# Development of Probability Model for Fatigue Crack Growth Using Response Surface Method

K.C. Chou, G.C. Cox & A.M. Lockwood
*University of Tennessee, Knoxville, TN, USA*

P.C. Hoffman & M.E. Hoffman
*Navy Air System Command, AIR-4.3.3, Patuxent River, MD, USA*

**Introduction and Background**
by Allison M. Lockwood

One of the biggest things I have learned during my time at the University of Tennessee is that no major is able to stand by itself. In fact, when working in the field the lines between one major and another seem to blur or disappear. I first discovered this when I spent a summer working for one of my civil engineering professors. She specialized in structural engineering, so I was expecting bridges and buildings. I was more than a little surprised when I found out we would be working with airplanes!

The focus of our work was the frame or body of the planes, which for this project were Navy fighter jets. After these planes are flown many times, tiny cracks begin to develop in their frames. These cracks are the result of fatigue caused by flying, pressure and temperature variations, loading and unloading of the aircraft, and many other factors. Over time these cracks will grow to a size that will compromise the structural integrity of the planes. Obviously it would be disastrous to have a plane fail while in use. As a result of this, the Navy has been discontinuing the use of these planes after they have flown a certain number of missions. The number of missions has just been an educated guess based on the condition of retired planes in the past, with a large factor of safety to compensate for the lack of information on this subject. However, the Navy wants a better estimate of when their planes should be retired. Military budgets have been significantly reduced over recent years, and there is now a focus on making what they have last longer.

This is where a structural engineer comes into the picture. Some studies have already been conducted to determine the size of cracks occurring over a given time interval. However, for this project we had to find out the time it took for a crack to reach critical size. This problem is magnified when one considers that it is totally infeasible to test this on actual aircraft.

I worked with Dr. Karen Chou and her master's student, Glenn Cox on this project. Dr. Chou came up with the premise of the project and worked with Glenn and I on how to solve it. Glenn focused on assimilating the data and finishing his thesis. My first responsibility was to generate a population set on which to conduct our research. I used the program MATLAB and prior research to create a program that would yield 10000 pseudo-population data while considering the randomness associated with this type of test. After generating 100 sets of 10000 data points each, I was ready to begin testing the data. Dr. Chou and I discussed many different statistical tests to use. We wanted to find tests that would apply to the type of research we were conducting and test the validity of our hypotheses. I generated more MATLAB programs to achieve this and also utilized Microsoft Excel.

Finally, after running numerous tests, we could begin to sort out the data. We evaluated all of our statistical tests and compiled the results into the following paper. This paper was accepted by the 2001 International Conference on Structural Safety and Reliability in Newport Beach, CA. I presented the paper at this conference in June 2001. I enjoyed working on this research. It gave me an opportunity to see a side of structural engineering that I would not have been exposed to in class. I would like to thank Dr. Chou for the chance to work with her, and Glenn Cox for helping me get through the hours of program writing.

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ABSTRACT: This paper summarizes the development of the RSM approach for crack growth life projections. The emphasis for the RSM model development was the quantification of the scatter in the number of load cycles N needed for a crack to propagate to a specific size. For complex geometry, the integration of the crack growth rate did not yield a closed form solution, it was more practical to develop the probability density function for N using response surface method (RSM). The study showed that the most workable regression function was a linear regression with all the variables transformed to the logarithmic values. The study also showed that, in addition to the number of samples, the response surface function was influenced by the range of the input values. By the Central Limit Theorem, P*n* N was approximated by Gaussian distribution. This Gaussian model compared well with the histograms of the number of load cycles generated from 100,000 simulated crack growth curves. The work is ongoing to validate the preliminary findings.

## 1 INTRODUCTION

The central aging aircraft issue is the evaluation of airframe service lives. A realistic analysis must consider the randomness inherent in life prediction. As part of an ongoing research and development effort, the response surface method (RSM) is being applied to probability crack growth predictions.

Studies of crack growth are numerous. There have also been extensive studies on crack growth due to cyclic loads. To study the randomness inherent in the load-resistance behavior, researchers have developed stochastic models between crack size and load cycles (for example, Wirsching, 1983, Wu & Wirsching, 1983). These studies primarily focused on the crack size for a given number of load cycles. The interest here, for life prediction, is the number of load cycles for a crack to grow from its initiation size to a critical size.

In this paper, development of the RSM in the number of load cycles needed for a crack to propagate to a specific size is presented. During the study, the crack growth rate was assumed to follow Paris Law. The stress intensity was assumed to be constant throughout the entire crack growth. The crack initiation size was defined as 0.01 in. and the critical size was defined as 0.3 in.

## 2 THEORETICAL DEVELOPMENT

### 2.1 *Crack growth model*

The crack growth model used in this study was based on Paris Law (Bannantine et al., 1990) where the rate of crack growth is defined as

$$\frac{da}{dN} = c(\Delta k)^m \tag{1}$$

where a = crack size, c = geometric parameter, m = material constant, Δk = stress intensity factor and is defined as

$$\Delta k = \beta(\Delta\sigma)\sqrt{\pi a} \tag{2}$$

in which β = correction factor and Δσ = stress intensity. For a complex geometry, the integration of Equation 1 does not yield a closed form solution. Numerical integration was used instead. In this study, the difference between crack initiation size and critical crack size was divided into 100 equal intervals. The number of load cycles ΔN needed to achieve each increment of crack length Δa was computed as,

$$\Delta N_j = \frac{c(\Delta k_j)^m}{\Delta a} = \frac{c\beta_j(\Delta\sigma)\sqrt{\pi a_j}}{\Delta a} \tag{3}$$

where

$$\Delta a = \frac{a_f - a_i}{100} \tag{4}$$

and $\beta_j$ and $a_j$ are, respectively, the correction factor and crack size at the j-th increment. The total number of cycles N equals to the summation of the incremental number of cycles computed by Equation 3.

### 2.2 *Response Surface Method*

Since the integration of the crack size rate (Eq. 1) does not yield a closed form expression for most cases, the probability model for the number of cycles N cannot be derived explicitly. Using numerical integration (Eqs. 3 & 4) and Monte Carlo simulations, it is possible to "empirically" develop the probability model. When actual experimental data are available on the number of load cycles needed for a crack to grow from initiation to critical size, Monte Carlo simulations would not be necessary.

Response surface method (RSM) requires a set of input and its corresponding output. In this case, for each input set of c, m and $\Delta\sigma$ values, there exists an output value N. Each set of information represents one experimental datum. Using these sets of input parameters and output values, one can define a relationship between the input and output using regression analysis. This relationship represents the response surface.

Once the total number of cycles N is defined, N = g(c, m, $\Delta\sigma$), the probability model for N can be derived. The mean and standard deviation can be computed using either the point estimate method (Rosenblueth, 1975, 1981; Harr, 1987) or Taylor series approximation (Benjamin & Cornell, 1970).

### 2.3 *Statistical Verification*

Any probability model developed based on experimental data requires statistical verifications. The statistical moments, such as mean and standard deviation (determined based on the response surface functions) were compared with the pseudo population mean using a minimum of 10,000 simulated crack growth curves. The Kologmorov-Smirnov (K-S) test was used to verify the model by comparing the probability model developed for the response surface function with the histograms compiled from the pseudo population.

## 3 ANALYSIS

Since N is a function of 3 variables, a typical linear regression analysis (the simplest regression relationship) yields 4 coefficients. Hence, a minimum of 5 sets of (c, m, $\Delta\sigma$ and N) values are required for the regression analysis. A computer software called Matlab and its statistical toolbox (by Mathwork) were used to perform the regression analysis based on the least-squared-error criteria. Both linear and nonlinear regression analyses were attempted to develop a response surface function for N as well as for $\ell n$ N. The most workable regression function was a linear regression with all the variables transformed to the logarithmic values,

$$\ell n\, N = b_1 \ell n\, c + b_2 \ell n\, m + b_3 \ell n\, (\Delta\sigma) + b_4 \tag{5}$$

where $b_i$ = coefficients obtained from the regression analysis.

A range of 5 to 729 sets of (c, m, $\Delta\sigma$, N) values ("experimental" sample points) were used in the study. It was found that the number of samples was not the only factor influencing the response surface (RS) function. The range of input values of (c, m, $\Delta\sigma$) also influenced the "quality" of the RS functions. Hence, various values of (c, m, $\Delta\sigma$) were considered as well. Table 1 summarizes the statistical data for the input parameters. These are not the statistics for any specific material or structure. But they are representative values of actual aircraft that the co-authors have experience with. For each regression analysis, the values of c, m and $\Delta\sigma$ were assumed to be at their mean values, or +/- some standard deviations from the mean. A detailed discussion on the combinations of the input parametric values is presented in Cox (2000).

Once the values for c, m and $\Delta\sigma$ were determined, the number of load cycles N was computed using Equations 3 and 4. The procedure was repeated for all crack growth curves as indicated by the sample size (second column in Table 2). A regression analysis was then performed. As stated earlier, the most workable regression is a linear regression given by Equation 5. In this study, a total of 39 RS functions were determined.

Table 1. Statistical data for input parameters.

| Parameter | Mean | Standard Deviation | Coeff. of Variation | Probability Distribution |
|---|---|---|---|---|
| c | 1.0E-8 | 2.3E-9 | 0.23 | lognormal |
| m | 3.0 | 0.1 | 0.0333 | normal |
| $\Delta\sigma$ | 13.9 | 1.39 | 0.1 | normal |

## 4 PROBABILITY MODEL

### 4.1 *Statistical Moments*

Since the response surface function for $\ell n$ N (Eq. 5) is linear, $\ell n$ N can be assumed to be normally (Gaussian) distributed by the Central Limit theorem. The statistics for $\ell n$ N can be computed using either the point estimate method (Rosenblueth, 1975, 1981; Harr, 1987) or the Taylor series (Benjamin & Cornell, 1970). In this paper, only the point estimate is presented. The results of Taylor series approximation can be found in Cox (2000).

The point estimate method requires 8 points for the 3 variables c, m and $\Delta\sigma$. These points are:
(c+, m+, $\Delta\sigma$+); (c+, m+, $\Delta\sigma$-); (c+, m-, $\Delta\sigma$+);
(c+, m-, $\Delta\sigma$-); (c-, m+, $\Delta\sigma$+); (c-, m+, $\Delta\sigma$-);
(c-, m-, $\Delta\sigma$+); (c-, m-, $\Delta\sigma$-)
in which "*+" = mean of * plus (+) one standard deviation of *; and "*-" = mean of * minus (-) one standard deviation of *. The $\ell n$ N values were computed for each of the points listed above using the response surface function given in Eq. 5. The mean and standard deviation of $\ell n$ N becomes:

$$E[\ell n\ N] = \frac{1}{8}\sum_{i=1}^{8} \ell n\ n_i \tag{6}$$

$$\sigma^2_{\ell n N} = \frac{1}{8}\sum_{i=1}^{8} (\ell n\ n_i)^2 - \left(E\ [\ell n\ N]\right)^2 \tag{7}$$

Table 2 summarizes the means and standard deviations computed using Equations 6 and 7.

To verify if the statistical moments obtained via the approximate methods presented here are representative of the "populations", 1000 sets of 10,000 simulated crack growth curves were generated for use as the basis for comparisons. These sets of simulated curves were considered as the pseudo population information. A 95% confidence interval (CI) was determined for the mean of each response surface (RS) function. The mean values obtained using the pseudo population were compared with this 95% confidence interval. Only 2 out of 39 RS function studied had pseudo population mean did not fall within the 95% CI. In those 2 RS functions, 1 had 4 out of 1000 pseudo populations whose means fell outside the CI and the other had 36 out of 1000 fell outside the CI. This result indicated that the estimated mean predicted the population mean with a 95% confidence level.

To further assess the probability model of $\ell n$ N, the coefficient of skewness and the coefficient of Kurtosis were examined. If $\ell n$ N follows a Gaussian distribution, then the coefficient of skewness should equal to zero and the coefficient of Kurtosis should equal to 3. One hundred sets of 10,000 "population" crack growth curves were used to determine these higher order statistical moments. The average coefficient of skewness from these 100 sets of data was found to be 0.0481 and the standard deviation was 0.0257. The average coefficient of Kurtosis from the same 100 sets of data was 3.0285 and the standard deviation was 0.0490. The average coefficient of skewness indicates that the $\ell n$ N values were skewed slightly to the right. Although the pseudo population indicates that the probability distribution of $\ell n$ N was not exactly symmetric, the coefficient of skewness and the coefficient of Kurtosis were very close to the target values of 0 and 3 respectively.

### 4.2 *Model Verification*

After the statistical moments were examined, the proposed probability model (Gaussian distribution) was verified. Figures 1 and 2 show a typical comparison between the proposed model and pseudo population. Figure 1 shows the fitting of the probability density function of a response surface (RS function no. 17 listed in Table 2 with 27 samples) to the histogram compiled from 10,000 simulated crack growth curves. Figure 2 shows the cumulative distribution function of the same response surface and 10,000 simulated values. Note that Figure 2 is plotted on normal probability scale. As can been seen from these 2 figures, the Gaussian distribution of the response surface fits the pseudo population well except at the extreme values.

Kolmogorov-Smirnov (K-S) test was used to test the acceptance of the Gaussian distribution as the probability model for $\ell n$ N. The critical value $D_{cr}$ depends on the number of "experimental" data (crack growth curves) that were used to develop the response surface (and hence the proposed probability model). Table 3 summarizes the K-S test for 39 response surface functions. Columns 5 and 6 are the comparison with one set of 10,000 simulated crack growth curves. Column 5 gives the maximum difference in cumulative distribution function. Column 6 gives the ratio of the difference (column 5) to the critical value $D_{cr}$ (column 4). Columns 7 and 8 and columns 9 and 10 are comparison with one set each of, respectively, 50,000 and 100,000 simulated crack growth curves. As can be seen from the Table, all the response surface functions passed the K-S test at the 5% significant level.

## 5 OPTIMAL SAMPLE SIZE AND CHARACTERIZATION FOR RESPONSE SURFACE DEVELOPMENT

The presentations in previous sections have demonstrated that a response surface function developed using a limited number of crack growth curves is sufficient to probabilistically model the number of critical size. The study also used a wide range of sample sizes, from 5 to 729, as well as wide range of input values, from 0.01 standard deviation from the mean to 4 standard deviations from the mean. It is desirable to identify the optimal sample size and range of input values used to develop a response surface function and eventually, a probability model can be derived.

Figure 3 shows the fraction of critical values $D_{cr}$, based on 50,000 simulated crack growth curves, as a function of range of input values. The abscissa gives the maximum number of standard deviations from the mean (column 3 in Table 3) used for the input parameter c, m and $\Delta\sigma$ to develop the response surface functions. The ordinate gives the values in column 8 of Table 3. The optimal was chosen as the one with the lowest fraction.

It was observed that as the sample size increases, the fraction of $D_{cr}$ increases. This does not imply that the more samples one has to determine the response surface function, the worse the approximation will be. The reason for this observation is that $D_{cr}$ decreases with sample size. Despite this observation, for a given sample size, there is a fairly distinct trend that if the range of input values is very narrow or very broad from the mean, the fraction of $D_{cr}$ would be larger. Based on this trend, it is recommended that the optimal choice of sample size and range of input values are respectively 27 and 2 to 2.5. That is a sample size of 27 and the number of standard deviations from the mean is about 2 to 2.5. Hence, RS function number 17 or 18 given in Table 3 would be the optimal RS functions for modeling the number of load cycles needed for a crack to grow from the initiation size of 0.01 in. to 0.3 in. and for crack growth rate with parametric values given in Table 1.

## 6 CONCLUSIONS

This study has demonstrated that,

- Response surface method is applicable for determining the number of load cycles N required for a crack to grow from initiation to critical size.
- Gaussian model is an adequate probability model to describe the inherent randomness of crack growth and more specifically the natural logarithmic of N, $\ell n$ N.
- When actual experimental data were used to determine the response surface function, a sample size of 27 with parametric values 2 to 2.5 standard deviations from the mean would yield an optimal response surface function.

The above observations were based on the assumption that the crack growth rate follows Paris Law. Furthermore, the stress intensity $\Delta\sigma$ was assumed to remain constant throughout the entire crack growth. In reality, $\Delta\sigma$ varies with time and differs with different pilots. An ongoing study is considering random stress intensity in developing the RS functions. The probabilistic information developed here was intended for the assessment of life expectancy of existing aircraft. However, the procedures presented here can be used in developing similar information for other structural systems.

## 7 ACKNOWLEDGEMENT

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Table 2. Means and standard deviations from point estimate method and results of the 95% confidence interval test.

| RS Function No. | Sample Size | Average Value | Standard Deviation | 95% Confidence Interval | | Pseudo Population | | Percent Acceptable |
|---|---|---|---|---|---|---|---|---|
| | | | | Lower Limit | Upper Limit | Passed | Failed | |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| 1 | 5 | 9.63 | 0.310 | 9.36 | 9.91 | 1000 | 0 | 100% |
| 2 | 5 | 9.65 | 0.457 | 9.25 | 10.1 | 1000 | 0 | 100% |
| 3 | 5 | 9.65 | 0.457 | 9.25 | 10.0 | 1000 | 0 | 100% |
| 4 | 5 | 9.65 | 0.459 | 9.24 | 10.0 | 1000 | 0 | 100% |
| 5 | 5 | 9.65 | 0.463 | 9.24 | 10.1 | 1000 | 0 | 100% |
| 6 | 5 | 9.65 | 0.468 | 9.24 | 10.1 | 1000 | 0 | 100% |
| 7 | 5 | 9.66 | 0.472 | 9.24 | 10.1 | 1000 | 0 | 100% |
| 8 | 5 | 9.66 | 0.477 | 9.25 | 10.1 | 1000 | 0 | 100% |
| 9 | 5 | 9.67 | 0.481 | 9.25 | 10.1 | 1000 | 0 | 100% |
| 10 | 8 | 9.64 | 0.454 | 9.33 | 9.96 | 1000 | 0 | 100% |
| 11 | 8 | 9.64 | 0.454 | 9.32 | 9.95 | 1000 | 0 | 100% |
| 12 | 8 | 9.63 | 0.453 | 9.32 | 9.94 | 1000 | 0 | 100% |
| 13 | 8 | 9.62 | 0.453 | 9.31 | 9.94 | 1000 | 0 | 100% |
| 14 | 8 | 9.61 | 0.452 | 9.30 | 9.92 | 1000 | 0 | 100% |
| 15 | 27 | 9.64 | 0.455 | 9.47 | 9.82 | 1000 | 0 | 100% |
| 16 | 27 | 9.64 | 0.455 | 9.47 | 9.81 | 1000 | 0 | 100% |
| 17 | 27 | 9.64 | 0.454 | 9.46 | 9.81 | 1000 | 0 | 100% |
| 18 | 27 | 9.63 | 0.453 | 9.46 | 9.80 | 1000 | 0 | 100% |
| 19 | 27 | 9.62 | 0.452 | 9.45 | 9.79 | 1000 | 0 | 100% |
| 20 | 27 | 9.61 | 0.452 | 9.44 | 9.78 | 1000 | 0 | 100% |
| 21 | 27 | 9.74 | 0.451 | 9.57 | 9.91 | 1000 | 0 | 100% |
| 22 | 125 | 9.64 | 0.455 | 9.56 | 9.72 | 1000 | 0 | 100% |
| 23 | 125 | 9.64 | 0.454 | 9.56 | 9.72 | 1000 | 0 | 100% |
| 24 | 125 | 9.63 | 0.453 | 9.55 | 9.71 | 1000 | 0 | 100% |
| 25 | 125 | 9.63 | 0.454 | 9.56 | 9.71 | 1000 | 0 | 100% |
| 26 | 125 | 9.63 | 0.453 | 9.55 | 9.71 | 1000 | 0 | 100% |
| 27 | 125 | 9.63 | 0.453 | 9.55 | 9.71 | 1000 | 0 | 100% |
| 28 | 125 | 9.63 | 0.453 | 9.55 | 9.71 | 1000 | 0 | 100% |
| 29 | 343 | 9.64 | 0.454 | 9.59 | 9.69 | 1000 | 0 | 100% |
| 30 | 343 | 9.64 | 0.454 | 9.59 | 9.69 | 1000 | 0 | 100% |
| 31 | 343 | 9.63 | 0.454 | 9.59 | 9.68 | 1000 | 0 | 100% |
| 32 | 343 | 9.63 | 0.453 | 9.58 | 9.68 | 1000 | 0 | 100% |
| 33 | 343 | 9.63 | 0.453 | 9.58 | 9.67 | 1000 | 0 | 100% |
| 34 | 729 | 9.64 | 0.454 | 9.61 | 9.67 | 964 | 36 | 96.4% |
| 35 | 729 | 9.64 | 0.454 | 9.60 | 9.67 | 996 | 4 | 99.6% |
| 36 | 729 | 9.63 | 0.453 | 9.60 | 9.67 | 1000 | 0 | 100% |
| 37 | 729 | 9.63 | 0.453 | 9.60 | 9.66 | 1000 | 0 | 100% |
| 38 | 729 | 9.63 | 0.453 | 9.60 | 9.66 | 1000 | 0 | 100% |
| 39 | 729 | 9.62 | 0.452 | 9.59 | 9.65 | 1000 | 0 | 100% |

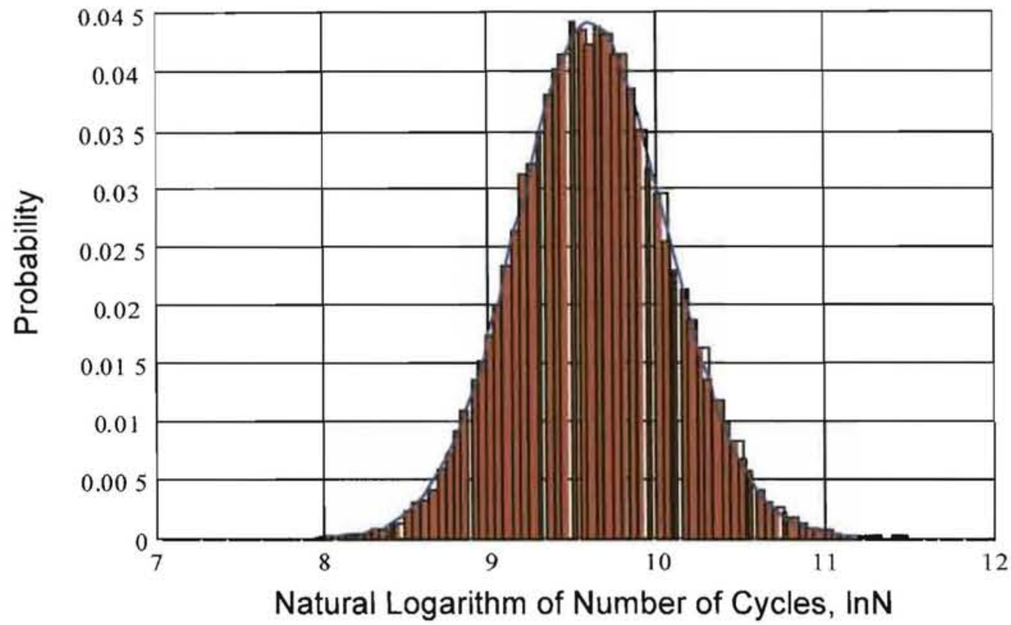


Figure 1. Typical comparison between the proposed probability model from a response surface function and histogram from the pseudo population.

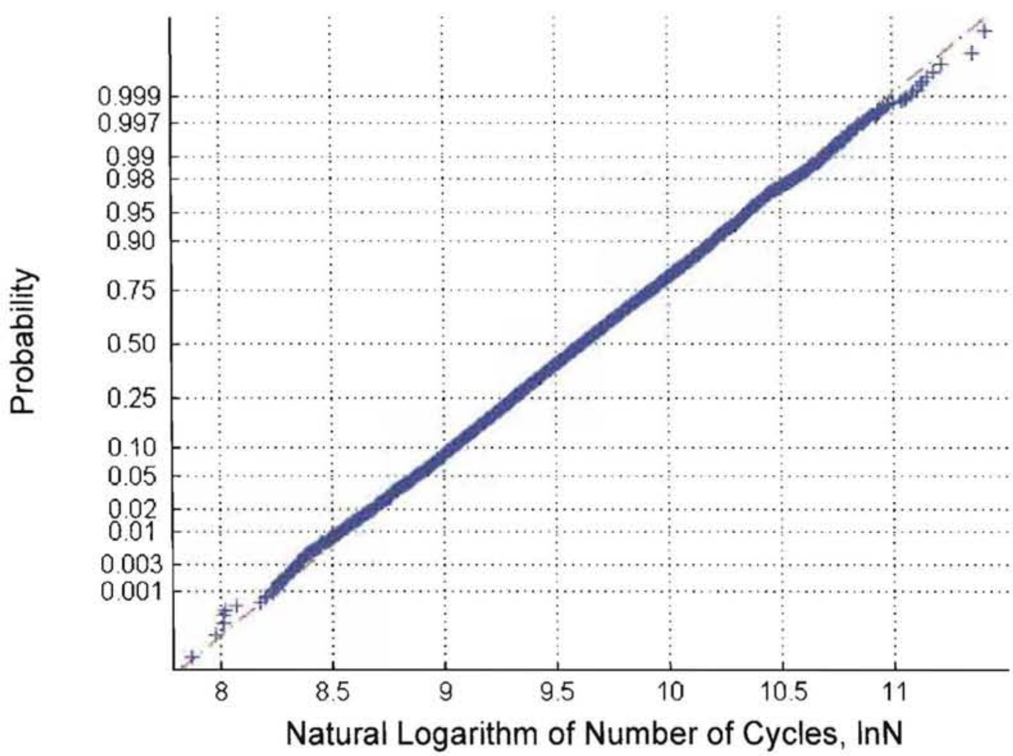


Figure 2. Typical comparison in cumulative distributions between a response surface function and the pseudo population.

Table 3. Results of the Kolmogorov-Smirnov test.

| RS Function No. | Sample Size | No. of Std. Dev. from Mean | $D_{cr}$ | Pseudo Population Size: 10,000 | | Pseudo Population Size: 50,000 | | Pseudo Population Size: 100,000 | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Difference | Fraction of $D_{cr}$ | Difference | Fraction of $D_{cr}$ | Difference | Fraction of $D_{cr}$ |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| 1 | 5 | 0.01 | 0.560 | 0.096 | 0.171 | 0.094 | 0.168 | 0.093 | 0.166 |
| 2 | 5 | 0.05 | | 0.020 | 0.035 | 0.011 | 0.020 | 0.008 | 0.014 |
| 3 | 5 | 0.1 | | 0.020 | 0.037 | 0.012 | 0.021 | 0.008 | 0.015 |
| 4 | 5 | 0.5 | | 0.022 | 0.039 | 0.013 | 0.023 | 0.010 | 0.018 |
| 5 | 5 | 1 | | 0.024 | 0.042 | 0.016 | 0.029 | 0.013 | 0.022 |
| 6 | 5 | 1.5 | | 0.027 | 0.049 | 0.020 | 0.036 | 0.016 | 0.029 |
| 7 | 5 | 2 | | 0.032 | 0.057 | 0.025 | 0.044 | 0.021 | 0.037 |
| 8 | 5 | 2.5 | | 0.037 | 0.066 | 0.030 | 0.054 | 0.026 | 0.046 |
| 9 | 5 | 3 | | 0.043 | 0.077 | 0.036 | 0.064 | 0.032 | 0.057 |
| 10 | 8 | 1 | 0.470 | 0.017 | 0.036 | 0.009 | 0.019 | 0.005 | 0.010 |
| 11 | 8 | 1.5 | | 0.013 | 0.027 | 0.005 | 0.010 | 0.003 | 0.006 |
| 12 | 8 | 1.5 | | 0.012 | 0.025 | 0.012 | 0.026 | 0.016 | 0.033 |
| 13 | 8 | 2 | | 0.009 | 0.020 | 0.007 | 0.016 | 0.008 | 0.017 |
| 14 | 8 | 3 | | 0.019 | 0.039 | 0.021 | 0.045 | 0.025 | 0.053 |
| 15 | 27 | 0.5, 0 | 0.258 | 0.020 | 0.077 | 0.009 | 0.035 | 0.009 | 0.035 |
| 16 | 27 | 1, 0 | | 0.018 | 0.071 | 0.007 | 0.028 | 0.007 | 0.029 |
| 17 | 27 | 2, 0 | | 0.011 | 0.044 | 0.005 | 0.019 | 0.004 | 0.016 |
| 18 | 27 | 2.5, 0 | | 0.010 | 0.037 | 0.010 | 0.038 | 0.007 | 0.027 |
| 19 | 27 | 3, 0 | | 0.011 | 0.044 | 0.016 | 0.062 | 0.013 | 0.051 |
| 20 | 27 | 3.5, 0 | | 0.016 | 0.061 | 0.023 | 0.090 | 0.021 | 0.080 |
| 21 | 27 | 4, 0 | | 0.024 | 0.092 | 0.031 | 0.121 | 0.029 | 0.112 |
| 22 | 125 | 1, 0.5, 0 | 0.122 | 0.019 | 0.155 | 0.008 | 0.065 | 0.008 | 0.065 |
| 23 | 125 | 2, 1, 0 | | 0.014 | 0.112 | 0.003 | 0.023 | 0.003 | 0.026 |
| 24 | 125 | 2.5, 1.5, 0 | | 0.009 | 0.072 | 0.007 | 0.061 | 0.005 | 0.043 |
| 25 | 125 | 2.5, 1, 0 | | 0.011 | 0.086 | 0.006 | 0.047 | 0.004 | 0.037 |
| 26 | 125 | 2.5, 2, 0 | | 0.009 | 0.078 | 0.010 | 0.080 | 0.007 | 0.056 |
| 27 | 125 | 3, 1.5, 0 | | 0.010 | 0.081 | 0.011 | 0.091 | 0.008 | 0.067 |
| 28 | 125 | 3.5, 2, 0 | | 0.010 | 0.083 | 0.018 | 0.145 | 0.015 | 0.123 |
| 29 | 343 | 1.5, 1, 0.5, 0 | 0.073 | 0.017 | 0.233 | 0.006 | 0.083 | 0.006 | 0.085 |
| 30 | 343 | 2, 1, 0.5, 0 | | 0.015 | 0.209 | 0.004 | 0.060 | 0.005 | 0.062 |
| 31 | 343 | 2.5, 2, 1, 0 | | 0.009 | 0.128 | 0.007 | 0.092 | 0.005 | 0.067 |
| 32 | 343 | 3, 2, 1, 0 | | 0.009 | 0.128 | 0.009 | 0.127 | 0.007 | 0.091 |
| 33 | 343 | 3.5, 2, 1, 0 | | 0.010 | 0.141 | 0.012 | 0.169 | 0.010 | 0.132 |
| 34 | 729 | 2, 1.5, 1, 0.5, 0 | 0.050 | 0.015 | 0.293 | 0.004 | 0.077 | 0.004 | 0.079 |
| 35 | 729 | 2.5, 2, 1, 0.5, 0 | | 0.012 | 0.232 | 0.005 | 0.091 | 0.004 | 0.077 |
| 36 | 729 | 3, 2, 1, 0.5, 0 | | 0.010 | 0.190 | 0.007 | 0.131 | 0.005 | 0.098 |
| 37 | 729 | 3.5, 2.5, 1, 0.5, 0 | | 0.010 | 0.196 | 0.011 | 0.211 | 0.008 | 0.156 |
| 38 | 729 | 3.5, 2, 1, 0.5, 0 | | 0.009 | 0.187 | 0.009 | 0.178 | 0.007 | 0.130 |
| 39 | 729 | 4, 3, 2, 1, 0 | | 0.012 | 0.241 | 0.018 | 0.361 | 0.016 | 0.308 |

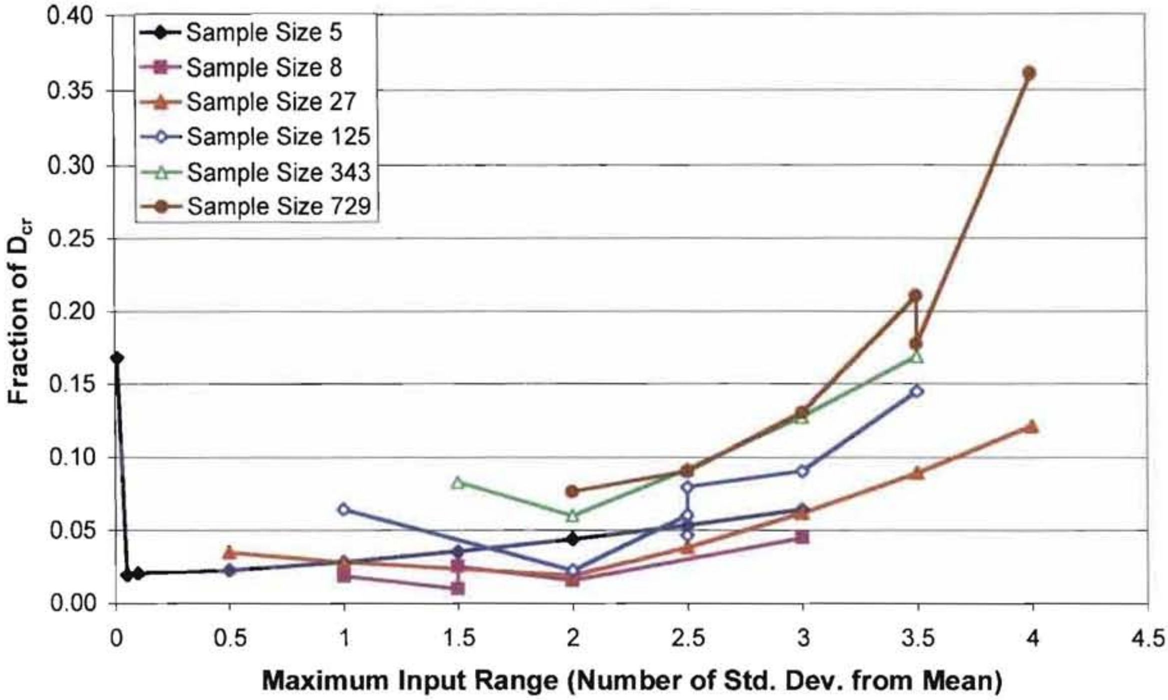


Figure 3. Comparison of fraction of $D_{cr}$ and maximum input range for a pseudo population size of 50,000.