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A STUDY OF THE PERTURBATIONS ON URANUS

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Deborah S. Brooks

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ABSTRACT

The aim of this investigation was to develop an analytical procedure which could predict the existence of a trans-Plutonian planet and to determine if numerical methods, implemented on an Apple IIe computer, could be used to estimate the orbital elements, mass and present position of the planet. The probe planet was chosen as Uranus because of its close proximity to the unknown planet and because the data set for Uranus is the best of the outer, known planets. The data were obtained from the U.S. Naval Observatory; they consist of the rsiduals of Uranus in right ascension and declination from 1690 to 1977. The residuals were calculated by the Jet Propulsion Laboratory. FORTRAN programs were written to implement the theoretical method. It was determined that the Apple IIe, single-precision calculations produced unacceptable round-off errors. Therefore, a larger, double-precision, computer is required to run the programs. Further, during the course of this investigation it was discovered that the mass uncertainty of the planet Neptune introduced errors which rendered the procedure developed here unusable. The uncertainty in the mass of Neptune will not improve until 1989 when the Voyager probe makes a close approach to the planet. Therefore, the prediction of a trans-Plutonian planet based on the perturbation analysis of the residuals of Uranus will not be feasible before 1989.

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NOMENCLATURE

a	semimajor axis of an orbit
AU	Astronomical unit, distance from the Sun to the Earth
d	mean distance to Uranus
DEC	declination
e	eccentricity
E	eccentric anomaly
$\underline{F}_{ji}$	force exerted on body i by body j
$\underline{g}$	gravity vector
$\hat{\underline{g}}$	approximate solution to the gravity vector
G	gravitational constant
$\underline{h}$	orbital angular momentum
i	inclination
$\underline{I}$	unit vector
m	mass of a planet
M	mean anomaly
M_0	reference mean anomaly
p	similatus rectum
P	orbital period
r	radius vector
RA	right ascension
t	time
$\underline{\underline{T}}$	transformation matrix
u	argument of latitude
$\underline{U}$	unit vector in the direction of $\underline{r}$

NOMENCLATURE (continued)

v_θ	transverse velocity
w	angular velocity
$\underline{W}$	unit vector in the direction of the orbital angular momentum
xyz	rectangular orbital coordinates
$x_0y_0z_0$	sun-centered reference rectangular coordinates
α	longitude of a planet
γ	angle between two radius vectors
δ_{jk}	Kronecker delta
$\underline{\delta\rho}$	perturbed radius vector
ξ	angle from the positive x-axis to the unit vector $\underline{I}$
θ	true anomaly
θ_i	latitude of Uranus at time i
μ	mass of a body times the gravitational constant
$\underline{\rho}$	unperturbed, two-body radius vector
σ	standard deviation of the elements
ϕ_i	colongitude of Uranus at time i
ω	argument of pericenter
$\tilde{\omega}$	longitude of perhelion
Ω	longitude of the ascending node
$\odot$	Sun
$♃$	Jupiter
$♄$	Saturn
$⛢$	Uranus

NOMENCLATURE (continued)

♆	Neptune
X	planet X

I. INTRODUCTION

The Solar System, before the mid 1700s, was seen as a static, completely determined system. The orbits of all six planets were known to within the accuracy of the observations and no thought was given to the existence of planets beyond Saturn. Then, in 1781, Uranus was discovered by William Herschel (Grosser, 1979). No one, not even Herschel, had dreamed of the possibility of discovering a new planet. Herschel's ambition was to measure the distribution of the stars in space and thus to extend the knowledge of orderly cosmic structure from the solar system to the entire stellar universe. In the course of this survey, on the night of March 13, 1781, he noticed an object with a magnifiable disk. Although he did not realize it at the time, Herschel had discovered the seventh planet, Uranus.

The new addition to the solar system was somewhat troublesome to astronomers because, within a relatively short time, it became obvious that Uranus would not stay in its prescribed orbit. It was being perturbed by something outside of the known solar system. Then, in 1845 and 1846, J. C. Adams and U. J. J. Leverrier (Hoyt, 1980) independently calculated the position of a trans-Uranian planet by solving the problem of inverse perturbation, that is, they derived the mass, the orbital elements, and the instantaneous position of an unknown planet by analyzing deviations seen in the orbit of Uranus. Although Adams and Leverrier predicted the position of the planet to within two degrees, their predicted elements were quite different from the elements which were calculated for Neptune just after its discovery

(Table I). An investigation of this discrepancy was led by astronomers Benjamin Peirce and Benjamin A. Gould (Hoyt, 1980). They concluded that there were two solutions to the problem--one corresponding to the predictions of Leverrier and Adams and the other to the actual orbit of Neptune.

The mathematical triumph of the prediction of Neptune became a model of analytical prestige. Between 1850 and 1904 many attempts were made to mathematically predict the existence of a trans-Neptunian planet. As a result, at least five planets were proposed with a variety of masses and radius vectors. None of the planets could be found observationally. In 1904, Percival Lowell began an investigation of the perturbations of Uranus and Neptune. Based on the magnitudes of the residuals, the deviations in position after the perturbing effects of all other known planets have been taken into account, Lowell predicted the existence of a trans-Neptunian planet. These residuals amount of 10 arc seconds as compared with an observational error of approximately 0.5 arc seconds.

In early 1905 Lowell initiated a search for the planet but due to its faintness (on the order of 15th magnitude) and the lack of suitable equipment, the planet was not found. Lowell devoted 13 years to the theory and search for a trans-Neptunian planet until his death in 1916. This brought the planet search to a complete halt for 13 years. Then, in 1929, Dr. V. M. Slipher and Clyde Tombaugh reinstated the observational search using the elements predicted by Lowell, a larger telescope and a blink-microscope-comparator. They were rewarded in

TABLE I

PREDICTED AND CALCULATED ORBITAL ELEMENTS OF URANUS

Elements	Calculated	Leverrier	Adams
d (mean distance, AU)	30.25	36.15	37.25
e (eccentricity)	0.00884	0.10761	0.12062
i (inclination)	1°54'54"	*	*
Ω (longitude of node)	131°17'38"	*	*
$\tilde{\omega}$ (longitude of perihelion)	0°12'25"	284°45'	299°11'
P (period, in years)	166.381	217.387	227.3
m (mass, Sun = 1)	0.0000666	0.0001073	0.000150
(longitude 1847.0)	328°7'55"	326°32'	329°57'

*Not predicted.

March of 1930 with the discovery of Pluto near the longitude and at the approximate distance predicted by Lowell but with different orbital elements. In fact, Pluto's unexpected faintness (less than the calculated value by nearly 3 magnitudes) implied insufficient mass to be the planet predicted by Lowell. Attempts to find a satellite failed and therefore a reliable means for determination of the mass was denied. It was not until 1978, when Pluto's moon Charon was discovered, that the mass of Pluto was determined to be less than that of Earth's moon. According to Lowell (and in agreement with other astronomers of the period) Pluto needed to have a mass of 6.6 to 7.0 Earth masses to produce the observed residuals of Uranus.

Within a few months from the discovery of Pluto, it became apparent to astronomers that Pluto could not be Lowell's predicted trans-Neptunian planet and so, the search was on again. Over a period of 14 years, Tombaugh blinked nearly the entire sky from declination -50 degrees to +50 degrees to magnitudes of 14 to 17. No planet was ever found. At the onset of World War II, the search by Tombaugh was ended on the note that it was extremely improbable, based on observational data, that a trans-Plutonian planet could exist.

Since the discovery of Pluto in 1930, there have been at least three reports of the discovery of a tenth planet. Two of the reports were observational but they could never be substantiated. The third was a theoretical prediction by J. L. Brady using the method of inverse perturbations on the residuals of Halley's comet. The planet predicted by Brady had a mass approximately three times greater than Saturn with an inclination of 120 degrees, i.e., a retrograde orbit

tipped at an angle of 60 degrees. No such planet could be found. This was not surprising, since comets experience nongravitational forces due to jets of gas being ejected from their interiors (Weissman, 1985), therefore, the exact orbit of a comet cannot be known and certainly cannot be analyzed for residuals.

In the present United States scientific community there are four primary people who have been involved with the prediction of a tenth planet: Dr. Robert Harrington, Dr. Thomas Van Flandern, Mr. Charles Kowal and Mr. Clyde Tombaugh. Harrington, the central figure in the search for planet X, is an astronomer at the U.S. Naval Observatory in Washington, D.C. He has used a Monte Carlo method to predict a trans-Plutonian planet (Harrington, 1986): given the residuals in the orbits of the outer planets, Uranus and Neptune, Harrington has tried different orbital elements and masses to see what solution most closely satisfies the observations. Both Uranus and Neptune have been studied in this way but the predictions for the orbit of planet X differ significantly. Harrington suggests that this difference could be due to the uncertainty in the masses of the two planets causing an uncertainty in the calculations of the residuals themselves. The residuals (and any observational error) are calculated by the Jet Propulsion Laboratory--they are taken as the difference between the observed angular positions and the angular positions predicted by a nine planet least-squares fit. Although the mass of Uranus is well known, the uncertainty in the mass of Neptune has been estimated as high as 10% (Harrington, 1986). This condition will not be alleviated

until 1989 when Voyager encounters Neptune. A further cause of the discrepancy between the Uranus- and Neptune-based predictions of planet X could be that observations of Neptune do not extend very far into the past. That is, since Neptune's discovery, it has completed just over three quarters of a revolution about the Sun (the sidereal period of Neptune is 164.8 years), whereas Uranus has completed more than two revolutions about the Sun (the sidereal period of Uranus is 84.01 years). Another possible explanation could be the existence of two trans-Plutonian planets rather than one. Although a rough analysis of the residuals of the right ascension (RA) indicates that this is highly unlikely, the noise in the observational data is too great to totally exclude this possibility.

Another avenue that Harrington has approached is based on his work with Clyde Tombaugh (Harrington, 1986). The two men looked at the oddities of Neptune and Pluto: Neptune has a moon in a retrograde orbit and one in a highly eccentric orbit and the Pluto/Charon system has a very high eccentricity and a mass less than Earth's moon. These unusual occurrances indicate that Neptune and Pluto had a close encounter with some massive outer body. In fact, it is reasonable to theorize that Pluto was once a moon of Neptune and was perturbed into its present orbit by this unknown body. Harrington and Tombaugh assumed an orderly moon system of Neptune with Pluto as a moon of Neptune. They then assumed the existence of a tenth planet in some orbit which would bring it close to Neptune. Harrington again used the Monte Carlo approach, allowing the orbit and mass of the unknown

planet to vary, to see if there was some scenerio in which the two planets would perturb each other to the extent that the present Neptune/Pluto configuration would result. Harrington did find a solution which would put Neptune and Pluto in their present orbits and it would shift planet X into a high inclination (inclination i approximately 30 degrees) orbit with a large eccentricity. This theory is partially substantiated by the clear perturbations in the observed declination (DEC) of Uranus (and Neptune) which indicates that the outer planets are being perturbed by a body which is somewhat outside the ecliptic plane. Given that planet X may have a high eccentricity, Harrington and Tombaugh now agree that when Tombaugh initially did his search for the trans-Plutonian planet, it was probably near the constellation Sagittarius, an area of the sphere which was unaccessible to Tombaugh at the time of his blinking study.

Two other people have been involved in the prediction of a tenth planet (Harrington, 1986; Powell, 1986). The first is Charles Kowal, an observational astronomer who uses a blink-microscope to make a slow systematic scan of the celestial sphere. The second is Van Flandern who worked with Harrington at the U.S. Naval Observatory. Van Flandern approached the problem in a more analytical way. He had developed a general theory to predict planet X but he was not able to implement it during his residence at the Observatory.

Figures 1 and 2 show plots of the averaged residuals in right ascension and declination for Uranus from 1900 to 1978. Prior to 1900 the observational errors were so great that they obscured the

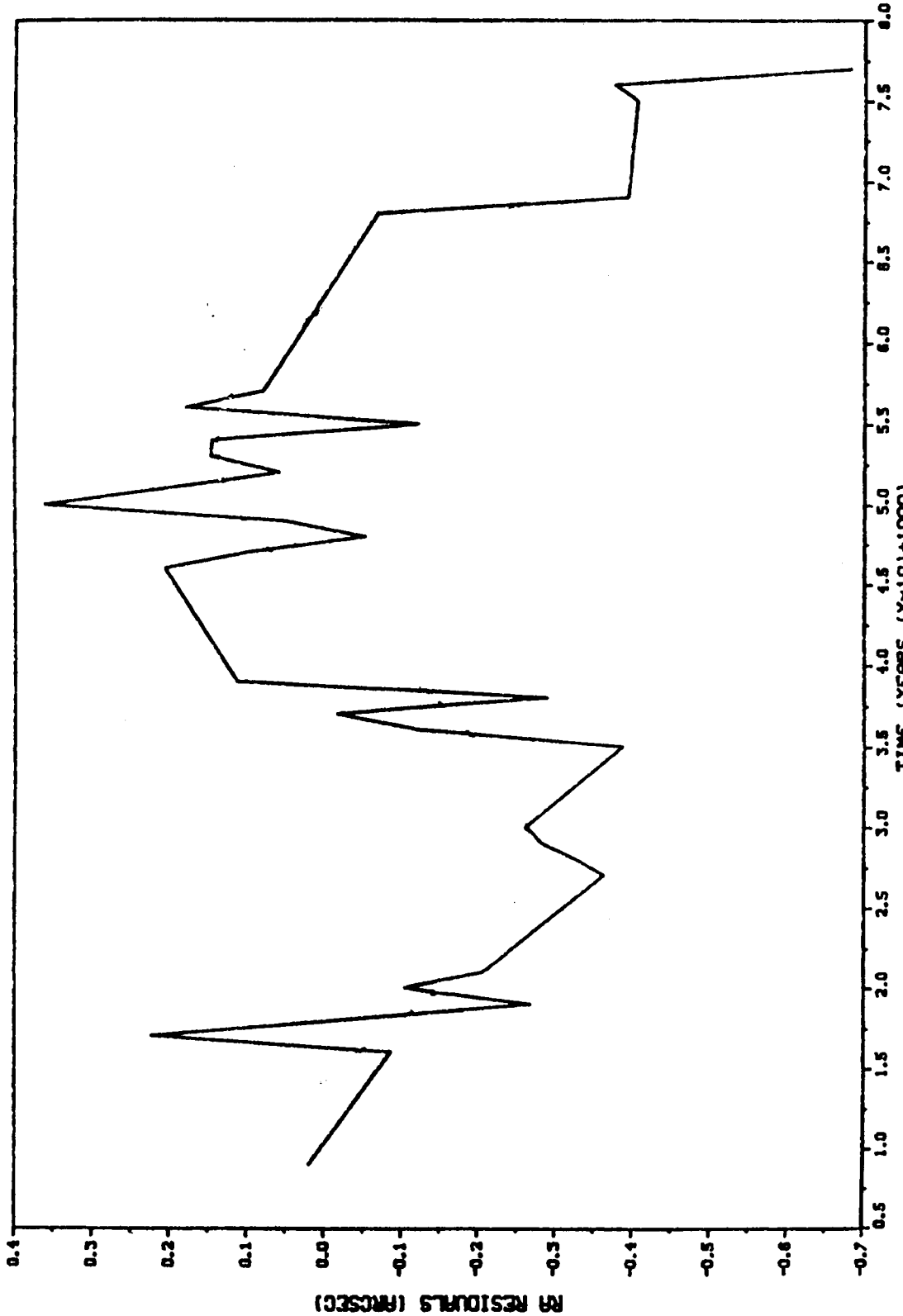


Figure 1. Right ascension residuals.

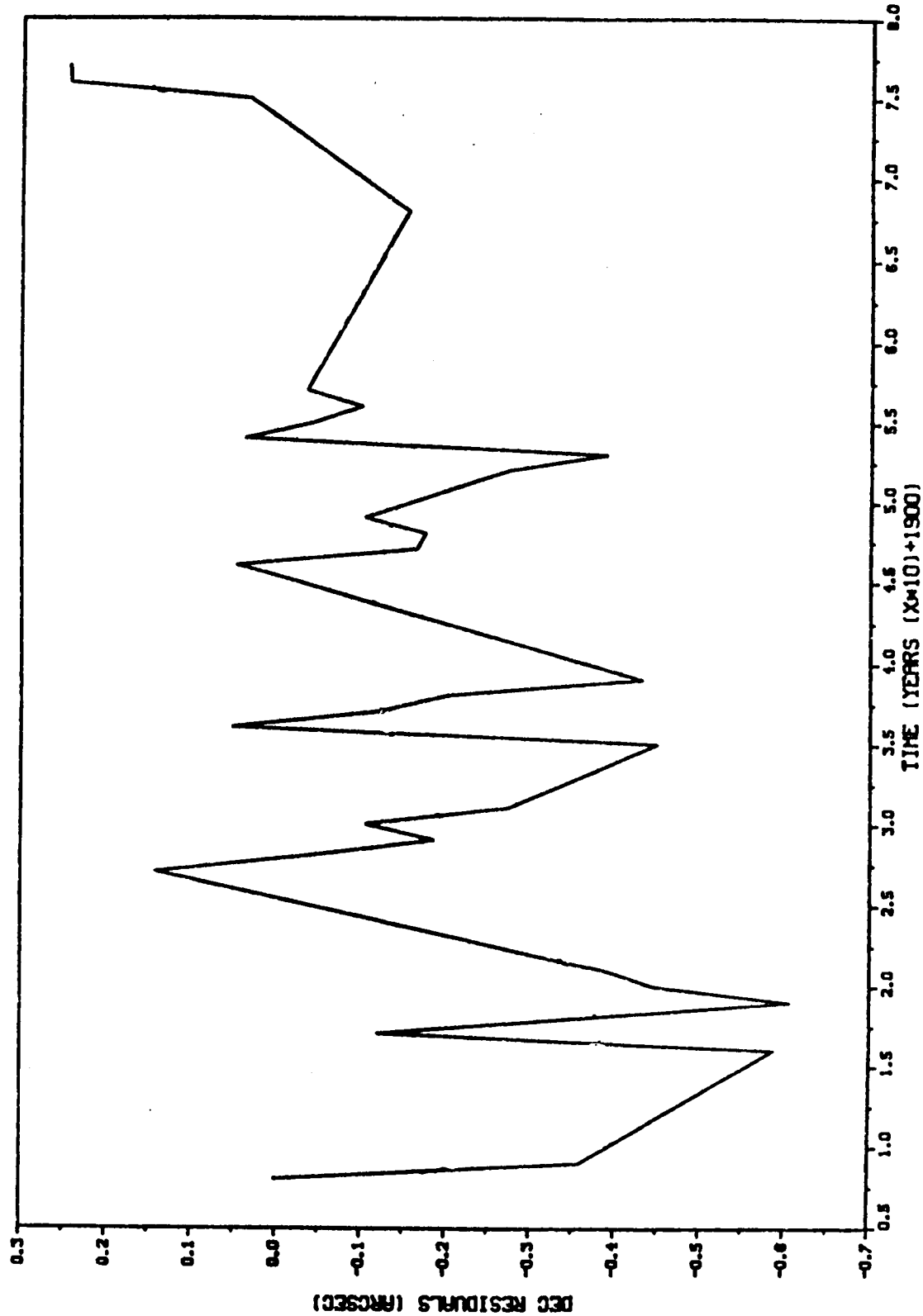


Figure 2. Declination residuals.

residuals in the data. Each point in the figures represents the average, calculated residual for all observations taken in one year. That is, given N observations of Uranus in one year (N ranges from 1 to 50), the average residual of an observation is the sum of the residuals divided by N. As expected, the residuals in RA are greater than those in DEC, that is, the magnitude of the perturbations in the ecliptic plane is greater than those outside of the plane. Figure 1 shows strong evidence of the existence of a planet beyond Pluto because the residuals seem to follow a smooth periodic curve indicative of a synodic period between Uranus and some other body.

Although the initial observational evidence presented by Tombaugh suggested that a trans-Plutonian planet did not exist, the more recent work on the subject refutes this conclusion. The residuals in the Uranian orbit calculated by Lowell still exist unexplained and the only way to effectively determine the origin of the residuals is to perform a perturbation analysis on the available data. The most likely explanation continues to be the existence of a tenth planet (Harrington, 1986; Hoyt, 1980; Powell, 1986).

The aim of this investigation was to develop a new analytical procedure which could predict the existence of a trans-Plutonian planet and to determine if numerical methods, implemented on an Apple IIe computer, could be used to estimate the orbital elements, mass and present position of the planet. Characteristics of the computer hardware and software used in this investigation are listed in Table II.

This approach differs from previous attempts to predict a trans-Plutonian planet in several ways. First, the method of data

TABLE II

COMPUTER CHARACTERISTICS

Computer system	Apple IIe
Internal memory	64K
Additional hardware	Z80 card CP/M card with 64K onboard memory
Software language	Fortran IV single precision

analysis is unique in that the residuals are averaged over a year and a polynomial curve is fitted to these averaged residuals. The curve fit clearly indicates how the residuals change with time and supports the idea of a synotic period between Uranus and another body. Second, range estimates to planet X are obtained using information from the residual gravity vector. The range estimate calculations are important because the initial range must be reasonably close to the true range for the calculations to converge to the correct solution (rather than converging to some local minimum). Third, this approach uses information provided by the gravity vector at Uranus to calculate initial estimates of the orbital elements of planet X.

The concept for this procedure as well as many of the generalized FORTRAN programs were developed by Dr. Conley Powell. Individual parts of the method are based on classical, orbital dynamics. In this work, the general orbital programs have been assembled into a set of interdependent routines which can determine if a trans-Plutonian planet is the cause of the observed perturbations in the orbit of Uranus. A method of minimizing the initial error in the residuals of Uranus was developed so they can be used with more confidence as inputs to the programs. Finally, it was discovered during the course of this investigation that there was serious doubt about the value of the mass of Neptune.

This report consists of a theoretical description of the procedure which was developed to determine if a trans-Plutonian planet could cause the observed perturbations in the orbit of Uranus. The

data which will be used in this procedure is presented. Data smoothing techniques are also discussed because they are required to manipulate the data into a form which can be input to the FORTRAN programs. The programs written to implement the theory are discussed along with the results of the program development. Program documentation is presented in an Appendix. To make this report as stand-alone as possible, orbit prediction, orbit determination and coordinate transformation methods are presented as Appendixes.

II. THEORETICAL DISCUSSION

A. A Qualitative Description of Method

The method chosen to predict a trans-Plutonian planet revolves about the calculation of the gravity vector at Uranus based on the observed angular positions (Figure 3). The total gravity vector, i.e., the total forces exerted on Uranus, is derived from the six Lagrange planetary equations which equate the rates of change of the orbital elements of Uranus (or any planet) to functions involving the heliocentric distance and true anomaly of Uranus, the orbital elements, and the total gravity vector. Next, the perturbing gravity vectors of the largest and closest of the known planets (Jupiter, Saturn, and Neptune) are found from Newton's law and the calculated radius vectors of the planets. It is assumed that the interior planets and Pluto produce negligible perturbations on Uranus. The calculated, perturbing gravity vectors are then subtracted from the total gravity vector to isolate the remaining perturbing gravity, i.e., the residual, vector due to planet X.

The residual gravity vector is used to calculate a unit vector pointing from Uranus to planet X (Figure 4); this provides the direction to planet X. The unit vector is utilized by a two-body orbit determination method (with the Sun as the primary and planet X as the secondary) which calculates estimates of the instantaneous orbital elements of planet X from three angular positions and two range estimates. Once the orbital elements are known, the orbit is determined.

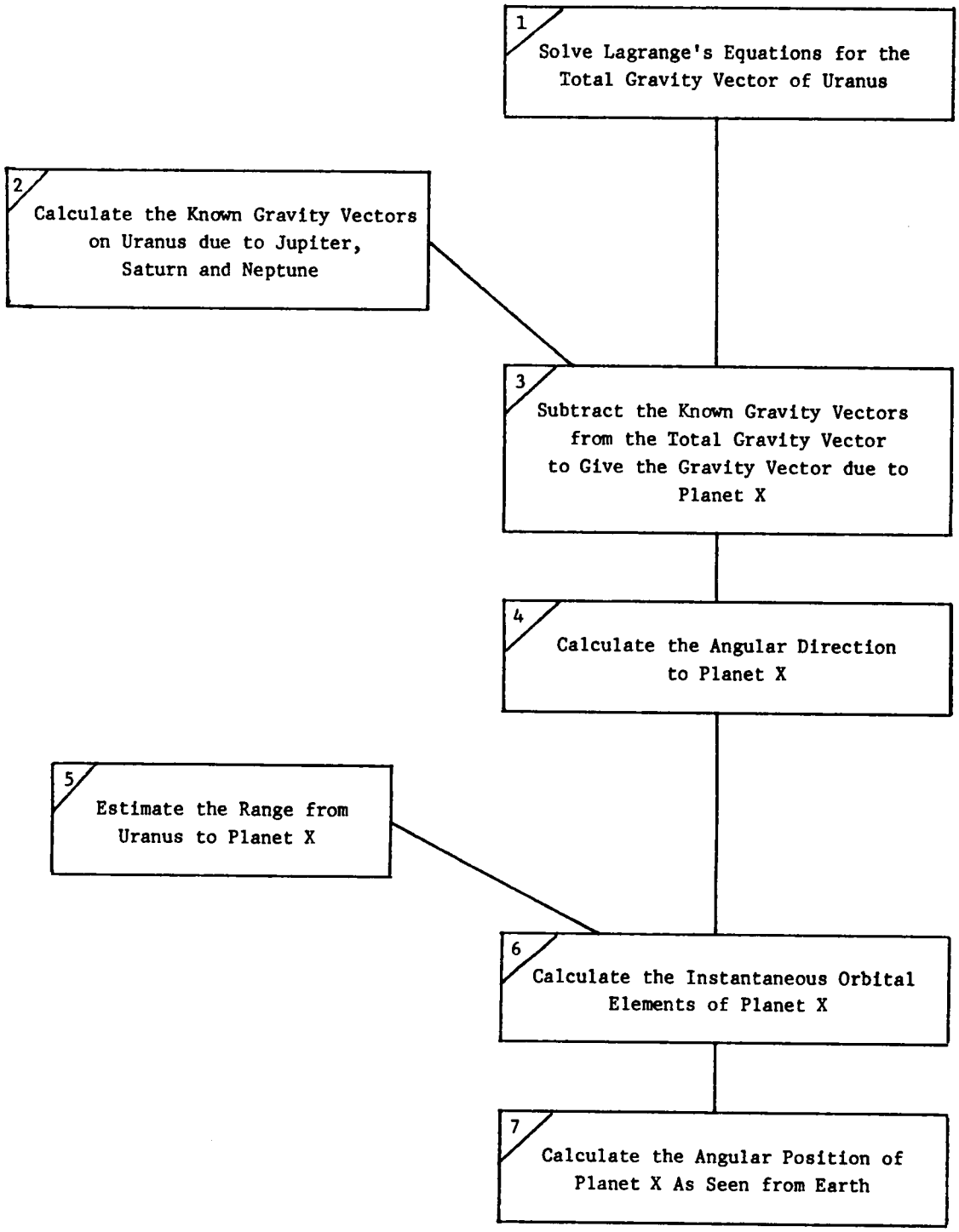
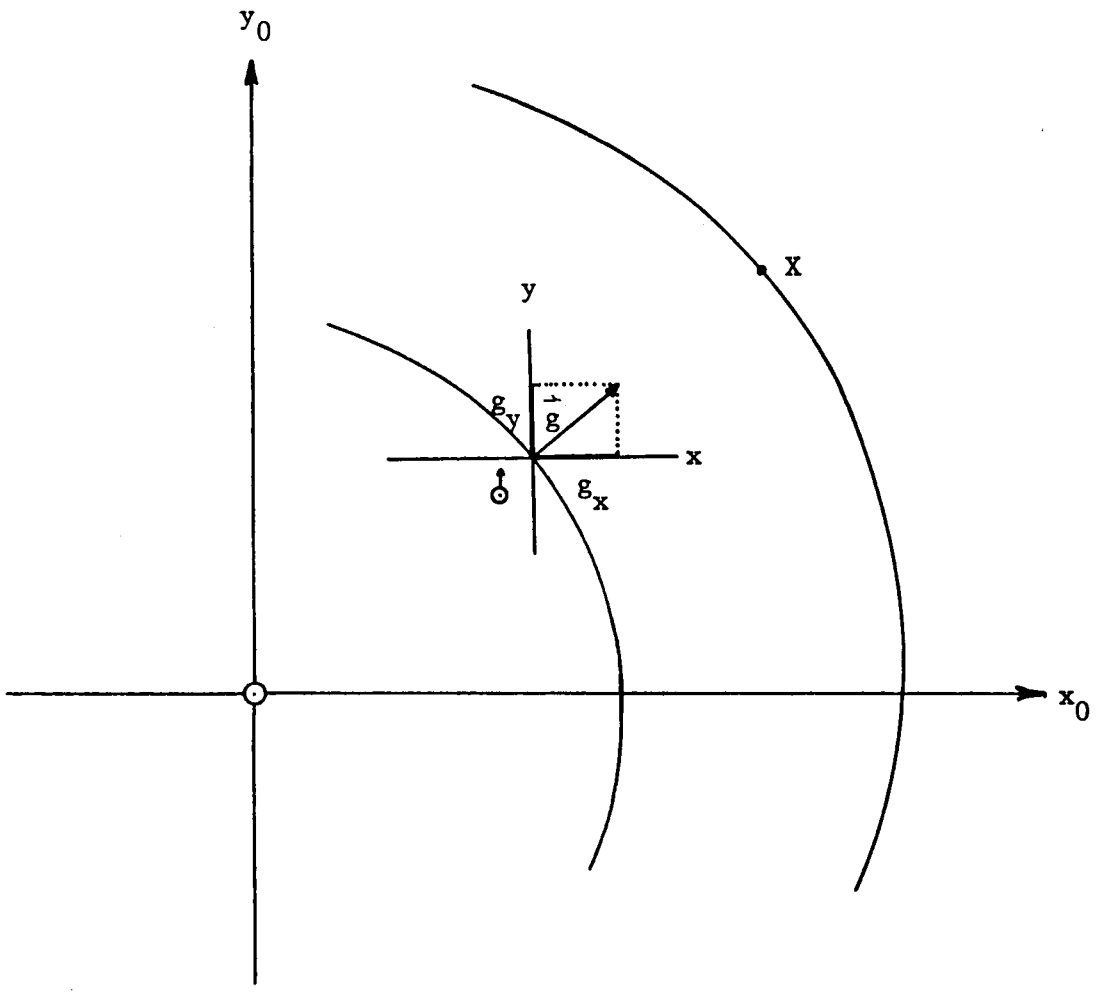


Figure 3. Description of method.



⊙ Sun

⊙ Uranus

X Planet X

x_0 y_0 Sun-centered, reference, rectangular coordinates in the eccliptic plane

x y Uranus-centered, orbital, rectangular coordinates

$\vec{g}$ Gravity vector at Uranus due to Planet X

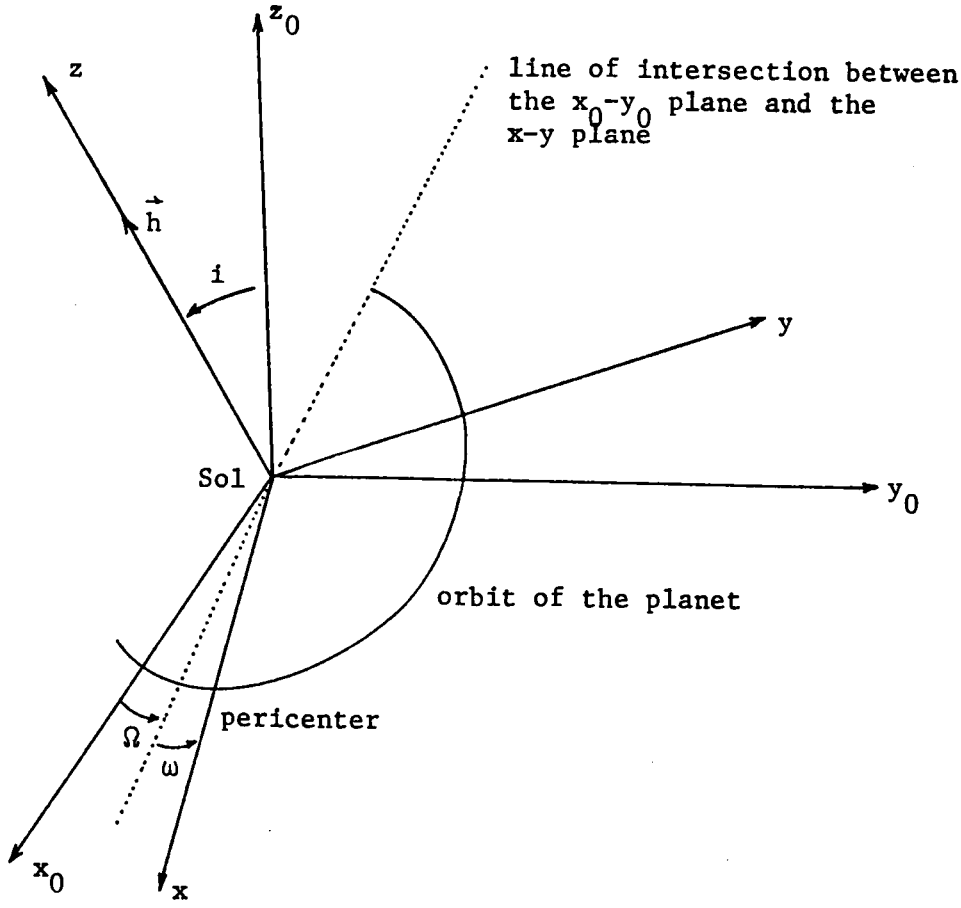
Figure 4. Gravity vector at Uranus due to planet X.

The range estimates to planet X are calculated from a set of equations derived from the observed angular positions of Uranus and the gravity vector at Uranus due to planet X. These equations relate the range and mass of planet X to the x-component of the gravity vector from Uranus to planet X at two times in the orbit. Once the equations have been set up, they are solved using an iterative method. Since these ranges are only estimates, two simplifying assumptions are made: Uranus and planet X are in circular, co-planar orbits and no perturbations are involved.

Once the elements have been estimated, a curve-fit routine optimizes the orbital elements by calculating the best fit of the elements based on the observed angular positions. The optimized elements of planet X are then translated into angular positions which can be used to point an Earth-bound telescope toward the vicinity of the tenth planet.

B. Calculate the Total Gravity Vector of Uranus

A two-body orbit of a planet is determined by six constant parameters called the osculating orbital elements (Figures 5 and 6). The eccentricity e determines the shape of the orbit, the similatus rectum p determines the size, the mean anomaly at some reference time M_0 locates the initial position of the body, and the angle of inclination i, the longitude of the ascending node Ω, and the argument of pericenter ω determine the orientation of the orbit in space. The last three elements correspond to the Euler angles.



The orbit is in the x-y plane.

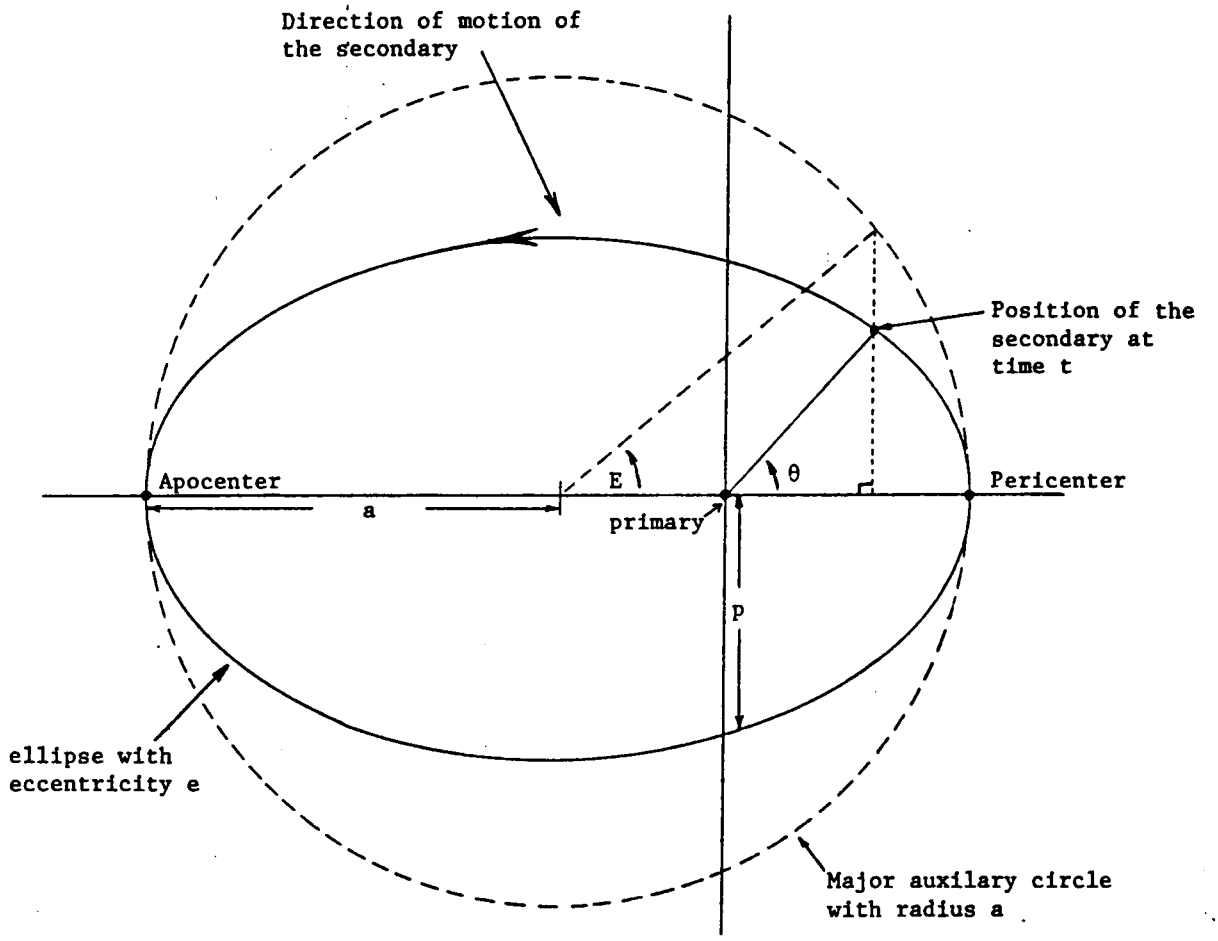
x_0-y_0 is in the plane of the ecliptic and z_0 is in the direction of the zenith.

i = inclination

Ω = longitude of the ascending node

ω = argument of pericenter

Figure 5. Orbit orientation.



e = eccentricity

p = similatus rectum

θ = true anomaly

E = eccentric anomaly

M = mean anomaly

≡ E - e sin E

M_0 = reference mean anomaly at time t = t_0

Figure 6. Orbit geometry.

When more than one planet orbits about the central or primary body, each planet not only feels the force from the primary, but also the perturbing forces from the other orbiting bodies. These perturbing forces cause the orbital elements to vary as functions of time. At any given moment in time, the orbit is represented by a set of instantaneous elements. These instantaneous elements of a body and the rates of change of those elements with respect to time are coupled to the total force exerted on the body via the Lagrange planetary equations (Powell, 1983):

$$\frac{de}{dt} = \frac{(1 - e^2)^{3/2}}{\left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2}} \left[\frac{g_r \sin\theta}{p} + \frac{p^2 - (1 - e^2)r^2 g_\theta}{ep^2 r}\right] \qquad (1)$$

$$\frac{dp}{dt} = \frac{2rg_\theta}{p\left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2}} (1 - e^2)^{3/2} \qquad (2)$$

$$\frac{dM_0}{dt} = \frac{(1 - e^2)^2}{\left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2} ep^2} [(p\cos\theta - 2er)g_r - (p + r)\sin\theta\, g_\theta]$$

$$+ \left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2} 3t \left[\frac{1}{2p}\frac{dp}{dt} + \frac{e}{1 - e^2}\frac{de}{dt}\right] \qquad (3)$$

$$\frac{di}{dt} = \frac{1}{\left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2} p^2} [(1 - e^2)^{3/2}\, r\cos(\omega + \theta)\, g_z] \qquad (4)$$

$$\frac{d\Omega}{dt} = \frac{1}{\left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2} p^2 \sin i} [(1 - e^2)^{3/2} r \sin(\omega + \theta) g_z] \quad (5)$$

$$\frac{d\omega}{dt} = \frac{(1 - e^2)^{3/2}}{\left(\mu \frac{|1 - e^2|}{p^3}\right)^{1/2}} \left[- \frac{g_\theta \cos\theta}{ep} + \frac{(2 + e\cos\theta)\sin\theta\, g_\theta}{(1 + e\cos\theta)ep} - \frac{r \sin(\omega + \theta) g_z}{p^2 \tan i} \right] \quad (6)$$

where t is the time at which the equations are being evaluated, r is the magnitude of the radius vector at the instant the elements are calculated, μ is the mass of the body times the gravitational constant, $\underline{g}$ is the total gravity vector of the body in cylindrical, orbital coordinates, and θ is the true anomaly of the body at time t.

In this case, the body under consideration is Uranus. To solve Lagrange's equations for the gravity vector, the osculating elements, the true anomaly and the range to Uranus are calculated using the orbit determination method outlined in Appendix A.

Solution to the Lagrange Planetary Equations

The Lagrange planetary equations form a set of six overdetermined equations with three unknowns, the gravity vector components. They are of the form,

$$Y_i^n(x^n) = F_i(\underline{g}, x^n) \qquad i = 1, \ldots, 6 \quad (7)$$

where $\underline{g}$ is the total gravity vector at Uranus, the x's are the known variables, the orbital elements of Uranus, the magnitude of the radius vector, and time, and $Y_i^n(x^n)$ are the rates of change of the elements with respect to time at the n^{th} time in the orbit (n = 1 to N).

Overdetermined Equations

Using the method of least squares, define

$$\varepsilon(\underline{g}) = \sum_{i=1}^{6} \sum_{n=1}^{N} [F_i(\underline{g}, x^n) - y_i^n(x^n)]^2 \qquad (8)$$

such that $\underline{g}$ is chosen to minimize $\varepsilon(\underline{g})$. Now let $\hat{\underline{g}}$ be an approximate solution, expanding $F_i(\underline{g},x^n)$ in a Taylor series about $\hat{\underline{g}}$ gives (Bevington, 1969)

$$F_i(\underline{g}, x) \simeq F_i(\hat{\underline{g}}, x) + \sum_{k=1}^{3} (g_k - \hat{g}_k) \left[\frac{\partial F_i(\underline{g}, x)}{\partial g_k} \bigg|_{g_k=\hat{g}_k} \right] \qquad (9)$$

If ε is a minimum, then

$$\frac{\partial \varepsilon(\underline{g})}{\partial g_i} = 0 \qquad j = 1, 2, 3 \qquad (10)$$

and,

$$0 = \frac{\partial \varepsilon(\underline{g})}{\partial g_j} = \sum_{i=1}^{6} \sum_{n=1}^{N} 2[F_i(\hat{\underline{g}}, x^n) - Y_i^n(x^n)] \frac{\partial F_i(\underline{g}, x^n)}{\partial g_j}$$

$$\simeq \sum_{i=1}^{6} \sum_{n=1}^{N} 2 \left\{ \left[F_i(\hat{\underline{g}}, x^n) + \sum_{k=1}^{3} (g_k - \hat{g}_k) \left(\frac{\partial F_i(\underline{g}, x^n)}{\partial g_k} \bigg|_{g_k=\hat{g}_k} \right) \right. \right.$$

$$- Y_i^n(x^n)\Bigg] \quad \frac{\partial}{\partial g_j}\left[F_i(\hat{g}, x^n) + \sum_{k=1}^{3}(g_k - \hat{g}_k)\left(\frac{\partial F_i(g, x^n)}{\partial g_k}\bigg|_{g_k=\hat{g}_k}\right)\right]\Bigg\} \tag{11}$$

To evaluate this expression, consider the last two terms:

$$\frac{\partial}{\partial g_i} F_i(\hat{g}, x^n) = 0 \tag{12}$$

and,

$$\sum_{k=1}^{3} \frac{\partial}{\partial g_j}\left[(g_k - \hat{g}_k)\left(\frac{\partial F_i(g, x^n)}{\partial g_k}\bigg|_{g_k=\hat{g}_k}\right)\right] = \delta_{jk}\left(\frac{\partial F_i(g, x^n)}{\partial g_k}\bigg|_{g_k=\hat{g}_k}\right) \tag{13}$$

Completing the multiplication and dividing through by 2 gives,

$$\sum_{i=1}^{6}\sum_{n=1}^{N}[Y_i^n(x^n) - F_i(\hat{g}, x^n)]\left(\frac{\partial F_i(g, x^n)}{\partial g_j}\bigg|_{g_j=\hat{g}_j}\right)$$

$$= \sum_{i=1}^{6}\sum_{n=1}^{N}\sum_{k=1}^{3}(g_k - \hat{g}_k)\left(\frac{\partial F_i(g, x^n)}{\partial g_k}\bigg|_{g_k=\hat{g}_k}\right)\left(\frac{\partial F_i(g, x^n)}{\partial g_j}\bigg|_{g_j=\hat{g}_j}\right) \tag{14}$$

This equation is now a linear equation of the form

$$\sum_{k=1}^{3} K_{jk}(\hat{g})(g_k - \hat{g}_k) = \alpha_j(\hat{g}) \tag{15}$$

where

$$K_{jk}(\hat{\underline{g}}) = \sum_{i=1}^{6} \sum_{n=1}^{N} \left(\frac{\partial F_i(\underline{g}, x^n)}{\partial g_j} \bigg|_{g_j=\hat{g}_j} \right) \left(\frac{\partial F_i(\underline{g}, x^n)}{\partial g_k} \bigg|_{g_k=\hat{g}_k} \right) \quad (16)$$

and

$$\alpha_j(\hat{\underline{g}}) = \sum_{i=1}^{6} \sum_{n=1}^{N} [Y_i^n(x^n) - F_i(\hat{\underline{g}}, x^n)] \left(\frac{\partial F_i(\underline{g}, x^n)}{\partial g_j} \bigg|_{g_j=\hat{g}_j} \right) \quad (17)$$

The derivatives of F with respect to g are calculated by means of a five point central difference approximation (Powell, 1983):

$$\frac{\partial F(y)}{\partial y} \simeq \frac{1}{12\delta y} [F(y - 2\delta y) - 8F(y - \delta y) + 8F(y + \delta y) - F(y - 2\delta y)] \quad (18)$$

where δy is a small difference in the parameter y.

Given an initial approximation $\hat{\underline{g}}$, equation (18) can be solved iteratively using Gauss elimination until the difference between successive approximations, $(g_k - \hat{g}_k)$, is smaller than some chosen tolerance level. Once the iterative loop stops, the total gravity vector of Uranus will be found for each of the N points in the orbit.

Coordinate Transformation of the Total Gravity Vector

The gravity vector calculated via the Lagrange planetary equations is in cylindrical, orbital coordinates. To subtract the known components of the gravity vector, it must be in reference rectangular coordinates. Rectangular coordinates are obtained from the relations

$$g_x = g_r \cos \theta - g_\theta \sin \theta \tag{19}$$

$$g_y = g_r \sin \theta + g_\theta \cos \theta \tag{20}$$

$$g_z = g_z \tag{21}$$

where θ is the true anomaly of Uranus.

Reference rectangular coordinates are then calculated from the equation

$$\begin{bmatrix} g_{x_0} \\ g_{y_0} \\ g_{z_0} \end{bmatrix} = \underline{T} \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \tag{22}$$

where $\underline{T}$ is defined in Appendix B.

C. Determine the Known, Perturbing Gravity Vectors

The radius vectors of the largest and closest of the known planets Jupiter, Saturn, and Neptune are calculated using the method of Encke (Brouwer and Clemence, 1961; Powell, 1983) outlined below. Once the radius vectors are known, the gravity vectors are calculated using Newtonian mechanics. Figure 7 shows the simplified system under consideration. The contributions of the small terrestrial planets, Mercury, Venus, Earth, Mars and Pluto, to the equations of motion of the outer gas giants is assumed to be negligible.

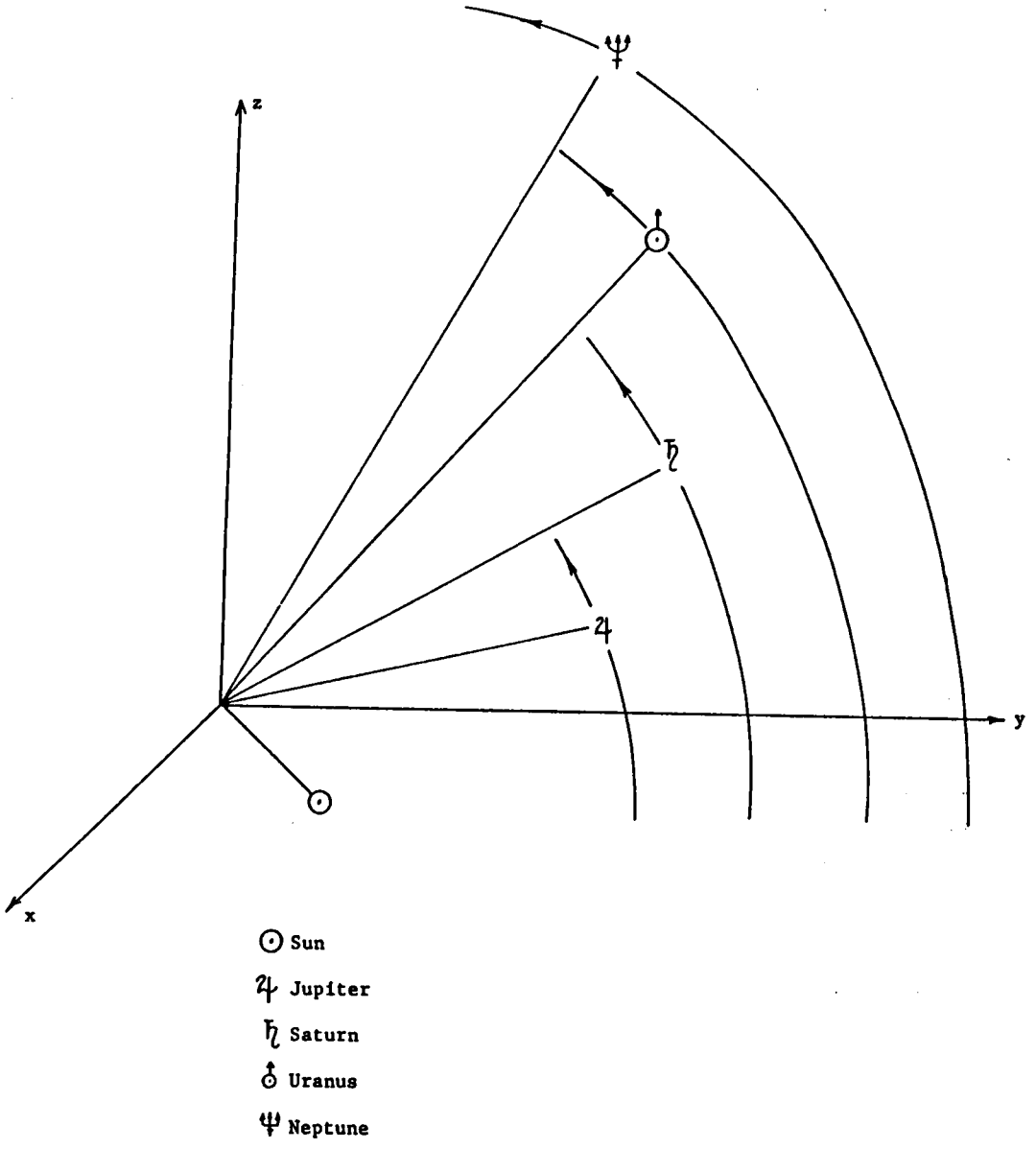


Figure 7. The Sun and four outer planets.

Encke's Method

To compute the position vectors of a planetary system, it is necessary to separate the two-body positions of a planet from the perturbations due to the remaining planets in the system. The technique of separating the perturbations allows the position vectors to be calculated with much greater precision. Encke's method assumes that the perturbations can be separated from the two-body positions and then solves the equations of motion for the perturbed part of the positions. The total radius vectors of the planets are formed by adding the perturbations to the two-body positions.

Let the Sun be represented as body 0, and Jupiter, Saturn, Uranus, and Neptune as bodies 1, 2, 3, and 4, respectively. The equations of motion of the i^{th} body are given by (Powell, 1983)

$$\frac{d^2\underline{R}_i}{dt^2} = \sum_{j=0}^{4} \frac{\underline{F}_{ji}}{m_i} = -\sum_{j=0}^{4} \mu_j \frac{\underline{R}_i - \underline{R}_j}{|\underline{R}_i - \underline{R}_j|^3} \qquad i = 0, \ldots, 4;\ i \neq j \qquad (23)$$

where $\underline{F}_{ji}$ is the force produced by body j on body i, m_i is the mass of body i, the R's are the radius vectors of the bodies, and

$$\mu_j = Gm_j \qquad (24)$$

where G is the gravitational constant, and m_j is the mass of body j.

Equation (23) represents absolute accelerations with respect to an arbitrary origin. To find the acceleration of the planets relative to the Sun, let

$$\underline{r}_j = \underline{R}_j - \underline{R}_0 \tag{25}$$

Then,

$$\frac{d^2\underline{r}_i}{dt^2} = -\frac{(\mu_0 + \mu_j)\,\underline{r}_i}{r_i^3} - \sum_{j=1}^{4} \mu_j \left[\frac{\underline{r}_j}{r_j^3} + \frac{\underline{r}_i - \underline{r}_j}{|\underline{r}_i - \underline{r}_j|^3} \right] \quad i = 1, \ldots, 4 \tag{26}$$

where the first term on the right is the acceleration of the i^{th} planet relative to the Sun and the second term is the relative acceleration of planet i due to the other planets in the system. This second term represents the perturbing effects of the outer planets on planet i.

Now, if the force exerted by the Sun on body i is much greater than the other forces on body i--that is, if

$$\frac{\mu_0}{r_i^2} >> \frac{\mu_j}{|\underline{r}_i - \underline{r}_j|^2} \qquad i, j = 1, \ldots, 4 \tag{27}$$

then each orbit approximates a two-body orbit, at least for a limited time. Let

$$\underline{r}_i = \underline{\rho}_i + \delta\underline{\rho}_i \tag{28}$$

were $\underline{\rho}_i$ satisfies the two-body equations of motion of planet i:

$$\frac{d^2\underline{\rho}_i}{dt^2} = \frac{-(\mu_0 + \mu_i)\underline{\rho}_i}{\rho_i^3} \tag{29}$$

and the perturbing acceleration on the i^{th} planet is:

$$\frac{d^2}{dt^2}(\delta\underline{\rho}_i) = -(\mu_0 + \mu_i)\left[\frac{\underline{\rho}_i + \delta\underline{\rho}_i}{|\underline{\rho}_i + \delta\underline{\rho}_i|^3} - \frac{\underline{\rho}_i}{\underline{\rho}_i^3}\right]$$

$$-\sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_j\left[\frac{\underline{\rho}_j + \delta\underline{\rho}_j}{|\underline{\rho}_j + \delta\underline{\rho}_j|^3} + \frac{\underline{\rho}_i + \delta\underline{\rho}_i - (\underline{\rho}_j + \delta\underline{\rho}_j)}{|\underline{\rho}_i + \delta\underline{\rho}_i - (\underline{\rho}_j + \delta\underline{\rho}_j)|^3}\right] \qquad (30)$$

This equation can be solved numerically for $\underline{\delta\rho}_i$ using the procedure given below.

Equation (30) is a second order differential equation of the form

$$y''(t) = f(t,y') \qquad a \leq t \leq b \qquad (31)$$

where $y(t) = \underline{\delta\rho}_i$.

The initial conditions of the system can be chosen such that the perturbations and the rates of change of the perturbations at time t_o are zero:

$$\begin{aligned} &\underline{\delta\rho}_i(t_o) = 0 \\ &\qquad\qquad i = 1, \ldots, 4 \\ &\frac{d}{dt}\underline{\delta\rho}_i(t_o) = 0 \end{aligned} \qquad (32)$$

These are in the form

$$\begin{aligned} y(a) &= \alpha_1 \\ y'(a) &= \alpha_2 \end{aligned} \qquad (33)$$

To convert this second order differential equation to a set of first order differential equations, let,

$$\begin{aligned} U_1(t) &= y(t) \\ U_2(t) &= y'(t) \\ U_3(t) &= y''(t) \end{aligned} \tag{34}$$

Then $U_1(a) = \alpha_1$ and $U_2(a) = \alpha_2$ and

$$\frac{dU_1}{dt} = y'(t) = U_2 = f_1(t, U_1, U_2) \tag{35}$$

$$\frac{dU_2}{dt} = y''(t) = U_3 = f_2(t, U_1, U_2) \tag{36}$$

Equations (35) and (36) now form a set of first order differential equations with given initial conditions. They can be solved numerically using the Adams fourth order predictor-corrector method described in Appendix C.

So the perturbation, $\underline{\delta\rho}_i$, on the orbit of the i^{th} planet due to the remaining planets in the system can be solved given the two-body positions of the planets, $\underline{\rho}_i$, at each point in the orbit.

The two-body positions of the planets in rectangular orbital coordinates at each time in the orbit are calculated from the orbital elements using the equations:

$$\theta = 2 \text{ atan } \left[\left(\frac{e+1}{e-1}\right)^{\frac{1}{2}} \quad \tan E/2\right] \tag{37}$$

$$\rho = \frac{P}{1 + e \cos \theta} \tag{38}$$

where the elements and the eccentric anomaly, E, are determined using the orbit prediction method of Appendix C. So,

$$\begin{aligned} \rho_x &= \rho \cos \theta \\ \rho_y &= \rho \sin \theta \\ \rho_z &= 0 \end{aligned} \tag{39}$$

Again, the positions in reference rectangular coordinates are obtained from the transformation given in Appendix B.

Finally, the total radius vectors of the planets:

$$\underline{r}_i = \underline{\rho}_i + \delta\underline{\rho}_i \tag{40}$$

can be calculated.

Gravity Vector Calculation

The known, perturbing gravity vector is now obtained using the Newtonian equation

$$\underline{g}_{k,p} = \sum_{i=1}^{3} \frac{\underline{F}_{ui}}{m_u} = \sum_{i=1}^{3} \mu_i \frac{\underline{r}_i - \underline{r}_u}{|\underline{r}_i - \underline{r}_u|^3} \tag{41}$$

where $\underline{r} = \underline{\rho} + \underline{\delta\rho}$ in reference rectangular coordinates, $\underline{r}_u$ is the radius vector of Uranus, $\underline{r}_i$ is the radius vector of planet i, m_u is the mass of Uranus, $\underline{F}_{ui}$ is the force exerted on Uranus by the i^{th} planet,

and μ_i is the mass of the i^{th} planet times the gravitational constant. Figure 8 illustrates the Sun-centered four-planet system.

The two-body acceleration of Uranus is

$$\underline{g}_{2\text{-body}} = \frac{\mu_s \underline{r}_u}{r_u^3} \tag{42}$$

where μ_s is the mass of the Sun times the gravitational constant.

So finally, the perturbing gravity vector on Uranus due to planet X is:

$$\underline{g}(X) = \underline{g}_{total} - \underline{g}_{k,p} - \underline{g}_{2\text{-body}} \tag{43}$$

where $\underline{g}_{total}$ is the total gravity vector found from solving the Lagrange planetary equations.

D. Calculate the Orbital Elements of Planet X

The next step in this procedure is to calculate the instantaneous orbital elements of planet X using the residual or perturbing gravity vector on Uranus due to planet X, $\underline{g}(X)$. The orbit determination method to do this is very similar to the procedure described in Appendix A except that the gravity vector of planet X is used to construct the unit vector from Uranus to planet X:

$$\underline{I} = \frac{g_x}{|\underline{g}(X)|}\,\hat{i}_o + \frac{g_y}{|\underline{g}(X)|}\,\hat{j}_o + \frac{g_z}{|\underline{g}(X)|}\,\hat{k}_o \tag{44}$$

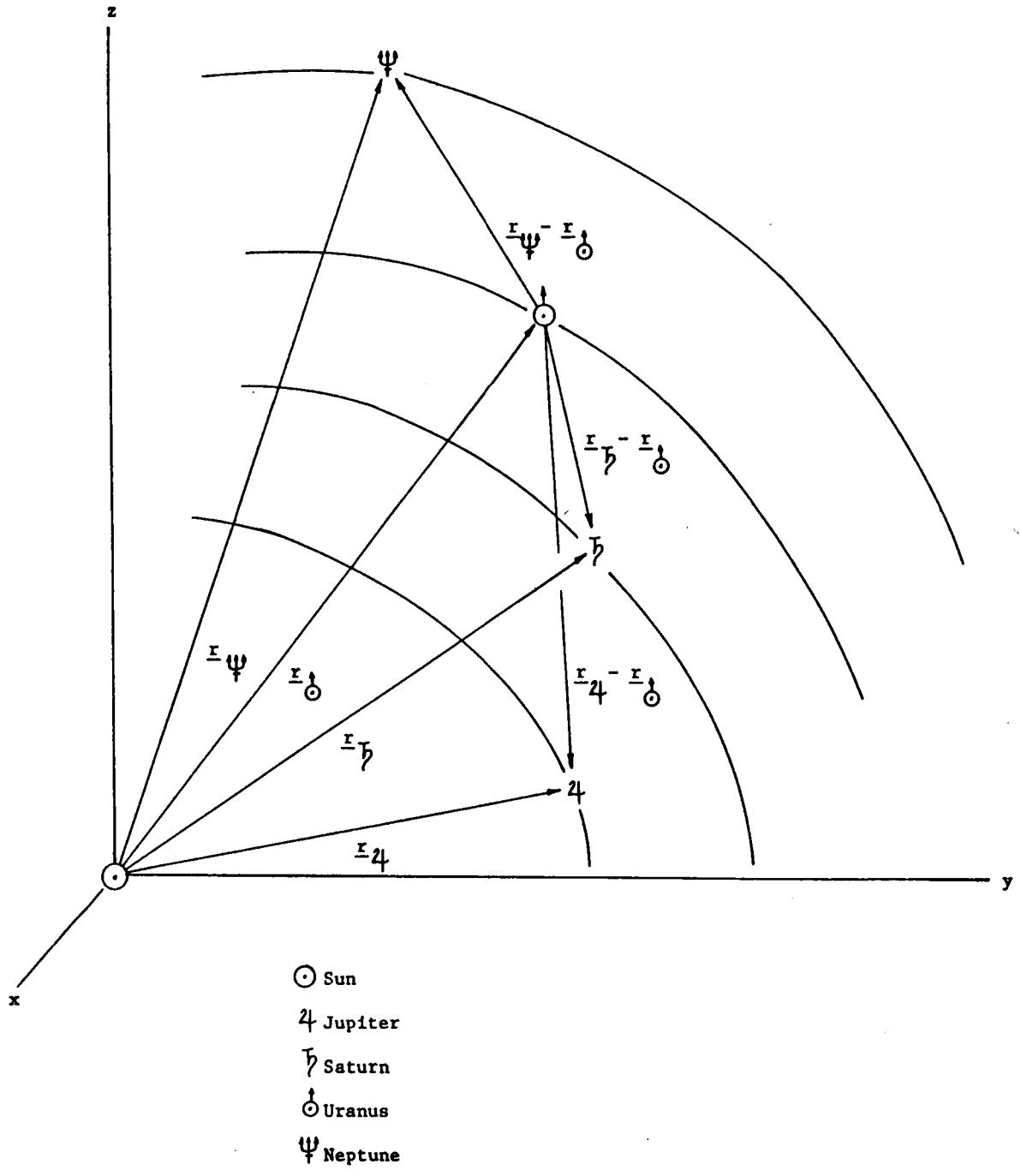


Figure 8. Sun-centered four-planet system.

where $\underline{I}$ is the unit vector from Uranus to planet X at time t in the $x_0y_0z_0$ reference rectangular coordinate system. So the radius vector from the Sun to planet X at time t is given by

$$\underline{r} = \underline{r}_u + R\,\underline{I} \tag{45}$$

where $\underline{r}_u$ is the radius vector from the Sun to Uranus at time t, and R is the estimated distance between Uranus and planet X at time t.

Determine the Range Estimate to Planet X

Before equation (45) can be solved, the estimated range from Uranus to planet X must be determined. To do this, the following assumptions are made:

1. No perturbations are involved, that is, the orbits of Uranus and planet X approximate two-body orbits with the Sun as the primary.
2. Uranus and planet X are in circular, co-planar orbits.

The first assumption is made to simplify the system description. Because the range estimate is calculated for a very short time interval, the orbital elements can be taken as constants for that interval. Hence, this assumption is reasonable.

The second assumption is certainly reasonable for Uranus because the orbit is known to have a small eccentricity (0.0505) and angle of inclination (0.0135 radians). In fact, this assumption could be made for all the known planets but Pluto. The validity of assumption two is more difficult to argue for planet X for the simple reason that its

orbit is unknown. If plant X follows the norm for the solar system, then it will be in a near-circular orbit near the plane of the ecliptic. But, if the preliminary analysis of Harrington and Tombaugh described in the Introduction is correct, then planet X would have a high inclination orbit (i approximately 0.52 radians) with a large eccentricity. In this case, the error introduced by the assumption that Uranus and planet X are in co-planar orbits may be as high as 5 or 10%. Given this possibility, the solution to the range to planet X must be solved using an iterative method which will minimize this error.

The calculation of the instantaneous orbital elements produces an improved value for the range to planet X. If the improved range differs significantly from the initial estimate, then the elements would be iteratively calculated using the improved range estimate until the difference between successive approximations becomes acceptable. So, although the error introduced by applying assumption two to planet X may be large, iterating the orbital elements would significantly decrease this error.

With assumptions one and two, the initial rough estimate of the range to planet X can be made. The components of the radius vector of Uranus at time t are

$$\begin{aligned} x_u &= r_u \cos(\alpha_u + w_u t) \\ y_u &= r_u \sin(\alpha_u + w_u t) \end{aligned} \tag{46}$$

where α_u is the longitude of Uranus at some reference time t_o, w_u is the angular velocity of Uranus, r_u is the range from the Sun to Uranus, and

$$w_u = \left(\frac{\mu_s}{r_u^3}\right)^{\frac{1}{2}} \tag{47}$$

where μ_s is the mass of the Sun times the gravitational constant. Similarly, the components of the radius vector of planet X at time t are

$$\begin{aligned} x_X &= r_X \cos(\alpha_X + w_X t) \\ y_X &= r_X \sin(\alpha_X + w_X t) \end{aligned} \tag{48}$$

And a unit vector pointing from Uranus to planet X is given by

$$\underline{I} = \frac{(x_X - x_u)\hat{i} + (y_X - y_u)\hat{j}}{[(x_X - x_u)^2 + (y_X - Y_u)^2]^{\frac{1}{2}}} = \frac{(x_X - x_u)\hat{i} + (y_X - y_u)\hat{j}}{[r_X^2 + r_u^2 - 2x_X x_u - 2y_X y_u]^{\frac{1}{2}}} \tag{49}$$

Let the angle from the positive x-axis to the unit vector $\underline{I}$ be ξ. Then the x-component of the unit vector is

$$\begin{aligned} \cos\xi &= \frac{x_X - x_u}{[r_X^2 + r_u^2 - 2x_X x_u - 2y_X y_u]^{\frac{1}{2}}} \\ &= \frac{r_X \cos(\alpha_X + w_X t) - r_u \cos(\alpha_u + w_u)}{[r_X^2 + r_u^2 - 2r_X r_u \cos(\alpha_u - \alpha_X + (w_u - w_X)t)]^{\frac{1}{2}}} \end{aligned} \tag{50}$$

Choose $t = 0$ at opposition, that is, when Uranus and planet X are closest together and in line with the Sun. Further, choose the x-axis to be along the line of opposition such that ξ is zero at opposition. Then equation (50) becomes

$$1 = \frac{r_X - r_u}{[r_X^2 + r_u^2 - 2r_X r_u]^{\frac{1}{2}}} \tag{51}$$

But, this is a trivial equation; when solved, it simply states that $1 = 1$. Physically, equation (51) says that Uranus and planet X are on the x-axis, but it does not specify where on the axis planet X is located. So more information is required to uniquely determine the orbit of planet X. Therefore, look at the magnitude of the gravity vector on Uranus due to planet X at time t:

$$|g(X)| = \frac{|\underline{F}_{uX}|}{m_u} = \left| \frac{\mu_X(\underline{r}_u - \underline{r}_X)}{|\underline{r}_u - \underline{r}_X|^3} \right| = \frac{\mu_X}{|\underline{r}_u - \underline{r}_X|^2} \tag{52}$$

where $\underline{F}_{uX}$ is the force on Uranus due to planet X, r_u is the radius vector of Uranus, m_u is the mass of Uranus, r_X is the radius vector of planet X, and μ_X is the mass of planet X times the gravitational constant. So,

$$\cos \xi = \frac{g_x}{|g(X)|} = \frac{g_x \, |\underline{r}_u - \underline{r}_X|^2}{\mu_X} \tag{53}$$

where g_x is the x-component of the gravity vector.

The gravity vector is known at each time in the orbit. Choosing two times, t_1 and t_2 gives

$$\frac{g_X(t_i)|\underline{r}_u - \underline{r}_X|^2}{\mu_X} = \frac{r_X \cos(w_X t_i) - r_u \cos(w_u t_i)}{\{r_u^2 + r_X^2 - 2r_u r_X \cos[(w_u - w_X)t_i]\}^{\frac{1}{2}}} \qquad i = 1, 2 \tag{54}$$

So, there are two equations which can be solved for the unknowns, the range r_X and mass m_X of planet X. Equations (54) are of the form

$$y_i^n(x^n) = F_i(z, x^n) \tag{55}$$

where the x's are the known variables, and the z's are the unknown variables r_X and m_X. Equation (55) can now be solved using the variation on the method of least squares described in Section IIB.

Once the range to planet X has been found, the radius vector from the Sun to planet X can be calculated from equation (45) and the instantaneous orbital elements of planet X can be calculated using the procedure described in Appendix A.

Optimize the Orbital Elements of Planet X

The instantaneous orbital elements of planet X can be calculated for some time t. It is expected that the error produced during these calculations will be larger than any perturbations planet X will experience from the known planets. Therefore, the elements can be treated as constants for all times. The estimates of the orbital elements are improved by utilizing an orbit determination procedure which finds the

best fit to the elements based on the observational data. This curve-fit routine uses the method of overdetermined equations outlined in Section IIB to optimize the elements of planet X. For this case, the least-squares criterion becomes

$$\varepsilon(c) = \sum_{n=1}^{N} [F(c,\underline{g}^n) - Y^n(\underline{g}^n)]^2 \qquad (56)$$

where $Y^n(\underline{g}^n)$ are the angular positions of planet X, calculated from the gravity vector, at the n^{th} point in the orbit, $F(c,\underline{g}^n)$ are the angular positions calculated from the orbital elements of planet X, and the c's are the orbital elements of planet X.

Given the initial approximation to the elements $\hat{c}$, the curve-fit routine improves the fit to the elements until the difference between successive approximations becomes sufficiently small. The accuracy of the fit to the elements is driven by the precision of the computer being used.

Once the elements have been optimized, the standard deviation of the elements are given by (Bevington, 1969)

$$\sigma_i = \left[\frac{E_{kk}(c)\ \varepsilon(c)}{MN - K} \right]^{\frac{1}{2}} \qquad i = 1, \ldots, 7 \qquad (57)$$

where M = 2, the number of dependent variables per observation, K = 7, the number of orbital elements plus one for the mass, N is the number of observations, $\varepsilon(c)$ is calculated using the optimized elements, and

$E_{kk}(c)$ is the inverse of the curvature matrix K_{jk} defined in equation (16). E_{kk} is called the error matrix. The standard deviation for each of the orbital elements and mass of planet X is calculated to give an estimate of the error associated with the solution.

To further improve the orbital elements, the optimization procedure is performed for four or five sets of elements calculated at equally spaced times in the orbit. The final elements will be taken as the average of the optimized elements.

E. Calculate the Angular Position of Planet X

Once the average orbital elements are known, they can be used in the orbit prediction calculation described in Appendix D to calculate the radius vector from Earth to planet X.

The radius vector for the time t = present is related to the angular positions declination DEC and right ascension RA by the equations:

$$\begin{aligned} r_x &= r \cos (DEC) \cos (RA) \\ r_y &= r \cos (DEC) \sin (RA) \\ r_z &= r \sin (DEC) \end{aligned} \tag{58}$$

To find the right ascension and declination look at

$$r \cos (DEC) = (r_x^2 + r_y^2)^{\frac{1}{2}} \tag{59}$$

Using the identity

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} \tag{60}$$

the declination can be found:

$$\text{DEC} = 2 \text{ atan} \left[\frac{r_z}{r + (r_x^2 + r_y^2)^{\frac{1}{2}}} \right] \tag{61}$$

Now,

$$\cos (\text{RA}) = \frac{r_x}{r \cos (\text{DEC})} \tag{62}$$

and

$$\sin (\text{RA}) = \frac{r_y}{r \cos (\text{DEC})} \tag{63}$$

So,

$$\text{RA} = 2 \text{ atan} \left[\frac{r_y}{r \cos (\text{DEC}) + r_x} \right] \tag{64}$$

The right ascension and declination calculated for the present time can then be used to point an Earth-bound telescope in the direction of the tenth planet.

III. ANALYSIS OF THE RESIDUALS OF URANUS

It is important to examine the available data and determine whether these data are of sufficient quality to be used in the method described in Section II. The input data for this method consist of the residuals in right ascension and declination for Uranus. The residuals were obtained by subtracting angular positions calculated from a nine-planet curve fit from the observed angular positions. The calculations were done by the Jet Propulsion Laboratories. Since the curve fit was made for a completely determined, nine-planet system, the difference between the calculated and observed positions represents the deviation in the orbit due to (presumably) an unknown body in the solar system.

A. Angular Positions of Uranus

Throughout the century there has been no long term effort by an observatory to measure the angular positions of Uranus. Therefore, the irregular observations of several sources must be combined to form a complete set. This task was performed by the U.S. Naval Observatory. To estimate the magnitude of the residuals and observational error, the angular positions of Uranus were predicted using a least-squares fit for a nine-planet system. The predicted angular positions were then subtracted from the observed positions to give the magnitude of the error and the residuals for each time.

Since Uranus and planet X move relatively little during the course of a year, the residuals remain nearly constant for that time

while the error fluctuates randomly. To decrease the magnitude of the error with respect to the residuals, the residuals were averaged for each year observations were made. Figures 1 and 2 (pp. 8 and 9) show the averaged residuals in right ascension and declination for Uranus as functions of time. The standard deviation of each point in the figures is typically two seconds of arc.

It is obvious that, despite the averaging, there is still a significant amount of error in the residuals. To decrease this still further, the averaged residuals were curve fitted to a fifth-order polynomial. The results are shown in Figures 9 and 10. From the curve fits one can see that the residuals of Uranus measure approximately 0.5 arc seconds in right ascension and 0.35 arc seconds in declination. These are small deviations compared to the residuals of Neptune which measure approximately 8 arc seconds. Although Neptune's residuals are larger than Uranus', it was decided to use Uranus as the probe planet rather than Neptune for two reasons. First, the usable data for both planets begins at about 1900; prior to this time the observational error is significantly greater. The improved data is due to the modern use of photography for measuring planetary positions. Between 1900 and 1980 Uranus has completed nearly one revolution (the orbital period of Uranus is 84.01 years) while Neptune has completed only half a revolution (the orbital period of Neptune is 164.8 years). Therefore, Uranus' data is more complete than Neptune's in the sense that the residuals vary over a complete cycle. The second reason Uranus was chosen as the probe is because the orbit of

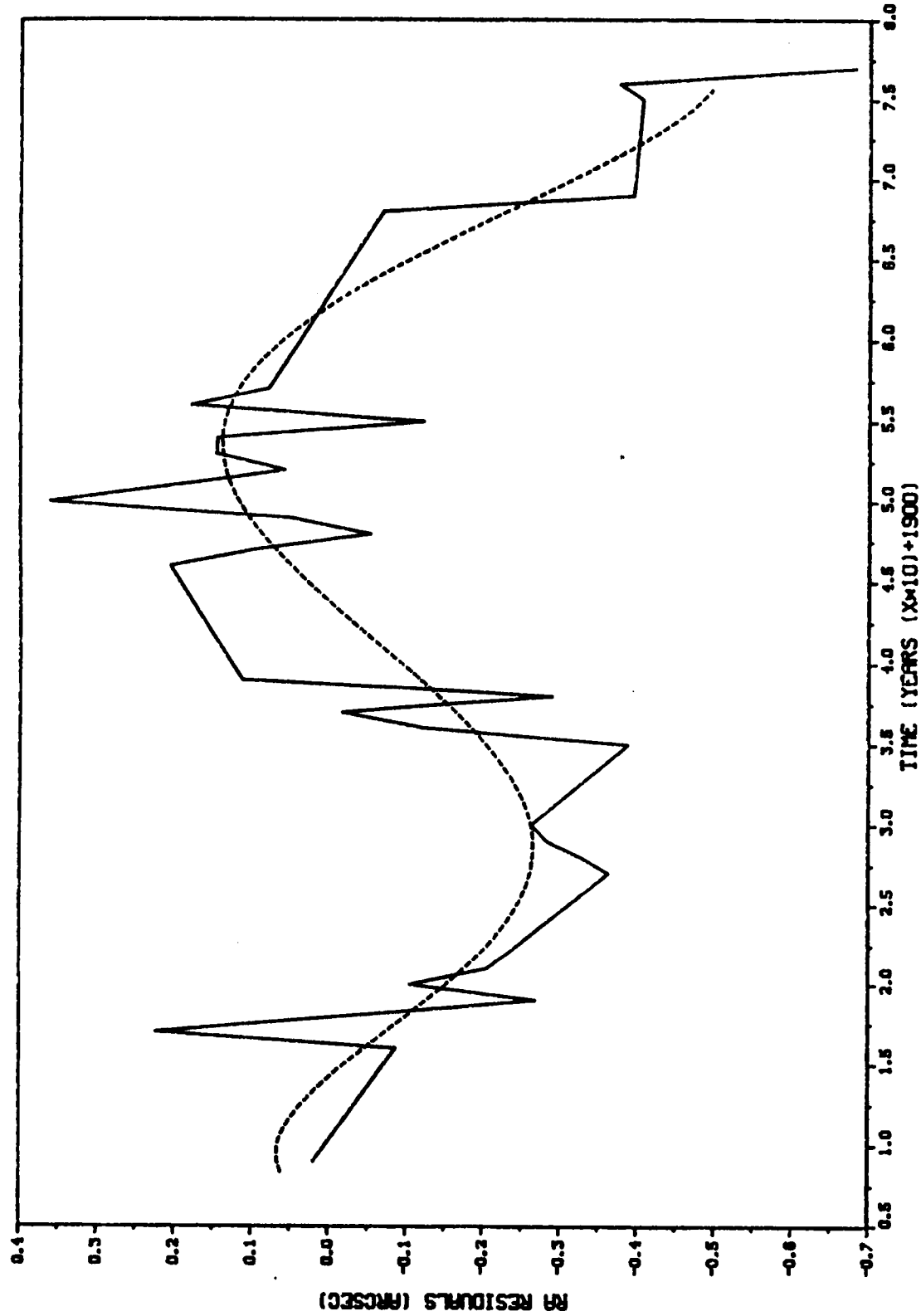


Figure 9. Polynomial curve-fit of right ascension residuals.

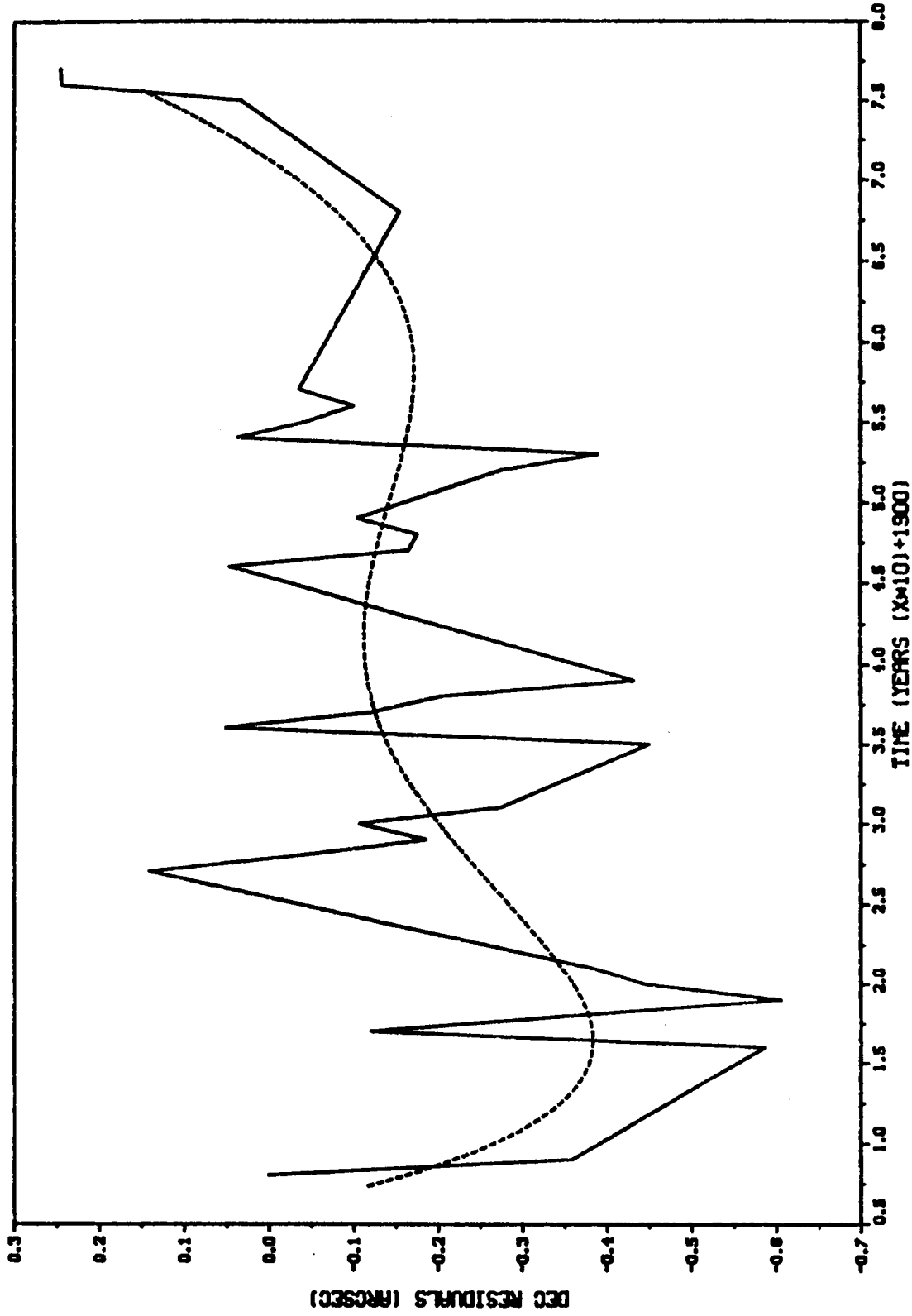


Figure 10. Polynomial curve-fit of declination residuals.

Neptune is not as well known and thus the calculated residuals would contain larger errors.

The program which calculates the total gravity vector of Uranus, SLE, requires the angular positions of Uranus at equally spaced times in the orbit. Clearly, the irregularly spaced times Uranus was actually observed and the errors associated with each measurement precludes the use of the original data. Therefore, angular positions of Uranus at equally spaced times will be generated by first calculating the predicted positions of Uranus using Encke's method. The residuals will then be calculated using the polynomial curve fits of Figures 9 and 10. The final set of data will be the predicted angular positions plus the residuals at equally spaced points in the orbit.

B. Mass Uncertainties

The creation of the data set for the SLE program requires that the masses of the known planets be well defined. This assumption is made in calculating the predictions of the planets both with the nine-planet least-squares fit and the Encke position determination. Toward the end of this investigation the author learned that the mass of Neptune has not been well determined. Typically, the mass of Neptune is said to be known to within approximately 1%, but recent work by Harrington and Tombauch (Harrington, 1986) indicates strongly that the error may be as high as 10%. The mass of a planet is determined from observations of the orbits of its moons. Neptune has two moons, Triton and Nereid, but only Triton can be used to measure Neptune's

mass. There are two reasons why Nereid cannot be used. First, Nereid is so far away from Neptune that solar perturbations are significant. Second, Nereid's orbital period is on the order of one year so it has only been observed during a portion of its orbit. That is, each time Earth is in a position to view Nereid, the moon traverses nearly the same segment of its orbit. These two circumstances make it impossible to know the orbit of Nereid well enough for it to be useful as a mass determination probe. Triton, on the other hand, is close enough but its orbit lies directly in the equatorial plane of Neptune. This makes it impossible to separate the perturbations due to the oblateness of the planet. If there were two useful satellites, then the oblateness could be separated from the two-body orbits. Clearly, this cannot be done here. The mass of Neptune will be determined more accurately in 1989 when the Voyager probe makes a close approach to Neptune. Because the method of orbit prediction described in Section II is based upon the calculation of the gravity vectors, it is very sensitive to mass uncertainties. Clearly, a mass error of Neptune of 10% would create uncertainties in the gravity vector on the order of, or larger than, the observed residuals. Therefore, the prediction of a tenth planet based on the calculation of the gravity vector of Uranus will not be feasible until 1989 when the mass of Neptune will be more accurately known.

IV. NUMERICAL IMPLEMENTATION OF METHOD

FORTRAN programs have been written to implement the theoretical method described in Section II. Although the program listings will not be presented here, descriptions of the programs and their associated subroutines are presented in Appendix E. Program listings will be made available to interested readers through the University of Tennessee Space Institute Library. A flow chart of the programs is shown in Figure 11.

To determine if the programs correctly and accurately modeled the planetary system shown in Figure 8 (p. 33), a set of verifiable test data for the SLE program was generated from the ENCKE and OP programs. The test data required for the SLE program consists of a set of generated angular positions (right ascension and declination) of Uranus as seen from Earth. This data will be used to generate an output from SLE that is the gravity vector at Uranus. To verify this output, the generated gravity vector will be used to recalculate the angular positions of Uranus which, in turn, will be checked against the initial inputs.

Using the orbital elements of Jupiter, Saturn, Uranus, and Neptune obtained from the 1980 Ephemeris, ENCKE produced the Sun-centered radius vectors of Uranus at 20 points in the orbit. The times were chosen in three month intervals beginning at 1980 and going backward in time. The three month interval was dictated by the orbit of Jupiter, the planet with the greatest angular velocity. If the time difference was much larger than three months, then the ENCKE program could not accurately calculate Jupiter's orbit and hence the

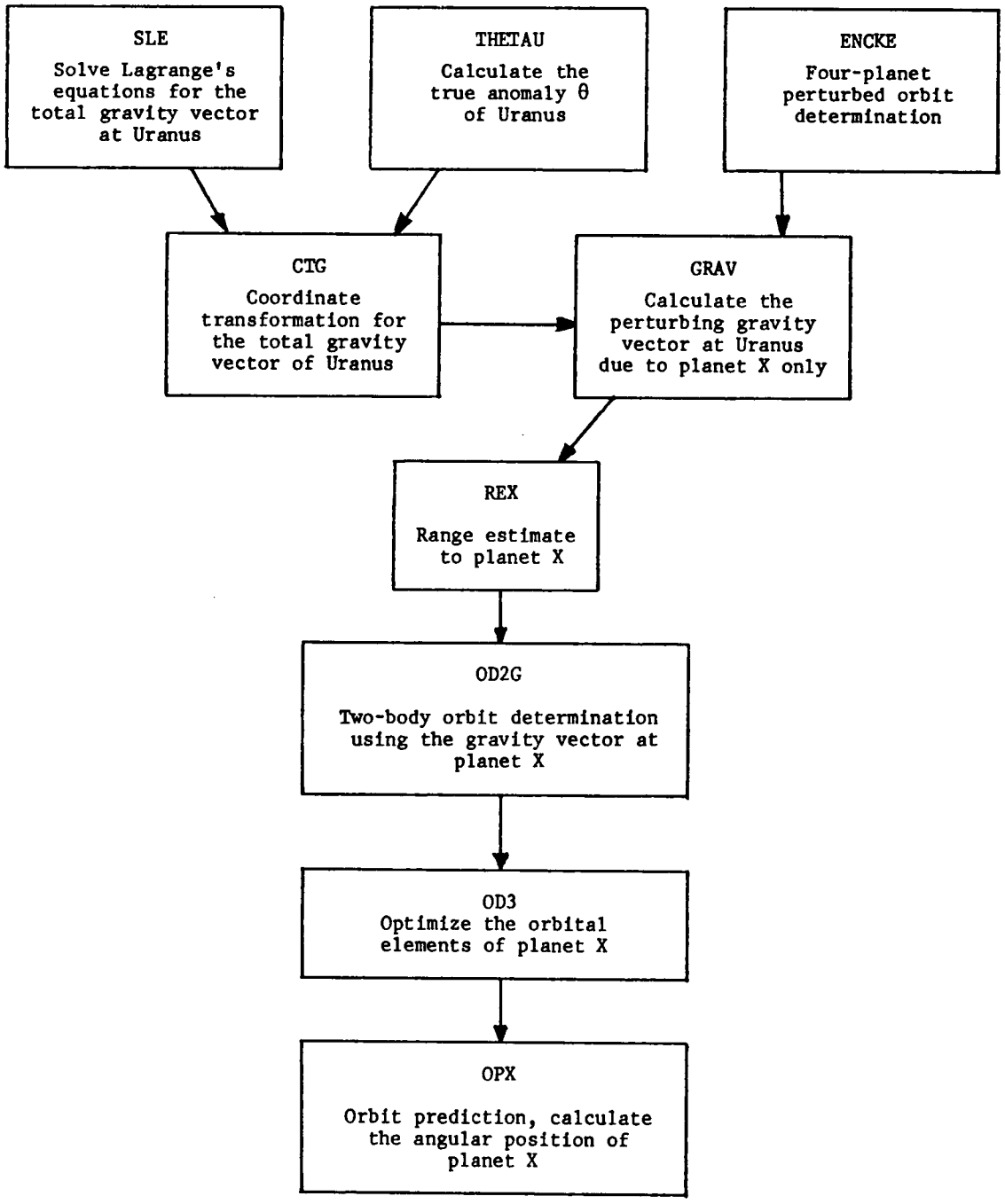


Figure 11. Program flow chart.

perturbations caused by Jupiter. If the time difference was much less than three months, then Uranus and Neptune barely moved with respect to Jupiter and Saturn. So the three month time difference was taken as a compromise between the two ends.

Next, the heliocentric position of Earth was calculated for the same time intervals using the OP program. Again, Earth's orbital elements were obtained from the 1980 Ephemeris. Then, the radius vector from Earth to Uranus at each point was found from the equation

$$\underline{r}_{eu_i} = \underline{r}_{su_i} - \underline{r}_{se_i} \tag{65}$$

where $\underline{r}_{eu_i}$ is the radius vector from Earth to Uranus at time i, $\underline{r}_{su_i}$ is the Sun-centered radius vector of Uranus, $\underline{r}_{se_i}$ is the Sun-centered radius vector of Earth. And finally, the latitude and colongitude of Uranus were calculated using the equations

$$\theta_i = \pi/2 - DEC_i \tag{66}$$

$$\phi_i = 2\pi + RA_i \tag{67}$$

where the right ascension RA and the declination DEC were calculated using equations (61) and (64).

At this point it is important to note the units of time and distance which were chosen. Because of the matrix algebra used in the numerical procedures, it was necessary to choose the time and distance units such that the array elements were of, or near the same order of magnitude. The time unit was chosen as 10^{12} seconds and distance was

measured in units of 10^{13} meters. Planetary distances were approximately 1 unit and time varied from 0 to 10^{-3} units.

The first attempts to run SLE failed. It was determined that the problem lay in subroutine NRNE which optimized the value of the ranges to Uranus using the orbit determination theory of Section IIB (Subroutine OD2). The Newton-Raphson range iteration diverged before the first approximation was obtained because the range difference used in the approximation was too large. Ordinarily, a range difference of 0.01AU would be sufficiently small compared to the range of Uranus (approximately 19AU). But in this case it was not. Decreasing the range difference to 10^{-7}AU (the smallest step size allowed by single precision calculations) prevented the solution from diverging, but instead, it oscillated about the true solution and never converged sufficiently. An error analysis of the program output led to the conclusion that the orbit determination method became extremely sensitive to range values when the observing station (Earth) neared opposition with Uranus, that is, when Uranus was observed in retrograde motion. Since the time interval between observations was three months, Earth would complete one revolution every four time intervals. Hence, retrograde motion is encountered every fourth time. Although theoretically there is no problem with retrograde motion, the numerical calculation of the unit vector in the direction of the orbital angular momentum becomes very sensitive because it involves the difference of nearly equal quantities. During retrograde motion, the unit vector must be calculated much more accurately than with direct

motion to calculate the radius vector at the third time. Since the unit vector is crucial to the calculation of the orbital elements, no solution could be obtained.

It is important to note that the orbit determination program has been tested successfully when the observing station was taken as Jupiter and the planets were not in opposition. Therefore, this program could still be used to predict the orbital elements and range of planet X as long as Uranus and planet X are not in opposition. The time at opposition is reasonably easy to determine since the approximate synodic period of Uranus and planet X is known from looking at the rate-of-change of the residuals in right ascension with respect to time (Figure 1, p. 8).

In summary, SLE could not be run using the orbit determination subroutine OD2 to calculate the orbital elements of Uranus at each point in the orbit. The OD2 routine is designed to predict the orbital elements of a body from three angular observations and two range estimates. But since the orbit of Uranus is known, good initial estimates of the instantaneous orbital elements can be obtained relatively easily. Therefore, subroutine OD2 in the program SLE can be replaced by a routine which optimizes the initial estimate of the elements rather than calculating them from first principles as OD2 does. The new routine, called OD3, optimizes the elements of Uranus by fitting them to functions involving the observed angular measurements. This curve fit method is described in Section IID. Also, subroutine OD3 will not suffer from problems due to retrograde motion because the

calculations are not dependent upon adjoining points in the orbit. Attempts to run OD3 using Nevada Fortran failed due to compile-time internal stack overflow errors. The program has been successfully implemented on an Apple II+ system which uses SuperSoft FORTRAN with an 8088 central processor and an 8087 arithmetic processor.

Since the OD3 subroutine is the only alternative program which can replace subroutine OD2, it was decided that the computer system described in Table II, p. 11, was not sufficient for this problem. Further, analysis of the precision in the outputs of the programs implemented to date show that double-precision calculations are required for this procedure. Although each program in this process has relative small errors, each step removed from the actual observational data increases the error until it becomes unacceptable. That is, by the time the gravity vector on Uranus due to planet X is calculated, the errors introduced by single-precision mathematics will be larger than, or on the same order of, the gravity vector itself. Therefore, it was determined that a double- or triple-precision computer, such as the VAX, was required to run the programs described in Appendix E.

The final status of the programs is that all programs shown in Figure 11, have been written, compiled and documented. The primary routines, DCF, OP, THETAU, and ENCKE have been run with test input and checked to make sure they will give the correct output. Although the SLE program could not be checked out in its entirety because of the internal stack difficulty with the OD3 subroutine, each subroutine in

SLE has been checked separately either on the Apple IIe computer or the Apple II+ system. Several of the programs which require the actual data, that is, programs whose inputs are created from the output of a previous program (CTG, OD2G, and OPX) have been checked by creating a modified program which had deleted subject-specific source code. CTG was based on a general coordinate transformation program CT which was written, verified and utilized in ENCKE. OD2G was a modification of OD2 which was run successfully for two planets which were not near opposition. OPX was a modification of OP which was run and checked for both Earth and Uranus.

Although the complete REX program could not be checked because it required the gravity vector on Uranus due to planet X, the subroutines called by REX, OE and LE, have been run and verified independently. The only remaining routine shown in Figure 11 is GRAV. The program itself has not been verified because it is based solely upon manipulations of the calculated gravity vector. Despite this situation, no problems with GRAV are anticipated because the calculations performed by the routine are standard Newtonian physics.

V. CONCLUSIONS

A procedure was developed to determine if a trans-Plutonian planet could cause the observed perturbations in the orbit of Uranus. The method involved the calculation of the gravity vector at Uranus which was then used to estimate the mass, orbital elements, and angular position of the planet. FORTRAN programs were written to implement the procedure. They were, where possible, successfully tested using trial inputs.

Two problems surfaced during the course of this investigation. First, it was determined that the programs must be run on a double-precision computer to obtain the accuracy required by this method. Second, it was found that the uncertainty in the mass of Neptune would introduce unacceptable errors in the calculations. The first problem can be alleviated by using a larger, double-precision computer such as the VAX. The second problem cannot be overcome until 1989 when the Voyager probe makes a close approach to Neptune and determines its mass more accurately. Therefore, the prediction of a trans-Plutonian planet based on this perturbation analysis of the residuals of Uranus is not feasible before 1989. After 1989, newly calculated residuals will be obtained which will use the corrected mass of Neptune in the nine-planet least-squares fit. The new data will also cover a longer time span (approximately 90 years) than was available for this study. The additional data should improve the curve fit and lend more confidence to the period determination. Once the data has been curve fitted, the program inputs will be configured and then the programs will

be run and their outputs checked. At this time it will be determined if the residuals of Uranus are large enough to predict the existence of a tenth planet. If the residuals are not large enough then more consideration must be given to the initial assumptions made to create this data set. That is, a different curve fit or interpretation of the curve fit of the residuals may be necessary or perhaps, Neptune should be reconsidered for the probe planet.

LIST OF REFERENCES

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1. Allen, C. W., Astrophysical Quantities, Second Edition, The Athlone Press, London, England, 1955.

2. Bevington, P. R., Data Reduction and Error Analysis for the Physical Sciences, McGraw-Hill Book Company, New York, 1969.

3. Brouwer, D., and Clemence, G. M., Methods of Celestial Mechanics, Academic Press, New York, 1961.

4. Burden, R. L., Faires, J. D., and Reynolds, A. C., Numerical Analysis, Second Edition, Prindle, Weber and Schmidt, Boston, Massachusetts, 1981.

5. Danby, J. M. A., Fundamentals of Celestial Mechanics, The Macmillan Company, New York, 1962.

6. Duncombe, R. L., "Planetary Ephemerides," in Tapley, B. D., and Szebehely, V. (eds.), Recent Advances in Dynamical Astronomy, pp. 289-308, D. Reidel Publishing Company, Dordrecht-Holland, 1973.

7. Grosser, M., The Discovery of Neptune, Dover Publications, Inc., New York, 1979.

8. Harrington, R., personal communication, 1986.

9. Hoyt, W. G., Planets X and Pluto, The University of Arizona Press, Tucson, Arizona, 1980.

10. Powell, C., personal communication, 1986.

11. Powell, C., astrodynamics course notes, 1983.

12. Sterne, T. E., An Introduction to Celestial Mechanics, Interscience Publishers, Inc., New York, 1960.

13. U.S. Nautical Almanac Office, The American Ephemeris and Nautical Almanac for the Year 1980, U.S. Government Printing Office, Washington, D.C., 1979.

14. Weissman, P. R., "Cometary Dynamics," Space Science Reviews, 41, 1985, 299-349.

APPENDIXES

APPENDIX A

ORBIT DETERMINATION

The instantaneous orbital elements of Uranus are calculated from three observed angular positions of Uranus, two range estimates from Earth to Uranus, and the radius vector from the Sun to Earth at the times Uranus is observed. The orbit is calculated in the reference $x_0y_0z_0$ coordinate system.

Let $\underline{I}_i$ be a unit vector pointing from Earth to Uranus at time t_i for i = 1, 2 or 3. Let θ_i be the latitude and ϕ_i, the colongitude of Uranus at t_i (Figure 12). Then,

$$\underline{I}_i = \sin\theta_i \cos\phi_i \hat{i}_o + \sin\theta_i \sin\phi_i \hat{j}_o + \cos\theta_i \hat{k}_o \qquad (A1)$$

Uranus' radius vector from the Sun to Earth at time t_i is given by

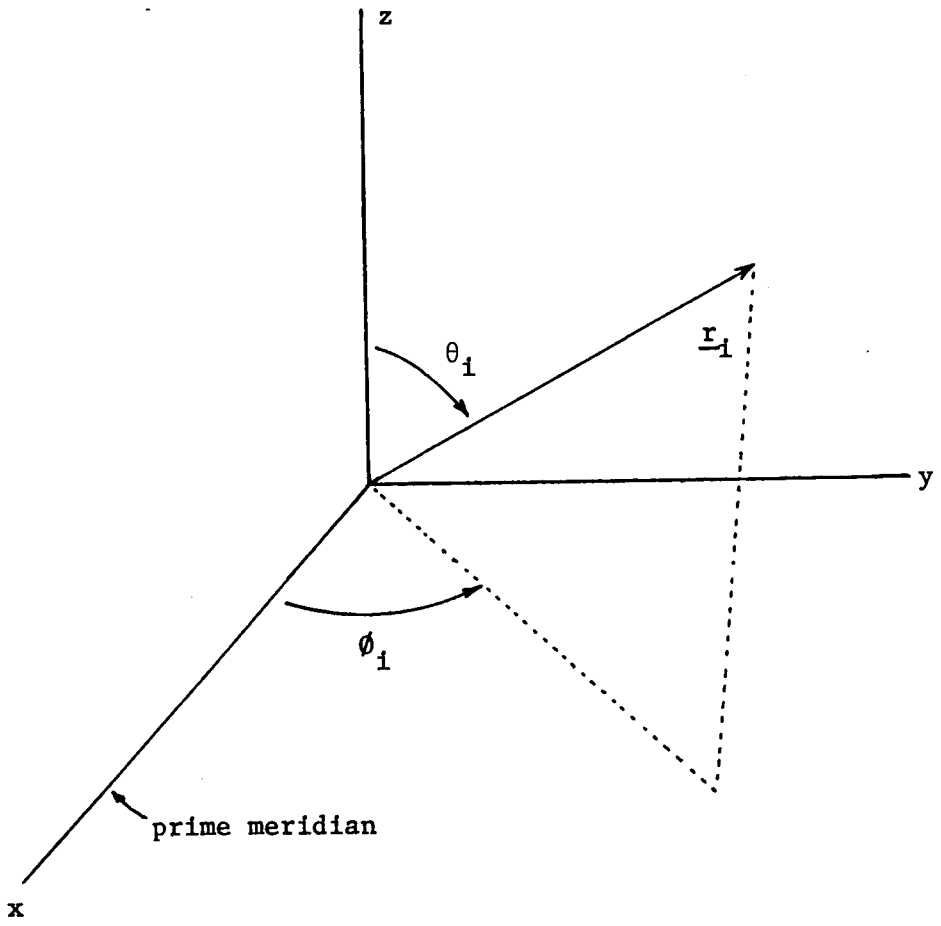
$$\underline{r}_i = \underline{r}_{ei} + R_i \underline{I}_i \qquad (A2)$$

where $\underline{r}_{ei}$ is the radius vector from the Sun to Earth at time t_i, and R_i is the distance between the Earth and Uranus at time t_i.

Now, R_3 is calculated using $\underline{W}$, a unit vector in the direction of the orbital angular momentum. $\underline{W}$ is given by

$$\underline{W} = \frac{\underline{r}_1 \times \underline{r}_2}{r_1 r_2 \sin\gamma} \qquad (A3)$$

where γ is the angle from $\underline{r}_1$ to $\underline{r}_2$ measured in the direction of Uranus' motion (Figure 13).



θ_i = latitude at time i

ϕ_i = colongitude at time i

Figure 12. Latitude and colongitude definitions.

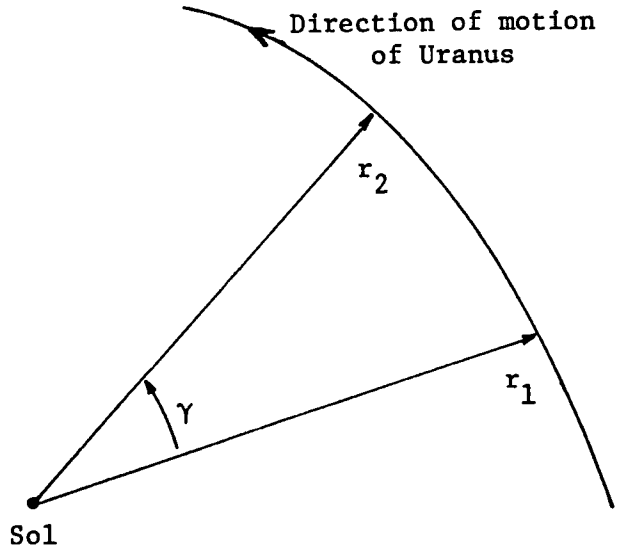


Figure 13. The orbit of Uranus.

Since $\underline{W}$ is normal to the orbital plane,

$$\underline{r}_3 \cdot \underline{W} = 0 \tag{A4}$$

Therefore,

$$R_3 = - \frac{\underline{r}_{e3} \cdot \underline{W}}{\underline{I}_3 \cdot \underline{W}} \tag{A5}$$

where $\underline{r}_{e3}$ is the radius vector from the Sun to Earth at time t_3 and,

$$\underline{r}_3 = \underline{r}_{e3} + R_3 \underline{I}_3 \tag{A6}$$

The next step is to find the instantaneous orbital elements of Uranus. The second component of the unit vector in the direction of the orbital angular momentum is,

$$W_3 = \cos i \left(\frac{\underline{r}_1 \times \underline{r}_2}{r_1 r_2 \sin \gamma} \right) \hat{k}_0 \tag{A7}$$

Therefore, the inclination is,

$$i = \text{arc cos } W_3 \tag{A8}$$

The longitude of the ascending node Ω is calculated using $\underline{W}$ and i. $\underline{W}_0$, the unit vector in the direction of the orbital angular momentum in the Sun centered reference rectangular coordinate system (Figure 5, p. 18) is determined by applying the transformation equation:

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \underline{\underline{T}} \begin{bmatrix} W_{1_0} \\ W_{2_0} \\ W_{3_0} \end{bmatrix} = \underline{\underline{T}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin i \sin \Omega \\ -\sin i \cos \Omega \\ \cos i \end{bmatrix} \tag{A9}$$

The transformation theory and the array $\underline{\underline{T}}$ are given in Appendix B.

Using the identity

$$\tan \frac{\Omega}{2} = \frac{\sin \Omega}{1 + \cos \Omega} \tag{A10}$$

a relation between $\underline{W}$, i and Ω is obtained:

$$\tan \frac{\Omega}{2} = \frac{W_1}{\sin i - W_2} \tag{A11}$$

or

$$\Omega = 2 \operatorname{atan} \left[\frac{W_1}{\sin i - W_2} \right] \tag{A12}$$

The remainder of the orbital elements are calculated using the unit vector in the direction of r:

$$\underline{U} = \frac{\underline{r}}{|\underline{r}|} \tag{A13}$$

Using the transformation equation, a relation between U, θ, i, ω and Ω is obtained:

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \frac{1}{|\underline{r}|} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{\underline{T}}{|\underline{r}|} \begin{bmatrix} r_{x_0} \\ r_{y_0} \\ r_{z_0} \end{bmatrix} = \frac{\underline{T}}{|\underline{r}|} \begin{bmatrix} r \cos\theta \\ r \sin\theta \\ 0 \end{bmatrix} = \underline{T} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{A14}$$

Let u, the argument of latitude, be defined as

$$u = \omega + \theta \tag{A15}$$

and using the identities

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \tag{A16}$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \tag{A17}$$

the expressions for U_x and U_y become:

$$U_x = -\cos i \sin\Omega \sin u + \cos\Omega \cos u \tag{A18}$$

$$U_y = \cos i \cos\Omega \sin u + \sin\Omega \cos u \tag{A19}$$

Solving these two equations for sin u and cos u and using the half-angle identity,

$$\tan\alpha/2 = \frac{\sin\alpha}{1 + \cos\alpha} \tag{A20}$$

gives

$$\tan\left(\frac{u}{2}\right) = \frac{-U_x \sin\Omega + U_y \cos\Omega}{1 + U_x \cos\Omega + U_y \sin\Omega} \quad \frac{1}{\cos i} \tag{A21}$$

or

$$u = 2 \text{ atan} \left[\frac{-r_x \sin \Omega + r_y \cos \Omega}{|\underline{r}| + r_x \cos \Omega + r_y \sin \Omega} \quad \frac{1}{\cos i} \right] \tag{A22}$$

Now, r and θ are defined by the equations,

$$r = |\underline{r}| = \frac{p}{1 + e \cos \theta} \tag{A23}$$

$$\theta = u - \omega \tag{A24}$$

so,

$$p = [1 + e (\sin u \sin \omega) + \cos u \cos \omega)]r \tag{A25}$$

Now, p is a constant for all r and u. Look at p at two different times t_1 and t_2:

$$p = r_1 + er_1 \sin u_1 \sin \omega + er_1 \cos u_1 \cos \omega \tag{A26}$$

$$p = r_2 + er_2 \sin u_2 \sin \omega + er_2 \cos u_2 \cos \omega \tag{A27}$$

Subtracting the second equation from the first gives,

$$0 = (r_1 - r_2) + e \sin \omega (r_1 \sin u_1 - r_2 \sin u_2) + e \cos \omega (r_1 \cos u_1 - r_2 \cos u_2) \tag{A28}$$

A similar equation can be obtained for times t_2 and t_3:

$$0 = (r_2 - r_3) + e \sin \omega (r_2 \sin u_2 - r_3 \sin u_3) \quad \text{(A29)}$$
$$+ e \cos \omega (r_2 \cos u_2 - r_3 \cos u_3)$$

To simplify equations (A28) and (A29), make the following definitions:

$$c_1 = r_1 - r_2 \quad \text{(A30)}$$

$$c_2 = r_2 - r_3 \quad \text{(A31)}$$

$$x_1 = e \sin \omega \quad \text{(A32)}$$

$$x_2 = e \cos \omega \quad \text{(A33)}$$

$$a_{11} = r_1 \sin u_1 - r_2 \sin u_2 \quad \text{(A34)}$$

$$a_{12} = r_1 \cos u_1 - r_2 \cos u_2 \quad \text{(A35)}$$

$$a_{21} = r_2 \sin u_2 - r_3 \sin u_3 \quad \text{(A36)}$$

$$a_{22} = r_2 \cos u_2 - r_3 \cos u_3 \quad \text{(A37)}$$

Then,

$$0 = c_1 + x_1 a_{11} + x_2 a_{12} \quad \text{(A38)}$$

$$0 = c_2 + x_1 a_{21} + x_2 a_{22} \quad \text{(A39)}$$

Solving for x_1 and x_2 gives:

$$x_1 = \frac{a_{12}c_2 - a_{22}c_1}{a_{11}a_{22} - a_{12}a_{21}} \tag{A40}$$

$$x_2 = \frac{a_{21}c_1 - a_{11}c_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{A41}$$

Using the identities,

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{A42}$$

$$\tan \alpha/2 = \frac{\sin \alpha}{1 + \cos \alpha} \tag{A43}$$

expressions for the eccentricity e and the argument of pericenter ω are derived:

$$e = (x_1^2 + x_2^2)^{\frac{1}{2}} \tag{A44}$$

$$\omega = 2 \text{ atan } \left(\frac{x_1}{e + x_2}\right) \tag{A45}$$

To this point, e, p, i, Ω, and ω, have been found. It only remains to calculate the mean anomaly at time t_1. The orbit of Uranus is elliptic, so the eccentric anomaly at time t_1 (Figure 6, p. 19) is related to the true anomaly by the equation,

$$\tan \left(\frac{E_1}{2}\right) = \left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}} \tan \left(\frac{\theta_1}{2}\right) \tag{A46}$$

so,

$$E_1 = 2 \text{ atan } \left[\left(\frac{1 - e}{1 + e} \right)^{\frac{1}{2}} \tan \left(\frac{u_1 - \omega}{2} \right) \right] \tag{A47}$$

and

$$M_1 = E_1 - e \sin_1 E \tag{A48}$$

which is the final orbital element.

Once the elements have been determined, the distance and true anomaly are found from the equations,

$$\theta = 2 \text{ atan } \left[\left(\frac{1 + e}{1 - e} \right)^{\frac{1}{2}} \tan \left(\frac{E}{2} \right) \right] \tag{A49}$$

$$r = \frac{P}{1 + e \cos \theta} \tag{A40}$$

where E is the eccentric anomaly at time t.

The derivatives of the osculating elements are found using a five-point central difference approximation:

$$\frac{dc_i}{dt} = \frac{1}{12h_i} \left[c_{i-2} - 8c_{i-1} + 8c_{i+1} - c_{i+2} \right] \tag{A51}$$

where h_i is the step size or difference in the approximation.

APPENDIX B

COORDINATE TRANSFORMATION

The coordinate transformation of a vector $\underline{A}$ from rectangular orbital coordinates x y z to reference rectangular coordinates $x_0 y_0 z_0$ is given by

$$\begin{bmatrix} A_{y_0} \\ A_{y_0} \\ A_{z_0} \end{bmatrix} = \underline{T} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{B1}$$

where

$$\underline{T} = \begin{bmatrix} \cos\Omega\cos\omega - \cos i \sin\Omega \sin\omega & -\cos\Omega\sin\omega - \cos i \sin\Omega\cos\omega & \sin i \sin\Omega \\ \sin\Omega\cos\omega - \cos i \cos\Omega \sin\omega & -\sin\Omega\sin\omega - \cos i \cos\Omega\cos\omega & -\sin i \cos\Omega \\ \sin i \sin\omega & \sin i \cos\omega & \cos i \end{bmatrix} \tag{B2}$$

and i is the angle of inclination, Ω is the longitude of the ascending node, and ω is the argument of pericenter.

Figure 5 (p. 18) shows the spatial relation between the x y z and the $x_0 y_0 z_0$ systems and shows how the angles i, Ω and ω are oriented.

APPENDIX C

NUMERICAL SOLUTION TO DIFFERENTIAL EQUATIONS

The Adams fourth order predictor-corrector method of solving first order differential equations (Brouwer and Clemence, 1961) is summarized below.

The approximate solution of the initial value problem:

$$\begin{aligned} y'(t) &= f(t,y) \\ y(a) &= \alpha \end{aligned} \qquad a \leq t \leq b \tag{C1}$$

at $N + 1$ equally spaced points in the interval $[a,b]$, begins with initial approximations, W_0, W_1, W_2, W_3 to y at the first four values of t. W_0 is taken as:

$$W_0 = y(a) = 0 \tag{C2}$$

and W_1, W_2 and W_3 are obtained via the fourth order Runge-Kutta method:

$$W_i = W_{i-1} + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad i = 1, 2, 3 \tag{C3}$$

where

$$h = (b - a)/N \tag{C4}$$

$$K_1 = hf(t_{i-1}, W_{i-1}) \tag{C5}$$

$$K_2 = hf(t_{i-1} + h/2, W_{i-1} + \frac{K_1}{2}) \tag{C6}$$

$$K_3 = hf(t_{i-1} + h/2,\ W_{i-1} + \frac{K_2}{2}) \tag{C7}$$

$$K_4 = hf(t_{i-1} + h,\ W_{i-1} + K_3) \tag{C8}$$

and

$$t_i = a + ih \tag{C9}$$

Next, an approximation $W_4^{(0)}$, to $y(t_4)$ is calculated using the Adams-Bashforth method:

$$W_4^{(0)} = W_3 + \frac{h}{24}[55f(t_3,W_3) - 59f(t_2,W_2) + 37f(t_1,W_1) - 9f(t_0,W_0)] \tag{C10}$$

This approximation to $y(t_4)$ is then improved using the three-step Adams-Moulton method:

$$W_4^{(1)} = W_3 + \frac{h}{24}[9f(t_4,W_4^{(0)}) + 19f(t_3,W_3) - 5f(t_2,W_2) + f(t_1,W_1)] \tag{C11}$$

The technique is repeated to find $W_5^{(0)}$ and $W_5^{(1)}$, and so on, until the final approximation to $y(t_{N+1})$ is obtained.

APPENDIX D

ORBIT PREDICTION

Orbit Prediction (OP) calculates the position of a planet orbiting about the Sun, given the orbital elements (Powell, 1983).

The first task of this orbit prediction routine is to find the time it takes the planet to move from the time at pericenter, t_p, to time t. To do this, look at the magnitude of the orbital momentum of the planet:

$$h = |\underline{r} \times \underline{v}| = rv_\theta = r^2 \frac{d\theta}{dt} \tag{D1}$$

where θ is the true anomaly and v_θ is the transverse velocity normal to $\underline{r}$. The similatus rectum, the second orbital element p, is defined by the equation

$$p = \frac{h^2}{\mu} \tag{D2}$$

where $\mu = G(m_s + m_p)$, m_s is the mass of the Sun, and m_p is the mass of the planet.

Solving equation (D2) for h and substituting into equation (D1) gives:

$$(\mu p)^{\frac{1}{2}} = r^2 \frac{d\theta}{dt} \tag{D3}$$

Now,

$$r = \frac{p}{1 + e \cos \theta} \tag{D4}$$

So,

$$dt = \left(\frac{p^3}{\mu}\right)^{\frac{1}{2}} \frac{d\theta}{[1 + e \cos \theta]^2} \tag{D5}$$

Equation (D5) relates the time to the true anomaly. To solve it, look at the geometric relation between the eccentric and true anomalies at time t:

$$\tan\left(E/2\right) = \left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}} \quad \tan\left(\theta/2\right) \tag{D6}$$

Differentiating both sides gives:

$$\frac{dE}{\cos^2\left(E/2\right)} = \left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}} \frac{d\theta}{\cos^2\left(\theta/2\right)} \tag{D7}$$

Using the trigonometric identities

$$\cos^2\left(\alpha/2\right) = \frac{1}{1 + \tan^2\left(\alpha/2\right)}$$
$$\tan^2\left(\alpha/2\right) = \frac{1 - \cos \alpha}{1 + \cos \alpha} \tag{D8}$$

and

$$\cos^2\left(\alpha/2\right) = \frac{1 - \cos \alpha}{2}$$

Equation (D7) becomes

$$d\theta = (1 - e^2)^{\frac{1}{2}} \frac{dE}{1 - e \cos \theta} \tag{D9}$$

and

$$\cos \theta = \frac{\cos E - e}{1 - e \cos E} \tag{D10}$$

Substituting these expressions into equation (D5) gives:

$$dt = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} (1 - e \cos E)dE \tag{D11}$$

where a, the semimajor axis of the orbit is

$$a = \frac{P}{1 - e^2} \tag{D12}$$

Integrating equation (D11) from the time at pericenter, t_p, to a time t gives:

$$t - t_p = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} (E - e \sin E) \tag{D13}$$

Define the mean anomaly at time t by

$$M = E - e \sin E \tag{D14}$$

The time it takes the planet to move from pericenter to time t is

$$t - t_p = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} M \tag{D15}$$

At some reference time t_o

$$t_o - t_p = \left(\frac{a^3}{\mu}\right) \quad M_o \tag{D16}$$

where M_o is the mean anomaly at t_o, the third orbital element.

Eliminating t_p from equations (D15) and (D16) and solving for M gives:

$$M = M_o + (t - t_o)\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} \tag{D17}$$

Once M is known, equation (D14) can be solved for E using Newton-Raphson iteration (Burden, Faires and Reynolds, 1981). That is, given a set of equations of the form

$$F_i(x) = 0 \qquad i = 1, \ldots, N \tag{D18}$$

The $(k+1)^{th}$ approximation to x is

$$x^{k+1} = x^k - \frac{F(x^k)}{\frac{d}{dx} F(x^k)} \tag{D19}$$

Here,

$$F_i(E) = E_i - e \sin E_i - M_i = 0 \qquad i = 1, \ldots, N \tag{D20}$$

The first approximation to E is $E^1 = M$. And the $(i+1)^{th}$ approximation is

$$E^{i+1} = E^{i} - \frac{E^{i} - e \sin E^{i} - M}{1 - e \cos E^{i}} \tag{D21}$$

Once E is known, θ is found through equation (D6) and finally, the radius vector at time t_i is

$$\begin{aligned} r_x &= r \cos \theta \\ r_y &= r \sin \theta \\ r_z &= 0 \end{aligned} \tag{D22}$$

where r is found from equation (D4).

APPENDIX E

PROGRAM DOCUMENTATION

ENCKE Documentation

Subroutine ENCKE uses Encke's method to calculate the perturbed positions of a four-body system. For this calculation, the system consists of Jupiter, Saturn, Uranus, and Neptune. It is assumed that the effects of the minor planets and Pluto on this system are too small to be seen, that is, their contributions to the perturbations are well within the "noise" of the observational data.

ENCKE is first used to generate test input for SLE to make sure the programs work as they were intended. In the final procedure to predict planet X, ENCKE is required because the Subroutine REX, Range Estimates to Planet X, needs the x component of the gravity vector on Uranus due to planet X only. Subroutine SLE calculates the total gravity vector on Uranus due to the other planets in this system. To find the gravity vector due to planet X, the contributions due to the other planets must be subtracted from the total vector. Subroutine ENCKE is the first step in this procedure. The true positions of the planets, given by ENCKE, will be used in the Subroutine GRAV to find the forces (and hence the gravity vector) on Uranus due to Jupiter, Saturn, and Neptune. GRAV will then subtract these contributions from the total gravity vector to give the gravity vector on Uranus due to planet X only.

ENCKE is composed of four subroutines:

1. Subroutine OP, Orbit Prediction, calculates the two-body position for each planet at each time, N, using the constant unperturbed values of the orbital elements of the planets.

2. Subroutine CT, Coordinate Transformation, transforms these positions from elliptic coordinates to reference rectangular coordinates for further calculations.

3. Subroutine F, Function, sets up a set of equations which relate the second derivatives of the perturbations of the i^{th} body to the perturbed positions of all the bodies in the system. It is assumed here that the positions can be expressed in terms of an unperturbed part and a perturbed part.

4. Subroutine AM, Adams-Moulton, uses the functions given by F to solve for the perturbations of each planet and the derivatives of those perturbations. AM is based on the fourth-order Adams-Bashforth method as a predictor and one iteration of the Adams-Moulton method as a corrector. The starting values are obtained via the fourth order Runge-Kutta method. It is also sometimes called the Adams fourth order predictor-corrector method.

After the perturbations are calculated by the Subroutines described above, ENCKE calls OP again to calculate the two-body positions of the four planets. Then ENCKE adds the perturbations to these positions to obtain the true positions of each of the planets.

OP Documentation

OP calculates the two-body position or radius vector of a planet using the constant, unperturbed values of the orbital elements. OP is

used to calculate the position of Earth for the test input to SLE.

SLE Documentation

Subroutine SLE solves the six Lagrange equations for the components (in cylindrical orbital coordinates) of the total gravity vector of the planet Uranus. The Lagrange planetary equations equate the rates of change of the orbital elements to an expression involving the orbital elements themselves and the three components of the total gravity vector at Uranus. SLE first calls the Subroutine CEU to calculate the osculating orbital elements of the Uranian orbit and then it calls the Subroutine DA to calculate the rates of change of those elements with respect to time. SLE's final act is to call the Subroutine OE to solve this set of six overdetermined equations for the three components of the gravity vector at Uranus.

Subroutine CEU

Subroutine CEU calculates the elements of Uranus. CEU's first task is to set up several working arrays to use in the Subroutine OD2. Since OD2 uses a set of three data points to calculate the two-body orbital elements, CEU must divide up the entire data set into subsets of three points each; then it assigns the values in the three point data set to a group of working arrays. After OD2 calculates the orbital elements, CEU assigns them to the second point in the data set of three and then performs the entire procedure again for the next subset. It is important to note here that each subset is three fold redundant. That is, adjacent subsets overlap by two data points:

```
  1   2   3   4   5   6   7   8   9
| subset 1 |
    | subset 2 |
        | subset 3 | , etc.
```

Therefore, except for the first and last data points, separate orbital elements for Uranus are calculated for each point in the original data set.

Subroutine OD2

Subroutine OD2 performs an orbit determination for a two-body system. Given three angular positions at three times and two range estimates, Subroutine OD2 will find the osculating orbital elements of the secondary body. After setting up its working arrays, OD2 first calls the Subroutine NRNE to calculate the best values of the ranges. In this case, the inputted range "estimates" are the calculated distances from Earth to Uranus and hence are very good. But, these distances are found by a polynomial approximation to a nine-planet system and therefore, they will be different from the values calculated here. After the best possible values for the ranges are found, OD2 then proceeds to calculate the orbital elements based on the equations of classical celestial mechanics.

Subroutine NRNE

Subroutine NRNE will solve a set of two equations (provided by the Subroutine FOD2) for two unknowns given estimates for those unknowns. In this case, the unknowns are the ranges from Earth to

Uranus at two times and the equations relate the time from pericenter to the mean motion and reference mean anomaly of the Uranian orbit. NRNE's first task is to calculate the approximate derivatives of the functions using a five-point central difference formula. Next, it calls FOD2 to calculate the actual values of the functions. Then, LENRNE is called to solve for the difference, DX, between the N^{th} and $(N+1)^{st}$ approximations to the ranges and, if DX is smaller than 10^{-6}AU, the calculations are stopped and the range values are given back to OD2. If DX is larger than 10^{-6}AU then another iteration will be made.

Subroutine FOD2

Subroutine FOD2 will set up two equations of the form $Y(x)\simeq 0$. These equations relate the time from pericenter to the mean motion and reference mean anomaly of the Uranian orbit. FOD2 will then pass the equations back to NRNE to solve for the best values of the ranges from Earth to Uranus at two different times.

Subroutine LENRNE

Subroutine LENRNE uses simple matrix algebra to solve a set of linear equations of the form AX = C, where A and C are constant arrays and X is an array of unknowns. Here, the components of A are the derivatives of the functions (given by FOD2) with respect to the unknowns. The components of C are the functions themselves and the X's are the differences between the N^{th} and $(N+1)^{st}$ approximations to the ranges from Earth to Uranus at two times.

Subroutine DA

Given a set of orbital elements associated with each data point, DA will find the rates of change of the elements of Uranus with respect to time using a simple five-point central difference approximation.

Subroutine OE

Subroutine OE is called by SLE. It will take a set of six nonlinear equations and compute a least-squares solution for three unknowns. The equations, given by Subroutine F, are the Lagrange planetary equations and the unknowns OE will solve for are the three components of the total gravity vector at Uranus. The Lagrange planetary equations are functions of the radius, R, and true anomaly, THETA, of the Uranian orbit (both calculated in the Subroutine RTHETA), the orbital elements of Uranus (calculated in the Subroutine CEU), the rates of change of the orbital elements (calculated in the Subroutine DA) and finally, the three components of the total gravity vector at Uranus. To solve the nonlinear Lagrange equations, OE will first formulate a set of linear equations which are derived from the Lagrange equations. OE will then go into an iterative loop to solve the linear equations for successive approximations to the gravity vector components (this is very similar to the Newton-Raphson method of solving equations). In this iteration loop, OE calls the Subroutine LE to solve for the difference between the N^{th} and $(N+1)^{st}$ approximations to the components of the gravity vector. When this difference becomes small enough, that is, when the solution has converged

sufficiently to suit the user, then the loop will end and the total gravity vector at Uranus will have been found.

Subroutine RTHETA

Subroutine RTHETA will find the heliocentric distance and the true anomaly of Uranus at a given time, using the equations of classical celestial mechanics.

Subroutine F

Subroutine F will set up the set of six Lagrange planetary equations of the form $Y(x) \simeq 0$. It will then pass these equations back to OE to solve for the components of the total gravity vector at Uranus.

Subroutine LE

Subroutine LE solves a set of linear equations for the difference between the N^{th} and $(N+1)^{st}$ approximations to the gravity vector at Uranus.

THETAU Documentation

THETAU finds the true anomaly of Uranus for each point in the data set. The true anomaly of Uranus is required for the calculations in the CTG program. THETAU uses those equations which are defined in the Subroutine OD2.

CTG Documentation

The SLE program calculates the components of the gravity vector in cylindrical orbital coordinates. But, it is necessary to have them

in reference rectangular coordinates for the next series of calculations. The CTG program performs this transformation.

The first DO loop in the routine changes the cylindrical orbital coordinates to rectangular orbital coordinates. The transformed array is called GREF, which stands for the reference gravity vector. The second DO loop changes the arbitrary rectangular coordinates to the reference rectangular coordinates. The array name stays the same: GREF.

GRAV Documentation

The function of GRAV is to find the gravity vector at Uranus due to planet X only. First it takes the true positions of the planets Jupiter, Saturn, Neptune, and Uranus in reference rectangular coordinates (given by ENCKE) to calculate the accelerations of Uranus due to the other planets. Basic Newtonian physics is used.

GRAV then subtracts these acceleration contributions from the total perturbing gravity vector (calculated by SLE and CTG) on Uranus to get the perturbing gravity vector due to planet X only.

REX Documentation

The body of REX consists of a single CALL statement: CALL OE. OE will do all of the actual work for REX. The first thing required by OE is an equation provided by the Subroutine F. This equation relates (1) the x component of the gravity vector on Uranus due to planet X, (2) the mass of Uranus, (3) the radius, R, and the true anomaly, THETA of Uranus, and (4) Sol's mass with the two unknowns:

x(1) = mass of planet X

x(2) = magnitude of the radius of planet X's orbit

(Note: the first values of x must be input by the user.)

OE first sets up a set of linear equations which are dependent upon the unknowns. OE then calls the Subroutine LE (Linear Equations) to solve for the unknowns using matrix algebra and back substitution. Once the mass and range to planet X have been found, REX's job is finished.

OD2G Documentation

OD2G is very similar to the Subroutine OD2 except that it uses the gravity vector of planet X to construct the unit vector from Earth to X. With this information, an estimate of the range to X (from REX), and the direction to the planet at three times, OD2G will calculate the two-body instantaneous elements of planet X.

OD3 Documentation

OD3 is an orbit determination routine which uses a least-squares fit to optimize the orbital elements of a planet given an initial approximation to those elements. Once the elements have been optimized, an estimate of the error associated with the solution is made by calculating the standard deviation of the elements.

OPX Documentation

OPX calculates the two-body position (i.e., the radius vector) of planet X using the constant unperturbed values of the orbital elements of planet X. The orbital elements of X are given to OPX by the Sub-

routine OD2G. OPX calculates the position of X at one time only. For the first run, this time will be the present (i.e., OPX will predict where planet X is in the solar system today).

VITA

Deborah Susan Brooks was born in Hamilton, Ohio, on November 3, 1958. As an undergraduate, she attended the University of Cincinnati, and in June 1981 she received a Bachelor of Science degree in physics. In the summer of 1981 she accepted a position as a research engineer with Calspan Corporation, AEDC Division at Arnold Air Force Station in Tennessee. She began her graduate studies at the University of Tennessee Space Institute in the fall of 1981. She was awarded a Master's degree in physics in June 1986.