero-divisor graph. For example, $\Gamma(\mathbb{Z}_9)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ are both the connected graph on two vertices.

Given a graph G , a *path* between vertices x and y is a sequence of edges joining x and y . We say G is *connected* if every two vertices of G can be joined by a path. The *distance* between two vertices x and y , typically denoted $d(x, y)$, is the number of edges in the shortest path joining x and y , with the convention that $d(x, y) = \infty$



Figure 1.1: $\Gamma(\mathbb{Z}_9)$

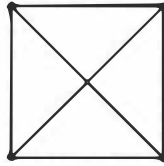


Figure 1.2: $\Gamma(\mathbb{Z}_{25})$

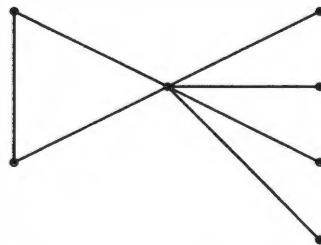


Figure 1.3: $\Gamma(\mathbb{Z}_{16})$

if there is no path between x and y .

We then define the *diameter* of a graph, denoted $\text{diam}(G)$, by $\text{diam}(G) = \sup\{d(x, y) : x, y \text{ vertices of } G\}$. To avoid trivialities, we use the convention that the diameter of a non-connected graph is infinity. The *girth* of a graph G is the length of the shortest cycle in G , again with the conventions that the girth of an acyclic graph is infinity, and the girth of a graph that consists of a single vertex is 0.

We say a graph is *complete* if for all distinct vertices x and y , there is an edge joining x and y . A *clique* of a graph G is a complete subgraph of G . We use K_n to denote a complete graph on n vertices. We define the *clique number* of G , $\omega(G)$, by $\omega(G) = \sup\{n : K_n \subseteq G\}$. A graph is said to be *complete bipartite* if its vertices can be partitioned into two disjoint sets V and W so that an edge joins x and y in G if and only if $x \in V$ and $y \in W$. A complete bipartite graph with bipartition into sets of cardinalities m and n will be denoted $K_{m,n}$. A $K_{1,n}$ is said to be a *star*.

A *subdivision* of an edge $x - y$ is two edges of the form $x - z - y$ (where z is a vertex different than x and y). We say that a graph H is a subdivision of G if H can be obtained from G via a sequence of subdivisions of edges of G . We say that two graphs G and H are *graph isomorphic* (or *homeomorphic*) if there is a bijection f from the vertex set of G to the vertex set of H so that an edge joins $f(x)$ and $f(y)$ in H if and only if an edge joins x and y in G .

If G is a graph, we will sometimes denote the set of vertices of G by $v(G)$, and the set of edges of G by $E(G)$. If G and H are graphs, we sometimes will want to speak of the *union* of G and H , denoted $G \cup H$. $G \cup H$ is the graph whose vertex set is $v(G) \cup v(H)$ and whose edge set is $E(G) \cup E(H)$.

In Chapter 1, we are able to completely classify those finite commutative rings R for which $\Gamma(R)$ is a planar graph. In our approach, we are also able to classify all the non-local rings whose zero-divisor graphs have clique number 4. Our approach also yields a handful of interesting corollaries. Namely, we prove that a commutative ring R with 28 or more elements or 10 or more zero-divisors cannot have a planar zero-divisor graph. The question as to which infinite commutative rings have planar zero-divisor graphs is also investigated. Namely, we show that if R is an infinite commutative ring and $\Gamma(R)$ is planar, then $\Gamma(R)$ must belong to one of three classes of graphs.

In [8], it is pointed out that since the definition of $\Gamma(R)$ only fully uses the multiplicative structure of R , one can talk about the zero-divisor graph of a commutative (multiplicative) semigroup (with 0). In Chapter 2, we investigate several different possibilities for how to define the zero-divisor graph of a noncommutative semigroup. We show that many of the graph-theoretic properties of $\Gamma(R)$ also carry over to the noncommutative semigroup setting. If S is a noncommutative semigroup, we also investigate the possible shapes of $\Gamma(S)$.

There is some question as to how one should define the zero-divisor graph of a noncommutative ring. In [17] and [18], S. Redmond discussed several ways to define the zero-divisor graph of a noncommutative ring. In [20], U. Vishne defines a graph whose vertices are the proper ideals of R . We attempt to generalize this construction. We give many examples, it is proved that certain classes of rings have connected graphs, and we generalize some results from the commutative setting.

The *circumference* of a graph G is the length of the longest cycle in G . In [3], the authors ask what is the diameter of $\Gamma(K^n)$, where K is a finite field. This is a surprisingly difficult problem, even when $K = \mathbb{Z}_2$. In Chapter 3, we give some partial

results to this question.

Chapter 2

Planar Zero-Divisor Graphs

We should mention that we have published many of the results in sections two and three in a somewhat condensed form in [19]. In that paper, proofs of minor technical details tend to be omitted.

1 Introduction

There are many graph-theoretic features of $\Gamma(R)$ which can be studied. One feature that we shall study in some detail is the notion of *planarity* of the graph. Informally, one thinks of a graph as being *planar* if it can be drawn in such a way that ‘none of its edges cross’. To be precise, we say that a graph G is planar if there is a topological embedding of G into $\mathbb{R} \times \mathbb{R}$.

In [3], Anderson, Frazier, Lauve, and Livingston determined which rings of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ have planar zero-divisor graphs and posed the question as to which finite commutative rings have planar zero-divisor graphs. In this chapter, we will classify all such rings (Theorem 2.25). We also obtain several results which

are interesting in their own right; for instance, a (finite) commutative local ring (not a field) with 28 or more elements (10 or more zero-divisors) cannot have a planar zero-divisor graph.

In this chapter, R will always denote a commutative ring (with $1 \neq 0$) and $\Gamma(R)$ will denote its zero-divisor graph. We will typically denote the clique number of $\Gamma(R)$ by $\omega(R)$. Observe that the clique number of an integral domain is zero. As usual, $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{F}_m$ will denote the integers, the integers mod n , and the field with m elements, respectively. Also, in this chapter, k or k_i will always denote a finite field.

As mentioned previously, in [6] and [2], a slightly different definition of the zero-divisor graph is used, as each element of R is allowed to be a vertex in the graph. However, one can easily show that the clique number of said graph is $\omega(R) + 1$ (with $\omega(R)$ as defined above)(cf. [3, Section 3]).

In [1], Akbari, Maimani, and Yassemi have also investigated when $\Gamma(R)$ is planar, and some of the results in their paper parallel some of ours. However, our interest lies mainly in classifying those rings whose zero-divisor graphs are planar, which is not explicitly done in their paper.

In the case where R is finite, we may write R as the product of finitely many local rings. So, in this setting, we shall consider two cases, depending on whether R is local or R decomposes into a product of two or more finite local rings. These cases are covered separately in sections two and three, respectively.

In section four, we investigate the problem as to which planar graphs can be realized as $\Gamma(R)$ in the setting where R is an infinite commutative ring. In particular we show that there are only three types of planar graphs that can be realized as $\Gamma(R)$ in this setting.

2 The Non-Local Case

We shall see that to classify those finite commutative R with $\Gamma(R)$ planar, it will suffice to examine only those rings with $\omega(R) \leq 4$. It is known which finite commutative rings R have $\omega(R) \leq 3$. Thus, in the non-local case, our approach will be to classify those finite commutative R with clique number 4, and then determine the planarity of those R with $\omega(R) \leq 4$. We have cause to make extensive use of the famous theorem of Kuratowski (Theorem 2.4). Note that $\omega(R) = 0$ if and only if R is an integral domain.

Lemma 2.1. *(Beck) Let R be a finite commutative ring. Then $\omega(R) = 1$ if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.*

Proof. A proof can be found in [6, Proposition 2.2]. □

Lemma 2.2. *(Beck) Let R be a finite commutative ring, and k_1 and k_2 finite fields. Then $\omega(R) = 2$ if and only if R is isomorphic to one of:*

$\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2), \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2-2), k_1 \times \mathbb{Z}_4, k_1 \times \mathbb{Z}_2[X]/(X^2),$
or $k_1 \times k_2$.

Proof. A proof can be found in [6, page 226]. □

Lemma 2.3. *(D.D. Anderson and Nasseer) Let R be a finite commutative ring, and k_1, k_2 , and k_3 finite fields. Then $\omega(R) = 3$ if and only if R is isomorphic to one of:*

$\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2), k_1 \times k_2 \times k_3,$
 $k_1 \times k_2 \times \mathbb{Z}_4, k_1 \times k_2 \times \mathbb{Z}_2[X]/(X^2), k_1 \times \mathbb{Z}_8,$
 $k_1 \times \mathbb{Z}_9, k_1 \times \mathbb{Z}_3[X]/(X^2), k_1 \times \mathbb{Z}_2[X]/(X^3),$
 $k_1 \times \mathbb{Z}_4[X]/(2X, X^2-2), \mathbb{Z}_{16}, \mathbb{Z}_2[X]/(X^4),$

$\mathbb{Z}_4[X]/(2X, X^3 - 2), \mathbb{Z}_4[X]/(X^2 - 2), \mathbb{Z}_4[X]/(X^2 + 2X + 2), \mathbb{F}_4[X]/(X^2),$
 $\mathbb{Z}_4[X]/(X^2 + X + 1), \mathbb{Z}_2[X, Y]/(X, Y)^2, \mathbb{Z}_4[X]/(2, X)^2,$
 $\mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X),$
 $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY), \mathbb{Z}_2[X, Y]/(X^2, Y^2), \mathbb{Z}_8[X]/(2X - 4, X^2),$
 $\mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X),$
 $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y), \text{ or } \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y).$

Proof. A proof can be found in [2, Theorem 4.4]. $\square$

As mentioned previously, the above three results are stated somewhat differently in the cited papers since those authors used a slightly different definition of $\Gamma(R)$. However, it is easy to verify that they translate as above (cf. [3, Example 3.1 and Theorem 3.2]). For other results on $\omega(R)$, see [6], [2], and [3, Section 3].

We will often use the following famous result as well; a proof can be found in [21, pp. 259–264]. One immediate consequence of Theorem 2.4 will be that if $\omega(R) \geq 5$, then $\Gamma(R)$ is not planar.

Theorem 2.4. (*Kuratowski*) *A graph is planar if and only if it contains no subdivision homeomorphic to K_5 or $K_{3,3}$.*

Since we will be considering the zero-divisor graph of a finite product, we need to know how the clique number behaves with respect to products. This is addressed in the following results.

Proposition 2.5. *Let $R_1, \dots, R_n$ be finite commutative rings, R a finite commutative ring which is not an integral domain, and D an integral domain. Then:*

- (a) $\omega(R_1 \times \dots \times R_n) \geq \omega(R_1) + \dots + \omega(R_n).$
- (b) $\omega(R \times D) = \omega(R) + 1.$

Proof. To prove (a), we may take $m = 2$, the general result follows by induction. Say $\omega(R_1) = j$ and $\omega(R_2) = l$, and let $\{r_1, \dots, r_j\}$ be a j -clique in R_1 and $\{s_1, \dots, s_l\}$ be an l -clique in R_2 . Then

$$\{(r_i, 0) \mid i = 1, \dots, j\} \cup \{(0, s_i) \mid i = 1, \dots, l\}$$

is a clique in $R_1 \times R_2$; so $\omega(R_1 \times R_2) \geq \omega(R_1) + \omega(R_2)$.

To see (b), say $\omega(R) = m$, and let $\{x_1, \dots, x_m\}$ be an m -clique in R . Then,

$$\{(x_i, 0) \mid i = 1, \dots, m\} \cup \{(0, 1)\}$$

is an $m + 1$ clique in $R \times D$; so, $\omega(R \times D) \geq m + 1$. To see that strict inequality is impossible, note that since D is a domain, a clique can not contain two elements of the form $(x, d_1), (y, d_2)$, where d_1 and d_2 are two nonzero elements of D . Thus an $m + 2$ clique in $R \times D$ would give rise to an $m + 1$ clique in R , contrary to $\omega(R) = m$. $\square$

Note that strict inequality may hold in (a) above. For example, $\omega(\mathbb{Z}_4) = 1$, but $\omega(\mathbb{Z}_4 \times \mathbb{Z}_4) = 3$. This example motivates a slight strengthening of the above proposition.

Proposition 2.6. *Let R and S be finite commutative rings, $\omega(R) = m$ with $\{r_1, \dots, r_m\}$ an m -clique in R , and $\omega(S) = n$ with $\{s_1, \dots, s_n\}$ an n -clique in S . If $r_i^2 = 0$ for some i and $s_j^2 = 0$ for some j , then $\omega(R \times S) > \omega(R) + \omega(S)$.*

Proof. Certainly " $\geq$ " holds since

$$C = \{(r_1, 0), \dots, (r_m, 0), (0, s_1), \dots, (0, s_n)\}$$

is an $m + n$ clique. However, since $r_i r_k = 0$ for all k , and $s_j s_k = 0$ for all k , we see that $C \cup \{(r_i, s_j)\}$ is an $m + n + 1$ clique in $R \times S$. $\square$

It is also possible to strengthen the above result somewhat further by considering the number of 2-nilpotents, but we will not need this for our purposes. To finish the preliminaries, we state an often-used corollary.

Corollary 2.7. *Let $n \geq 2$ be an integer and $R_1, \dots, R_n$ be finite commutative rings. Then $\omega(R_1 \times \dots \times R_n) \geq n$ with equality if and only if either*

(a) each R_i is a finite field, or

(b) exactly one factor is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and all the other factors are finite fields.

Proof. The key observation is that if R and S are integral domains, then $\Gamma(R \times S)$ is a complete bipartite graph, and thus $\omega(R \times S) = 2$. The corollary then follows from this observation and Propositions 2.5 and 2.6. $\square$

For the remainder of this section, R will denote a finite, non-local ring, and $R_1 \times \dots \times R_n$ will be its local decomposition.

We will prove the main result of this section (Theorem 2.16) via a handful of somewhat technical lemmas. To greatly reduce the number of cases in certain arguments, we will often want to be able to say that if R and S have isomorphic zero-divisor graphs, then for a given ring T , $R \times T$ and $S \times T$ also have isomorphic zero-divisor graphs. This is not quite true as stated, but if the 2-nilpotents in R and S ‘agree’, then the statement is true. This is the motivation for the following result.

Lemma 2.8. *Let R and S be finite commutative rings. If there is a bijection $f : R \rightarrow S$ which maps $Z(R)$ onto $Z(S)$, maps 2-nilpotent elements of R to 2-nilpotents in S , maps 0 to 0, and preserves adjacency in the zero-divisor graph (in the sense that $f(xy) = 0$ if and only if $f(x)f(y) = 0$) then for a given finite commutative ring T , $\Gamma(R \times T) \cong \Gamma(S \times T)$.*

Proof. We begin by fixing some notation. Given a graph G , let $v(G)$ denote the vertex set of G . Let $f : R \rightarrow S$ be a bijection between R and S which has the stated effect on zero-divisors and 2-nilpotents. Thus, if $0 \neq r \in R$ so that $r^2 = 0$, we have $(f(r))^2 = 0$. Now define a map $\phi : v(\Gamma(R \times T)) \rightarrow v(\Gamma(S \times T))$ by $\phi(r, t) = (f(r), t)$. Since f is a bijection, ϕ is as well. It suffices to show that ϕ preserves adjacency. So pick adjacent vertices $(r_1, t_1), (r_2, t_2) \in v(\Gamma(R \times T))$. Note $r_1 r_2 = 0$ and $t_1 t_2 = 0$. If $r_1 \neq r_2$, then $f(r_1) f(r_2) = 0$ since f preserves adjacency, and hence $\phi(r_1, t_1) \phi(r_2, t_2) = (f(r_1) f(r_2), t_1 t_2) = (0, 0)$, as needed. It remains only to examine the case where $r_1 = r_2$. In this case, r_1 is a 2-nilpotent, and hence $\phi(r_1, t_1) \phi(r_1, t_2) = ((f(r_1))^2, t_1 t_2) = (0, 0)$, since f preserves 2-nilpotents. Thus, ϕ is a graph isomorphism. $\square$

Remark 2.9. The hypothesis that 2-nilpotents be mapped to 2-nilpotents cannot be dropped, for one can have $\Gamma(R) \cong \Gamma(S)$, but for a particular T , $\Gamma(R \times T)$ and $\Gamma(S \times T)$ may not be graph-isomorphic. To see this, let $R = \mathbb{Z}_9$ and $S = \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $\Gamma(R) \cong \Gamma(S)$, both graphs being the connected graph on 2 vertices. However, R has 2-nilpotents, while S does not. Then $\Gamma(R \times \mathbb{Z}_2) \cong \Gamma(\mathbb{Z}_{18})$ (a graph with 11 vertices) and $\Gamma(S \times \mathbb{Z}_2) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. However, the latter graph has only 6 vertices, so there can be no graph isomorphism between $\Gamma(R \times \mathbb{Z}_2)$ and $\Gamma(S \times \mathbb{Z}_2)$.

To utilize this result, we note that many of the rings listed in Lemmas 2.1, 2.2, and 2.3 have isomorphic (homeomorphic) zero-divisor graphs and corresponding 2-nilpotents. Many of the graphs are included in the appendix.

Lemma 2.10. *For each family of commutative rings listed below, there are bijections between each member of the family that preserve zero-divisors and 2-nilpotents.*

(a) $\mathbb{Z}_4$ and $\mathbb{Z}_2[X]/(X^2)$

- (b) $\mathbb{Z}_9$ and $\mathbb{Z}_3[X]/(X^2)$
- (c) $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$ and $\mathbb{Z}_4[X]/(2X, X^2 - 2)$
- (d) $\mathbb{Z}_{16}$, $\mathbb{Z}_2[X]/(X^4)$, $\mathbb{Z}_4[X]/(2X, X^3 - 2)$, $\mathbb{Z}_4[X]/(X^2 - 2)$ and $\mathbb{Z}_4[X]/(X^2 + 2X + 2)$
- (e) $\mathbb{F}_4[X]/(X^2)$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$ and $\mathbb{Z}_4[X]/(2, X)^2$
- (f) $\mathbb{Z}_{27}$, $\mathbb{Z}_3[X]/(X^3)$, $\mathbb{Z}_9[X]/(X^2 - 3, 3X)$ and $\mathbb{Z}_9[X]/(X^2 - 6, 3X)$
- (g) $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY)$ and $\mathbb{Z}_8[X]/(2X - 4, X^2)$
- (h) $\mathbb{Z}_4[X]/(X^2)$, $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$ and $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$
- (i) $\mathbb{Z}_4[X]/(X^2 - 2X)$ and $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y)$

Remark 2.11. Observe that all of the rings in parts (h) and (i) have isomorphic zero-divisor graphs, but the 2-nilpotents of the rings in part (h) do not correspond to the 2-nilpotents of the rings in part (i). For each ring in part (h) has the property that each of its zero-divisors is a 2-nilpotent, but each ring in part (h) has a zero-divisor which is not 2-nilpotent. To see this, take the cosets represented by X and Y in $\mathbb{Z}_4[X]/(X^2 - 2X)$ and $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y)$, respectively.

Lemma 2.12. Let $R = R_1 \times R_2 \times R_3$, where each R_i is a finite commutative, local ring, and let k_1 and k_2 be finite fields. If $\omega(R) = 4$, then R is isomorphic to either

- (a) $k_1 \times (\text{one of the rings in Lemma 2.1}) \times (\text{one of the rings in Lemma 2.1})$,

or

- (b) $k_1 \times k_2 \times (\text{one of the local rings in Lemma 2.2})$.

Proof. Since $\omega(R) = 4$, not all the factors can be fields. If exactly one factor, say R_1 , is a field, then

$$\omega(R_1 \times R_2 \times R_3) = \omega(R_2 \times R_3) + 1.$$

Since any local ring with clique number 2 contains a maximal clique with a 2-nilpotent (see Lemma 2.2), we must have $\omega(R_2) = \omega(R_3) = 1$. Thus R_2 and R_3 must be isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Then, one can see that

$$\omega(k_1 \times \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)) = \omega(k_1 \times \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_4) = \omega(k_1 \times \mathbb{Z}_4 \times \mathbb{Z}_4) = 4.$$

If exactly two factors are fields, similar reasoning shows that R is isomorphic to $k_1 \times k_2 \times$ (one of the local rings listed in Lemma 2.2). If none of the factors is a field, then since each of the local rings listed in Lemmas 2.1-2.3 contains a maximal clique with 2-nilpotents, we must have each R_i isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, since each of the R_i 's must have clique number 1. But then $\omega(R_1 \times R_2 \times R_3) > 4$. Thus the only possibilities for R are as stated. $\square$

Lemma 2.13. *Let $R = R_1 \times R_2$, where each R_i is a finite, commutative, local ring, and let k be a finite field. If $\omega(R) = 4$, then R is isomorphic to either*

- (a) $k \times$ (one of the local rings in Lemma 2.3), or
- (b) (one of the rings in Lemma 2.1) $\times (\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \text{ or } \mathbb{Z}_4[X]/(2X, X^2 - 2))$.

Proof. Certainly, any ring R as in (a) has clique number 4.

By Propositions 2.5 and 2.6, we are left to consider only those R_1 and R_2 (not fields) with $\omega(R_1) + \omega(R_2) = 3$, that is, only those R_1 as in Lemma 2.1, and only those local R_2 as in Lemma 2.2.

We claim that $\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) = 4$. To see this, the subgraph induced by

$$((\mathbb{Z}_4 - \{0\}) \times \{0\}) \cup (\{0\} \times (\mathbb{Z}_8 - \{0\}))$$

contains a $K_{3,7}$. Any zero-divisor of the form $(2, x)$, where $x \in (\mathbb{Z}_8 - \{0\})$ with x odd, is adjacent only to $(2, 0)$. Similarly, $(1, 4)$ and $(3, 4)$ are adjacent only to $(0, 2)$,

$(0, 4)$, and $(0, 6)$. And, $(1, 2)$, $(3, 2)$, $(1, 6)$, and $(3, 6)$ are adjacent only to $(0, 4)$. Since the only other zero-divisors in $\mathbb{Z}_4 \times \mathbb{Z}_8$ capable of contributing to a clique with more than 2 elements are $(2, 2)$, $(2, 4)$, and $(2, 6)$, then certainly $\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) \leq 5$. However, since in $\mathbb{Z}_8$, $\text{ann}(2) = \text{ann}(6) = \{0, 4\}$, we see that $\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) \leq 4$. Since $\{(2, 0), (2, 2), (0, 4), (2, 4)\}$ is a 4-clique, we conclude that $\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) = 4$. Similarly, $\omega(\mathbb{Z}_4 \times (\mathbb{Z}_2[X]/(X^3) \text{ or } \mathbb{Z}_4[X]/(2X, X^2 - 2))) = 4$ and $\omega(\mathbb{Z}_2[X]/(X^2) \times (\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \text{ or } \mathbb{Z}_4[X]/(2X, X^2 - 2))) = 4$.

However, $\omega(\mathbb{Z}_4 \times \mathbb{Z}_9) \geq 5$, as $\{(2, 0), (0, 3), (0, 6), (2, 3), (2, 6)\}$ is a 5-clique. Similarly, $\mathbb{Z}_4 \times \mathbb{Z}_3[X]/(X^2)$, $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_3[X]/(X^2)$, and $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_9$ all have clique number at least 5. $\square$

We next summarize our previous results.

Proposition 2.14. *Let R be a finite, commutative, non-local ring with $\omega(R) = 4$, and let k_1, k_2, k_3 , and k_4 be finite fields. Then R is isomorphic to one of the following:*

- (a) $k_1 \times k_2 \times k_3 \times k_4$,
- (b) $k_1 \times (\text{one of the rings in Lemma 2.1}) \times (\text{one of the rings in Lemma 2.1})$,
- (c) $k_1 \times k_2 \times (\text{one of the local rings in Lemma 2.2})$,
- (d) $k_1 \times (\text{one of the local rings in Lemma 2.3})$,
- (e) $(\text{one of the rings in Lemma 2.1}) \times (\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \text{ or } \mathbb{Z}_4[X]/(2X, X^2 - 2))$, or
- (f) $k_1 \times k_2 \times k_3 \times (\text{one of the rings in Lemma 2.1})$.

Some remarks are in order. Virtually all of the rings listed above can immediately be seen to have nonplanar zero-divisor graphs. Observe that if $R = R_1 \times R_2$, then $\Gamma(R)$ contains a $K_{|R_1|-1, |R_2|-1}$. Hence, if R_1 and R_2 each have 4 or more elements, then $\Gamma(R_1 \times R_2)$ contains a $K_{3,3}$, and will thus (by Kuratowski's Theorem) be

nonplanar. Hence the only rings listed above whose zero-divisor graphs could possibly be planar are found in part (d) of the above proposition. However, we will show that the zero-divisor graphs of these rings all fail to be planar as well. To this end, we will need the following easy, but useful, lemma.

Lemma 2.15. *Let R be a commutative ring and D an integral domain. If $\Gamma(R)$ contains a 3-clique C with the property that for all $x, y \in C$ (x and y not necessarily distinct) we have $xy = 0$, then $\Gamma(D \times R)$ is not planar.*

Proof. If $\{x, y, z\}$ is such a 3-clique, then the subgraph of $\Gamma(D \times R)$ spanned by $\{(0, x), (0, y), (0, z), (1, x), (1, y), (1, z)\}$ contains a $K_{3,3}$. Thus, $\Gamma(D \times R)$ is not planar. $\square$

Since each of the local rings listed in Lemma 2.3 (with the exception of $\mathbb{Z}_{27}$, $\mathbb{Z}_3[X]/(X^3)$, $\mathbb{Z}_9[X]/(X^2 - 6, 3X)$, and $\mathbb{Z}_9[X]/(X^2 - 3, 3X)$) has such a clique, we need only consider products of $\mathbb{Z}_2$ or $\mathbb{Z}_3$ with these rings.

Fortunately, since $\mathbb{Z}_{27}$, $\mathbb{Z}_3[X]/(X^3)$, $\mathbb{Z}_9[X]/(X^2 - 6, 3X)$, and $\mathbb{Z}_9[X]/(X^2 - 3, 3X)$ satisfy the hypotheses of Lemma 2.8, one sees that $k \times \mathbb{Z}_{27}$, $k \times \mathbb{Z}_3[X]/(X^3)$, $k \times \mathbb{Z}_9[X]/(X^2 - 6, 3X)$, and $k \times \mathbb{Z}_9[X]/(X^2 - 3, 3X)$ all have isomorphic zero-divisor graphs as well.

Thus, it will suffice to examine the planarity of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{27})$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{27})$. We show that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{27})$ is nonplanar; thus, since $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{27})$ contains $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{27})$ as a subgraph, both of the rings in question have nonplanar zero-divisor graphs.

To establish the nonplanarity of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{27}) \cong \Gamma(\mathbb{Z}_{54})$, one simply notes that the subgraph spanned by $\{9, 27, 45\} \cup \{6, 18, 36\}$ is a $K_{3,3}$.

We are now left only to consider the planarity of the zero-divisor graphs of those rings listed in Lemmas 2.1-2.3. One can easily check that all of the zero-divisor

graphs of the local rings in question have planar embeddings. By Lemma 2.8, all that remains is to consider the planarity of the zero-divisor graphs of $k \times \mathbb{Z}_4$, $k_1 \times k_2$, $k_1 \times k_2 \times k_3$, $k \times \mathbb{Z}_8$, $k \times \mathbb{Z}_9$, and $k \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$, where k is $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and k_1, k_2 , and k_3 are chosen so the zero-divisor graph does not contain a $K_{3,3}$.

It is easy to see that $\Gamma((\mathbb{Z}_2 \text{ or } \mathbb{Z}_3) \times (\text{any finite field}))$ is planar. As indicated in previous remarks, it therefore suffices to consider only the zero-divisor graphs of $k \times \mathbb{Z}_4$, $k_1 \times k_2 \times k_3$, $k \times \mathbb{Z}_8$, and $k \times \mathbb{Z}_9$, where k, k_1, k_2 , and k_3 are $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

It is easy to check that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_{12})$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_9) \cong \Gamma(\mathbb{Z}_{18})$, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_9)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ have planar embeddings. Some of these can be found in the appendix.

One then finds that, of the indicated rings, only $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_3 \times \mathbb{Z}_8 \cong \mathbb{Z}_{24}$ have nonplanar zero-divisor graphs. To see this, one could invoke [13, Proposition 2.2], or proceed directly as follows.

In $\Gamma(\mathbb{Z}_{24})$, the subgraph spanned by $\{4, 8, 16\} \cup \{6, 12, 18\}$ is a $K_{3,3}$. It is then enough to note that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ is nonplanar, because this graph contains a subdivision of K_5 as indicated in Figure 2.1.

Our next result summarizes the above statements.

Theorem 2.16. *Let R be a finite, non-local, commutative ring, and k a finite field. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\begin{aligned} & k \times \mathbb{Z}_2, k \times \mathbb{Z}_3, \\ & \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2), \\ & \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2), \\ & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \\ & \mathbb{Z}_2 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2), \end{aligned}$$

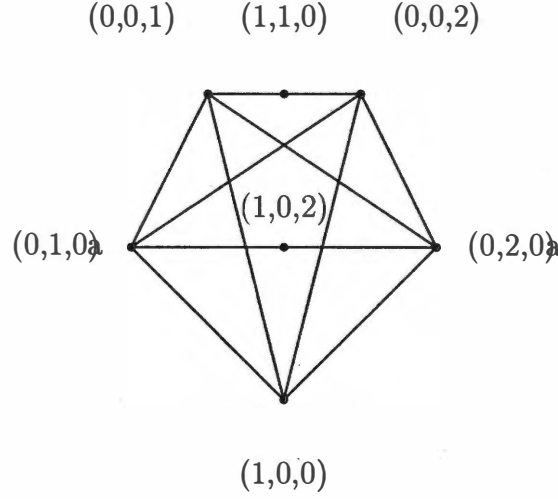


Figure 2.1: Subdivision of K_5 in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$

$$\mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2),$$

$$\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3), \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2).$$

It is interesting to note that all of the rings listed in Theorem 2.16 have cardinality either p^n or $p^n + 1$ for some prime integer p and integer $n \geq 1$. Conversely, note that $|\Gamma(\mathbb{F}_{p^n} \times \mathbb{Z}_2)| = p^n$ and $|\Gamma(\mathbb{F}_{p^n} \times \mathbb{Z}_3)| = p^n + 1$.

3 The Local Case

We begin by fixing some notation. In this section, unless otherwise indicated, (R, M) will denote a (commutative) finite local ring with (unique) nonzero maximal ideal M . We first state two basic facts which are easy, yet extremely useful.

Lemma 2.17. *Let R be a commutative ring and $x \in R$. Then $Rx \cong R/(0 : x)$.*

Proof. Define a map $\phi : R \rightarrow Rx$ by $\phi(r) = rx$. Then $\phi(r) = rx = 0$ if and only if $r \in (0 : x)$. ϕ is onto, so by the First Isomorphism Theorem, $Rx \cong R/(0 : x)$. $\square$

Proposition 2.18. *Let (R, M) be a commutative local ring. If there is an integer n so that $M^n = 0$ and $M^{n-1} \neq 0$, then for each $1 \leq i \leq n - 1$, M^i/M^{i+1} is an R/M - vector space.*

Proof. All that needs to be done is to show that for each i , M^i/M^{i+1} is a R/M - module, via $(r + M)(x + M^{i+1}) = rx + M^{i+1}$. It suffices to show this multiplication is well-defined, but this is easy, for if $r + M = s + M$, then $r - s \in M$. Hence $(r - s)(x + M^{i+1}) = (r - s)x + M^{i+1} = 0 + M^{i+1}$, since $(r - s)x \in M^{i+1}$. Thus, $(r + M)(x + M^{i+1}) = (s + M)(x + M^{i+1})$, as needed. $\square$

Since R is finite, there is an integer $n \geq 2$ so that $M^n = 0$ and $M^{n-1} \neq 0$. Our next result gives a bound on n .

Proposition 2.19. *Let (R, M) be a finite, commutative local ring. If $\Gamma(R)$ is planar, then n (as above) satisfies $n \leq 5$.*

Proof. Note that if $n \geq 6$, then $2n - 6 \geq n$; so that $(M^{n-3})^2 = 0$, i.e., $M^{n-3} - \{0\}$ is a clique. Therefore, if $\Gamma(R)$ is to be planar, we must have $|M^{n-3}| \leq 5$, else $\Gamma(R)$ will contain a K_5 , and will thus be nonplanar. But

$$|M^{n-3}| = |M^{n-3}/M^{n-2}| |M^{n-2}/M^{n-1}| |M^{n-1}/M^n| \geq 2^3 = 8,$$

which is a contradiction. $\square$

As in the previous section, we will obtain the main result via a sequence of cases, as to whether n is 2, 3, 4, or 5.

Lemma 2.20. *If (R, M) is a finite, commutative local ring with $0 = M^5 \subsetneq M^4 \subsetneq M^3 \subsetneq M^2 \subsetneq M$, then $\Gamma(R)$ is nonplanar.*

Proof. Suppose that $\Gamma(R)$ is planar. Then $(M^3)^2 = 0$; so $M^3 - \{0\}$ is a clique. Thus $|M^3| = 2, 3, 4$, or 5 . Since $M^4 \neq 0$, necessarily $|M^4| \geq 2$. Since $|M^4| \mid |M^3|$, necessarily $|M^3| = 4$. But $M^2M^3 = 0$. Since $|M^3 - \{0\}| = 3$, and $|M^2 - M^3| \geq 3$ in any event, $\Gamma(R)$ will thus contain a $K_{3,3}$, and hence will be nonplanar by Kuratowski's Theorem, a contradiction. $\square$

The $n = 3$ and $n = 4$ cases are somewhat involved, but the $n = 2$ case is easy with the help of the following proposition from [2, Proposition 4.2]

Proposition 2.21. *Let (R, M) be a finite, commutative local ring, not a field, with $M^2 = 0$, $\text{char}(R/M) = p$, and $s = \dim_{R/M} M$.*

(a) *If $\text{char}(R) = p$, then $R \cong (R/M)[X_1, \dots, X_s]/(X_1, \dots, X_s)^2$.*

(b) *If $\text{char}(R) = p^2$, then $R \cong \mathbb{Z}_{p^2}[X_1, \dots, X_s]/((p, X_2, \dots, X_s)^2 + (f(X_1)))$,*

where $f(X_1) \in \mathbb{Z}_{p^2}[X_1]$ is irreducible mod p with $\mathbb{Z}_p[X_1]/(\bar{f}(X_1)) \cong R/M$.

As an immediate corollary of the above, we have:

Corollary 2.22. *Let (R, M) be a finite, commutative local ring with $\omega(R) = 4$ and $M^2 = 0$. Then R is isomorphic to either $\mathbb{Z}_{25}$ or $\mathbb{Z}_5[X]/(X^2)$. Thus $\Gamma(R) = K_4$, and hence $\Gamma(R)$ is planar.*

Proof. First, observe that $\Gamma(R)$ is complete since $M^2 = 0$ (and since the vertices of $\Gamma(R)$ are precisely the nonzero elements of M). Further, $\omega(R) = 4 = |M - \{0\}|$. Thus we have $5 = |M| = |R/M|^s$, which in turn yields $s = 1$ and $|R/M| = 5$. It then follows that $|R| = 25$, and from the previous proposition, the only possibilities for R can be the two given rings, as $\text{char}(R) = 5$ or $\text{char}(R) = 25$. $\square$

Note that the previous proposition can also be used to classify those finite, commutative local rings (R, M) with $M^2 = 0$ and $\omega(R) = 1, 2$, or 3 . Namely, $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[X]/(X^2)$, $\mathbb{F}_4[X]/(X^2)$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$, and $\mathbb{Z}_4[X]/(2, X)^2$ can be seen to have this property. Each of these rings has a planar zero-divisor graph; thus there are 10 local rings with $M^2 = 0$ and $M \neq 0$ whose zero-divisor graph is planar.

Lemma 2.23. *Let (R, M) be a finite, commutative local ring with $0 = M^4 \subsetneq M^3 \subsetneq M^2 \subsetneq M$. If $\Gamma(R)$ is planar, then $\omega(R) = 3$. Thus R is isomorphic to one of the following five rings:*

$$\mathbb{Z}_{16}, \mathbb{Z}_2[X]/(X^4), \mathbb{Z}_4[X]/(2X, X^3-2), \mathbb{Z}_4[X]/(X^2-2), \text{ or } \mathbb{Z}_4[X]/(X^2+2X+2).$$

Proof. Since $0 = M^4 \subsetneq M^3 \subsetneq M^2 \subsetneq M$, we may pick $x \in M - (0 : M)$. This is possible since $M^2 \neq 0$. We claim that

$$((0 \rtimes M) \cup \{x\}) - \{0\}$$

is a clique. To see this, if $z \in (0 : M)$, then certainly $xz = 0$ (since $x \in M$), and if $y \in (0 \rtimes M) - \{z\}$, then y is a zero-divisor; so $y \in M$, and hence $yz = 0$. Thus $|(0 : M)| = |((0 : M) \cup \{x\}) - \{0\}| \leq 4$. Observe that the equality holds since $x \notin (0 \rtimes M)$.

Now, observe that R/M injects into $(0 \rtimes M)$. To see this, pick $0 \neq z \in (0 \rtimes M)$, and define $\phi : R \rightarrow (0 \rtimes M)$ by $\phi(r) = rz$. Note that if $r \in \ker(\phi)$, i.e., $rz = 0$, then r is a zero-divisor, and so $r \in M$. Thus $\ker(\phi) = M$, and hence

$$R/M = R/\ker(\phi) \cong \text{im}(\phi) \subseteq (0 : M).$$

This tells us that $|R/M| \leq |(0 : M)| \leq 4$. So the proof splits into cases, as $|R/M|$ equals 2, 3, or 4.

If $|R/M| = 3$, then $|M^3| \geq |R/M| = 3$. So, certainly, $|M^2| \geq 6$ and $|M^2 - \{0\}| \geq 5$. However, $M^2 - \{0\}$ is a clique, so that $\Gamma(R)$ contains a K_5 , and is nonplanar. The same conclusion obviously holds if $|R/M| = 4$.

If $|R/M| = 2$, then $|M^3| = 2^d$ for some integer $d \geq 1$. If $\Gamma(R)$ is planar, then necessarily $|M^2| = 4$. For 2 divides $|M^2|$, and if $|M^2| \geq 6$, then as above, $\Gamma(R)$ would contain a K_5 .

Thus, necessarily, $|M^3| = 2$, $|M^2| = 4$ (and $|M^2/M^3| = 2$). We claim that M^2 is principal. To see this, say

$$M^3 = \{0, x\}, M^2 = \{0, x, b, y\}.$$

We claim that $M^2 = Rb$. To see why, pick $m \in M$ so that $mM^2 \neq 0$. Then $mb \in M^3$, and we can take $mb = x$. Now $b + x = y$ ($b + x \neq 0$ since $b \in M^2 - M^3$, and obviously $b + x \neq x$ and $b + x \neq b$). So $y = b + x = b + mb = b(1 + m)$, and hence $M^2 = Rb$, as claimed.

Since $b \in M^2$, $b^2 = 0$, and thus $(0 : b) \supseteq Rb = M^2$. If $Rb = (0 : b)$, then Anderson and Nasseer (see [2, p. 511]) show that $|R| = 16$, $\omega(R) = 3$, and R is one of the rings in the statement of the lemma (each of which has a planar zero-divisor graph).

If $(0 : b) \supsetneq Rb = M^2$, pick $t \in (0 : b) - Rb$. Then $tb = 0$, so that $t(Rb) = tM^2 = 0$. So, $M^2 \subsetneq (0 : M^2) \subseteq M$, since $t \notin M^2$. Thus $|(0 : M^2)| \geq 8$ since $|M^2| = 4$. Therefore $|(0 : M^2) - M^2| \geq 4$, and one then sees that the subgraph of $\Gamma(R)$ induced by $(0 : M^2)$ and $M^2 - \{0\}$ contains a $K_{3,3}$, which forces $\Gamma(R)$ to be nonplanar. $\square$

Some of the statements in the above proof could be obtained using Nakayama's Lemma, but we have opted to keep the arguments as elementary and self-contained as possible.

The point of our next result is somewhat similar; we will see that if $0 = M^3 \subsetneq M^2 \subsetneq M$ and $\Gamma(R)$ is planar, then $\omega(R) \leq 3$. In effect, if all we are concerned with is the planarity of the zero-divisor graph, we need not try to explicitly classify all the local rings with clique number 4. In fact, the 2 rings in Corollary 2.22 are the only finite local rings with $\omega(R) = 4$ and $\Gamma(R)$ planar. The next result applies when $M^3 = 0$, $M^2 \neq 0$, and $\omega(R) \geq 3$. Using Lemmas 2.1 and 2.2 one can check that precisely the local rings $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, and $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ have $M^3 = 0$, $M^2 \neq 0$, and $\omega(R) \leq 2$, and these rings all have planar zero-divisor graphs. This fact, together with the next lemma, will give us precisely 14 local rings with $M^3 = 0$ and $M^2 \neq 0$ whose zero-divisor graphs are planar.

Lemma 2.24. *Let (R, M) be a finite, commutative local ring with $0 = M^3 \subsetneq M^2 \subsetneq M$ and $\omega(R) \geq 3$. If $\Gamma(R)$ is planar, then $\omega(R) = 3$ and R is isomorphic to one of the following eleven rings:*

$$\begin{aligned} &\mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X), \\ &\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY), \mathbb{Z}_2[X, Y]/(X^2, Y^2), \mathbb{Z}_8[X]/(2X - 4, X^2), \\ &\mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X), \\ &\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y), \text{ or } \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y). \end{aligned}$$

Proof. First, observe that each of the rings listed above has $M^3 = 0$ and $M^2 \neq 0$.

Since $M^2 - \{0\}$ is a clique, necessarily $|M^2 - \{0\}| \leq 4$, i.e., $|M^2| \leq 5$. If $|M^2| = 4$, then $|M| \geq 8$. Thus $|M - M^2| \geq 4$, and the subgraph of $\Gamma(R)$ spanned by $(M^2 - \{0\}) \cup (M - M^2)$ contains a $K_{3,3}$. The same conclusion holds if $|M^2| = 5$.

So the proof splits into cases, as $|M^2| = 2$ or 3.

Case 1. $|M^2| = 3$.

Pick $x \in M - M^2$, and consider $(0 : x)$. Certainly, $M^2 \subseteq (0 : x) \subseteq M$, and so 3 divides $|(0 : x)|$. If $|(0 : x)| = 3$, then Anderson and Nasseer (see [2, p. 512]) prove that R is isomorphic to one of $\mathbb{Z}_{27}$, $\mathbb{Z}_3[X]/(X^3)$, $\mathbb{Z}_9[X]/(X^2 - 3, 3X)$, or $\mathbb{Z}_9[X]/(X^2 - 6, 3X)$. Also note that each of these rings has a planar zero-divisor graph whose clique number is 3.

$|(0e x)| = 6$ is impossible, as each M^i/M^{i+1} is an R/M -vector space.

If $|(0e x)| \geq 9$, then

$$|(0 : x) - (\{x\} \cup M^2)| \geq 5;$$

so pick (nonzero) $x_1, x_2, x_3 \in ((0e x) - (\{x\} \cup M^2))$. Then the subgraph of $\Gamma(R)$ spanned by

$$\{x_1, x_2, x_3, x\} \cup (M^2 - \{0\})$$

contains a $K_{3,3}$.

Thus, if $|M^2| = 3$, then either R is one of the rings listed above, or $\Gamma(R)$ is nonplanar.

Case 2. $|M^2| = 2$.

As in [2, p. 513], we may pick $x \in M - M^2$ with the property that $x^2 = 0$. Thus $Rxe - \{0\}$ is a clique, and consequently $|Rx| \leq 5$. Since $|M^2| = |R/M| = 2$ (and thus $|R|$ is even), we see that $|Rx| = 2$ or 4. In [2, p. 513], Anderson and Nasseer show that $|Rx| = 2$ is impossible. Thus $|Rx| = 4$. Since $x^2 = 0$, we have $Rx \subseteq (0e x)$. So either $|(0e x)| = 4$ or $|(0e x)| \geq 8$. However, in the latter case, since one may pick 3 nonzero elements of Rx and 3 elements of $(0e x) - Rx$, $\Gamma(R)$ will contain a $K_{3,3}$.

So it remains only to consider the case where $|(0e x)| = 4$. Since $Rx \subseteq (0e x)$, and $|Rx| = 4 = |(0e x)|$, we must have $Rx = (0e x)$. Now M^2 contains one non-zero

element; for the remainder of this proof let us denote this element by y . Observe that since $M^3 = 0$, $y \in (0 : x)$; so that $(0 \in x) = \{0, y, x, x + y\}$ (recall $x \notin M^2$, so $x \neq y$).

Note that since $|Rx| = |R|/(0 : x)|$ and $|R/M| = |M^2| = 2$, we must have $|R| = 16$ and $|M| = 8$. So pick $m \in M - (0 : x)$, and note that M is generated by x and m . To see this, observe that $x + m \neq 0$, else $m = -x \in (0 : x)$, and thus $x + m \in M - (0 : x)$; similarly $y + m, x + y + m \in M - (0 : x)$. Hence $M = \{0, y, x, x + y, m, m + x, m + y, m + x + y\}$.

Observe that $xm \in M^2$, but $xm \neq 0$, as $m \notin (0 : x)$. So $xm = y$ since $M^2 = \{0, y\}$. Also note that $m^2 = 0$ or $m^2 = y = xm$.

Since $|R| = 16$, the proof now splits into cases, as $\text{char}(R)$ equals 16, 8, 4, or 2.

If $\text{char}(R) = 16$, then $R \cong \mathbb{Z}_{16}$. Again, note that this ring has a planar zero-divisor graph with clique number 3, but $M^3 \neq 0$.

If $\text{char}(R) = 2$, then Anderson and Nasseer [2, p. 513] have shown that $R \cong \mathbb{Z}_2[X, Y]/(X^2, Y^2)$ or $R \cong \mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY)$, as $m^2 = 0$ or $m^2 = xm$, respectively. Again, both of these rings have a planar zero-divisor graph whose clique number is 3.

If $\text{char}(R) = 8$, note that $2 \in M - M^2$ since $2^2 = 4$, but $2 \in M^2$ would imply that $2^2 = 0$. Then M is generated by 2 and x (recall x was chosen in $M - M^2$ with $x^2 = 0$). Then $2x = 4$ or 0, and if $2x = 4$, in [2, p. 513] it is shown that $R \cong \mathbb{Z}_8[X]/(2X - 4, X^2)$ (whose zero-divisor graph is planar with clique number 3). If $2x = 0$, one notes that

$$\text{ann}(2) = \text{ann}(6) = \text{ann}(x + 2) \supseteq \{0, 4, x, x + 4\},$$

which gives rise to a $K_{3,3}$ in $\Gamma(R)$. In fact, it is not too hard to see that in this case

$R \cong \mathbb{Z}_8[X]/(2X, X^2)$. To see this, take $\phi : \mathbb{Z}_8[X] \rightarrow R$ by $\phi(X) = x$. One notes that ϕ is onto and that $\ker(\phi) = (2X, X^2)$.

Lastly, if $\text{char}(R) = 4$, there are many cases to consider. Certainly $2 \in M$. So either $2 \in M^2$ or $2 \notin M^2$. If $2 \in M^2$, then Anderson and Nasseer [2, pp. 513-514] show that R is isomorphic to either $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$ or $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y)$. Both of these rings have a planar zero-divisor graph whose clique number is 3.

If $2 \notin M^2$, as before we may choose m so that $M = (2, m)$ (where $m \in M - \text{ann}(2)$). If $2m \neq 0$, Anderson and Nasseer [2, p. 513] show R is isomorphic to $\mathbb{Z}_4[X]/(X^2)$ or $\mathbb{Z}_4[X]/(X^2 - 2X)$, as $m^2 = 0$ or $m^2 \in (M^2 - \{0\}) = \{2m\}$. Once again, each of these rings has a planar zero-divisor graph whose clique number is 3.

So we are left to consider the case where $2m = 0$. To refresh the reader's memory, we have m so that $M = (2, m)$ and $M^2 - \{0\} = \{y\}$.

Case 1. $2m = 0$ and $m^2 = 0$.

In this case, note that

$$\text{ann}(2) = \{0, 2, m, 2 + m, y, y + 2, y + m, y + m + 2\} = M.$$

Observe that $2^2 = 0$ since $\text{char}(R) = 4$, $2m = 0$ by hypothesis, and $2y \in M^3 = 0$. Similarly, $\text{ann}(y) = \text{ann}(m) = M$, and thus the subgraph of $\Gamma(R)$ spanned by $\{2, y, m, 2 + m, 2 + y, y + m\}$ contains a $K_{3,3}$.

Case 2. $2m = 0$ and $m^2 \neq 0$ (i.e., $m^2 = y$).

Here, just as above, one checks that $\text{ann}(y) = \text{ann}(2) = \text{ann}(y + 2) = M$, again giving rise to a $K_{3,3}$ in $\Gamma(R)$, and the proof is complete. $\square$

Our work up to this point proves the following theorem.

Theorem 2.25. *Let R be a finite commutative ring (not a field), and k a (finite) field. Then $\Gamma(R)$ is a planar graph if and only if R is isomorphic to one of the following 44 types of rings:*

$\mathbb{Z}_2 \times k, \mathbb{Z}_3 \times k, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2),$
 $\mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,$
 $\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2),$
 $\mathbb{Z}_2 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2),$
 $\mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2),$
 $\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2),$
 $\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2 - 2), \mathbb{Z}_{16}, \mathbb{Z}_2[X]/(X^4),$
 $\mathbb{Z}_4[X]/(2X, X^3 - 2), \mathbb{Z}_4[X]/(X^2 - 2), \mathbb{Z}_4[X]/(X^2 + 2X + 2), \mathbb{F}_4[X]/(X^2),$
 $\mathbb{Z}_4[X]/(X^2 + X + 1), \mathbb{Z}_2[X, Y]/(X, Y)^2, \mathbb{Z}_4[X]/(2, X)^2,$
 $\mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X),$
 $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY), \mathbb{Z}_2[X, Y]/(X^2, Y^2), \mathbb{Z}_8[X]/(2X - 4, X^2),$
 $\mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X),$
 $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y), \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y),$
 $\mathbb{Z}_{25}, \text{ or } \mathbb{Z}_5[X]/(X^2).$

To conclude this section, we state a corollary and some remarks which are immediate, yet interesting in their own right.

Corollary 2.26. *Let R be a finite, commutative local ring (not a field) with either $|R| \geq 28$ or $|Z(R)| \geq 10$. Then $\Gamma(R)$ is not planar. In particular, if $\Gamma(R)$ is planar, then $|\Gamma(R)| \leq 8$.*

Proof. One need only examine the previous list of rings. □

Remark 2.27. Corollary 2.26 is best possible since $|\mathbb{Z}_{27}| = 27$ and $|Z(\mathbb{Z}_{27})| = 9$. Also note that all finite commutative rings R with $|\Gamma(R)| \leq 4$ have planar zero-divisor graphs (since $\Gamma(R)$ can not contain a K_5 or a $K_{3,3}$), and each such graph can be realized by a local ring when $|\Gamma(R)| \leq 3$ (cf. [3, Example 2.1]). As a consequence of [4, Theorem 2.10], one can see that if $|\Gamma(R)| = 5$, then $\Gamma(R)$ is planar. Further, it is shown that $\Gamma(R)$ can not be realized by a local ring (for then M would have 6 elements). And, one can easily find examples of finite commutative rings R with $6 \leq |\Gamma(R)| \leq 8$ having both planar and nonplanar zero-divisor graphs. In the next example, we illustrate these possibilities. In this case, such planar zero-divisor graphs can be realized by local rings only when $|\Gamma(R)| = 7$ or 8. Also, observe that a local ring R (not a field) having a planar zero-divisor graph necessarily has 4, 8, 16, 9, 27, or 25 elements (and $|\Gamma(R)| = 1, 3, 7, 2, 8$, or 4, respectively).

Example 2.28. As mentioned above, it is possible to produce examples of finite commutative rings R with $6 \leq |\Gamma(R)| \leq 8$ with both planar and nonplanar zero-divisor graphs. For instance, among graphs on 6 vertices, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5)$ is a planar graph (as it is a $K_{2,4}$) while $\Gamma(\mathbb{F}_4 \times \mathbb{F}_4)$ is nonplanar (for this graph is a $K_{3,3}$). On 7 vertices, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_7)$ is planar, being a star graph, but $\Gamma(\mathbb{F}_4 \times \mathbb{Z}_5)$ is nonplanar (as this graph is a $K_{3,4}$ and thus contains a $K_{3,3}$). And on 8 vertices, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_7)$ is planar (it is a $K_{2,6}$), but $\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5)$ is a $K_{4,4}$ and is thus nonplanar.

Remark 2.29. It is well-known that not all connected graphs on a given number of vertices can be realized as $\Gamma(R)$ for some appropriate R . For instance, an octagon cannot be $\Gamma(R)$, since in [4, Theorem 2.3], it is shown that any two vertices of $\Gamma(R)$ can be joined by a path of length at most 3. However, the previous theorem lets us find examples of graphs that satisfy this requirement, yet cannot be realized as

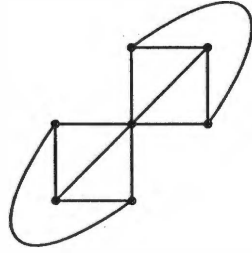


Figure 2.2: A planar graph with diameter 2 and girth 3 not realizable as $\Gamma(R)$

$\Gamma(R)$. For example, consider the graph obtained by pasting together two copies of K_4 at a ‘corner’ (see Figure 2.2). In this graph, any two vertices can be joined by a path of length 2, and this graph has *girth* 3 (and is planar!), but is not one of the graphs arising from the previous list, and hence cannot be realized as $\Gamma(R)$ for any commutative ring R .

Remark 2.30. As a final remark, $\Gamma(R)$ can be an infinite planar graph; for example, $\Gamma(\mathbb{Z} \times \mathbb{Z}_2)$ is a planar graph. This prompts the question as to when $\Gamma(R)$ is planar in the case when R is an infinite commutative ring. We study this problem in the next section.

4 The Infinite Case

The final remark in the preceding section prompts the question as to which *infinite* commutative rings have planar zero-divisor graphs. Careful examination of the proofs in the last section yields the following proposition.

Proposition 2.31. *If (R, M) is a commutative, Artinian local ring, then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the (finite) rings listed in Theorem 2.25.*

Proof. If one carefully examines the arguments in the previous section, one finds that in those arguments, the finiteness of R was only used to ensure that $M^n = 0$ for some positive integer n . $\square$

As might be expected, it is easy to generate examples of infinite rings with nonplanar zero-divisor graphs, as in the following proposition.

Proposition 2.32. *Let R be a commutative ring which is not an integral domain. Then $\Gamma(R[X])$ is nonplanar.*

Proof. Suppose r and s are distinct nonzero zero-divisors of R . Then, the subgraph of $\Gamma(R[X])$ spanned by $\{r, rX, rX^2, s, sX, sX^2\}$ contains a $K_{3,3}$. If R has only one nonzero zero-divisor z , then the subgraph of $\Gamma(R[X])$ spanned by $\{z, zX, zX^2, zX^3, zX^4\}$ is a K_5 , since $z^2 = 0$. $\square$

If R can be decomposed into a product, we can determine which such R have planar zero-divisor graphs. To motivate the results, we consider the following example.

Proposition 2.33. *Let R be an integral domain with $|R| \leq c$. Then $\Gamma(R \times \mathbb{Z}_2)$ and $\Gamma(R \times \mathbb{Z}_3)$ are planar graphs.*

Proof. Since $|R| \leq c$, there is an injection $R \rightarrow [0, 1]$. With this fact in hand, it suffices to show that $\Gamma(R \times \mathbb{Z}_3)$ is planar; the planar embedding is demonstrated in Figure 2.3. Observe that the condition $|R| \leq c$ cannot be dropped. $\square$

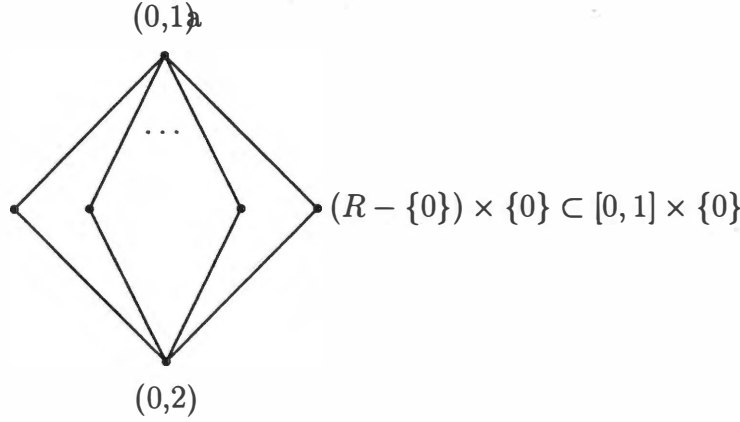


Figure 2.3: $\Gamma(R \times \mathbb{Z}_3)$, where $|R| \leq c$.

In the case where R decomposes (nontrivially) as a product $R \cong \bigoplus_{\alpha \in A} R_\alpha$ (note that A must be a finite set), we will see that the rings described in the above proposition are the only rings whose zero-divisor graph is planar. Recall that if $R \cong S \times T$, then $\Gamma(R)$ contains a $K_{|S|-1, |T|-1}$. Moreover, in Section 1 it was proved that if $R \cong \bigoplus_{\alpha \in A} R_\alpha$ and $\Gamma(R)$ is planar, then $|A| \leq 3$, for $|A| \geq 4$ would force $\Gamma(R)$ to contain a $K_{3,3}$. Thus, if $R \cong \bigoplus_{\alpha \in A} R_\alpha$ we may assume A is finite, and that one of the factors is an infinite ring. This, together with the previous comments, forces $|A| = 2$, i.e., $R \cong R_1 \times R_2$, where without loss R_1 is infinite and R_2 is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

In the case where $R \cong R_1 \times \mathbb{Z}_3$, we can almost immediately see that if $|R| = \infty$ and $\Gamma(R)$ is planar, then R_1 must be a domain. We first prove a very easy lemma.

Lemma 2.34. *Let R be a commutative ring. If $\Gamma(R \times \mathbb{Z}_3)$ is planar, then $\Gamma(R)$ contains no vertices of degree 3 or more.*

Proof. Suppose $0 \neq r$ is such a vertex, with (nonzero) x, y , and z adjacent to r .

Then in $\Gamma(R \times \mathbb{Z}_3)$, the subgraph spanned by $\{(0, 1), (0, 2), (r, 0), (x, 0), (y, 0), (z, 0)\}$ contains a $K_{3,3}$. $\square$

Proposition 2.35. *Let R_1 be an infinite commutative ring. If $\Gamma(R_1 \times \mathbb{Z}_3)$ is planar, then R_1 is an integral domain with $|R_1| \leq c$.*

Proof. Suppose not, that $\Gamma(R_1 \times \mathbb{Z}_3)$ is planar, but R_1 contains nonzero zero-divisors. In [10] and [11] it is shown, among other things that if S is an infinite commutative ring which is not an integral domain, then $Z(S)$ is also infinite. So R_1 has infinitely many zero-divisors. However, by the previous lemma, every element of $\Gamma(R_1)$ must have degree 1 or 2. But, $\Gamma(R_1)$ is connected, so therefore $\Gamma(R_1)$ must contain an infinite ‘line graph’. However, this then forces $\text{diameter}(\Gamma(R_1)) = \infty$, contradicting the fact that for any commutative ring R , $\text{diameter}(\Gamma(R)) \leq 3$ (cf. [4, Theorem 2.3]). $\square$

One could say the key element of the above proof was certain ‘degree restrictions’. Our next result, which seems mildly interesting in its own right, is somewhat similar in this regard.

Lemma 2.36. *Let R be an infinite commutative ring with $\Gamma(R)$ planar. Then $\text{degree}(x) = 1, 2$, or ∞ for each $x \in \Gamma(R)$.*

Proof. Pick $0 \neq x \in Z(R)$ with $\text{degree}(x)$ finite and consider $(0 : x)$. Recall $\text{degree}(x) = |(0 : x)| - 1$ if x is not a 2-nilpotent, and $\text{degree}(x) = |(0 : x)| - 2$ if x is a 2-nilpotent. Since $|Rx| = |R|/|(0 : x)|$, if $|(0 : x)| < \infty$, then $|Rx| = \infty$, and thus $|Rx - (0e x)| = \infty$. So the subgraph of $\Gamma(R)$ spanned by $Rx - (0e x)$ and $(0e x) - \{0\}$ contains a $K_{|Rx - (0e x)|, |(0e x) - \{0\}|}$. Thus, $|(0 : x) - \{0\}| \leq 2$ since $\Gamma(R)$ is planar, and hence $\text{degree}(x) \leq |(0e x) - \{0\}| - 1 = |(0e x) - \{0\}| \leq 2$. $\square$

Proposition 2.37. *Let R_1 be an infinite commutative ring. If $\Gamma(R_1 \times \mathbb{Z}_2)$ is planar, then R_1 is an integral domain with $|R_1| \leq c$.*

Proof. Suppose that $\Gamma(R_1 \times \mathbb{Z}_2)$ is planar, and let $0 \neq r$ be a zero-divisor in R_1 . By Lemma 2.36, the degree of $(r, 0)$ in $\Gamma(R_1 \times \mathbb{Z}_2)$ must be 1, 2, or ∞ . Since $Z(R_1)$ is an infinite set and $\Gamma(R_1)$ is connected, there exists $s \in R - \{0, r\}$ so that $rsa = 0$, and thus $(r, 0)$ is adjacent to $(a, 0)$, $(0, 1)$, and $(s, 1)$. Thus $\text{Degree}(r, 0)$ cannot be 1 or 2.

So necessarily $\text{degree}(r, 0) = \infty$. However, we will show that this forces $\Gamma(R_1 \times \mathbb{Z}_2)$ to be nonplanar. If the degree of $(r, 0)$ in $\Gamma(R_1 \times \mathbb{Z}_2)$ is infinite, then the degree of r (in $\Gamma(R_1)$) is infinite. So suppose that $(0 :_{R_1} r) \setminus \{0, r\} \supseteq \{x_1, x_2, x_3, \dots\}$. Then the subgraph of $\Gamma(R_1 \times \mathbb{Z}_2)$ spanned by $\{(r, 0), (r, 1), (0, 1)\} \cup \{(x_1, 0), (x_2, 0), (x_3, 0)\}$ will contain a $K_{3,3}$, and $\Gamma(R_1 \times \mathbb{Z}_2)$ is nonplanar, a contradiction. Hence, R_1 can contain no nonzero zero-divisors. $\square$

We now summarize some of the above results.

Theorem 2.38. *Suppose R is an infinite commutative ring and $R \cong \bigoplus_{\alpha \in A} R_\alpha$ with $|A| \geq 2$. If $\Gamma(R)$ is planar, then $|A| = 2$ and further, $R \cong R_1 \times \mathbb{Z}_2$ or $R \cong R_1 \times \mathbb{Z}_3$ where R_1 is an infinite integral domain such that $|R_1| \leq c$.*

On the basis of this result, one might hope to prove that if R is infinite and $\Gamma(R)$ is planar, then R is isomorphic to one of the two types of rings listed in the above theorem. Unfortunately, it is not possible to make such a strong statement, as the following example illustrates.

Example 2.39. Let A be a set with $|A| = c$ such that A does not contain the element 1, and let $S = \mathbb{Z}_2[X_1, \{X_1 X_i : i \in A\}]$, $J = (X_1^2, \{X_i : i \in A\})$, and $R = S/J$. Note that $Z(R) = \{f + \alpha X_1 + J : f \in \mathbb{Z}_2[\{X_i : i \in A\}], \alpha \in \mathbb{Z}_2\}$.

Thus R is an infinite commutative ring with $\Gamma(R)$ a star graph centered at X_1 . Further note that $|Z(R)| = |\mathbb{Z}_2[\{X_i\}_{i \in A}]| = |X_1 + \mathbb{Z}_2[\{X_i\}_{i \in A}]| = c$. To see this, any monomial in $\mathbb{Z}_2[\{X_i\}_{i \in A}]$ can be thought of as a tuple in $\bigoplus_{\omega} A$. Thus, $|\mathbb{Z}_2[\{X_i\}_{i \in A}]| = |\bigcup_{n \in \mathbb{N}} (A^{\omega})^n| = \omega(2^{\omega})^{\omega} = \omega 2^{\omega} = c$. However, R is not isomorphic to a ring of the form $(\mathbb{Z}_2 \text{ or } \mathbb{Z}_3) \times (\text{a domain})$, since a ring of this form has no nonzero 2-nilpotents, while R does (namely X_1 is a nonzero 2-nilpotent). Since $|\Gamma(R)| = c$, this is a maximal example of a planar star graph.

Remark 2.40. It is known in general when $\Gamma(R)$ is a star graph on infinitely many vertices. See [15, Theorem 2.2]. Also, in [4, page 439], the authors give an example of R with $\Gamma(R)$ a star graph with $|\Gamma(R)| = \omega$. Namely, $R = \mathbb{Z}[X]/(2X, X^2)$. This was the motivation for Example 2.39.

If $\Gamma(R)$ is infinite and contains a vertex of degree 1, we can say quite a bit.

Theorem 2.41. *Let R be an infinite commutative ring with $\Gamma(R)$ planar. If $\Gamma(R)$ contains a vertex x of degree 1, then either R has the form $R_1 \times \mathbb{Z}_2$, where R_1 is an integral domain with $|R_1| \leq c$, or Rx is an annihilator ideal.*

Proof. Suppose $\deg(x) = 1$, so that $(0 \neq x)$ has 2 or 3 elements. Note $x^2 \neq 0$. For if $x^2 = 0$, then $Rx - \{0\}$ is a clique. But $|Rx| = |R|/|(0 \neq x)| = \infty$. So $\Gamma(R)$ contains a K_5 , and is nonplanar. Hence $(0 \neq x) = \{0, z\} = Rz$ for some $z \in R - \{x\}$. Thus $z^2 = 0$ or $z^2 = z$. The proof then splits into cases accordingly.

Case 1. If $z^2 = z$, then $R \cong Rz \times R(1 - z) \cong \mathbb{Z}_2 \times R(1 - z)$. And by Theorem 2.38, it follows that $R(1 - z)$ is an integral domain.

Case 2. If $z^2 = 0$, then since $Rz = \{0, z\}$, and since $|(0 \neq x)| = 2$,

$$2 = |(0 \neq x)| = |R|/|Rx| = |R/Rx|.$$

We claim that $Rx = (0 : z)$. ' $\subseteq$ ' is clear. To see ' $\supseteq$ ', let $as \in (0 : z)$, and suppose for contradiction that $as \notin Rx$. Then $s - 1 \in Rx$ and thus $(s - 1)z = 0$. So, $sz - z = 0$, but $sz = 0$, so that $z = 0$, a contradiction. Thus $Rx = (0 : z)$, and hence Rx is an annihilator ideal. $\square$

Remark 2.42. One can have both $R \cong R_1 \times \mathbb{Z}_2$, and $\text{degree}(x) = 1$ with Rx an annihilator ideal simultaneously. In $R = \mathbb{Z} \times \mathbb{Z}_2$, $(1, 0)$ has degree 1, and one can easily check that the ideal generated by $(1, 0)$ is an annihilator ideal. Namely, this ideal is the annihilator of $(0, 1)$. Further, $\Gamma(R)$ is a star graph centered at $(0, 1)$.

Next, we investigate the structure of $\Gamma(R)$ when the graph contains a vertex x of degree two. Some facts are immediate. Since $Rx \cong R/(0 : x)$, if x is 2-nilpotent, then $\Gamma(R)$ is nonplanar since in this case $|Rx| = \infty$ (and Rx is a clique). Note that if D is a domain, $\Gamma(D \times \mathbb{Z}_3)$ contains many vertices of degree two. There are rings that do not take this form, yet have planar zero-divisor graphs and vertices of degree two. See Example 2.44. However, in the next proposition, we show that if R has vertices of degree two and is reduced, then R must take the form described above.

Proposition 2.43. *Let R be an infinite commutative ring with $\Gamma(R)$ planar, and suppose that $\Gamma(R)$ contains a vertex x of degree two. If R is reduced, then $R \cong S \times \mathbb{Z}_3$, where S is an infinite integral domain with $|S| \leq c$.*

Proof. Suppose x is adjacent to precisely a and b in $\Gamma(R)$, so that in particular $a \neq b$. Consider $a + b$. Since $(a + b)x = 0$ and R is reduced, $a + b = 0, a$, or b . Clearly $a + b = a$ and $a + b = b$ are impossible. Thus $a = -b$.

Now consider ab . Note $(ab)x = 0$, so that $ab = 0, a$, or b . If $ab = a$ then $-a^2 = a$, so that $(a^2)^2 = (-a^2)^2 = a^2$. Note that $a^2 = -a \neq 0$, so that $R \cong Ra^2 \times R(1 - a^2)$.

Thus, R may be written as a product $S \times T$ and by Theorem 2.38, $T \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ and S is an infinite integral domain. However, since $\Gamma(S \times \mathbb{Z}_2)$ is an infinite star graph if S is an infinite integral domain (and thus has no vertices of degree two), we must have $R \cong S \times \mathbb{Z}_3$. The same result holds if $ab = b$. The $ab = 0$ case is impossible, for then $0 = ab = a(-a) = -a^2$ so that a is a nonzero 2-nilpotent. $\square$

In the above proposition, $\Gamma(R)$ is a $K_{2,|S|-1}$. If we drop the requirement that R be reduced, we can generate examples so that $\Gamma(R)$ is planar and contains vertices of degree two, but $\Gamma(R)$ is not complete bipartite.

Example 2.44. Let $R = \mathbb{Z}[X]/(X^2, 3X)$. Then every vertex of $\Gamma(R)$ has degree two (or has infinite degree) and $\Gamma(R)$ is planar. The planar embedding is demonstrated in Figure 2.4.

One might wonder if it is possible to ‘mix’ vertices of degree one and two in an infinite zero-divisor graph which is planar. Our next proposition answers this question in the negative.

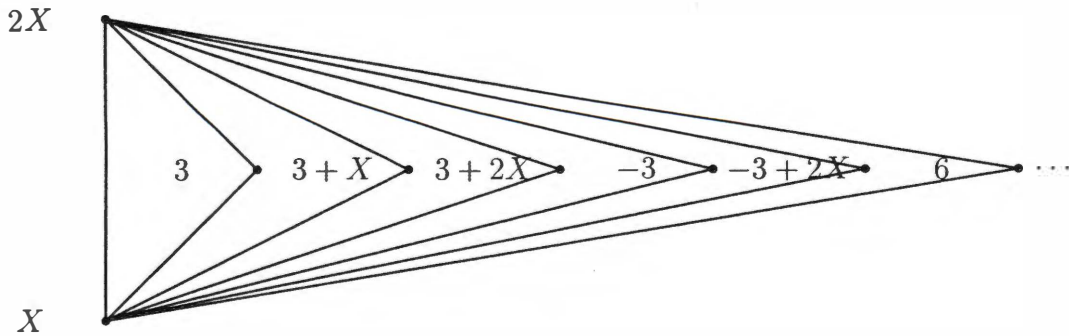


Figure 2.4: $\Gamma(\mathbb{Z}[X]/(X^2, 3X))$

Proposition 2.45. *Let R be an infinite commutative ring. If $\Gamma(R)$ is planar, then $\Gamma(R)$ does not contain vertices of both degree one and degree two.*

Proof. Suppose that $\Gamma(R)$ contains vertices y and x of degree one and two respectively, and $\Gamma(R)$ is planar. First note that $Rx \cong R/(0 : x)$ implies that x cannot be a 2-nilpotent. Similarly, y cannot be 2-nilpotent. Let $(0 : x) = \{0, a, b\}$ and let $(0 : y) = \{0, c\}$. Then, as in previous arguments, since $(a+b)x = 0$, we must have $a + b = 0$, and hence $b = -a$. Now, consider a^2 . Since $(a^2)x = 0$, we must have $a^2 = a, -a$ or 0 . If $a^2 = a$, then $R \cong Ra \times R(1-a)$, and thus by Theorem 2.38, R must take the form $R \cong D \times (\mathbb{Z}_2 \text{ or } \mathbb{Z}_3)$, where D is an (infinite) integral domain. But then, $\Gamma(R)$ contains no vertices of degrees two and one respectively, which is a contradiction. A similar contradiction occurs if $a^2 = -a$. Thus, we may assume that $a^2 = 0$. Similar reasoning shows that $c = -c$ and that we may take $c^2 = 0$ and $3a = 0$.

Next we consider ac . The proof will split into cases, as $ac = 0$ or $ac \neq 0$. If $ac = 0$, simply note that the subgraph of $\Gamma(R)$ spanned by $\{a, -a, a+c, -a+c, c\}$ is a K_5 since all of the elements in this set are distinct.

We now show that $ac \neq 0$ is impossible. For if $ac \neq 0$, then $(ac)y = 0$. But $(0a)y = \{0, c\}$, so $ac = c$. But this forces $cax = (ac)x = c(ax) = 0$, so that $c \in (0 : x)$. But $(0a)x = \{0, a, -a\}$, a contradiction. $\square$

Our next result is somewhat similar.

Proposition 2.46. *Let R be an infinite commutative ring with $\Gamma(R)$ is planar. If x and y are two distinct vertices of degree 2 in $\Gamma(R)$, then $(0a)x = (0a)y$.*

Proof. As before, if $\Gamma(R)$ is planar, then neither x nor y can be 2-nilpotent. So, let $(0 : x) = \{0, a, b\}$ and let $(0 : y) = \{0, c, d\}$. As before, we must have $b = -a$ and

$d = -c$, and we may assume $a^2 = 0 = c^2$. If $a \in \{c, -c\}$, then $(0 : x) = (0 : y)$. So we may suppose that $a \notin \{c, -c\}$, and we will obtain a contradiction. If $ac = 0$, then we claim that the subgraph of $\Gamma(R)$ spanned by $\{a, -a, c, -c, a + c\}$ is a K_5 . Note that $a + c \neq -a$. For $a + c = -a$ implies that $c = -2a$. But $-2a \in (0 : x)$, and thus $-2a = 0$ or $-2a = b$. This would then imply that $c \in \{0, b\} = \{0, -a\}$, a contradiction. Similarly $a + c = -c$. If $ac \neq 0$, then $(ac)x = 0 = (ac)y$. $(ac)x = 0$ implies that $ac = a$ or $-a$, while $(ac)x = 0$ implies that $ac = c$ or $-c$. But since $a, -a, c$, and $-c$ are distinct, we have a contradiction. Thus $(0 : x) = (0 : y)$. $\square$

Our next two results will enable us to completely determine the structure of those infinite zero-divisor graphs which are planar and contain a vertex of degree two.

Lemma 2.47. *Let R be an infinite commutative ring such that $\Gamma(R)$ is planar and contains a vertex x of degree two, with $(0 : x) = \{0, a, -a\}$. If $y, z \in (0 : a) - \{0\}$ with $y, z \notin \{a, -a\}$ and y, z distinct, then $yz \neq 0$.*

Proof. As in previous arguments, we may assume that $a^2 = 0$ (see Proposition 2.45) and that $2a = -a$. Suppose that $yz = 0$. Then, the subgraph of $\Gamma(R)$ spanned by $\{a, -a, y, z\}$ is a K_4 . So consider $a + y$. The proof splits into cases, as $a + y \neq z$ or $a + y = z$.

If $a + y \neq z$, we consider two subcases. First we consider the case where $y^2 = 0 = z^2$. It is obvious that $a + y \notin \{0, a, y, z\}$. If $a + y = -a$, then $y = -2a = a$, contrary to the choice of y . Then the subgraph of $\Gamma(R)$ spanned by $\{a, -a, y, z, a + y\}$ is a K_5 .

If $a + y \neq z$ and y^2 or z^2 is nonzero, as above we may assume that $a, -a, y, z$, and $a + y$ are distinct and nonzero. So consider $-a + z$. Certainly, $-a + z \notin \{0, -a, z, y\}$. If $-a + z = a$, then $z = 2a = -a$, contrary to our choice of z . If $-a + z = a + y$,

then $z = 2a + y = -a + y$. Then $yz = 0$ implies that $0 = yz = y(-a + y) = -ay + y^2 = y^2$. Similarly, this assumption forces $z^2 = 0$. Thus we may assume that $a, -a, z, y, a + y, -a + z$ are distinct and nonzero. But the subgraph of $\Gamma(R)$ spanned by $\{a, -a, y, z, a + y\} \cup \{-a + z\}$ contains a subdivision of K_5 .

If $a + y = z$, then as above we must have $y^2 = 0 = z^2$. Now $a + z = a + (a + y) = 2a + y = -a + y$. Note $(a + z) \notin \{0, a, -a, z\}$. Also, $a + z \neq y$; for if $a + z = y$, then $-a + y = a + z$ by the previous comments. However, $a + z = y$ by hypothesis. Thus $-a + y = y$, so that $-a = 0$, which is a contradiction. Thus $(a + z) \notin \{0, a, -a, y, z\}$, and the subgraph of $\Gamma(R)$ spanned by $\{a, -a, y, z, a + z\}$ is a K_5 .

□

Our next result extends the above lemma.

Lemma 2.48. *Let R be an infinite commutative ring with $\Gamma(R)$ planar. Suppose that $\Gamma(R)$ contains a vertex x of degree 2 with $(0 : x) = \{0, a, -a\}$. If $y \notin \{0, a, -a\}$ is another nonzero element of $(0 : a)$, then $\deg(y) = 2$.*

Proof. Let $z \in R - \{0, a, -a, y\}$ so that $yz = 0$. First, by Proposition 2.45 note that the degree of z cannot be 1. If $\deg(z) = 2$, then since z is assumed to be adjacent to exactly two (nonzero) vertices, say y and k , that $y + k$ must be 0, and hence $k = -y$, and thus $y \neq -y$ in particular. But as in previous arguments, $y^2 = 0$, y , or $-y$. The latter two cases imply that R decomposes as a product, and thus if $\Gamma(R)$ is planar, we must have $R \cong D \times (\mathbb{Z}_2 \text{ or } \mathbb{Z}_3)$, where D is a domain. So, we may assume that $a^2 = 0 = y^2$. But then the subgraph of $\Gamma(R)$ spanned by $\{a, -a, y, -y, a + y\}$ is a K_5 .

Therefore, we may take $\deg(z) = \infty$. So we may pick distinct $k_1, k_2, \dots$, with $k_i \in (0 : z) - \{0, a, -a, y, z\}$. Now, since $\text{diameter}(\Gamma(R)) \leq 3$ and x is adjacent

precisely to a and $-a$, for each i , we may obtain a path from k_i to a of length at most 2. We will consider cases according to the nature of such a path.

First, note that if $k_i a = 0$ for at least three i , then the subgraph spanned by $\{k_1, k_2, k_3\} \cup \{a, -a, z\}$ contains a $K_{3,3}$. Also, note that by the previous lemma that z and a cannot be adjacent. Thus, we may assume that $k_i a = 0$ for at most two i .

Now, we consider the case when $k_i y = 0$ for all i . In this case, examine $y + z$. Certainly $y + z \neq y$ and $y + z \neq z$. Also note we may take $y + z \neq 0$. For $y + z = 0$ implies that $z = -y$ and hence $z \in (0 : a)$ and $zy = 0$ which is forbidden by the previous lemma. It is possible that $y + z$ could equal k_i for some i ; thus, reindexing if necessary, we may suppose $y + z \notin \{k_1, k_2, k_3\}$. Then, the subgraph spanned by $\{k_1, k_2, k_3\} \cup \{y, z, y + z\}$ contains a $K_{3,3}$.

In the above paragraph, we showed that $k_i y = 0$ for at least four i forces $\Gamma(R)$ to be nonplanar. Thus, we may assume that $k_i y = 0$ for at most three k_i . So, we may throw out any k_i such that $k_i y = 0$ and reindex, if necessary. Note that the set of all such indices $i \in I$ is still an infinite set. Next, we show that there cannot be 2 distinct paths of length two joining vertices in $\{k_i | i \in I\}$ to a that do not pass through the vertex y . Observe that since x is adjacent to only a and $-a$ in $\Gamma(R)$, and since $\text{diameter}(\Gamma(R)) \leq 3$, the distance d in $\Gamma(R)$ from each k_i to a must satisfy $d(k_i, a) \leq 2$. Thus, there must be paths from each k_i to a of length at most two. Previous remarks show that we may assume each k_i and y are nonadjacent, and as mentioned above, the previous lemma forces z and a to be nonadjacent. So, suppose there were two such paths, as indicated in Figure 2.5. In this case, the subgraph spanned by $\{l_1, l_2, y, k_1, k_2\} \cup \{a, -a, z\}$ contains a subdivision of $K_{3,3}$. See Figure 2.7.

Thus, as is indicated in Figure 2.6, we may assume that there is exactly

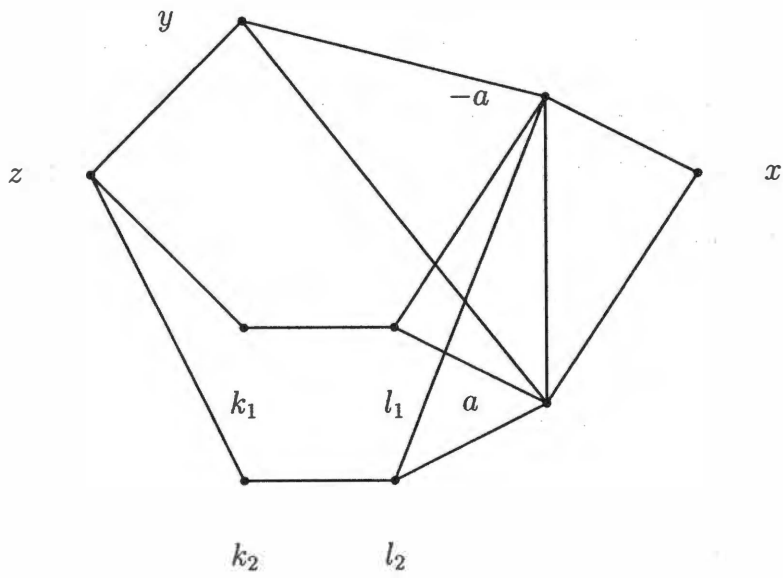


Figure 2.5: Figure 1 for proof of Lemma 2.48

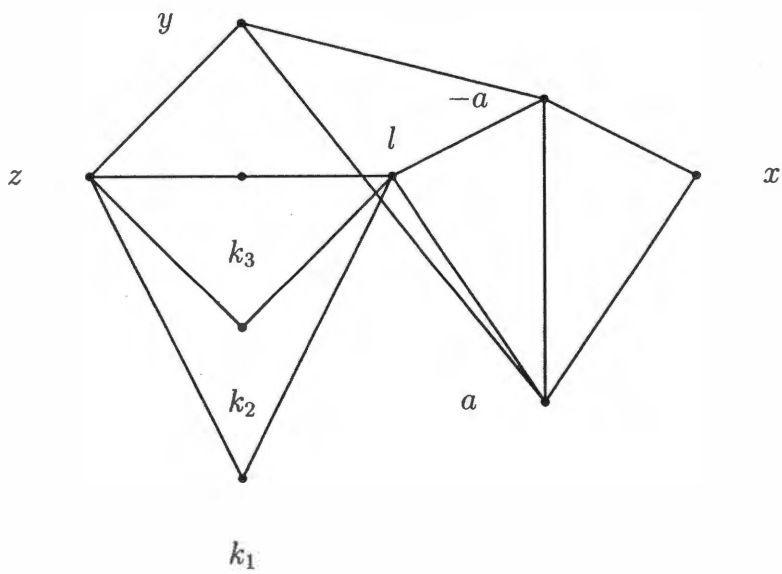


Figure 2.6: Figure 2 for proof of Lemma 2.48

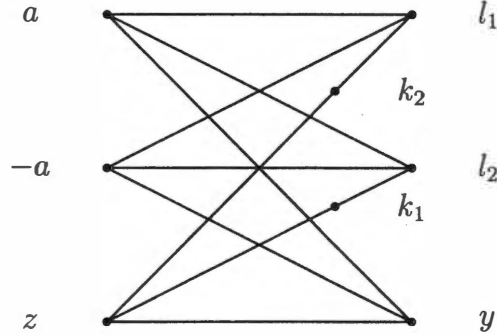


Figure 2.7: Subdivision of $K_{3,3}$ indicated in Lemma 2.48

one path of length two joining vertices in $\{k_i | i \in I\}$ to a that does not pass through the vertex y .

Consider $l + z$, and as before we may, reindexing if necessary, assume that $(l + z) \notin \{k_1, k_2, k_3\}$. Note that $l + z \neq 0$, for $l + z = 0$ would imply that $z = -l$, so that $la = 0$ implies that $z \in (0 : a)$. But $za \neq 0$, as in the third paragraph of this proof. Then, the subgraph spanned by $\{l, z, l + z\} \cup \{k_1, k_2, k_3\}$ contains a $K_{3,3}$.

So, we have shown that paths joining vertices of $\{k_i | i \in I\}$ to a can pass through at most one common point. However, since $|I| = \infty$, at least four k_i must neighbor one of these 2 points, which the above paragraphs show is a contradiction.

□

With these two lemmas and Proposition 2.46 in hand, the following theorem follows easily.

Theorem 2.49. *Let R be an infinite commutative ring such that $\Gamma(R)$ is planar and contains a vertex x of degree two. Denote the elements of $(0 : x) - \{0\}$ by a and $-a$. Then $\Gamma(R)$ is graph-isomorphic to either $K_{2,\alpha}$ (where $\omega \leq \alpha \leq c$) or the graph*

obtained from such a $K_{2,\alpha}$ by adding an edge joining a and $-a$.

Proof. Let x be a vertex of degree two in $\Gamma(R)$. As before, note that if $x^2 = 0$, then $Rx \cong R/(0 : x)$ will force $\Gamma(R)$ to contain a K_5 . As before, we may assume $(0 \in x) = \{0, a, -a\}$, where $a \notin \{0, x\}$. By Proposition 2.45, $\Gamma(R)$ contains no vertices of degree one. Thus all other vertices of $\Gamma(R)$ have degree two or are of infinite degree. If y is another vertex of degree two, then by Proposition 2.46, $(0 : x) = (0 : y)$. Further, since by Lemma 2.47, $xy \neq 0$, the subgraph of $\Gamma(R)$ spanned by $\{x, y, a, -a\}$ is either a $K_{2,2}$ or the graph obtained by taking this $K_{2,2}$ and adding an edge between a and $-a$, as $a^2 = a$ or $a^2 = 0$ respectively. If S denotes the vertices of degree two in $\Gamma(R)$, similar reasoning shows that the subgraph of $\Gamma(R)$ spanned by $S \cup \{a, -a\}$ is either a $K_{2,|S|}$ or the graph obtained from this $K_{2,|S|}$ and adding an edge joining a and $-a$,

All that remains is to show that a and $-a$ are the only vertices of infinite degree in $\Gamma(R)$. But this is easy; for if $z \notin \{a, -a\}$ with the degree of z infinite, then z is not adjacent to a or $-a$, since by Lemma 2.48, such a vertex would necessarily have degree two.

Moreover, z can not be adjacent to any neighbors of a or $-a$, since by Lemma 2.48 once again, neighbors of a and $-a$ have degree precisely two. Thus, since $\Gamma(R)$ is connected, there can be no such vertex z , and $\Gamma(R)$ must be one of the two types of graphs as stated. $\square$

Remark 2.50. The previous theorem allows us to generate additional examples of planar graphs which can not be realized as $\Gamma(R)$ for any (infinite) commutative ring

R . For example, let $G = K_{2,\alpha}$, where $\alpha \leq c$. Let a_1 and a_2 denote the vertices of infinite degree in G . Denote by H_1 the graph formed by taking G and adding an edge joining a_1 and a_2 . Let H_2 denote another copy of H_1 , and denote the vertices of infinite degree in H_2 by b_1 and b_2 . Let H be the graph formed by from H_1 and H_2 by adding edges between a_1 and b_1 , and between a_2 with b_2 . For a representation of H , see Figure 2.8. Then H is a planar graph. To see this, note that H_1 and H_2 are both planar graphs. For if $\alpha \leq c$, then $K_{2,\alpha}$ is a planar graph and can hence be embedded in $(-\infty, 0] \times \mathbb{R}$, and we may place $a_1, a_2 \in \{0\} \times \mathbb{R}$. Thus, the graph obtained by adding to this $K_{2,\alpha}$ the line segment joining a_1 and a_2 is planar, i.e., H_1 is planar. Similarly, H_2 can be embedded in $[1, \infty) \times \mathbb{R}$ and thus H is planar as well. Also, H has diameter 3 and girth 4, but by the previous theorem this graph cannot be realized as $\Gamma(R)$ for any (infinite) commutative ring R .

Example 2.51. It is possible to construct planar graphs where every vertex has infinite degree. To see how this is done, we proceed as follows. Let G_1 denote a copy

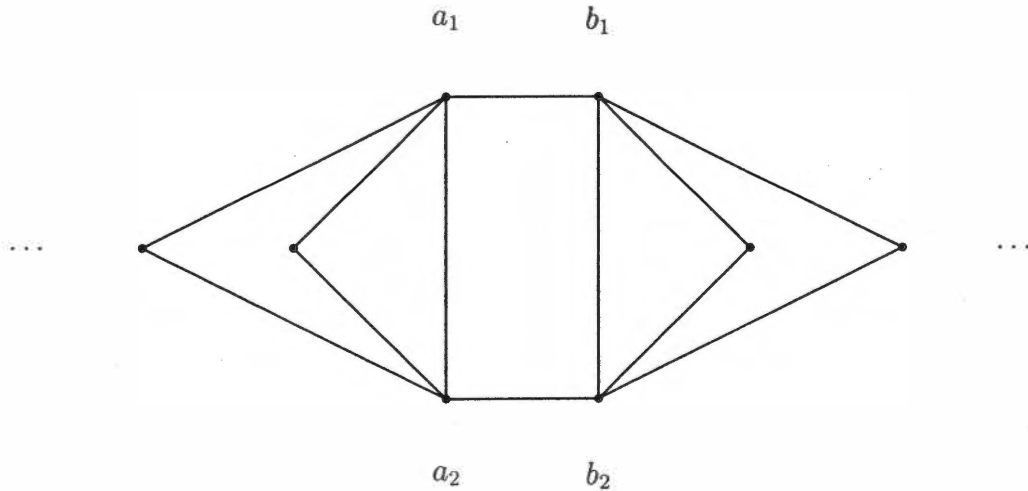


Figure 2.8: An infinite planar graph not realizable as $\Gamma(R)$

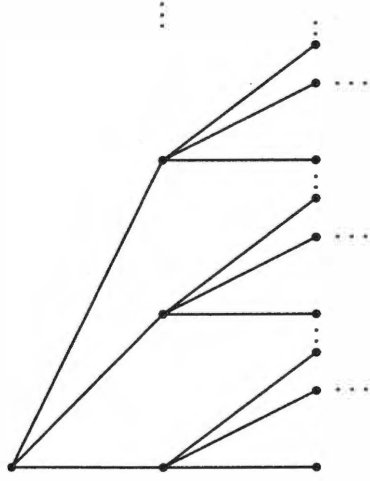


Figure 2.9: A planar graph where each vertex has infinite degree

of $K_{1,\omega}$. Note that G_1 is a planar graph. Let x be the vertex of infinite degree in G_1 and let $x_1, x_2, \dots$ denote the vertices of degree 1 of this graph. Then, let G_2 be the graph obtained by taking G_1 and attaching a (graph-isomorphic) copy of G_1 at each of the vertices of degree 1. Inductively, we then define G_{n+1} to be the graph obtained by taking G_n and attaching an isomorphic copy of G_1 at each of the vertices of degree 1 in G_n . Then, let $G = \cup_{i=1}^{\infty} G_i$. A representation of G is given in Figure 2.9.

We claim that G is planar. To see this, note that we may embed G_1 in the plane as follows. Place the vertex of infinite degree at $(0,0)$. Then, as above, if $x_1, x_2, \dots$ denote the vertices of degree 1 in this graph, place x_i at the point $(1,i)$. Note that this argument shows that not only is G_1 planar, but it can in fact be embedded in $[n, n+1] \times I$ where $I \subset \mathbb{R}$ is any open interval and n is any positive integer.

Similarly, we can show that G_2 is planar. Let H_i denote the copy of G_1 which

is attached to vertex $x_i \in V(G_1)$. As mentioned, we may assume that x_i is placed at $(1, i) \in \mathbb{R} \times \mathbb{R}$. Then, if we let $\{x_{ij} | j \in \mathbb{N}\}$ denote the vertices of degree 1 in H_i , placing vertex x_{ij} at $(2, i + \frac{2^{j-1}-1}{2^j-1})$ realizes G_2 as a planar graph. Then, by induction, since any graph isomorphic to G_1 can be embedded in $[n, n+1] \times I$, where I is any open interval in $\mathbb{R}$, G_n is planar for each positive integer n , and hence G is planar. Note that for each i , $V(G_i)$ and $E(G_i)$ are countable sets, and since a countable union of countable sets is countable, we have $|V(G_i)| = \omega$ and $|E(G_i)| = \omega$.

However, this graph can not be realized as $\Gamma(R)$ for any commutative ring R . For in the graph in question, given any positive integer n , there exist a pair of points at distance n . Since for any commutative ring R , we have $\text{diameter}(\Gamma(R)) \leq 3$, the shown graph cannot be $\Gamma(R)$ for any commutative ring R .

This example motivates the following proposition.

Proposition 2.52. *Let R be an infinite commutative ring. If every vertex of $\Gamma(R)$ has infinite degree, then $\Gamma(R)$ is nonplanar.*

Proof. First, if the diameter of $\Gamma(R)$ is 1, then $\Gamma(R)$ will contain a K_5 , and be thus nonplanar. Therefore, since $\Gamma(R)$ is connected, we may assume that $\Gamma(R)$ contains two vertices x and y at distance 2. The proof now splits into cases, as $(0 \mid x) = (0 \mid y)$ or $(0 : x) \neq (0 : y)$.

Case 1. $(0 : x) = (0 \mid y)$.

Since $(0 : x)$ is an infinite set, we may pick distinct $x_1, x_2, x_3 \in (0 : x) - \{0, x, y, x+y, xy\}$. Then, the subgraph of $\Gamma(R)$ spanned by $\{x, y, x+y\} \cup \{x_1, x_2, x_3\}$ contains a $K_{3,3}$, provided that $x+y \neq 0$. Note that in this case, we did not use the hypothesis that x and y are at distance two.

In the above paragraph, if $x + y = 0$, then $y = -x$, and hence $(0 : x) = (0 : y)$. Then examine xy . Since x and y are at distance two, we have $xy \neq 0$. Note that if $xy = x$, then we have $-x^2 = x$, and thus R will decompose into a product, a case which was covered earlier in this section. Thus we may assume $xy \neq x$ and by similar reasoning, $xy \neq y$. Then, the subgraph of $\Gamma(R)$ spanned by $\{x, y, xy\} \cup \{x_1, x_2, x_3\}$, with x_1, x_2 and x_3 as in the above paragraph will contain a $K_{3,3}$.

Case 2. $(0 : x) \neq (0 : y)$. Note $x + y \neq 0$, for $x + y = 0$ implies that $(0 : x) = (0 : y)$.

Consider xy . Since x and y are at distance 2, we have $xy \neq 0$. We may assume that $xy \neq x$. For $xy = x$ implies that $(0 : x) \supset (0 : y)$ which as in the previous case forces $\Gamma(R)$ to contain a $K_{3,3}$. To see this, let $y_1, y_2, y_3 \in (0 : y) - \{0, x, y, x + y\}$. Then the subgraph of $\Gamma(R)$ spanned by $\{x, y, x + y\} \cup \{y_1, y_2, y_3\}$ contains a $K_{3,3}$. Similarly, we may assume that $xy \neq y$. Since x and y both have infinite degree, Rx and Ry must both have two or fewer nonzero elements, else $\Gamma(R)$ would contain a $K_{3,3}$. To see this, we may without loss suppose Rx has at least three nonzero elements. Let S be a set containing exactly three distinct nonzero elements of Rx . But since the degree of x is infinite, certainly $(0 : x)$ is an infinite set. Then, we may pick three distinct nonzero elements from $(0 : x) - S$. Let T denote such a collection. Then, the subgraph of $\Gamma(R)$ spanned by S and T contains a $K_{3,3}$. Therefore, we may assume that $Rx = \{0, x, xy\}$ and $Ry = \{0, y, xy\}$. However, this is a contradiction, for $|Rx \cap Ry|$ must divide $|Rx|$.

□

Next we prove that in the case where R is an infinite commutative ring, if $\Gamma(R)$ contains a vertex of degree one and $\Gamma(R)$ is planar, then $\Gamma(R)$ must be a star

graph. For more results on when $\Gamma(R)$ is a star graph, see [15].

First, we recall some earlier observations. If x is a vertex of degree one and $\Gamma(R)$ is planar, then certainly $x^2 \neq 0$ (since $Rx - \{0\}$ is a clique). Thus, $(0 : x) = \{0, z\}$ for some $z \neq x$. Then, since $(0 : x)$ is an ideal, z^2 must equal 0 or z . If $z^2 = z$, then R decomposes as a product, and we have determined which such rings have planar zero-divisor graphs. And, by the proof of Theorem 2.41 (case 2), if $z^2 = 0$, we have $Rx = (0 \mathfrak{a} z)$.

Lemma 2.53. *Let R be an infinite commutative ring with $\Gamma(R)$ planar. Suppose $\Gamma(R)$ contains a vertex x of degree one, with $(0 \mathfrak{a} x) = \{0, \mathfrak{x}\}$. Then if $z^2 = 0$, the subgraph of $\Gamma(R)$ spanned by $Rx \cup \{z\}$ is a star graph centered at z .*

Proof. We proceed by contradiction. Suppose there are $rx \neq sx$ in $Rx - \{0, x, z\}$ with $(rx)(sx) = 0$. Then, since the degree of x is one and $\Gamma(R)$ is planar, by Proposition 2.45 the degree of rx and sx must be infinite. So, denote $(0 : rx) - \{0, z, sx\} = \{a_\alpha | \alpha \in A\}$ and $(0 \mathfrak{a} sx) - \{0, z, rx\} = \{b_\beta | \beta \in B\}$, where A and B are infinite sets.

First consider $rx + sx$. Suppose $rx + sx = 0$. Then $rx = -sx$, and hence $(0 \mathfrak{a} rx) = (0 \mathfrak{a} sx)$. Then, examine $rx - sx$. Since rx and sx were assumed distinct, $rx - sx \neq 0$ and certainly $rx - sx$ does not equal rx , since sx is assumed nonzero. If $rx - sx = sx$, then $rx = 2sx$. Now, if $|R(sx)| \geq 4$, then if $S \subset (0 : rx) - R(sx)$, with $|S| \geq 3$, the subgraph of $\Gamma(R)$ spanned by $R(sx) \cup S$ will contain a $K_{3,3}$. If $|R(sx)| \leq 3$, then $R(sx) = \{0, rx, sx\}$. Now consider $(sx)^2 \in R(sx)$. If $(sx)^2 = sx$, then $R \cong R(sx) \times R(1 - sx)$. But then since $|R(sx)| \geq 3$ and $|R(1 - sx)| = \infty$, $\Gamma(R)$ does not contain any elements of degree one, a contradiction. Similarly, if $(sx)^2 = rx$, then $(sx)^2 = -sx$, and so $(sx)^2$ is an idempotent, and R will decompose as a product. We are left to examine the case where $(sx)^2 = 0$. In this case, consider $rx + z$. Note

that $rx + z \neq 0$. For if $rx + z = 0$, then $rx = -z$ and thus $(0\alpha z) = (0\alpha rx) = (0\alpha sx)$. Then the subgraph of $\Gamma(R)$ spanned by $\{z, rx, sx\} \cup ((0\alpha rx) - \{0, rx, sx, z\})$ contains a $K_{3,3}$. Similarly, $rx + z \neq sx$. To see this, if $rx + z = sx$, then $z = sx - rx$. But since $R(sx) = \{0, rx, sx\}$, it follows that $z = rx$, which is contrary to our original choice of z . Thus we may assume that $z, rx, sx, rx + z$, and $sx + z$ are all distinct and nonzero. Since $z^2 = 0$, the subgraph of $\Gamma(R)$ spanned by $\{rx, sx, z, rx + z, sx + z\}$ is a K_5 .

So we may assume $rx + sx \neq 0$. Now note that $|(0 : rx) \cap (0 : sx)| < 7$ since otherwise, $\Gamma(R)$ would contain a $K_{3,3}$. Namely, if $z_1, z_2, z_3 \in ((0 : rx) \cap (0 : sx)) - \{0, rx, sx, rx + sx\}$, then the subgraph of $\Gamma(R)$ spanned by $\{rx, sx, rx + sx\} \cup \{z_1, z_2, z_3\}$ contains a $K_{3,3}$. In particular $(0 : rx) \cap (0 : sx)$ is a finite set. Pick $a_1 \in (0 : rx) - \{0, rx, sx, rx + sx\}$. Consider $\{a_1 b_\beta | \beta \in B\}$. This set must be finite, being a subset of $(0\alpha rx) \cap (0\alpha sx)$. So, we may pick a sequence $b_1, b_2, \dots \in B$ so that $a_1 b_1 = a_1 b_2 = a_1 b_3 = \dots$. Thus, $a_1(b_1 - b_i) = 0$ for all $i = 1, 2, \dots$. If $i \neq j$, then $b_1 - b_i \neq b_1 - b_j$. Now $b_1 - b_i$ could equal $rx, sx, rx + sx$, or a_1 , but since each of these terms is distinct, without loss we may assume that $a_1(b_1 - b_2) = 0 = a_1(b_1 - b_3)$, and $(b_1 - b_2), (b_1 - b_3) \notin \{0, rx, sx, rx + sx, a_1\}$. Note that the subgraph of $\Gamma(R)$ spanned by $\{rx, b_1 - b_2, b_1 - b_3\} \cup \{a_1, sx\}$ is a $K_{2,3}$.

For notational convenience, denote $b'_2 = b_1 - b_2$ and $b'_3 = b_1 - b_3$. Exactly as in the above paragraph, we may then find a_2 in $(0 : rx) - \{0, rx, sx, rx + sx, a_1, b'_2, b'_3\}$ so that $a_2 b'_2 = 0$. Now, as in the above paragraph, since $a_1(b_1 - b_i) = 0$ for all i , and if $i \neq j$, we have $(b_1 - b_i) \neq (b_1 - b_j)$, as above, we may actually choose a *sequence* of distinct $b'_4, b'_5, \dots$ so that $b_k \notin \{0, a_1, b'_2, b'_3, sx, rx, a_2\}$ for each $k \geq 4$ with the feature that $a_2 b'_k = 0$ for all $k \geq 4$.

We now want to produce a $K_{3,3}$ subgraph of $\Gamma(R)$. Exactly as before, note that for each $k \geq 3$, we have $a_2 b'_k \in (0 : rx) \cap (0 : sx)$. Since this is a finite set, we

may pick a subsequence $b_{k_1}, b_{k_2}, \dots$ so that $a_2 b_{k_1} = a_2 b_{k_2} = \dots$. And thus as in the above paragraph, we may then obtain $b'' \notin \{0, a_1, a_2, rx, sx, b'_2\}$ so that $0 = b''a_1 = b''sx = b''a_2$. Then, the subgraph of $\Gamma(R)$ spanned by $\{ax, b'_2, b''\} \cup \{a_1, sx, a_2\}$ is a $K_{3,3}$, and the proof is complete. $\square$

Lemma 2.54. *Let R be an infinite commutative ring with x a vertex of degree one and $\Gamma(R)$ planar. Further, let x be adjacent to precisely z in $\Gamma(R)$. If $a \notin (\{0\} \cup Rx)$ with $ay = 0$ for some $y \in Rx - \{0, z\}$, then $a = z$.*

Proof. Suppose that $a \neq z$. Then, as in previous arguments, we may assume that $z^2 = 0$ and $(0 : z) = Rx$. Also, $Rx \cong R/(0 : x)$ shows that $x^2 \neq 0$. Note that since $|R/Rx| = |(0 : x)| = 2$ and since $a \notin Rx$ by hypothesis, we must have $a - 1 \in Rx$. Also $z(Rx) = 0$ implies that $z(a - 1) = 0$, i.e., $za = z$. Now since $y \in Rx$ is adjacent to a and z , and since $\Gamma(R)$ contains a vertex of degree one, we must have that the degree of y is infinite. So, by Lemma 2.53, since the subgraph of $\Gamma(R)$ spanned by $Rx \cup \{z\}$ is a star, we may pick distinct $a_1, a_2, a_3 \in (0 : y) - [\{0, a, z, y\} \cup Rx]$. Then the same reasoning as above shows that $za_1 = za_2 = za_3 = z$, i.e., $z(a - a_1) = z(a - a_2) = z(a - a_3) = 0$. So, for $i = 1, 2$, and 3 , we have $(a - a_i) \in Rx$. Now, $a - a_i$ could equal y or z , but since if $j \neq k$, $a - a_j \neq a - a_k$, we may without loss assume that $a - a_1$ does not equal $0, y$ or z . But then $(a - a_1)y = 0$ and $a - a_1 \in Rx - \{0, y, z\}$ which is a contradiction to Lemma 2.53. $\square$

The above results, together with previous facts, thus give the following theorem.

Theorem 2.55. *Let R be an infinite commutative ring with $\Gamma(R)$ planar. Then $\Gamma(R)$ is graph-isomorphic to either a star graph, a $K_{2,\alpha}$ where $\alpha \leq c$, or the graph*

obtained by taking such a $K_{2,\alpha}$ and adding an edge between the two vertices of infinite degree.

Proof. From previous results, we have shown that if $\Gamma(R)$ is planar, then given a vertex x of $\Gamma(R)$, the degree of x must be 1, 2, or the degree of x is infinite (Lemma 2.36). We also have shown that if $\Gamma(R)$ is planar, then the graph does not contain vertices of both degree 1 and degree 2 (Proposition 2.45). In the event where $\Gamma(R)$ contains a vertex of degree 2, we have shown that $\Gamma(R)$ must be one of the two latter graphs above (Theorem 2.49). Since it is impossible to have $\Gamma(R)$ planar if every vertex has infinite degree (Proposition 2.52), it suffices to examine the case where $\Gamma(R)$ contains a vertex of degree 1.

If $\Gamma(R)$ contains a vertex x of degree one (note that $Rx \cong R/(0 : x)$ shows that $x^2 \neq 0$) with $(0 : x) = \{0, z\}$, then $z^2 = z$ implies that $R \cong \mathbb{Z}_2 \times D$, where D is an (infinite) integral domain by Theorem 2.38, and hence $\Gamma(R)$ is a star graph. If $z^2 = 0$, then by Lemma 2.53, $Rx = (0 : z)$, and the subgraph of $\Gamma(R)$ generated by $Rx \cup \{z\}$ is a star graph. If $Z(R) = Rx \cup \{z\}$, then $\Gamma(R)$ is thus a star graph. If not, then pick $a \in Z(R) \setminus [Rx \cup \{z\}]$. Then since $\Gamma(R)$ is connected, there must be a path from a to z . Since $a \notin Rx = (0 : z)$, a and z cannot be adjacent. Thus we must have $by \neq 0$ for some $b \notin (Rx \cup \{z\})$ and some $y \in Rx$. But by the previous lemma, this is impossible. Therefore the only possibilities for $\Gamma(R)$ are as listed. $\square$

Remark 2.56. Note that if $\Gamma(R)$ is to be planar, we must have $|V(\Gamma(R))| \leq c$. Note that if F is a field of cardinality α , where α is an infinite cardinal, then $\Gamma(F \times \mathbb{Z}_2)$ and $\Gamma(F \times \mathbb{Z}_3)$ are a $K_{1,\alpha}$ and a $K_{2,\alpha}$ respectively, and each of these is a graph on α vertices. Also, if $R = \mathbb{Z}[X]/(X^2, 3X)$, then $\Gamma(R)$ has the form of a $K_{2,\beta}$ with one additional edge. Note that this is a graph on ω vertices.

Let $R = \mathbb{Z}[X_0, \{X_\gamma | \gamma \in S\}] / (X_0^2, 3X_0, \{X_0X_\gamma | \gamma \in S\})$, where S is a nonempty set with $|S| \leq c$ and $0 \notin S$. Then $\Gamma(R)$ has the form of a $K_{2,\beta}$ with one additional edge. This is a graph on $\max\{\omega, |S|\}$ vertices. Thus every graph as listed in Theorem 2.55 can be realized as $\Gamma(R)$ for an appropriate (infinite) commutative ring R .

We end this chapter by posing two questions which may be of some interest.

Question. In the first section, we were able to classify all finite local commutative rings whose zero-divisor graph is planar. In doing so, we did not need to explicitly classify those local rings R with $\omega(R) = 4$. So, which finite commutative local rings have $\omega(R) = 4$?

Question. Is it possible to completely classify those infinite commutative rings R with $\Gamma(R)$ isomorphic to a star graph, a $K_{2,\alpha}$, or the graph obtained from adding an edge between the two vertices of infinite degree in such a $K_{2,\alpha}$?

Chapter 3

Zero-Divisor Graphs of Noncommutative Structures

1 Introduction

It is not quite clear what the best way is to extend the notion of the zero-divisor graph to noncommutative rings. This question has been studied in [17] and [18]. We will suggest other ways to define the zero-divisor graph of a non-commutative ring. As an upshot, we will also be able to prove new results about the constructions in the cited works as corollaries.

We will begin by examining how to define the zero-divisor graph of a non-commutative semigroup. In [8], DeMeyer, McKenzie, and Schneider noted that the mere definition of the zero-divisor graph of a commutative ring does not use the full ring structure; all one really needs is a (multiplicative) semigroup. It is somewhat surprising that many results about $\Gamma(R)$ translate verbatim into the semigroup case. For example, if S is a commutative semigroup, it is shown that $\Gamma(S)$ is connected

with diameter ≤ 3 [8, Theorem 1.2] and that if $\Gamma(S)$ contains a cycle, then its girth is ≤ 4 [8, Lemma 1.1, Lemma 1.4].

In [20], Vishne defines a zero-divisor graph Γ of a commutative ring where the vertices of Γ is a subset of the proper ideals of R . Namely, there is an edge joining I and J iff $IJ = 0$. Informally, since ‘if R is commutative, one studies elements and if R is noncommutative, one studies ideals’, this seems to us a natural construction for the zero-divisor graph of a noncommutative ring. In section 2, we give several interesting results, and prove a handful of analogues from the commutative setting.

2 The Zero-Divisor Graph of a Noncommutative Semigroup

We begin by fixing some notation. In this section, S will denote a (multiplicative) semigroup with a zero element, denoted 0 . Define

$$Z_R(S) = \{x \in S : ax = 0 \text{ for some } 0 \neq a \in S\},$$

$$Z_L(S) = \{x \in S : xa = 0 \text{ for some } 0 \neq a \in S\}, \text{ and}$$

$$Z(S) = Z_L(S) \cup Z_R(S).$$

Also, given any (nonempty) subset T of S , define $T^* = T - \{0\}$. And lastly, given $s \in S$, define

$$\text{ann}_L(s) = \{r \in S : rs = 0\} \text{ and}$$

$$\text{ann}_R(s) = \{r \in S : sr = 0\}.$$

There are many ways to define a zero-divisor graph of a noncommutative semigroup; we will primarily focus on those given by the following definitions.

Define a graph $\Gamma(S)$ whose vertices are $Z(S)^*$ and there is a (directed) edge

$x \rightarrow y$ iff $xy = 0$.

Define a graph $\Gamma'(S)$ whose vertices are $Z(S)^*$ and there is an (undirected) edge $x-y$ iff $xy = yx = 0$.

As mentioned earlier, there are many other structures that can be studied. See [17]

We will begin by stating conditions that guarantee that these graphs are connected. Recall that an undirected graph G is connected if and only if every two vertices of G can be joined by a path. If G happens to be directed, one often says that G is *strongly connected* if given any two (distinct) vertices x and y of G , there are (directed) paths from x to y and from y to x . We begin by stating [17, Theorem 5], suitably generalized.

Proposition 3.1. *(S. Redmond) Let S be a (noncommutative) semigroup. Then $\Gamma(S)$ is (strongly) connected if and only if $Z_L(S) = Z_R(S)$.*

Proof. As mentioned, the proof is given in [17, Theorem 5]. One need only note that the result is stated there for noncommutative rings. However, all of the arguments go through for a semigroup as well. $\square$

There is a nice condition that will guarantee connectivity of $\Gamma'(S)$.

Proposition 3.2. *Let S be a (noncommutative) semigroup. If $\text{ann}_L(s) = \text{ann}_R(s)$ for every $s \in S$, then $\Gamma'(S)$ is connected.*

Proof. The proof is easy. One notes that $\text{ann}_L(s) = \text{ann}_R(s)$ for every $s \in S$ if and only if for every $x, y \in S$, we have $xy = 0$ if and only if $yx = 0$. Thus, in this setting, $Z_L(S) = Z_R(S)$ and hence by the previous proposition, $\Gamma'(S)$ is connected. $\square$

$*$	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	b	0
c	0	0	c	0

Figure 3.1: Multiplication table for S as in Example 3.5

The converse to the above proposition fails, as the following example illustrates.

Example 3.3. Let S be a noncommutative semigroup with 3 nonzero elements, denoted a, b , and c . Suppose that all products in S are zero, except for the product $ab = c$. One can then check that the multiplication in S is associative, and that $\Gamma'(S)$ is the ‘line graph’ $a—c—b$. $\Gamma'(S)$ is connected, but $\text{ann}_L(a) = \{0, a, b, c\}$, while $\text{ann}_R(a) = \{0, a, c\}$.

Some remarks are in order.

Remark 3.4. Note that there is an edge joining x and y in $\Gamma'(S)$ if and only if there are directed edges from x to y and from y to x in $\Gamma(S)$. This leaves the possibility that $\Gamma(S)$ could be connected, while $\Gamma'(S)$ is not connected. We investigate this situation in the next example.

Example 3.5. Consider a semigroup S which contains 0 as well as three nonzero elements, denoted a, b , and c . Let the multiplication table for S be as in Figure 3.1.

One can then check that the multiplication in S is associative. $\Gamma(S)$ and $\Gamma'(S)$ are given in Figure 3.2. Note that $\Gamma(S)$ is connected, while $\Gamma'(S)$ is not.



Figure 3.2: $\Gamma(S)$ and $\Gamma'(S)$, where S is as in Figure 3.1

As in the above example, if $\Gamma(S)$ were a directed three-cycle, then $\Gamma'(S)$ would be the completely disconnected (directed) graph on three vertices. Thus, one might wonder if $\Gamma(S)$ can actually be a directed three-cycle. We answer this question in the negative shortly.

In Theorem 2.3 of [4], Anderson and Livingston show that for a commutative ring R , the diameter of $\Gamma(R)$ is less than or equal to 3. In Theorem 2 of [8], DeMeyer, McKenzie and Schneider show that the same inequality holds for the zero-divisor graph of a commutative semigroup. We have seen that if S is a noncommutative semigroup, $\Gamma(S)$ may not be connected. However, if $\Gamma(S)$ happens to be connected, then the diameter of $\Gamma(S)$ must be less than or equal to 3. Since in this setting, $\Gamma(S)$ is a directed graph, we take a moment to recall the definition of the diameter of a directed graph before proving this result.

Definition 3.6. Let G be a graph (possibly directed). The *diameter* of G , denoted $\text{diam}(G)$, is given by $\text{diam}(G) = \sup\{d(u, v) : u, v \in V(G)\}$, where $d(u, v)$ denotes the length of the shortest path in G between u and v , and with the convention that $\text{diam}(G) = \infty$ if G is not connected.

Proposition 3.7. *Let S be a (noncommutative) semigroup. If $\Gamma(S)$ is (strongly)*

connected, then $\text{diam}(\Gamma(S)) \leq 3$.

Proof. The proof could essentially be obtained by examining the proof of Theorem 2 in [8] and invoking Proposition 3.1, but we provide the details.

Let x and y be two distinct vertices of $\Gamma(S)$. Since $\Gamma(S)$ is strongly connected, there are paths $x \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \rightarrow y$ and $y \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_m \rightarrow x$. Thus, $0 = xa_1 = a_1a_2 = \cdots = a_ny$ and $0 = yb_1 = b_1b_2 = \cdots = b_mx$. If $xy = 0 = yx$, there are directed edges from x to y and from y to x . If $xy \neq 0$ and $yx = 0$, there is a directed edge $y \rightarrow x$. If $a_1a_n = 0$ then we have a path $x \rightarrow a_1 \rightarrow a_n \rightarrow y$ of length at most 3; otherwise $x \rightarrow a_1a_n \rightarrow y$ is a path of length at most 2. The argument for the case where $xy = 0$ and $yx \neq 0$ and the case where $xy \neq 0$ and $yx \neq 0$ are completely similar. $\square$

At least in the case where $\Gamma(S)$ is strongly connected, the diameter of the graph agrees with the commutative ring setting. In our next two results, we show there are some differences between the two settings. If R is a commutative ring with $1 \neq 0$, $\Gamma(R)$ can be a three-cycle or a four cycle. To see this, note that $\Gamma(\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2))$ is a three-cycle and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a four-cycle. However, in our next two results, we show that neither a directed three-cycle nor directed four-cycle can be realized as $\Gamma(S)$ when S is a (noncommutative) semigroup.

Proposition 3.8. *Let H denote a directed cycle on three vertices. Then H cannot be realized as $\Gamma(S)$ for any (noncommutative) semigroup S .*

Proof. Suppose $H = \Gamma(S)$. Then, if we denote the (nonzero) vertices of $\Gamma(S)$ by a, b , and c , we must have, without loss, $0 = ab = bc = ca$. Also, since $\Gamma(S)$ has no other directed edges, we must also have $ba, cb, ac \in S - \{0\}$.

So consider ac . Note that $c(ac) = (ca)c = 0c = 0$. Thus, ac must equal c or a (note that c could be 2-nilpotent). So, we consider cases as $ac = a$ or $ac = c$.

If $ac = a$, then $ab = 0$ implies that $0 = (ac)b = a(cb)$, and hence cb must equal b or a . If $cb = b$, consider $ba \neq 0$. Now $0 \neq ba = b(ac) = (ba)c$ implies that ba equals a or c . Further, $0 \neq ba = (cb)a = c(ba)$ implies that ba equals b or c . Thus $ba = c$. But then, $b = cb = (ba)b = b(ab) = b(0) = 0$, which is a contradiction.

The case where $ac = a$ and $cb = a$ is impossible. For then $a = ac = (cb)c = c(bc) = c(0) = 0$.

If $ac = c$, the arguments are somewhat similar. For then $bc = 0$ implies that $0 = b(ac) = (ba)c$ and hence ba equals b or c . If $ba = b$, consider $cb \neq 0$. Then $0 \neq cb = c(ba) = (cb)a$ implies that cb equals a or b . Similarly, $0 \neq cb = (ac)b = a(cb)$ implies that cb equals a or c . Thus $cb = a$. Then, we have $b = ba = b(cb) = (bc)b = 0b = 0$, a contradiction.

Lastly, the case where $ac = c$ and $ba = c$ is impossible also. For then $c = ac = a(ba) = (ab)a = 0a = 0$. Thus, $\Gamma(S)$ cannot be a directed three-cycle. $\square$

Proposition 3.9. *Let H denote a directed cycle on four vertices. Then H cannot be realized as $\Gamma(S)$ for any (noncommutative) semigroup S .*

Proof. Suppose $\Gamma(S)$ is a directed four-cycle; the arguments will be somewhat similar to the previous proof. Without loss we may assume that we have $a, b, c, d \in S - \{0\}$ with $0 = ab = bc = cd = da$, and these are the only nontrivial zero products in S . Since $d(ad) = 0$, we must have either $ad = a$ or $ad = d$. Note that since $\Gamma(S)$ is a directed four-cycle, $ad \neq 0$.

Now, consider db . Since $c(db) = 0$, we must have that db equals c or d . But $(db)c = 0$ implies that db equals b or c . Thus $db = c$. Similarly, it is possible to show

that $ca = b$. The proof then splits into cases as $ad = a$ or $ad = d$.

If $ad = a$, then $ab = 0$ implies that $0 = ab = (ad)b = a(db) = ac$, and thus there is a directed edge from a to c , which is a contradiction. Similarly, if $ad = d$, $cd = 0$ implies that $0 = cd = c(ad) = (ca)d = bd$, and thus there is a directed edge from b to d , a contradiction.

□

However, $\Gamma(S)$ can be a single directed edge as the following example illustrates.

Example 3.10. Let S be a multiplicative semigroup with 0 and two nonzero elements a and b . Let $a^2 = 0 = ab$, $ba = a$, and $b^2 = b$. Then it is easy to verify that the multiplication in S is associative, and that $\Gamma(S)$ consists of a single directed edge joining a and b .

We conclude this section with two questions.

Question: In the case where R is a commutative ring, the diameter of $\Gamma(R)$ must be less than or equal to 3 (cf. [3, Theorem 2.2]). This result carries over to $\Gamma(S)$, where S is a commutative semigroup (cf. [8, Theorem 1.2]). And, we have shown that in the case where S is an arbitrary semigroup with $\Gamma(S)$ strongly connected, the same bound holds (Proposition 3.7). In [8, Theorem 1.6], DeMeyer, McKenzie and Schnieder show that if S is a commutative semigroup, then the girth of $\Gamma(S)$ is less than or equal to 4. It would be interesting to see how this result generalizes to the noncommutative case.

Question: In the case where R is a commutative ring, one quickly sees that for many classes of rings, $\Gamma(R)$ possesses a high degree of symmetry; specifically, in [4, Theorem 3.2] Anderson and Livingston show that $\text{Aut}(\Gamma(R))$ is actually a product

of symmetric groups for a large class of rings. It would be interesting to attempt to generalize this result to the semigroup case.

3 A Zero-Divisor Graph of Noncommutative Rings

In this section, we will consider a number of possible structures for the zero-divisor graph of a (not necessarily commutative) ring. Namely, we will take the two structures defined in Section 1, and vary the semigroup in question. Given a noncommutative ring R , we list several semigroups one can obtain from R .

Given R , let

$S_L = \{I \mid I \text{ is a proper left ideal of } R \text{ with } IK = 0 \text{ or } KI = 0 \text{ for some nonzero left ideal } K \text{ of } R\}$,

$S_R = \{I \mid I \text{ is a proper right ideal of } R \text{ with } IK = 0 \text{ or } KI = 0 \text{ for some nonzero right ideal } K \text{ of } R\}$,

$S_T = \{I \mid I \text{ is a proper two-sided ideal of } R \text{ with } IK = 0 \text{ or } KI = 0 \text{ for some nonzero two-sided ideal } K \text{ of } R\}$.

One might worry whether S_L , S_R , and S_T are actually semigroups. The next proposition addresses this concern. We will show that S_L is a (multiplicative) semigroup. Similar arguments prove that S_R and S_T are semigroups as well.

Proposition 3.11. *S_L as defined previously is a multiplicative semigroup.*

Proof. Let $I, J \in S_L$. We need only show that $IJ \in S_L$. If I and J are both annihilated on the left (right) by left ideals K and L , then $K(IJ) = (KI)J = 0$, so that $IJ \in S_L$ (analogously, $(IJ)L = I(JL) = 0$). Thus we need only check the case where $IK = LJ = 0$. Consider cases as $IJ = 0$ or $IJ \neq 0$.

If $IJ = 0$, there is nothing to check. If $IJ \neq 0$, then since I and J are left ideals of R , $IJ \subseteq J$ so that $L(IJ) \subseteq LJ = 0$. $\square$

We now give an example to illustrate several zero-divisor graph structures.

Example 3.12. Let $R = \{M \in M_2(\mathbb{Z}_2) : M \text{ is upper triangular}\}$. For ease of notation, let

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then, observe that $ER = BR = \{0, A, B, E\}$ and $RC = RD = \{0, A, C, D\}$. Using the three semigroups indicated above, one can obtain the graphs in Figures 3.3, 3.4, and 3.5. Observe that $S_L = \{0, AR (= RA), BR (= ER), RB, RC (= RD), RE\}$, $S_R = \{0, AR (= RA), BR (= ER), CR, DR, RC (= RD)\}$, and $S_T = \{0, AR (= RA), BR (= ER), RC (= RD)\}$. In these situations, we will sometimes refer to the elements of S_L , S_R , and S_T as *zero-divisor ideals*.

Observe that $\Gamma'(S_L)$, $\Gamma'(S_R)$, and $\Gamma'(S_T)$ can be easily obtained from the above graphs. For there is an (undirected) edge $x - y$ in Γ' iff there are directed edges in Γ from x to y and from y to x . With the above graphs in mind, we will mostly study $\Gamma(S)$, and vary the semigroup in question.

Further notice that in the previous example, none of the graphs in question are connected graphs. This prompts the question as to when these constructions yield connected graphs. As it turns out, there is a large class of rings for which $\Gamma(S_L)$, $\Gamma(S_R)$, and $\Gamma(S_T)$ are connected graphs. In particular, we will show that for any ring that is *semiprime*, these graphs are connected. To proceed, we will need to prove some basic facts about semiprime rings.

Recall that an ideal I of R is said to be *semiprime* if, for any ideal J of R , $J^2 \subseteq I$ implies that $J \subseteq I$. A ring R is said to be semiprime if the zero ideal is

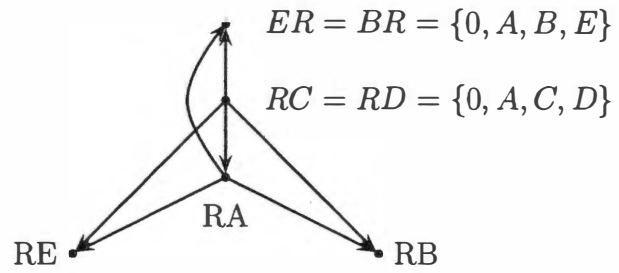


Figure 3.3: $\Gamma(S_R)$ as in Example 3.12

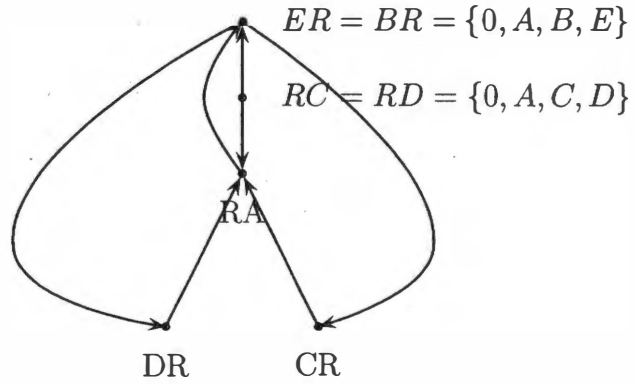


Figure 3.4: $\Gamma(S_L)$ as in Example 3.12

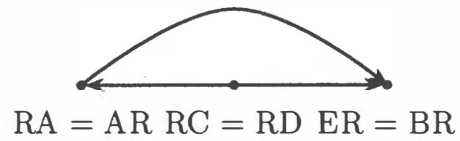


Figure 3.5: $\Gamma(S_T)$ as in Example 3.12

a semiprime ideal. To proceed further, we state some well-known basic facts about semiprime rings.

Lemma 3.13. *Let R be a ring. Then R is semiprime if and only if for any ideals I, J of R with $IJ = 0$, then $I \cap J = 0$.*

Proof. Suppose that R is semiprime, and I, J are two-sided ideals of R so that $IJ = 0$. Observe that $(I \cap J)^2 \subseteq IJ = 0$. Thus, since R is semiprime, $I \cap J \subseteq 0$.

The converse is just as easy. For if $I^2 = 0$, then by hypothesis $I \cap I = 0$; that is, $I = 0$, and thus the zero ideal is semiprime. $\square$

Remark 3.14. Lemma 3.13 was stated in terms of two-sided ideals. By examining the proof, one sees that ‘two-sided ideal’ can actually be replaced by ‘left ideal’ or ‘right ideal’.

Lemma 3.15. *Let R be a semiprime ring. Then for all left ideals I, J of R , $IJ = 0$ if and only if $JI = 0$.*

Proof. Suppose R is semiprime and $IJ = 0$. Then

$$(JI)^2 = (JI)(JI) = J(IJ)I = J(0)I = 0.$$

But since the zero ideal is semiprime, we must have $JI = 0$. The argument for the converse is completely analogous. $\square$

Remark 3.16. Again, note that Lemma 3.15 holds if ‘left ideal’ is replaced by ‘right ideal’ or ‘two-sided ideal’.

Lemma 3.17. *Let R be a semiprime ring, and let I be a two-sided ideal of R . Then $\text{ann}_L(I) = \text{ann}_R(I)$.*

Proof. Note that $(\text{ann}_L(I))I = 0$; so by Lemma 3.15, $I(\text{ann}_L(I)) = 0$. That is, $\text{ann}_L(I) \subseteq \text{ann}_R(I)$. Repeating this argument, noting that $I(\text{ann}_R(I)) = 0$, we similarly obtain that $\text{ann}_R(I) \subseteq \text{ann}_L(I)$, and hence $\text{ann}_L(I) = \text{ann}_R(I)$. $\square$

Observe that if R is a ring and $S = S_L, S_R$, or S_T , then by Proposition 3.1, $\Gamma(S)$ is connected if and only if $Z_L(S) = Z_R(S)$. We thus have the following theorem.

Theorem 3.18. *Let R be a ring, and let $S = S_L, S_R$, or S_T . If R is semiprime, then $\Gamma(S)$ is (strongly) connected.*

Proof. With the right tools, the proof is now easy. We prove the case where $S = S_L$, the other cases are analogous. By Proposition 3.1, it is enough to show that $Z_R(S) = Z_L(S)$. So say $I \in Z_R(S)$ is a left ideal, with $JI = 0$, for some left ideal J of R . Then by Lemma 3.15, $IJ = 0$; that is, $I \in Z_L(S)$ and hence $Z_R(S) \subseteq Z_L(S)$. A symmetric argument shows that $Z_L(S) \subseteq Z_R(S)$, and thus $\Gamma(S)$ is connected. $\square$

There is also a companion theorem to Theorem 3.18. Before we proceed, recall that a ring R is said to be *left-primitive* (*right-primitive*) if R has a faithful left (right) R -module, that is to say $\text{ann}_L(M) = 0$ ($\text{ann}_R(M) = 0$) for some left (right) R -module M . We will simply say that R is *primitive* if R is either left-primitive or right-primitive.

Theorem 3.19. *Let R be a ring and T denote the set of zero-divisors of R . If R is semiprime, then either $\Gamma(T)$ is (strongly) connected (and in fact T is a semigroup), or R is primitive.*

Proof. Again, by Proposition 3.1, it is enough to show that any left zero-divisor of R is a right zero-divisor of R (and conversely). So let $0 \neq r \in Z_L(R)$ such that $rz = 0$ for some nonzero $z \in R$. Then $(rR)\text{ann}_R(rR) = 0$, and $\text{ann}_R(rR)$ and

rR are right ideals. Thus, by Lemma 3.15 $\text{ann}_R(rR)(rR) = 0$. Since $1 \in R$, this implies that $\text{ann}_R(rR)r = 0$. If $\text{ann}_R(rR) \neq 0$, then $r \in Z_R(R)$. If $\text{ann}_R(rR) = 0$, then R is (right) primitive. So we have shown that either $Z_L(R) \subseteq Z_R(R)$ or R is right-primitive. Similarly, one can show that either $Z_R(R) \subseteq Z_L(R)$ or else R is left-primitive and the statement of the theorem follows. $\square$

As might be expected, the converse to the above Theorem 3.18 fails. We give two examples to illustrate.

Example 3.20. Let G be any abelian group with additive identity 0. Give G a ring structure by declaring all products in G to be 0. Then, $T = G - \{0\}$. Further, $\Gamma(S_L)$ is vacuously connected, being the empty graph. However, with the given ring structure, G is certainly not semiprime. In this somewhat trivial example, G is commutative, and has no multiplicative identity.

For the next example, we need to recall some basic facts. A noncommutative ring R is said to be local if $R/\text{rad}R$ is a division ring, or alternately, R has a unique maximal left (right) ideal. Another equivalent condition is R is local iff the nonunits of R form an ideal. For more equivalences, see [12, Theorem 19.1].

Example 3.21. For a slightly less trivial example, let R be any division ring with 1. Then R is local, and thus $R[[X]]$ is a local ring with maximal ideal (X) . Then, let $R_1 = R[[X]]/(X^2)$. Then R_1 is a local ring, and we have that $\Gamma(S_L)$ is a complete graph. Thus, $\Gamma(S_L)$ is connected in particular, but R_1 is not semiprime. For $I = (X)/(X^2)$ is an ideal of R_1 with the feature that $I^2 = 0$, but $I \neq 0$.

Since the graphs under study are connected for a large class of rings, one might suspect that it should be possible to prove many analogues of theorems from the commutative case. To motivate the next result, we state Theorem 2.5 from [4].

Theorem 3.22. (*D.F. Anderson and Livingston*) *Let R be a commutative ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain, or $Z(R)$ is an annihilator ideal.*

We seek a companion theorem of this result for $\Gamma(S_R)$. We investigate the situation where $\Gamma(S_R)$ contains a vertex I adjacent to all other vertices, in the sense that there is a directed edge from I to every other vertex and there is an edge from each other vertex to I .

Theorem 3.23. *Let R be a ring with $1 \neq 0$. Let I be a nonzero right zero-divisor ideal of R such that $IJ = JI = 0$ for all right zero-divisor ideals $J \neq I$ of R . Then either $Z_R(R)$ is an annihilator ideal, I is generated by one element, or R is primitive.*

Proof. Let I be such an ideal. Then, either $I^2 = 0$ or $I^2 \neq 0$.

We first consider the case where $I^2 = 0$. Then $I \subseteq \text{ann}_R(I)$. We claim that either $\text{ann}_R(I) = Z_R(R)$ or I is principal.

To see this, observe that certainly $\text{ann}_R(I) \subseteq Z_R(R)$, for if $z \in \text{ann}_R(I)$ then $Iz = 0$. Conversely, let $0 \neq x \in Z_R(R)$. Note that $xR \neq 0$. Since $(xR)\text{ann}_R(xR) = 0$, either $\text{ann}_R(xR) = 0$ and R is primitive, or $\text{ann}_R(xR) \neq 0$, in which case xR is a (right) zero-divisor ideal. If $xR = I$, then I is generated by one element. If $xR \neq I$, then by hypothesis $I(xR) = (xR)I = 0$. Since $1 \in R$, we have $xI = Ix = 0$, and thus $x \in \text{ann}_R(I)$.

If $I^2 \neq 0$, then $\text{ann}_R(I) \neq I$. So pick $y \in I - \text{ann}_R(I)$. Note $yR \subseteq I$. If $yR = I$, then I is generated by one element. If $yR \subsetneq I$, then as above either R is primitive or yR is a right zero-divisor ideal, and thus by hypothesis $I(yR) = 0$. However, this implies that $y \in \text{ann}_R(I)$, contrary to our choice of y . $\square$

Remark 3.24. Note that there is an analogue of Theorem 3.23 for $\Gamma(S_L)$ (i.e., in terms of left ideals).

We conclude this section by posing two questions.

Question. The previous proof and many of the results in this section strongly make use of the fact that $1 \in R$. Can these results be generalized to the case where $1 \notin R$?

Question. Since the definition of the zero-divisor graph of a semigroup S does not technically use the fact that S is multiplicatively closed, it might be interesting to study how well results about $\Gamma(S)$ translate into a situation where the closure properties of the underlying structure are relaxed somewhat. To see the motivation for this situation, it would be tempting to let $S = \{I \text{ a proper left, right or two-sided ideal of } R \mid IJ = 0 \text{ for some nonzero left, right or two sided ideal of } R\}$, and study $\Gamma(S)$. It is conceivable that S might not be closed under multiplication, yet $\Gamma(S)$ seems a natural object to study.

Chapter 4

The Cycle Structure of $\Gamma(K^n)$

1 Introduction

We begin with some definitions. Given a graph G with vertex set $V(G)$, a *cycle* in G is a subset $\{v_1, v_2, \dots, v_n\}$, where an edge joins v_i and v_{i+1} for each i , and also there is an edge joining v_n and v_1 . It will often be notationally convenient to say that ' $v_1 - \dots - v_2 - \dots - \dots - v_n$ is a cycle'. We may also sometimes simply say that ' $\{v_1, v_2, \dots, v_n\}$ is a cycle' if the ordering of $v_1, \dots, v_n$ is clearly understood. We will say such a cycle has *length* n or that said cycle is an *n -cycle*. The *circumference* of G , $c(G)$, is the length of the longest cycle in G . We will use the convention that if G contains an n -cycle for each integer $n \geq 3$, then $c(G) = \infty$, and that $c(G) = 0$ if G contains no cycles. We will sometimes say that a graph G on n vertices is *Hamiltonian* if G contains a cycle of length n . Such a cycle is said to be a *Hamilton Cycle*.

In [3], the authors ask what is the circumference of $\Gamma(K^n)$, where K is an arbitrary (finite) field. There are few results in this direction, even when $K = \mathbb{Z}_2$.

2 Lower and Upper Bounds

We begin by establishing some notation and basic facts. We will denote $c(\Gamma(\mathbb{Z}_2^n))$ by simply c_n . Since $\Gamma(\mathbb{Z}_2^n)$ has many vertices of degree one (namely, those having ones in all but one ‘slot’) and degree one vertices are not of interest if cycles are under consideration, we will denote by Γ_n the graph obtained by deleting all of the vertices of degree one from $\Gamma(\mathbb{Z}_2^n)$. And, given a set A , we will denote the complement of A by A^c .

There is another way to formulate the above problem. Given a positive integer n , we may define a graph on the power set of $[n] = \{1, 2, \dots, n\}$ (denoted $P([n])$) as follows. Declare two (proper, nonempty) subsets A and B of $[n]$ to be adjacent if and only if $A \cap B = \emptyset$. Then the resulting graph is isomorphic to $\Gamma(\mathbb{Z}_2^n)$. As above, subsets of cardinality $n - 1$ cannot possibly contribute to a cycle, so we may delete these vertices freely. We will refer to this graph as Γ_n as well, as there will be no danger of confusion.

Yet another formulation of this problem that we will not have cause to use was given in [3]. One can see that for $n \geq 2$, c_n is the largest integer such that there is an $n \times c_n$ matrix $A = [a_{i,j}] \in M(\mathbb{Z}_2)$ so that A has distinct nonzero columns and $a_{i,j} \neq 0$ implies that $a_{i,j+1} = 0$ for $j < c_n$ and $a_{i,c_n} \neq 0$ implies that $a_{i,1} = 0$ for all i .

We will refer to vertices of $\Gamma(\mathbb{Z}_2^n)$ which contain ones in exactly i slots as *type i vertices* and similarly we will refer to elements of $P([n])$ of cardinality i as *i -sets*.

One can immediately see that $n \leq c_n \leq 2^n - n - 2$. For $\{1\}, \{2\}, \dots, \{n\}$ is an n -cycle in Γ_n . If we delete $\emptyset$ and $[n]$ from $P([n])$, as well as complements of singleton subsets, we obtain the upper bound.

In general, finding upper bounds on c_n is more difficult than finding lower

bounds, for if Γ_n contains a cycle of length k , then $k \leq c_n$. Our first proposition shows that one can obtain a slightly sharper upper bound on c_n .

Proposition 4.1. *For $n \geq 5$, $c_n \leq 2^n - 2 - \binom{n}{2} + n$.*

Proof. The key is the observation that for an $(n-2)$ -set A , A has degree 3 in Γ_n . For A^c consists of two elements, say $A^c = \{x, y\}$. Then A is adjacent to precisely A^c , $\{x\}$, and $\{y\}$. Since such a set A in a cycle must be adjacent to at least one singleton set, at most $2n$ $(n-2)$ -sets could appear in a cycle, and we necessarily have

$$\begin{aligned} c_n &\leq \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-3} + 2n \\ &= [2^n - \binom{n}{n} - \binom{n}{n-1} - \binom{n}{n-2} - \binom{n}{0}] + 2n \\ &= 2^n - 2 - \binom{n}{n-2} + n \\ &= 2^n - 2 - \binom{n}{2} + n. \end{aligned}$$

□

We will note a bit later that Proposition 4.1 is not best possible for $n \leq 5$. In fact, we will see that for $n \leq 5$, $c_n \leq 2^n - n - 2$.

Our next result shows how lower bounds on c_n can be established by recurrences.

Proposition 4.2. *Suppose that Γ_n contains a cycle of length k . If k is even, then Γ_{n+1} contains a cycle of length $2k+1$. If k is odd, then Γ_{n+1} contains a cycle of length $2k+2$. Thus, if c_n is even, then $c_{n+1} \geq 2c_n + 1$, and if c_n is odd, then $c_{n+1} \geq 2c_n + 2$.*

Proof. Suppose that $A_1, \dots, A_k \subset [n]$ with $A_1, \dots, A_k$ a cycle in Γ_n . If k is odd, then A_1 (viewed as a subset of $[n+1]$), $A_2 \cup \{n+1\}$, A_3 , $A_4 \cup \{n+1\}$, $\dots$, A_k , $\{n+1\}$, A_1^c (viewed as a subset of $[n]$), $A_1 \cup \{n+1\}$, A_2 , $A_3 \cup \{n+1\}$, $\dots$, $A_k \cup \{n+1\}$ forms a $(2k+2)$ -cycle in Γ_{n+1} .

If k is even, then $A_1 \cup \{n+1\}$, A_2 , $A_3 \cup \{n+1\}$, $\dots$, A_k , A_1 , $A_2 \cup \{n+1\}$, A_3 , $\dots$, A_{k-1} , $\{n+1\}$, A_1^c (the complement taken in $[n]$), $A_1 \cup \{n+1\}$ is a $(2k+1)$ -cycle in Γ_{n+1} . $\square$

Note that in Γ_n , singleton sets have very high degree. Namely, the degree of any singleton set in Γ_n is precisely $2^{n-1} - 2$. A reasonable conjecture might thus be that any maximal cycle in Γ_n must contain all singleton sets. We prove this fact in our next two results.

Lemma 4.3. *Suppose $C = \{S_1, \dots, S_k\}$ is a maximal cycle in Γ_n . Then S_i is a singleton set for some $i = 1, \dots, k$.*

Proof. Suppose not, that is, C contains no singleton sets. Then C can contain no sets of cardinality $(n-2)$, for an $(n-2)$ -set A is adjacent to precisely $A^c = \{a, b\}$, $\{a\}$, and $\{b\}$.

Now, pick a triple $S_i - S_{i+1} - S_{i+2}$. Necessarily $S_i \cap S_{i+1} = \emptyset$, and the proof splits into cases, as $S_i \cup S_{i+1} = [n]$ or $S_i \cup S_{i+1} \subsetneq [n]$.

Case 1. $S_i \cup S_{i+1} = [n]$. Then $S_i \cap S_{i+1} = \emptyset$ implies that $S_{i+1}^c = S_i$. Also, S_{i+1} and S_{i+2} are adjacent, so $S_{i+2} \subseteq S_{i+1}^c = S_i$. But C was assumed to be a cycle, so $S_{i+2} \subsetneq S_i$. Thus, we may pick $x \in S_i - S_{i+2}$. Then

$$\{S_1, \dots, S_i, S_{i+1}, \{x\}, S_{i+2}, \dots, S_k\}$$

is a cycle of longer length than C , violating its maximality.

Case 2. $S_i \cup S_{i+1} \subsetneq [n]$. Then pick $y \in [n] - (S_i \cup S_{i+1})$. Then

$$\{S_1, \dots, S_i, S_{i+1}, \{y\}, S_{i+2}, \dots, S_k\}$$

is a longer cycle than C . □

With this result in hand, we can now show that a cycle of maximum length in Γ_n must contain *all* singleton subsets of $[n]$.

Proposition 4.4. *Suppose $C = \{S_1, \dots, S_k\}$ is a cycle of maximum length in Γ_n . Then for each $x \in [n]$, $\{x\} = S_i$ for some $1 \leq i \leq k$.*

Proof. By the previous lemma, C must contain some singleton set. For notational ease and without loss, we may therefore assume C contains $\{1\}$ and that hence C takes the form

$$\{1\} - - - S_2 - - - S_3 - - - \dots - - - S_k - - - \{1\}.$$

We will show that C must then contain every other singleton set. Again, it will suffice to show that C must contain $\{2\}$. Again, note that the $(n-2)$ -set $\{3, 4, \dots, n-1, n\} = T$ is adjacent in Γ_n precisely to the sets $\{1\}$, $\{2\}$, and $\{1, 2\}$. So the proof will be by contradiction, and splits into cases, as $T \in C$ or $T \notin C$.

Case 1. Suppose $T \in C$, but $\{2\} \notin C$.

Since $\{2\} \notin C$, T must be either S_2 or S_k . By symmetry, we may suppose that $T = S_2$. Thus C has the form

$$\{1\} - - - \{3, 4, \dots, n-1, n\} = S_2 - - - S_3 - - - \dots - - - S_k - - - \{1\}.$$

Hence, $\{2\}$ may be inserted between $\{1\}$ and S_2 to produce a longer cycle, thus contradicting the maximality of C .

Case 2. Suppose $T \notin C$ and $\{2\} \notin C$.

As above, we may assume that $2 \in S_2$ but $2 \notin S_3$. So C has the form

$$\{1\} - - - (2 \in) S_2 - - - (2 \notin) S_3 - - - \dots - - - S_k - - - \{1\}.$$

But then

$$\{1\} - - - \{3, 4, \dots, n-1, n\} - - - \{2\} - - - S_3 - - - \dots - - - S_k - - - \{1\}$$

is a longer cycle than C . Thus by symmetry, $\{1\} \in C$ forces $\{m\} \in C$ for each $m = 2, 3, \dots, n$, and the proof is complete. $\square$

In Proposition 4.2, some recurrences were established for bounding c_n below. To use these, it would be useful to know the exact value of c_n for small values of n . This is done in the next few results.

Proposition 4.5. $c_3 = 3$, $c_4 = 10$, and $c_5 = 25$.

Proof. The first statement is obvious. To see the second statement, note that Γ_4 has 10 vertices, so it will suffice to find a 10-cycle in Γ_4 . So, note that

$$\begin{aligned} &\{1\} - - - \{2, 4\} - - - \{1, 3\} - - - \{2\} - - - \{4\} - - - \\ &\{2, 3\} - - - \{1, 4\} - - - \{3\} - - - \{1, 2\} - - - \{3, 4\} \end{aligned}$$

is the desired 10-cycle.

For the last statement, $c_5 \leq 25$ follows from Proposition 4.1. Thus, to establish equality we need only produce a 25-cycle in Γ_5 .

$$\begin{aligned}
& \{1, 2\} - - - \{3, 4, 5\} \text{a} - - \{1\} - - - \{2, 4, 5\} \text{a} - - \{1, 3\} \text{a} - - \\
& \{2, 5\} \text{a} - - \{1, 3, 4\} \text{a} - - \{2\} \text{a} - - \{1, 3, 5\} \text{a} - - \{2, 4\} \text{a} - - \\
& \{3, 5\} - - - \{1, 2, 4\} \text{a} - - \{5\} \text{a} - - \{2, 3, 4\} \text{a} - - \{1, 5\} \text{a} - - \\
& \{3, 4\} - - - \{1, 2, 5\} \text{a} - - \{3\} - - - \{1, 4, 5\} - - - \{2, 3\} - - - \\
& \{4, 1\} \text{a} - - \{2, 3, 5\} \text{a} - \text{a} \{4\} - - - \{1, 2, 3\} \text{a} - - \{4, 5\}
\end{aligned}$$

is the needed cycle.

□

For $n = 3, 4$, and 5 , Γ_n contains the largest cycle that it possibly could; see Proposition 4.5 and the comments after Proposition 4.1. However, our next result shows that for $n > 6$ this is never the case. To this end, we need an easy lemma.

Lemma 4.6. *If $n \geq 6$, then $\binom{n}{3} > 2[\binom{n}{2} - n]$.*

Proof. The proof is just simple algebra. For $\binom{n}{3} > 2[\binom{n}{2} - n]$ if and only if $n^3 - 9n^2 + 20n > 0$, and the latter statement holds for $n \geq 6$ by elementary calculus.

□

Proposition 4.7. *For $n \geq 6$, $c_n < 2^n - 2 - \binom{n}{2} + n$.*

Proof. Note by Proposition 4.1, $c_n \leq 2^n - 2 - \binom{n}{2} + n$. We suppose there were a cycle of length $2^n - 2 - \binom{n}{2} + n$ and derive a contradiction. As in Proposition 4.1, to construct a cycle of length $2^n - 2 - \binom{n}{2} + n$, begin by placing the $2n(n-2)$ -sets which would have to be used in the construction of such a cycle. These sets will be represented by an 'X' in the diagrams to follow.

— — — — — X — — — — — X — — — — — X — — — — — X — — — — — X — — — — —

Next, in a cycle of the desired length, all n singleton sets must be placed somewhere. In the diagrams, 'O' will represent a singleton set. Further, by permuting the cycle if necessary we may assume the cycle begins with a singleton set.

Further, observe that since there are n singletons available and $2n$ $(n-2)$ -sets in this cycle, and each $(n-2)$ -set is adjacent to certain singletons as well as a unique 2-set, after placing these sets our cycle must necessarily have the form as indicated in Figure 4.1. Also note that every $(n-2)$ -set *must* be adjacent to a singleton.

Said differently, the cycle can be partitioned into blocks, each of the form $((n-2)\text{-set})\text{---}(\text{singleton})\text{---}((n-2)\text{-set})$. Note that since each $(n-2)$ -set is adjacent to a unique 2-set, in the cycle as shown the singletons and $(n-2)$ -sets must be adjacent. Next, we are forced to place n 2-sets to have any hope of creating this

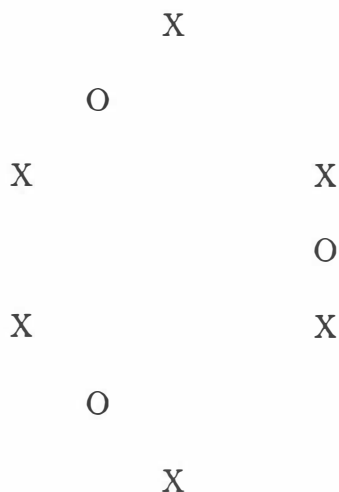


Figure 4.1: Construction of maximal cycle in Γ_n

cycle. Further, the placement of these sets is determined by the placement of the $(n - 2)$ -sets. See the diagram below. These sets will be represented by a ‘*’ in the subsequent diagrams. Note that in the diagram below, the first and last singleton sets should be the same set. A ‘—’ indicates a place in the cycle where we may still place sets at some later point in the construction.

$$O_1X * - - - *XOX * - - - *XOX * - - - *XOX * - - - *XO_1$$

At this point in the construction, there are now $\binom{n}{2} - n$ 2-sets left to place. In the diagram below, these sets will be indicated by a ‘+’.

$$O_1X * - + - - *XOX * + - - + *XOX * - - - *XOX * - + - *XO_1$$

After making all of these placements, let

$$S_1 = \{A | A \text{ is a 2-set in } \Gamma_n \text{ so that } A \text{ is immediately preceded by an } (n - 2)\text{-set at this point in the construction} \}$$

$$S_2 = \{A | A \text{ is a 2-set in } \Gamma_n \text{ so that } A \text{ is immediately followed by an } (n - 2)\text{-set at this point in the construction} \}$$

$$S_3 = \{A | A \text{ is a 2-set in } \Gamma_n \text{ so that } A \text{ is neither immediately preceded nor immediately followed by an } (n - 2)\text{-set at this point in the construction} \}.$$

If the 2-sets placed at this point in the construction are indexed $A_1, A_2, \dots, A_k$, let

$$B = \{(A_i, A_j) | j = i + 1 \text{ and } (A_i, A_j) \in (S_1 \times S_2) \cup (S_1 \times S_3) \cup (S_3 \times S_2)\}.$$

Informally, think of the ordered pairs in B as determining the boundaries of *blocks* where sets still to be placed in the construction of this cycle could (and must) still be put. After placing all of the 1- and 2-sets, there will be $n + \left[\binom{n}{2} - 2n\right] = \binom{n}{2} - n$ of these blocks available to fill. We then turn to the problem of trying to place the $\binom{n}{3}$ $(n-3)$ -sets which must appear in this cycle. Note that each $(n-3)$ -set is adjacent to a unique 3-set, certain 1- and 2-sets, and nothing else, and all of the 1- and 2-sets are accounted for (and may begin and end some of the indicated blocks). Thus, at most *two* $(n-3)$ -sets could possibly be placed in each of the indicated blocks. However, since $n \geq 6$, by the previous lemma, the number of $(n-3)$ -sets to be placed is strictly greater than (2 times the number of available blocks), since the left and right sides of the equation in that lemma counts these items, respectively. Thus by the pigeonhole principle, at least one compartment must receive at least three $(n-3)$ -sets, which is impossible by the above argument.

□

Corollary 4.8. $c_6 = 52$.

Proof. The proof is immediate. Proposition 4.2 and the fact that $c_5 = 25$ implies that $52 \leq c_6$. The previous proposition gives that $c_6 < 53$. Thus $c_6 = 52$. □

Some remarks are in order. Since $c_5 = 25$, in some sense Proposition 4.1 is best possible for $n \geq 5$. And, as Γ_5 contains a 25-cycle, but Γ_6 contains a 52-cycle, but no 53-cycle, in that sense Proposition 4.2 is best possible as well.

We conclude this section with two conjectures and two questions.

Conjecture: Suppose Γ_n contains a k -cycle. Then Γ_n contains a $(k-1)$ cycle, and hence contains an m -cycle for each $3 \leq m \leq k$.

Conjecture: Suppose C is a maximal cycle in Γ_n . Then for each $1 \leq i \leq (n-3)$, every i -set in $[n]$ appears in C .

Question: A problem which is somewhat related to the problem of finding long cycles in a graph G is to determine the *toughness* of G . A graph G with vertex set V is said to be *t-tough* if $|S| \geq t \cdot (\text{the number of components of } (G - S))$ for every cutset $S \subseteq V$. The toughness of G is defined to be the maximum t so that G is *t-tough*. Can anything be said about the toughness of Γ_n ?

Question: How do any of the results proved so far translate to $\Gamma(K^n)$, where K is an arbitrary finite field, or to $\Gamma(K_1 \times \cdots \times K_n)$, where $K_1, \dots, K_n$ are finite fields?

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Appendices

Appendix A

Planar Embeddings of Certain Graphs

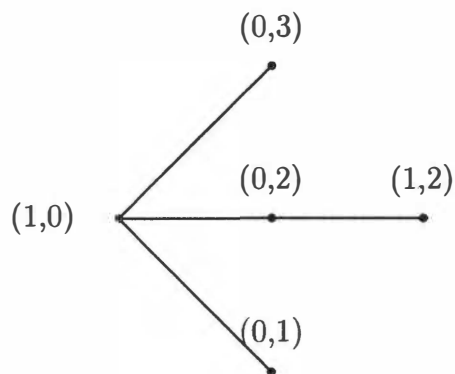


Figure A.1: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$

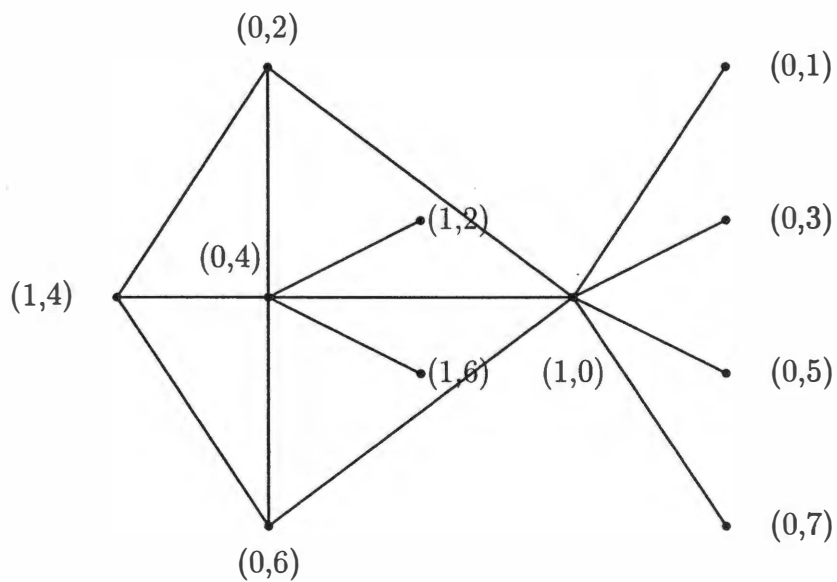


Figure A.2: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8)$

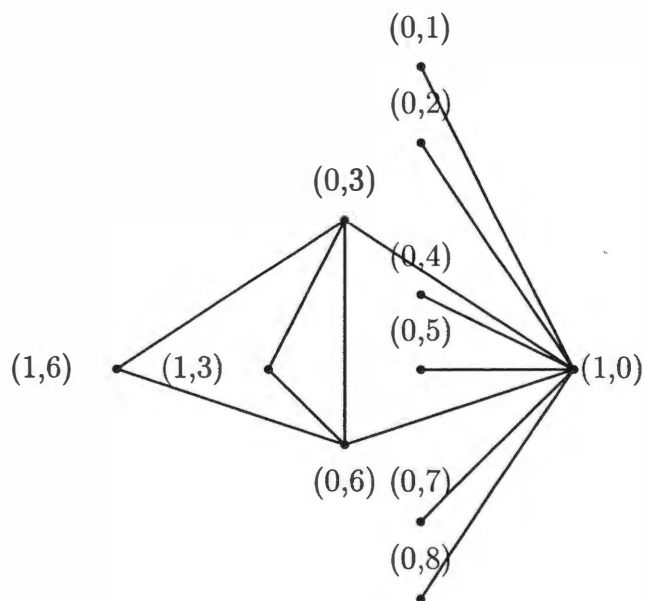


Figure A.3: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_9)$

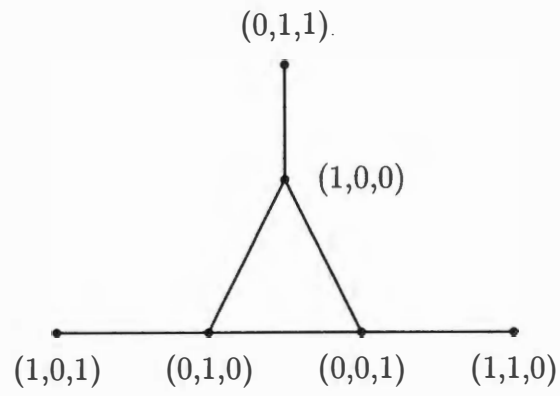


Figure A.4: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

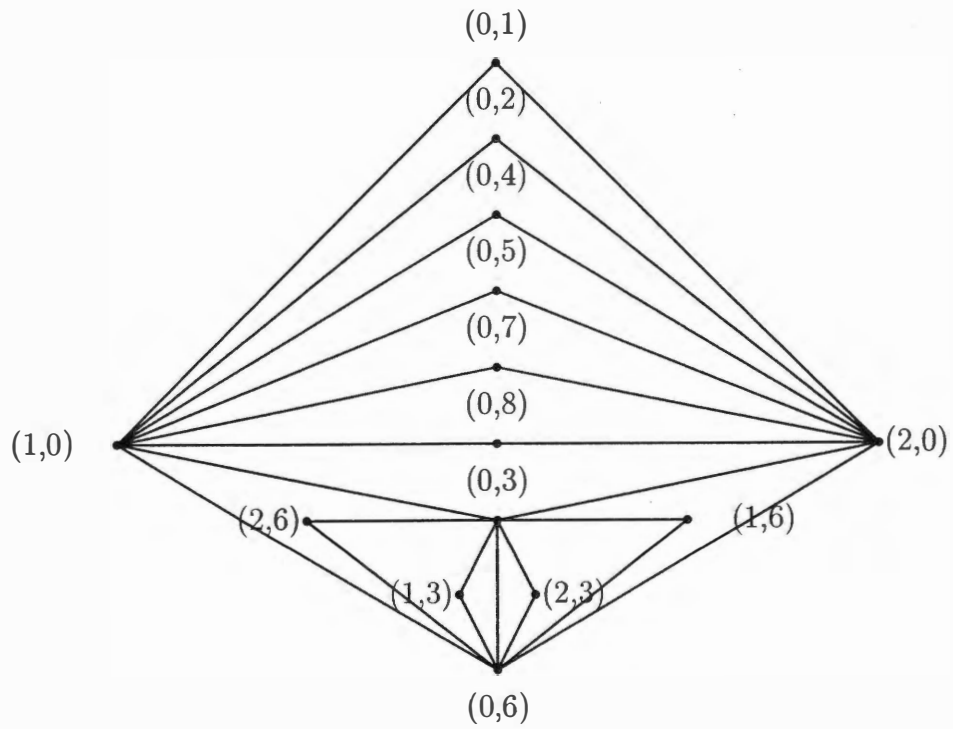


Figure A.5: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_9)$

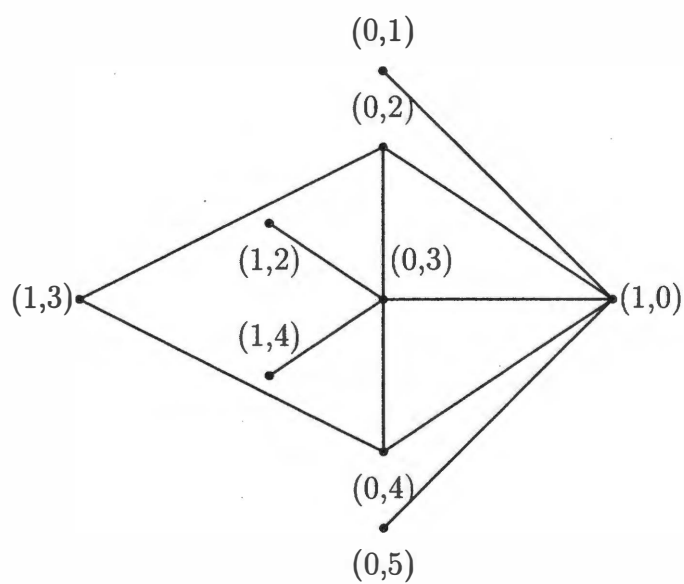


Figure A.6: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_6)$

Appendix B

Zero-Divisor Graphs of Certain Families of Rings

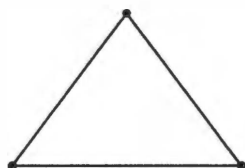


Figure B.1: $\Gamma(R)$ for those R in Lemma 2.10 (e)

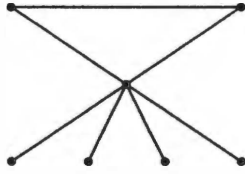


Figure B.2: $\Gamma(R)$ for those R in Lemma 2.10 (d)

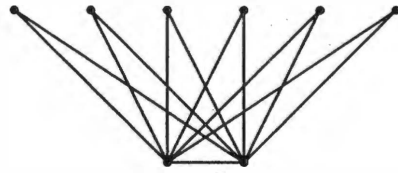


Figure B.3: $\Gamma(R)$ for those R in Lemma 2.10 (f)

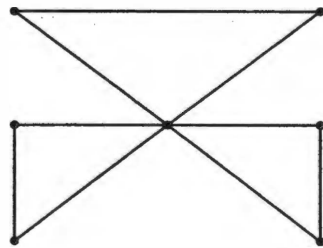


Figure B.4: $\Gamma(R)$ for those R in Lemma 2.10 (h) and (i)

Vita

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