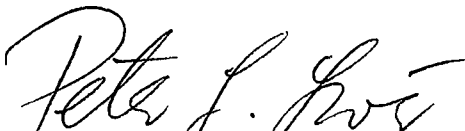


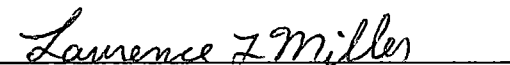
To the Graduate Council:

I am submitting herewith a master's thesis written by Mat Mustafa Tamimi entitled "Bayesian Analysis of a Radiation Detection Problem." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with major in Nuclear Engineering.


Dr. Peter Groer, Major Professor

We have read this thesis
and recommend its acceptance:





Accepted for the Council:


Associate Vice Chancellor and
Dean of The Graduate School

**BAYESIAN ANALYSIS OF
A RADIATION
DETECTION PROBLEM**

A
Thesis
Presented for the
Master of Science Degree
The University of Tennessee, Knoxville

Mat M. Tamimi

December 1999

ACKNOWLEDGMENTS

I would like to thank Dr. P. Groer for this project and his patient guidance throughout this work.

I would also like to thank Dr. L. Miller and Dr L. Townsend for their support and the University of Tennessee Radiation Safety Office , in particular the R.S.O. Roy P. Osborne, for the graduate assistantship during the course of my studies.

ABSTRACT

Radioactive decay is a random process. As a result any measurement based on observing the radiation emitted during a nuclear decay is subject to statistical fluctuations. These fluctuations represent an unavoidable source of uncertainty in all nuclear measurements and often are the predominant source of imprecision or error [Knoll, 1989].

Probability provides the best formalism for quantifying uncertainty. Probability distributions are used to express uncertainty [Morgan, 1990].

In a nuclear counting experiment fluctuations are due to two major sources; the statistical nature of the radioactive decay process and the properties of the nuclear counting equipment (detector). Appropriate probability models were assigned to describe the uncertainty due to each source.

The following is an outline of our analysis:

a. Statistical modeling of the detection problem:

1. The binomial distribution was selected to model the process of radioactive decay, where the initial sample contains N atoms and the number of atoms decaying is n .
2. The binomial distribution was also selected to model the probability distribution for the number of atoms m a

nuclear counter with efficiency ϵ will detect from the n atoms that actually decayed.

3. Uncertainty about ϵ was described through analysis of the calibration procedure. Parts of this analysis, similar to the procedures of step one and two, were used to model the calibration of the standard source. We derived a statistical model for the absolute detector efficiency ϵ .
 4. The uncertainty due to background was also considered and an expression for net count rate was derived.
- b. Results from the steps above were combined. The outcome is a probability distribution that estimates the activity and therefore the number of atoms in a sample from a given detector reading.

TABLE OF CONTENTS

CHAPTER	PAGE
1 INTRODUCTION	1
1.1 OBJECTIVE	1
1.2 WHAT IS MEANT BY RADIATION?	2
1.3 STATISTICAL NATURE OF RADIATION EMISSION	3
1.4 DEFINITION OF PROBABILITY AND BASIC THEOREMS	5
1.4.1 BASIC DEFINITIONS OF PROBABILITY	5
1.4.2 ADDITION LAW OF PROBABILITY	7
1.4.3 MULTIPLICATION LAW OF PROBABILITY	8
1.4.4 BAYES' THEOREM	8
1.5 PROBABILITY DISTRIBUTIONS AND RANDOM VARIABLES	10
1.6 STATISTICAL MODELS	12
1.6.1 THE BINOMIAL DISTRIBUTION	12
1.6.2 THE POISSON DISTRIBUTION	15
1.6.3 THE NORMAL (GAUSSIAN) DISTRIBUTION	17
1.6.4 UNIFORM DISTRIBUTIONS	18
2 UNCERTAINTY ANALYSIS OF THE DETECTION PROBLEM	20
2.1. ASSUMPTIONS	21
2.2. UNCERTAINTY ANALYSIS OF THE DETECTION PROBLEM PARTS	22
2.2.1. RADIOACTIVE DECAY PROCESS	22
2.2.2. THE COUNTING PROCESS	24
2.2.3. INSTRUMENT CALIBRATION USING A STANDARD SOURCE	25
2.2.4. BACKGROUND RADIATION	26
2.3. SUMMARY OF UNCERTAINTY ANALYSIS RESULTS	27

3	STATISTICAL MODELING OF A COUNTING EXPERIMENT	29
3.1.	STATISTICAL MODELING OF THE RADIOACTIVE DECAY PROCESS	29
3.2.	STATISTICAL MODELING OF THE NUCLEAR COUNTING PROCESS	30
3.3	STATISTICAL MODELING OF THE CALIBRATION PROCESS	35
3.3.1.	MODELING RADIOACTIVE DECAY OF THE STANDARD SOURCE	36
3.3.2.	MODELING THE DETECTION PROCESS USING A STANDARD SOURCE	36
3.3.3.	MODELING THE UNCERTAINTY ABOUT THE DETECTION EFFICIENCY ϵ	42
3.3.4.	MODELING OF THE PRIOR DENSITY FOR THE DETECTION EFFICIENCY	43
3.4.	BACKGROUND RADIATION	50
3.4.1	MODELING OF THE BACKGROUND COUNTS	51
3.4.2	MODELING OF THE GROSS COUNTS	52
3.4.3	DERIVATION OF THE LIKELIHOOD AND THE POSTERIOR DENSITY FOR THE BACKGROUND COUNT RATE β_0 AND NET COUNT RATE Δ	53
4	DERIVATION OF THE POSTERIOR DENSITIES FOR SAMPLE ACTIVITY AND NUMBER OF ATOMS	56
5	NUMERICAL ILLUSTRATIONS	64
5.1	RADIOACTIVE DECAY MODEL	64
5.2	DETECTION MODEL	65
5.3	DETECTOR COUNTS	65
5.4	STANDARD SOURCE MODEL	66

5.5	POSTERIOR DENSITY OF THE ABSOLUTE EFFICIENCY	67
5.6	JOINT POSTERIOR DISTRIBUTION FOR THE BACKGROUND COUNT RATE AND THE NET COUNT RATE	69
5.7	POSTERIOR DENSITIES FOR SAMPLE ACTIVITY AND NUMBER OF ATOMS	71
	DISCUSSION AND CONCLUDING REMARKS	73
	FUTURE RESEARCH	75
	BIBLIOGRAPHY	76
	APPENDIXES	80
1.	LIST OF SYMBOLS	81
2.	BINOMIAL AND POISSON DISTRIBUTIONS	84
3.	RADIOACTIVE DECAY MODEL	89
4.	DETECTION MODEL	106
5.	DETECTOR COUNTS	118
6.	STANDARD SOURCE MODEL	132
7.	POSTERIOR DENSITY OF THE ABSOLUTE EFFICIENCY	136
8.	JOINT POSTERIOR DISTRIBUTION FOR THE BACKGROUND COUNT RATE AND THE NET COUNT RATE	145
9.	POSTERIOR DENSITIES FOR SAMPLE ACTIVITY AND NUMBER OF ATOMS	155
	VITA	168

LIST OF FIGURES

FIGURE	PAGE
1.1 Binomial distribution for $N=25$ and $p=0.45$	13
1.2 Binomial distribution for $N=500$ and $p=0.45$	15
1.3 Uniform distribution for random variable X	19
3.1 Illustration of the angle θ_0	47
3.2 Prior distribution for the detector efficiency ϵ	48
5.1 Standard source distribution for $\mu_s = 76.00$ dpm and $\sigma_s = 3.00$ dpm	67
5.2 Posterior density for detector absolute efficiency using two numerical integration methods	68
5.3 Joint distribution for background and net count rate for $b = 3$ cpm and $s = 21$ cpm	69
5.4 Joint distribution for background and net count rate for $b = 3$ cpm and $s = 183$ cpm	70
5.5 Joint distribution for background and net count rate for $b = 3$ cpm and $s = 21$ cpm	70
5.6 Posterior density for sample activity for $m_s = 14$ cpm	72
5.7 Posterior density for sample number of atoms for Rn-210 with $T_{1/2} = 144$ minutes	72

LIST OF FIGURES

TABLE	PAGE
5.1 Calculated expectation for n for several isotopes	65
5.2 Calculated Poisson mean for several values Of Efficiency ϵ , number of atoms N and isotope	66

CHAPTER 1

INTRODUCTION

1.1 OBJECTIVE

It is customary to estimate the activity in a sample with two measurements using a detector with a known efficiency ϵ :

- 1) Background count rate is obtained by counting a blank.
- 2) Gross count rate is obtained by counting the sample in question.
- 3) Net count rate is calculated by taking the difference between the gross count rate and background count rate.
- 4) Estimated sample activity is the quotient of the net count rate and the detector efficiency.

Assuming that the half-life of the radionuclide and the counting periods for background and the sample are precisely known, the estimated number of atoms in the sample is the quotient of the estimated sample activity and the decay constant. Frequency based confidence intervals (c.i.) for the estimated quantities can be calculated [Currie, 1968]. These c.i.s are based on hypothetical repetitions of the counting experiment.

This commonly used approach fails to deal with several sources of uncertainty and its uncertainty estimates are based on hypothetical repetitions of the experiment. It does not describe the uncertainty of the result conditional on the observed data. It underestimates the uncertainty of the sample activity.

Using Bayesian analysis we can describe the uncertainty for each step of the detection process and derive a probability density for the sample activity.

1.2 WHAT IS MEANT BY RADIATION?

The word "radiation" commonly refers to the whole electromagnetic spectrum as well as to all the atomic and subatomic particles that have been discovered.

Different types of radiation are grouped together as ionizing and nonionizing radiation. The word ionizing refers to the ability of the radiation to ionize the medium it passes through. Ionizing radiation includes x-rays, γ rays, and all the atomic and subatomic particles, for example electrons, positrons, protons, alpha, neutrons and heavy ions. Nonionizing radiation is electromagnetic radiation such as radiowaves, microwaves, visible light, and ultraviolet light [Tsoulfanidis, 1981]. In

this thesis, our analysis will deal only with ionizing radiation. Thereafter radiation means ionizing radiation.

1.3 STATISTICAL NATURE OF RADIATION EMISSION

Radiation emission is nothing more than release of energy by an atomic or subatomic system as it completes the transition from one state to another. According to classical physics the amounts of energy exchanged or released vary continuously, i.e., any amount of energy, no matter how small, may be exchanged as long as the exchange is consistent with conservation laws. The fate of a system is determined by the initial conditions and its equations of motion. Classical physics gives a "deterministic" view of the world.

Quantum theory changed all that. According to quantum theory, energy can be exchanged only in discrete amounts when a system moves from one state to another. The fate of a system is not determined exactly if initial conditions and equations of motion are known. With the introduction of quantum theory, the study of the physical world changed from "deterministic" to "probabilistic".

The emission of atomic and nuclear radiation obeys the rules of quantum theory. Consequently, one can talk only about the probability that a reaction will take place or that a particle will be emitted. If one

attempts to measure the number of particles emitted by a nuclear reaction, that number is not constant in time; it has a statistical uncertainty because of the probabilistic nature of the phenomenon under study.

Consider a radioactive source emitting electrons and assume that one attempts to measure the number of electrons per unit time emitted by the source. For every atom of the source, there is a probability that an electron will be emitted during the next unit of time. One can never measure the "exact" number. The number of particles emitted per unit time is different for successive units of time. Therefore, one can only estimate the average number of particles emitted. That estimate, like any estimate, carries with it uncertainty. The determination of this uncertainty is an integral part of any radiation measurement procedure [Tsoulfanidis, 1981].

A measurement is an attempt to determine the value of a certain parameter or quantity. Any one attempting a measurement should keep in mind the following facts regarding the results of the measurement:

- 1) No measurement yields a result without uncertainty.
- 2) The result of a measurement is almost worthless unless the uncertainty associated with that result is also reported.

Sometimes the term error is used to describe the following concepts:

1. The difference between measured quantity and true parameter value.
2. The estimated uncertainty of the measured quantity.

Uncertainty may also arise from incomplete information, use of simplifications and approximations in an experiment. Measuring and quantifying uncertainty is commonly done using probability, where probability distributions are used to express the uncertainty about the quantities in question. The probability of an event is viewed, as the degree of belief that a person has that the event will occur, given all the relevant information currently known to that person. This makes probability a function of the event and the state of the information [Morgan, 1990].

1.4 DEFINITION OF PROBABILITY AND BASIC THEOREMS

1.4.1 BASIC DEFINITIONS OF PROBABILITY

An "event" is an outcome of one or more experiments and it is defined by the experimenter. Assume that one repeats an experiment many times and observes whether a certain event x is the outcome. The event is a specific result defined by the experimenter. If the experiment

was performed N times, and n results were event x , the probability $P(x)$ that any single event will be of type x is equal to:

$$P(x) = \lim_{N \rightarrow \infty} \frac{n}{N} \quad (1.3.1.1)$$

This is the relative frequency definition of probability. The ratio n/N is called the relative frequency of occurrence of x in the first N trials. This definition for the probability of an event requires the performance of an infinite number of experiments, which is impossible. Instead the experiment is repeated N times, and if the event x occurs n times out of N , the probability $P(x)$ is approximately:

$$P(x) \approx n/N \quad (1.3.1.2)$$

Given information about an event, one can estimate the probability of that event occurring. In some cases, an event may consist of simpler components, and one would like to know how to calculate the probability of the complex event from the probabilities of its components.

For two events x and y , we can define the following probabilities:

$$P(x) = \text{Probability that } x \text{ will occur}$$

$P(y)$ = Probability that y will occur

$P(x+y)$ = Probability that either x or y will occur

$P(xy)$ = Probability that both x and y will occur

$P(x|y)$ = conditional probability that x will occur
given that y has occurred

$P(y|x)$ = conditional probability that y will occur
given that x has occurred

Some of the probabilities defined earlier can be combined using the following relations:

1.4.2 ADDITION LAW OF PROBABILITY

$$P(x+y) = P(x) + P(y) - P(xy) \quad (1.3.2.a)$$

In the case where x and y are mutually exclusive, i.e. they cannot occur simultaneously, then $P(xy) = 0$ and the addition law simplifies to:

$$P(x+y) = P(x) + P(y) \quad (1.3.2.b)$$

1.4.3 MULTIPLICATION LAW OF PROBABILITY

$$P(xy) = P(y|x) P(x) = P(x|y) P(y) \quad (1.3.3.a)$$

In the case where the probability of x is independent of whether or not y occurs, and vice versa, then $P(y|x) = P(y)$ and $P(x|y) = P(x)$ and the product rule become:

$$P(xy) = P(x) P(y) \quad (1.3.3.b)$$

1.4.4 BAYES' THEOREM

Since there are many problems involving conditional probability, it is important to have a relationship which expresses $P(y|x)$ in terms of $P(x|y)$. Starting with the product rule one can deduce an expression for $P(y|x)$ in terms of $P(y|x)$ or visa versa. The mathematical formula is:

$$P(y|x) = \frac{P(x|y) P(y)}{P(x)} \quad (1.3.4.1)$$

This result is known as Bayes' Theorem. One can view y in this formula as a parameter. Since $P(x)$ is dependent only on x , it will be considered as a constant. This will allow us to simplify the expression by omitting $P(x)$ and replacing the equal sign with the proportional sign $\propto$. The result is known as the proportional form of Bayes' Theorem. This form is shown below:

$$P(y | x) \propto P(x | y) P(y) \quad (1.3.4.2)$$

This is a very useful form since it enables us to reduce the amount of computation necessary to arrive at an expression for $P(y | x)$. It should be noted that for the parameter y , $P(y | x)$ is called the posterior density; since it is the density after (posterior to) the use of the available data x . $P(x | y)$ is called the likelihood. It is the probability of the experimental data given the probability model. $P(y)$ is called the prior density. It describes all your prior knowledge about the parameter y .

1.5 PROBABILITY DISTRIBUTIONS AND RANDOM VARIABLES

When an experiment is repeated many times under identical conditions, the results of the measurement will not necessarily be identical. In fact, as a rule rather than as an exception, the results will be different. Therefore, it is very desirable to know if there is a law that governs the individual outcomes of the experiment. Such a law, if it exists and it is known, would be helpful in two ways. First, from a small number of measurements, the experimenter may obtain information about expected results of subsequent measurements. Second, a series of measurements may be checked for faults. If it is known that the results of an experiment obey a certain law and a given series of outcomes of such an experiment does not follow that law, then that series of outcomes is suspect and should be thoroughly investigated before it is accepted.

A quantity x that can be determined quantitatively and that, in successive but similar experiments, can assume different values is called a random variable. There are two types of random variables, discrete and continuous.

The discrete random variable takes a set of discrete values. One may define a function

$f(x_i)$ = probability that the value of the random
variable is x_i
 $i=1,2,\dots,N$

Where N = number of possible (discrete) values of x . Since x takes only one value at a time, the events represented by the probabilities $f(x_i)$ are mutually exclusive; therefore,

$$\sum_{i=1}^N f(x_i) = 1$$

A continuous random variable can take any value within a certain interval $[a, b]$ ($a \leq x \leq b$). Then,

$f(x) dx$ = probability that the value of x lies between x and $x+dx$.

One should notice this difference in interpretation for continuous variables. The normalization of $f(x)$ is expressed by:

$$\int_a^b f(x) dx = 1$$

The function $f(x_i)$ or $f(x)$ is called the probability density function or abbreviated (*pdf*). A great deal of information can be inferred about a

random variable x if its probability distribution function is known [Tsoulfanidis, 1981].

1.6 STATISTICAL MODELS

The outcome of experiments is a distribution of results. The results can be thought of as a distribution for a random variable x . There are many models for the probability density function (*pdf*) for a random variable x . Here we are interested in the following models only.

1.6.1 THE BINOMIAL DISTRIBUTION

The binomial distribution is a discrete probability distribution function that applies under the following conditions:

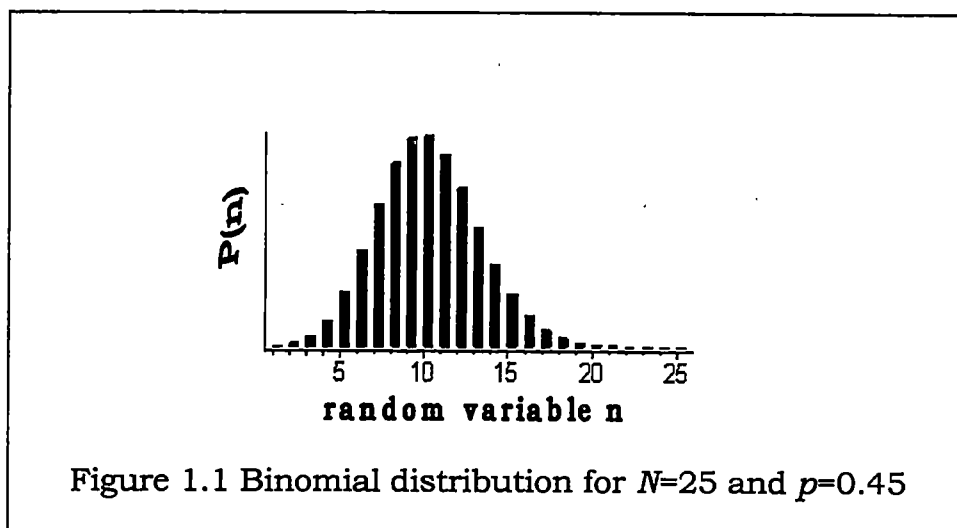
1. The experiment consist of N trials (e.g., N atoms each having a chance to decay)
2. The experiment result has two possible outcomes: success or failure (e.g. in the case of radioactive decay it's decaying or not decaying).
3. The occurrence of an event in any given observation does not affect the probability that the event will occur again in subsequent observations. The trials are independent.

4. The probability of the outcome is constant and independent of the number of observations [Turner, 1989].

If $p =$ is the probability of success, then $q = 1-p =$ is the probability of failure. If N is the number of trials for which each trial has a success probability p , then the probability of counting n successes is given by:

$$P_n = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \quad (1.5.1.1)$$

Figure 1.1 is a binomial distribution with $N = 25$, $p = 0.45$.



It is easy to show that the binomial distribution satisfies:

$$\sum_{n=0}^N P_n = \sum_{n=0}^N \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} = 1$$

The mean $E(n|p)$ is equal to:

$$E(n|p) = \sum_{n=0}^N n P_n = pN \quad (1.5.1.3)$$

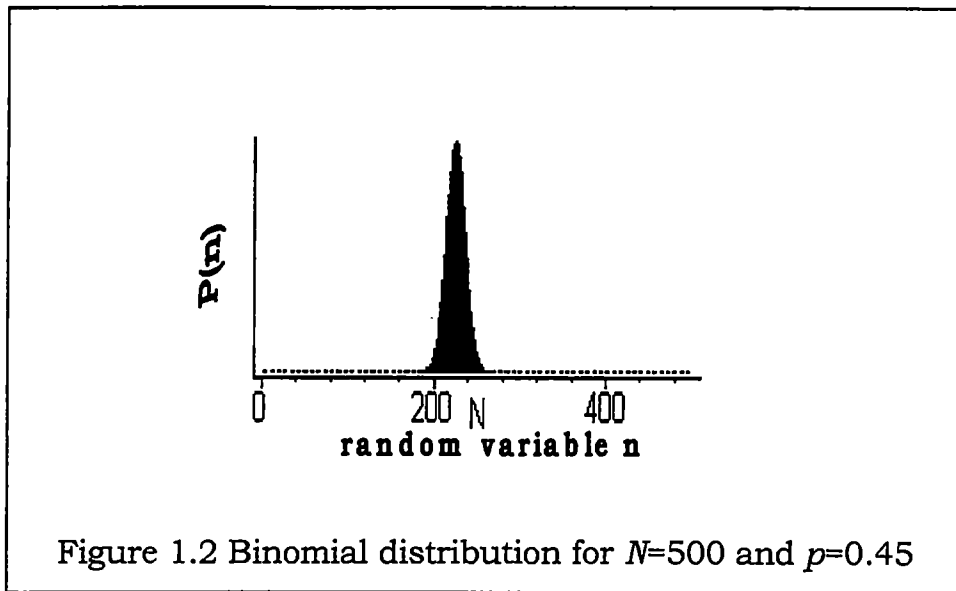
The variance is:

$$V(n|p) = \overline{(E(n|p) - n)^2} = \sum_{n=0}^N (E(n|p) - n)^2 P_n = pN(1-p) \quad (1.5.1.4)$$

The standard deviation:

$$\sigma = \sqrt{V(n|p)} = \sqrt{pN(1-p)} \quad (1.5.1.5)$$

It should be noted that the binomial distribution could be represented as a continuous distribution for large N . In this case the probability for n and $n-1$ is so close that the resulting histogram will look like a continuous plot. This is illustrated in the figure 1.2.



1.6.2 THE POISSON DISTRIBUTION

The Poisson distribution is a discrete distribution that applies to random occurring events with probability of occurrence that is small or constant. It can be derived from the binomial distribution by letting $N \rightarrow \infty$ and $p \rightarrow 0$ in such a way that the average $\mu = Np$ stays constant [Turner, 1989]. The mathematical formula for a Poisson distribution is:

$$P_n = \frac{\mu^n}{n!} e^{-\mu} \quad (1.5.2.1)$$

P_n is the probability of observing the outcome n when the mean is μ .

The Poisson distribution satisfies:

$$\sum_{n=0}^{\infty} p_n = 1$$

The mean is equal to:

$$E(n | \mu) = \mu = \sum_{n=0}^{\infty} n P_n \quad (1.5.2.2)$$

The variance is:

$$V(n | \mu) = \overline{(E(n | \mu) - n)^2} = \sum_{n=0}^{\infty} (E(n | \mu) - n)^2 P_n = \mu \quad (1.5.2.3)$$

The standard deviation is:

$$\sigma = \sqrt{V(n | \mu)} = \sqrt{\mu} \quad (1.5.2.4)$$

1.6.3 THE NORMAL (GAUSSIAN) DISTRIBUTION

Both Binomial and Poisson distributions apply to discrete variables whereas other random variables involved in experiments are continuous. Sometimes it is also convenient to approximate a discrete distribution by a continuous distribution. It would be desirable, therefore, to have a *pdfs* that apply to continuous variables [Tsoulfanidis, 1981]. One such a distribution is the normal or Gaussian distribution. The normal distribution $G(x|\mu, \sigma)$ is given by:

$$G(x|\mu, \sigma)dx = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx \quad (1.5.3.1)$$

$G(x|\mu, \sigma) dx$ = probability that the value of x lies between x and $x+dx$,

μ = mean of the distribution,

σ^2 = variance of the distribution.

In general, the probability of finding the value of x between any two limits x_1 and x_2 is given by:

$$G(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} G(x|\mu, \sigma)dx \quad (1.5.3.2)$$

The mean or expectation of the distribution is:

$$E(x | \mu, \sigma) = \int_{-\infty}^{\infty} xG(x)dx \quad (1.5.3.3)$$

The variance is:

$$V(x | \mu, \sigma) = \sigma^2 = \int_{-\infty}^{\infty} (x - E(x | \mu, \sigma))^2 G(x | \mu, \sigma) dx \quad (1.5.3.4)$$

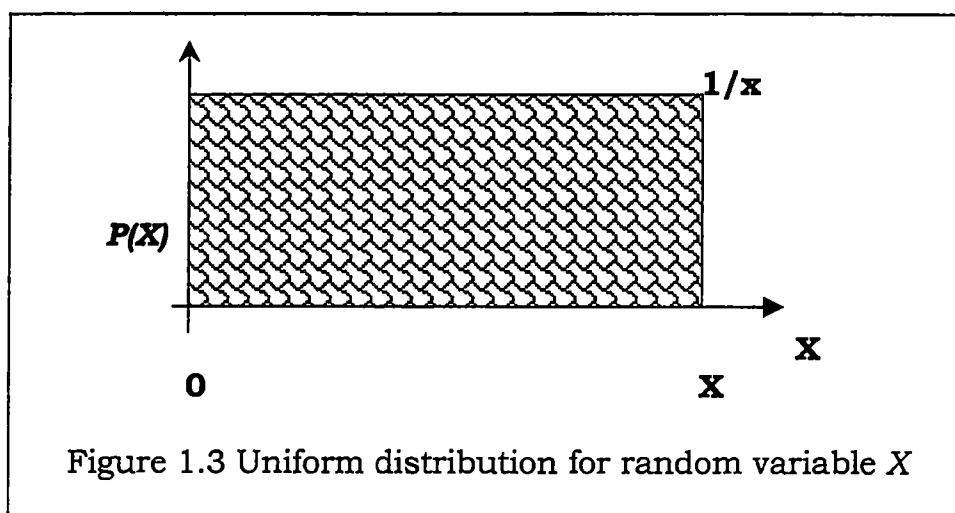
The standard deviation is:

$$\sigma = \sqrt{V(x | \mu, \sigma)} \quad (1.5.3.5)$$

1.6.4 UNIFORM DISTRIBUTIONS

The uniform distribution has special properties. Describing a random variable using a uniform distribution implies equal probability for all the possible values for the random variable. The plot of the random variable versus its probability is a rectangle. Since this is a probability density, the total area under the curve is equal to one.

Thus, the length of the rectangle should be equal to the reciprocal of its height, illustrated in the Figure 1.3.



This distribution is used when the analyst does not have much information about the parameter under investigation.

CHAPTER 2

UNCERTAINTY ANALYSIS OF THE DETECTION PROBLEM

Quantifying the amount of radioactivity in a sample requires a counting experiment. The experimenter places the sample under a calibrated counter with a known efficiency and obtains counts in a pre-set time interval. The sample activity is then estimated by:

$$A = \frac{\Delta}{\epsilon}$$

where A represent the activity in the sample, Δ is the net counts obtained from the counting experiment and ϵ is the efficiency of the counter for the particular radiation emitted by the sample [NCRP # 58, 1978].

If the above nuclear counting experiment is repeated many times under the same conditions, a distribution of values results. This is due to the statistical nature of the decay process. As mentioned before probability is the best way to deal with the statistical fluctuations. This is done by assigning a statistical distribution for each step of the analysis. The goal is to describe all the uncertainties in the experiment.

Assignment of these probability distributions is done by studying the statistical nature of each step. These probability distributions for each quantity are then used in place of the quantities in the mathematical formulae. The result is a probability density that describes the remaining uncertainty about the unknown activity. This density gives an estimate for the number of atoms in a sample.

In order to reach our goal we will look at each step of the analysis separately. These steps are:

1. Radioactive decay process.
2. Nuclear counting process using a detector with efficiency ϵ .
3. Calibration using standard source.
4. Background radiation.

A study of each of these steps follows. The mathematical representation and formulation are given in the next chapter. We will begin our study by stating the assumptions we are making through out this work.

2.1. ASSUMPTIONS

Our analysis assumes that the following conditions are true:

1. The radioactive decay process is a spontaneous and random process.

2. The counting time is fixed and there is no uncertainty about the measurement of time. I am assuming that we are able to measure time with high precision. Thus time and half-life are the only certain quantities in our analysis.
3. The radionuclide under observation has a large half-life. Therefore, only an insignificant amount of it decays while under observation.
4. The sample is counted for a specified length of time that is short in comparison to the nuclide's half-life.
5. All nuclei under observations are identical and their decay is independent of each other.
6. We have a large number of nuclei under observation.
7. The background counting time and sample counting time is not equal.

2.2. UNCERTAINTY ANALYSIS OF THE DETECTION PROBLEM

2.2.1. RADIOACTIVE DECAY PROCESS

Radioactive decay is a truly random process. If there is a certain amount of a radioisotope at hand, there is no guarantee that in the next

unit of time “ so many nuclei will decay” or “none will decay”. One can talk only about the probability that a nucleus will decay in a certain period of time [Tsoulfanidis, 1981]. The rate of decay of a nucleus per unit time is the decay constant. It is denoted by λ . The so called radioactive decay law is:

$$N(t) = N_0 e^{-\lambda t} \quad (2.2.1.1)$$

Where $e^{-\lambda t}$ can be interpreted as the probability that an atom survives time t without disintegrating:

$$q = \text{probability of survival} = e^{-\lambda t} . \quad (2.2.1.2)$$

It follows that:

$$p = \text{probability of decay} = 1 - q = 1 - e^{-\lambda t} \quad (2.2.1.1)$$

The probability of decay is constant from a trial to trial and the decay “event” for one nucleus is independent of the decay event of another nucleus. The binomial distribution applies under these conditions. Therefore, we can model the radioactive decay using a

binomial distribution. The mathematical description of this process is done in the next chapter.

Activity is the number of atoms that decay per unit time. Activity is sometimes also referred to as the source strength. For $N(t)$ nuclei present at time t , each having decay constant λ , the expected number of nuclei decaying per unit time is:

$$A(t) = \lambda N(t) \quad (2.2.1.4)$$

[Cember, 1996] [Turner,1989].

2.2.2. THE COUNTING PROCESS

Similarly, one can talk about the probability of a detector registering a count when a decay happens in a certain period of time or the probability of not registering a count in that same time interval. For a detector with efficiency ϵ the probability of observing a count would be given by:

$$p^* = \text{probability of detection} = \epsilon \quad (2.2.2.1)$$

and the probability of not observing a count would be given by:

$$q^* = \text{probability of not detecting} = 1 - \epsilon \quad (2.2.2.2)$$

Observing counts with a given detector is again a binary process. The probability of detection is constant from trial to trial and the detection of a decay event is independent of other decay events. Again these are the conditions under which the binomial distribution applies. Therefore, we can model the detection process using the binomial distribution [Turner, 1989]. The mathematical description of this process is done in the next chapter.

2.2.3. INSTRUMENT CALIBRATION USING A STANDARD SOURCE

Detectors are calibrated using a standard source with known activity to estimate the detection efficiency ϵ . This will introduce uncertainty into the activity estimation since the detector efficiency ϵ is not precisely known. In order to describe this uncertainty we use a statistical model for the strength of the standard source. We can accomplish this by careful consideration of the information provided with the standard source.

Every standard source is sold with a calibration certificate. In this certificate the source manufacturer commonly reports the source activity in the form of a mean measurement $\pm$ a standard deviation. The standard deviation is a statement regarding the uncertainty of the source activity. It is reasonable to assume that the activity of the standard source is normally distributed. Therefore, the standard source strength can be modeled using a normal distribution with its mean and standard deviation taken from the manufacturer's calibration statement.

2.2.4. BACKGROUND RADIATION

Background radiation needs to be considered since it plays a role in every radiation measurement. Background radiation has also statistical fluctuations and introduces additional uncertainty. The counts from a sample registered in a detector are due to background radiation and sample disintegrations. The counts due to the sample will have a distribution that will be derived [little, 1982]. The counts due to background radiation can be modeled using Poisson distribution. The reasoning for this modeling will be supported with a mathematical derivation. This derivation is forth coming in the next chapter.

2.3. SUMMARY OF UNCERTAINTY ANALYSIS RESULTS

The following are the result of the uncertainty analysis study in this chapter:

1. Radioactive decay is well described with a binomial distribution. The model describes the number of atoms n decaying given the initial number of atoms N .
2. The detection process is also modeled with a binomial distribution. In this model the fraction of the n atoms that actually decayed and will be seen by a detector with efficiency ϵ is m .
3. The uncertainty about the detector efficiency ϵ is derived from a careful study of the calibration process. When a detector is calibrated we estimate its efficiency ϵ . Since this estimation is done using a standard source, we first modeled the standard source using a normal distribution. This modeling is consistent with the manufacturer's reporting about the source. Then this result and result from previous parts are used. The result is an expression describing the detector absolute efficiency ϵ .
4. Background radiation is considered. The background count rate and gross count rate will be modeled using a Poisson distribution. From this modeling, we derive an expression describing net count rate. So

when we are given a gross count rate we can describe the uncertainty due to background radiation in the net count rate.

5. The results from step 3 and 4 will be combined and an expression to estimate the activity in a sample, therefore the number of atoms, given a detector reading will be derived. This final step will be explained in detail in chapter 4.

CHAPTER 3

STATISTICAL MODELING OF A COUNTING EXPERIMENT

3.1. STATISTICAL MODELING OF THE RADIOACTIVE DECAY PROCESS

A measurement can be defined as counting the number of successes resulting from a given number of trials. Each trial is assumed to be a binary process with two possible results. The radioactive decay is a binary process and a trial consists of observing a radioactive nucleus for a period of time t . The number of trials is equivalent to the number of nuclei in the sample under observation N , and the measurement consists of counting those nuclei n that undergo decay. [Knoll, 1989].

Earlier the probability of a radionuclide decaying (defined to be the probability of success) was shown equal to $(1 - e^{-\lambda t})$. Therefore, the probability of not decaying is $e^{-\lambda t}$. λ is the decay constant of the radioactive nuclide in the sample. We have N nuclei that have the same probability of decay, the decay of one nuclei is independent of the decay of any other nuclei. The probability of success is constant for all trials.

These are the conditions under which the binomial model applies. Therefore, the binomial distribution is the best model to describe this binary event. Then for a sample containing N atoms the probability distribution for the number of atoms n decayed is:

$$p(n | (1 - e^{-\lambda t}), N) = \binom{N}{n} (1 - e^{-\lambda t})^n (e^{-\lambda t})^{N-n} \quad (3.1.1)$$

This result is interpreted as the conditional probability of n atoms decaying given that the initial sample contains N atoms. Each atom in the sample have a constant decay probability during time t equal to $(1 - e^{-\lambda t})$ [Turner, 1989]. It should be noted that $n \leq N$.

3.2. STATISTICAL MODELING OF THE NUCLEAR COUNTING PROCESS

Radioactive decay usually takes the nuclide from an excited state to a more stable state. The energy released in this process is carried out by the particles emitted. These particles have the ability to interact with matter; therefore, releasing the energy they carry. If this interaction occurs within the active volume of a detector, electrical charges are created. Collecting these charges produces an electrical pulse. This

pulse is registered as a count using electronic circuitry. The accumulation of counts over time is assumed to be proportional to the number of particles emitted from the decaying nuclide. A detector does not register all the particles emitted from the decaying nuclide. The ratio of the fraction of particles seen by a detector to the total number of particles emitted from the decaying nuclide defines the detector efficiency. Detector efficiency is denoted by the symbol ϵ . Detector efficiency ϵ is interpreted as the probability to register a count. The probability of not registering a count is simply $1-\epsilon$. Detection of one particle is independent of detection of another, and the probability of detection is assumed constant from trial to trial. These are the conditions under which the binomial distribution applies. Therefore, the binomial distribution is used to describe this process. The probability distribution that describes the number of registered counts m in a detector given n decayed atoms is:

$$p(m|\epsilon, n) = \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} ; \quad m \leq n \quad (3.2.1)$$

Using the concept of a compound process, the probability that m counts are registered is:

$$p(m|\epsilon, N, \lambda) = \sum_{n=m}^N p(m|\epsilon, n) p(n|(1-e^{-\lambda t}), N) \quad (3.2.2)$$

Substituting expression (3.1.1) and (3.2.1) in (3.2.2), we get

$$p(m|\epsilon, N, \lambda) = \sum_{n=m}^N \left(\binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \right) \left(\binom{N}{n} (1-e^{-\lambda t})^n (e^{-\lambda t})^{N-n} \right) \quad (3.2.3)$$

Expression (3.2.3) can be rewritten under the following assumptions:

1. When the nuclide under observation has a large half-life in comparison to the counting time ($\lambda t \ll 1$), the term $e^{-\lambda t}$ can be approximated as $(1 - \lambda t)$.

2. In the case of long-lived nuclide $N \gg n$. Then $\binom{N}{n}$ can be simplified

to be $N^n / n!$. This is shown below:

$$\begin{aligned} \binom{N}{n} &= \frac{N!}{n!(N-n)!} = \frac{N(N-1)(N-2) \dots (N-n+1)(N-n)!}{n!(N-n)!} \\ &= \frac{N(N-1)(N-2) \dots (N-n+1)}{n!} \cong \frac{N^n}{n!} \end{aligned}$$

Substituting the two approximations back into formula (3.2.3)

$$p(m|\epsilon, N, \lambda) = \sum_{n=m}^N \left(\binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \right) \left(\frac{N^n}{n!} (1-(1-\lambda t))^n (e^{-\lambda t})^N \right)$$

$$= \sum_{n=m}^N \left(\binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \right) \left(\frac{N^n}{n!} (\lambda t)^n e^{-\lambda t N} \right)$$

$$= \sum_{n=m}^N \left(\binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \right) \left(\frac{(N\lambda t)^n}{n!} e^{-\lambda t N} \right)$$

Further simplification can be accomplished by letting $B = N \lambda t$; and

$q = 1 - \epsilon$ and substituting back in the last expression. The result is the following expression:

$$\begin{aligned} p(m|\epsilon, N, \lambda) &= \frac{\epsilon^m e^{-B}}{q^m} \sum_{n=m}^N \binom{n}{m} q^n \frac{B^n}{n!} \\ &= \frac{\epsilon^m e^{-B}}{m! q^m} \sum_{n=m}^N \frac{(qB)^n}{(n-m)!} \end{aligned} \tag{3.2.4}$$

Utilize the power series for e^x [Fogiel, 1980]:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We can further simplify expression (3.2.4). First, we let $j = n - m$; solving for n we get $n = j + m$ and substituting back in (3.2.4) we get:

$$\begin{aligned} p(m|\epsilon, N, \lambda) &= \frac{\epsilon^m e^{-B}}{m! q^m} \sum_{j=0}^N \frac{(qB)^{j+m}}{j!} \\ &= \frac{\epsilon^m e^{-B} (qB)^m}{m! q^m} \sum_{j=0}^N \frac{(qB)^j}{j!} \end{aligned}$$

Using the power series identity described before, this last expression will become:

$$\begin{aligned} p(m|\epsilon, N, \lambda) &= \frac{\epsilon^m e^{-B} (qB)^m e^{qB}}{m! q^m} \\ &= \frac{\epsilon^m B^m e^{B(q-1)}}{m!} \end{aligned}$$

Substituting back $B = N \lambda t$; and $q = 1 - \epsilon$ in the above result we get the following important expression:

$$p(m | \epsilon, N, \lambda) = \frac{(\epsilon N \lambda t)^m e^{-\epsilon N \lambda t}}{m!} \quad (3.2.5)$$

This formula is interpreted as the conditional probability that m counts will register in a detector given that the detector efficiency is ϵ and the initial number of atoms in the sample is N . Note that only m decays out of the n actual decays from a sample containing N atoms initially registering in the detector. The derived probability distribution for the detected decays is a Poisson distribution. The mean is given by $\epsilon N \lambda t$.

3.3 STATISTICAL MODELING OF THE CALIBRATION PROCESS

The main objective in this part of the analysis is to describe the uncertainty due to the detector efficiency. This objective is accomplished in several steps. Since calibration is performed using a standard source, we start by describing the uncertainty due to the radioactive decay of the standard source, and then we describe the uncertainty due to the detection process. Bayes' theorem is applied, and we derive a posterior density for the absolute detector efficiency.

3.3.1. MODELING RADIOACTIVE DECAY OF THE STANDARD SOURCE

For a standard source containing N_s atoms each having probability of decay equal to $(1 - e^{-\lambda_s t})$, the probability distribution that describes the number of atoms decaying n_s is:

$$p(n_s | (1 - e^{-\lambda_s t}), N_s) = \binom{N_s}{n_s} (1 - e^{-\lambda_s t})^{n_s} (e^{-\lambda_s t})^{N_s - n_s} \quad (3.3.1)$$

It should be noted that $n_s \leq N_s$.

3.3.2. MODELING THE DETECTION PROCESS USING A STANDARD SOURCE

In section 3.2 we deduced expression (3.2.1) that describes the statistical model for the detector's ability to register counts. In the same manner for a standard source with N_s initial atoms, one can write an expression for the detection process using a detector with efficiency ϵ . The expression will describe the probability distribution that will result from a detector with efficiency ϵ detecting only m_s atoms out of the n_s

atoms. This probability distribution is binomial and is mathematically represented as follows:

$$p(m_s | \epsilon, n_s) = \binom{n_s}{m_s} \epsilon^{m_s} (1 - \epsilon)^{n_s - m_s} ; \quad m_s \leq n_s \quad (3.3.2.1)$$

Using the concept of a compound process, the probability m_s counts registered is:

$$p(m_s | \epsilon, N_s, \lambda_s) = \sum_{n_s=m_s}^{N_s} p(m_s | \epsilon, n_s) p(n_s | (1 - e^{-\lambda_s t}), N_s) \quad (3.3.2.2)$$

Substituting expression (3.3.1) and (3.3.2.1) in (3.3.2.2), we get:

$$p(m_s | \epsilon, N_s, \lambda_s) = \sum_{n_s=m_s}^{N_s} \left(\binom{n_s}{m_s} \epsilon^{m_s} (1 - \epsilon)^{n_s - m_s} \right) \left(\binom{N_s}{n_s} (1 - e^{-\lambda_s t})^{n_s} (e^{-\lambda_s t})^{N_s - n_s} \right) \quad (3.3.2.3)$$

Expression (3.3.2.3) can be rewritten using the following:

1. For a nuclide with long lived half life in comparison to the counting time ($\lambda_s t \ll 1$); the term $e^{-\lambda_s t}$ can be approximated as $(1 - \lambda_s t)$.

2. Also for the case of long-lived nuclide $N_s \gg n_s$. The $\binom{N_s}{n_s}$ is easily

simplified to be $N_s^{n_s} / n_s!$. This is shown below:

$$\begin{aligned} \binom{N_s}{n_s} &= \frac{N_s!}{n_s!(N_s - n_s)!} = \frac{N_s(N_s - 1)(N_s - 2) \dots (N_s - n_s + 1)(N_s - n_s)!}{n_s!(N_s - n_s)!} \\ &= \frac{N_s(N_s - 1)(N_s - 2) \dots (N_s - n_s + 1)}{n_s!} = \frac{N_s^{n_s}}{n_s!} \end{aligned}$$

Substituting back these results in formula (3.3.2.3)

$$\begin{aligned} p(m_s | \epsilon, N_s, \lambda_s) &= \sum_{n_s=m_s}^{N_s} \left(\binom{n_s}{m_s} \epsilon^{m_s} (1-\epsilon)^{n_s-m_s} \right) \left(\frac{N_s^{n_s}}{n_s!} (1-(1-\lambda_s t))^{n_s} (e^{-\lambda_s t})^{N_s} \right) \\ &= \sum_{n_s=m_s}^{N_s} \left(\binom{n_s}{m_s} \epsilon^{m_s} (1-\epsilon)^{n_s-m_s} \right) \left(\frac{N_s^{n_s}}{n_s!} (\lambda_s t)^{n_s} e^{-\lambda_s t N_s} \right) \\ &= \sum_{n_s=m_s}^{N_s} \left(\binom{n_s}{m_s} \epsilon^{m_s} (1-\epsilon)^{n_s-m_s} \right) \left(\frac{(N_s \lambda_s t)^{n_s}}{n_s!} e^{-\lambda_s t N_s} \right) \end{aligned}$$

Further simplification can be accomplished by letting $B = N_s \lambda_s t$; and

$q = 1 - \epsilon$ and substituting back in the last expression to yield the following:

$$\begin{aligned}
 p(m_s | \epsilon, N_s, \lambda_s) &= \frac{\epsilon^{m_s} e^{-B}}{q^{m_s}} \sum_{n_s=m_s}^{N_s} \binom{n_s}{m_s} q^{n_s} \frac{B^{n_s}}{n_s!} \\
 &= \frac{\epsilon^{m_s} e^{-B}}{m_s! q^{m_s}} \sum_{n_s=m}^{N_s} \frac{(qB)^{n_s}}{(n_s - m_s)!}
 \end{aligned} \tag{3.3.2.4}$$

Notice that we can utilize the power series identity $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to further simplify expression (3.3.2.4). First we let $j = n_s - m_s$, solving for n_s we get $n_s = j + m_s$ and substituting back in (3.3.2.4) we get :

$$\begin{aligned}
 p(m_s | \epsilon, N_s, \lambda_s) &= \frac{\epsilon^{m_s} e^{-B}}{m_s! q^{m_s}} \sum_{j=0}^{N_s} \frac{(qB)^{j+m_s}}{j!} \\
 &= \frac{\epsilon^{m_s} e^{-B} (qB)^{m_s}}{m_s! q^{m_s}} \sum_{j=0}^{N_s} \frac{(qB)^j}{j!}
 \end{aligned} \tag{3.3.2.5}$$

Using the power series identity described before our expression becomes:

$$p(m_s | \epsilon, N_s, \lambda_s) = \frac{\epsilon^{m_s} e^{-B} (qB)^{m_s} e^{qB}}{m_s! q^{m_s}} = \frac{\epsilon^{m_s} B^{m_s} e^{B(q-1)}}{m_s!} \quad (3.3.2.6)$$

Substituting back in expression (3.2.6) $B = N_s \lambda_s t$; and $q = 1 - \epsilon$ we get the following important expression:

$$p(m_s | \epsilon, N_s, \lambda_s) = \frac{(\epsilon N_s \lambda_s t)^{m_s} e^{-\epsilon N_s \lambda_s t}}{m_s!} \quad (3.3.2.7)$$

m_s are the counts registered in a detector with efficiency ϵ when a standard source with decay constant equal to λ_s and containing N_s atoms observed for a period equal t . Expression (3.3.2.7) is the likelihood for the data m_s .

Counts are converted into activity via the detection efficiency ϵ . We assume that the radioactive source strength is normally distributed with the given mean and standard deviation. We denote this mean by μ_s and the standard deviation by σ_s . we can write the probability distribution that will describe the uncertainty about the standard source activity A_s , as follows:

$$p(A_s | \mu_s, \sigma_s) = \frac{1}{\sqrt{2\pi}\sigma_s} e^{-\left(\frac{A_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2} \quad (3.3.2.8)$$

The distribution for the number of atoms in standard source is obtained from expression (3.3.2.8) using a transformation of variable as follows:

For functions of random variables of the form

$$N_s = \frac{1}{\lambda_s} A_s$$

For $1/\lambda_s > 0$ the distribution of the random variable N_s is given by:

$$p(N_s) = \lambda_s p_A(\lambda_s N_s | \mu_s, \sigma_s)$$

Or simply:

$$p(N_s | \mu_s, \sigma_s) = \frac{\lambda_s}{\sqrt{2\pi}\sigma_s} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2} \quad (3.3.2.9)$$

3.3.3.MODELING THE UNCERTAINTY ABOUT THE DETECTION EFFICIENCY ϵ

In the previous section, we concluded that expression (3.3.2) is the likelihood for the data m_s under the condition that the detection efficiency ϵ and the initial standard source strength N_s are known. We can use this fact and Bayes' theorem to derive the posterior distribution for the detection efficiency. We use the proportional form of Bayes' theorem. The result of this step is shown below:

$$p(\epsilon|m_s, N_s, \lambda_s) \propto p(m_s|\epsilon, N_s, \lambda_s) p(\epsilon) \quad (3.3.3)$$

The left side of the proportionality is the posterior density for the detection efficiency ϵ given m_s and N_s . The right side of the proportionality, defined above, is the likelihood times the prior density for the detector efficiency. We find the prior density for the detection efficiency $p(\epsilon)$ by studying the detection process. This is explored in the next section.

3.3.4.MODELING OF THE PRIOR DENSITY FOR THE DETECTION EFFICIENCY

When counting a standard source, the measured rate r is related to the standard source strength S through factors representing the effects of the experimental setup.

$$r = f_1 f_2 f_3 \cdots f_n S \quad (3.3.4.1)$$

These factors may be grouped into three categories [Tsoulfanidis, 1981]:

1. Geometry effects. This term refers to the size and shape of the source, the size and shape of the detector aperture, and the distance between the source and the detector.
2. Source effects. The size and the way the source is made have an effect on the measurement.
3. Detector effects. The detector may affect the measurement in two ways. First, the size and thickness of the detector window determine how many particles enter the detector and how much energy they lose as they traverse the window. Second, particles entering the detector will not necessarily be counted. The fraction of particles that is recorded depends on the efficiency of the detector.

It is convenient to subdivide counting efficiencies into two classes: absolute and intrinsic [Knoll, 1989]. Absolute efficiencies are defined as:

$$\epsilon_{abs} = \frac{\text{number of pulses recorded}}{\text{number of radiation quanta emitted by source}} \quad (3.3.4.2)$$

Absolute efficiencies are dependent on detector properties and the details of the counting geometry (primarily the distance from the source to the detector). The intrinsic efficiency is defined as:

$$\epsilon_{int} = \frac{\text{number of pulses recorded}}{\text{number of radiation quanta incident on detector}} \quad (3.3.4.3)$$

intrinsic efficiency no longer includes the solid angle subtended by the detector as an implicit factor. Intrinsic efficiency depends on the following [Tsoulfanidis, 1981]:

1. Density and size of detector material. The probability of interaction per unit distance traveled is proportional to the density of the material.
2. Type and energy of radiation. For charged particles, the detector efficiency is practically independent of particle energy except for very low energies, when the particles may be stopped by the detector

window. Photons and neutrons traversing a medium show exponential attenuation, which means that there is always a nonzero probability for a photon or a neutron to traverse any thickness of material without interaction. Because of this property, detectors for photons or neutrons have efficiency less than 100 percent regardless of detector size and energy of the particle.

3. Electronics. The effect of electronics on the counter efficiency is indirect. When a particle interacts in the detector and produces a signal, that particle will be recorded only if the signal is recorded. The signal will be registered only if it is higher than the discriminator level that is determined by the electronic noise of the counting system. Thus, the counting efficiency may increase if the level of electronic noise is decreased.

The two efficiencies are simply related for isotropic sources by

$$\epsilon_{\text{abs}} = \epsilon_{\text{int}} \left(\frac{\Omega}{4\pi} \right) \quad (3.3.4.4)$$

Where Ω is the solid angle of the detector seen from the actual source position [Knoll, 1989]. The intrinsic efficiency is assumed to remain constant since the experimental setup will not change during counting.

Therefore, from equation (3.3.4.4) the absolute efficiency will depend only on the solid angle.

In the general case of an extended source, the solid angle Ω is defined by:

$$\Omega = \frac{\text{number of particles/ sec emitted inside the space defined by the contours of the source and the detector aperture}}{\text{number of particles /sec emitted by the source}} \quad (3.3.4.5)$$

The portion of particles emitted by a source entering a detector is estimated by the fractional solid angle subtended by the detector at the location of the source. The expression that is used to calculate the solid angle is given by [Tsoulfanidis, 1981]:

$$\Omega = \frac{1}{4\pi A_s} \int_{A_s} dA_s \int_{A_d} dA_d \frac{\cos \omega}{r^2} \quad (3.3.4.6)$$

Where A_s is the area of a plane source, A_d is the detector area and $\cos \omega = \frac{\hat{n} \cdot \mathbf{r}}{r}$, where $\hat{n}$ is a unit vector normal to the surface of the detector aperture. Since Ω is equal to the fractional solid angle, Ω takes values from zero to one ($0 \leq \Omega \leq 1$). The most frequently encountered case of obtaining a solid angle is that of an isotropic point source at a certain

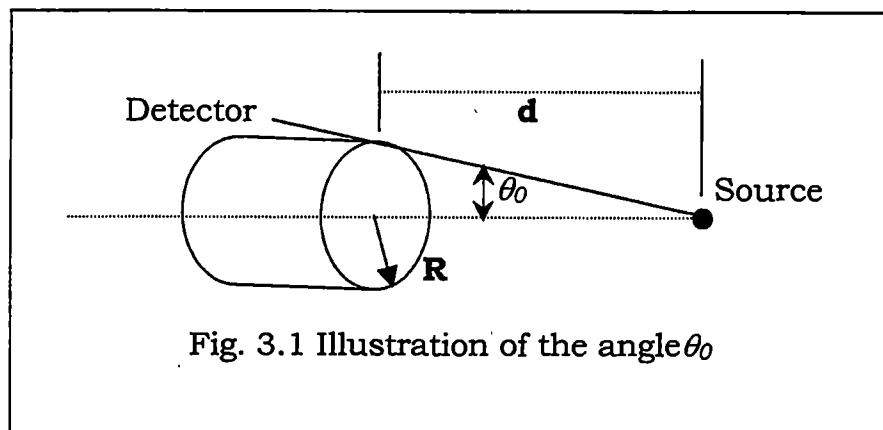
distance away from a detector with a circular aperture. For this situation the general formula reduces to [Tsoulfanidis, 1981]:

$$\Omega = \frac{1}{2}(1 - \cos \theta_0) \quad (3.3.4.7)$$

Where $\cos \theta_0$ is given by:

$$\cos \theta_0 = \frac{d}{\sqrt{d^2 + R^2}} \quad (3.3.4.8)$$

Illustration of this angle is shown in Figure 3.1:



If the radioactive source is on the detector surface θ_0 is 90° . This means that the solid angle is $\frac{1}{2}$ for this case.

In view of this result and to simplify our analysis, we assume that the efficiency has the uniform prior distribution shown in figure 3.2.

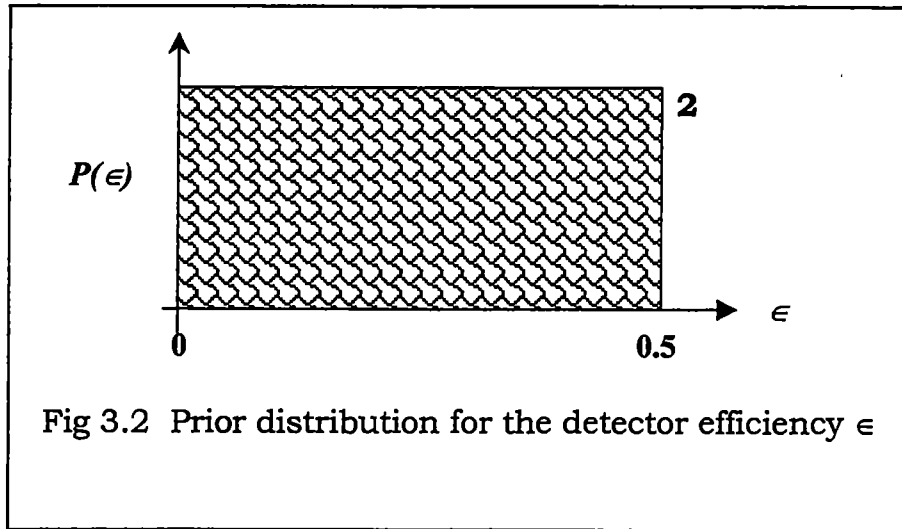


Fig 3.2 Prior distribution for the detector efficiency ϵ

Since the constant can be dropped the posterior distribution for the detector efficiency reduces to:

$$p(\epsilon | m_s, N_s, \lambda_s) \propto p(m_s | \epsilon, N_s, \lambda_s) \quad (3.3.4.9)$$

Substituting expression (3.3.2.7), we get the posterior density for the detector efficiency given m_s and N_s in proportional form:

$$p(\in|m_s, N_s, \lambda_s) \propto \frac{(\in N_s \lambda_s t)^{m_s}}{m_s!} e^{-\in N_s \lambda_s t} \quad (3.3.4.10)$$

The normalized form for fixed m_s and N_s is obtained by finding the normalization constant c :

$$c = \int_0^{0.5} \frac{(\in N_s \lambda_s t)^{m_s}}{m_s!} e^{-\in N_s \lambda_s t} d \in \quad (3.3.4.10a)$$

Then:

$$p(\in|m_s, N_s, \lambda_s) = \frac{1}{c} \frac{(\in N_s \lambda_s t)^{m_s}}{m_s!} e^{-\in N_s \lambda_s t} \quad (3.3.4.11)$$

N_s in this formula is a discrete variable. We will approximate this discrete variable N_s by a continuous variable with a normal distribution of expression (3.3.1). Then the posterior density for the absolute detector efficiency is the result of integrating equation (3.3.4.11) over all possible values of N_s as shown below:

$$p(\epsilon|m_s, \mu_s, \sigma_s, \lambda_s) = \int_0^\infty \frac{\lambda_s (\epsilon N_s - \lambda_s t)^{m_s}}{c m_s! \sigma_s \sqrt{2\pi}} e^{-\epsilon N_s - \lambda_s t} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2} dN_s \quad (3.3.4.12)$$

So for a given standard source with reported mean μ_s , reported standard deviation σ_s , and detector counts m_s we can obtain a probability distribution that describes the uncertainty in the detector efficiency ϵ .

3.4. BACKGROUND RADIATION

Background radiation is inevitable. It is always present in every radiation measurement. In this part of the analysis, we will show how to deal with background radiation statistically.

Background radiation can be statistically described by a Poisson distribution. We begin by modeling background count rate and gross count rate using the Poisson distribution. Since every radiation measurement consists of background counts and sample counts, I will refer to the difference between sample counts and background counts as net counts. The gross count rate model is constructed using the fact that gross count rate is equal to background count rate plus net counting rate. A joint distribution for the background count rate and

net count rate is developed [little, 1982]. One can obtain an expression for the background count rate or the net count rate from the joint distribution by integration with respect to other quantity. Those steps are illustrated below.

3.4.1 MODELING OF THE BACKGROUND COUNTS

This analysis uses the earlier conclusion from equation (3.2.5). Detector counts are described using a Poisson model. Therefore, we can write

$$p(m | \mu) = \frac{\mu^m e^{-\mu}}{m!} \quad (3.4.1)$$

Where m represents the observed number of counts and μ represents the mean of the Poisson distribution.

Then the probability of observing b background counts for a background distribution with rate β_0 (Poisson mean = $\beta_0 t_0$) during time interval t_0 is given by:

$$p(b | \beta_0 t_0) = \frac{(\beta_0 t_0)^b e^{-\beta_0 t_0}}{b!} \quad (3.4.1)$$

3.4.2 MODELING OF THE GROSS COUNTS

Similarly for the gross count distribution with mean γt_1 , the probability of observing s gross count during time interval t_1 can be mathematically represented by:

$$p(s|\gamma, t_1) = \frac{(\gamma t_1)^s e^{-\gamma t_1}}{s!} \quad (3.4.2)$$

However, the gross counting rate is the sum of net and background counting rate. The background rate was defined as β_0 . If we define net counts rate as Δ , we can rewrite $\gamma = \beta_0 + \Delta$. Expression (3.4.2) becomes:

$$p(s|(\beta_0 + \Delta)t_1) = \frac{((\beta_0 + \Delta)t_1)^s e^{-(\beta_0 + \Delta)t_1}}{s!} \quad (3.4.3)$$

3.4.3 DERIVATION OF THE LIKELIHOOD AND THE POSTERIOR DENSITY FOR THE BACKGROUND COUNT RATE β_0 AND NET COUNT RATE Δ

Since the distribution for background counts is independent of the sample counts, their joint distribution will be:

$$p(b,s|\beta_0,\Delta) = p(b|t_0\beta_0) p(s|t_1(\beta_0 + \Delta)) \quad (3.4.3.1)$$

$$p(b,s|\beta_0,\Delta) = \frac{(\beta_0 t_0)^b e^{-\beta_0 t_0}}{b!} \frac{((\beta_0 + \Delta)t_1)^s e^{-(\beta_0 + \Delta)t_1}}{s!} \quad (3.4.3.2)$$

Since b and s are data,

$$p(b,s|\beta_0,\Delta) = \frac{(\beta_0 t_0)^b e^{-\beta_0 t_0}}{b!} \frac{((\beta_0 + \Delta)t_1)^s e^{-(\beta_0 + \Delta)t_1}}{s!} \quad (3.4.3.3)$$

Is the likelihood of the data. The proportional form for Bayes' theorem is applied to get the joint density for β_0 and Δ . If we assume a uniform prior density for β_0 and Δ , we get:

$$p(\beta_0, \Delta | D) \propto \frac{(\beta_0 t_0)^b e^{-\beta_0 t_0}}{b!} \frac{((\beta_0 + \Delta) t_1)^s e^{-(\beta_0 + \Delta) t_1}}{s!} \quad (3.4.3.4)$$

Because of the proportionality, we can drop the constants t_0^b , t_1^s , $b!$ and $s!$. Combining the exponents for the exponential function, we get the joint distribution for background rate β_0 and net rate Δ given the observed background and observed gross counts:

$$p(\beta_0, \Delta | D) \propto \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1 + t_0) + \Delta t_1)} \quad (3.4.3.5)$$

Dividing equation (3.4.3.5) by c_2 yields the normalized form where c_2 is:

$$c_2 = \int_0^\infty \int_0^\infty \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1 + t_0) + \Delta t_1)} d\beta_0 d\Delta$$

Therefore, the normalized form for the joint distribution for background rate β_0 and net rate Δ given the observed background and observed gross counts is as follows:

$$p(\beta_0, \Delta | D) = \frac{1}{c_2} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1 + t_0) + \Delta t_1)} \quad (3.4.3.6)$$

Note that we can get the probability distribution for background rate β_0 from expression (3.4.3.6) by integrating over Δ . In the same manner, we can get the probability distribution for net rate Δ from (3.4.3.6) expression by integrating over β_0 . The two formulas are shown below:

$$p(\beta_0 | D) = \frac{1}{c_2} \int_0^{\infty} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)} d\Delta$$

$$p(\Delta | D) = \frac{1}{c_2} \int_0^{\infty} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)} d\beta_0$$

CHAPTER 4

DERIVATION OF THE POSTERIOR DENSITIES FOR SAMPLE ACTIVITY AND NUMBER OF ATOMS

In a counting experiment, the sample is placed under the counter and observed for a period of time. Only some of the decays register as counts in the detector. To estimate the activity, the detector absolute efficiency ϵ is used. Mathematically, ϵ is the ratio between the number of counts registered in the counter to the actual decays in the sample. If the detector efficiency ϵ is known, we can estimate the activity A by:

$$A = \frac{\Delta}{\epsilon} \quad (4.1)$$

Δ in the above expression is the net count rate from the detector.

Earlier we discussed sources of uncertainty in the nuclear counting experiment. We developed statistical models that describe the uncertainty in various parts of the counting experiment. Now we use

these models to arrive at an expression for the posterior density of the sample activity.

The uncertainty in expression (4.1) due to Δ and ϵ needs to be described. We started by deriving a probability distribution for the efficiency. This was done in the following steps:

1. We first examined radioactive decay and modeled it using the binomial distribution. The result was equation (3.1.1). It gave us the probability distribution for the number of atoms n that will actually decay from N initial atoms with each atom having the same decay probability $(1 - e^{-\lambda t})$.
2. Next, we modeled the detection process. It was modeled again using a binomial distribution. The result was equation (3.2.1). This equation gave us the probability distribution for the number of atoms m registering a count in a detector with efficiency ϵ from the n atoms decayed in a sample containing N atoms.
3. The results of step one and two were combined to arrive at the likelihood. The result was equation (3.2.3). To simplify this equation, we used the facts that a long lived radioisotope will have the following effect:

A. We can approximate $e^{-\lambda t}$ as $1 - \lambda t$

B. We can approximate $\binom{N}{n}$ as N^n .

The result of these approximations was equation (3.2.5). This equation represents a Poisson distribution with mean equal to $\epsilon N \lambda t$.

$$p(m|\epsilon, N) = \frac{(\epsilon N \lambda t)^m e^{-\epsilon N \lambda t}}{m!}$$

4. In the next step, we looked at the calibration process using a standard source. The radioactive decay for the standard source was modeled. The detection process using this standard source was also modeled. Using the same rational as for step 3 we arrived at expression (3.3.2.7). This equation is the likelihood of the data m_s . Applying Bayes' theorem, the result was equation (3.3.3). To obtain the probability distribution for the absolute efficiency, we need a prior distribution for the efficiency. We assumed that the efficiency has the uniform prior density shown in figure 3.2. This led us to the proportional form of the posterior density describing the absolute efficiency, equation (3.3.4.10). The normalized form is equation (3.3.4.11).

Using the fact that a radioactive source manufacturer reports the source activity with error, the standard source activity was modeled using a normal distribution, we arrived at equation

(3.3.2.8). Applying transformation of variables, we arrive at equation (3.3.2.9). This distribution describes the number of atoms in the standard source, and approximates the parameter N_s using a continuous variable.

Equation (3.3.4.11) is dependent on N_s , and equation (3.3.2.9) describes the uncertainty about N_s . The compound distribution integrated over all N_s is the absolute efficiency model for a detector using standard source. The result is equation (3.3.4.12).

$$p(\epsilon | m_s, \mu_s, \sigma_s, \lambda_s) = \int_0^\infty \frac{\lambda_s (\epsilon N_s \lambda_s t)^{m_s}}{c m_s! \sigma_s \sqrt{2\pi}} e^{-\epsilon N_s \lambda_s t} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2} dN_s \quad (4.2)$$

Where,

$$c = \int_0^\infty \frac{(\epsilon N_s \lambda_s t)^{m_s}}{m_s!} e^{-\epsilon N_s \lambda_s t} d\epsilon$$

c needs to be evaluated numerically. Equation (4.2) is the posterior density for the absolute detector efficiency. This result shows that we can determine the efficiency of a detector from a single detector reading using a source standard with given mean

and standard deviation. In the next chapter, we will give numerical examples for illustration.

5. Looking at background radiation, we derived the likelihood for the background count rate and net count rate (equation (3.4.3.3)). The proportional form of the joint density describing the background count rate and the net count rate is equation (3.4.3.4). It was derived by applying the proportional form of Bayes' theorem, assuming that background count rate and net count rate have a uniform prior density. Equation (3.4.3.4) is rewritten as equation (3.4.3.5) by dropping all the constants. The normalized form of equation (3.4.3.5) is equation (3.4.3.6). This equation was already given:

$$p(\beta_0, \Delta | D) = \frac{1}{c_2} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)} \quad (4.4)$$

Where,

$$c_2 = \int_0^\infty \int_0^\infty \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)} d\beta_0 d\Delta$$

The resultant joint density will be used to obtain an expression for the distribution of the net counting rate. This equation is arrived at by numerically integrating (4.4) over β_0 . This is represented mathematically as follows:

$$p(\Delta | D) = \frac{1}{c_2} \int_0^{\infty} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)} d\beta_0 \quad (4.5)$$

This equation describes the distribution of the net rate.

Now we have all the pieces we need to derive the probability distribution for the sample activity. We use Equation (4.1). The left hand side of the equation, A , is a distribution because the right hand side is the quotient of two distributions $p(\Delta | D)$ and $p(\epsilon | m_s)$. To produce a distribution that will describe A , we need a transformation of variables. The following is the derivation of the distribution for $p(A)$ [Springer, 1979]:

The random variable activity, A , (as in equation (4.1)) is:

$$A = \frac{\Delta}{\epsilon}$$

This equality may be written as a system of equalities:

$$\Delta = A \epsilon \quad \epsilon = \epsilon$$

The joint distribution of $p(\Delta, \epsilon) = p(\Delta|D) p(\epsilon|m_s)$ is transformed into $g(A, \epsilon)$, where:

$$g(A, \epsilon) = p(A\epsilon|D) p(\epsilon|m_s) |J|$$

J being the Jacobian of the inverse transformation defined as:

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial \Delta}{\partial A} & \frac{\partial \Delta}{\partial \epsilon} \\ \frac{\partial \epsilon}{\partial A} & \frac{\partial \epsilon}{\partial \epsilon} \end{vmatrix} \\ &= \begin{vmatrix} \epsilon & A \\ 0 & 1 \end{vmatrix} \\ &= \epsilon \end{aligned}$$

Substituting this result in the expression $g(A, \epsilon)$ and integrating the result with respect to ϵ will give the distribution for the sample activity:

$$p_A(A) = \int_0^{0.5} p(A \in |D) p(\in |m_s) d\in \quad (4.6)$$

The activity in a sample is related to the number of atoms N by:

$$A = \lambda N$$

N is the number of atoms in the sample and λ is the decay constant. To obtain the distribution for N one has to consider the relation between N and A [Fisz, 1963].

$$N = \frac{1}{\lambda} A$$

For $1/\lambda > 0$ the distribution of the random variable N is given by

$$p(N) = \lambda p_A(\lambda N) \quad (4.7)$$

$p_A(\lambda N)$ is given by expression (4.6), the distribution for the sample activity $A = \lambda N$. In the next chapter we will give numerical examples, obtain plots of distributions, and predict activities.

CHAPTER 5

NUMERICAL ILLUSTRATIONS

In the previous chapter, we summarized the general results from our analysis. In this chapter, we will present numerical examples and graphical illustrations of those results. The Maple V computer code that was used to do the calculations and to generate the graphs is given in appendices 2 through 9.

5.1 RADIOACTIVE DECAY MODEL

$$p(n | (1 - e^{-\lambda t}), N) = \binom{N}{n} (1 - e^{-\lambda t})^n (e^{-\lambda t})^{N-n}$$

The above formula models the radioactive decay. In Appendix 3 several isotopes with different values of $T_{1/2}$ were used to generate several plots for this formula. To obtaining these plots and calculate the expectation of n , $N=300$ (atoms) and $t=1$ (minute) were used. The expected value for n is calculated using the following:

$$E(n | p) = N * (1 - e^{-\lambda t})$$

A summary of the results is listed in the table below.

Table 5.1 Calculated expectation for n for several isotopes

	Half-Life $T_{1/2}$					
	Rn-204 1.24 min	Rn-206 5.7 min	Rn-208 24.4 min	Rn-209 29 min	Rn-223 43 min	Rn-224 108 min
EXPECTATION OF n	128.5	34.35	8.40	7.08	4.80	1.92

5.2 DETECTION MODEL

$$p(m | \epsilon, n) = \binom{n}{m} \epsilon^m (1 - \epsilon)^{n-m} ; \text{ for } m \leq n$$

Appendix 4 contains plots for the detection model using different efficiency values. I used $n=300$ atoms in generating the plots. The expected value of m . was calculated using the following equation:

$$E(m | \epsilon) = n * \epsilon$$

5.3 DETECTOR COUNTS

$$p(m | \epsilon, N, \lambda) = \frac{(\epsilon N \lambda t)^m e^{-\epsilon N \lambda t}}{m!}$$

In appendix 5 several plots for different values for the efficiency ϵ , N number of atoms present, $T_{1/2}$ isotope half-life (in minutes) and

observation time t (in minutes) were generated. The Poisson mean is calculated using the following equation:

$$E(m | \epsilon, N, \lambda) = \epsilon N \lambda t$$

A summary is given in the table below:.

Table 5.2 Calculated Poisson mean for several values

Of Efficiency ϵ , number of atoms N and isotope

ϵ	0.10	.25	0.50	0.50	0.50
N	300	300	300	3000	3000
$T_{1/2}$	Rn-208 24.4 minute				Rn-222 5505.84
t	1	1	1	1	1
Poisson mean	0.85	2.13	4.26	42.6	0.19

5.4 STANDARD SOURCE MODEL

$$p(N_s | \mu_s, \sigma_s, \lambda_s) = \frac{\lambda_s}{\sqrt{2\pi\sigma_s}} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2}$$

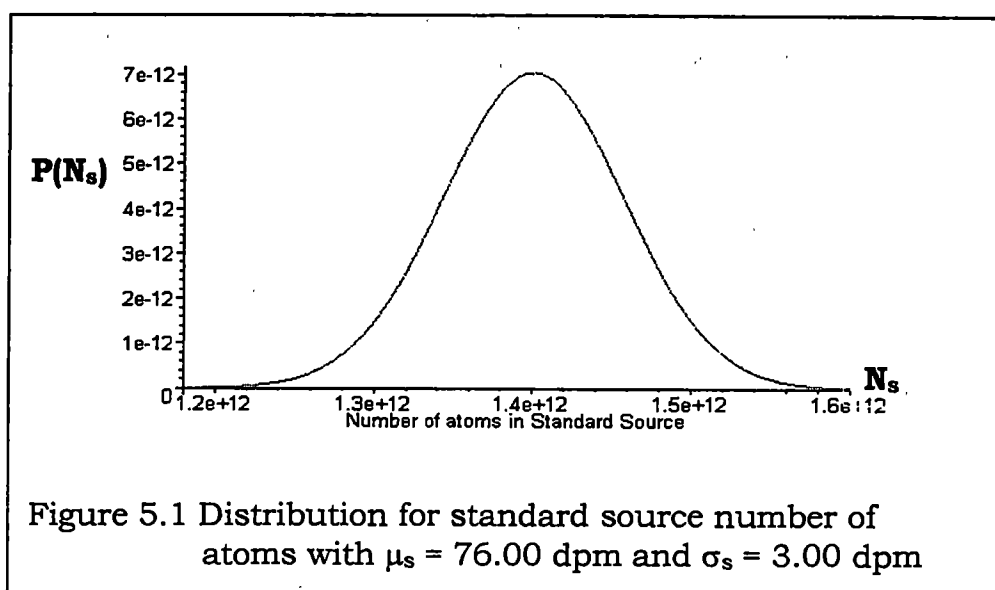
Appendix 6 contains the computer code that generated the plot of the standard source number of atoms distribution.

Figure 5.1 is the number of atoms distribution for a Pu-239 standard source with the following values:

$$\mu_s = 76.00 \text{ dpm}$$

$$\sigma_s = 3.00 \text{ dpm}$$

$$T_{1/2} = 2.411 \times 10^4 \text{ years}$$



5.5 POSTERIOR DENSITY OF THE ABSOLUTE EFFICIENCY

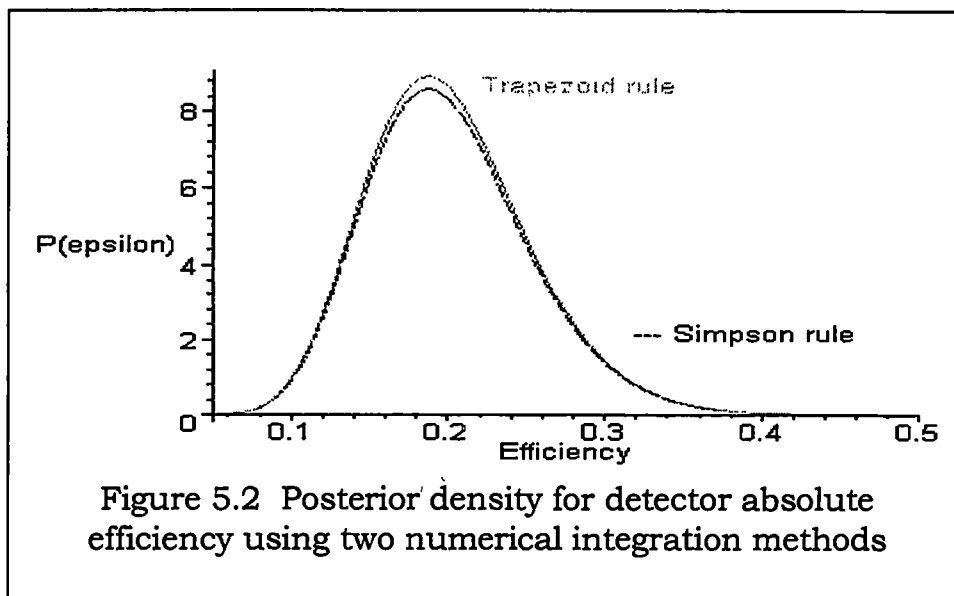
$$p(\epsilon | m_s, \mu_s, \sigma_s, \lambda_s) = \int_0^{\infty} \frac{\lambda_s (\epsilon N_s \lambda_s t)^{m_s}}{c m_s! \sigma_s \sqrt{2\pi}} e^{-\epsilon N_s \lambda_s t} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2} dN_s$$

The above formula is the posterior density for the absolute efficiency. Modeling using Pu-239 standard source is done in steps:

1. Modeling of Pu-239 standard source where, $\lambda_s = 2.411 \times 10^4$ years, $\mu_s = 76.00$ dpm and $\sigma_s = 3.00$ dpm
2. Modeling the detection process where, $ms = 14$ cpm
3. Combining the results from 1 and 2 is the posterior density for ϵ .

In appendix 7 you will find the computer code that generated the posterior density for the absolute efficiency using two different numerical integration procedures. Figure 5.2 below is the results. The calculated expectation for the absolute efficiency using the Simpson and Trapezoid rules was as follows:

- 1: Simpson rule was = 0.1980
2. Trapezoid rule was = 0.1970



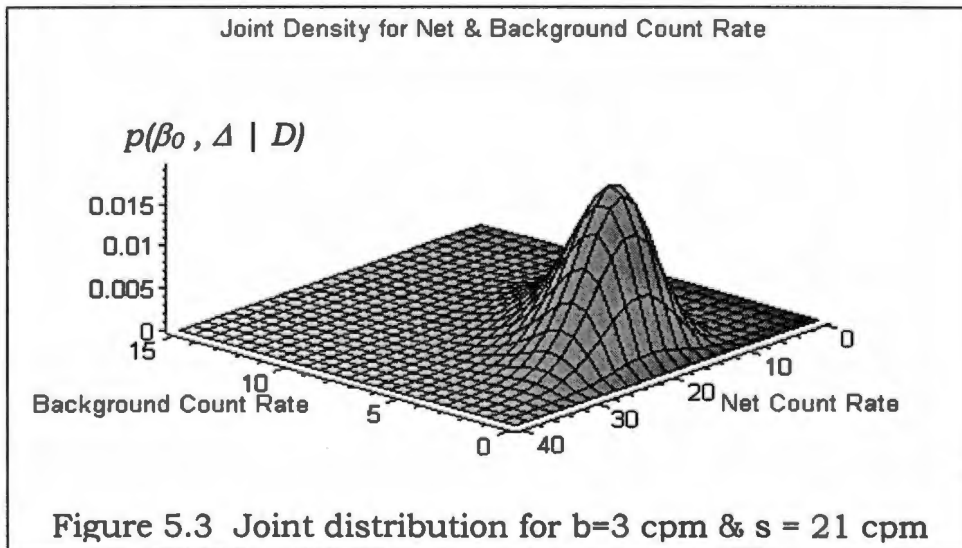
5.6 JOINT POSTERIOR DISTRIBUTION FOR THE BACKGROUND COUNT RATE AND THE NET COUNT RATE

$$p(\beta_0, \Delta | D) = \frac{1}{c_2} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1 + t_0) + \Delta t_1)}$$

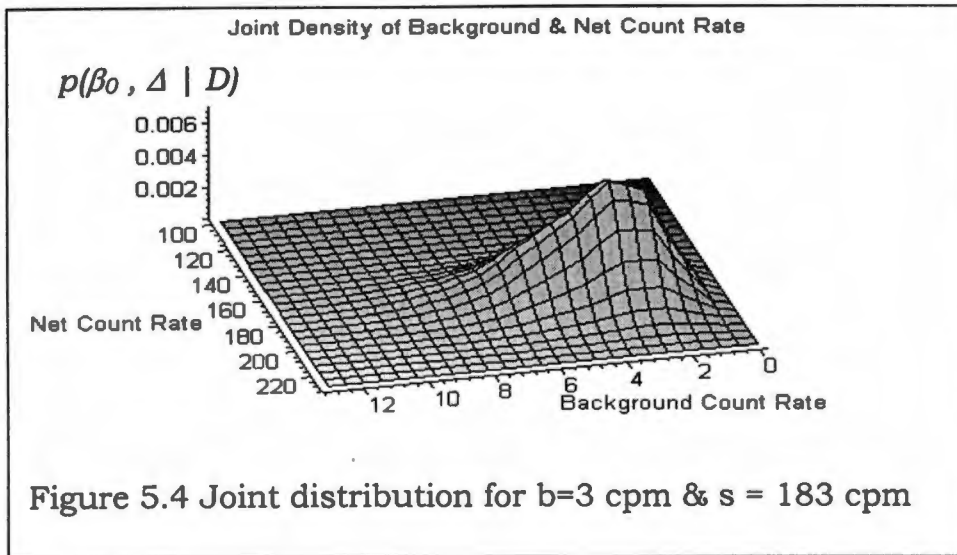
$$c_2 = \int_0^\infty \int_0^\infty \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1 + t_0) + \Delta t_1)} d\beta_0 d\Delta$$

In appendix 8 contain the computer code that generated the joint distribution, posterior densities for background count rate and net count rates for several cases. Below are the plots for the joint distribution where:

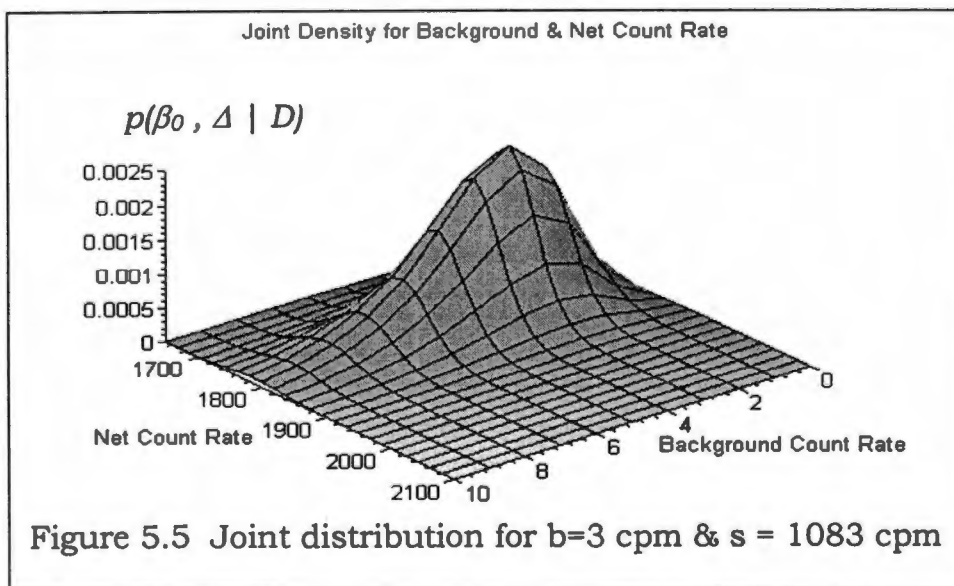
1. The background counts = 3 cpm and the sample counts = 21 cpm



2. The background counts = 3 cpm and the sample counts = 183 cpm



3. The background counts = 3 cpm and the sample counts = 1083 cpm



5.7 POSTERIOR DENSITIES FOR SAMPLE ACTIVITY & NUMBER OF ATOMS

In chapter 4 we derived the posterior density for the sample activity and sample number of atoms. Those expressions were:

$$p(A) = \int_0^{0.5} p(A \in | D) p(\epsilon | m_s) d\epsilon$$

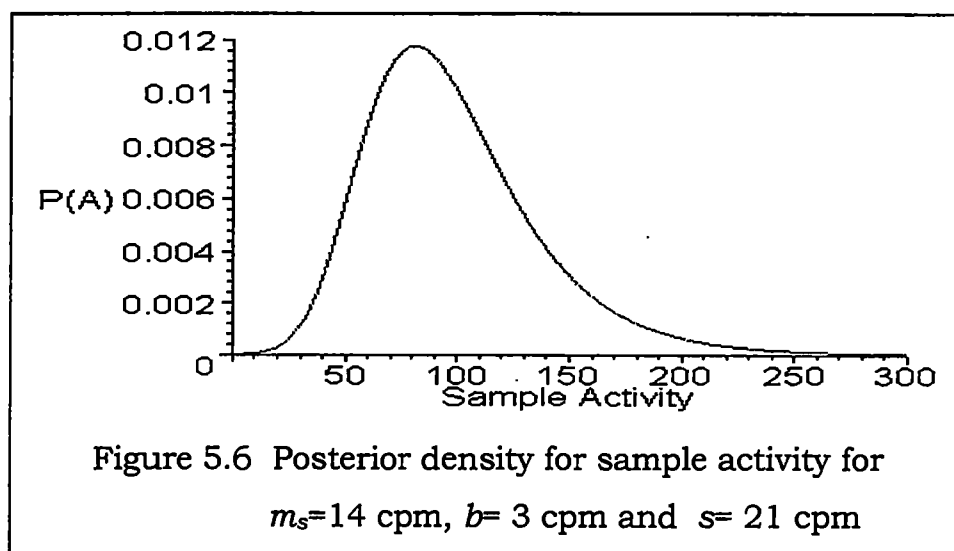
and

$$p(N) = \lambda p_A(\lambda N)$$

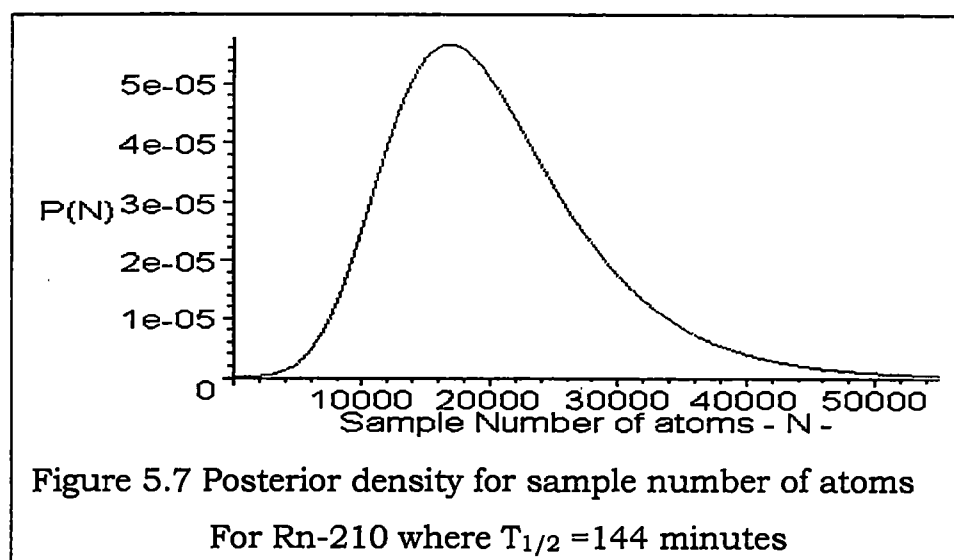
Appendix 9 contains the computer code for the calculations and plots. A summary of the result from appendix 9 follows:

1. A Pu-239 standard source was used with $\lambda_s = 2.411 \times 10^4$ years,
 $\mu_s = 76.00$ dpm, $\sigma_s = 3.00$ dpm and $m_s = 14$ cpm. As a result an expected value for the absolute efficiency ϵ was calculated = 0.1980
2. Sample count rate = 21 cpm and background count rate = 3 cpm, the calculated expectation for the net count rate (Δ) was 17.99 cpm and standard deviation 5.089 cpm. The conventional method yields the following results:
 - a. The sample net rate is $\Delta = 21/1 - 3/1 = 18$ cpm.
 - b. The variance is $\sigma^2 = 21/1^2 + 3/1^2 = 24$; therefore the standard deviation is $\sigma = 4.899$ cpm

The calculated expectation for sample activity is = 96.78 dpm. A plot of the posterior density for the sample activity is shown in Figure 5.6:



3. For a sample containing Rn-210 with $T_{1/2} = 2.4$ hours = 144 minutes. The calculated expectation for the number of atoms in a sample was 19880 atoms. A plot showing the distribution of number of atoms in a sample given in figure 5.7:



CHAPTER 6

DISCUSSION AND CONCLUDING REMARKS

Using a detector with a known efficiency ϵ , the sample activity is usually estimated as follows:

- 1) Background count rate is estimated by counting a blank.
- 2) Gross count rate is estimated by counting the sample.
- 3) Net count rate is calculated by taking the difference between the estimated gross and background count rate.
- 4) Estimated sample activity is the quotient of the estimated net count rate and the detector efficiency.

Assuming that the half-life of the radionuclide and the counting periods for the blank and sample are precisely known, the estimated number of atoms in the sample is the quotient of the estimated sample activity and the decay constant. Frequency based confidence intervals (c.i.) for the estimated quantities can be calculated. These c.i.s are based on long run hypothetical repetitions of the counting experiment [Currie, 1968].

This commonly used approach fails to deal with several sources of uncertainty. Its uncertainty estimates are based on hypothetical

repetitions of the experiment. It does not describe the uncertainty of the result conditional on the observed data and it underestimates the uncertainty of the sample activity. Therefore, we tackled this problem differently. Using Bayesian analysis, we evaluated the uncertainty for each step of the detection process by:

- 1) Statistically modeling radioactive decay.
- 2) Statistically modeling the detection process.
- 3) Combining the results from steps 1 and 2 and derive an expression that describes the distribution of counts .
- 4) Using the results from steps 1,2 and 3 above a posterior distribution for the detection efficiency is determined. It describes the uncertainty about the detection efficiency ϵ .
- 5) Considering background radiation and deriving an expression that describes the net count rate.
- 6) Taking the results from step 4 and 5 and using a transformation of variables, we derived an expression for the posterior distribution of the sample activity.
- 7) Deriving an expression for the posterior distribution of the number of atoms in a sample using transformation of variables and the result of step 6.

This novel method of analysis accounts for all the uncertainty due to all stochastic quantities involved in the detection process and describes

the remaining uncertainty about the estimated activity conditional only on the observed data with the help of probability densities.

FUTURE RESEARCH

The following is a list of possible future research topics:

1. Consider half-life uncertainty.
2. Consider a different prior density for the detector efficiency. One can think about the case when the sample is placed at a distance d from the detector surface $\theta_0 < 90^\circ$, then the solid angle will be < 0.5 resulting in a different prior density for the detector efficiency.
3. Use the Binomial and not the Poisson distribution for modeling gross and background counts.
4. Investigate the influence of different prior densities for the background and gross count rates on the end result. If we have prior knowledge about the background and gross count rates an appropriate gamma density can be used to model their prior densities [Little, 1982].

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Bailey, N. T. "The Elements of Stochastic Processes with Applications to the Natural Sciences". John Wiley and son, New York, 1964.
2. Box, George E. P. and Tiao, George C. "Bayesian Inference in Statistical Analysis". Addison-Wesley Publishing Company, Massachusetts, 1973.
3. Brase, Charles Henry and Brase, Corrinne Pellillo "Understanding Basic Statistics". Houghton Mifflin company, New York, 1997.
4. Cember, Herman "Introduction to Health Physics- 3rd ed.". McGraw-Hill Health Professions Division, New York 1996.
5. Cox, D. R. and Miller, H. D. "The Theory of Stochastic Processes". John Wiley and son, New York, 1965.
6. Currie, L. A. "Limits for Qualitative Detection and Quantitative determination". Analytical Chemistry. 40:586-593; 1968.
7. Debertin, Klaus and Helmer, Richard G. "Gamma- and X-ray Spectrometry with Semiconductor Detectors". North - Holland Publishing Company, Amsterdam, 1988.
8. Dr M. Fogiel, Director and Staff of Research and Education Association "Handbook of Mathematical Scientific, and Engineering Formulas, Tables, Functions, Graphs, Transforms" United States, 1980.

9. Feller, W. "An Introduction to Probability Theory and Its Applications" second edition. Wiley, New York, 1957.
10. Fisz, Marek "Probability Theory and Mathematical Statistics". John Wiley and son, New York, 1963.
11. Freund, John E. "Modern Elementary Statistics". Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1984.
12. Frohner, F. H. "Assigning Uncertainties to Scientific Data". Nuclear Science & Engineering: 126, 1-18 (1997).
13. Giri, Narayan C. "Introduction to Probability and Statistics Part I: Probability". Marcel Decker, Inc., New York 1974.
14. Handloser, John S. "Health Physics Instrumentation". Pergamon Press, New York, 1959.
15. Knoll, Glenn F. "Radiation Detection and Measurement". John Wiley & Son Inc., New York, 1989.
16. Krane, Kenneth S. "Introductory Nuclear Physics". John Wiley & Son Inc., New York, 1987.
17. Little, R. J. A. "Statistical Analysis of Low-Level Radioactivity". Health Phys. 43:693-703:1982.
18. Morgan, M. Granger and Henrion, Max "Uncertainty". Cambridge University Press, Cambridge, 1990.

19. Myers, Buddy L. and Enrik, Norbert L. "Statistical Functions a Source of Practical Derivations Based on Elementary Mathematics". The Science Press Inc., United states, 1970.
20. NCRP Report No. 58 "A Handbook of Radioactivity Measurements Procedures". Recommendations of the National Council on Radiation Protection and Measurements, Washington, November 1, 1978.
21. Price, William J. "Nuclear Radiation Detection - 2nd ed.". McGraw-Hill Book Company, New York .
22. Springer, M.D. " The Algebra of Random Variables". John Wiley and son, New York, 1979.
23. Tsoulfanidis, Nicholas, date. "Measurement and Detection of Radiation". Hemisphere publishing corporation, Washington & McGraw-Hill book company, New York, 1981.
24. Turner, J.E. (James Edward) "Atoms, Radiation, and Radiation Protection/ James E. Turner- 2nd ed.". John Wiley & Son Inc., New York, 1989.
25. Wadsworth, George P. and Bryan, Joseph G. "Applications of Probability and Random Variables 2nd edition". McGraw-Hill , Inc., New York, 1974.

APPENDIXES

APPENDIX 1

LIST OF SYMBOLS

LIST OF SYMBOLS

A	= sample activity
Δ	= net count rate
ϵ	= Detector absolute efficiency = probability of detection
N_0	= initial number of atoms in a sample
N	= number of atoms in a sample
q	= probability of survival = probability of not decaying
p	= probability of decay
λ	= decay constant
t	= counting time
n	= number of atoms decayed
m	= number of atoms in a sample that decayed and registered a count in a detector.
N_s	= number of atoms in a standard source
μ_s	= mean of N_s of the standard source
σ_s	= reported standard deviation of standard source
λ_s	= decay constant
m_s	= number of atoms in a standard source that decayed and registered a count in a detector.
Ω	= solid angle
f	= a combination of correction factors

θ_0 = angle defined by the detector edge and the line passing through the source perpendicular to the detector surface.

β_0 = background count rate

b = observed background counts

t_0 = time taken to count background

t_1 = time taken to count a sample – to get gross count

s = observed gross counts

γ = gross count rate

APPENDIX 2

BINOMIAL AND POISSON

DISTRIBUTIONS

BINOMIAL AND POISSON DISTRIBUTIONS

$$P_n = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$$

The following is Maple V code that generated the following Binomial distribution using the above formula and the following values:

➤ **Binomial distribution with N = 25 and p= 0.45**

> restart:with(plots):

with(linalg):

bi:=binomial(25,j)*(0.45^j)*((0.55)^(25-

j));c:=linalg[matrix](25,1,[seq(bi,j=1..25)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90, 90],view=

[1..25,1..1.5,0..0.2], axes=frame, labels = ['N','P(N)'], title='Binomial

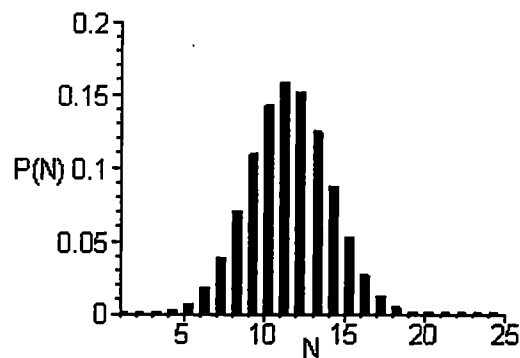
Distribution for N=25 & p=0.45',color=black);

Warning, new definition for norm

Warning, new definition for trace

$bi := \text{binomial}(25, j) \cdot 0.45^j \cdot 0.55^{(25-j)}$

Binomial Distribution for N=25 & p=0.45



➤ **Binomial distribution with N = 500 and p = 0.45**

> restart:with(plots):

with(linalg):

bi:=binomial(500,j)*(0.45^j)*((0.55)^(500-

j));c:=linalg[matrix](500,1,[seq(bi,j=1..500)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=

[190..270,1..1.5,0..0.04],axes=frame, labels=['N','','P(N)'],title='Binomial

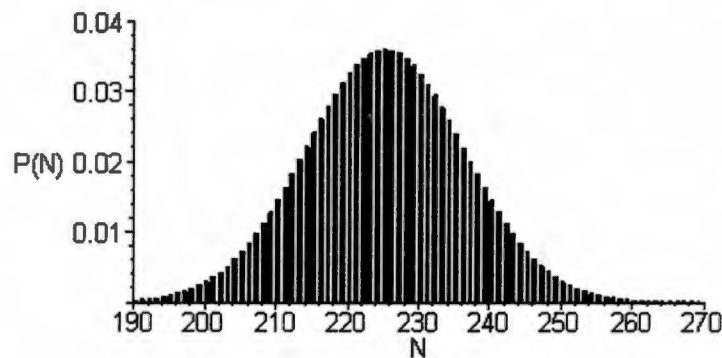
Distribution for N=500 & p=0.45',color=black);

Warning, new definition for norm

Warning, new definition for trace

$bi := \text{binomial}(500, j) \cdot 0.45^j \cdot 0.55^{(500-j)}$

Binomial Distribution for N=500 & p=0.45



$$P_n = \frac{\mu^n}{n!} e^{-\mu}$$

The following is the code lines that generated the Poisson distribution using the above formula with the following values:

➤ **Poisson Distribution with mean = 5**

```
> restart:with(plots):
```

```
with(linalg):
```

```
psson:=(5^j)*(exp(-5))/(j!); c:=linalg[matrix](20,1,[seq(evalf(psson),j=1..20)]):
```

```
> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90])
```

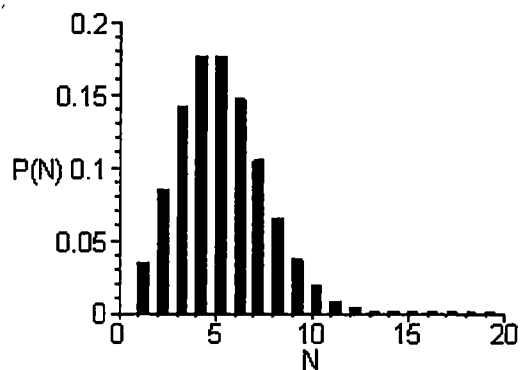
```
,axes=frame,view= [0..20,1..1.5,0..0.2], labels =['N','','P(N)'], title='Poisson
Distribution with mue=5', color=black);
```

Warning, new definition for norm

Warning, new definition for trace

$$p_{sson} := \frac{5^j e^{(-5)}}{j!}$$

Poisson Distribution with mue=5



APPENDIX 3

RADIOACTIVE DECAY MODEL

RADIOACTIVE DECAY MODEL

$$p(n|(1-e^{-\lambda t}), N, \lambda) = \binom{N}{n} (1-e^{-\lambda t})^n (e^{-\lambda t})^{N-n}$$

Using several isotopes (each having different half-life), $N=300$ (atoms) and counting time $t=1$ (minute) we generated several plots for the above formula. The purpose is to show the effect of those variables on the radioactive decay distribution.

(1) The first set of values is:

1. Rn-204 half-life = 1.24 minute
2. Number of atoms present = 300 atoms
3. Counting interval length = 1 minute

MAPLE CODE LINES

> restart:

> Number_of_Atoms_in_a_sample:=N;Number_of_atoms_Decayed:=n;

Decay_Constant:=lambda;Half_life:=half_life;Observation_time:=t;Observed_Number_of_atoms:=m;

Number_of_Atoms_in_a_sample := N

Number_of_atoms_Decayed := n

Decay_Constant := λ

Half_life := half_life

Observation_time := t

Observed_Number_of_atoms := m

```
> restart:Digits:=4:assume(lambda,real):assume(half_life,real):
assume(t,real):assume(N,integer):assume(n,integer):assume(lambda>0,N>0,n>0,ha
lf_life>0,t>0):with(student):

> `Rn_204_half_life_1.24_minute`;half_life:=1.24; N:=300;t:=1;
lambda:=0.693/half_life;

> Probability_Decay:=1-exp(-lambda*t);p:=%.Probability_Not_Decay:=exp(-
lambda*t);

> Decay_Model:=(binomial(N,n))*(p^n)*((1-p)^(N-n));
Decay_Model:=(convert(Decay_Model,factorial)): EXPECT_MEAN:=N*p;

> with(plots):
```

with(linalg):

c:=linalg[matrix]((N-100),1,[seq(evalf(subs(n=j,Decay_Model)),j=1..(N-100))]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=[100..160,1..1.5,0..0.05],axes=frame,labels=['# atoms',``,'P(N)`],tickmarks=[10,3,10],title='Decay Model for Rn-204 N=300 & p=0.4282',color=black);

Rn_204_half_life_1.24_minute

half_life := 1.24

N := 300

t := 1

λ := .5589

Probability_Decay := .4282

Probability_Not_Decay := .5718

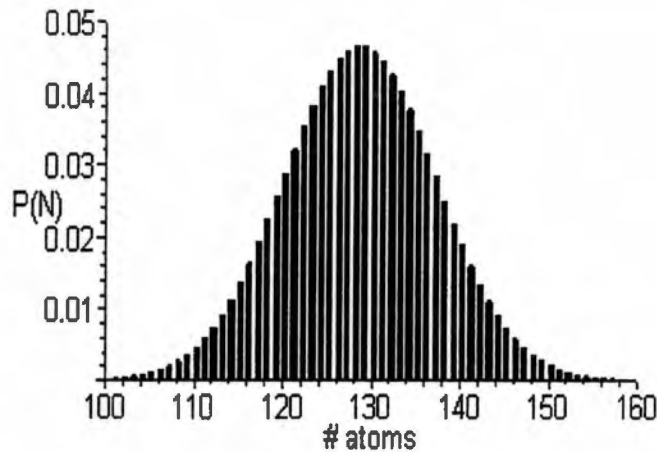
Decay_Model := binomial(300, n~) .4282^{n~} .5718^(300 - n~)

EXPECT_MEAN := 128.5

Warning, new definition for norm

Warning, new definition for trace

Decay Model for Rn-204 $N=300$ & $p=0.4282$



(2) The second set of values is:

1. Rn-206 half-life = 5.7 minutes
2. Number of atoms present = 300 atoms
3. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4; assume(lambda,real); assume(half_life,real):  
assume(t,real); assume(N,integer); assume(n,integer); assume(lambda>0,N>0,n>0,ha  
lf_life>0,t>0): with(student):
```

```

> `Rn_206_half_life_5.7_minute`;half_life:=5.7;N:=300;t:=1;

lambda:=0.693/half_life;

> Probability_Decay:=1-exp(-lambda*t);p:=%:Probability_Not_Decay:=exp(-
lambda*t);

> Decay_Model:=(binomial(N,n))*(p^n)*((1-p)^(N-
n));Decay_Model:=(convert(Decay_Model,factorial)): EXPECT_MEAN:=N*p;

> with(plots):

with(linalg):

c:=linalg[matrix]((N-225),1,[seq(evalf(subs(n=j,Decay_Model)),j=1..(N-225))]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=
[15..62,1..1.5,0..0.08],axes=frame,labels=[`#
atoms`,``,`P(N)`],tickmarks=[10,3,10],title=`Decay Model for Rn-206 N=300 &
p=0.1145`,color=black);

```

Rn_206_half_life_5.7_minute

half_life = 5.7

N := 300

t := 1

λ = .1216

Probability_Decay = .1145

Probability_Not_Decay = .8855

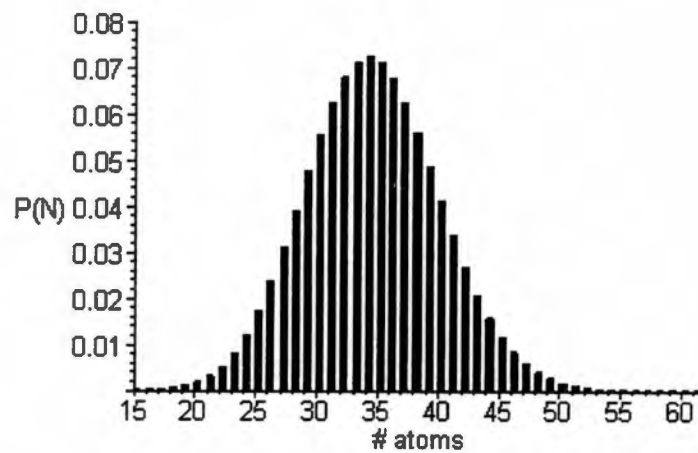
Decay_Model := binomial(300, *n~*) .1145^{*n~*} .8855^(300 - *n~*)

EXPECT_MEAN = 34.35

Warning, new definition for norm

Warning, new definition for trace

Decay Model for Rn-206 N=300 & p=0.1145



(3) The third set of values is:

1. Rn-208 half-life = 24.4 minutes
2. Number of atoms present = 300 atoms
3. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4: assume(lambda, real): assume(half_life, real):  
assume(t, real): assume(N, integer): assume(n, integer): assume(lambda>0, N>0, n>0, ha  
lf_life>0, t>0): with(student):  
  
> `Rn_208_half_life_24.4_minute`; half_life:=24.4; N:=300; t:=1;  
lambda:=0.693/half_life;  
  
> Probability_Decay:=1-exp(-lambda*t); p:=%; Probability_Not_Decay:=exp(-  
lambda*t);  
  
> Decay_Model:=(binomial(N,n))*(p^n)*((1-p)^(N-  
n)); Decay_Model:=(convert(Decay_Model, factorial)): EXPECT_MEAN:=N*p;  
  
> with(plots):  
with(linalg):  
c:=linalg[matrix]((N-275), 1, [seq(evalf(subs(n=j, Decay_Model)), j=1..(N-275))]):
```

```
> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-
90,90],axes=frame,labels=['# atoms',``,'P(N)`'],tickmarks=[10,3,10],title='Decay
Model for Rn-208 N=300 & p=0.0280',color=black);
```

Rn_208_half_life_24.4_minute

half_life := 24.4

N := 300

t := 1

λ := .02840

Probability_Decay := .0280

Probability_Not_Decay := .9720

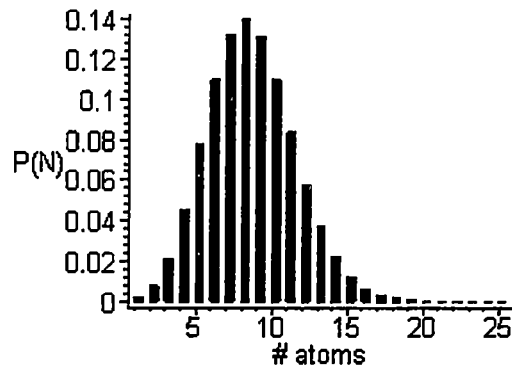
Decay_Model := binomial(300, $n \sim$) $.0280^{n \sim} .9720^{(300 - n \sim)}$

EXPECT_MEAN = 8.400

Warning, new definition for norm

Warning, new definition for trace

Decay Model for Rn-208 $N=300$ & $p=0.0280$



(4) The forth set of values is:

1. Rn-204 half-life = 29 minutes
2. Number of atoms present = 300 atoms
3. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4; assume(lambda,real); assume(half_life,real);
assume(t,real); assume(N,integer); assume(n,integer); assume(lambda>0,N>0,n>0,ha
lf_life>0,t>0); with(student):

> `Rn_209_half_life_29_minute`; half_life:=29; N:=300; t:=1;
```

```
lambda:=0.693/half_life;
```

```
> Probability_Decay:=1-exp(-lambda*t);p:=%:Probability_Not_Decay:=exp(-
lambda*t);
```

```
> Decay_Model:=(binomial(N,n))*(p^n)*((1-p)^(N-
n));Decay_Model:=(convert(Decay_Model,factorial)): EXPECT_MEAN:=N*p;
```

```
> with(plots):
```

```
with(linalg):
```

```
c:=linalg[matrix]((N-275),1,[seq(evalf(subs(n=j,Decay_Model)),j=1..(N-275))]):
```

```
> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-
90,90],axes=frame,labels=['# atoms`,`','P(N)`'],tickmarks=[10,3,10],title=`Decay
Model for Rn-209 N=300 & p=0.0236`,color=black);
```

Rn_209_half_life_29_minute

half_life := 29

N := 300

t := 1

λ := .02390

Probability_Decay := .0236

Probability_Not_Decay := .9764

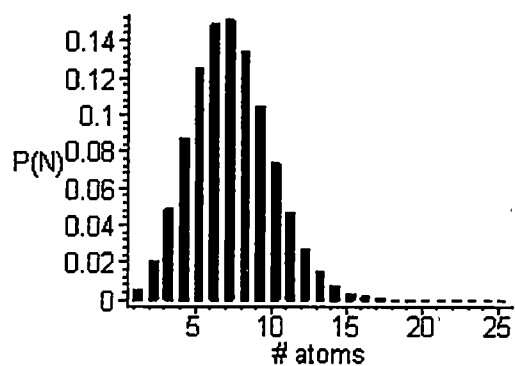
$$\text{Decay_Model} := \text{binomial}(300, n \sim) .0236^{n \sim} .9764^{(300 - n \sim)}$$

$$\text{EXPECT} = 7.080$$

Warning, new definition for norm

Warning, new definition for trace

Decay Model for Rn-209 N=300 & p=0.0236



(5) The fifth set of values is:

1. Rn-223 half-life = 43 minute
2. Number of atoms present = 300 atoms
3. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart:Digits:=2:assume(lambda,real):assume(half_life,real):
assume(t,real):assume(N,integer):assume(n,integer):assume(j,integer):assume(lam
bda>0,N>0,n>0,j>0,half_life>0,t>0):with(student):

> `Rn_223_half_life_43_minute`;half_life:=43;N:=300;t:=1;
lambda:=0.693/half_life;

> Probability_Decay:=1-exp(-lambda*t);p:=%:Probability_Not_Decay:=exp(-
lambda*t);

> Decay_Model:=(binomial(N,n))*(p^n)*((1-p)^(N-
n));Decay_Model:=(convert(Decay_Model,factorial)):

> with(plots):
EXPECT_MEAN:=N*p;
with(linalg):c:=linalg[matrix](25,1,[seq(evalf(subs(n=j,Decay_Model)),j=1..25)]:

> matrixplot(c,heights=HISTOGRAM,gap=0.5, orientation=[-90,90],view=
[1..15,0.1..1,0..0.17],axes=frame,labels=['#
atoms',``,`P(N)`],tickmarks=[10,3,10],title=`Decay Model for Rn-223 N=300 &
p=0.02`,color=black);
```

Rn_223_half_life_43_minute

half_life := 43

$$N := 300$$

$$t := 1$$

$$\lambda := .016$$

$$\text{Probability_Decay} := .02$$

$$\text{Probability_Not_Decay} := .98$$

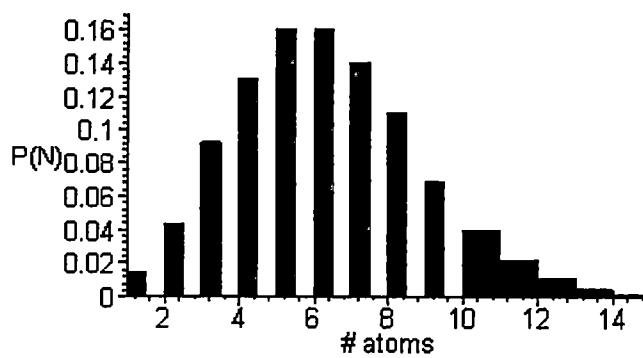
$$\text{Decay_Model} := \text{binomial}(300, n \sim) .02^{n \sim} .98^{(300 - n \sim)}$$

$$\text{EXPECT_MEAN} = 6.0$$

Warning, new definition for norm

Warning, new definition for trace

Decay Model for Rn-223 N=300 & p=0.02



(6) The sixth set of values is:

1. Rn-224 half-life = 108 minute
2. Number of atoms present = 300 atoms
3. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4; assume(lambda,real): assume(half_life,real):
assume(t,real): assume(N,integer): assume(n,integer): assume(lambda>0,N>0,n>0,ha
lf_life>0,t>0): with(student):

> `Rn_224_half_life_108_minute`; half_life:=108; N:=300; t:=1;
lambda:=0.693/half_life;

> Probability_Decay:=1-exp(-lambda*t); p:=%: Probability_Not_Decay:=exp(-
lambda*t);

> Decay_Model:=(binomial(N,n))*(p^n)*((1-p)^(N-
n)); Decay_Model:=(convert(Decay_Model,factorial)): EXPECT_MEAN:=N*p;

> with(plots):

with(linalg):

c:=linalg[matrix](20,1,[seq(evalf(subs(n=j,Decay_Model)),j=1..20)]):
```

```
> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=
[1..10,1..1.5,0..0.35],axes=frame,labels=['#
atoms`,`P(N)`],tickmarks=[10,3,10],title='Decay Model for Rn-224 N=300 &
p=0.0064`,color=black);
```

Rn_224_half_life_108_minute

half_life := 108

N := 300

t := 1

λ := .006417

Probability_Decay := .0064

Probability_Not_Decay := .9936

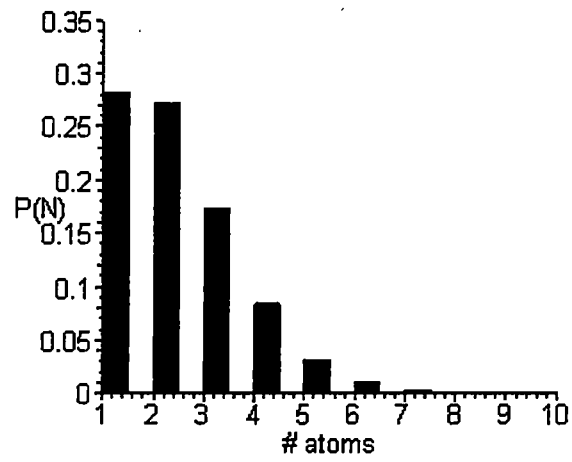
Decay_Model := binomial(300, n~) .0064^{n~} .9936^(300 - n~)

EXPECT_MEAN := 1.920

Warning, new definition for norm

Warning, new definition for trace

Decay Model for Rn-224 $N=300$ & $p=0.0064$



APPENDIX 4

DETECTION MODEL

DETECTION MODEL

$$p(m|\epsilon, n) = \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m}; \quad m \leq n$$

Using several sets of values for the efficiency and n we generated several plots for the above formula. The purpose is to show the effect of those variables on the detection distribution.

(1) First set of values:

1. Efficiency = Epsilon = 0.05
2. Number of atoms that actually decay = 300 atoms

MAPLE CODE LINES

```
> restart; Digits:=4; assume(epsilon, real); assume(n, integer); assume(m, integer);  
assume(epsilon>0, n>0, m>0); with(student):  
  
> n:=300; epsilon:=.05;  
  
> Probability_Registering:=epsilon; epsilon:=%:
```

```

Probability_Not_Registering_Count:=1-epsilon;

> Detection_Model:=(binomial(n,m))*(epsilon^m)*((1-epsilon)^(n-
m));Detection_Model:=(convert(Detection_Model,factorial));m_EXPECT_MEAN:
= epsilon*n;

> with(plots):

with(linalg):

c:=linalg[matrix](35,1,[seq(evalf(subs(m=j,Detection_Model)),j=1..35)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-
90,90],axes=frame,labels=['observed #
atoms','`,`P(m)`'],tickmarks=[10,3,10],title=`Detection Model for n=300 &
efficiency of 5 %`,color=black);

```

$$n := 300$$

$$\epsilon := .05$$

$$Probability_Registering := .05$$

$$Probability_Not_Registering_Count := .95$$

$$Detection_Model := \text{binomial}(300, m) \cdot .05^m \cdot .95^{(300 - m)}$$

```

Detection_Model := 30605751221644063603537046129726862938858880417357699\
941677674125947653317671686746551529142247757334993914788870172636886\
426390775900315422684292790697455984122547693027195460400801221577625\

```

$$00000000000 \frac{.05^m \cdot .95^{(300-m)}}{m! (300-m)!}$$

Warning, new definition for norm

Detection Model for n=300 & efficiency of 5 %



(2) Second set of values:

1. Efficiency = Epsilon = 0.1
2. Number of atoms that actually decay = 300 atoms

MAPLE CODE LINES

```
> restart; Digits:=4: assume(epsilon, real): assume(n, integer): assume(m, integer):  
assume(epsilon>0, n>0, m>0): with(student):  
  
> n:=300; epsilon:=.1;  
  
> Probability_Registering:=epsilon; epsilon:=%:  
Probability_Not_Registering_Count:=1-epsilon;  
  
> Detection_Model:=(binomial(n,m))*(epsilon^m)*((1-epsilon)^(n-  
m)); Detection_Model:=(convert(Detection_Model, factorial)); m_EXPECT_MEAN:  
= epsilon*n;  
  
> with(plots):  
with(linalg):  
c:=linalg[matrix](55,1,[seq(evalf(subs(m=j,Detection_Model)),j=1..55)]):  
  
> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=  
[15..50,1..1.5,0..0.08],axes=frame,labels=['observed #  
atoms',``,`P(m)`'],tickmarks=[10,3,10],title=`Detection Model for n=300 &
```

```
efficiency of 10 %',color=black);
```

 $n = 300$

$x = 1$

Probability Registering = .1

Probability Not Registering Count = .9

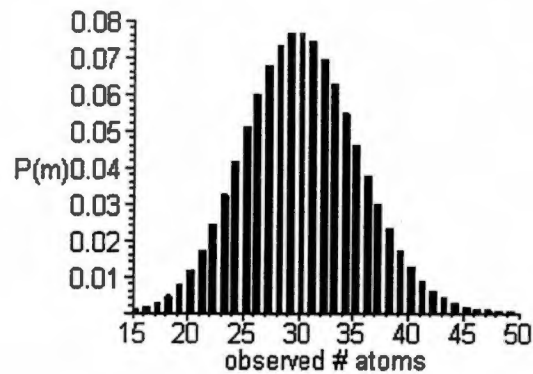
$$\text{Detection Model} := \text{binomial}(300, m_{\sim}) \cdot 1^{m_{\sim}} \cdot 9^{(300 - m_{\sim})}$$
[illegible]
$$00000000000 \frac{.1^m .9^{(300-m)}}{m!(300-m)!}$$

m EXPECT MEAN:= 30.0

Warning, new definition for norm

Warning, new definition for trace

Detection Model for $n=300$ & efficiency of 10 %



(3) Third set of values:

1. Efficiency = Epsilon = 0.2
2. Number of atoms that actually decay = 300 atoms

MAPLE CODE LINES

```
> restart; Digits:=4; assume(epsilon,real); assume(n,integer); assume(m,integer);  
assume(epsilon>0,n>0,m>0); with(student):  
  
> n:=300; epsilon:=.2;  
  
> Probability_Registering:=epsilon; epsilon:=%:  
Probability_Not_Registering_Count:=1-epsilon;
```

```

> Detection_Model:=(binomial(n,m))*(epsilon^m)*((1-epsilon)^(n-
m));Detection_Model:=(convert(Detection_Model,factorial));m_EXPECT_MEAN:
= epsilon*n;

> with(plots):

with(linalg):

c:=linalg[matrix](100,1,[seq(evalf(subs(m=j,Detection_Model)),j=1..100)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=
[38..85,1..1.5,0..0.06],axes=frame,labels=['observed #
atoms`,`P(m)`],tickmarks=[10,3,10],title='Detection Model for n=300 &
efficiency of 20 %`,color=black);

```

$$n := 300$$

$$\epsilon := .2$$

$$Probability_Registering := .2$$

$$Probability_Not_Registering_Count := .8$$

$$Detection_Model := \text{binomial}(300, m) \cdot 2^{m \sim} \cdot 8^{(300 - m \sim)}$$

```

Detection_Model := 30605751221644063603537046129726862938858880417357699\
941677674125947653317671686746551529142247757334993914788870172636886\
426390775900315422684292790697455984122547693027195460400801221577625\
217685425596535690350678872526432189626429936520457644883038890975394\

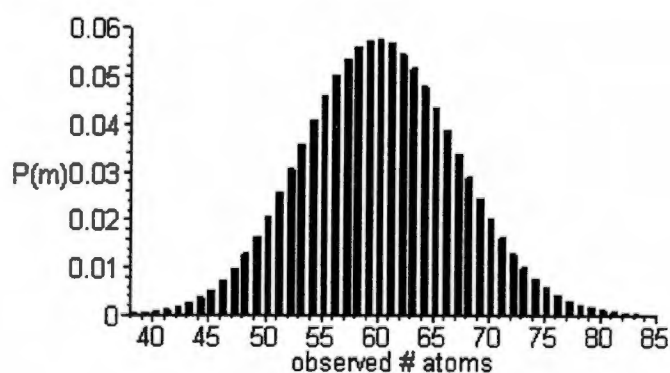
```

$$00000000000 \frac{.2^m \cdot .8^{(300-m)}}{m! (300-m)!}$$

Warning, new definition for norm

Warning, new definition for trace

Detection Model for $n=300$ & efficiency of 20 %



(4) Forth set of values:

1. Efficiency = Epsilon = 0.5
2. Number of atoms that actually decay = 300 atoms

MAPLE CODE LINES

```
> restart:Digits:=4:assume(epsilon,real):assume(n,integer):assume(m,integer):
assume(epsilon>0,n>0,m>0):with(student):

> n:=300;epsilon:=.5;

> Probability_Registering:=epsilon;epsilon:=%:
Probability_Not_Registering_Count:=1-epsilon;

> Detection_Model:=(binomial(n,m))*(epsilon^m)*((1-epsilon)^(n-
m));Detection_Model:=(convert(Detection_Model,factorial));m_EXPECT_MEAN:
= epsilon*n;

> with(plots):
with(linalg):
c:=linalg[matrix](180,1,[seq(evalf(subs(m=j,Detection_Model)),j=1..180)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.5,orientation=[-90,90],view=
[125..180,1..1.5,0..0.05],axes=frame,labels=['observed #
atoms',``,`P(m)`],tickmarks=[10,3,10],title=`Detection Model for n=300 &
```

efficiency of 50 %',color=black);

$$n := 300$$

$$\varepsilon := .5$$

$$Probability_Registering := .5$$

$$Probability_Not_Registering_Count := .5$$

$$Detection_Model := binomial(300, m\sim) \cdot 5^{m\sim} \cdot 5^{(300 - m\sim)}$$

Detection_Model := 30605751221644063603537046129726862938858880417357699\
941677674125947653317671686746551529142247757334993914788870172636886\
426390775900315422684292790697455984122547693027195460400801221577625\
217685425596535690350678872526432189626429936520457644883038890975394\
348962543605322598077652127082243763944912012867867536830571229368194\
364995646049816645022771650018517654646934011222603472972406633325858\
350687015016979416885035375213755491028912640715715483028228493795263\
658014523523315693648223343679925459409527682060806223281238738388081\
70496000000000000000000000000000000000000000000000000000000000000000\
0000000000

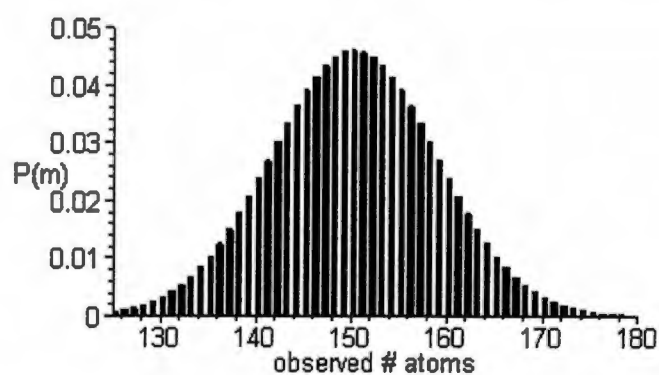
$$\frac{5^{m\sim} \cdot 5^{(300 - m\sim)}}{m\sim! (300 - m\sim)!}$$

$$m_EXPECT_MEAN := 150.0$$

Warning, new definition for norm

Warning, new definition for trace

Detection Model for $n=300$ & efficiency of 50 %



APPENDIX 5

DETECTOR COUNTS

DETECTOR COUNTS

$$p(m | \epsilon, N, \lambda) = \frac{(\epsilon N \lambda t)^m e^{-\epsilon N \lambda t}}{m!}$$

Using several sets of values for the efficiency, N , half-life and counting time we generated several plots for the above formula. The purpose is to show the effect of those variables on the distribution.

MAPLE CODE LINES

```
> restart: Number_of_atoms_in_sample := N; Half_life := half_life;  
Decay_Constant := lambda; Observation_time := t; Observed_Number_of_atoms := m;  
Detector_Efficiency := epsilon;
```

Number_of_atoms_in_sample := N

Half_life := half_life

Decay_Constant := λ

Observation_time := t

Observed_Number_of_atoms := m

Detector_Efficiency := ε

(1) The first set of values is:

1. Efficiency = 0.1
2. Number of atoms present = 300 atoms
3. Rn- 208 half-life = 24.4 minutes
4. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4: assume(epsilon,real): assume(N,integer): assume(half_life,real):  
assume(lambda,real): assume(t,real): assume(m,integer): assume(epsilon>0,N>0,half  
_life>0,lambda>0,t>0,m>0):  
  
> epsilon:=.1; N:=300; `Rn_208_half_life_24.4_minute`; half_life:=24.4; t:=1;  
  
> lambda:=0.693/half_life;  
  
> MEAN:=epsilon*N*lambda*t;  
  
> Probability_of_observing_count:=(MEAN^m)*(exp(-MEAN))/m!; p:=%:  
  
> with(plots):  
  
with(linalg):  
  
c:=linalg[matrix](10,1,[seq(evalf(subs(m=j,Probability_of_observing_count)),j=1..1  
0)]):
```

```
> matrixplot(c,heights=HISTOGRAM,gap=0.25,orientation=[-90,90],view=
[1..6,1..1.5,0..0.4],axes=frame,tickmarks=[10,3,10],labels=['Observed # atoms - m -
`,`P(m)`],title='Posterior Density for observed Number of atoms Rn-208 Eff.=
10%',color=black);
```

$$\varepsilon := .1$$

$$N := 300$$

$$Rn_208_half_life_24.4_minute$$

$$half_life := 24.4$$

$$t := 1$$

$$\lambda := .02840$$

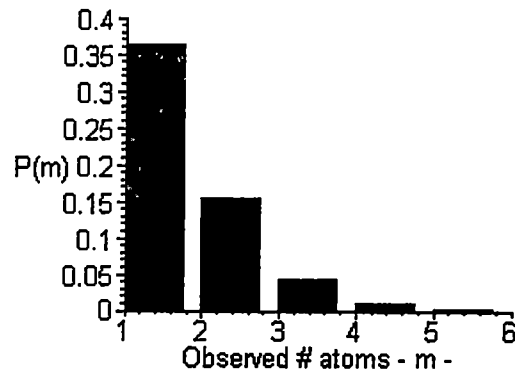
$$MEAN := .8520$$

$$Probability_of_observing_count := .4266 \frac{.8520^{m\sim}}{m\sim!}$$

Warning. new definition for norm

Warning. new definition for trace

Posterior Density for observed Number of atoms Rn-208 Eff.= 10%



(2) The second set of values is:

1. Efficiency = 0.25
2. Number of atoms present = 300 atoms
3. Rn-208 half-life = 24.4 minutes
4. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4; assume(epsilon,real): assume(N,integer): assume(half_life,real):
assume(lambda,real): assume(t,real): assume(m,integer): assume(epsilon>0,N>0,half
_life>0,lambda>0,t>0,m>0):

> epsilon:=.25; N:=300; `Rn_208_half_life_24.4_minute`; half_life:=24.4; t:=1;

> lambda:=0.693/half_life;
```

```

> MEAN:=epsilon*N*lambda*t;

> Probability_of_observing_count:=(MEAN^m)*(exp(-MEAN))/m!;p:=%;

> with(plots):

with(linalg):

c:=linalg[matrix](10,1,[seq(evalf(subs(m=j,Probability_of_observing_count)),j=1..10)]);

> matrixplot(c,heights=HISTOGRAM,gap=0.25,orientation=[-90,90],view=[1..6,1..1.5,0..0.3],axes=frame,tickmarks=[10,3,10],labels=['Observed # atoms - m - ',``,P(m)`],title='Posterior Density for observed Number of atoms Rn-208 Eff. 0.25`,color=black);

```

$$\epsilon := .25$$

$$N := 300$$

$$Rn_208_half_life_24.4_minute$$

$$half_life := 24.4$$

$$t := 1$$

$$\lambda := .02840$$

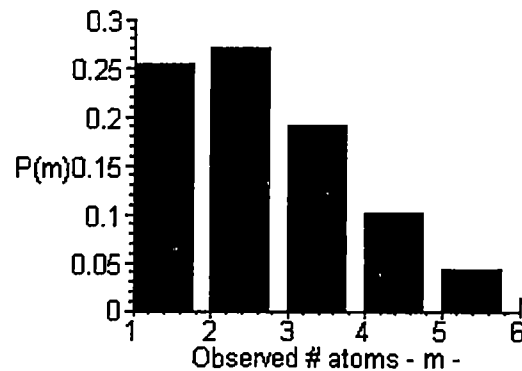
$$MEAN := 2.130$$

$$Probability_of_observing_count = .1188 \frac{2.130^{m\sim}}{m\sim!}$$

Warning, new definition for norm

Warning, new definition for trace

Posterior Density for observed Number of atoms Rn-208 Eff. 0.25



(3) The third set of values is:

1. Efficiency = 0.5
2. Number of atoms present = 300 atoms
3. Rn-208 half-life = 24.4 minutes
4. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart; Digits:=4; assume(epsilon,real): assume(N,integer): assume(half_life,real):
assume(lambda,real): assume(t,real): assume(m,integer): assume(epsilon>0,N>0,half
```

```

_life>0,lambda>0,t>0,m>0):

> epsilon:=.5;N:=300;`Rn_208_half_life_24.4_minute`;half_life:=24.4;t:=1;

> lambda:=0.693/half_life;

> MEAN:=epsilon*N*lambda*t;

> Probability_of_observing_count:=(MEAN^m)*(exp(-MEAN))/m!;p:=%:

> with(plots):

with(linalg):

c:=linalg[matrix](15,1,[seq(evalf(subs(m=j,Probability_of_observing_count)),j=1..1
5)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.25,orientation=[-90,90],view=
[1..12,1..1.5,0..0.22],axes=frame,tickmarks=[10,3,10],labels=['Observed # atoms -
m -`,`',`P(m)`],title=`Posterior Density for observed Number of atoms Rn-208 Eff.
50%`,color=black);

```

$$\varepsilon := .5$$

$$N := 300$$

$$Rn_208_half_life_24.4_minute$$

$$half_life := 24.4$$

$$t := 1$$

$$\lambda = .02840$$

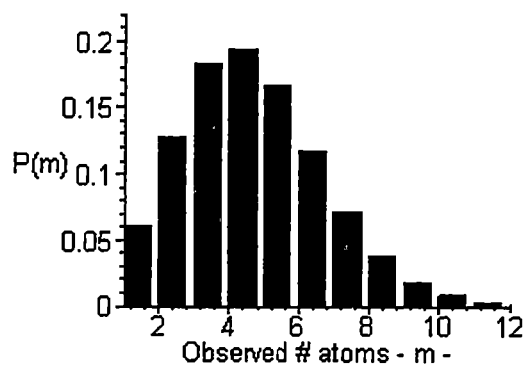
$$MEAN = 4.260$$

$$Probability_of_observing_count = .01412 \frac{4.260^{m\sim}}{m\sim!}$$

Warning, new definition for norm

Warning, new definition for trace

Posterior Density for observed Number of atoms Rn-208 Eff. 50%



(4) The fourth set of values is:

1. Efficiency = 0.5
2. Number of atoms present = 3000 atoms
3. Rn-208 half-life = 24.4 minutes
4. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart:Digits:=4:assume(epsilon,real):assume(N,integer):assume(half_life,real):
assume(lambda,real):assume(t,real):assume(m,integer):assume(epsilon>0,N>0,half
_life>0,lambda>0,t>0,m>0):

> epsilon:=.5;N:=3000;`Rn_208_half_life_24.4_minute`;half_life:=24.4;t:=1;

> lambda:=0.693/half_life;

> MEAN:=epsilon*N*lambda*t;

> Probability_of_observing_count:=(MEAN^m)*(exp(-MEAN))/m!;p:=%:

> with(plots):

with(linalg):

c:=linalg[matrix](200,1,[seq(evalf(subs(m=j,Probability_of_observing_count)),j=1..
200)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.4,orientation=[-90,90],view=
[25..62,1..1.5,0..0.07],axes=frame,tickmarks=[10,3,10],labels=[`Observed # atoms -
m -`,``,`P(m)`],title=`Posterior Density for observed Number of atoms Rn-208 Eff.
50%`,`color=black);
```

$$\varepsilon = .5$$

$$N = 3000$$

Rn_208_half_life_24.4_minute

half_life := 24.4

t := 1

λ := .02840

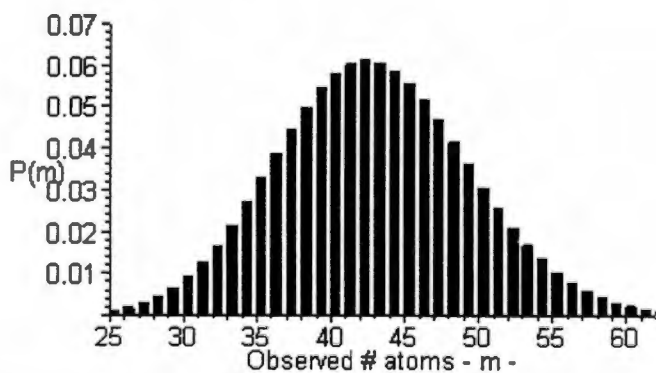
MEAN := 42.60

Probability_of_observing_count := .3155 $10^{-18} \frac{42.60^{m_{\sim}}}{m_{\sim}!}$

Warning, new definition for norm

Warning, new definition for trace

Posterior Density for observed Number of atoms Rn-208 Eff. 50%



(5) The fifth set of values is:

1. Efficiency = 0.5
2. Number of atoms present = 3000 atoms
3. Rn-222 half-life = 5505.84 minutes
4. Counting interval length = 1 minute

MAPLE CODE LINES

```
> restart:Digits:=4:assume(epsilon,real):assume(N,integer):assume(half_life,real):
assume(lambda,real):assume(t,real):assume(m,integer):assume(epsilon>0,N>0,half
_life>0,lambda>0,t>0,m>0):

> epsilon:=.5;N:=3000;`Rn_222_half_life_5505.84_minute`;half_life:=5505.84;
t:=1;

> lambda:=0.693/half_life;

> MEAN:=epsilon*N*lambda*t;

> Probability_of_observing_count:=(MEAN^m)*(exp(-MEAN))/m!;p:=%:

> with(plots):

with(linalg):

c:=linalg[matrix](25,1,[seq(evalf(subs(m=j,Probability_of_observing_count)),j=1..2
5)]):

> matrixplot(c,heights=HISTOGRAM,gap=0.25,orientation=[-90,90],view=
```

```
[1..8,1..1.5,0..0.17],axes=frame,tickmarks=[10,3,10],labels=[' Observed # atoms - m
-','`P(m)`'],title='Posterior Density for observed Number of atoms Rn-222 Eff.
50%`,color=black);
```

$$\varepsilon := .5$$

$$N := 3000$$

$$Rn_222_half_life_5505.84_minute$$

$$half_life := 5505.84$$

$$t := 1$$

$$\lambda := .0001258$$

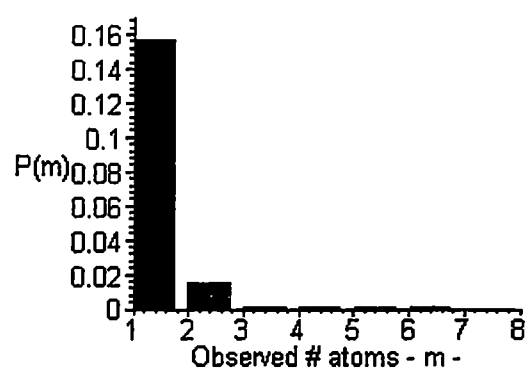
$$MEAN := .1887$$

$$Probability_of_observing_count := .8280 \frac{.1887^{m\sim}}{m\sim!}$$

Warning, new definition for norm

Warning, new definition for trace

Posterior Density for observed Number of atoms Rn-222 Eff. 50%



APPENDIX 6

STANDARD SOURCE MODEL

STANDARD SOURCE MODEL

$$p(N_s | \mu_s, \sigma_s) = \frac{\lambda_s}{\sqrt{2\pi\sigma_s}} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2\sigma_s}}\right)^2}$$

Using the above formula the distribution for the number of atoms in a standard source is plotted for the following values:

$$\mu_s = 76.00 \text{ dpm}$$

$$\sigma_s = 3.00 \text{ dpm}$$

$$T_{1/2} = 2.411 \times 10^4 \text{ years}$$

MAPLE CODE LINES

```
> restart; Digits:=4;
```

```
> Number_of_atoms_in_a_Standard_Source:=NS;
```

```
Standard_Source_Distribution:=nd; Decay_Constant:=lambda; Mean_of_Standard  
_Source:=mu; Standard_deviation_of_Standard_Source:=sigma;
```

```
> assume(NS, integer): assume(mu, real): assume(sigma, real):
```

```
assume(NS>0, mu>0, sigma>0):
```

```
> lambda:=0.693/(2.411e4*365*24*60);
```

```

> nd:=lambda*(exp(-0.5*(((lambda*NS)-
mu)/sigma))^2)/(sigma*sqrt(2*Pi));nd:=evalf(subs({mu=76.00,sigma=3.00},nd));
mu= 76.00 dpm`;sigma=3.00 dpm`;T1/2=2.411e4 years`;
plot(nd,NS=1.18e12..1.6e12,labels=['Number of atoms in Standard
Source`,`P(NS)`],title='Number of atoms Distribution for Standard Source
mu=76.00 dpm & sigma=3.00 dpm`);

```

$$\text{Number_of_atoms_in_a_Standard_Source} := NS$$

$$\text{Standard_Source_Distribution} := nd$$

$$\text{Decay_Constant} := \lambda$$

$$\text{Mean_of_Standard_Source} := \mu$$

$$\text{Standard_deviation_of_Standard_Source} := \sigma$$

$$\lambda := .5470 \cdot 10^{-10}$$

$$nd := .2735 \cdot 10^{-10} \frac{e^{\left(-0.5 \frac{(.5470 \cdot 10^{-10} NS - \mu)^2}{\sigma^2} \right)}}{\sigma \sqrt{\pi}}$$

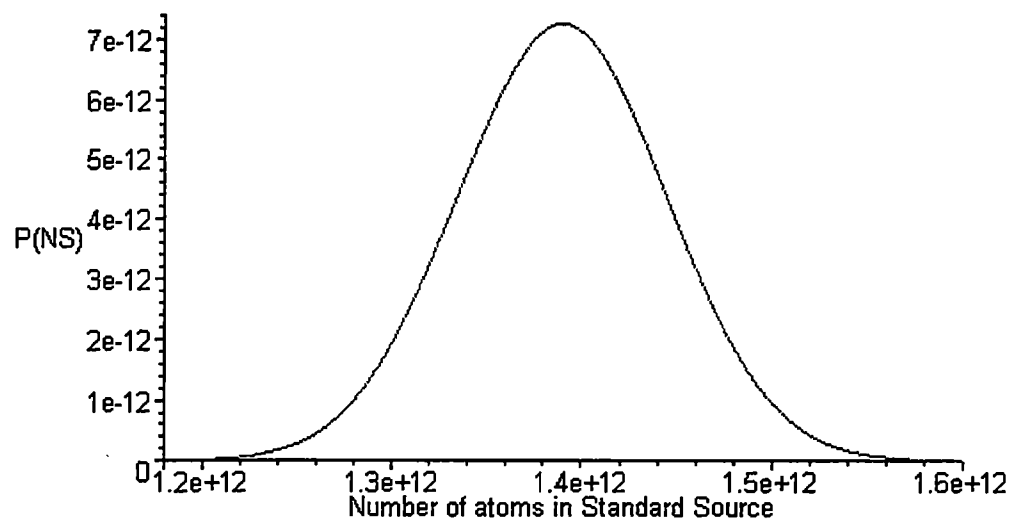
$$nd := .7270 \cdot 10^{-11} e^{\left(-0.05555 (.5470 \cdot 10^{-10} NS - 76.00)^2 \right)}$$

$$\mu = 76.00 \text{ dpm}$$

$$\sigma = 3.00 \text{ dpm}$$

$$T_{1/2} = 2.411 \times 10^4 \text{ years}$$

Number of atoms Distribution for Standard Source $\mu = 76.00 \text{ dpm}$ & $\sigma = 3.00 \text{ dpm}$



APPENDIX 7

POSTERIOR DENSITY OF THE

ABSOLUTE EFFICIENCY

EFFICIENCY MODELING

$$p(\epsilon | m_s, \mu_s, \sigma_s, \lambda_s) = \int_0^{\infty} \frac{\lambda_s (\epsilon N_s \lambda_s t)^{m_s}}{c m_s! \sigma_s \sqrt{2\pi}} e^{-\epsilon N_s \lambda_s t} e^{-\left(\frac{\lambda_s N_s - \mu_s}{\sqrt{2}\sigma_s}\right)^2} dN_s$$

The above formula is the posterior density for the absolute efficiency. The modeling of the absolute efficiency of a detector is done in steps:

1. Modeling of the detection process using standard source, where
 $m_s = 14$ dpm
2. Modeling of standard source, $\lambda_s = 2.411 \times 10^4$ years, $\mu_s = 76.00$ dpm
and $\sigma_s = 3.00$ dpm
3. Combining the above models to get the posterior density for the overall detector efficiency.

A plot of the standard source number of atoms distribution and joint density for N_s and Efficiency was produced. The posterior density numerically evaluated using two methods. The result is plotted and estimation of the efficiency was computed.

MAPLE CODE LINES

```
> restart:Digits:=4:with(student):

> assume(NS,integer):assume(ms,integer):assume(epsilon,real):
assume(mu,integer):assume(sigma,real):assume(NS>0,ms>0,NS>ms,epsilon>0,epsil
lon<=1,mu>0,sigma>0):

> Standard_Source_Number_of_atoms:=NS;
Observed_atoms_by_a_detector:=ms;Detector_Efficiency:=epsilon;Source_Decay
_Constant:=lambda;observation_time:=t;Detection_Probability_Distribution:=p;S
tandard_Source_number_of_atoms_Probability_Distribution:=nd;Distribution_of
_count_from_detector:=Count_Dist;Probability_Distribution_efficiency:=ANS;

> Source_Half_Life_in_minutes:=2.411e4*365*24*60;t:=1;

> lambda:=0.693/(Source_Half_Life);

> p:=((NS*epsilon*lambda*t)^ms)*(exp(-(NS*epsilon*lambda*t)))/ms!;

> nd:=lambda*(exp(-0.5*(((lambda*NS)-mu)/sigma))^2)/(sigma*sqrt(2*Pi));
`mu:=76.00 dpm`; `sigma=3.00 dpm`; nd:=subs({mu=76.00,sigma=3.00},nd);
plot(nd,NS=1.18e12..1.6e12,labels=['NS',`P(NS)`],title=`Number of atoms
Distribution Function for Standard with mu= 76.00 dpm & sigma=3.00 dpm`);

> Count_Dist:=simplify(nd*p);

> eval_count_Dist:=evalf(subs(ms=14,Count_Dist));
```

```

c:=evalf((value(simpson(value(simpson(eval_count_Dist,epsilon=0..1,12)),NS=1.18e
12..1.60e12,12)))));Normalized_joint_Dist:=simplify((eval_count_Dist)/c);

> plot3d(Normalized_joint_Dist,NS=1.18e12..1.6e12,epsilon=0..0.5,title=`Joint
Distribution NS & Efficiency`,labels=[`NS`,`Epsilon`,`Prob.`]);

> with(student,simpson):

ANS:=value(simpson(Normalized_joint_Dist,NS=1.12e12..1.60e12,54)):#
Evaluation of c using simpson

> with(student):

ANS2:=evalf(trapezoid(Normalized_joint_Dist,NS=1.18e12..1.6e12,140)):#
Evaluation of c using trapezoid

> with(plots): ANSplot:=plot(ANS,epsilon=0.05..0.5,linestyle=3):

ANS2plot:=plot(ANS2,epsilon=0.05..0.5,title=` Comparison of Numerical
Integration`,labels=[`Efficiency`,`P(epsilon)`],color=green):

p:=plots[display]({ANSplot,ANS2plot}):p1:=textplot([0.32,2,`- - Simpson
rule`],color=red,align={ABOVE,RIGHT}): p2:=textplot([0.21,8.5,`Trapezoid
rule`],color=green,align={ABOVE,RIGHT}):display({p,p1,p2});

> with(student,simpson);

Simpson_area_under_curve:=(evalf(simpson(ANS,epsilon=0..0.5,30)));t:=%: #
varification of the area under the curve

> with(student,trapezoid);

```

```

Trapezoid_area_under_curve:=(evalf(trapezoid(ANS2,epsilon=0..0.5,30)));t1:=%;
# varification of the area under the curve

> NANS:=(epsilon*ANS/t):NANS2:=(epsilon*ANS2/t1):

> with(student,simpson);
EXPECTED_epsilon_using_simpson:=evalf(simpson(NANS,epsilon=0.05..0.5,32));

> with(student,trapezoid);
EXPECTED_epsilon_using_trapezoid:=evalf(trapezoid(NANS2,epsilon=0.05..0.5,3
2));

```

Standard_Source_Number_of_atoms := NS~

Observed_atoms_by_a_detector := ms~

Detector_Efficiency := ϵ ~

Source_Decay_Constant := λ

observation_time := t

Detection_Probability_Distribution := p

Standard_Source_number_of_atoms_Probability_Distribution := nd

Distribution_of_count_from_detector := Count_Dist

Probability_Distribution_efficiency := ANS

$$\text{Source_Half_Life_in_minutes} := .1267 \cdot 10^{11}$$

$$t := 1$$

$$\lambda := .5470 \cdot 10^{-10}$$

$$p := \frac{(.5470 \cdot 10^{-10} \text{ NS} \sim \epsilon \sim)^{\text{ms} \sim} e^{(-.5470 \cdot 10^{-10} \text{ NS} \sim \epsilon \sim)}}{\text{ms} \sim !}$$

$$nd := .2735 \cdot 10^{-10} \frac{e^{\left(- .5 \frac{(.5470 \cdot 10^{-10} \text{ NS} \sim - \mu \sim)^2}{\sigma \sim^2} \right) \sqrt{2}}}{\sigma \sim \sqrt{\pi}}$$

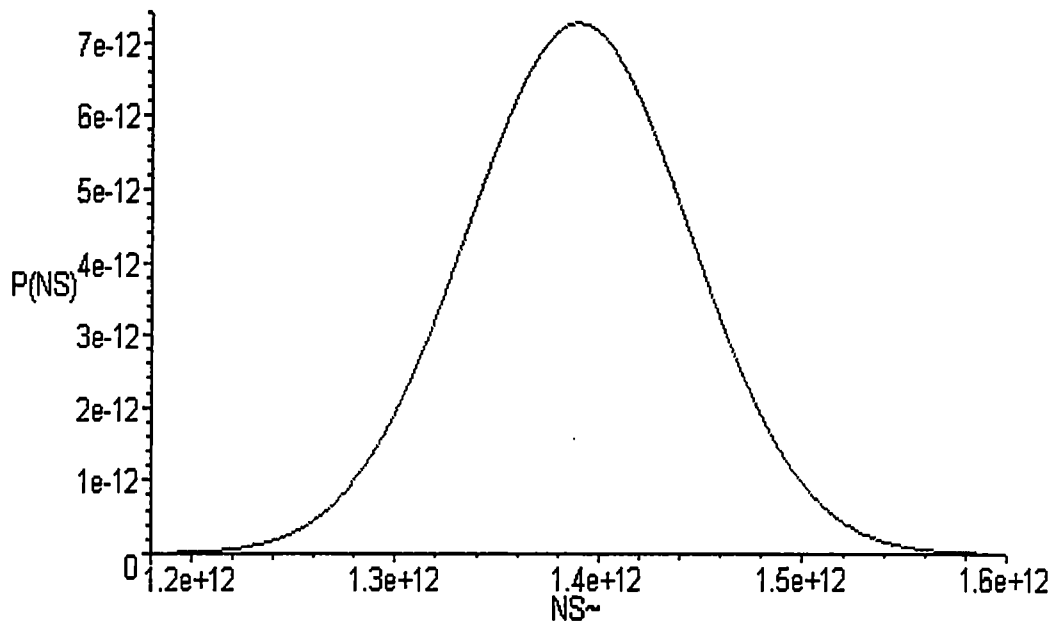
$$\mu := 76.00 \text{ dpm}$$

$$\sigma := 3.00 \text{ dpm}$$

$$T1/2 = 2.411 \cdot 10^4 \text{ years}$$

$$nd := .9116 \cdot 10^{-11} \frac{e^{\left(-.05555 (.5470 \cdot 10^{-10} \text{ NS} \sim - 76.00)^2 \right) \sqrt{2}}}{\sqrt{\pi}}$$

Number of atoms Distribution Function for Standard with $\mu=76.00$ dpm & $\sigma=3.00$ dpm



$$\text{Count_Dist} := .7270 \cdot 10^{-11}$$

$$e^{(-.1662 \cdot 10^{-21} \text{NS}^2 + .4618 \cdot 10^{-9} \text{NS} - 320.9 - 23.63 \text{ms} - .5470 \cdot 10^{-10} \text{NS} \cdot \epsilon)}$$

$$\text{NS}^{\text{ms}} \cdot \epsilon^{\text{ms}} / \Gamma(\text{ms} + 1.)$$

$$\text{eval_count_Dist} := .8339 \cdot 10^{-22}$$

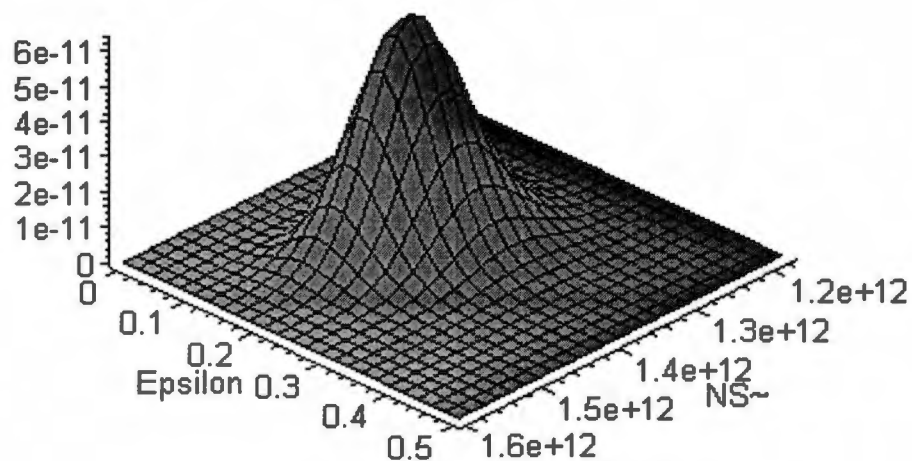
$$e^{(-.1662 \cdot 10^{-21} \text{NS}^2 + .4618 \cdot 10^{-9} \text{NS} - 651.7 - .5470 \cdot 10^{-10} \text{NS} \cdot \epsilon)} \text{NS}^{14} \epsilon^{14}$$

$$c := .01091$$

$$\text{Normalized_joint_Dist} := .7643 \cdot 10^{-20}$$

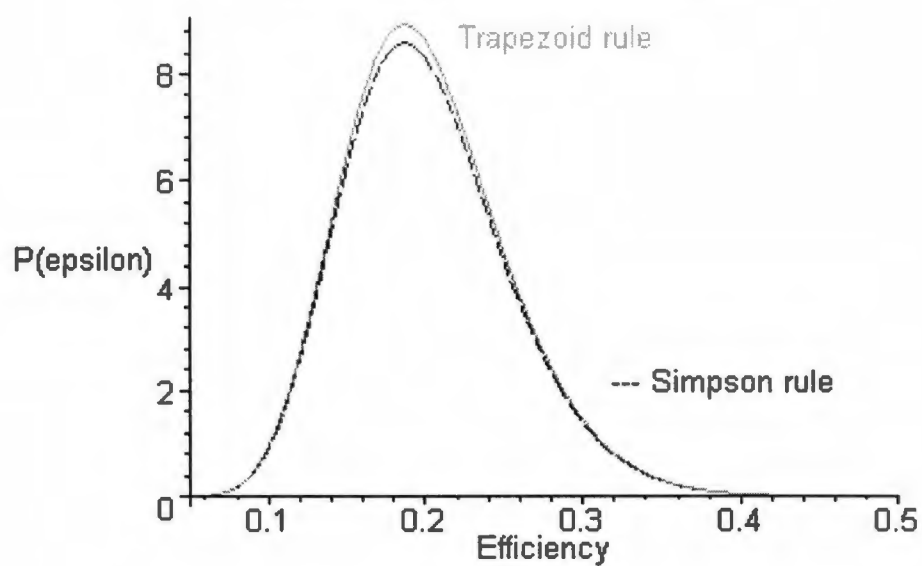
$$e^{(-.1662 \cdot 10^{-21} \text{NS}^2 + .4618 \cdot 10^{-9} \text{NS} - 651.7 - .5470 \cdot 10^{-10} \text{NS} \cdot \epsilon)} \text{NS}^{14} \epsilon^{14}$$

Joint Distribution NS & Efficiency



Warning, new definition for D

Comparison of Numerical Integration



[simpson]

Simpson_area_under_curve := 1.104

[trapezoid]

Trapezoid_area_under_curve := 1.033

[simpson]

EXPECTED_epsilon_using_simpson := .1980

[trapezoid]

EXPECTED_epsilon_using_trapezoid := .1970

APPENDIX 8

JOINT POSTERIOR DISTRIBUTION FOR THE BACKGROUND COUNT RATE AND NET COUNT RATE

MODELING OF THE BACKGROUND COUNT RATE

AND THE NET COUNT RATE

$$p(\beta_0, \Delta | D) = \frac{1}{c_2} \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)}$$

$$c_2 = \int_0^\infty \int_0^\infty \beta_0^b (\beta_0 + \Delta)^s e^{-(\beta_0(t_1+t_0)+\Delta t_1)} d\beta_0 d\Delta$$

Using the formulas above, we calculated the posterior densities for background count rate and net count rates for several cases. We also generated plots for the joint distributions, background count rate, and net count rate for each case.

> restart;Digits:=4:

> assume(beta>0,Delta>0):

> Net_Count_Rate:=Delta;Background_Count_Rate:=beta;

Joint_Density_Background_and_Net_Count_Rate:=w;

Net_Count_Rate := Δ~

$$\text{Background_Count_Rate} := \beta \sim$$

$$\text{Joint_Density_Background_and_Net_Count_Rate} := w$$

Case 1:-

- Background count rate = 3 cpm
- Sample count rate = 21 cpm

MAPLE CODE LINES

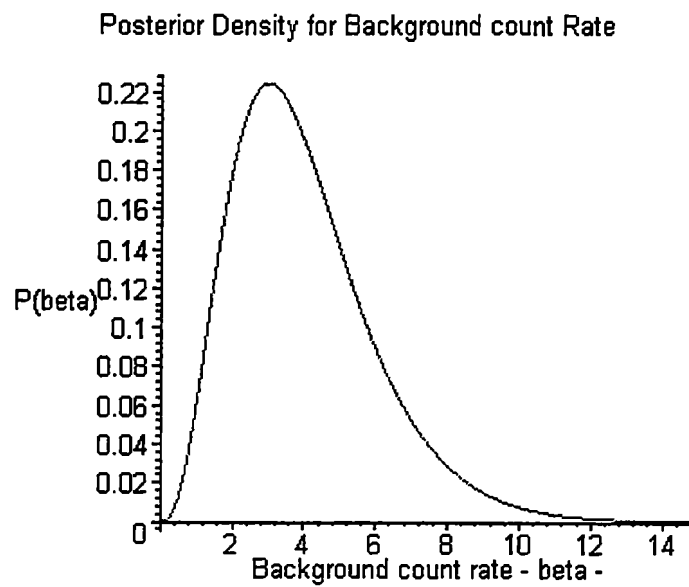
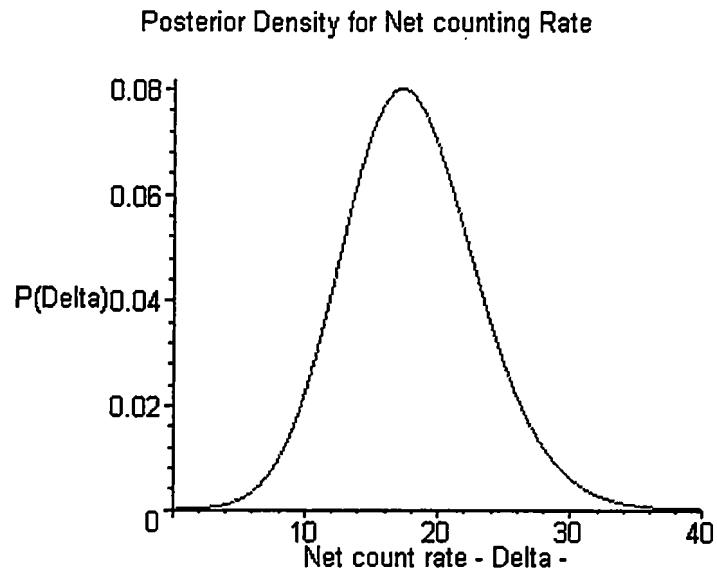
```
> w:=(beta^3)*((beta+Delta)^21)*(exp(-
(2*beta+Delta)));c:=evalf(int(int(w,beta=0..infinity),Delta=0..infinity));w:=w/c;
```

$$w := \beta^3 (\beta + \Delta)^{21} e^{(-2\beta - \Delta)}$$

$$c := .3065 \cdot 10^{21}$$

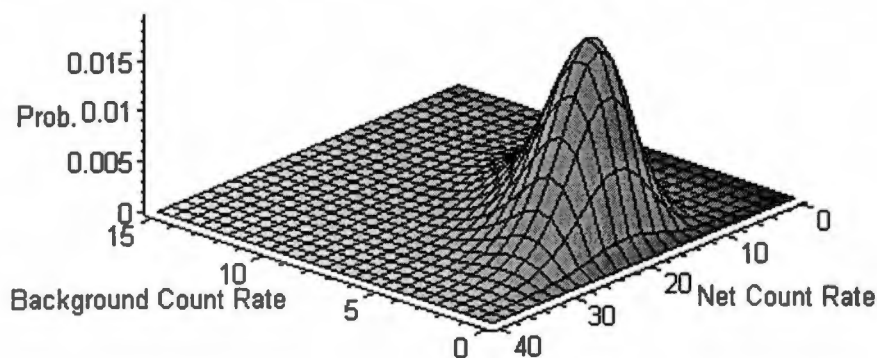
$$w := .3263 \cdot 10^{-20} \beta^3 (\beta + \Delta)^{21} e^{(-2\beta - \Delta)}$$

```
> with(plots):plot(int(w,beta=0..infinity),Delta=0..40,title='Posterior Density for
Net counting Rate',labels=['Net count rate - Delta -', 'P(Delta)']);
with(plots):plot(int(w,Delta=0..infinity),beta=0..15,title='Posterior Density for
Background count Rate', labels=['Background count rate - beta -', 'P(beta)']);
```



```
> plot3d(w,beta=0..15,Delta=0..40,title='Joint Density for Net & Background
Count Rate ',labels=['Background Count Rate','Net Count Rate','Prob.']);
```

Joint Density for Net & Background Count Rate



Case 2:-

- Background count rate = 3 cpm
- Sample count rate = 183 cpm

MAPLE CODE LINES

```
> restart; Digits:=4: assume(beta>0, Delta>0): with(student): with(plots):

> w:=(beta^3)*((beta+Delta)^183)*(exp(-(2*beta+Delta))); c:=evalf(simpson(evalf(simpson(w,beta=0..30,64)),Delta=0..250,64)); w:=w/c;

plot(simpson(w,beta=0..20,64),Delta=130..240,adaptive=true,numpoints=50,title='Posterior density for Net Count Rate', labels=['Net count rate - Delta -', 'P(Delta)']); plot(simpson(w,Delta=0..2000,64),beta=0..13,title='Posterior Density
```

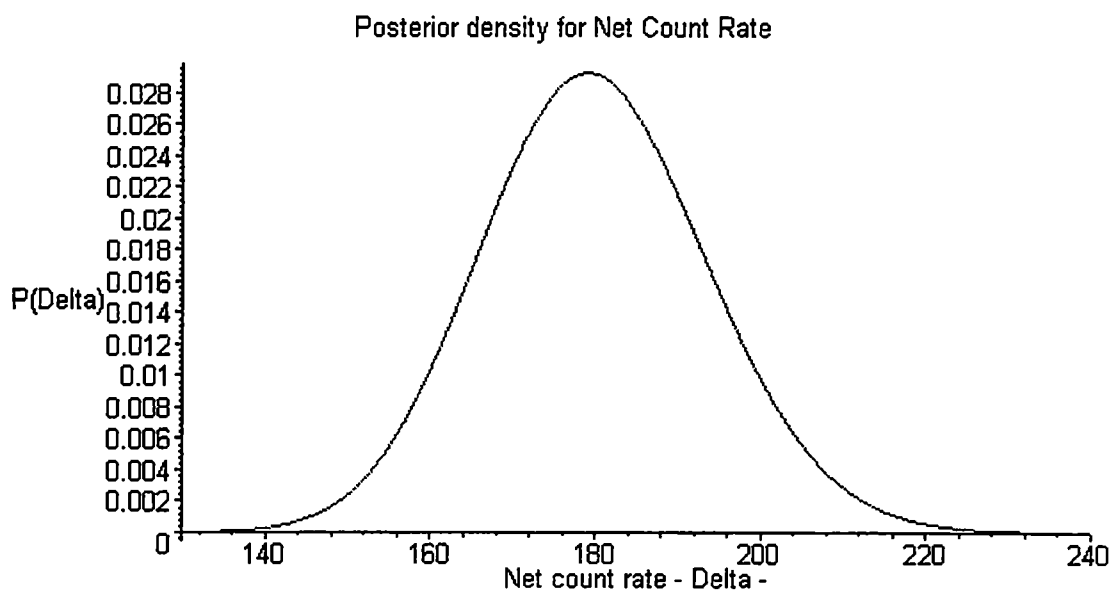
```
for Background Count Rate`,labels=['Background Count Rate - beta -`,`P(beta)`]);
```

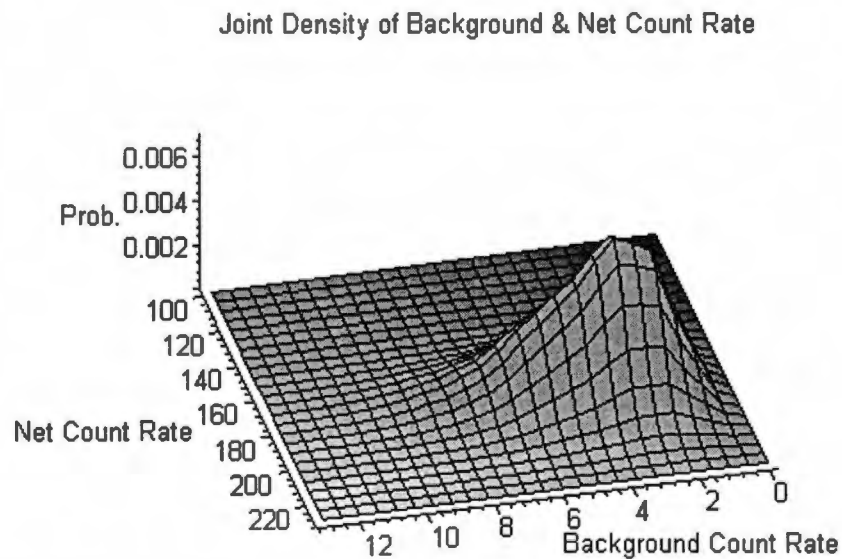
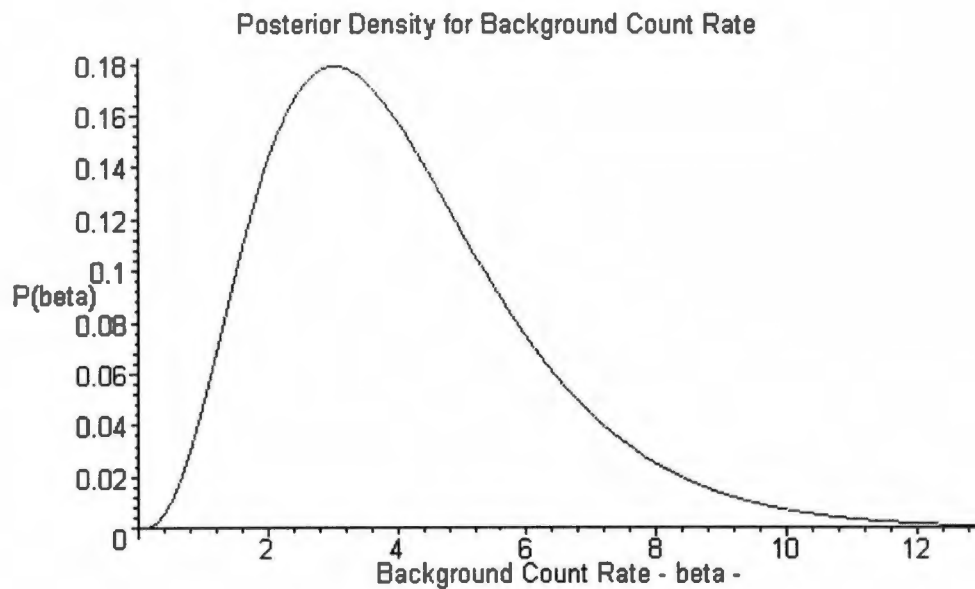
```
> plot3d(w,beta=0..13,Delta=100..230,title='Joint Density of Background & Net  
Count Rate`,labels=['Background Count Rate`,`Net Count Rate`,`Prob.`]);
```

$$w = \beta^3 (\beta + \Delta)^{183} e^{(-2\beta - \Delta)}$$

$$c = .7182 \cdot 10^{337}$$

$$w = .1392 \cdot 10^{-336} \beta^3 (\beta + \Delta)^{183} e^{(-2\beta - \Delta)}$$





Case 3:-

- Background count rate = 3 cpm
- Sample count rate = 1803 cpm

MAPLE CODE LINES

```
> restart; Digits:=4: assume(beta>0, Delta>0): with(student): with(plots):

> w:=(beta^3)*((beta+Delta)^1803)*(exp(-
(2*beta+Delta))); c:=evalf(simpson(evalf(simpson(w,beta=0..30,64)),Delta=1500..22
00,64)); w:=w/c;

plot3d(w,beta=0..10,Delta=1650..2100,grid=[11,25],view=[0..10,1650..2100,0..0.0025
],title='Joint Density for Background & Net Count Rate',labels=['Background
Count Rate','Net Count Rate','Prob.']);

> with(student): with(plots):

plot(simpson(w,beta=0..10,32),Delta=1650..2000,title='Posterior Density for Net
Count Rate',labels=['Net count rate - Delta - ', 'P(Delta)']);

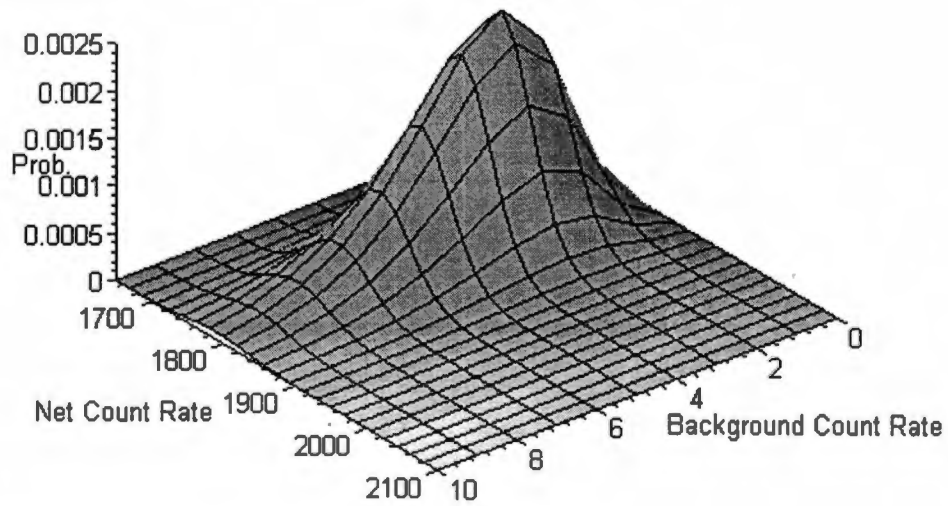
plot(simpson(w,Delta=0..2000,32),beta=0..15,adaptive=false,title='Posterior
Density for Background Count Rate',labels=['Background Count Rate - beta -
','P(beta)']);
```

$$w := \beta^3 (\beta + \Delta)^{1803} e^{(-2\beta - \Delta)}$$

$$c := .1895 \cdot 10^{5091}$$

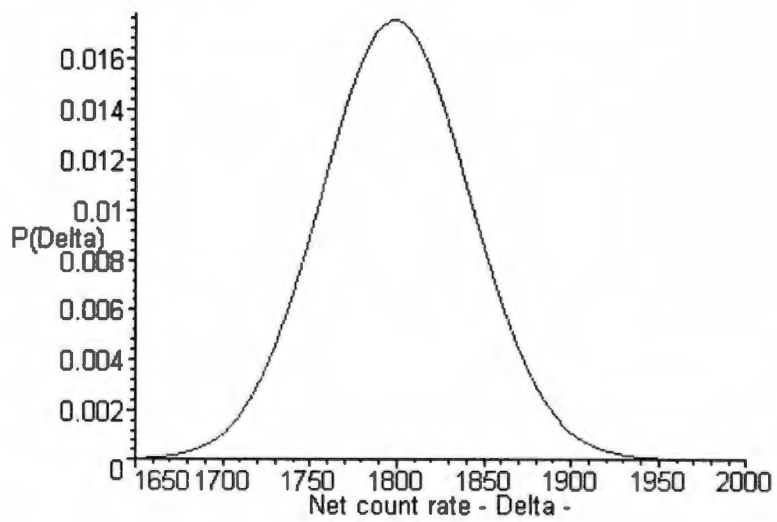
$$w := .5277 \cdot 10^{-5090} \beta^3 (\beta + \Delta)^{1803} e^{(-2\beta - \Delta)}$$

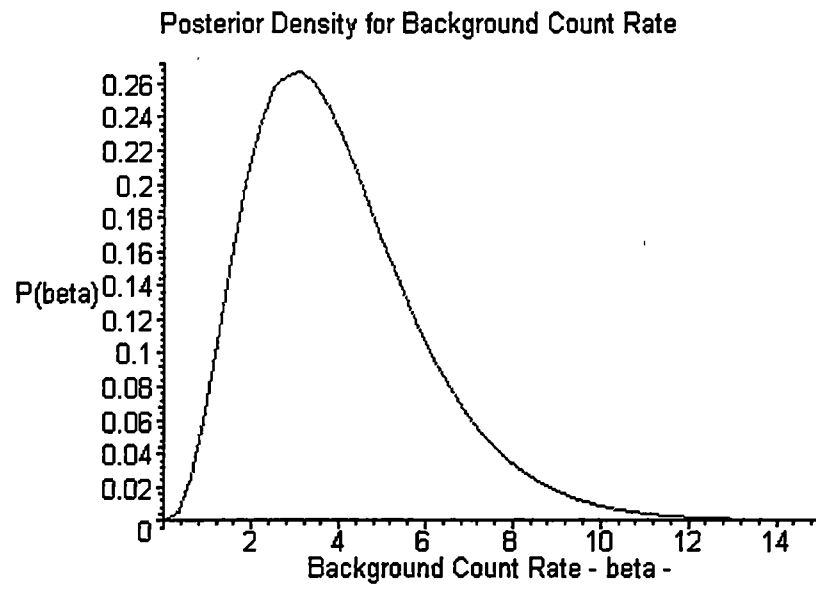
Joint Density for Background & Net Count Rate



Warning, new definition for D

Posterior Density for Net Count Rate





APPENDIX 9

POSTERIOR DENSITIES FOR

SAMPLE ACTIVITY AND NUMBER OF

ATOMS

POSTERIOR DENSITY FOR SAMPLE ACTIVITY AND NUMBER OF ATOMS

$$p(A) = \int_0^{\infty} p(A | D) p(m_s) dm_s$$

$$p(N) = \lambda p_A(\lambda N)$$

The efficiency distribution and net rate distribution was calculated in appendix 7 and 8. Those results were used to calculate the distributions for sample Activity then sample number of atoms using the above formulas.

MAPLE CODE LINES

```
> restart:Digits:=4:

> assume(NS,integer):assume(ms,integer):assume(epsilon,real):
assume(NS>0,ms>0,NS>ms,epsilon>0,epsilon<=1): with(student):with(plots):

> Number_of_atoms_in_a_Standard_Source:=NS;
Observed_atoms_in_a_Detector:=ms;Detector_Efficiency:=epsilon;Standard_Dec
```

```

ay_Constant:=lambda;Observation_time:=t;Detection_Probability_Distribution:=
p;Detection_Probability_Distribution_in_Factorial_form:=np;Standard_Source_P
robability_Distribution:=nd;Posterior_Density_for_overall_efficiency:=ANS;Num
ber_of_atoms_in_sample_in_question:=AN;

> lambda:=0.693/(2.411e4*365*24*60);t:=1;

> p:=((NS*epsilon*lambda*t)^ms)*(exp(-NS*epsilon*lambda*t))/ms!;

> nd:=lambda*(exp(-0.5*(((lambda*NS)-
mu)/sigma))^2))/(sigma*sqrt(2*Pi));`mu:=76.00 dpm`;`sigma=3.00
dpm`;nd:=subs({mu=76.00,sigma=3.00},nd);plot(nd,NS=1.18e12..1.60e12,labels=['
NS`,`P(NS)`],title=`Number of atoms Distribution for Standard with mu= 76.00
dpm & sigma=3.00 dpm`);

> efficiency_Dist:=(nd*p):

> eval_efficiency_Dist:=subs(ms=14,efficiency_Dist):

> with(student,simpson):
ANS:=evalf(simpson((eval_efficiency_Dist),NS=1.18e12..1.6e12,54));

> with(plots):
ANSplot:=plot(ANS/(evalf(simpson(ANS,epsilon=0..0.5,32))),epsilon=0.05..0.4,label
s=['Epsilon`,`P(epsilon)`]):p:=plots[display](ANSplot):p1:=textplot([0.2,8.75,`Poste
rior Distribution for
Efficiency`],color=red,align={ABOVE,CENTER}):display({p,p1});ANS:=ANS/(eva

```

If(simpson(ANS,epsilon=0..0.5,30)):EXPECTED_epsilon:=(evalf(simpson(ANS*epsilon,epsilon=0.05..0.5,30)));

Number_of_atoms_in_a_Standard_Source := NS~

Observed_atoms_in_a_Detector := ms~

Detector_Efficiency := ε~

Standard_Decay_Constant := λ

Observation_time := t

Detection_Probability_Distribution := p

Detection_Probability_Distribution_in_Factorial_form := np

Standard_Source_Probability_Distribution := nd

Posterior_Density_for_overall_efficiency := ANS

Number_of_atoms_in_sample_in_question := AN

$$\lambda := .5470 \cdot 10^{-10}$$

$$t := 1$$

$$p := \frac{(.5470 \cdot 10^{-10} \text{ NS~ } \epsilon\text{~})^{ms\text{~}} e^{(-.5470 \cdot 10^{-10} \text{ NS~ } \epsilon\text{~})}}{ms\text{~}!}$$

$$nd = .2735 \cdot 10^{-10} \frac{e}{\sigma \sqrt{\pi}} \left(-\frac{.5470 \cdot 10^{-10} NS_{\sim} - \mu}{\sigma^2} \right)^2 \sqrt{2}$$

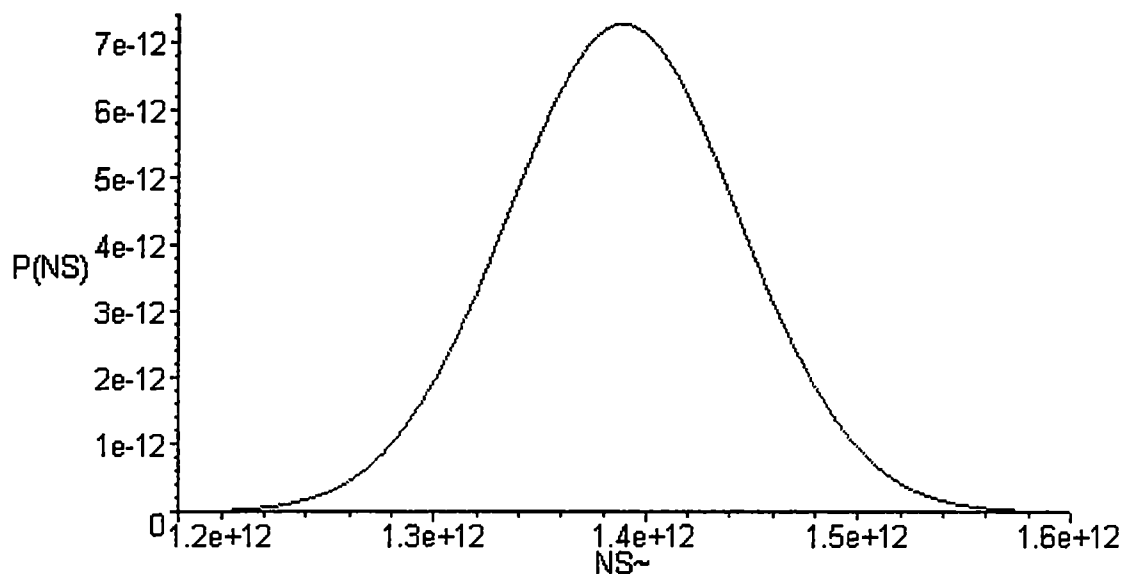
$$\mu := 76.00 \text{ dpm}$$

$$\sigma = 3.00 \text{ dpm}$$

$$T_{1/2} = 2.411 \cdot 10^4 \text{ years}$$

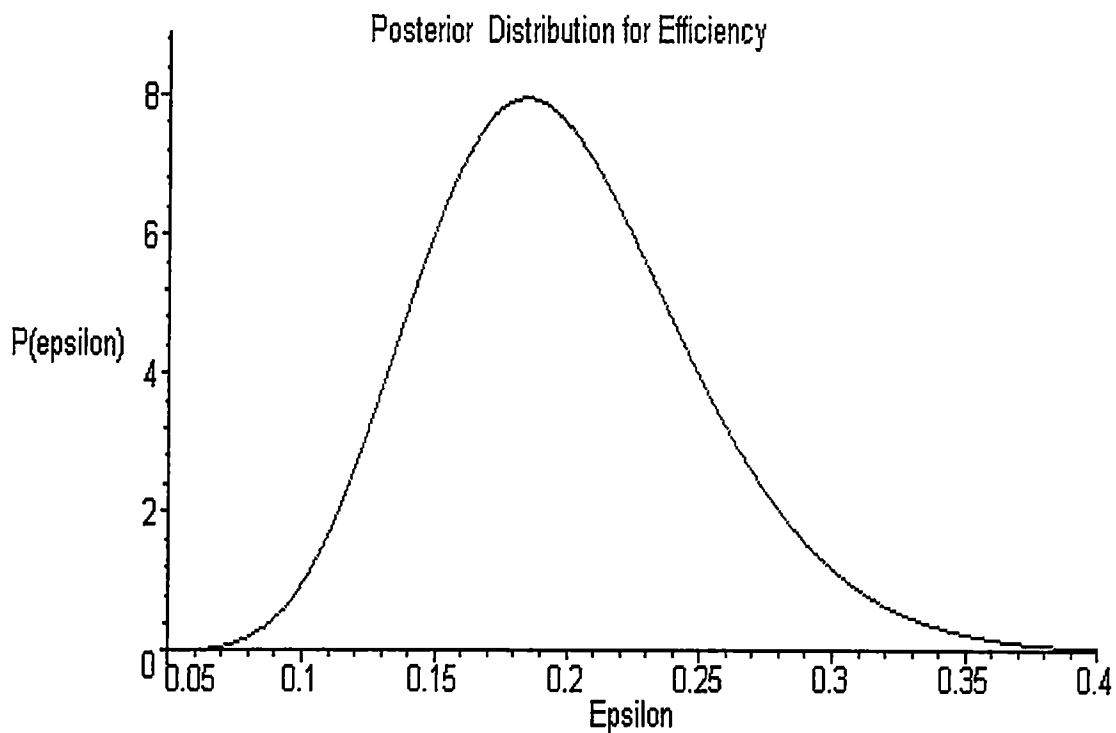
$$nd = .9116 \cdot 10^{-11} \frac{e}{\sqrt{\pi}} \left(-\frac{.05555 (.5470 \cdot 10^{-10} NS_{\sim} - 76.00)}{\sigma^2} \right)^2 \sqrt{2}$$

Number of atoms Distribution for Standard with $\mu = 76.00$ dpm & $\sigma = 3.00$ dpm



$$\begin{aligned}
ANS = & .7338 \cdot 10^{14} \varepsilon \sim^{14} e^{(-81.9822 \varepsilon \sim)} + .1072 \cdot 10^{15} \varepsilon \sim^{14} e^{(-81.1310 \varepsilon \sim)} \\
& + .2687 \cdot 10^{14} \varepsilon \sim^{14} e^{(-83.6844 \varepsilon \sim)} + .2367 \cdot 10^{11} \varepsilon \sim^{14} e^{(-64.9595 \varepsilon \sim)} \\
& + .6074 \cdot 10^{13} \varepsilon \sim^{14} e^{(-70.5039 \varepsilon \sim)} + .8569 \cdot 10^{14} \varepsilon \sim^{14} e^{(-75.6107 \varepsilon \sim)} \\
& + .4452 \cdot 10^{14} \varepsilon \sim^{14} e^{(-81.5686 \varepsilon \sim)} + .9985 \cdot 10^{14} \varepsilon \sim^{14} e^{(-76.4618 \varepsilon \sim)} \\
& + .9648 \cdot 10^{14} \varepsilon \sim^{14} e^{(-79.0152 \varepsilon \sim)} + .3889 \cdot 10^{13} \varepsilon \sim^{14} e^{(-69.2152 \varepsilon \sim)} \\
& + .1059 \cdot 10^{15} \varepsilon \sim^{14} e^{(-78.1641 \varepsilon \sim)} + .8418 \cdot 10^{13} \varepsilon \sim^{14} e^{(-70.0663 \varepsilon \sim)} \\
& + .8092 \cdot 10^{14} \varepsilon \sim^{14} e^{(-79.8664 \varepsilon \sim)} + .1160 \cdot 10^{14} \varepsilon \sim^{14} e^{(-71.3551 \varepsilon \sim)} \\
& + .6477 \cdot 10^{12} \varepsilon \sim^{14} e^{(-67.5129 \varepsilon \sim)} + .3305 \cdot 10^{14} \varepsilon \sim^{14} e^{(-73.0573 \varepsilon \sim)} \\
& + .2041 \cdot 10^{14} \varepsilon \sim^{14} e^{(-72.2062 \varepsilon \sim)} + .9824 \cdot 10^{13} \varepsilon \sim^{14} e^{(-84.1220 \varepsilon \sim)} \\
& + .1678 \cdot 10^{14} \varepsilon \sim^{14} e^{(-70.9175 \varepsilon \sim)} + .1765 \cdot 10^{14} \varepsilon \sim^{14} e^{(-83.2709 \varepsilon \sim)} \\
& + .5039 \cdot 10^{13} \varepsilon \sim^{14} e^{(-84.9732 \varepsilon \sim)} + .3080 \cdot 10^{14} \varepsilon \sim^{14} e^{(-71.7686 \varepsilon \sim)} \\
& + .6256 \cdot 10^{14} \varepsilon \sim^{14} e^{(-80.7175 \varepsilon \sim)} + .2920 \cdot 10^{14} \varepsilon \sim^{14} e^{(-82.4198 \varepsilon \sim)} \\
& + .1533 \cdot 10^{15} \varepsilon \sim^{14} e^{(-75.1731 \varepsilon \sim)} + .2381 \cdot 10^{13} \varepsilon \sim^{14} e^{(-85.8243 \varepsilon \sim)} \\
& + .2105 \cdot 10^{12} \varepsilon \sim^{14} e^{(-87.52 \varepsilon \sim)} + .3238 \cdot 10^{10} \varepsilon \sim^{14} e^{(-64.55 \varepsilon \sim)} \\
& + .4929 \cdot 10^{14} \varepsilon \sim^{14} e^{(-73.9085 \varepsilon \sim)} + .1037 \cdot 10^{13} \varepsilon \sim^{14} e^{(-86.6754 \varepsilon \sim)} \\
& + .5205 \cdot 10^{14} \varepsilon \sim^{14} e^{(-72.6197 \varepsilon \sim)} + .8101 \cdot 10^{14} \varepsilon \sim^{14} e^{(-73.4709 \varepsilon \sim)}
\end{aligned}$$

$$\begin{aligned}
& +.1161 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-74.3220 \, \varepsilon_{\sim})} +.1072 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-77.3130 \, \varepsilon_{\sim})} \\
& +.6769 \, 10^{14} \, \varepsilon_{\sim}^{14} \, e^{(-74.7596 \, \varepsilon_{\sim})} +.1654 \, 10^{13} \, \varepsilon_{\sim}^{14} \, e^{(-68.3641 \, \varepsilon_{\sim})} \\
& +.2153 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-77.7265 \, \varepsilon_{\sim})} +.7750 \, 10^{11} \, \varepsilon_{\sim}^{14} \, e^{(-65.8107 \, \varepsilon_{\sim})} \\
& +.1437 \, 10^{14} \, \varepsilon_{\sim}^{14} \, e^{(-84.5356 \, \varepsilon_{\sim})} +.2087 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-76.8754 \, \varepsilon_{\sim})} \\
& +.3220 \, 10^{13} \, \varepsilon_{\sim}^{14} \, e^{(-86.2378 \, \varepsilon_{\sim})} +.7089 \, 10^{13} \, \varepsilon_{\sim}^{14} \, e^{(-85.3867 \, \varepsilon_{\sim})} \\
& +.2046 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-78.5776 \, \varepsilon_{\sim})} +.1791 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-79.4288 \, \varepsilon_{\sim})} \\
& +.1348 \, 10^{13} \, \varepsilon_{\sim}^{14} \, e^{(-87.0890 \, \varepsilon_{\sim})} +.2208 \, 10^{11} \, \varepsilon_{\sim}^{14} \, e^{(-65.3971 \, \varepsilon_{\sim})} \\
& +.1444 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-80.2799 \, \varepsilon_{\sim})} +.1999 \, 10^{12} \, \varepsilon_{\sim}^{14} \, e^{(-67.0994 \, \varepsilon_{\sim})} \\
& +.6925 \, 10^{11} \, \varepsilon_{\sim}^{14} \, e^{(-66.2483 \, \varepsilon_{\sim})} +.2335 \, 10^{12} \, \varepsilon_{\sim}^{14} \, e^{(-66.6618 \, \varepsilon_{\sim})} \\
& +.5311 \, 10^{12} \, \varepsilon_{\sim}^{14} \, e^{(-67.9505 \, \varepsilon_{\sim})} +.4626 \, 10^{14} \, \varepsilon_{\sim}^{14} \, e^{(-82.8333 \, \varepsilon_{\sim})} \\
& +.1300 \, 10^{13} \, \varepsilon_{\sim}^{14} \, e^{(-68.8017 \, \varepsilon_{\sim})} +.2929 \, 10^{13} \, \varepsilon_{\sim}^{14} \, e^{(-69.6528 \, \varepsilon_{\sim})} \\
& +.1863 \, 10^{15} \, \varepsilon_{\sim}^{14} \, e^{(-76.0242 \, \varepsilon_{\sim})}
\end{aligned}$$



EXPECTED_epsilon := .1980

```

> assume(beta,real):assume(Delta,real):assume(b,integer): assume(s,integer):
assume(t1,real): assume(t2,real):assume(beta>0,Delta>0,b>0,s>0,t1>0,t2>0):

> Background_Mean:=beta;Background_count:=b;Sample_count:=s;
Net_count:=Delta;Time_to_count_Background:=t1;Time_to_count_Sample:=t2;N
et_count_Probability_Distribution:=Ncpd;

> w:=(beta^3)*((beta+Delta)^21)*(exp(-
(2*beta+Delta)));c:=evalf(int(int(w,beta=0..infinity),Delta=0..infinity));w:=w/c;Ncp
d:=int(w,beta=0..infinity);

```

```
> with(plots):plot(Ncpd,Delta=0..40,title='Posterior Density for Net counting
Rate',labels=['Net count rate - Delta - ','P(Delta)']);
EXPECTED_Net:=int(Delta*Ncpd,Delta=0..40);
```

$$\text{Background_Mean} := \beta \sim$$

$$\text{Background_count} := b \sim$$

$$\text{Sample_count} := s \sim$$

$$\text{Net_count} := \Delta \sim$$

$$\text{Time_to_count_Background} := t1 \sim$$

$$\text{Time_to_count_Sample} := t2 \sim$$

$$\text{Net_count_Probability_Distribution} := \text{Ncpd}$$

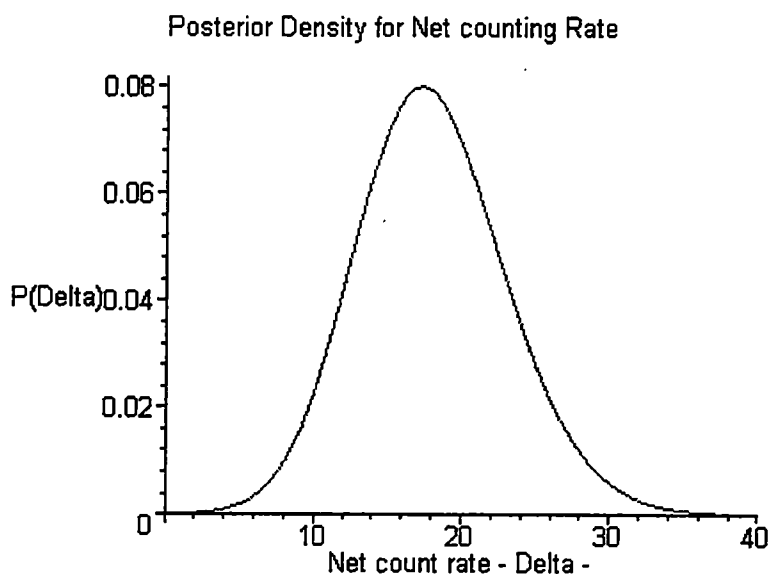
$$w := \beta \sim^3 (\beta \sim + \Delta \sim)^{21} e^{(-2 \beta \sim - \Delta \sim)}$$

$$c := .3065 \cdot 10^{21}$$

$$w := .3263 \cdot 10^{-20} \beta \sim^3 (\beta \sim + \Delta \sim)^{21} e^{(-2 \beta \sim - \Delta \sim)}$$

$$\begin{aligned} \text{Ncpd} := & .3263 \cdot 10^{-20} e^{(-1. \Delta \sim)} (.5865 \cdot 10^{13} \Delta \sim^9 + .3248 \cdot 10^{14} \Delta \sim^8 + .2361 \cdot 10^{16} \Delta \sim^5 \\ & + .1620 \cdot 10^{17} \Delta \sim^3 + .6943 \cdot 10^{16} \Delta \sim^4 + .2814 \cdot 10^{17} \Delta \sim^2 + 117800. \Delta \sim^{17} \\ & + .1602 \cdot 10^7 \Delta \sim^{16} + .1341 \cdot 10^{12} \Delta \sim^{11} + .1719 \cdot 10^{11} \Delta \sim^{12} + .1983 \cdot 10^{10} \Delta \sim^{13} \end{aligned}$$

$$\begin{aligned}
 &+.1923 \cdot 10^8 \Delta^{-15} + .2060 \cdot 10^9 \Delta^{-14} + .3750 \Delta^{-21} + .3236 \cdot 10^{17} \Delta^{-17} + 15.75 \Delta^{-20} \\
 &+.1578 \cdot 10^{15} \Delta^{-7} + 393.8 \Delta^{-19} + 7481. \Delta^{-18} + .6626 \cdot 10^{15} \Delta^{-6} + .1849 \cdot 10^{17} \\
 &+.9384 \cdot 10^{12} \Delta^{-10})
 \end{aligned}$$



EXPECTED_Net := 17.99

```

> ANS:=ANS:Ncpd:=subs(Delta=(A*epsilon),Ncpd):

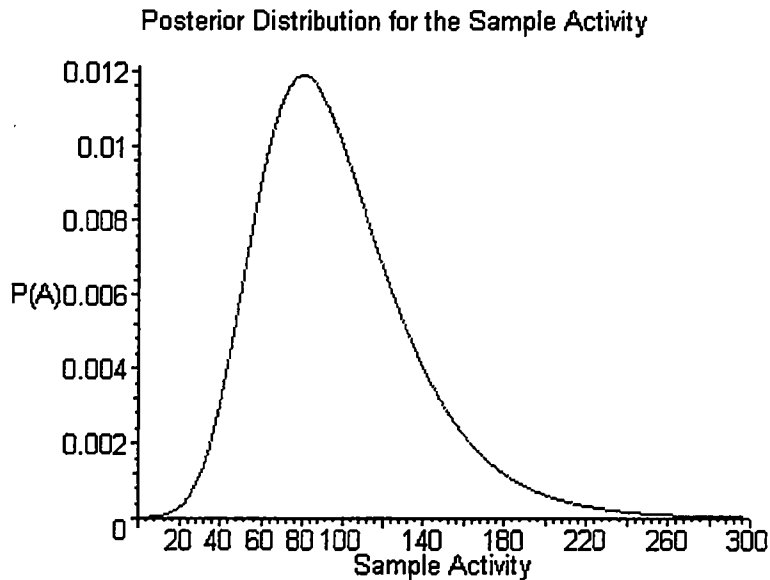
> AN:=Ncpd*ANS*epsilon: with(student):

pA:=value(simpson(AN,epsilon=0.05..0.5,24)): plot(pA,A=0..300,title=' Posterior
Distribution for the Sample Activity',labels=[' Sample Activity', 'P(A)']);

EXPECTED_ACTIVITY:=evalf(simpson(pA*A,A=0..300,24));

```

Warning, new definition for D



$$EXPECTED_ACTIVITY = 96.78$$

> **Rn-210 Half-life = 2.4 hours = 144**

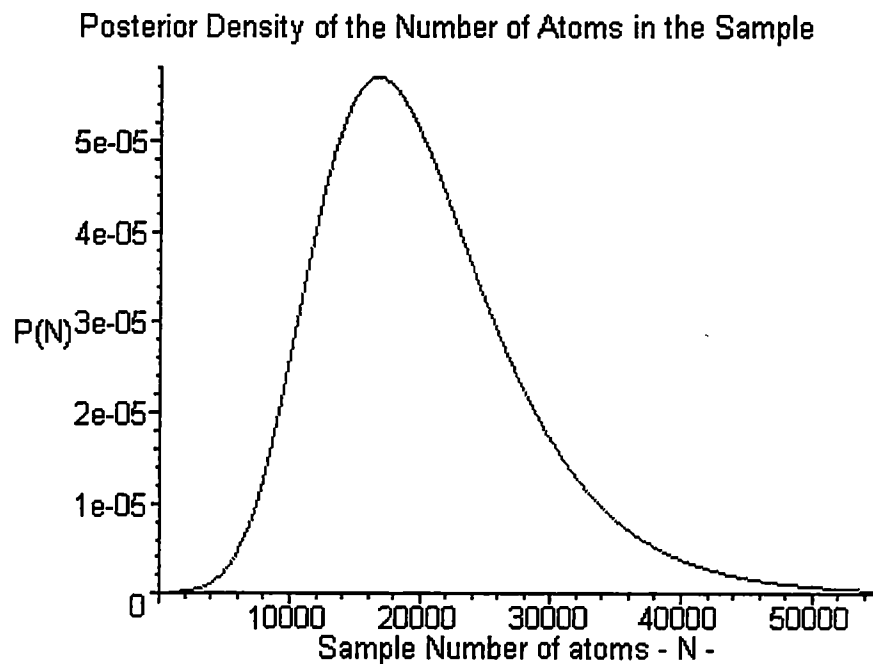
minutes; $pA := \text{subs}(A = \lambda_2 * N, pA)$; $pN := \lambda_2 * pA$:

$pN := \text{subs}(\lambda_2 = 0.693/144, pN)$; $\text{plot}(pN, N = 0..55000, \text{title} = \text{'Posterior Density of$

the Number of Atoms in the Sample'}, $\text{labels} = [\text{'Sample Number of atoms - N -$

'}, \text{'P(N)'}]); $EXPECTED_N := \text{evalf}(\text{simpson}(pN * N, N = 0..53000, 12))$;

$$Rn-210 \text{ Half-life} = 2.4 \text{ hours} = 144 \text{ minutes}$$



EXPECTED_N = 19880.

```
> EXPECTED_epsilon_2:=(evalf(simpson(ANS*(epsilon^2),epsilon
=0.05..0.5,30))); EXPECTED_epsilon:=EXPECTED_epsilon;

> SD_epsilon:=sqrt((EXPECTED_epsilon_2)-((EXPECTED_epsilon)^2));
```

EXPECTED_epsilon_2 = .04188

EXPECTED_epsilon = .1980

SD_epsilon = .05177

```
> EXPECTED_Net_2:=evalf(int((Delta^2)*npdc,Delta=0..40));
EXPECTED_Net:=EXPECTED_Net;SD_DELTA:=sqrt((EXPECTED_Net_2)-
```

((EXPECTED_Net)^2));

EXPECTED_Net_2 := 349.5

EXPECTED_Net := 17.99

SD_DELTA := 5.089

> EXPECTED_ACTIVITY_2:=evalf(simpson(pa*(A^2),A=0..300,24));

EXPECTED_ACTIVITY:=EXPECTED_ACTIVITY;SD_A:=sqrt((EXPECTED_ACTIVITY_2)-((EXPECTED_ACTIVITY)^2));

EXPECTED_ACTIVITY_2 := 10920.

EXPECTED_ACTIVITY := 96.78

SD_A := 39.42

> EXPECTED_N_2:=evalf(simpson(pN*(N^2),N=0..50000,16));

EXPECTED_N:=EXPECTED_N;SD_N:=sqrt((EXPECTED_N_2)-((EXPECTED_N)^2));

EXPECTED_N_2 := .4583 10⁹

EXPECTED_N := 19880.

SD_N := 7944.

VITA

Mustafa Tamimi was born in Kuwait city, Kuwait on September 11, 1966. He attended public schools in Kuwait, where he graduated from Abdula Al Salem Al Sobah High School in June 1984. He entered Hillsborough Community College, Florida during August 1984 where he completed his pre-Engineering courses. He started his Electrical Engineering studies at the University of South Florida in August 1986. At the end of 1986 he transferred to New Jersey Institute of Technology where he earned his Bachelor of Science in Electrical Engineering in January 1991. In January of 1997, he entered the University of Tennessee to pursue the Master's Degree in Nuclear Engineering with concentration in radiological engineering. During his studies at the University of Tennessee he worked as a graduate assistant in the University Radiation Safety Office.