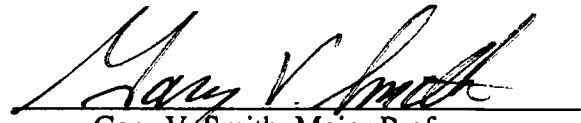
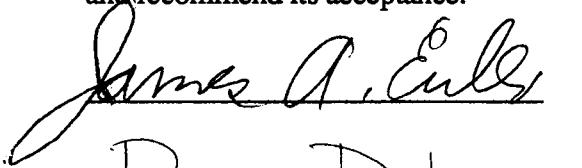
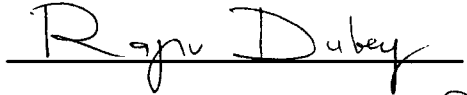
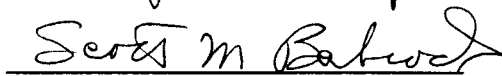


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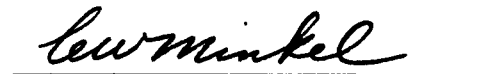
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Gary V. Smith, Major Professor

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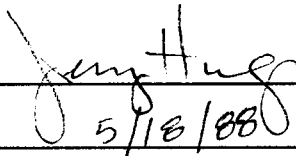
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MODELING AND CONTROL
OF A DIFFERENTIAL WRIST
WITH JOINT COMPLIANCE

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Jerry Chih-Lun Huang

June 1988

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To Ju-Chin and Pi-Chen Kuan Huang

ACKNOWLEDGMENTS

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ABSTRACT

The dynamics of a differential gear train employed on the compliant wrist drive of the CESAR manipulator was modeled mathematically. Expressions for the kinetic and potential energies of the compliant wrist drive train were developed and Lagrange's equations were applied to derive the equations of motion.

Three methods were used to control the compliant wrist drive train: pole-placement, PI-compensated pole-placement, and LQR design with asymptotic properties. The pole-placed design assumed the system was fourth-order, linearized, and decoupled. Feedback gains for the pole-placement design chosen so that the closed-loop characteristic equation corresponded to that of the binomial standard filter. The PI, pole-placement design involved choosing the PI gains so that the closed-loop characteristic roots were near the inner-loop characteristic roots. Inner-loop roots were then pole-placed so that the outer-loop characteristic equation would correspond to the binomial standard filter. For the LQR design, the compliant wrist drive train was assumed to be a linearized, coupled, eighth-order system. Using the method developed by Harvey and Stein, a set of weighting matrices was found so that the result of linear quadratic optimization would be a control structure with desired modal properties.

Simulation was performed using the coupled, nonlinear equations of motion with feedback gains determined by the three linear control techniques. Feedback gains for the pole-placement and PI designs were updated as position changed, while the gains for the LQR design were constants. Results showed that all three designs could achieve zero overshoot and minimal coupling under no load, but all failed with 50% of maximum load. The designs were then repeated assuming 50% of maximum load (15 lb) as a point mass on the wrist; pole-placement design still failed, while PI and LQR designs were able to keep overshoot to less than 5% while still minimizing coupled response.

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LIST OF SYMBOLS

SYMBOL(S)	DESCRIPTION
A, B, C, D	state matrices
$\mathbf{x}, \mathbf{u}$	state vector, input vector
Q, R	quadratic weighting matrices
ρ	input scaling factor
ϕ, ψ	pitch angle, yaw angle
θ_1, θ_2	motor angles
θ_3, θ_4	differential input angles
J_{m1}, J_{m2}	motor inertias
K_{m1}, K_{m2}	motor torque constants
B_{m1}, B_{m2}	motor damping constants
$I_{xx}, I_{yy}, I_{zz}, I_{xz}$	wrist inertia terms
k_{s1}, k_{s2}	cable spring rates
r_1, r_2, r_3, r_4	sheave radii
N_1, N_2	gear ratios
$K_{\theta 1}, K_{\theta 2}$	forward loop gains
K_{P1}, K_{P2}	motor position feedback gains
K_{V1}, K_{V2}	motor velocity feedback gains
K_{P3}, K_{P4}	differential input position feedback gains
K_{V3}, K_{V4}	differential input velocity feedback gains
K_{E1}, K_{E2}	steady-state error correction gains
K_P, K_I	proportional gain, integral gain
K_{PI}	PI-compensator gain (same as K_P)
ζ	ratio of integral gain to proportional gain

I. INTRODUCTION

In a majority of robotic systems, the location of the actuator (motor) of a given linkage is on the end of the previous linkage. By employing this serial method, the relative motion of that linkage with respect to the previous linkage can be specified by that single motor. The advantage of this method is that the resulting dynamics of this chain of drives are coupled only by Coriolis and centrifugal accelerations, which, at low tip speeds, are generally negligible. However, difficulties arise when working with small linkages such as those in the wrist of most manipulators. One problem is that the motion of the linkages are generally constrained by the presence of the motors; also, with the motors located on the wrist, the manipulator is required to carry the weight of the motor as well as the design load.

For these reasons, it is desirable to study a drive system with the actuators located away from these small linkages. An example of such drive systems is the Center for Engineering Systems Advanced Research Manipulator (CESARM) at Oak Ridge National Laboratories, as shown in Figure 1-1. By using a differential gear train to drive the wrist, the range of motion for the wrist is increased; furthermore, the motors are now located in the upper arm, allowing the wrist to carry a greater load without increasing the load requirements of the other joints.

While the differential wrist is a more compact design, its dynamics are more complicated than the serial method. This is due to the fact that two motor inputs are required to specify a single output, whereas only one motor is required in the serial method. Furthermore, joint compliance becomes an issue, since the motors must be connected to the wrist by some drive train of finite stiffness.

Because the differential wrist is a multi-input, multi-output (MIMO) system, classical single-input, single-output (SISO) techniques such as phase-lead, phase-lag, PID,

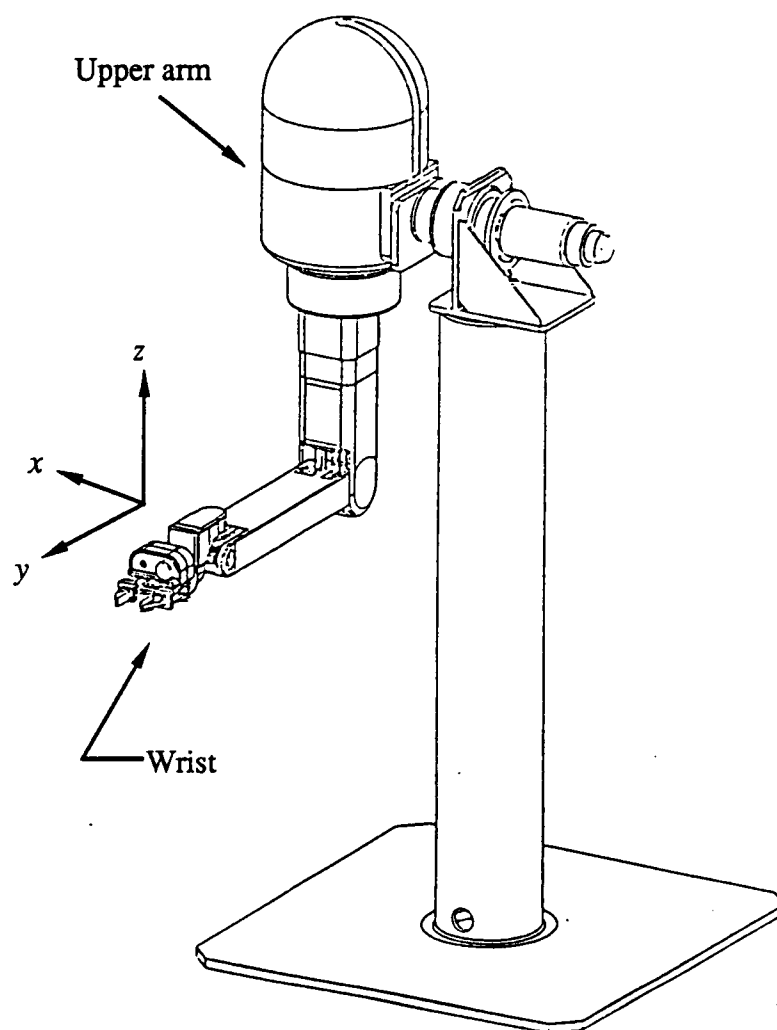


Figure 1-1. CESAR seven-degree-of-freedom manipulator.

and rate feedback control cannot be employed without either decoupling the system or operating at a point where the system is inherently uncoupled. As a result, the use of multivariable control methods may be required to improve dynamic response of the system.

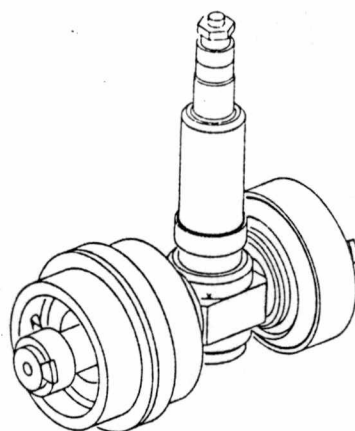
Description of the Problem

In Figure 1-2, the individual linkages that comprise the parts of the wrist are shown in detail, and a schematic of the compliant wrist drive system is shown in Figure 1-3. Two motors located in the cluster on the upper arm drive the wrist pitch and yaw. The motors are connected by tension cables and sheaved pulleys to the side gears of the differential, which are located on either side of the cross on the T-bar (link 8). The side gears, referred hereafter as the differential inputs, drive a top gear mounted on the base of the T-bar. Link 9 is an "L"-housing mounted on the top gear, and link 10 is a gripper tong located at the end of the "L"-housing.

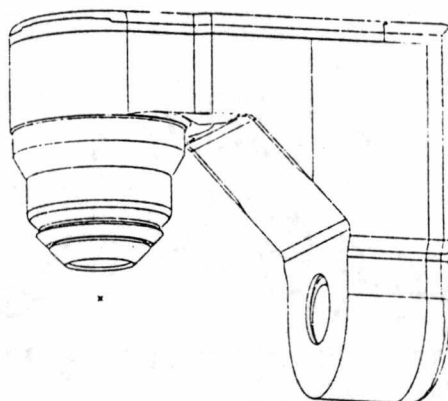
The pitch axis is the axis connecting the differential inputs (the cross of the "T") and the yaw axis is the axis through the top gear (the base of the "T"). It should be noted that the pitch axis is fixed on the arm, whereas the yaw axis varies with pitch position. The wrist roll is the axis through the gripper, and is driven independently through the yaw axis by a cable in the "L"-housing.

Thesis Objectives

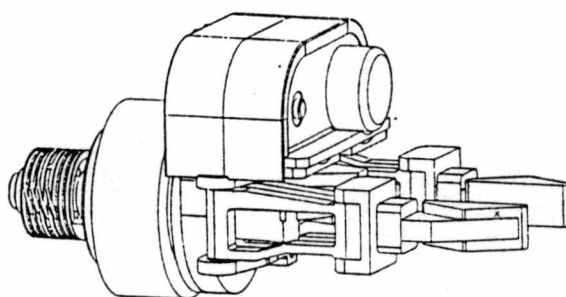
It is the purpose of this thesis to (1) develop a working model of the wrist dynamics, and (2) to apply classical and modern control design techniques to the compliant differential wrist and study the characteristics of each approach. The equations of motion are derived by writing the expressions for kinetic and potential energy expressions and



a. Link 8 ("T"-bar).



b. Link 9 ("L"-housing).



c. Link 10 (gripper tong).

Figure 1-2. Detail of wrist linkages.

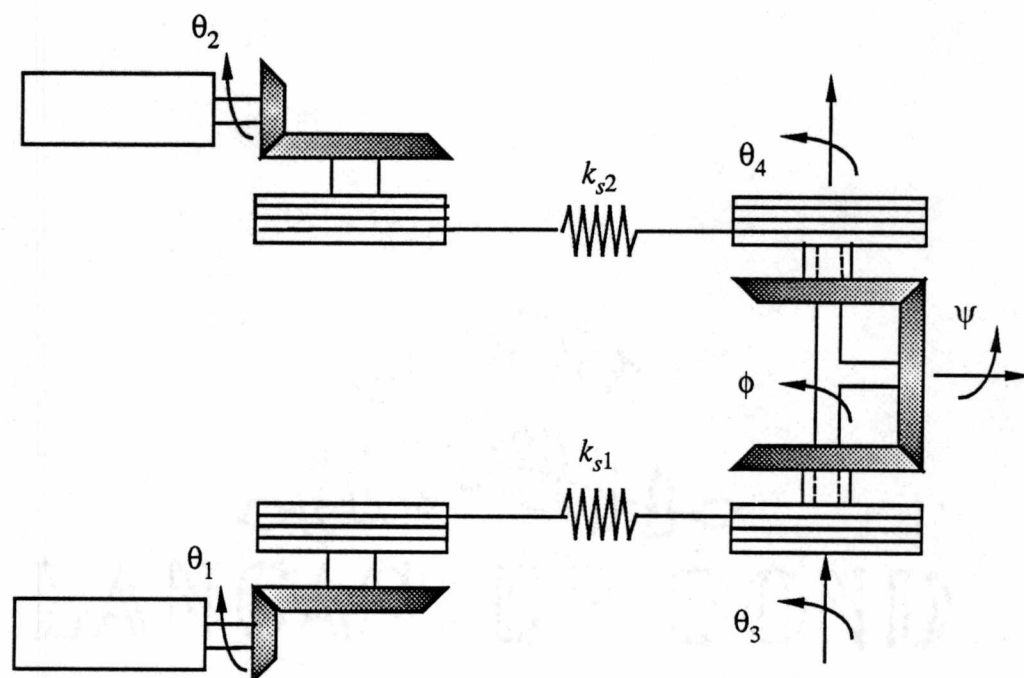


Figure 1-3. Schematic of CESAR Manipulator wrist drive system.

are used: pole-placement through full-state feedback, pole-placement with proportional-plus-integral (PI) compensation, and Linear Quadratic regulator (LQR) design for asymptotic modal properties. In addition, several analytical tools are employed: Advanced Continuous Simulation Language (ACSL) [16] is used to obtain simulation of step responses, MACSYMA [21] is used for verification of algebraic operations, and Matrix_x [12] is used to solve for regulator gains of the Linear Quadratic (optimal) problem.

Assumptions

Equations of motion

In writing the kinetic and potential energy expressions, the following assumptions are made:

1. The motors have linear torque constants, damping coefficients, and back emf constants, though a current loop in the system's circuitry makes the latter negligible along with the R-L time constant of the motor. The governing differential equation is second order linear.
2. The differential wrist is without backlash, friction, or gravitational load. Its dynamics are fourth order nonlinear with two inputs and two outputs.
3. Elements of the inertia tensor identified by [1] as less than 10^{-7} kg-m² are considered to be zero; thus, for the CESARM wrist, the only inertia terms present are I_{xx} , I_{yy} , I_{zz} , and I_{xz} .

4. The drive train is compliant with linear spring constants, and each cable drive is modeled as a spring-connected pulley. An equivalent analogy can be made by assuming an axial drive train connected by torsional rods.

Pole-Placement, PI design

For both the pole-placement model and the pole-placed PI model, the following assumptions are made:

1. The model involves only wrist motion (i.e., no shoulder or elbow motion in the wrist dynamics).
2. The mass matrix is assumed diagonal by ignoring the off- diagonal terms, and the remaining inertia terms are functions of the yaw angle, ψ .
3. The Coriolis and centrifugal acceleration terms are small compared to the inertial acceleration terms and, therefore, are considered negligible.
4. Linearization of the mass matrix is at discrete values of ψ , and is evaluated at static equilibrium conditions (i.e., tension in cables are equal).

LQR design

The same assumptions for the pole-placement and PI design apply to the LQR design, except that the mass matrix will not be assumed to be diagonal. Therefore, the overall system will be eighth order with two inputs (motors) and two outputs (pitch, ϕ , and yaw, ψ).

Design Procedure

Pole-placement, PI design

The design procedure for the pole-placement model through state feedback is as follows:

1. A feedback configuration is assumed for the closed-loop system, and the closed-loop characteristic equation for each drive train is determined by evaluation of the determinant of $s\mathbf{I}-\mathbf{A}$.
2. The coefficients of the closed-loop characteristic equation are matched up with those corresponding to a binomial standard filter of the form:

$$s^4 + 4\omega_0 s^3 + 6\omega_0^2 s^2 + 4\omega_0^3 s + \omega_0^4 \quad (1.1)$$

3. Four nonlinear equations are solved for four unknown gains in terms of the parameters of the compliant wrist drive train. These gain represent the closed-form solution of the nonlinear gains for the wrist.

The steady-state error of the system is compensated by two methods: by multiplying the input by the inverse of the steady-state response and by employing PI-compensation. The first is obtained by applying the final value theorem,

$$F_{ss} = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s), \quad (1.2)$$

to the linearized transfer function. The second is achieved via the following procedure:

1. Arbitrarily select a value of $\zeta = K_I/K_P$ such that it is greater than zero but much less than one.
2. Set the closed-loop attenuation factor F_1 initially to unity, pole-place the approximate closed-loop characteristic equation, and simulate the step response.
3. Decrease the value of F_1 according to the peak overshoot in the step response and iterate the above procedure until the desired overshoot is obtained.

LOR design

A viable alternative to pole-placement is to choose the feedback gains by optimizing a linear quadratic performance index of the form:

$$V = \int_0^T (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \rho \mathbf{u}^T \mathbf{R} \mathbf{u}) dt \quad (1.3)$$

where

$\mathbf{x}$ = state vector,

$\mathbf{u}$ = input vector,

$\mathbf{Q}, \mathbf{R}$ = weighting matrices,

ρ = input scaling factor.

In this design, the assumptions are identical to those in the pole-placement design, with the exception of the diagonal mass matrix assumption. Thus, an MIMO system is assumed, with the two motors as inputs and the pitch and yaw angles as outputs.

The technique for finding proper weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ for asymptotic modal behavior is described in detail in [11] and is summarized here as:

1. Select $n-m$ characteristic roots (eigenvalues) of the system, where n is the order of the system and m is the number of inputs.
2. Specify certain components of the eigenvectors corresponding to each selected eigenvalue, then finding a modal vector by best fit that falls in the subspace of $(s^0\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$, where s^0 is the selected eigenvalue.
3. Calculate a projection matrix $\mathbf{P}'$ from the modal matrix, $\mathbf{X}$, defined by:

$$\mathbf{P}' = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \mathbf{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \quad (1.4)$$

4. Construct the $\mathbf{H}_0$ -matrix from the projection matrix $\mathbf{P}'$ defined by:

$$\mathbf{H}_0 = \begin{bmatrix} P_{22}^{-1}P_{12}^T & \mathbf{I}_m \end{bmatrix} \quad (1.5)$$

5. Specify m "infinite" eigenvalues and corresponding eigenvector modes.

Once the $\mathbf{H}_0$ -matrix, the infinite eigenvalues, and the infinite modes are determined, the weighting matrices for the LQR problem will be completely defined. The final step is to solve for regulator gains by for several consecutively decreasing values of ρ until the desired modal properties are achieved.

Literature Survey

For the most part, MIMO systems have utilized the LQR and LQG solutions in different forms to obtain robust servomechanisms. In Davison and Scherzinger [3], a criterion is formulated in terms of arbitrarily good, approximate error regulation (AGAER); relaxation of requirements on regulation lead to a high-gain servomechanism with

arbitrarily good transient response. A key AGAER requirement, however, was for initial conditions to be zero.

On the other hand, others have adopted a generalization of the SISO frequency domain analysis for MIMO systems. Doyle and Stein [4] proposed this in 1981 and presented a loop transfer recovery (LTR) technique that, when used with LQG optimization, produced a robust servomechanism with a specific "phase margin". Later, Stein and Athans [19] showed that the LQG/LTR procedure was a trade-off in the frequency domain of H^2 -problem. In addition, they appended the LTR technique for use with non-minimal phase systems, though this method was only outlined for SISO systems.

Using Computer Aided Systems and Controls Analysis and Design Environment (CASCADE), Hansen and Birdwell [10] were able to apply the LQG/LTR technique on the PUMA 560 manipulator with reasonable success. Problems encountered but not discussed by either [4] or [19] were minimum and maximum singular values being of different system types and the lack of a definitive means of selecting cut-off frequencies. The first problem was resolved by orthogonal transformation while the second problem was circumvented by using trial-and-error.

Still another approach was used by Harvey and Stein [11] in which the asymptotic properties of the LQR problem are used to obtain specified eigenvalues and, to an extent, eigenvectors. Devised in 1978, the method of attack involved finding a set of weighting matrices that yielded desired frequencies and modes as the input scaling factor, ρ , tended to zero (i.e., cost of control is "cheap"). In addition, a simple flight control example was used to illustrate the application of this technique. In 1987, Fujii [7] devised a similar scheme, but with a different formulation of the problem. Instead of solving the LQR problem via the classical approach, a redefinition of the optimal condition is used. The result was an almost identical response to [11] without the necessity of solving the Riccati equation nor requiring the input scaling factor, ρ , to approach zero.

In the area of drive trains, most of the emphasis has been put on inclusion of actuator dynamics and compliance. In 1984, Good, Sweet, and Strobel [8] stressed the importance of modeling the drive train and illustrated the point with a detailed model of a real manipulator. The results were that frequency response up to 50 Hz could be verified experimentally. Later, Sweet and Good [20] stated that path errors within acceptable tolerances could be achieved if (1) the manipulator is modeled correctly, (2) realistic command trajectories are used in design, and (3) sensing equipment is used to compensate for any uncertainties encountered. Simulation results obtained by Forrest-Barlach and Babcock [5] in 1986 also supported the statements in [20]. Using inverse dynamics position control and modeling joint compliance and actuator dynamics, the study showed that only unmodeled resonant high frequencies and finite limits on motor torque restricted the bandwidth of the system.

In 1977, Khalil and Liegeois [13] presented guidelines for modeling cable driven manipulators, and offered a computer-aided technique for obtaining the dynamic equations. Based on Lagrange's equations, this technique not only obtained equations of motion in symbolic form, it also performed modal analysis of the model.

II. EQUATIONS OF MOTION

Review of Lagrange's Equations

Given the expression for kinetic energy, T , of a system, the equations of motion can be obtained by the application of Lagrange's equations. The derivation of Lagrange's equations can be found in [9] and are presented here as

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i, \quad \text{for } i = 1, 2, \dots, n. \quad (2.1)$$

where

q_i = generalized coordinates of the system,

Q_i = generalized forces acting on the system.

Various forms of Lagrange's equations

Defining the Lagrangian function, L , to be

$$L = T - V, \quad (2.2)$$

where V is the potential function, then Lagrange's equations can be written in the form:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q'_i, \quad \text{for } i = 1, 2, \dots, n, \quad (2.3)$$

where Q'_i are generalized forces which cannot be derived from a potential function (i.e. non-conservative forces such as motor torques, viscous damping). Furthermore, if the system is conservative, then Lagrange's equations take on the form of:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = - \frac{\partial V}{\partial q_i}, \quad \text{for } i = 1, 2, \dots, n \quad (2.4)$$

In the equation above, it can be seen that the terms on the right are generalized forces acting on the system; in fact, they are conservative forces since they are derivable from the potential function [9]. Also, the first term in the above equation is referred to as the generalized momenta because of its analogy to the Newtonian definition of momentum.

To illustrate the application of Lagrange's equation, consider the simple pendulum in Figure 2-1. The expressions for kinetic and potential energy for the simple pendulum are:

$$T = \frac{1}{2} m (l \dot{\theta})^2, \quad (2.5a)$$

$$V = mgl (1 - \cos \theta). \quad (2.5b)$$

The non-zero terms of Lagrange's equation are:

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \quad (2.6a)$$

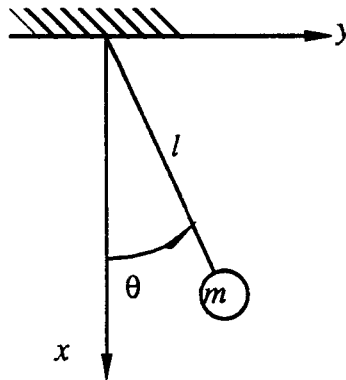


Figure 2-1. Simple pendulum example.

$$\frac{\partial V}{\partial \theta} = mgl \sin \theta. \quad (2.6b)$$

Thus,

$$ml^2 \ddot{\theta} + mgl \sin \theta = 0, \quad (2.7)$$

or for small displacements,

$$\ddot{\theta} + \frac{g}{l} \theta = 0. \quad (2.8)$$

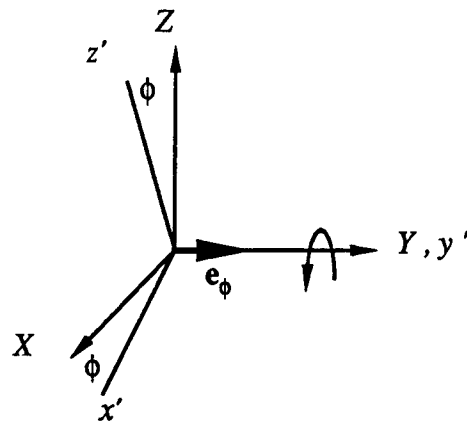
which is the well-known equation of motion for the simple pendulum.

The greatest advantage of Lagrange's equations is the simplicity of formulation; only velocities and displacements are needed to derive the equations of motion [9], whereas detailed kinematic analysis is required in the direct Newtonian formulation. Furthermore, scalar quantities (kinetic and potential energies) are used as opposed to vector quantities (force, moments, etc.) [6], which simplifies the analysis. However, as the system becomes more complex, the method becomes computationally inefficient when compared to Newtonian formulation [2].

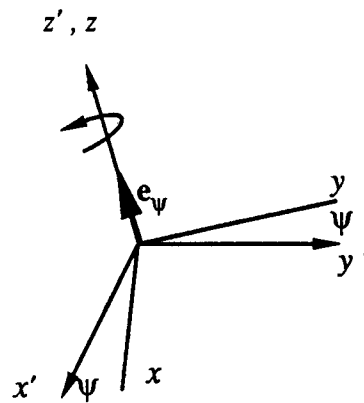
Kinematics

Rigid body kinematics

Consider the Euler rotations in Figure 2-2 used to obtain an arbitrary configuration of the wrist. The rotational matrices required to obtain the body fixed frame [9] are given by:



a. First rotation about pitch (Y) axis.



b. Second rotation about yaw (x') axis.

Figure 2-2. Euler rotations.

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad (2.9)$$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}. \quad (2.10)$$

where

ϕ = wrist pitch angle,

ψ = wrist yaw angle.

Relating the unit vectors, $\mathbf{e}_\phi$ and $\mathbf{e}_\psi$, to the body fixed unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ yields:

$$\vec{\omega} = \dot{\phi}(\sin \psi \mathbf{i} + \cos \psi \mathbf{j}) + \dot{\psi} \mathbf{k}. \quad (2.11)$$

Thus, the components of angular velocity are:

$$\begin{aligned} \omega_x &= \dot{\phi} \sin \psi, \\ \omega_y &= \dot{\phi} \cos \psi, \\ \omega_z &= \dot{\psi}. \end{aligned} \quad (2.12)$$

Differential kinematics

To obtain the input-output relationship of the differential, the two-step tabular method [15] was used. Analysis consisted of observing motion with (1) the train locked and (2) the arm locked, and adding the two results together. A schematic of the differential is shown in Figure 2-3, and the results of the analysis are given in Table 2.1. Solving for x and y as a function of the input angles yields:

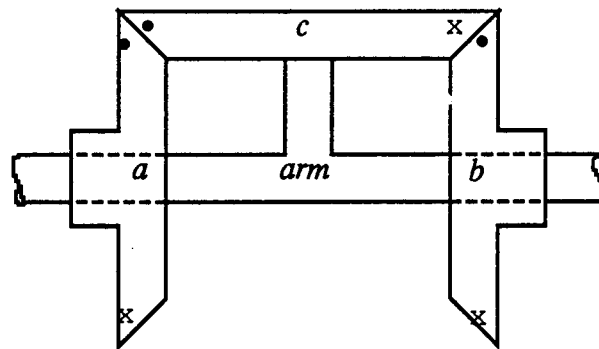


Figure 2-3. Schematic of differential gear train and labeled components.

Table 2.1. Kinematic Analysis of Differential

	a	b	c	arm
Train locked:	x	x	-	x
Arm locked	y	$-y$	y	0
Total:	$x+y$	$x-y$		

$$x = \theta_{pitch} = \frac{\theta_a + \theta_b}{2}, \quad (2.13a)$$

$$y = \theta_{yaw} = \frac{\theta_a - \theta_b}{2}. \quad (2.13b)$$

Kinetic and Potential Energies

Kinetic energy

For a rigid body rotating about a fixed point in space, the kinetic energy can be expressed as:

$$T = \frac{1}{2} [\omega_x \ \omega_y \ \omega_z] \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad (2.14a)$$

or more compactly:

$$T = \frac{1}{2} (I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{zx}\omega_z\omega_x). \quad (2.14b)$$

For the motors, the kinetic energy expression is:

$$T_{motors} = \frac{1}{2} J_{m1} \dot{\theta}_1^2 + \frac{1}{2} J_{m2} \dot{\theta}_2^2. \quad (2.15)$$

where

J_{m1}, J_{m2} = motor inertias,

θ_1, θ_2 = motor angles.

Substituting the rigid body kinematics of equation (2.12) into the second kinetic energy expression (2.14b) and adding motor kinetic energies (2.15) yields:

$$T_{Total} = \frac{1}{2} (I_{xx} \dot{\phi}^2 \sin^2 \psi + I_{yy} \dot{\phi}^2 \cos^2 \psi + I_{zz} \dot{\psi}^2 + 2I_{xz} \dot{\psi} \dot{\phi} \sin \psi + \frac{1}{2} J_{m1} \dot{\theta}_1^2 + \frac{1}{2} J_{m2} \dot{\theta}_2^2). \quad (2.16)$$

Introducing new inertia relationships $I_{xx} = I_1 + I_2$, $I_{yy} = I_1 + I_2$, and $I_{zz} = I_3$, the expression (2.16) can be further reduced to:

$$T_{Total} = \frac{1}{2} (I_1 \dot{\phi}^2 + I_2 \dot{\phi}^2 \cos 2\psi + I_3 \dot{\psi}^2 + 2I_{xz} \dot{\psi} \dot{\phi} \sin \psi + \frac{1}{2} J_{m1} \dot{\theta}_1^2 + \frac{1}{2} J_{m2} \dot{\theta}_2^2). \quad (2.17)$$

At this point, it is not clear whether or not to substitute the differential kinematics eq. (2.13) in place of the Euler angles. For a MIMO control system design, eq. (2.17) is perfectly acceptable, since ϕ and ψ are the desired outputs. However, if two SISO control systems are to be assumed, using eq. (2.13) allows essentially a one-to-one correspondence of motors and differential inputs. Replacing the angles θ_3 for θ_a , θ_4 for θ_b , ϕ for x , and ψ for y , eq. (2.13) can be substituted in the total kinetic energy expression to yield:

$$T_{Total} = \frac{1}{8} [I_1 (\dot{\theta}_3 + \dot{\theta}_4)^2 + I_2 (\dot{\theta}_3 + \dot{\theta}_4)^2 \cos(\theta_3 - \theta_4) + I_3 (\dot{\theta}_3 - \dot{\theta}_4)^2 + 2I_{xz} (\dot{\theta}_3^2 - \dot{\theta}_4^2) \sin\left(\frac{\theta_3 - \theta_4}{2}\right)] + \frac{1}{2} J_{m1} \dot{\theta}_1^2 + \frac{1}{2} J_{m2} \dot{\theta}_2^2. \quad (2.18)$$

where

θ_3, θ_4 = differential input angles.

Potential energy

For the drive train shown in Figure 2-4, if θ_1/N_1 is assumed to be greater than θ_3 , the deflection of the spring in the first drive train is:

$$\delta_1 = \frac{r_1}{N_1} \theta_1 - r_3 \theta_3. \quad (2.19)$$

Similarly for the second drive train, the deflection is:

$$\delta_2 = \frac{r_2}{N_2} \theta_2 - r_4 \theta_4. \quad (2.20)$$

Using Hooke's law, the potential energy of the drive train can be written as:

$$V = \frac{1}{2} \left[2k_{s1} \left(\frac{r_1}{N_1} \theta_1 - r_3 \theta_3 \right)^2 + 2k_{s2} \left(\frac{r_2}{N_2} \theta_2 - r_4 \theta_4 \right)^2 \right]. \quad (2.21)$$

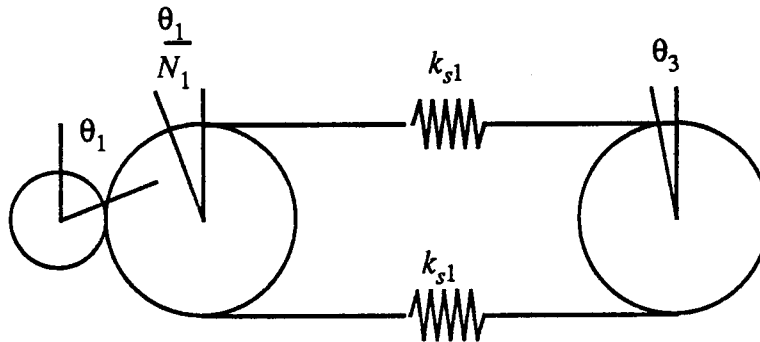


Figure 2.4. Simplified schematic of fourth-order drive train.

Application of Lagrange's Equations

Let us define the generalized coordinates, q_i , and the external forces, Q'_i of the system to be:

$$\{q_1, q_2, q_3, q_4\} = \{\theta_1, \theta_2, \theta_3, \theta_4\}, \quad (2.22)$$

$$\{Q'_1, Q'_2, Q'_3, Q'_4\} = \{T_{m1} - B_{m1}\dot{\theta}_1, T_{m2} - B_{m2}\dot{\theta}_2, 0, 0\}, \quad (2.23)$$

where

T_{m1}, T_{m2} = motor torques,

B_{m1}, B_{m2} = motor damping constants.

Lagrange's principle can now be applied to obtain the equations of motion of the differential wrist. Derivation of each of the Lagrangian terms are given in Appendix 1, and synthesis of the terms yields:

$$J_{m1}\ddot{\theta}_1 + 2k_{s1}\frac{r_1^2}{N_1^2}\theta_1 - 2k_{s1}\frac{r_1r_3}{N_1}\theta_3 = T_{m1} - B_{m1}\dot{\theta}_1, \quad (2.24)$$

$$J_{m2}\ddot{\theta}_2 + 2k_{s2}\frac{r_2^2}{N_2^2}\theta_2 - 2k_{s2}\frac{r_2r_4}{N_2}\theta_4 = T_{m2} - B_{m2}\dot{\theta}_2, \quad (2.25)$$

$$\frac{1}{4}\left[I_1 + I_2 \cos(\theta_3 - \theta_4) + I_3 + 2I_x \sin\left(\frac{\theta_3 - \theta_4}{2}\right)\right]\ddot{\theta}_3$$

$$+ \frac{1}{4}\left[I_1 + I_2 \cos(\theta_3 - \theta_4) - I_3\right]\ddot{\theta}_4$$

$$\begin{aligned}
& + \frac{1}{8} \left[-I_2 \sin(\theta_3 - \theta_4) + I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_3^2 \\
& + \frac{1}{8} \left[3I_2 \sin(\theta_3 - \theta_4) + I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_4^2 \\
& + \frac{1}{4} \left[I_2 \sin(\theta_3 - \theta_4) - I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_3 \dot{\theta}_4 \\
& + 2 k_{s1} r_3^2 \theta_3 - 2 k_{s1} \frac{r_1 r_3}{N_1} \theta_1 = 0,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
& \frac{1}{4} \left[I_1 + I_2 \cos(\theta_3 - \theta_4) + I_3 - 2I_{xx} \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \ddot{\theta}_4 \\
& + \frac{1}{4} \left[I_1 + I_2 \cos(\theta_3 - \theta_4) - I_3 \right] \ddot{\theta}_3 \\
& + \frac{1}{8} \left[I_2 \sin(\theta_3 - \theta_4) + I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_4^2 \\
& + \frac{1}{8} \left[-3I_2 \sin(\theta_3 - \theta_4) + I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_3^2 \\
& + \frac{1}{4} \left[-I_2 \sin(\theta_3 - \theta_4) - I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_3 \dot{\theta}_4 \\
& + 2 k_{s2} r_4^2 \theta_4 - 2 k_{s2} \frac{r_2 r_4}{N_2} \theta_2 = 0.
\end{aligned} \tag{2.27}$$

Inversion of Mass Matrix

At this point, let us use the following notation to denote sine and cosine functions:

$$\begin{aligned}\alpha_1 &= \sin(\theta_3 - \theta_4), \\ \alpha_2 &= \sin\left(\frac{\theta_3 - \theta_4}{2}\right), \\ \beta_1 &= \cos(\theta_3 - \theta_4), \\ \beta_2 &= \cos\left(\frac{\theta_3 - \theta_4}{2}\right).\end{aligned}\tag{2.28}$$

Now, if we consider only the wrist equations (2.26) and (2.27) , the mass matrix, $\mathbf{M}$, can be written as:

$$\mathbf{M} = \begin{bmatrix} J_1 & J_3 \\ J_3 & J_2 \end{bmatrix}.\tag{2.29}$$

where

$$J_1 = \frac{1}{4} \left[I_1 + I_2 \cos(\theta_3 - \theta_4) + I_3 + 2I_{xx} \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right],\tag{2.30a}$$

$$J_2 = \frac{1}{4} \left[I_1 + I_2 \cos(\theta_3 - \theta_4) + I_3 - 2I_{xx} \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right],\tag{2.30b}$$

$$J_3 = \frac{1}{4} [I_1 + I_2 \cos(\theta_3 - \theta_4) - I_3].\tag{2.30c}$$

Inverting the mass matrix yields:

$$\mathbf{M}^{-1} = \frac{\begin{bmatrix} J_2 & -J_3 \\ -J_3 & J_1 \end{bmatrix}}{|\mathbf{M}|},\tag{2.31}$$

where

$$|\mathbf{M}| = \frac{1}{4} \left[I_1(I_2 \cos(\theta_3 - \theta_4) + I_3) - I_{xx}^2 \sin^2\left(\frac{\theta_3 - \theta_4}{2}\right) \right]. \quad (2.32)$$

Therefore, the equations of motion (2.26) and (2.27) can be represented in matrix form as:

$$\begin{bmatrix} \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix} = \frac{1}{8|\mathbf{M}|} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{bmatrix} \begin{bmatrix} \theta_3^2 \\ \dot{\theta}_3 \dot{\theta}_4 \\ \theta_4^2 \end{bmatrix} + \frac{1}{|\mathbf{M}|} \begin{bmatrix} J_2 K_{13} & -J_3 K_{24} & -J_2 K_{33} & J_3 K_{44} \\ -J_3 K_{13} & J_1 K_{24} & J_3 K_{33} & -J_1 K_{44} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}, \quad (2.33)$$

where

$$\begin{aligned} v_{11} &= J_2(I_2 \alpha_1 - I_{xx} \beta_2) - J_3(3I_2 \alpha_1 - I_{xx} \beta_2), \\ v_{12} &= J_2(-2I_2 \alpha_1 + 2I_{xx} \beta_2) - J_3(2I_2 \alpha_1 + 2I_{xx} \beta_2), \\ v_{13} &= -J_2(3I_2 \alpha_1 + I_{xx} \beta_2) + J_3(I_2 \alpha_1 + I_{xx} \beta_2), \\ v_{21} &= -J_3(I_2 \alpha_1 - I_{xx} \beta_2) + J_1(3I_2 \alpha_1 - I_{xx} \beta_2), \\ v_{22} &= -J_3(-2I_2 \alpha_1 + 2I_{xx} \beta_2) + J_1(2I_2 \alpha_1 + 2I_{xx} \beta_2), \\ v_{23} &= J_3(3I_2 \alpha_1 + I_{xx} \beta_2) - J_1(I_2 \alpha_1 + I_{xx} \beta_2), \end{aligned} \quad (2.34)$$

$$\begin{aligned} K_{13} &= 2k_{s1} \frac{r_1 r_3}{N_1}, \\ K_{24} &= 2k_{s2} \frac{r_2 r_4}{N_2}, \end{aligned} \quad (2.35)$$

$$\begin{aligned} K_{33} &= 2k_{s1} r_3^2, \\ K_{44} &= 2k_{s2} r_4^2. \end{aligned} \quad (2.36)$$

Likewise, the motor dynamics can be expressed in matrix form as:

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \frac{-K_{11}}{J_{m1}} & \frac{K_{13}}{J_{m1}} & 0 & 0 \\ 0 & 0 & \frac{-K_{24}}{J_{m2}} & \frac{K_{44}}{J_{m2}} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} + \begin{bmatrix} T_{m1} - B_{m1}\dot{\theta}_1 \\ T_{m2} - B_{m2}\dot{\theta}_2 \end{bmatrix} \quad (2.37)$$

where

$$\begin{aligned} K_{11} &= 2k_{s1} \frac{r_1^2}{N_1^2}, \\ K_{22} &= 2k_{s2} \frac{r_2^2}{N_2^2}. \end{aligned} \quad (2.38)$$

III. POLE-PLACEMENT, PI DESIGN

Classical PI-Controller Theory

In fundamental single-input, single-output system analysis, the steady-state errors are characterized by the number of integrators in the denominator of its open-loop transfer function (i.e., the number of open-loop poles at the origin). Systems with no integrators, referred to as type-0 [14], track step inputs with finite error but cannot follow ramp inputs. In the same way, systems with one integrator (type-1) track step inputs with no error, ramp inputs with finite error, but cannot follow parabolic inputs. The results can be generalized to type- n systems, which track inputs of order $n-1$ or less with no error, n th order inputs with finite error, but cannot follow inputs of order $n+1$.

The conclusion of the above discussion is that systems with integration exhibit better steady-state response than systems without; therefore, the motivation for implementing proportional-plus-integral (PI) control is that the resulting system type is improved by one. Consider the first order example in Figure 3-1. The open-loop transfer function is:

$$G(s) = K_{Plant} \frac{K_P s + K_I}{s(s + a)} \quad (3.1)$$

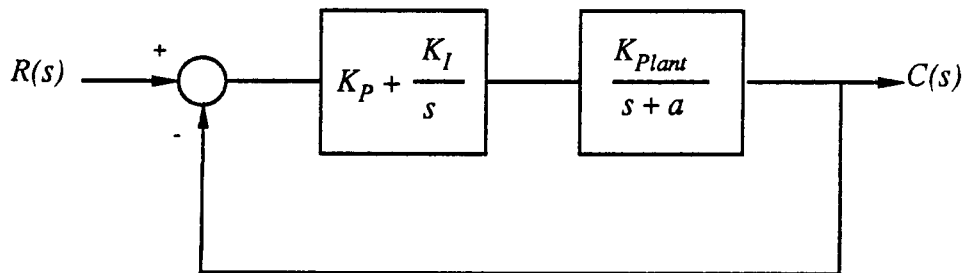


Figure 3-1. Block diagram of a first-order plant

Thus, the net effect of PI-control on the original system is the addition of a pole at the origin and a zero at $s = -K_I/K_P$. The choice of K_I and K_P depends on the purpose of the controller.

Stabilization of system

For systems with unstable roots, it is possible to choose values of K_I and K_P so that the resulting closed-loop system is stable. One method would be to close the loop with controller gains embedded and, via Routh-Hurwitz stability criterion, solve for bounds on K_I and K_P so that the system is stable. This technique is limited to low order systems, since only two parameters are available for design .

Improving dynamic response

Similarly, it is possible to choose K_I and K_P so that the dominant roots of the closed-loop characteristic equation will have desirable dynamic characteristics. Once again, this method is best suited for lower ordered systems.

Improving steady-state error

As implied before, PI-control can be employed to add integration to systems with desirable dynamic characteristics but exhibiting poor steady-state response. This can be done by choosing the ratio of $K_I/K_P \ll 1$ so that the zero is near the pole at the origin; by doing so, the root locus of the compensated system is almost the same as that of the original system. The result is that dynamic response is not greatly altered while increasing the system type by one [14].

The pitfalls of PI-control, however, are that (1) the overall order of the system is increased, and (2) the system is now susceptible to numerator dynamics (responses due to presence of s - terms in the numerator) [17].

State-Space Representation

For any given n th order ordinary differential equation, it is possible to write n first order ordinary differential equations; this is the fundamental basis of the state variable approach. In a general sense, we can represent such a differential equation as:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \end{aligned} \quad (3.2)$$

where

n = order of the differential equation,

m = number of external inputs,

$x_1, x_2, \dots, x_n$ = the designated states,

$u_1, u_2, \dots, u_m$ = the external inputs.

Furthermore, if the differential equation is linear, then the n equations can be written in a convenient matrix form:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad (3.3)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad (3.4a)$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix}, \quad (3.4b)$$

$$\mathbf{x} = \{x_1, x_2, \dots, x_n\}, \quad (3.4c)$$

$$\mathbf{u} = \{u_1, u_2, \dots, u_n\}. \quad (3.4d)$$

Generally the matrices **A** and **B** are referred to as the state matrix and the input matrix, respectively.

Likewise, the outputs of the system can be expressed as functions of the states and the inputs. Assuming there are as many outputs as there are inputs, the m output equations could be written as:

$$\begin{aligned} y_1 &= g_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \\ y_2 &= g_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \\ &\vdots \\ y_m &= g_m(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \end{aligned} \quad (3.5)$$

In the linear case, the matrix output equations look like:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \quad (3.6)$$

where

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}, \quad (3.7a)$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \dots & d_{mm} \end{bmatrix}. \quad (3.7b)$$

The matrices **C** and **D** are often referred to as the output matrices of the system, where **C** related the states to the output and **D** relates the inputs to the outputs directly. In almost every case, **D** is a zero matrix; nevertheless, it is included for completeness.

If the differential equation is represented in the *s*-domain and **D=0**, then an algebraic relationship exists between the transfer function and the matrices **A**, **B**, and **C**, which is given by:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B. \quad (3.8)$$

It is apparent that for any given differential equation, there are many different state-space forms. A few of importance are the phase-variable form [14], the block-Jordan form [18], and the second companion (observable) form [6].

Pole-Placement Techniques

In a classical single-input, single-output regulator problem, if the input is chosen to be:

$$u = -Kx, \quad (3.9)$$

then there will be as many feedback gains as there are roots of the characteristic equation. Thus, it is possible to specify all the closed-loop poles of a given system by some appropriate choice of gain matrix, **K**. Because various techniques exist for finding this feedback gain matrix, it is worth describing a few of them and discussing their attributes.

Transformation to phase-variable form

If a system is in phase-variable form, the state and input matrices have the form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (3.10)$$

Closing the loop reveals that the feedback gains are simply given by

$$k_i = a_i - \alpha_i, \quad (3.11)$$

where a_i 's are the coefficients of the desired characteristic equation (i.e. the binomial standard filter, ITAE filter, etc.), and α_i 's are the coefficients of the original characteristic equation. Thus, the only real problem is to find a transformation from the original state matrices to the phase-variable form. This can be done via the Bass-Gura formulation [6].

Brute Force Technique

This technique requires (1) embedding the feedback gains in the closed-loop system matrix, (2) finding the closed loop characteristic equation, (3) comparing the coefficients of this characteristic equation to some desired characteristic equation, and (4) solving n equations (generally nonlinear) simultaneously for the feedback gains, where n is the order of the system. For systems of low order, this technique allows a straightforward closed-form solution of the feedback gains in terms of system parameters. For higher order systems, however, the method becomes quite inefficient in light of better techniques such as the phase-variable transformation.

Pole-Placement Design

Linearized model

Neglecting Coriolis acceleration, centripetal acceleration, and the dynamic cross-coupling terms, eq. (3.18) and (3.19) can be simplified down to:

$$\begin{aligned} \frac{1}{4} \left[I_1 + I_2 \cos(\theta_3 - \theta_4) + I_3 + 2I_x \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \ddot{\theta}_3 \\ + 2k_{s1} r_3^2 \theta_3 - 2k_{s1} \frac{r_1 r_3}{N_1} \theta_1 = 0, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{1}{4} \left[I_1 + I_2 \cos(\theta_3 - \theta_4) + I_3 - 2I_x \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \ddot{\theta}_4 \\ + 2k_{s2} r_4^2 \theta_4 - 2k_{s2} \frac{r_2 r_4}{N_2} \theta_2 = 0. \end{aligned} \quad (3.13)$$

In general, eqs. (3.12) and (3.13) can be written in the form of

$\ddot{\theta}_3 = f(\theta_3, \theta_4, \ddot{\theta}_4)$ and $\ddot{\theta}_4 = g(\theta_3, \ddot{\theta}_3, \theta_4)$, and then linearized using the total differential form:

$$\delta \ddot{\theta}_3 = \left. \frac{\partial f}{\partial \theta_3} \right|_0 \delta \theta_3 + \left. \frac{\partial f}{\partial \theta_4} \right|_0 \delta \theta_4 + \left. \frac{\partial f}{\partial \ddot{\theta}_4} \right|_0 \delta \ddot{\theta}_4, \quad (3.14)$$

$$\delta \ddot{\theta}_4 = \left. \frac{\partial g}{\partial \theta_3} \right|_0 \delta \theta_3 + \left. \frac{\partial g}{\partial \ddot{\theta}_3} \right|_0 \delta \ddot{\theta}_3 + \left. \frac{\partial g}{\partial \theta_4} \right|_0 \delta \theta_4, \quad (3.145)$$

where the 0-subscript denotes nominal conditions. It should be noted that linearization could have been done with the centrifugal and Coriolis terms included; however, for nominal velocity conditions $\dot{\theta}_3 = 0$ and $\dot{\theta}_4 = 0$, these terms will be zero. Evaluating

equations (3.14) and (3.15) at the nominal conditions $\theta_3 = \theta_{30}$, $\theta_4 = \theta_{40}$, $\ddot{\theta}_3 = 0$, $\ddot{\theta}_4 = 0$, $\theta_1 = N_1 \theta_{30}$, and $\theta_2 = N_2 \theta_{40}$, the resulting equations of motion are:

$$J_{01} \ddot{\theta}_3 = -2k_{s1} r_3^2 \theta_3 + 2k_{s1} \frac{r_1 r_3}{N_1} \theta_1, \quad (3.16)$$

$$J_{02} \ddot{\theta}_4 = -2k_{s2} r_4^2 \theta_4 + 2k_{s2} \frac{r_2 r_4}{N_2} \theta_2, \quad (3.17)$$

where

$$J_{01} = \frac{1}{4} \left[I_1 + I_2 \cos(\theta_{03} - \theta_{04}) + I_3 + 2I_x \sin\left(\frac{\theta_{03} - \theta_{04}}{2}\right) \right],$$

$$J_{02} = \frac{1}{4} \left[I_1 + I_2 \cos(\theta_{03} - \theta_{04}) + I_3 - 2I_x \sin\left(\frac{\theta_{03} - \theta_{04}}{2}\right) \right].$$

Loop closure equations and transfer function

Assuming the feedback configuration in Figure 3-2, the equations at each summing node can be written as:

$$T_{m1} = K_{m1} K_{\theta 1} (\theta_{R1} - \theta_3 - K_{V3} \dot{\theta}_3 - K_{P1} \theta_1 - K_{V1} \dot{\theta}_1), \quad (3.18)$$

$$T_{m2} = K_{m2} K_{\theta 2} (\theta_{R2} - \theta_4 - K_{V4} \dot{\theta}_4 - K_{P2} \theta_2 - K_{V2} \dot{\theta}_2), \quad (3.19)$$

$$\ddot{\theta}_1 = \frac{1}{J_{m1}} [T_{m1} - K_{11} \theta_1 - B_{m1} \dot{\theta}_1 + K_{13} \theta_3], \quad (3.20)$$

$$\ddot{\theta}_2 = \frac{1}{J_{m2}} [T_{m2} - K_{22} \theta_2 - B_{m2} \dot{\theta}_2 + K_{24} \theta_4], \quad (3.21)$$

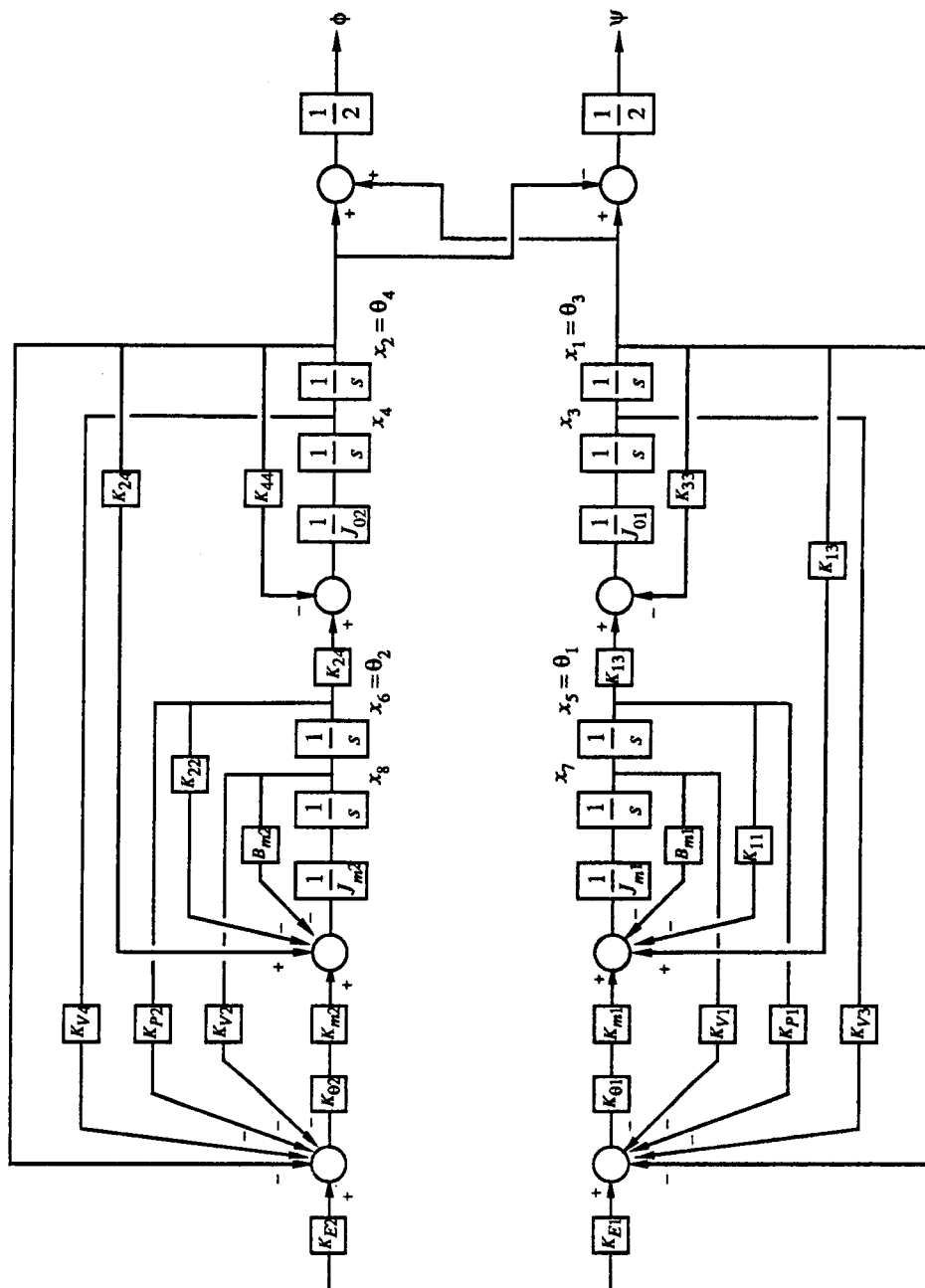


Figure 3-2. Block diagram of linearized, decoupled system

$$\ddot{\theta}_3 = \frac{1}{J_{01}} [-K_{33}\theta_3 + K_{13}\theta_1], \quad (3.22)$$

$$\ddot{\theta}_4 = \frac{1}{J_{02}} [-K_{44}\theta_4 + K_{24}\theta_2]. \quad (3.23)$$

Because of similarity in the two drive trains, the transfer function for the first drive train will be considered, and later, a generalization will be made to include the second drive train. Substituting the torque equation (3.18) in eq. (3.19), the motor angular acceleration can be written as:

$$\begin{aligned} \ddot{\theta}_1 = \frac{1}{J_{m1}} [& K_{m1}K_{\theta1}\theta_{R1} - (K_{11} + K_{m1}K_{\theta1}K_{P1})\theta_1 - (B_{m1} + K_{m1}K_{\theta1}K_{V1})\dot{\theta}_1 \\ & - (K_{m1}K_{\theta1} - K_{13})\theta_3 - K_{m1}K_{\theta1}K_{V3}\dot{\theta}_3]. \end{aligned} \quad (3.24)$$

If the state variables and input variables are:

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} \theta_3 \\ \dot{\theta}_3 \\ \theta_1 \\ \dot{\theta}_1 \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{R1} \end{Bmatrix}, \quad (3.25)$$

then, the state matrices for the fourth order system can be written as:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_{33}}{J_{01}} & 0 & \frac{K_{13}}{J_{01}} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_{F1} - K_{13}}{J_{m1}} & -\frac{K_{F1}K_{V3}}{J_{m1}} & -\frac{K_{11} + K_{F1}K_{P1}}{J_{m1}} & -\frac{B_{m1} + K_{F1}K_{V1}}{J_{m1}} \end{bmatrix}, \quad (3.26a)$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_{F1}}{J_{m1}} \end{bmatrix}, \quad K_{F1} = K_{m1} K_{\theta1}. \quad (3.26b), (3.26c)$$

For this fourth order system, the output is simply θ_3 ; thus, the output equations have the form:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \quad (3.6)$$

where

$$\mathbf{C} = [1 \ 0 \ 0 \ 0], \quad \mathbf{D} = [0].$$

Using eq. (3.8) and substituting the $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ matrices for the drive train, the transfer function relationship obtained is given by:

$$H(s) = \frac{K_{m1} K_{\theta1} K_{13}}{J_{m1} J_{10} |sI - \mathbf{A}|}, \quad (3.27)$$

where

$$\begin{aligned} |sI - \mathbf{A}| = & s^4 + \frac{B_{m1} + K_{m1} K_{\theta1} K_{V1}}{J_{m1}} s^3 + \left[\frac{K_{11} + K_{m1} K_{\theta1} K_{P1}}{J_{m1}} + \frac{K_{33}}{J_{01}} \right] s^2 \\ & + \left[\frac{K_{13} K_{m1} K_{\theta1} K_{V3} + K_{33} (B_{m1} + K_{m1} K_{\theta1} K_{V1})}{J_{m1} J_{01}} \right] s \\ & + \left[\frac{K_{13} (K_{m1} K_{\theta1} K_{V3}) + K_{33} (K_{11} + K_{m1} K_{\theta1} K_{P1})}{J_{m1} J_{01}} \right]. \end{aligned} \quad (3.28)$$

Solution of feedback gains

With the feedback gains embedded in the closed-loop transfer function, it is now possible to choose the poles of the characteristic equation in any manner. It is desired to have no overshoot, so a logical choice is a critically damped response with all the poles at $s = -\omega_0$, where ω_0 is the chosen bandwidth of the system. Thus, for a fourth order system, the characteristic equation desired is:

$$s^4 + 4\omega_0 s^3 + 6\omega_0^2 s^2 + 4\omega_0^3 s + \omega_0^4, \quad (1.1)$$

which is the binomial standard filter.

Comparing eqs. (3.28) and (1.1), it is obvious that in order for the two to be identical, the feedback gains must satisfy the following constraints:

$$\frac{B_{m1} + K_{m1} K_{\theta1} K_{V1}}{J_{m1}} = 4\omega_0, \quad (3.29)$$

$$\frac{K_{11} + K_{m1} K_{\theta1} K_{P1}}{J_{m1}} + \frac{K_{33}}{J_{01}} = 6\omega_0^2, \quad (3.30)$$

$$\frac{K_{13} K_{m1} K_{\theta1} K_{V3} + K_{33} (B_{m1} + K_{m1} K_{\theta1} K_{V1})}{J_{m1} J_{01}} = 4\omega_0^3, \quad (3.31)$$

$$\frac{K_{13} (K_{m1} K_{\theta1} K_{V3}) + K_{33} (K_{11} + K_{m1} K_{\theta1} K_{P1})}{J_{m1} J_{01}} = \omega_0^4. \quad (3.32)$$

The existence of such a solution requires that the system (A, B) be controllable; in this case, the feedback gains are simple to solve for, and are presented here as:

$$K_{\theta 1} = \frac{J_{01}^2 J_{m1} \omega_0^4 - 6J_{01} J_{m1} K_{33} \omega_0^2 + J_{m1} K_{33}^2 + J_{01} K_{13}^2}{J_{01} K_{13} K_{m1}}, \quad (3.33)$$

$$K_{P1} = \frac{6J_{01} J_{m1} \omega_0^2 - J_{m1} K_{33} - J_{01} K_{11}}{J_{01} K_{m1} K_{\theta 1}}, \quad (3.34)$$

$$K_{V1} = \frac{4J_{m1} \omega_0 - B_{m1}}{K_{m1} K_{\theta 1}}, \quad (3.35)$$

$$K_{V3} = \frac{4J_{m1} \omega_0 (J_{01} \omega_0^2 - K_{33})}{K_{m1} K_{\theta 1} K_{13}}. \quad (3.36)$$

Lastly, the system must be compensated for any steady-state error. Using the final value theorem,

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s), \quad (1.2)$$

the steady-state response to a step input is given by:

$$\lim_{s \rightarrow 0} s H(s) \left(\frac{1}{s} \right) = \frac{K_{m1} K_{\theta 1} K_{13}}{K_{13} (K_{m1} K_{\theta 1} - K_{13}) + K_{33} (K_{11} + K_{m1} K_{\theta 1} K_{P1})}. \quad (3.37)$$

As discussed previously, this constant attenuation of the output is characteristic of a system with no integration (type-0). One way of compensating for this steady-state error is to amplify the input signal by a gain of:

$$K_{E1} = \frac{1}{H_{ss}} = \frac{K_{13} (K_{m1} K_{\theta 1} - K_{13}) + K_{33} (K_{11} + K_{m1} K_{\theta 1} K_{P1})}{K_{m1} K_{\theta 1} K_{13}}. \quad (3.38)$$

$$K_{\theta 2} = \frac{J_{02}^2 J_{m2} \omega_0^4 - 6J_{02} J_{m2} K_{44} \omega_0^2 + J_{m2} K_{44}^2 + J_{02} K_{24}^2}{J_{02} K_{24} K_{m2}}, \quad (3.38)$$

$$K_{P2} = \frac{6J_{02} J_{m2} \omega_0^2 - J_{m2} K_{44} - J_{02} K_{22}}{J_{02} K_{m2} K_{\theta 2}}, \quad (3.39)$$

$$K_{V2} = \frac{4J_{m2} \omega_0 - B_{m2}}{K_{m2} K_{\theta 2}}, \quad (3.40)$$

$$K_{V4} = \frac{4J_{m2} \omega_0 (J_{02} \omega_0^2 - K_{44})}{K_{m2} K_{\theta 2} K_{24}}. \quad (3.41)$$

PI Design

Loop closure and transfer-function

In the previous section, the steady-state error was corrected by compensating the input signal. The short-comings of this technique is that the system may become saturated if the gains are too high; furthermore, if the model parameters change, then the input signal gain must also change. Thus, an obvious alternative to this type of error compensation would be to implement proportional-plus-integral control. Referring to Figure 3-3, the form of the compensator is:

$$G_c(s) = K_{PI} \frac{s + \zeta}{s}. \quad (3.42)$$

where ζ is K_I/K_P . In general, the wrist drive (plant) can be represented as a fourth order transfer function of the the form:

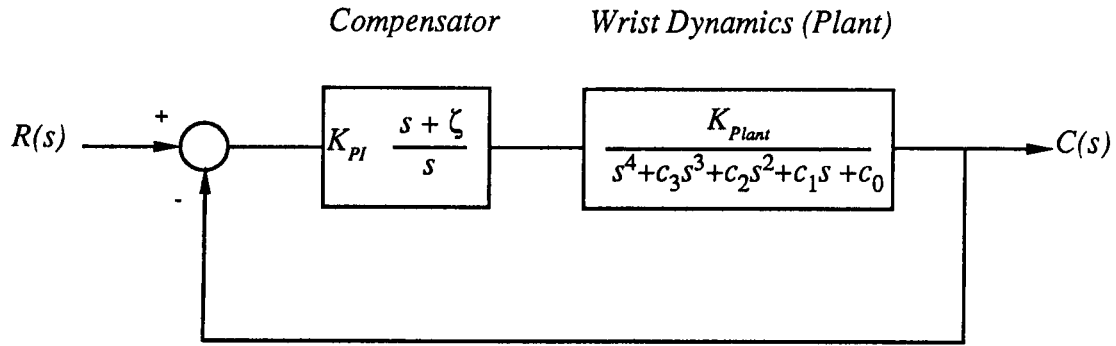


Figure 3-3. Simplified block-diagram of PI-compensator with pole-placed system.

where ζ is K_I/K_P . In general, the wrist drive (plant) can be represented as a fourth order transfer function of the the form:

$$G_p(s) = \frac{K_{Plant}}{s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0}. \quad (3.44)$$

Note that in the last chapter, the c_i 's were identically the coefficients of the binomial standard filter, though in this design, the c_i 's will be different.

The closed loop transfer function for the entire system can be expressed as:

$$H(s) = \frac{K_{Plant} K_{PI} (s + \zeta)}{s^5 + c_3 s^4 + c_2 s^3 + c_1 s^2 + (c_0 + K_{Plant} K_{PI}) s + \zeta K_{Plant} K_{PI}}. \quad (3.45)$$

Since the uncompensated dynamic response is desirable, the purpose of the PI-controller is to improve the system type. Therefore, if ζ is chosen to be much less than one, the system can be approximated by the transfer function

$$H(s) \approx \frac{K_{Plant} K_{PI}}{s^4 + c_3 s^3 + c_2 s^2 + c_1 s + (c_0 + K_{Plant} K_{PI})}. \quad (3.46)$$

Solution of feedback gains

Because the compensated system can be approximated by the transfer function of eq. (3.46), it follows that placing the poles of eq. (3.46) will be almost equivalent to placing the poles of the actual system. If the forward gain K_{PI} is arbitrarily chosen to be:

$$K_{PI} = \frac{F_1 \omega_0^4}{K_{Plant}}, \quad (4.47)$$

equation (3.45) can be written as:

$$H(s) \approx \frac{F_1 \omega_0^4}{s^4 + c_3 s^3 + c_2 s^2 + c_1 s + (c_0 + F_1 \omega_0^4)}. \quad (4.48)$$

Now the pole-placement technique used in the previous section is employed, but the constraints are slightly different. If the c_i 's are chosen to be:

$$\begin{aligned} c_3 &= 4\omega_0, \\ c_2 &= 6\omega_0^2, \\ c_1 &= 4\omega_0^3, \\ c_0 &= (1 - F_1) \omega_0^4. \end{aligned} \quad (3.49)$$

then the feedback gains are identical to the ones derived for the previous pole-placement model, with the exception of $K_{\theta 1}$ and $K_{\theta 2}$, which are given by:

$$K_{\theta 1} = \frac{J_{01}^2 J_{m1} (1 - F_1) \omega_0^4 - 6J_{01} J_{m1} K_{33} \omega_0^2 + J_{m1} K_{33}^2 + J_{01} K_{13}^2}{J_{01} K_{13} K_{m1}}, \quad (3.50)$$

$$K_{\theta 2} = \frac{J_{02}^2 J_{m2} (1 - F_1) \omega_0^4 - 6J_{02} J_{m2} K_{44} \omega_0^2 + J_{m2} K_{44}^2 + J_{02} K_{24}^2}{J_{02} K_{24} K_{m2}}. \quad (3.51)$$

Determination of F_1

To find the appropriate value of F_1 , the "steady-state" response of the approximate system is needed. (The steady-state response of a PI-compensated system to a step input will always be unity; however, for conservative values of ζ , the system tends to a "steady-state" value, then creeps toward unity). Applying the final value theorem to eq. (3.47), it can be seen that:

$$H_{ss} \rightarrow F_1. \quad (3.52)$$

It is tempting to set $F_1 = 1$ to obtain a response that tends to unity. However, the response of the real system contains a great deal of overshoot, due primarily to numerator dynamics. Therefore, F_1 should be considerably less than unity to reduce the overshoot of the actual system.

Because no analytical relationship exists for finding the peak overshoot of a fourth order system, the design procedure for determining F_1 consists of (1) simulating the time response with $F_1 = 1$, (2) checking the peak overshoot of the system, (3) reducing the value of F_1 accordingly and, (4) iterating the above procedure until the desired overshoot is obtained.

It should be noted that if the value of F_1 is greater than one, the inner loop characteristic equation will be:

$$s^4 + 4\omega_0 s^3 + 6\omega_0^2 s^2 + 4\omega_0^3 s + (1 - F_1)\omega_0^4, \quad (3.53)$$

which implies that the plant will be unstable. Because this controller design requires the original system to be stable, F_1 must be restricted to values less than or equal to unity.

IV. LQR DESIGN

Optimal Control Theory

For multi-input, multi-output systems of degree n and with m inputs, there are $n \times m$ feedback gains available for control. Thus if n roots of the characteristic equation are to be determined by pole-placement, the possible sets of feedback gains satisfying this condition would be infinite. Instead, if the gains are selected to optimize a performance index such as

$$V = \int_0^T (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \rho \mathbf{u}^T \mathbf{R} \mathbf{u}) dt, \quad (1.3)$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector, ρ is the input scaling factor, and $\mathbf{Q}$ and $\mathbf{R}$ are weighting matrices, then it is possible to find a unique set of gains that will minimize V . For this reason, this unique set of gains which minimizes its performance index, V , is called an optimal controller.

The Ricatti Equation

To obtain a set of control gains optimizing V , it is necessary to solve the matrix differential equation [6]

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P} = \dot{\mathbf{P}}, \quad (4.1)$$

which is known as the Riccati equation. If a matrix $\mathbf{P}$ satisfies the Riccati equation, then the optimal feedback gain can be expressed as

$$\mathbf{K} = \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}. \quad (4.2)$$

Furthermore, if the upper limit on the performance index is infinity, then the right hand term is zero in steady-state, and eq. (4.1) is reduced to an algebraic relationship,

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P} = \mathbf{0}, \quad (4.3)$$

which is known as the algebraic Riccati equation (ARE). The solution to the Riccati equation results in a positive, semidefinite matrix, $\mathbf{P}$. The requirement for existence of a solution is that the open-loop system must be either (1) asymptotically stable or (2) controllable and observable [6].

As mentioned before, the difficulty with this type of control is selection of the matrices $\mathbf{Q}$ and $\mathbf{R}$, which directly influence only state variables and inputs in the performance index. Thus, time domain parameters such as damping ratio, overshoot, and rise time cannot be directly manipulated as in pole-placement. In the past, this meant that designers used trial-and-error with different weights until a suitable time response could be found. In [18], a relationship was derived for choosing weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ in terms of the open-loop and closed-loop characteristic equations, though the relationship itself was no simpler to evaluate.

Asymptotic Regulator Properties

In order to better predict the response obtained from solving the LQR problem, the input scaling factor, ρ , is allowed to approach zero (i.e., cost of control is "cheap"). By studying the properties of this asymptotic solution, relationships between closed-loop eigenvalues and eigenvectors to the quadratic weighting matrices, $\mathbf{Q}$ and $\mathbf{R}$ can be

developed. These theorems are given along with their proofs in [11] and are summarized here for completeness.

Eigenvalues, eigenvectors, and state matrices

1. The vectors, x_i , and complex numbers, s_i , are eigenvectors and eigenvalues, respectively, of the closed-loop system if and only if there exists m -dimensional vectors, μ_i , such that

$$(s_i \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \mu_i = x_i, \quad (4.4a)$$

$$\mu_i = -\mathbf{K}x_i, \quad \text{for } i = 1, 2, \dots, n. \quad (4.4b)$$

2. For the return difference matrix defined by

$$\mathbf{T}(s) = \mathbf{I} + \mathbf{K}(s \mathbf{I} - \mathbf{A})^{-1} \mathbf{B}, \quad (4.5)$$

those same m -dimensional vectors are identical (except for magnitude) to the vectors, v_i , defined by

$$\mathbf{T}(s_i) v_i = 0, \quad \text{for } i = 1, 2, \dots, n - m. \quad (4.6)$$

3. Optimizing the performance index in eq. (4.1) will yield the same gain matrix as optimizing the performance index

$$V' = \int_0^{\infty} (\mathbf{y}^T \mathbf{y} + \rho \mathbf{u}^T \mathbf{R} \mathbf{u}) dt, \quad (4.7)$$

where $y = Hx$. (The author uses H instead of C for output matrix.) Thus, the form of the weighting matrix, Q , is given by

$$Q = H^T H . \quad (4.8)$$

Furthermore, H can be expressed in the form

$$H = WH_0 , \quad (4.9)$$

where

W = a nonsingular $m \times m$ matrix,

$$H_0 = [H_{11} \quad I_m] .$$

Asymptotic relationships

1. As $\rho \rightarrow 0$, there are $n-m$ eigenvalues, each of which tend toward some finite value, s_i^0 , and are associated to eigenvectors x_i , such that

$$x_i(\rho) \rightarrow (s_i^0 I - A)^{-1} B v_i^0 , \quad (4.10)$$

where s_i^0 and v_i^0 are defined by

$$H(s_i^0 I - A)^{-1} B v_i^0 . \quad (4.11)$$

2. As $\rho \rightarrow 0$, there are m eigenvalues of the form

$$s_j(\rho) \rightarrow s_j^\infty / \sqrt{\rho} , \quad (4.12a)$$

$$|s_j^\infty| < \infty, \quad \text{for } j = 1, 2, \dots, m, \quad (4.13b)$$

with associated eigenvectors

$$x_j(\rho) \rightarrow Bv_i^\infty, \quad (4.14)$$

where s_i^∞ and v_i^∞ are defined by

$$R = N^{-T} S^{-2} N^{-1}, \quad (4.15)$$

$$W^T W = (H_0 B)^{-T} N^{-T} N^{-1} (H_0 B)^{-1}, \quad (4.16)$$

N = matrix of infinite eigenvectors,

S = diagonal matrix of infinite eigenvalues.

Linearized Model

The same assumptions used in the pole-placement design are used in the LQR design, except that off-diagonal terms are not assumed to be zero. Thus, the resulting model will have two inputs and two outputs, and the governing differential equation of the overall system is eighth order. Using the same state variable assignment as in the pole-placement model, the state matrices are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{J_2 K_{33}}{|M|} & \frac{J_3 K_{44}}{|M|} & 0 & 0 & \frac{J_2 K_{13}}{|M|} & -\frac{J_3 K_{24}}{|M|} & 0 & 0 \\ \frac{J_3 K_{33}}{|M|} & -\frac{J_1 K_{44}}{|M|} & 0 & 0 & -\frac{J_3 K_{13}}{|M|} & \frac{J_1 K_{24}}{|M|} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{K_{13}}{J_{m1}} & 0 & 0 & 0 & -\frac{K_{11}}{J_{m1}} & 0 & -\frac{B_{m1}}{J_{m1}} & 0 \\ 0 & \frac{K_{24}}{J_{m2}} & 0 & 0 & 0 & -\frac{K_{22}}{J_{m2}} & 0 & -\frac{B_{m2}}{J_{m2}} \end{bmatrix}, \quad (4.17a)$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_{m1}}{J_{m1}} & 0 \\ 0 & \frac{K_{m2}}{J_{m2}} \end{bmatrix}. \quad (4.17b)$$

Using the numerical parameters of the compliant wrist drive train and linearizing about the position $\psi = 0$, the numerical values for the state matrices are given by

$$\mathbf{A} = \begin{bmatrix} 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\ -1708.1 & 13.8 & 0. & 0. & 427.1 & -3.5 & 0. & 0. \\ 13.8 & -1708.1 & 0. & 0. & -3.5 & 427.1 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. \\ 77104. & 0. & 0. & 0. & -19278. & 0. & -70.8 & 0. \\ 0. & 77104. & 0. & 0. & 0. & -19278. & 0. & -70.8 \end{bmatrix}, \quad (4.18)$$

$$\mathbf{B} = \begin{bmatrix} 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 202.12 & 0. \\ 0. & 202.12 \end{bmatrix}.$$

Construction of the H_0 -Matrix

Using the procedure outlined in [11], the quadratic weights of the LQR problem can be determined to yield desired eigenvalues and modes as the scaling factor, ρ , tends to zero. The form of the weighting matrices, Q and R , to be used is given by

$$Q = H^T H, \quad (4.8)$$

$$R = N^{-T} S^{-2} N^{-1}, \quad (4.16)$$

where the matrices H , N , and S are the same as those previously defined.

The primary objective is to find the appropriate H_0 matrix yielding the desired modal properties. First, the assumption is made that a transformation exists so that the B -matrix has the form

$$B = \begin{bmatrix} 0 \\ B_{21} \end{bmatrix}. \quad (4.19)$$

The next step of the procedure is to select the $n-m$ desired finite eigenvalues where n is the order of the system and m is the number of system inputs. Since the desired responses are to be first-order lags, all the eigenvalues are chosen to be real and are summarized in Table 4.1. Note that the eigenvalues for pitch and yaw position are not equal; this is to illustrate the flexibility of the method and is not a constraint on design. Also, there is a 2:1 bandwidth ratio between wrist velocity and wrist position, as well as motor position and wrist velocity. The reason for this will be given later in the Section.

Table 4.1 Desired Eigenvalues of Wrist Drive Train

Associated Motion	Eigenvalue
Pitch position:	-20
Yaw position:	-15
Pitch rate:	-40
Yaw rate:	-30
Motor pitch:	-80
Motor yaw:	-60

Next, the associated eigenvectors are chosen to correspond to the desired eigenvalues. Because these eigenvectors must satisfy the constraints of eq. (4.10), not all components of the eigenvector may be specified. For this problem, the selected eigenvectors are given by

$$\begin{aligned}
 x_1^* &= [1 \quad 1 \quad x \quad x \quad x \quad x \quad x \quad x], \\
 x_2^* &= [1 \quad -1 \quad x \quad x \quad x \quad x \quad x \quad x], \\
 x_3^* &= [x \quad x \quad 1 \quad 1 \quad x \quad x \quad x \quad x], \\
 x_4^* &= [x \quad x \quad 1 \quad -1 \quad x \quad x \quad x \quad x], \\
 x_5^* &= [x \quad x \quad x \quad x \quad 1 \quad 1 \quad x \quad x], \\
 x_6^* &= [x \quad x \quad x \quad x \quad 1 \quad -1 \quad x \quad x],
 \end{aligned} \tag{4.20}$$

where the x's are the unspecified components of the eigenvectors. In each eigenvector, the first two components are associated with differential positions; the next pair are differential velocities, the third pair are motor positions, and the last pair motor velocities. Thus, the

eigenvectors with a pair of components the same (1,1) are associated with pitch, and those with a pair of components opposite (1,-1) are associated with yaw.

The eigenvalues and eigenvectors specified previously must satisfy the relationship in eq. (4.10). Partitioning the selected vectors as

$$x_i^* = \begin{Bmatrix} \gamma_i^* \\ - \\ x \end{Bmatrix}, \quad (4.21)$$

where γ_i^* 's represent the specified eigenvector components, the condition of eq. (4.10) can be expressed as

$$\begin{Bmatrix} \gamma_i^* \\ - \\ x \end{Bmatrix} = \{ (s_i^0 I - A)^{-1} B \}^{R_i} v_i^0, \quad (4.22)$$

where

R_i = a row reordering operation [11],

v_i^0 = m -dimensional vectors in eq. (4.10).

In order to obtain the values of v_i^0 , the right hand side of eq. (4.22) can be partitioned into

$$\{ (s_i^0 I - A)^{-1} B \}^{R_i} = \begin{Bmatrix} L \\ - \\ M \end{Bmatrix}, \quad (4.23)$$

so that eq. (4.22) can be written as two separate matrix relations,

$$\gamma_i^* = L v_i^0, \quad (4.24a)$$

$$\{x\} = M v_i^0. \quad (4.24b)$$

The left hand side of eq. (4.24a) is completely known; thus, all that is required is to solve for v_i^0 , is to invert the matrix L . In general, though, L will not be a square matrix; one way of obtaining a solution would be to premultiply by L^T and inverting the matrix on the right hand side of (4.24a). Performing the implied operations yields

$$v_i^0 = [L^T L]^{-1} L^T \gamma_i^*. \quad (4.25)$$

The values for the x 's can now be obtained by back-substituting the values of v_i^0 into eq. (4.24b); thus, the numerical values for the attainable eigenvectors are given by

$$\begin{aligned} x_1^0 &= [\quad 1 \quad \quad 1 \quad -20. \quad -20. \quad 4.944 \quad 4.944 \quad -98.875 \quad -98.875], \\ x_2^0 &= [\quad 1 \quad \quad -1 \quad -15. \quad 15. \quad 4.522 \quad -4.522 \quad -67.832 \quad 67.832], \\ x_3^0 &= [-.025 \quad -.025 \quad 1 \quad 1 \quad -.194 \quad -.194 \quad 7.776 \quad 7.776], \\ x_4^0 &= [-.033 \quad .033 \quad 1 \quad -1 \quad -.203 \quad .203 \quad 6.090 \quad -6.090], \\ x_5^0 &= [.052 \quad .052 \quad -4.19 \quad -4.19 \quad 1 \quad 1 \quad -80. \quad -80.], \\ x_6^0 &= [.081 \quad -.081 \quad -4.85 \quad 4.85 \quad 1 \quad -1 \quad -60. \quad 60.]. \end{aligned} \quad (4.26)$$

By inspection, it can be seen that the ratio of some of the vector components is equal to its corresponding eigenvalue. In other words, if the pitch position eigenvalue was chosen to be the same as the pitch velocity eigenvalue, the same eigenvector would result. Therefore, in order to obtain a unique eigenvector mode, it is necessary to associate a unique eigenvalue with it.

Next, the projection matrix P' is to be determined from the matrix relation

$$\mathbf{P}' = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \mathbf{I}_n - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T. \quad (1.4)$$

Once the $\mathbf{P}'$ matrix is determined, the $\mathbf{H}_0$ matrix can be obtained by the following relationship

$$\mathbf{H}_0 = [\mathbf{P}_{22}^{-1} \mathbf{P}_{12}^T | \mathbf{I}_m]. \quad (1.5)$$

The numerical values for the $\mathbf{H}_0$ of this particular system are given by

$$\mathbf{H}_0 = \begin{bmatrix} -383.05 & -25.81 & 6.27 & 2.95 & 122.5 & 17.5 & | & 1 & 0 \\ -25.81 & -383.05 & 2.95 & 6.27 & 17.5 & 122.5 & | & 0 & 1 \end{bmatrix}. \quad (4.27)$$

Determination of Quadratic Weights

Once the $\mathbf{H}_0$ matrix has been determined, all that is required to completely determine the matrices, $\mathbf{Q}$ and $\mathbf{R}$, is to specify the m infinite eigenvalues and infinite modes. From eq. (4.14), it can be seen that the infinite modes are related to the $\mathbf{B}$ -matrix and, therefore, are associated with the actuators (motors) [11]. Only m components are needed and the choice of eigenvectors and eigenvalues are completely arbitrary. For mathematical convenience, let the infinite eigenvectors be defined as

$$\mathbf{N} = \begin{bmatrix} \frac{1}{202.12} & \frac{1}{202.12} \\ \frac{1}{202.12} & -\frac{1}{202.12} \end{bmatrix}. \quad (4.28)$$

Using eq. (5.3), the $\mathbf{W}$ -matrix can be obtained by performing the necessary matrix operations, resulting in

$$\mathbf{W} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (4.29)$$

Selecting the infinite eigenvalues to have a one-to-one ratio, the $\mathbf{S}$ -matrix is given by

$$\mathbf{S} = \mathbf{I}_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.30)$$

Finally, the $\mathbf{R}$ -matrix can be found from eq. (4.16) and is given by

$$\mathbf{R} = 2(101 \ 06)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.31)$$

Solution of Feedback Gains

Using the analytical tool, Matrix_x, the LQR gains are obtained by solving the Riccati equations using different values of the scaling factor ρ , and the results are shown in Table 4.2. It is obvious that as ρ decreases, the closed loop eigenvalues and eigenvectors approach the design values. However, the feedback gains become unreasonably high as ρ approaches zero; thus, there exists a practical limit on the attainable control system.

V. DISCUSSION OF RESULTS

Using the equations developed in Section II as the mathematical model for the wrist, the simulation of the step responses of the closed loop system can be performed. The analytical tool, ACSL, was used to aid in the simulating and plotting of the step responses as well as evaluation of the performance of each system. In all the simulations performed, the following facts pertain:

1. The integration scheme used was fourth-order Runge-Kutta method; the integration step size, communication interval, and the simulation period were .0005 seconds, .005 seconds, and 1 second, respectively.
2. The nonlinear equations of motion were simulated using controllers designed for the linearized system with all states were assumed observable; the pole-placed and PI systems updated feedback gains as position changed, whereas the LQR system used constant feedback gains.
3. The maximum load the wrist is required to carry is 30 pounds. For simulation purposes, the load will be considered as a point mass located at the end-effector.
4. The wrist is assumed to be extended straight out initially (i.e., the pitch and yaw angles are $\phi = 0$ and $\psi = 0$, respectively).

For each controller design, all three design objectives were studied: peak overshoot, sensitivity to load variation, and system decoupling. To evaluate overshoot and

system decoupling, a unit step (pitch or yaw) was applied to the wrist, and the responses of both pitch and yaw were plotted vs time. In addition, the primary responses (i.e., pitch response to pitch step input) for each controller were plotted together for better comparison. The index used to evaluate the performance of each controller was the integral of the error-squared (IES). Thus, for each simulated step response, the performance index was evaluated over the simulation period (one second). Performance curves were then obtained by evaluating the performance index of each controller first at no-load conditions and subsequently in increments of 3 pounds (10% of the arm's maximum capacity). In all, 22 runs were made for each controller (11 runs for pitch and 11 runs for yaw).

The numerical values of the parameters used in the simulation (and LQR design) are summarized in Table 5.1. As mentioned before, inertia terms identified as less than 10^{-7} kg-m² by [1] were considered to be zero and motor parameters used were obtained from manufacturer's specifications. Because the characteristics of the controllers were so different, the dominant roots (eigenvalues) were adjusted for each design to try to obtain similar time responses. The values selected for comparison are summarized in Table 5.2.

Design with No Load Assumed

Peak overshoot

In Figures 5-1 through 5-6, the simulated step responses are shown for each controller design. With the exception of the yaw response in Figure 5-2, all response curves exhibited no overshoot, and the rise times and settling times all appeared about the same in Figures 5-7 and 5-8. One interesting note was that although the bandwidth for the PI controller design was quite high, the resulting rise time was still slower than either the pole-placement or LQR systems. This result was due mainly to the fact that the PI compensator is essentially a low-pass filter [14]. Nevertheless, under the assumption of

Table 5.1 Physical Parameters Used in Simulation and Design

Parameter(s)	Symbol(s)	Value
Motor inertias	J_{m1}, J_{m2}	.000706 kg-m ²
	I_{xx}	.25288 kg-m ²
Wrist inertias	I_{yy}	.25700 kg-m ²
	I_{zz}	.01003 kg-m ²
Cable stiffness	k_{s1}, k_{s2}	300,000 N/m
Motor torque constants	K_{m1}, K_{m2}	.1427 N-m/V
Motor damping	B_{m1}, B_{m2}	.05 N-m-s/rad
Sheave radii	r_1, r_2, r_3, r_4	.01905 m
Gear ratio	N_1, N_2	4

Table 5.2. Summary of Design Parameters

Method of Design	Dominant Poles (Pitch, Yaw)	ζ	F_1
Pole-Placement	-35, -28	-	-
PI	-50, -50	0.2	0.39
LQR	-20, -15	-	-

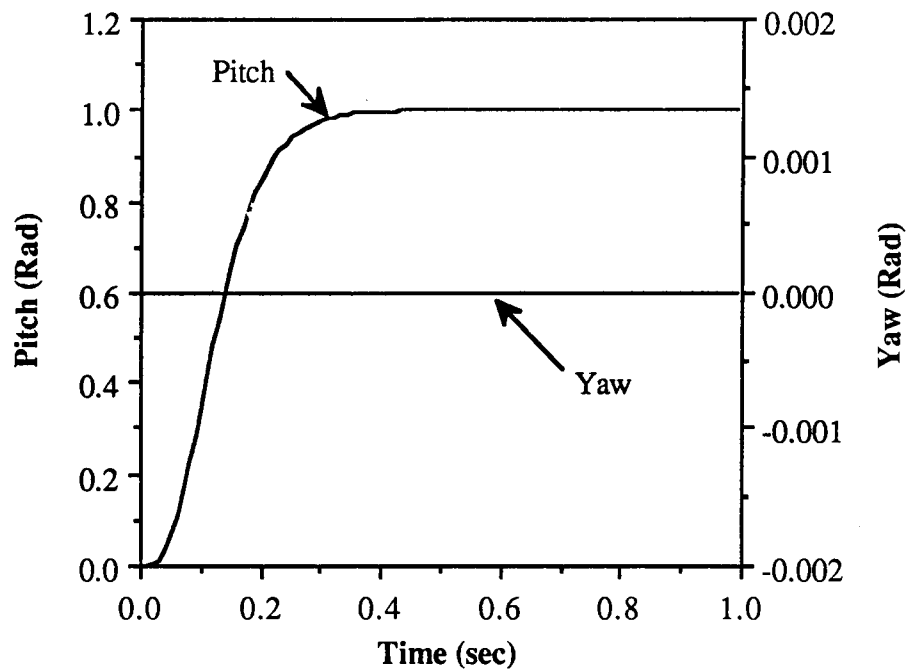


Figure 5-1. Response of pole-placed system to step pitch at no load (no load assumed).

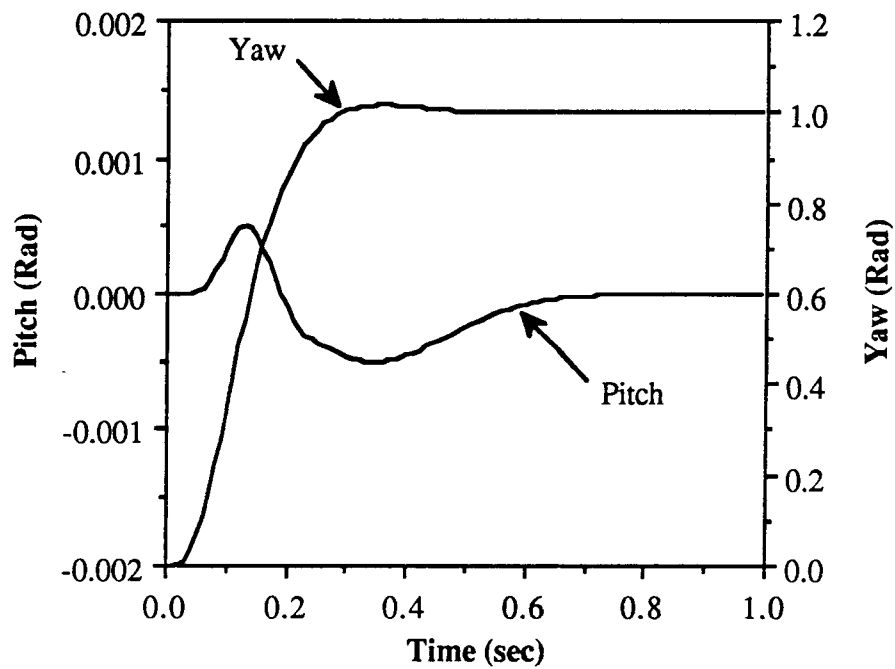


Figure 5-2. Response of pole-placed system to step yaw at no load (no load assumed).

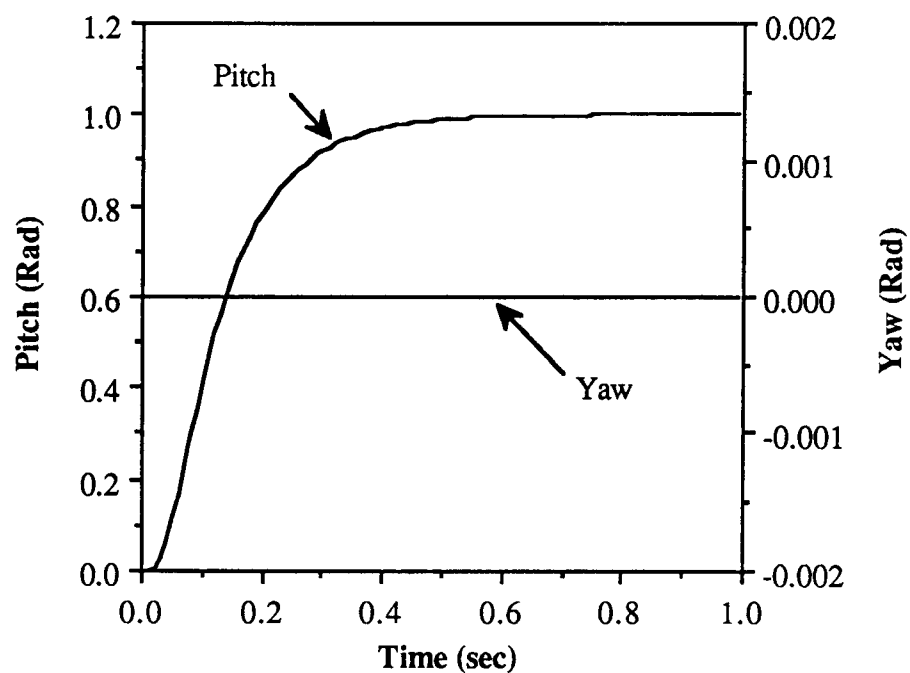


Figure 5-3. Response of PI system to step pitch at no load (no load assumed).

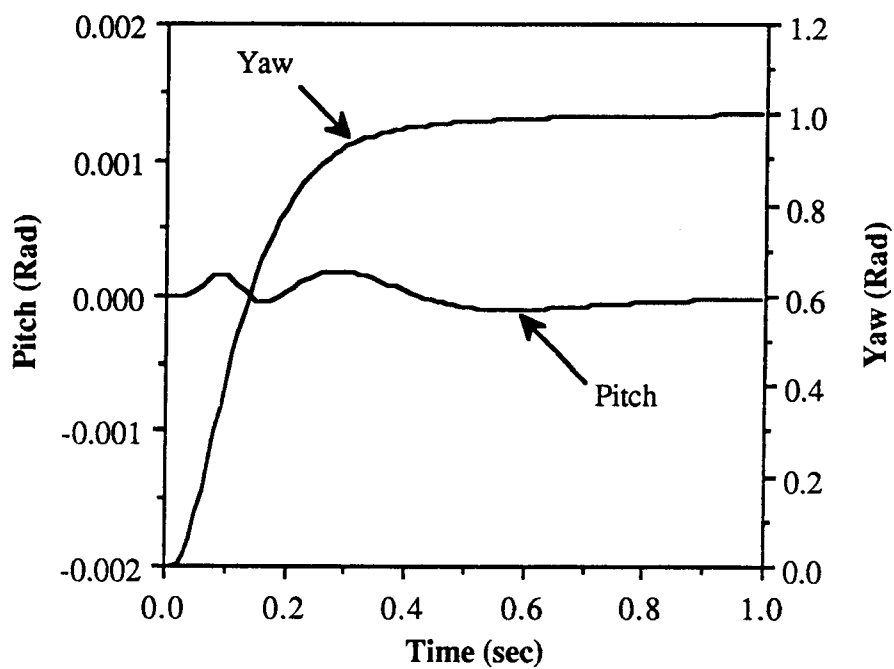


Figure 5-4. Response of PI system to step yaw at no load (no load assumed).

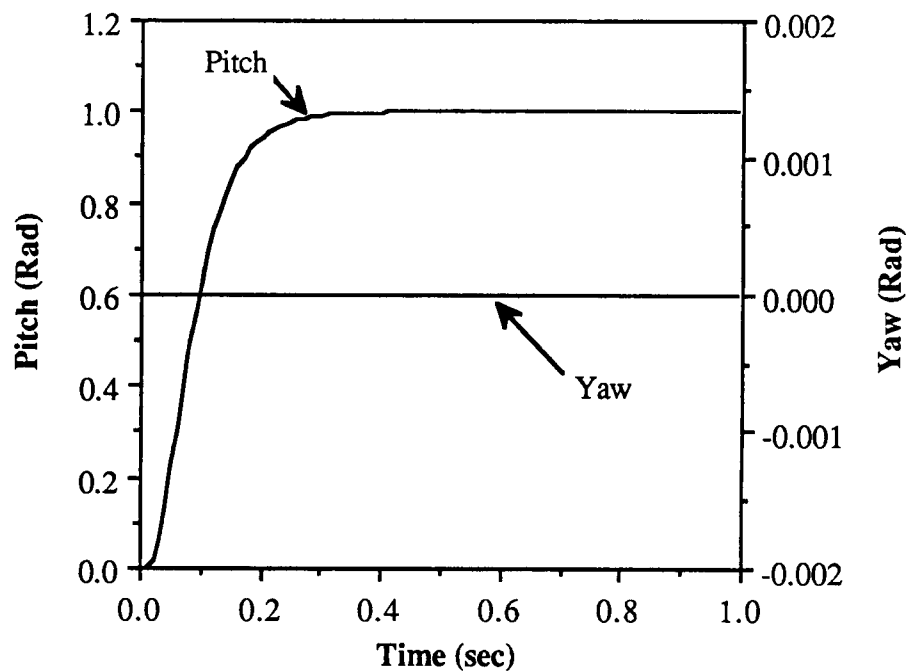


Figure 5-5. Response of LQR system to step pitch at no load (no load assumed).

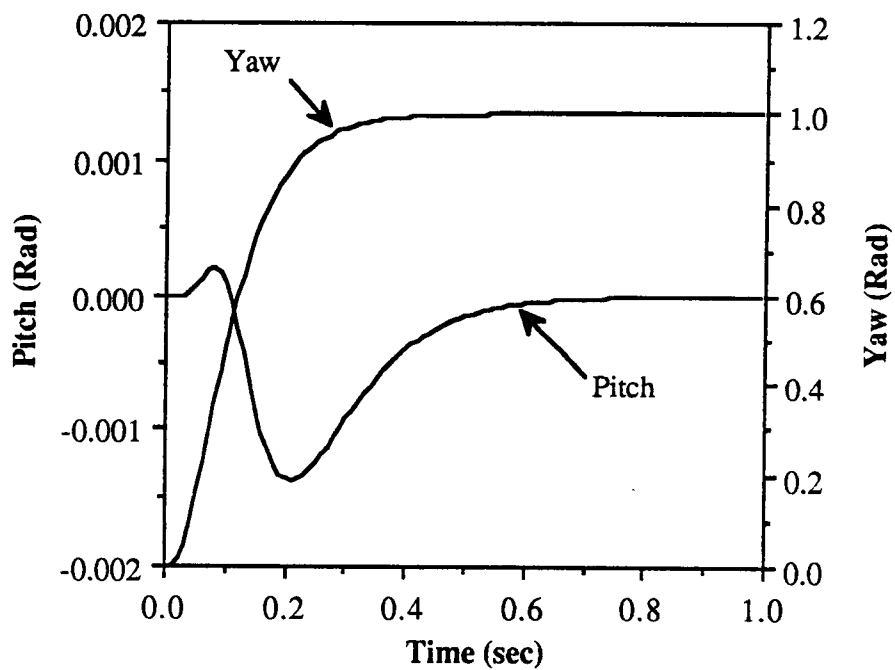


Figure 5-6. Response of LQR system to step yaw at no load (no load assumed).

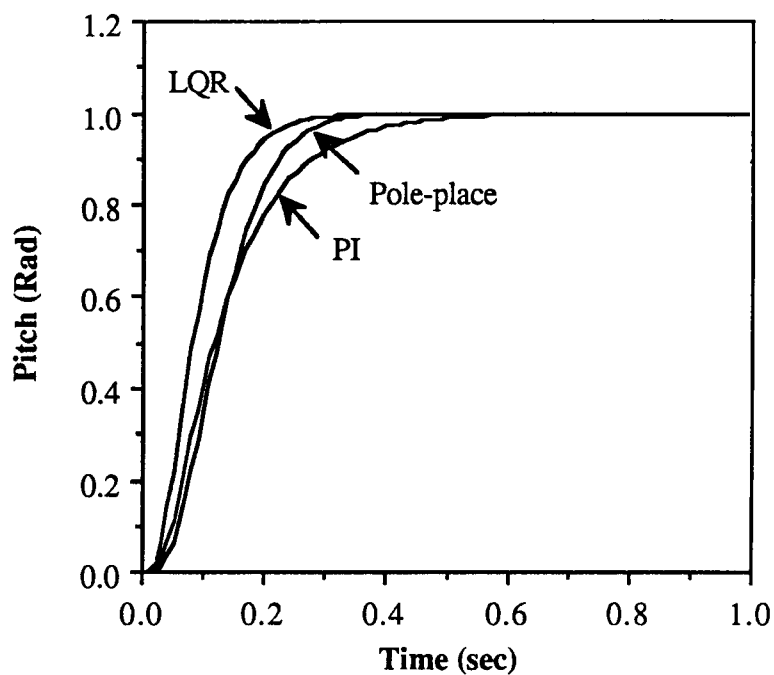


Figure 5-7. Comparison of pitch responses at no load (no load assumed).

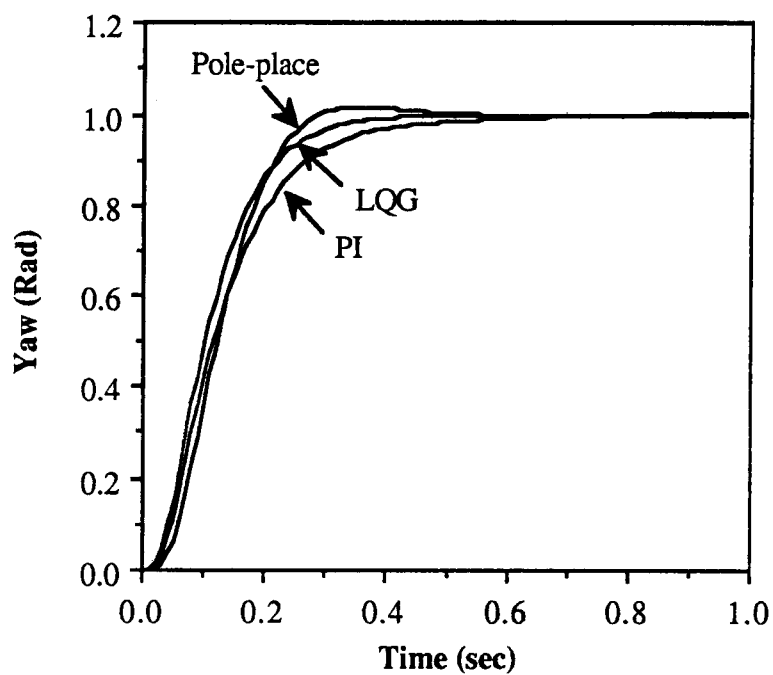


Figure 5-8. Comparison of yaw responses at no load (no load assumed).

no load and simulating with no load, all three approaches met zero overshoot specifications.

In Figures 5-9 through 5-14, the step responses of the same systems under 50% loading conditions are shown, with a comparison of the primary responses in Figures 5-15 and 5-16. Here, it is evident that all failed to meet zero overshoot criterion. This fact can be attributed to the load to weight ratio of the wrist. While the wrist itself weighs roughly five pounds, the load is almost three times the weight of wrist. It should be noted that despite not meeting overshoot requirements, the LQR design appeared to settle within the one second simulation period, whereas the others did not.

System decoupling

From the pitch response curves (Figures 5-1, 5-3, and 5-5), it can be deduced that the pitch mode has no effect on the yaw motion. However, the yaw response curves (Figures 5-2, 5-4, and 5-6) show that the yaw mode is slightly present in the pitch motion. From an analytical point of view, this can be seen by solving the wrist equations of motion in terms of ϕ and ψ . If the centrifugal and Coriolis terms are neglected, the resulting equations are given by

$$\begin{aligned} \frac{1}{4}[I_1 + I_2 \cos 2\psi]\ddot{\phi} + \frac{1}{4}[I_{xz} \sin \psi]\ddot{\psi} + (k_{s1}r_3^2 + k_{s2}r_4^2)\phi \\ + (k_{s1}r_3^2 - k_{s2}r_4^2)\psi - k_{s1}\frac{r_1r_3}{N_1}\theta_1 - k_{s2}\frac{r_2r_4}{N_2}\theta_2 = 0, \end{aligned} \quad (5.1)$$

$$\begin{aligned} \frac{1}{4}[I_3 + I_{xz} \sin \psi]\ddot{\psi} + (k_{s1}r_3^2 + k_{s2}r_4^2)\psi \\ + (k_{s1}r_3^2 - k_{s2}r_4^2)\phi - k_{s1}\frac{r_1r_3}{N_1}\theta_1 + k_{s2}\frac{r_2r_4}{N_2}\theta_2 = 0. \end{aligned} \quad (5.2)$$

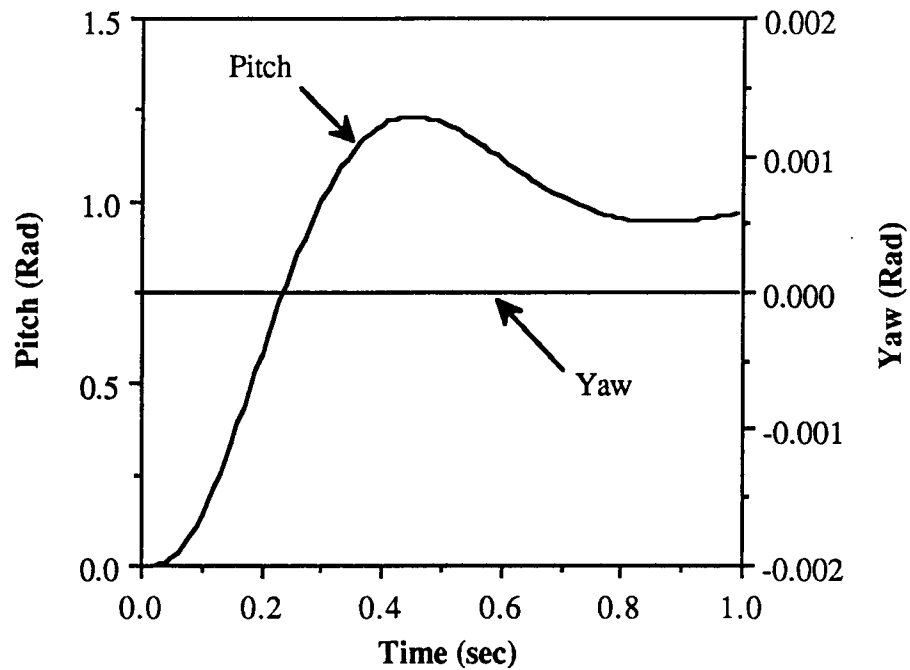


Figure 5-9. Response of pole-placed system to step pitch at 50% load (no load assumed).

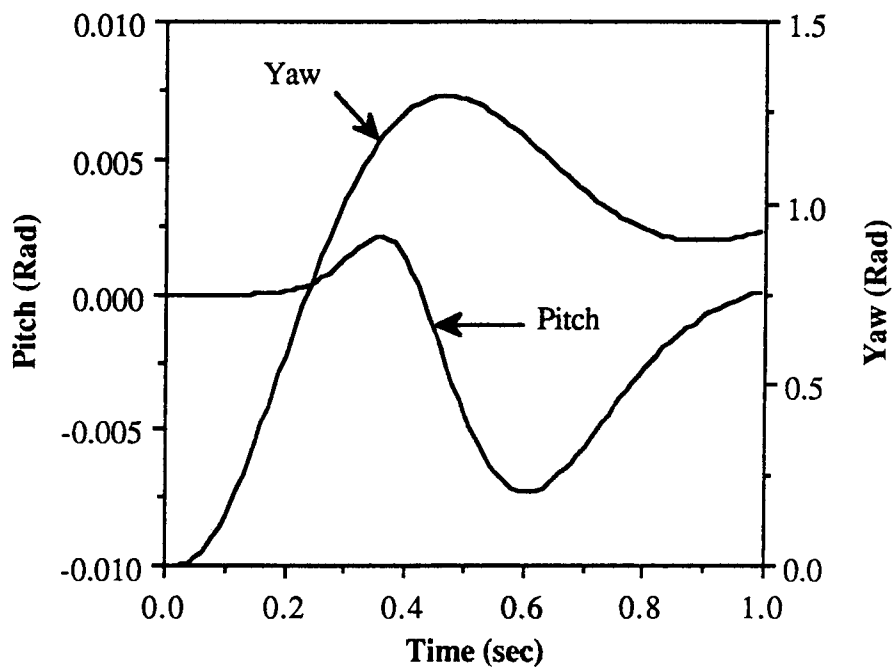


Figure 5-10. Response of pole-placed system to step yaw at 50% load (no load assumed).

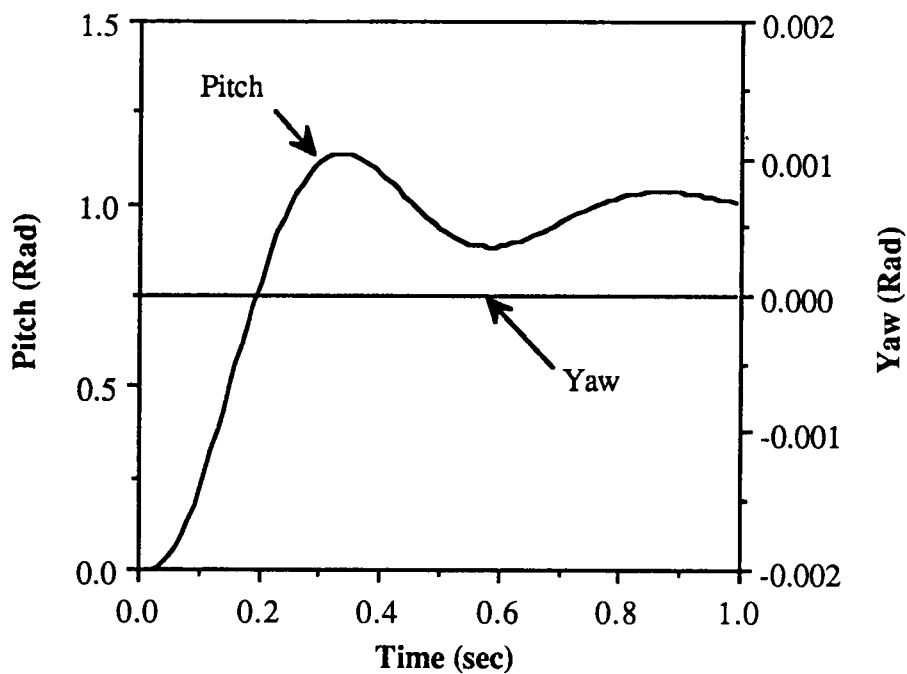


Figure 5-11. Response of PI system to step pitch at 50% load (no load assumed).

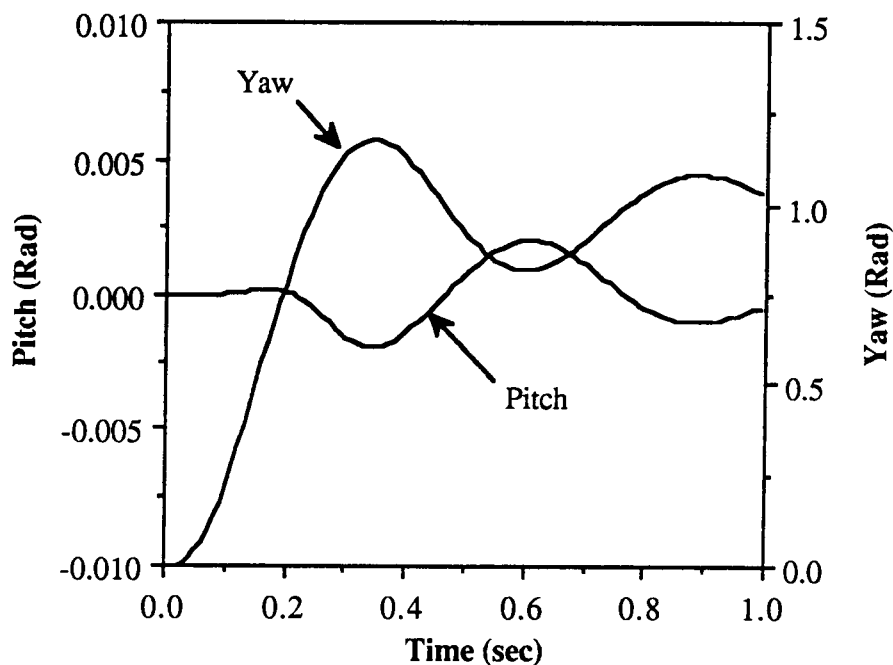


Figure 5-12. Response of PI system to step yaw at 50% load (no load assumed).

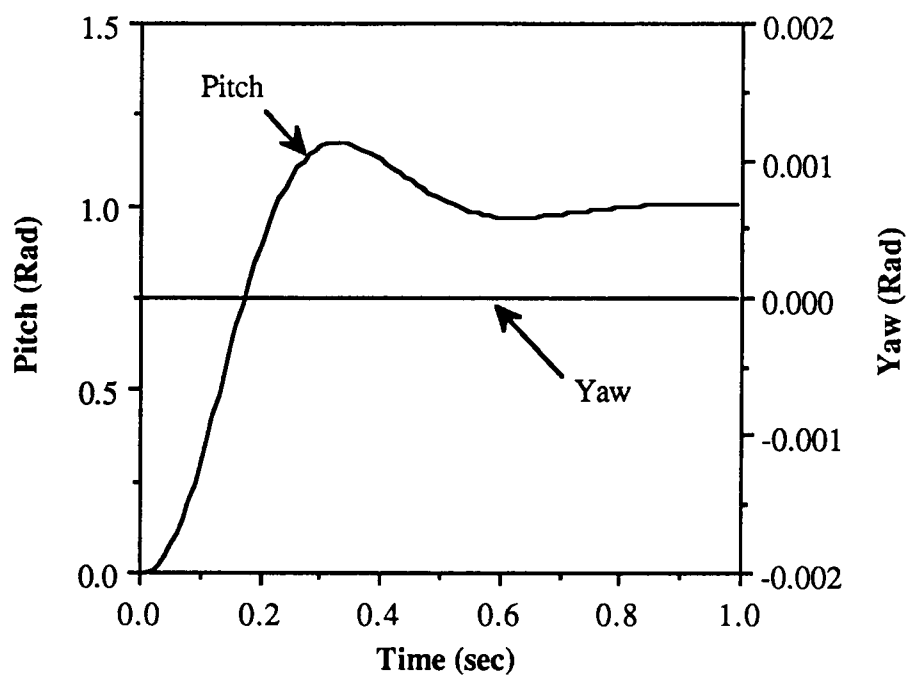


Figure 5-13. Response of LQR system to step pitch at 50% load (no load assumed).

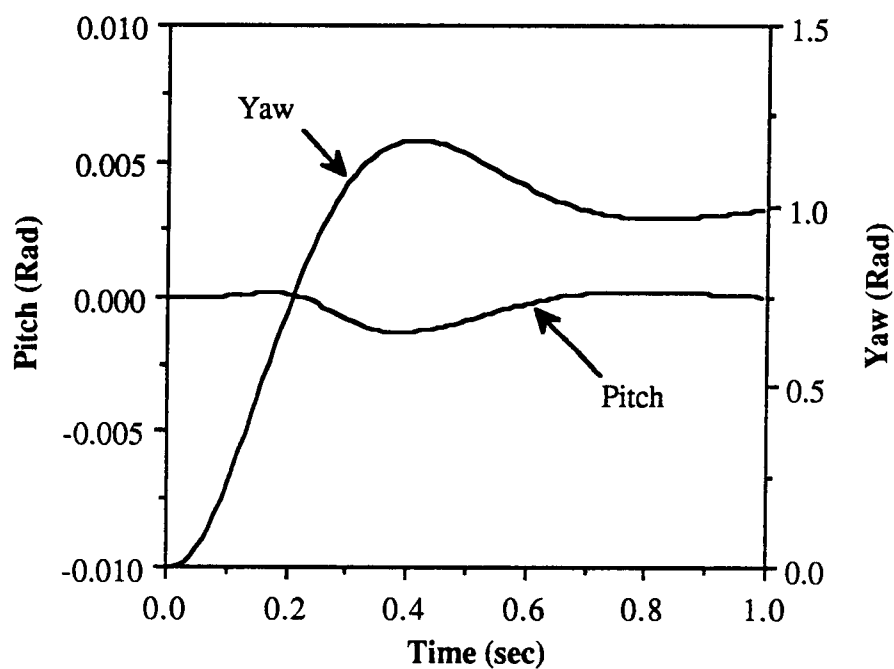


Figure 5-14. Response of LQR system to step yaw at 50% load (no load assumed).

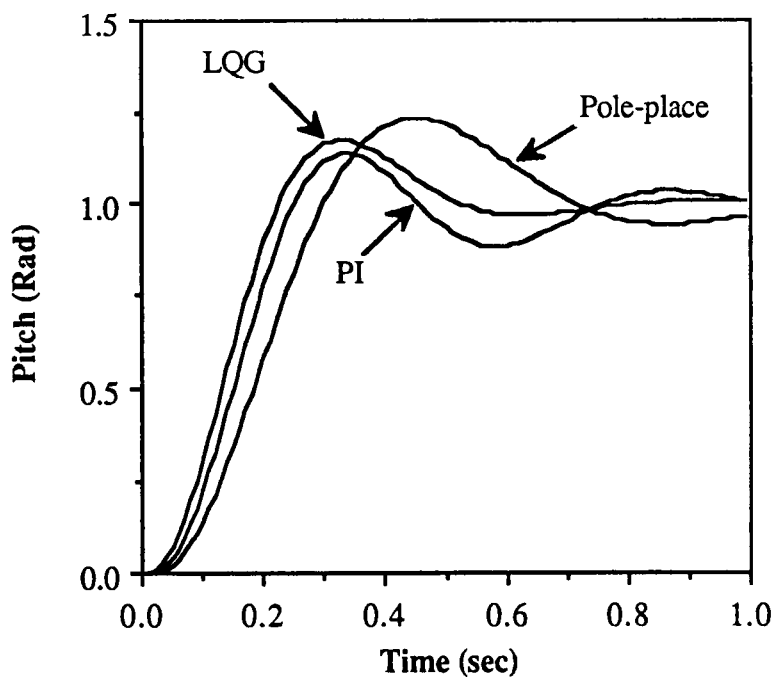


Figure 5-15. Comparison of pitch responses at 50% load (no load assumed).

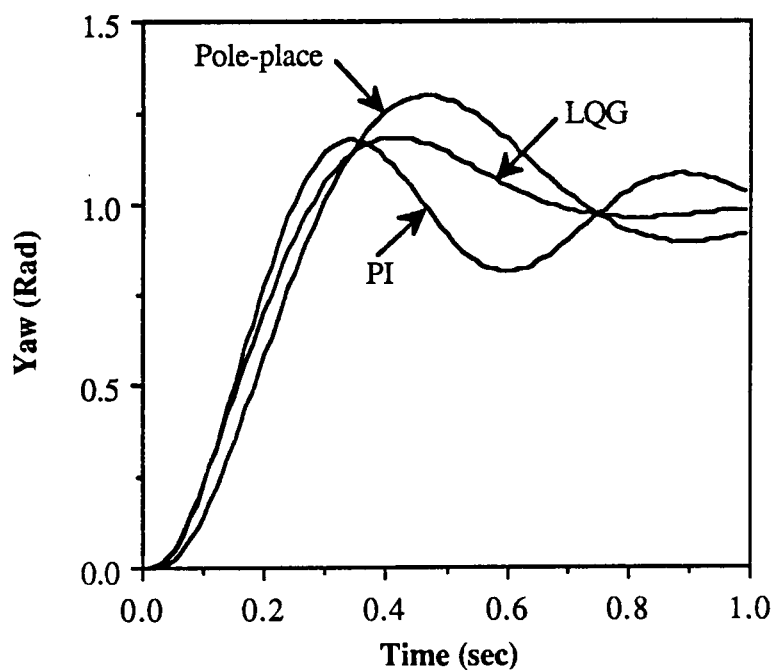


Figure 5-16. Comparison of yaw responses at 50% load (no load assumed).

If the spring rates are approximately equal, then the wrist yaw motion, eq.(5.2), will be dependent only on yaw angle, ψ . However, the wrist pitch motion, eq. (5.1) will be dependent on the pitch angle ϕ and the yaw angle ψ , and the yaw acceleration. Thus, the results obtained from the simulated step responses agree with the analytical equations of motion.

In all three no-loading cases, the order of magnitude of the coupling response was about .001 radians. However, under 50% loading conditions (Figures 5-9 through 5-16), the coupled response varied greatly among the controllers. Of the three, the pole-placement exhibited the worst response, with about .07 radians of yaw motion to a unit step pitch. The PI controller showed the worst change in coupling, with only .0002 radians of yaw at no load but .02 radians with 50% load. Meanwhile, the LQR design showed no change, with secondary response of about .0125 radians at both no load and 50% load. One reason is obviously that the cross-coupling terms have been included in the LQR design, whereas the terms were neglected in both the pole-placement and the PI designs. More importantly, the pole-placement and PI techniques depend greatly on accurate system parameters and are extremely sensitive to any parameter variation or drift.

System performance

In Figures 5-17 and 5-18 , the performance curves for each type of controller design (assuming no payload was present) are shown. Because the criterion used is IES, a smaller performance index implies a better transient response. In both pitch and yaw motions, it could be seen that the LQR performance index was always lower than either the pole-placed design or the PI design. It is worth noting that while the performance of the LQR and pole-placed systems varied linearly with load, the performance of the PI-controlled system appeared to vary nonlinearly with load. This is due primarily to the fact that PI controllers introduce a pole at the origin and a zero at $s = -K_I/K_P$ in the open-

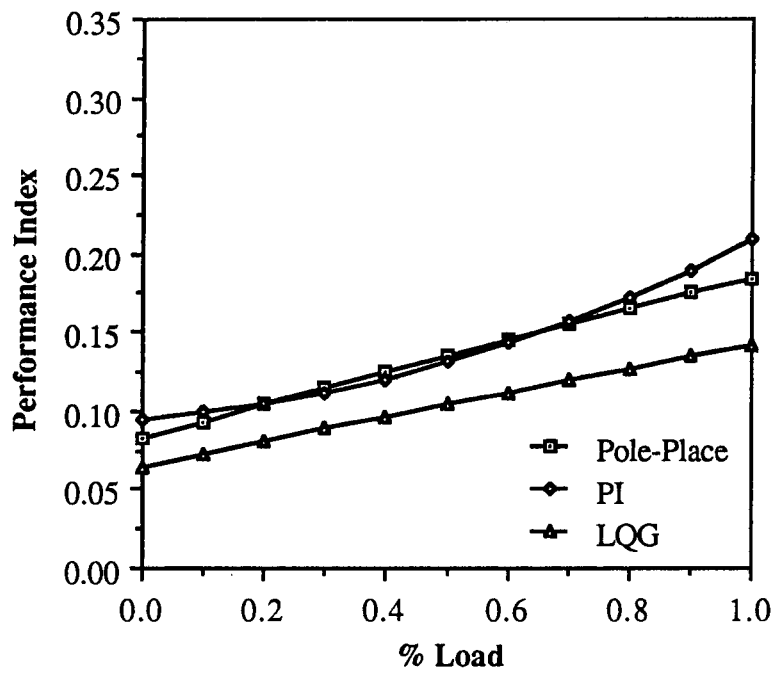


Figure 5-17. Comparison of pitch performance vs. load.

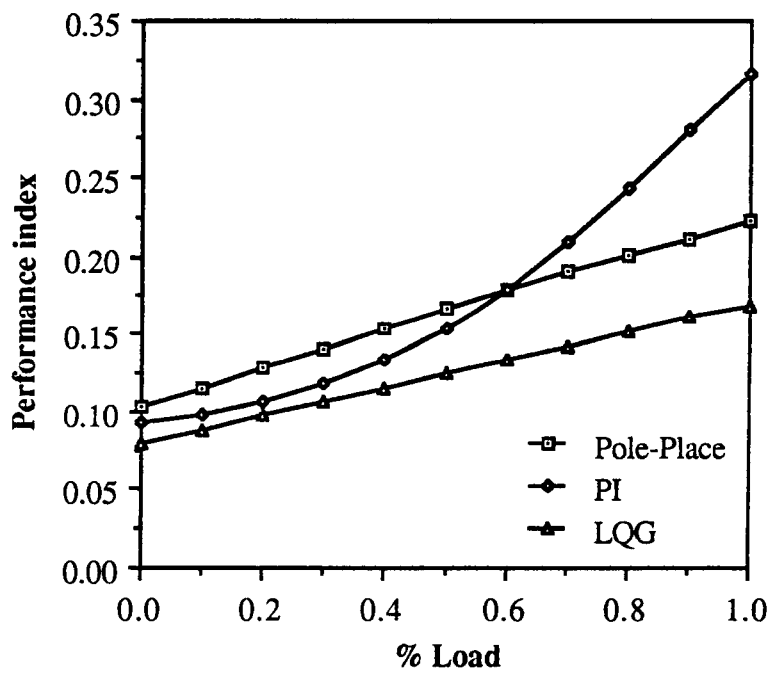


Figure 5-18. Comparison of yaw performance vs. load.

loop transfer function. Thus, variations in system parameters meant that the location of this zero moved away from the pole at the origin (for which it is supposed to cancel out).

To study the effect of the input scaling factor on system performance, the LQR controller was simulated using three different values of ρ : 4.5×10^{-6} , 5×10^{-7} , and 5×10^{-8} . Performance curves for each of these designs are then plotted in Figures 5-19 and 5-20. Surprisingly, the performance was essentially unaffected by varying ρ , though the slope of the curves decreased slightly with decreasing ρ . Therefore, the results seem to imply that once modal behavior is asymptotically obtained, further reduction in the value of ρ does not improve system performance.

The previous finding seems to contradict the intuition that as ρ increases, the sensitivity to load decreases. The reason for such a limiting performance curve is because the motors have a physical torque limit in real life. To illustrate the effect that torque limiting has on system performance, simulation was performed on the LQR system using $\rho = 5 \times 10^{-7}$, and the results were plotted in Figures 5-21 and 5-22. As expected, the slope of the performance curve decreased as the torque limit is increased; this time, however, the performance is limited by the finite choice of ρ . Therefore, in order to obtain a perfectly flat performance curve, (1) the value of ρ must approach zero, and (2) the torque limit must approach infinity.

Design with 50% Load Assumed

As seen in Figures 5-9 through 5-14, the time responses for the all systems designed assuming no payload (i.e., only wrist inertias) are quite poor. Since the design objective is to achieve a critically damped response, measures must be taken to obtain better response profiles. A viable solution would be to assume 50% of the total capacity load (15 pounds) to be a known point mass on the wrist. Time responses of the compliant wrist at

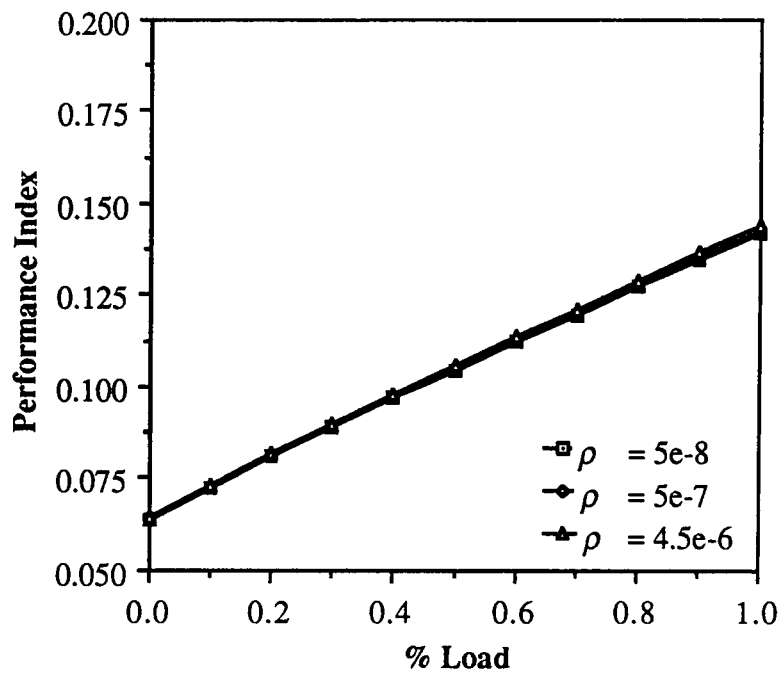


Figure 5-19. Effect of input scaling factor on pitch performance.

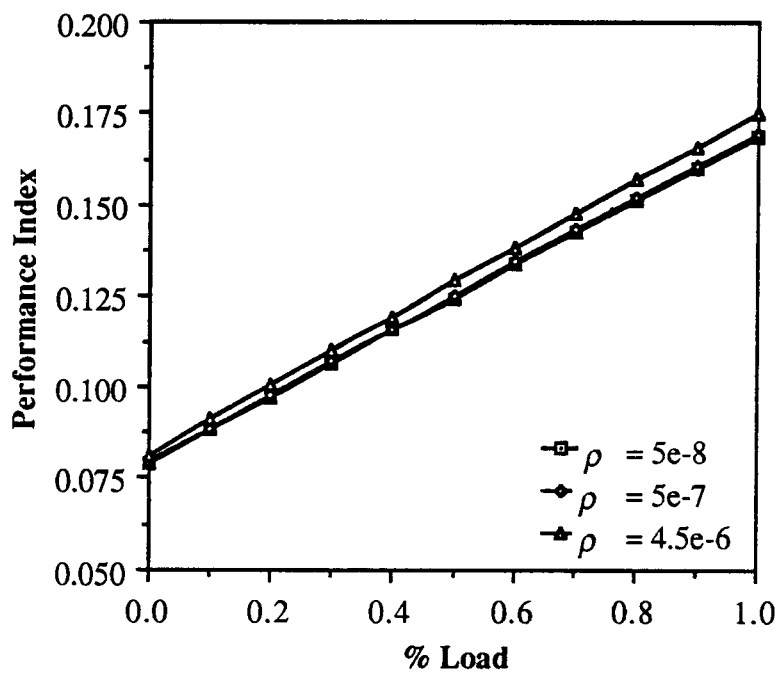


Figure 5-20. Effect of input scaling factor on yaw performance.

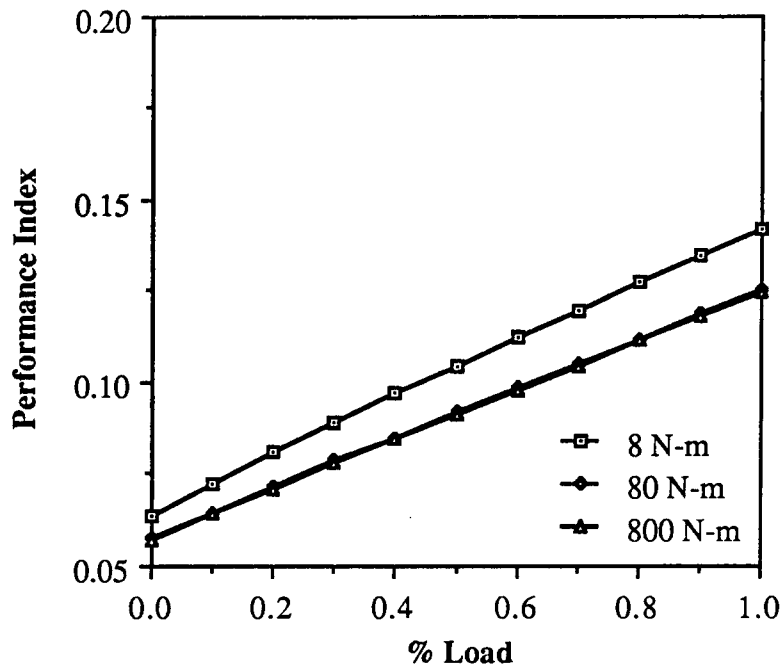


Figure 5-21. Effect of torque limit on pitch performance.

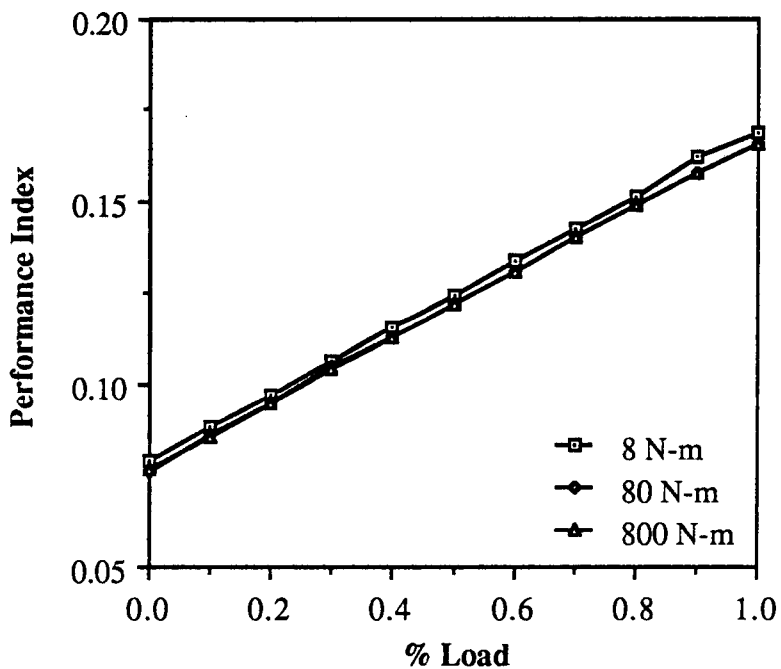


Figure 5-22. Effect of torque limit on yaw performance.

no load and at full load would then be compared. Because this assumption intuitively will improve performance curves, only peak overshoot and system decoupling will be studied.

Peak overshoot

In Figures 5-23 through 5-28, the step responses for the no-load case are shown and the primary responses are summarized in Figures 5-29 and 5-30. Once again, all three designs were successful, with no overshoot in the response profiles. However, the responses of the pole-placed and PI controllers showed some waviness or chopiness; the reason for this was that the characteristic roots of the system were being moved away from those of the binomial standard filter towards some location where a higher-order response was prevalent. At the same time, the response profile of the LQR system remained unaltered, retaining its first-order appearance to a certain degree.

In Figures 5-31 through 5-36, the step responses with full load are given, with the overall comparisons shown in Figures 5-37 and 5-38. While all three exhibited some overshoot in the response, the magnitudes were not as severe as those depicted in Figures 5-9 through 5-14. Not surprisingly, the pole-placed controller failed, with a peak overshoot of 10% in pitch and 20% in yaw. On the other hand, the PI controller had no overshoot in pitch response and only 2% overshoot in yaw response, though the yaw response did exhibit some "undershoot". As for the LQR design, it showed 3% overshoot in pitch response and 2% overshoot in yaw response; however, the 2% settling time for the LQR system was .5 seconds, whereas the PI system was .7 seconds.

System decoupling

While assuming 50% of the load as part of the wrist inertia improved the primary responses of all the systems, the same was not true of the secondary responses in Figures 5-24, 5-26, and 5-28. For the pole-placed system, the pitch response to a unit step

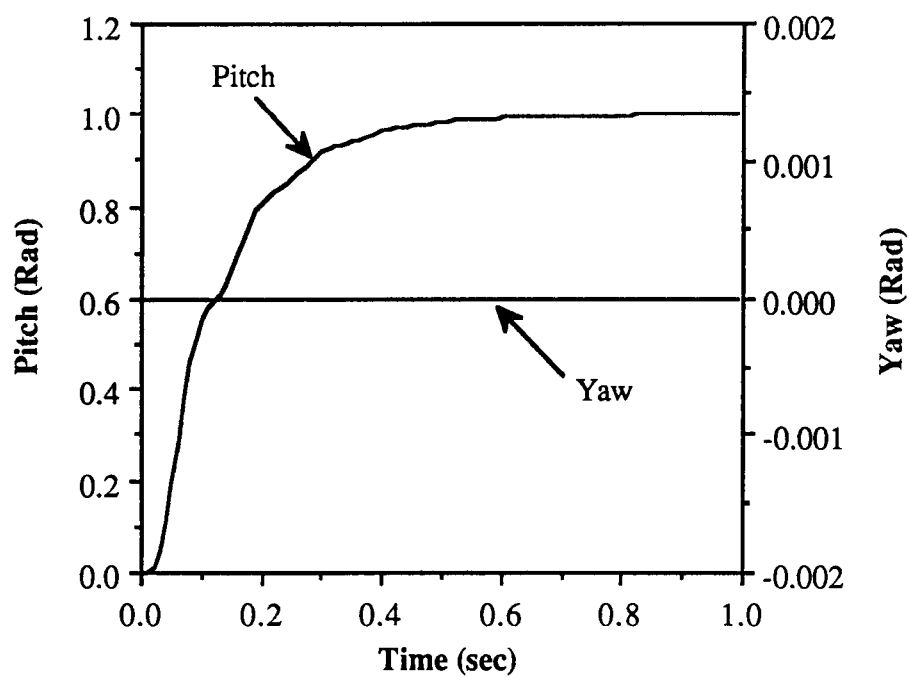


Figure 5-23. Response of pole-placed system to step pitch at no load (50% assumed).

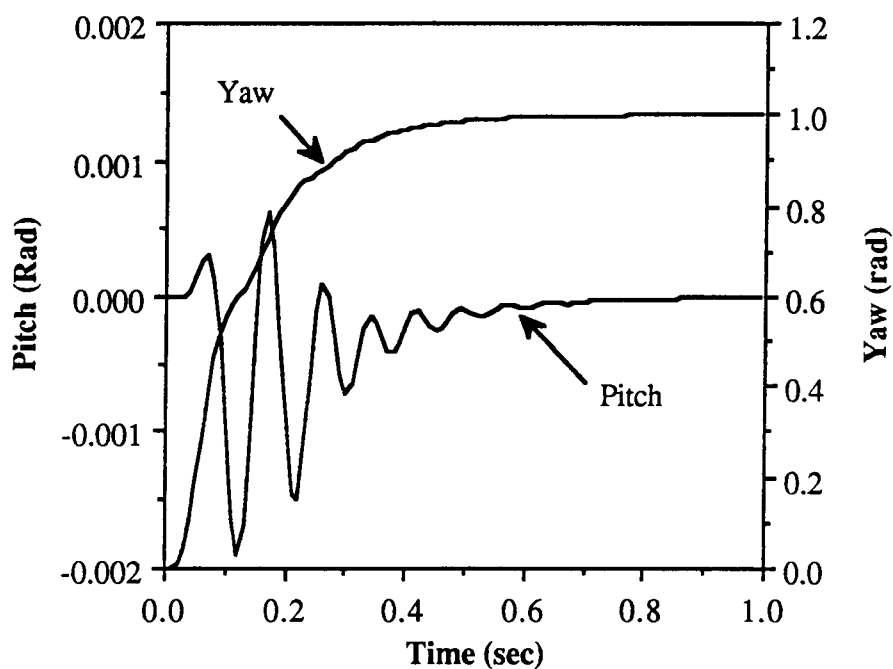


Figure 5-24. Response of pole-placed system to step yaw at no load (50% assumed).

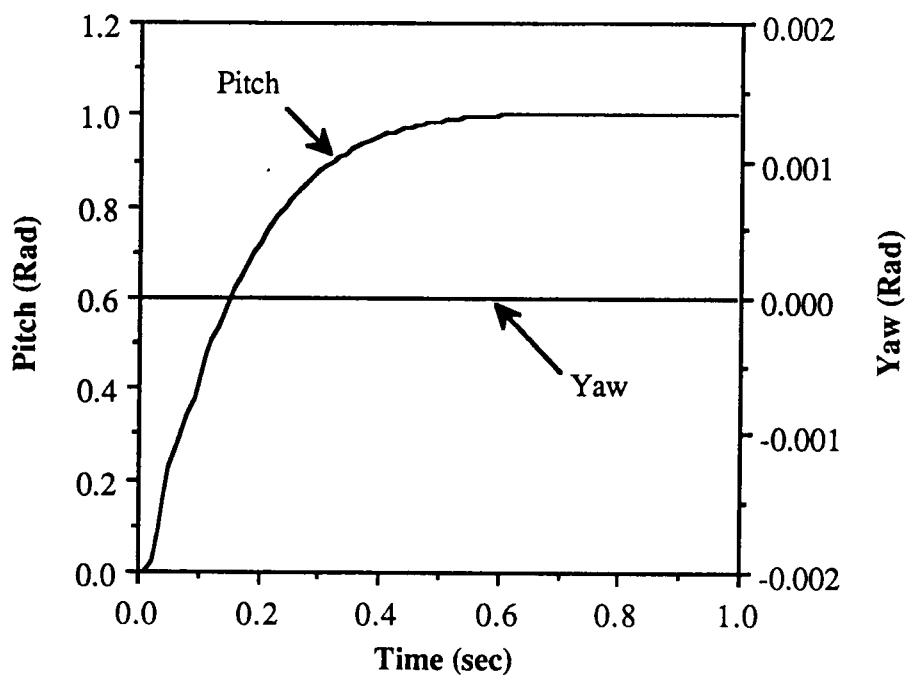


Figure 5-25. Response of PI system to step pitch at no load (50% assumed).

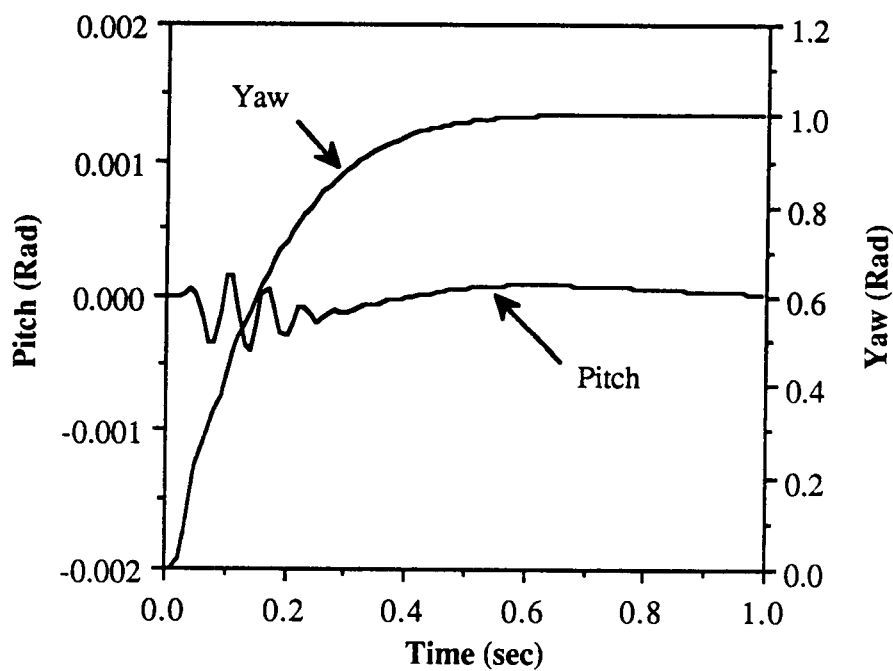


Figure 5-26. Response of PI system to step yaw at no load (50% assumed).

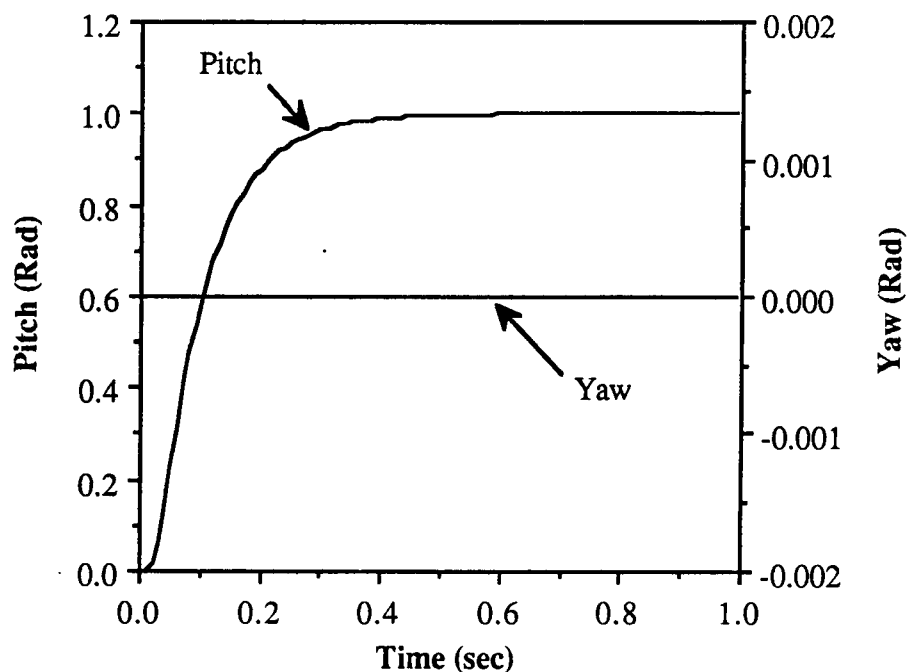


Figure 5-27. Response of LQR system to step pitch at no load (50% assumed).

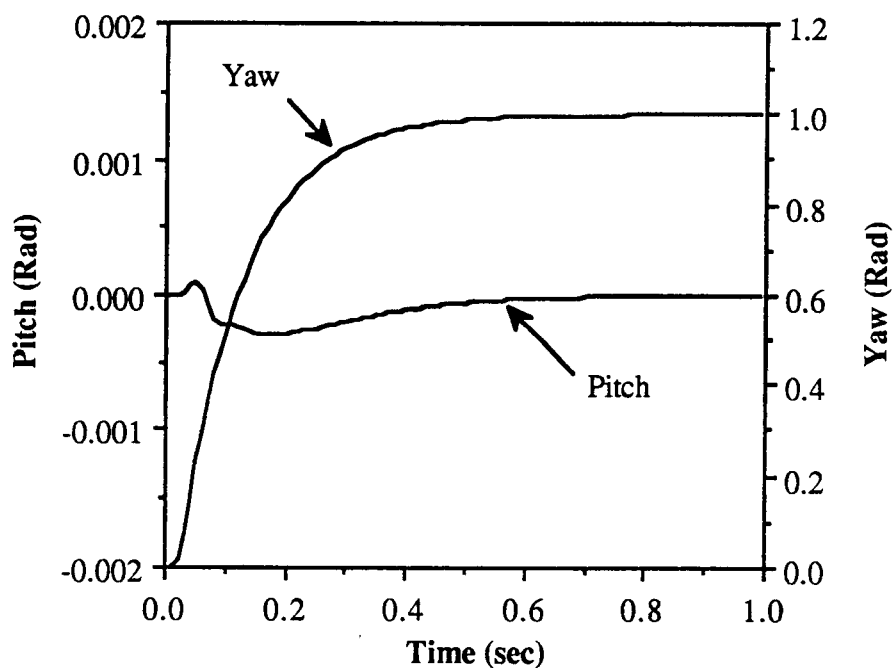


Figure 5-28. Response of LQR system to step yaw at no load (50% assumed).

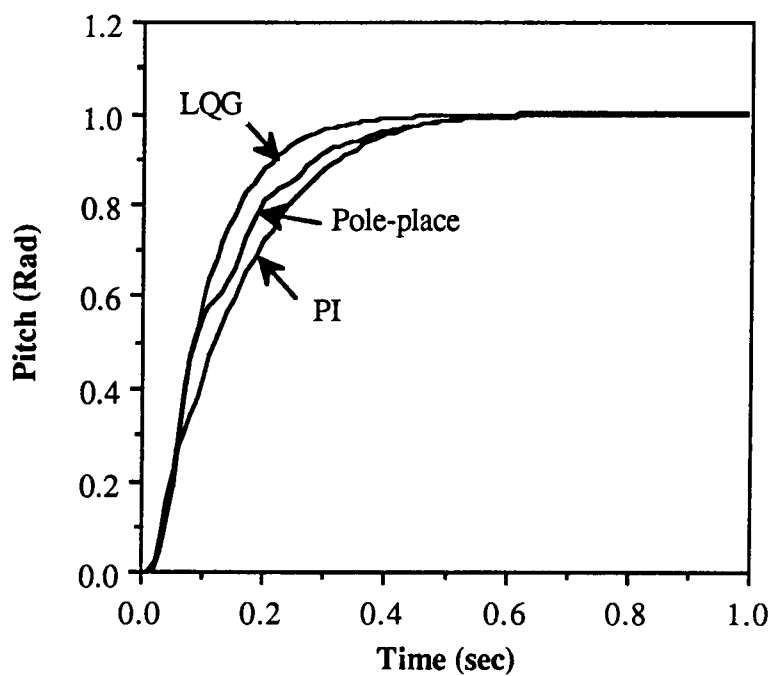


Figure 5-29. Comparison of pitch responses at no load (50% assumed).

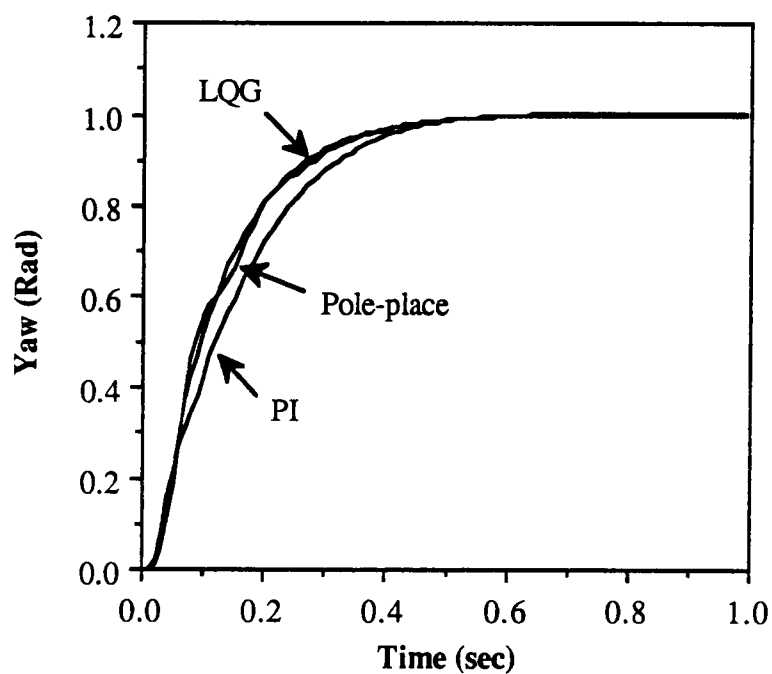


Figure 5-30. Comparison of yaw responses at no load (50% assumed).

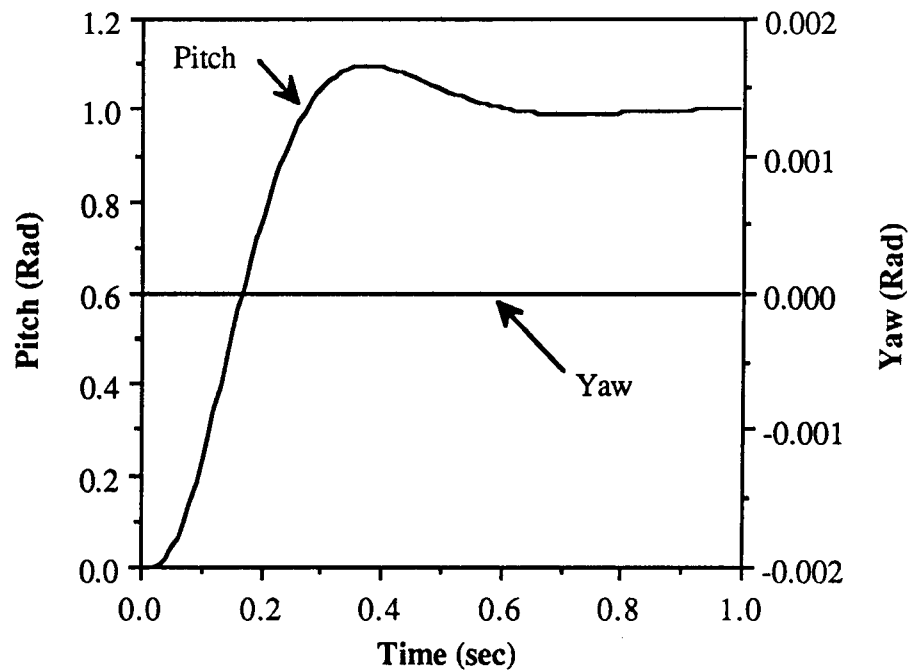


Figure 5-31. Response of pole-placed system to step pitch at full load (50% assumed).

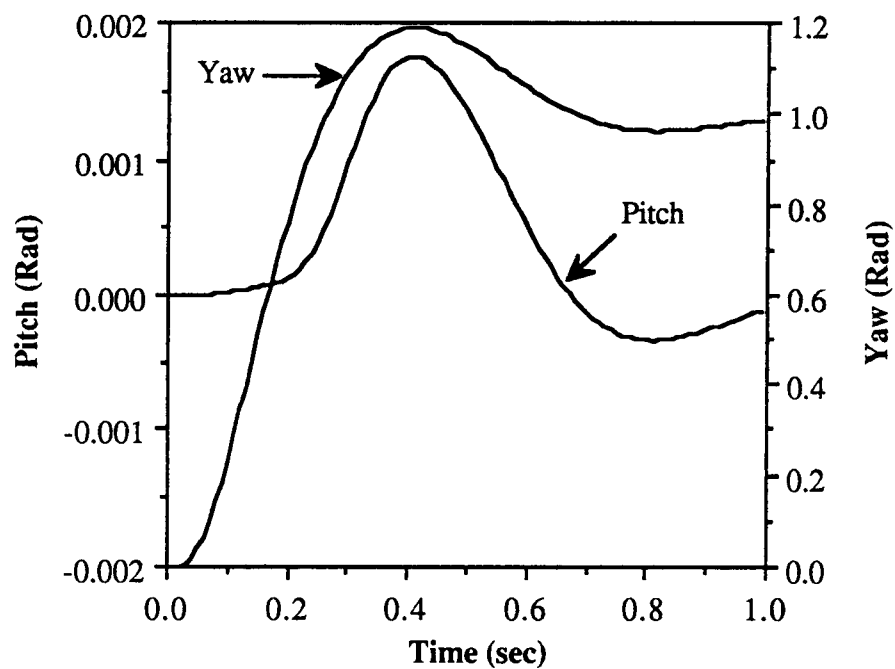


Figure 5-32. Response of pole-placed system to step yaw at full load (50% assumed).

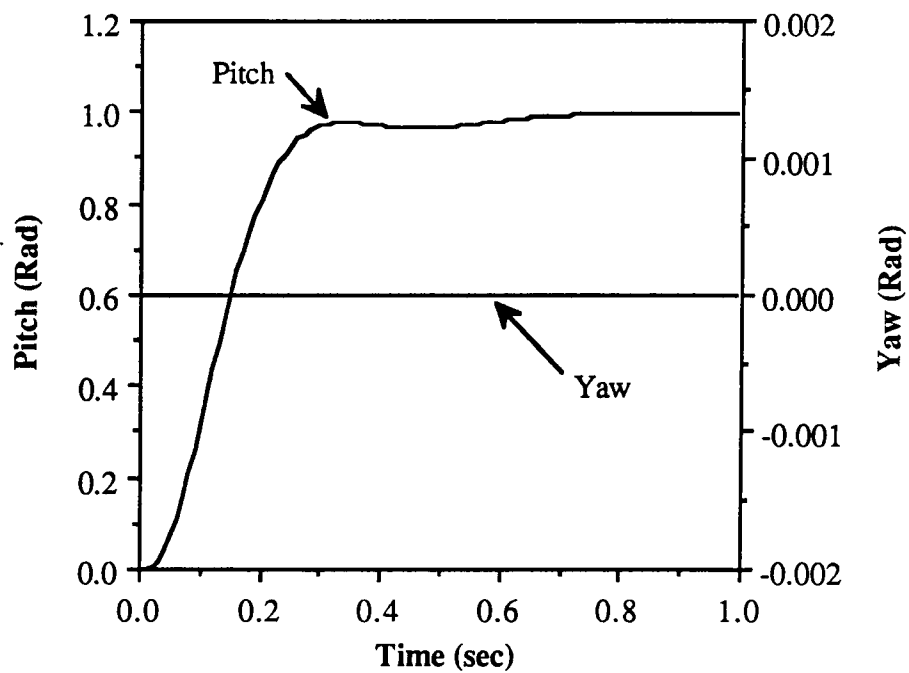


Figure 5-33. Response of PI system to step pitch at full load (50% assumed).

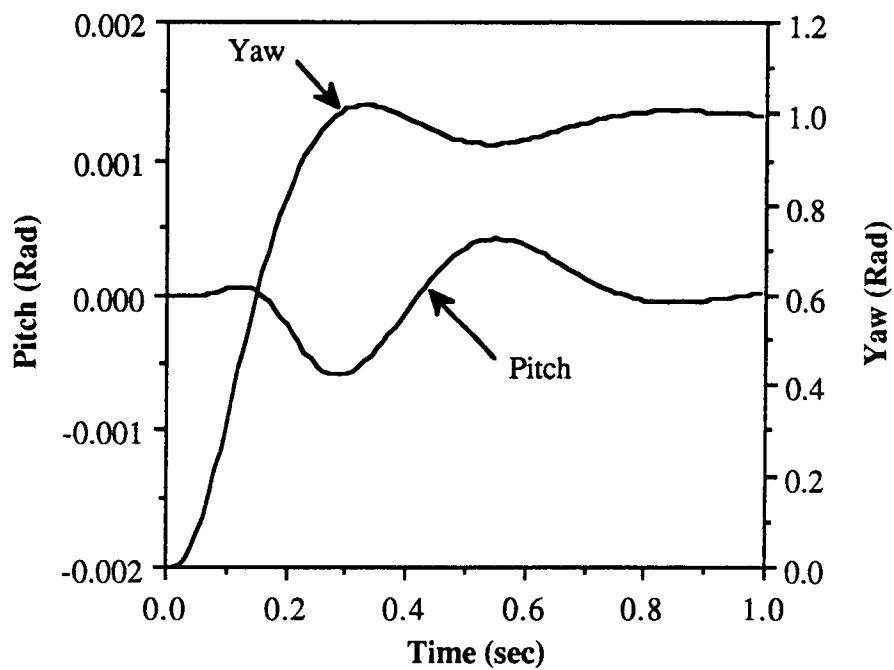


Figure 5-34. Response of PI system to step yaw at full load (50% assumed).

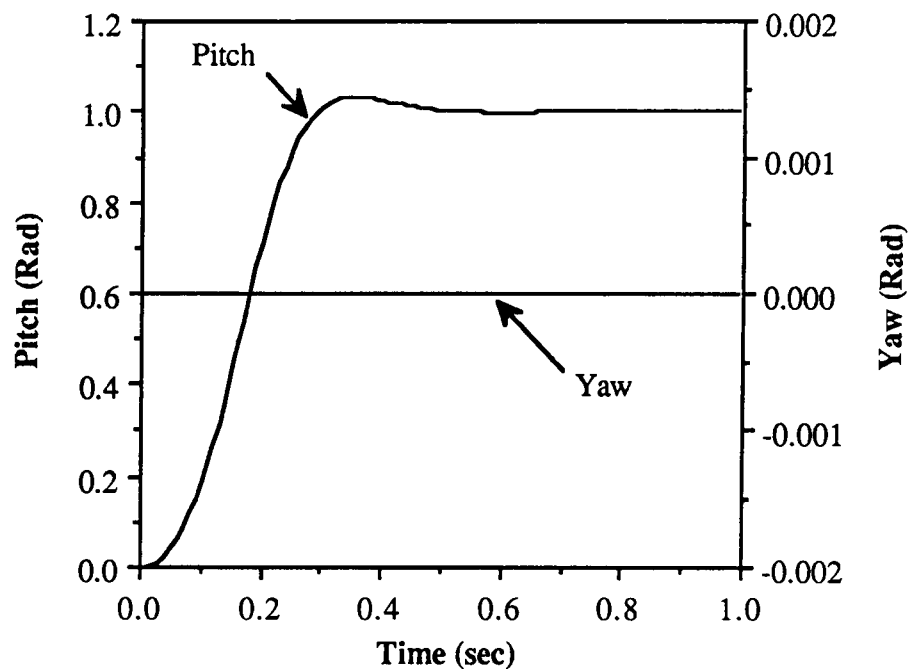


Figure 5-35. Response of LQR system to step pitch at full load (50% assumed).

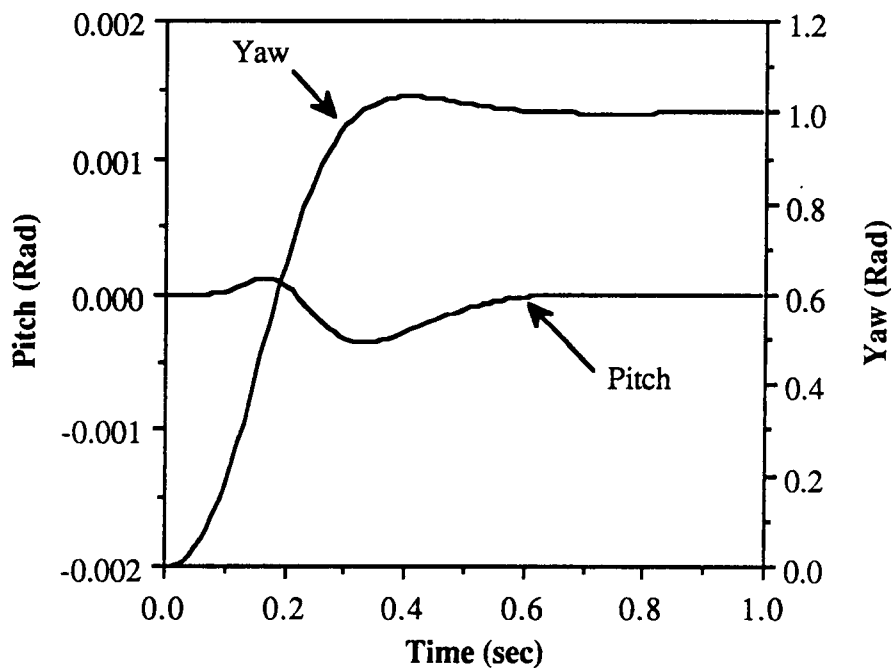


Figure 5-36. Response of LQR system to step yaw at full load (50% assumed).

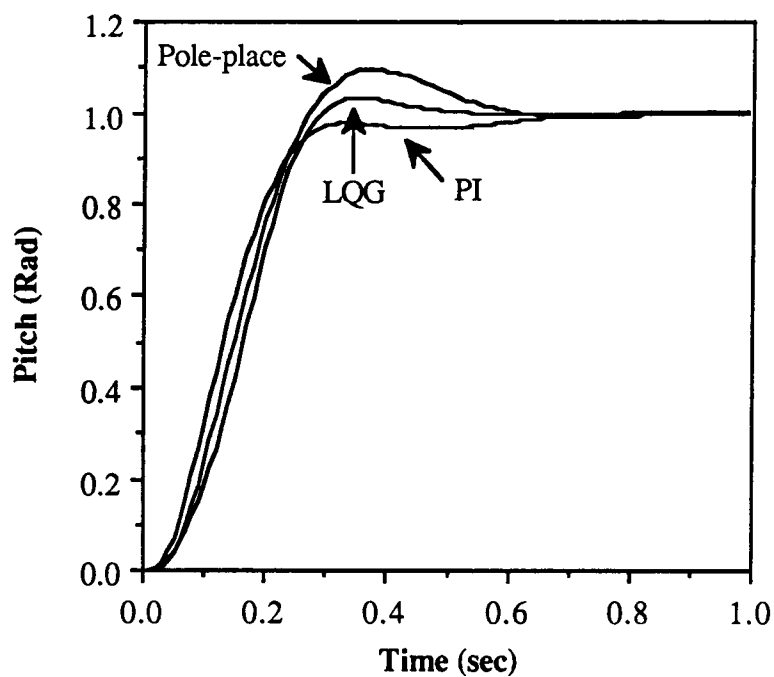


Figure 5037. Comparison of pitch responses at full load (50% assumed).

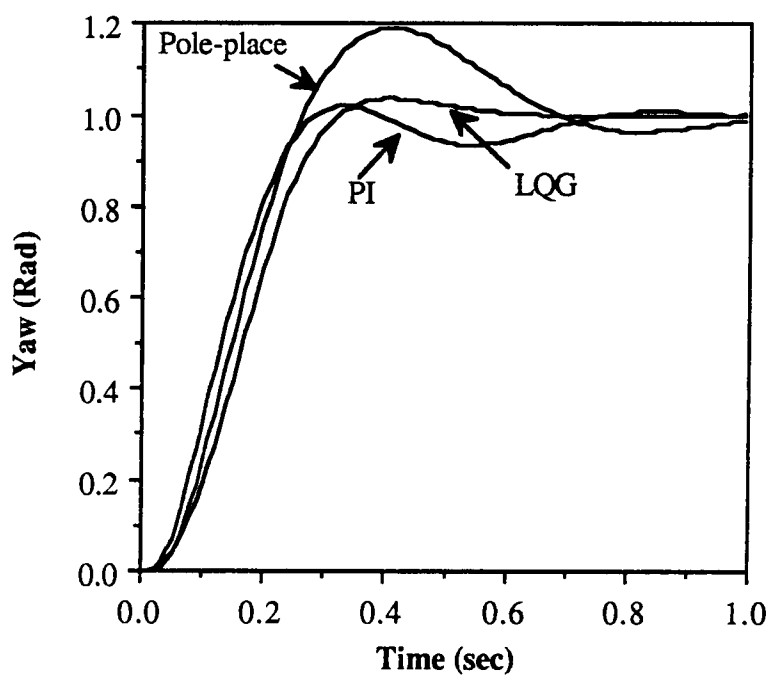


Figure 5-38. Comparison of yaw responses at full load (50% assumed).

yaw was quite oscillatory, with a peak response of about .002 radians; also the response required up to .8 seconds to return to zero. While the PI system registered no more than .0005 radians of secondary response, it still exhibited rather oscillatory response which did not diminish until one second later. As for the LQR controller, its secondary response was not only less than .0005 radians but also required just .6 seconds to dissipate transients. Furthermore, its response appeared critically damped, as opposed to the pole-placed and PI controllers, which appeared underdamped.

For the fully-loaded cases depicted in Figures 5-32, 5-34, and 5-36, the secondary responses were not oscillatory, but the peak responses were much worse than those for the no-load case. The worst of the secondary responses was the pole-placed design's response; not only did its system yaw .0017 radians when given a unit step pitch, the response did not dissipate within the one second simulation period. While the PI controller's yawed only .0006 radians, it did not return to zero until .8 seconds later. In the end, the LQR showed the best decoupled response, yawing only .0004 radians when given a unit step pitch and taking only .6 seconds to return to zero.

VII. CONCLUSIONS AND RECOMMENDATIONS

Summary of Modeling

Because of the relatively low number of degrees of freedom (four) of the compliant wrist drive train, the application of Lagrange's equation was quite feasible, and offered a straight-forward approach to obtaining the dynamic equations of motion. In equations (3.17) and (3.18), it can be seen that the inertial terms J_1 and J_2 dominated the inertial coupling term J_3 by about 10:1 at $\psi = 0$; however, as ψ approaches 90° , the ratio of inertias dwindled down to 3:1. Furthermore, if the system is loaded with mass, the ratio may decrease even more, bringing the two terms closer together.

From equations (5.1) and (5.2), the relationship between pitch and yaw is clearly shown by the absence of yaw terms in the pitch dynamic equation and the presence of a pitch term in the yaw dynamics. The influence of yaw motion on pitch motion can be explained as an inertial torque about the pitch axis back-driving the wrist. The nature of the inertial force resulted from the asymmetry of the wrist in its coordinate frame of rotation, causing somewhat of a rotational unbalance. Though it is not clearly evident in eqs. (5.1) and (5.2), a pitch motion would also yield some yaw motion if the initial yaw angle is not equal to zero. (In all the case studies, a step pitch was performed with the yaw angle initially at zero).

Recommendations

Though much detail was put into the modeling of the compliant wrist drive train, some nonlinearities were omitted due to time constraints. Those details not included but worthy of study were as follows:

1. The gravitational effects on the wrist dynamics.
2. The inclusion of Coloumb friction in damping terms.
3. The inclusion of wrist roll motion with the wrist kinematics.
4. The simulation of effects of digitization and quantization.

The addition of gravitational effects can be easily done by writing the potential energy associated with gravity as a function of the Euler angles, ϕ and ψ , and performing the required operation for Lagrange's equations. The second and fourth recommendation can be added through ACSL statements, and a comparison made to the model without these effects. Only the third recommendation requires a great deal more work, since a third Euler angle is required to complete the analysis. This additional rotation complicates the model by a factor of two.

Summary of Simulations

Design assuming no load

Using only the identified wrist inertias in the mass matrix, all three designs were able to achieve the zero overshoot criterion when simulating with no external load. With 50% of its total capacity load added (i.e., 15 pounds), however, none of the designs were able to meet the overshoot requirement.

On the issue of coupling, all the controllers designed assuming no load were able to keep secondary yaw responses at about .001 radians when simulating with no load. With 50% load, the controller that exhibited the most coupled response was that of the pole-placement design (.07 radians of pitch per unit step yaw); the PI controller showed the greatest change in coupled response (.0002 radians at no load to .02 radians at 50% load). The LQR design showed essentially no change (.00125 radians for both cases).

Using the integral of error-squared (IES) index, the system performance could be studied as the load on the compliant wrist drive train was increased from 0% to 100% of maximum capacity load. Results indicated that the LQR design was consistently better (i.e., lower performance index) than the pole-placed or PI designs; while the LQR and pole-placed models varied almost linearly with load, the PI design showed a nonlinear variation to load (poorer performance at higher loads). Studying the LQR performance in detail, it was concluded that perfectly flat performance required both the input scaling factor, ρ , to approach zero and the torque limit of the motors to be infinite.

Design with 50% assumed

To try to improve the transient responses, all three controllers were re-designed assuming 50% of the maximum capacity load to be a point mass on the wrist. Simulating the new controllers with no load, all three designs were able to meet the no-overshoot condition. With full load, however, the pole-placed controller had 10% overshoot in pitch and 20% overshoot in yaw. Both the PI and LQR controllers kept overshoot under 5% (2% for PI and 3% for LQR), but the LQR system settled under 2% faster than the PI system. (0.5 seconds for LQR and 0.7 seconds for PI).

Studying the secondary responses of the systems with 50% of the load assumed and simulated with no load, it can be seen that both the pole-placed and the PI controllers were quite oscillatory. Meanwhile, the LQR controller appeared to be more damped, oscillating for less than one cycle. The peak response of the pole-placed system was .002 radians of pitch per unit step yaw. The PI system was slightly better, with a peak response of .0005 radians, but took one second to dissipate. The LQR system had the same secondary response as the PI system (.0005 radians) and needed only .6 seconds to dissipate. Simulating at full load, the secondary responses of the systems designed with the 50% load assumption were not oscillatory, but their magnitudes were almost the same

as those in the simulations with no load. Pole-placement design had the worst response (.0017 radians of pitch per unit step yaw), while the PI was second with only .0006 radians. The best decoupling was exhibited by the LQR system, with .0004 radians of pitch per unit step yaw.

Remarks

1. Despite assuming 50% of the load as part of the wrist inertia, the pole-placement design proved to be too sensitive to parameter variation. While no-load responses were satisfactory, the overshoot in the full load responses proved to be intolerable.
2. Within certain limits of tolerance, both the PI and the LQR designs were able to meet the overshoot criterion, as well as exhibiting minimal cross-coupled response. Both controllers did not overshoot by more than 5% when assuming 50% of the load as a point mass on the wrist, and both had secondary responses of roughly .0005 radians.
3. It must be considered that both the pole-placement and PI designs required gain programming to obtain the results in the simulations. On the otherhand, the LQR controller, which was linearized about the nominal position $\psi = 0$, was valid over the entire range of motion from 1 radian to -1 radian. Thus the LQR controller not only outperformed the other SISO designs, but it also required less computation to implement.

Recommendations

1. While the LQR design works in theory, it is important to be able to apply the technique to a real system. Thus, implementation on actual hardware would be the next step in verifying the design.
2. The assumption of perfect measurements and complete observability is key in applying the LQR design technique. However, all real-life sensors have noise in their signals; thus, it would be wise to develop a stochastic model of the sensor noise.
3. Unfortunately, the LQR design does not give a measure of guaranteed relative stability (i.e., phase margin) of the system. Therefore, research into more elaborate techniques such as the LQG/LTR design, would allow control designers to have a better idea of the relative stability of the system.

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APPENDICES

APPENDIX A

DERIVATION OF LAGRANGIAN TERMS

In Section II, the equations of motion were presented without any details of the application of Lagrange's equations. Thus, it is the purpose of this appendix to present the formulation of the equations of motion from the kinetic and potential energy expressions.

Kinetic energy:

$$T_{Total} = \frac{1}{8} \left[I_1 (\dot{\theta}_3 + \dot{\theta}_4)^2 + I_2 (\dot{\theta}_3 + \dot{\theta}_4)^2 \cos(\theta_3 - \theta_4) + I_3 (\dot{\theta}_3 - \dot{\theta}_4)^2 + 2 I_{xz} (\dot{\theta}_3^2 - \dot{\theta}_4^2) \right] + \frac{1}{2} J_{m1} \dot{\theta}_1^2 + \frac{1}{2} J_{m2} \dot{\theta}_2^2. \quad (2.18)$$

Potential energy:

$$V = \frac{1}{2} \left[2 k_{s1} \left(\frac{r_1}{N_1} \theta_1 - r_3 \theta_3 \right)^2 + 2 k_{s2} \left(\frac{r_2}{N_2} \theta_2 - r_4 \theta_4 \right)^2 \right]. \quad (2.21)$$

First kinetic terms:

$$\frac{\partial T}{\partial \dot{\theta}_1} = J_{m1} \dot{\theta}_1, \quad \frac{\partial T}{\partial \dot{\theta}_2} = J_{m2} \dot{\theta}_2,$$

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}_3} = \frac{1}{4} & \left[I_1 (\dot{\theta}_3 + \dot{\theta}_4) + I_2 (\dot{\theta}_3 + \dot{\theta}_4) \cos(\theta_3 - \theta_4) \right. \\ & \left. + I_3 (\dot{\theta}_3 - \dot{\theta}_4) + 2 I_{xz} \dot{\theta}_3 \sin \left(\frac{\theta_3 - \theta_4}{2} \right) \right], \end{aligned}$$

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}_4} = \frac{1}{4} & \left[I_1 (\dot{\theta}_3 + \dot{\theta}_4) + I_2 (\dot{\theta}_3 + \dot{\theta}_4) \cos(\theta_3 - \theta_4) \right. \\ & \left. - I_3 (\dot{\theta}_3 - \dot{\theta}_4) - 2 I_{xz} \dot{\theta}_3 \sin \left(\frac{\theta_3 - \theta_4}{2} \right) \right]. \end{aligned}$$

Second kinetic terms:

$$\frac{\partial T}{\partial \dot{\theta}_1} = 0,$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = 0,$$

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}_3} &= \frac{1}{8} \left[-I_2 (\dot{\theta}_3 + \dot{\theta}_4)^2 \sin(\theta_3 - \theta_4) + I_{xx} (\dot{\theta}_3^2 - \dot{\theta}_4^2) \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \\ &= \frac{1}{8} \left[-I_2 \sin(\theta_3 - \theta_4) + I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_3^2 \\ &\quad + \frac{1}{8} \left[-I_2 \sin(\theta_3 - \theta_4) - I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_4^2 - \frac{I_2}{4} \sin(\theta_3 - \theta_4) \dot{\theta}_3 \dot{\theta}_4, \end{aligned}$$

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}_4} &= \frac{1}{8} \left[I_2 (\dot{\theta}_3 + \dot{\theta}_4)^2 \sin(\theta_3 - \theta_4) - I_{xx} (\dot{\theta}_3^2 - \dot{\theta}_4^2) \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \\ &= \frac{1}{8} \left[I_2 \sin(\theta_3 - \theta_4) - I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_3^2 \\ &\quad + \frac{1}{8} \left[I_2 \sin(\theta_3 - \theta_4) + I_{xx} \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right] \dot{\theta}_4^2 + \frac{I_2}{4} \sin(\theta_3 - \theta_4) \dot{\theta}_3 \dot{\theta}_4. \end{aligned}$$

Time derivative of first kinetic terms:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} = J_{m1} \ddot{\theta}_1,$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} = J_{m2} \ddot{\theta}_2,$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_3} &= \frac{1}{4} \left\{ I_1 (\ddot{\theta}_3 + \ddot{\theta}_4) + I_2 [(\ddot{\theta}_3 + \ddot{\theta}_4) \cos(\theta_3 - \theta_4) - (\dot{\theta}_3^4 - \dot{\theta}_4^4) \sin(\theta_3 - \theta_4)] \right. \\ &\quad \left. + I_3 (\ddot{\theta}_3 - \ddot{\theta}_4) + 2I_{xx} \left[\ddot{\theta}_3 \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right] + \left(\frac{\dot{\theta}_3^2 - \dot{\theta}_4^2}{2} \right) \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right\}, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_4} = \frac{1}{4} \left\{ I_1 (\ddot{\theta}_3 + \ddot{\theta}_4) + I_2 (\ddot{\theta}_3 + \ddot{\theta}_4) \cos(\theta_3 - \theta_4) - (\dot{\theta}_3^4 - \dot{\theta}_4^4) \sin(\theta_3 - \theta_4) \right. \\ \left. - I_3 (\ddot{\theta}_3 - \ddot{\theta}_4) - 2 I_{xz} \left[\ddot{\theta}_4 \sin\left(\frac{\theta_3 - \theta_4}{2}\right) \right] + \left(\frac{\dot{\theta}_3 \dot{\theta}_4 - \dot{\theta}_4^2}{2} \right) \cos\left(\frac{\theta_3 - \theta_4}{2}\right) \right\}. \end{aligned}$$

Potential energy terms:

$$\frac{\partial V}{\partial \theta_1} = 2 k_{s1} \frac{r_1^2}{N_1^2} \theta_1 - 2 k_{s1} \frac{r_1 r_3}{N_1} \theta_3, \quad \frac{\partial V}{\partial \theta_2} = 2 k_{s2} \frac{r_2^2}{N_2^2} \theta_2 - 2 k_{s2} \frac{r_2 r_4}{N_2} \theta_4,$$

$$\frac{\partial V}{\partial \theta_3} = -2 k_{s1} \frac{r_1 r_3}{N_1} \theta_1 + 2 k_{s1} r_3^2 \theta_3, \quad \frac{\partial V}{\partial \theta_4} = -2 k_{s2} \frac{r_2 r_4}{N_2} \theta_2 + 2 k_{s2} r_4^2 \theta_4.$$

External, nonconservative forces:

$$Q'_1 = T_{m1} - B_{m1} \dot{\theta}_1, \quad Q'_2 = T_{m2} - B_{m2} \dot{\theta}_2,$$

$$Q'_3 = 0, \quad Q'_4 = 0.$$

All that is required is to synthesize the terms of Lagrange's equations given by

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q'_i, \quad \text{for } i = 1, 2, \dots, n$$

Algebraic collection of the terms will yield the equations of motion given in equations (2.24) through (2.27).

APPENDIX B

SAMPLE ACSL SIMULATION PROGRAM

The following is a sample listing of the ACSL program used to simulate the time responses of the wrist using the pole-placement controller design. Similar programs exist for simulating the time responses using the PI-compensated pole-placement design and the LQ-regulator design, and those statements have been included at the end of the program listing. One note of importance is that the equations do not exactly match those derived in Section II; this is due to the fact that the positive pitch direction was assumed to be rotation downward. The right-hand rule, however requires assuming positive pitch upward; thus, the equations of motion in Section II have been re-derived using this assumption.

Another difference in the equations of motion and the simulation program is the choice of orientation of reference axes. In the simulation, the pitch axis is x , the yaw axis is y , and the roll axis is z ; however, the choice of axes adopted for the CESARM was x for pitch, z for yaw, and y for roll. Thus, when re-deriving the equations of motion, the system of nomenclature adopted for CESARM was used. In either case, the dynamics of the wrist are the same.

```

"-----"
"
"      CESARM 2-DOF WRIST MODEL
"
"      Programmer: Jerry Huang
"
"-----"
PROGRAM 2-DOF WRIST

" ---- DEFINITIONS OF STATE VARIABLES AND CONSTANTS ---- "
"
"      XR1 = MOTOR 1 REFERENCE POSITION [V]
"      XR2 = MOTOR 2 REFERENCE POSITION [V]
"
"      TM1 = MOTOR 1 TORQUE OUTPUT [N-M]
"      TM2 = MOTOR 2 TORQUE OUTPUT [N-M]
"
"      E1 = MOTOR 1 POSITION ERROR SIGNAL [V]
"      E2 = MOTOR 2 POSITION ERROR SIGNAL [V]
"
"      X1 = MOTOR 1 POSITION [RAD]
"      DX1 = MOTOR 1 VELOCITY [RAD/S]
"      DDX1 = MOTOR 1 ACCELERATION [RAD/S**2]
"
"      X2 = MOTOR 2 POSITION [RAD]
"      DX2 = MOTOR 2 VELOCITY [RAD/S]
"      DDX2 = MOTOR 2 ACCELERATION [RAD/S**2]
"
"      ** MOTOR 1 IS CONNECTED TO DIFFERENTIAL **
"      INPUT X3 (DIRECT ACTUATOR)
"
"      X3 = DIFFERENTIAL INPUT POSITION [RAD]
"      DX3 = DIFFERENTIAL INPUT VELOCITY [RAD/S]
"      DDX3 = DIFFERENTIAL INPUT ACCELERATION [RAD/S**2]
"
"      ** MOTOR 2 IS CONNECTED TO DIFFERENTIAL **
"      INPUT X4 (DIRECT ACTUATOR)
"
"      X4 = DIFFERENTIAL INPUT POSITION [RAD]
"      DX4 = DIFFERENTIAL INPUT VELOCITY [RAD/S]
"      DDX4 = DIFFERENTIAL INPUT ACCELERATION [RAD/S**2]
"
"      PITCH = PITCH ANGLE
"      YAW = YAW ANGLE
"
"      *** KINEMATIC RELATIONSHIP ***
"
"      X3 + X4      X3 - X4
"      PITCH = -----  YAW = -----
"          2          2
"
"      OR INVERSELY:
"
"      X3 = PITCH + YAW      X4 = PITCH - YAW
"

```

```

" CONSTANTS:
"
"      JM1,JM2 = MOTOR INERTIAS [KG-M**2]
"      IXX = WRIST INERTIA ABOUT X-AXIS
"              (YAW AXIS) [KG-M**2]
"      IYY = WRIST INERTIA ABOUT Y-AXIS
"              (PITCH AXIS) [KG-M**2]
"      IZZ = WRIST INERTIA ABOUT Z-AXIS
"              (ROLL AXIS) [KG-M**2]
"      IXZ = WRIST PRODUCT INERTIA IN XZ-PLANE [KG-M**2]
"
"      KS1,KS2 = CABLE SPRING RATES
"      KM1,KM2 = MOTOR TORQUE CONSTANTS(?) [N-M/VOLT]
"      BM1,BM2 = MOTOR DAMPING [N-M-S/RAD]
"      R1,R2,R3,R4 = SHEAVE RADII [M]
"      N1,N2 = GEAR RATIO [DIMENSIONLESS]
"      W1,W2 = DESIRED POLE FREQUENCY [RAD/S]
"      PSTP = MAGNITUDE OF PITCH STEP INPUT
"      YSTP = MAGNITUDE OF YAW STEP INPUT
"
"-----"

" ----- SET LENGTH OF SIMULATION TIME ----- "

CONSTANT          TSTP = .99

" ----- SET COMMUNICATION INTERVAL ----- "

CINTERVAL          CINT = .005

" ----- INITIALIZE CONSTANTS AND STATE VARIABLES ----- "
" ----- ( 0 - SUBSCRIPT DENOTES INITIAL VALUES ) ----- "

INITIAL

CONSTANT          JM1 = .000706,          JM2 = .000706 ...
                  , IXX = .25288,          IYY = .25700...
                  , IZZ = .01003,          IXZ = .00844 ...
                  , KS1 = 1000000,         KS2 = 1000000 ...
                  , KM1 = .1427,           KM2 = .1427 ...
                  , BM1 = .0500,           BM2 = .0500 ...
                  , R1 = .01905,           R2 = .01905 ...
                  , R3 = .01905,           R4 = .01905 ...
                  , N1 = 4,                 N2 = 4 ...
                  , W1 = 20,                W2 = 20 ...
                  , PSTP = .75,             YSTP = 1.0

" ----- LINEARIZE ABOUT INITIAL POINT ----- "

CONSTANT          PITCH0 = .25,           YAW0 = 0.0

X03 = PITCH0 + YAW0
X01 = X03*N1
X04 = PITCH0 - YAW0
X02 = X04*N2

```

" ----- DEFINE NEW CONSTANTS ----- "

K11 = 2*KS1*R1*R1/(N1*N1)
 K13 = 2*KS1*R1*R3/N1
 K33 = 2*KS1*R3*R3

K22 = 2*KS2*R2*R2/(N2*N2)
 K24 = 2*KS2*R2*R4/N2
 K44 = 2*KS2*R4*R4

" ----- DEFINE LINEARIZED INERTIAS ----- "

" I1 = IXX
 " I2 = (IYY + IZZ)/2
 " I3 = (IYY - IZZ)/2

" *** INVERSE OF MASS MATRIX: ***

" [J2 - J3]
 " [J2 - J3]
 " -1 [-J3 J1]
 " [M] = -----
 " DET [M]

" WHERE:

" J1 = I1 + I2 + I3 COS(X3-X4) - IXZ SIN($\frac{X3-X4}{2}$)

" J2 = I1 + I2 + I3 COS(X3-X4) + IXZ SIN($\frac{X3-X4}{2}$)

" J3 = -I1 + I2 + I3 COS(X3-X4)

" M = DET [M]

" = I1 (I2 + I3 COS(X3-X4) + IXZ² SIN($\frac{X3-X4}{2}$)²)

I1 = IXX
 I2 = (IYY + IZZ)/2
 I3 = (IYY - IZZ)/2

J01 = (I1 + I2 + I3*COS(X03-X04) - 2*IXZ*SIN((X03-X04)/2))/4
 J02 = (I1 + I2 + I3*COS(X03-X04) + 2*IXZ*SIN((X03-X04)/2))/4
 J03 = (-I1 + I2 + I3*COS(X03-X04))/4

J1 = J01
 J2 = J02
 J3 = J03

M = (I1*(I2 + I3*COS(X03 - X04)) ...
 - (IXZ*SIN((X03 - X04)/2))**2)/4

```

"----- CALCULATE SINES & COSINES -----"

A1 = SIN(X03 - X04)
B1 = COS(X03 - X04)
A2 = SIN((X03 - X04)/2)
B2 = COS((X03 - X04)/2)

" ----- EVALUATE FEEDBACK GAINS ----- "
"
"   KT1 = FORWARD LOOP POSITION GAIN (MOTOR 1)
"   KP1 = MOTOR 1 POSITION FEEDBACK GAIN
"   KV1 = MOTOR 1 VELOCITY FEEDBACK GAIN
"   KV3 = DIFFERENTIAL VELOCITY FEEDBACK GAIN
"   KE1 = STEADY-STATE POSITION ERROR GAIN (#1)
"
"   KT2 = FORWARD LOOP POSITION GAIN (MOTOR 2)
"   KP2 = MOTOR 2 POSITION FEEDBACK GAIN
"   KV2 = MOTOR 2 VELOCITY FEEDBACK GAIN
"   KV4 = DIFFERENTIAL VELOCITY FEEDBACK GAIN
"   KE2 = STEADY-STATE POSITION ERROR GAIN (#2)
"
"   THE DESIRED CHARACTERISTIC EQUATION:
"
"       4       3       2 2       3       4
"   S  + 4 WO S  + 6 WO S  + 4 WO S + WO  = 0
"
"   OR
"
"           4
"       (S + WO)  = 0
"
"   THUS YIELDING MULTIPLE ROOTS AT S = - WO;
"
"   THE RESPONSE DESIRED IS CRITICALLY DAMPED
"
" ----- "

KT1 = (JM1*J01*W1**4 - 6*JM1*K33*W1**2 + JM1/J01*K33**2 ...
      + K13**2)/(KM1*K13)

KP1 = (6*JM1*W1**2 - JM1/J01*K33 - K11) ...
      /(KM1*KT1)

KV1 = (4*JM1*W1 - BM1) ...
      /(KM1*KT1)

KV3 = 4*JM1*W1*(J01*W1**2 - K33) ...
      /(K13*KM1*KT1)

KE1 = (K13*(KT1*KM1 - K13) + K33*(K11 + KP1*KT1*KM1)) ...
      /(K13*KT1*KM1)

KT2 = (JM2*J02*W2**4 - 6*JM2*K44*W2**2 + JM2/J02*K44**2 ...
      + K24**2)/(KM2*K24)

KP2 = (6*JM2*W2**2 - JM2/J02*K44 - K22) ...
      /(KM2*KT2)

```



```

"      + --- [      ] [      ] [      ] = [      ] "
"      M [-J3   J1 ] [ 0      K24   0   -K44 ] [ X3 ]   [ 0 ] "
"                                           [      ] "
"                                           [ X4 ] "

```

```

" WHERE:

```

```

"                                     X3-X4
"      C11 = I3*SIN(X3-X4) + IXZ*COS( ----- )
"                                     2
"
"                                     X3-X4
"      C12 = 2*(-I3*SIN(X3-X4) - IXZ*COS( ----- ))
"                                     2
"
"                                     X3-X4
"      C13 = -3*I3*SIN(X3-X4) + IXZ*COS( ----- )
"                                     2
"
"                                     X3-X4
"      C21 = 3*I3*SIN(X3-X4) + IXZ*COS( ----- )
"                                     2
"
"                                     X3-X4
"      C22 = 2*(I3*SIN(X3-X4) - IXZ*COS( ----- ))
"                                     2
"
"                                     X3-X4
"      C23 = -I3*SIN(X3-X4) + IXZ*COS( ----- )
"                                     2
"
" -----

```

```

DDX3 = (1/8*((J2*(I3*A1 + IXZ*B2) - J3*(3*I3*A1 + IXZ*B2))*DX3*DX3 ...
+ (J2*2*(-I3*A1 - IXZ*B2) - J3*2*(I3*A1 - IXZ*B2))*DX3*DX4 ...
+ (J2*(-3*I3*A1 + IXZ*B2) - J3*(-I3*A1 + IXZ*B2))*DX4*DX4) ...
+ J2*K13*X1 - J3*K24*X2 - J2*K33*X3 + J3*K44*X4)/M

```

```

DDX4 = (1/8*((-J3*(I3*A1 + IXZ*B2) + J1*(3*I3*A1 + IXZ*B2))*DX3*DX3 ...
+ (-J3*2*(-I3*A1 - IXZ*B2) + J1*2*(I3*A1 - IXZ*B2))*DX3*DX4 ...
+ (-J3*(-3*I3*A1 + IXZ*B2) + J1*(-I3*A1 + IXZ*B2))*DX4*DX4) ...
- J3*K13*X1 + J1*K24*X2 + J3*K33*X3 - J1*K44*X4)/M

```

```

DX3 = INTEG (DDX3,0.0)

```

```

DX4 = INTEG (DDX4,0.0)

```

```

X3 = INTEG (DX3,X03)

```

```

X4 = INTEG (DX4,X04)

```

```

" ----- DEFINE OUTPUT ----- "

```

```

PITCH = (X3 + X4)/2

```

```

YAW = (X3 - X4)/2

```

```

END $ "DERIVATIVE"

```

```

PROCEDURAL

```

```

"----- UPDATE SINES & COSINES ----- "

A1 = SIN(X3 - X4)
B1 = COS(X3 - X4)
A2 = SIN((X3 - X4)/2)
B2 = COS((X3 - X4)/2)

" ----- UPDATE INERTIA TERMS ----- "

J1 = (I1 + I2 + I3*B1 - 2*IXZ*A2)/4
J2 = (I1 + I2 + I3*B1 + 2*IXZ*A2)/4
J3 = (-I1 + I2 + I3*B1)/4
M = (I1*(I2 + I3*B1) - IXZ*IXZ*A2*A2)/4

" ----- UPDATE FEEDBACK GAINS ----- "
" ----- (ON-LINE GAIN CALCULATION) ----- "

KT1 = (JM1*J1*W1**4 - 6*JM1*K33*W1**2 + JM1/J1*K33**2 ...
      + K13**2)/(KM1*K13)

KP1 = (6*JM1*W1**2 - JM1/J1*K33 - K11) ...
      /(KM1*KT1)

KV1 = (4*JM1*W1 - BM1) ...
      /(KM1*KT1)

KV3 = 4*JM1*W1*(J1*W1**2 - K33) ...
      /(K13*KM1*KT1)

KE1 = (K13*(KT1*KM1 - K13) + K33*(K11 + KP1*KT1*KM1)) ...
      /(K13*KT1*KM1)

KT2 = (JM2*J2*W2**4 - 6*JM2*K44*W2**2 + JM2/J2*K44**2 ...
      + K24**2)/(KM2*K24)

KP2 = (6*JM2*W2**2 - JM2/J2*K44 - K22) ...
      /(KM2*KT2)

KV2 = (4*JM2*W2 - BM2) ...
      /(KM2*KT2)

KV4 = 4*JM2*W2*(J2*W2**2 - K44) ...
      /(K24*KM2*KT2)

KE2 = (K24*(KT2*KM2 - K24) + K44*(K22 + KP2*KT2*KM2)) ...
      /(K24*KT2*KM2)

END $ "PROCEDURAL"

TERMT(T.GE.TSTP)

END $ "DYNAMIC"

END $ "PROGRAM"

```

For simulating the PI controller design, the following statements are substituted in the "INITIAL" section under feedback gains:

$$KT1 = (JM1*J01*(1 - F1)*W1**4 - 6*JM1*K33*W1**2 \dots \\ + JM1/J01*K33**2 + K13**2)/(KM1*K13)$$

$$KP1 = (6*JM1*W1**2 - JM1/J01*K33 - K11) \dots \\ / (KM1*KT1)$$

$$KV1 = (4*JM1*W1 - BM1) \dots \\ / (KM1*KT1)$$

$$KV3 = 4*JM1*W1*(J01*W1**2 - K33) \dots \\ / (K13*KM1*KT1)$$

$$KPI1 = F1*W1**4*JM1*J01/(K13*KM1*KT1)$$

$$KT2 = (JM2*J02*(1 - F2)*W2**4 - 6*JM2*K44*W2**2 \dots \\ + JM2/J02*K44**2 + K24**2)/(KM2*K24)$$

$$KP2 = (6*JM2*W2**2 - JM2/J02*K44 - K22) \dots \\ / (KM2*KT2)$$

$$KV2 = (4*JM2*W2 - BM2) \dots \\ / (KM2*KT2)$$

$$KV4 = 4*JM2*W2*(J02*W2**2 - K44) \dots \\ / (K24*KM2*KT2)$$

$$KPI2 = F2*W2**4*JM2*J02/(K24*KM2*KT2)$$

and the following statements are used in the "DYNAMIC" section in place of the motor torque equations:

" ----- DEFINE INTEGRAL CONTROLLER ----- "

$$EC1 = XR1 - X3 \\ EC2 = XR2 - X4$$

$$ER1 = KPI1*(EC1 + ZTA1*W1*INTEG(EC1,0.0)) \\ ER2 = KPI2*(EC2 + ZTA2*W2*INTEG(EC2,0.0))$$

" ----- CALCULATE MOTOR TORQUES ----- "

$$TM1 = KM1*KT1*(ER1 - KP1*X1 - KV1*DX1 - KV3*DX3 - X3) \\ TM2 = KM2*KT2*(ER2 - KP2*X2 - KV2*DX2 - KV4*DX4 - X4)$$

For simulating the LQ-regulator design, the following statements are substituted in the "INITIAL" section under feedback gains:

KP1 = 778.90	\$ KP2 = 778.90
KV1 = 7.2	\$ KV2 = 7.2
KP3 = -2367.2	\$ KP4 = -2367.2
KV3 = 42.30	\$ KV4 = 42.30
KC1 = 127.10	
KC2 = 0.085	
KC3 = -199.0	
KC4 = 20.5	

and the following statements are substituted in the "DYNAMIC" section for motor torques:

```
" ----- CALCULATE MOTOR TORQUES ----- "
```

$$TM1 = KM1 * (- KP1 * X1 - KV1 * DX1 - KV3 * DX3 - KP3 * (X3 - XR1) \dots$$

$$- KC1 * X2 - KC2 * DX2 - KC4 * DX4 - KC3 * (X4 - XR2))$$

$$TM2 = KM2 * (- KP2 * X2 - KV2 * DX2 - KV4 * DX4 - KP4 * (X4 - XR2) \dots$$

$$- KC1 * X1 - KC2 * DX1 - KC4 * DX3 - KC3 * (X3 - XR1))$$

APPENDIX C

MISCELLANEOUS SAMPLE OUTPUT

On the following pages are sample output from MACSYMA and Matrix_x. The solution of the pole-placed feedback gains in Section III symbolically is shown on page 110. On page 111, the LQ-regulator capability of Matrix_x is shown; inputs to the program are state matrices, **A** and **B**, weighting matrices, **Q** and **R**, and outputs are the closed-loop eigenvalues and regulator gains.

This is REX MACSYMA Release 309.5.

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Type describe(trade_secret); to see Trade Secret notice.

```
(c1) t: [(bml+kv1*kt1*km1)/jml = 4*wo, (k11+kp1*kt1*km1)/jml+k33/j1 =
6*wo^2, (-k13*kv3*kt1*km1+k33*(bml+kv1*kt1*km1))/(j1*jml) =
4*wo^3, (k13*(kt1*km1-k13)+k33*(k11+kp1*kt1*km1))/(j1*jml) = wo^4];
```

$$(d1) \left[\frac{km1 \cdot kt1 \cdot kv1 + bml}{jml} = 4 \cdot wo, \frac{km1 \cdot kp1 \cdot kt1 + k11}{j1} + \frac{k33}{j1} = 6 \cdot wo^2, \right.$$

$$\left. \frac{k33 \cdot (km1 \cdot kt1 \cdot kv1 + bml) - k13 \cdot km1 \cdot kt1 \cdot kv3}{j1 \cdot jml} = 4 \cdot wo^3, \right.$$

$$\left. \frac{k33 \cdot (km1 \cdot kp1 \cdot kt1 + k11) + k13 \cdot (km1 \cdot kt1 - k13)}{j1 \cdot jml} = wo^4 \right]$$

```
(c2) solve(t, [kt1, kp1, kv1, kv3]);
```

$$(d2) \left[[kt1 = \frac{j1^2 \cdot jml^4 \cdot wo^2 - 6 \cdot j1 \cdot jml^2 \cdot k33 \cdot wo^2 + jml^2 \cdot k33^2 + j1 \cdot k13^2}{j1 \cdot k13 \cdot km1}, \right.$$

$$kp1 = \frac{6 \cdot j1 \cdot jml^2 \cdot k13 \cdot wo^2 - jml^2 \cdot k13 \cdot k33 - j1 \cdot k11 \cdot k13}{j1^2 \cdot jml^4 \cdot wo^2 - 6 \cdot j1 \cdot jml^2 \cdot k33 \cdot wo^2 + jml^2 \cdot k33^2 + j1 \cdot k13^2},$$

$$kv1 = \frac{4 \cdot j1 \cdot jml \cdot k13 \cdot wo - bml \cdot j1 \cdot k13}{j1^2 \cdot jml^4 \cdot wo^2 - 6 \cdot j1 \cdot jml^2 \cdot k33 \cdot wo^2 + jml^2 \cdot k33^2 + j1 \cdot k13^2},$$

$$kv3 = - \frac{4 \cdot j1^2 \cdot jml \cdot wo^2 - 4 \cdot j1 \cdot jml^3 \cdot k33 \cdot wo^3}{j1^2 \cdot jml^4 \cdot wo^2 - 6 \cdot j1 \cdot jml^2 \cdot k33 \cdot wo^2 + jml^2 \cdot k33^2 + j1 \cdot k13^2}]]$$

```

*****
*                                     *
*                                     *
*               (TM)                 *
*          MATRIX      V6.0          *
*               X                   *
*                                     *
* (C) Copyright 1982,1983,1984,1985,1986 *
*                                     *
*          Integrated Systems, Inc.    *
*                                     *
*****

```

Type: WHO for list of variables.
 WHAT for list of functions and commands.
 BRIEF for brief topical descriptions.
 HELP for full topical information.

< A

```

A      =
1.0D+04 *
0.0000  0.0000  0.0001  0.0000  0.0000  0.0000  0.0000  0.0000
0.0000  0.0000  0.0000  0.0001  0.0000  0.0000  0.0000  0.0000
-0.1708  0.0014  0.0000  0.0000  0.0427 -0.0003  0.0000  0.0000
0.0014 -0.1708  0.0000  0.0000 -0.0003  0.0427  0.0000  0.0000
0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0001  0.0000
0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0001
7.7104  0.0000  0.0000  0.0000 -1.9278  0.0000 -0.0071  0.0000
0.0000  7.7104  0.0000  0.0000  0.0000 -1.9278  0.0000 -0.0071

```

< B

```

B      =
1.0D+05 *
0.0000  0.0000
0.0000  0.0000
0.0000  0.0000
0.0000  0.0000
0.0000  0.0000
0.0000  0.0000
2.5443  0.0000
0.0000  2.5443

```

< Q

```

Q      =
1.0D+04 *
7.3697  0.9887 -0.1239 -0.0646 -2.3688 -0.4933 -0.0192 -0.0013
0.9887  7.3697 -0.0646 -0.1239 -0.4933 -2.3688 -0.0013 -0.0192
-0.1239 -0.0646 0.0024 0.0018 0.0410 0.0236 0.0003 0.0001
-0.0646 -0.1239 0.0018 0.0024 0.0236 0.0410 0.0001 0.0003
-2.3688 -0.4933 0.0410 0.0236 0.7656 0.2144 0.0061 0.0009
-0.4933 -2.3688 0.0236 0.0410 0.2144 0.7656 0.0009 0.0061
-0.0192 -0.0013 0.0003 0.0001 0.0061 0.0009 0.0000 0.0000
-0.0013 -0.0192 0.0001 0.0003 0.0009 0.0061 0.0000 0.0000

```

< R

```

R      =

```

```

1.0D+04 *

2.0426    0.0000
0.0000    2.0426
<> RHO

RHO      =

5.0000D-06
<> [EV, KR]=REGULATOR(A, B, Q, RHO*R)

KR      =

-847.2808  -57.1270   13.8716   6.5271  270.9800   38.7256   2.2128   0.0001
-57.1270 -847.2808   6.5271  13.8716   38.7256  270.9800   0.0001   2.2128

EV      =

1.0D+05 *

-5.6296
-5.6296
-0.0008
-0.0006
-0.0004
-0.0003
-0.0002
-0.0001

```

VITA

Jerry Huang was born on June 14, 1964 in Gainesville, Florida to Ju-Chin and Pi-Chen Kuan Huang. In 1965, he and his family moved to Cookeville, Tennessee, where his father taught Engineering Science and Mechanics and Civil Engineering at Tennessee Technological University. He is the second of three children, all of which studied Mechanical Engineering at The University of Tennessee, Knoxville. Sister, Sybil received her Bachelor of Science degree in 1983; Jerry received his B.S. in 1986, and brother Johnny is due to receive his B.S. in 1989.

Jerry obtained his high school degree from Cookeville High School, graduating ninth in a class of over 400. His school activities included varsity tennis and sports editorship of *The Charger*, Cookeville High School's nationally honored newspaper. He received his Master of Science degree in Mechanical Engineering from The University of Tennessee, Knoxville in June of 1988, and will be subsequently employed by G.E. Aircraft Engines in Cincinnati, Ohio.

He enjoys tennis, golf, snow skiing, and other assorted sporting activities. His personal interests are major league baseball, playing acoustic guitar, and trivia pursuit.