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I am submitting herewith a dissertation written by Subhashini B. Chitta entitled “Nonlinear Localized Dissipative Structures for Long-Time Solution of Wave Equation.” I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Aerospace Engineering.

John S. Steinhoff, Major Professor

We have read this dissertation
and recommend its acceptance:

Dr. K. C. Reddy

Dr. Lloyd M. Davis

Dr. Gary A. Flandro

Accepted for the Council:

Carolyn

R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Nonlinear Localized Dissipative Structures for Long-Time Solution of Wave Equation

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Presented for the
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Subhashini B. Chitta
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DEDICATION

This thesis is dedicated to my father, Dr. Chandra S. Chitta, who has been my inspiration and who has taught me to keep up the spirit all the time and not let any failure obstruct the ambitions. I also dedicate this to my mother, Achyuta L. Chitta, who taught me the value of education and raised me to be the person I am today.

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ABSTRACT

In this dissertation, a new numerical method, “Wave Confinement” (WC), is developed to efficiently solve the linear wave equation. This is similar to the originally developed “Vorticity Confinement” method for fluid mechanics problems. It involves modification of the discrete wave equation by adding an extra nonlinear term that can accurately propagate the pulses for long distances without numerical dispersion/diffusion. These pulses are propagated as stable codimension-one surfaces and do not suffer phase shift or amplitude exchange in spite of nonlinearity. The pulses remain thin unlike conventional higher order numerical schemes, which only converge as N (number of grid cells across the pulse) becomes large. The additional term does not interfere with conservation of the important integral quantities such as total amplitude, centroid. Also, properties like varying index of refraction, diffraction, multiple reflections are included and tested.

The generated short pulses can be best described as solitary waves, which can recover the shape after a collision due to non destructive interaction between the pulses. Within the pulse, the dissipative effects due to the numerical errors are balanced with those of nonlinearity and the pulse will retain its original form and speed even after many collisions. The pulse is also used as a carrier wave to propagate other properties such as direction. Wave equation solutions in the high frequency approximation can be generated using the carrier wave approach. WC, together with Keller’s Approximation is then used to capture diffraction effects from a straight edge.

Scattering over complex bodies can be modeled with no use of complicated adaptive grid generation schemes around the bodies. The confinement term smoothes the boundary and prevents stair casing effects but the boundary remains thin. Validation studies have been performed for a number of real flow models and compared to the exact solutions. It is observed that the solutions match quite well with the exact solution. A new approximation for long range propagation of high frequency waves, the “Local Parabolic Method”, is introduced. There is a wide range of applications such as radio wave propagation, cell phone communications, target detection, etc. This approximation has a number of advantages over the existing paraxial approximation used to simulate radio wave propagation.

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Chapter 1

INTRODUCTION

The main objective of this dissertation is to develop a new and efficient algorithm to accurately solve the linear wave equation. The idea comes from an already existing method “Vorticity Confinement” (VC), which was developed for a vast range of fluid dynamics problems [1]. Hence, the new method “Wave Confinement” (WC) is named after the existing approximation (VC). The linear wave equation can describe different classes of wave phenomena. Such classes include acoustics, electromagnetics, microwave theory, etc. This dissertation focuses on scalar wave propagation over long distances. Effects to be accommodated include variation of the index of refraction, multiple scattering from complex surfaces, and some cases of diffraction.

1.1 Motivation and Historical Background

To begin with, a brief historical background of the exploration and exploitation of the wave equation is discussed. The grasp over the wave equation has been continually broadened by great minds to make it more and more powerful. A lot of effort has been expended to handle the problems of computing the wave equation. The basic nature of the wave equation is propagation of a disturbance through space and time. There has been a lot of research done to derive solutions to the wave equation, which goes way back to the 1700’s. A solution for wave equation was first published by Jean le Rond d’Alembert in 1747. His article contains the first appearance of the wave equation in print. A little later, in 1748, Leonhard Euler presented a new approach in deriving a solution. He first noticed that the solution to the wave equation is solely dependent on the initial conditions. He extended the concept of the Taylor series (developed by Brook Taylor in 1712) by deducing a solution by superposition of Taylor overtones. This led Daniel Bernoulli, in 1753 to claim that a general solution can also be expressed by an infinite trigonometric series. Above all, Jean-Baptiste-Joseph Fourier achieved the breakthrough. In 1807,

he used trigonometric series and developed functions in the series named after him. There are many other milestones in the later history of wave equation solutions.

One of the most significant milestones has come with the advent of modern computers, numerical techniques to solve the wave equation were first brought into practice around the 1970s. A lot of effort has been expended to solve complex problems for which analytical solutions do not exist. For many cases involving variations in properties, the partial differential equation (pde) becomes too complicated to be solved by explicit analytic formulae. The inability to derive closed form solutions for the wave equation is overcome by numerical approximations, which have been very useful for propagating quantitative information and also to understand the solutions qualitatively. It is important to understand that numerical methods are not just about solving finite difference equations for a discrete system. Much attention has to be paid to the essential physics of the problem. For example, the solution has to conserve the important quantities that describe the propagation. Higher order schemes have been introduced to reduce the error estimates but the error depends strongly on the wavelength. The accuracy of higher order schemes increases as the number of points across the pulse becomes relatively large. As the present study is more involved in treating a number of short waves, propagating long distances on feasible computational grids, the propagating functions cannot be more than 2–3 cells wide, and hence higher order schemes are not useful. A new approximation has been derived to solve the linear wave equation, while keeping in mind the important quantities such as total amplitude and centroid speed. Also, the new method can propagate thin pulses on a grid with no discretization errors (produced due to discretization of partial differential equation).

1.2 Related Methods

Some of the already existing methods that are in use to treat wave propagation problems are briefly discussed to provide an introduction to the newly developed WC with respect to the m. This is required to understand the limitations of existing methods and at the same time, to provide advantages of WC under certain circumstances when other methods fail.

In the high frequency approximation, Eikonal methods give fairly good results. Multiple arrival times and caustics can also be accommodated using recent developments [2]. These methods require extra independent variables and complex data management schemes. On the other hand, WC can treat short wavelength effects beyond the capabilities of Eikonal methods. Multiple reflections, including multiple scattering from surfaces and ducts (formed due to a varying index of refraction) are accurately computed. In many such cases, the source (antenna) will be relatively small compared to the domain of importance. To accurately capture the effects of antennae, the physical space has to be divided into separate regions that can be computed sequentially – such as the region near the antenna and the far field. Propagation through multiple grids is achieved with no anomalous reflections from grid interfaces. This feature can be used by WC to easily decompose the computational domain into separate domains, which can be treated sequentially, thereby greatly reducing memory requirements and allowing parallel computation. This does not appear to be feasible with straightforward implementations of the Lippmann Schwinger integral equation approaches [3], where each region of space is connected to other regions by a Green's function (assuming varying index of refraction).

Another high frequency approximation is the Ray Tracing technique, which is a Lagrangian method, treating all surfaces as a large number of markers, which are propagated by solving ordinary differential equations or, for constant index of refraction, by using algebraic functions [4]. Ray Tracing methods become complex due to the difficulty in capturing diverging functions. Also, these methods cannot capture interference and diffraction effects. A convenient structured computational grid is used in WC, which does not have any special data handling requirements. This grid is suitable for representing variations in the index of refraction, topographic boundary conditions and the geometries of scattering objects. Also, transmission and scattering amplitudes and phases, if they are to be useful, should be represented on a convenient, structured grid. This will be necessary to accomplish one of our objectives – to develop a transmission “map” that can be used in real-time to interpolate these quantities on to an aircraft flight path that is not known beforehand. Ray tracing methods do not lend themselves easily to such interpolation [5]. Also, they are, of course, limited to the high frequency limit.

1.3 Plan of Dissertation

To start with, a brief plan of this dissertation is provided. The new method “Wave Confinement” is developed to propagate thin waves as solitary waves over arbitrary distances accurately.

First, in Chapter 2, the WC method is examined using the simplest one sided wave equation. A qualitative and quantitative study of parameters that affect the convergence is also described. A nonlinear ϵ pde for m of WC is derived and investigated. Also, physical and numerical properties of WC are studied. Test cases are used to demonstrate the capability (pros and cons) of this new method.

WC is then applied to the full wave equation in Chapter 3. Numerical experiments have been performed to study the behavior of WC. Important phenomena of wave propagation such as wave interactions and wave reflections are investigated. Formulations in higher dimensions and their solutions are discussed.

In Chapter 4, various test cases involving the treatment of realistic problems such as scattering, refraction, and reflection from complex boundaries are simulated and tested. The WC results are compared to Lagrangian markers computed using Ray Tracing techniques for validation purpose. The advantages of WC over Ray Tracing methods are examined. Also, transition of the WC to and from Ray Tracing is studied and discussed (This may not be necessary but may be desired for use with existing codes). Results are presented to demonstrate the smooth transition.

The time domain wave fronts were used to generate constant phase contours. The procedure to reproduce them for the Helmholtz equation is explained in Chapter 5. Also, the formulation required to capture the arrival times is presented and discussed. The new method is also used to propagate important wave properties by means of a carrier wave, and the results are discussed in Chapter 5. The carrier wave acts as a wave packet, which has the necessary information to describe the propagated quantities. This property is used to generate sharp shadows and, along with Keller’s approximation, knife edge diffraction is simulated. These properties are also used in another new approximation: “Locally Parabolic Method” (LPM), to capture interference. Simple test cases such as propagation at higher angles and diffraction from a single slit are simulated and tested. The new approximation is developed from an already existing Standard Parabolic or Paraxial Method [6]. The existing paraxial

approximation, which can only propagate information in a pre-defined direction, is improved and extended to treat arbitrary propagation directions. The results are demonstrated and discussed in Chapter 5.

Atmospheric properties such as varying index of refraction and multiple reflections from complex boundaries are included in the formulation. Finally, even though a lot of study has been already done, there is still scope for future developments due to its vast applications in various fields. Techniques used in this project should lead to new, more efficient applications for a number of other problems.

Chapter 2

APPROACH

A new method to propagate scalar “sheets” with thin delta function-like profiles is developed. The idea is to “capture” thin features directly on a computational grid by using discretized equations. This is similar to shock capturing (as opposed to “shock fitting” that uses markers to define the geometry), which goes back to the work of Lax and others [7]. The main objective of this chapter is to provide the basic derivation and principles of the newly developed numerical method, “Wave Confinement” (WC), in order to better understand the numerical and physical properties of the method. These thin features are treated as nonlinear solitary waves that can propagate on the computational lattice indefinitely with no numerical dissipation. Both for the sake of brevity and simplicity, the linear wave equation in one dimension is first studied. However, applications to higher dimensions are provided in later chapters.

The one dimensional wave equation for a travelling wave of speed, c , is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}. \quad 2.1$$

The wave equation may then be written in the factored form as

$$\left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \phi = 0 \quad 2.2$$

with general solution (d’Alembert’s solution) as $\phi = f(x + ct) + g(x - ct)$ where f and g are arbitrary functions advecting in the forward and backward directions respectively. Because the wave equation is linear, each of these functions can in turn be the sum of many functions of the arguments $x \pm ct$. To make it more straightforward and easy to understand the characteristics of the method, it is important to first start the study of wave phenomena with the one direction (one sided) wave equation. Solution to the full wave equation is discussed in Chapter 3.

The emphasis in this chapter is primarily on presenting the newly developed method (WC) in detail.

2.1 Advection Equation

2.1.1 Introduction

Let us consider a scalar field, ϕ , which satisfies the one sided wave equation (advection equation) advecting with constant speed c :

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = 0 \quad 2.3$$

Evidently, if the above equation is numerically solved using a conventional, discrete finite difference scheme, it tends to develop discretization errors. The conventional Eulerian schemes [8] suffer dissipation no matter what the order is. Many higher order schemes have been proposed to reduce the numerical diffusion/dispersion, but they only reduce the error if the number of grid points across the pulse, N is relatively large. As the present work involves treating thin pulses $\sim 2-3$ grid cells in size, the method is not necessarily going to be more accurate with increasing order. To keep the pulse confined on the discrete domain, numerical dissipation/dispersion must be counter-acted by adding a new term to the advection equation that will not interfere with the essential properties of the pulse. This term is called the “Confinement” term and hence the method is named “Wave Confinement” (WC). Ideas of adding extra terms to prevent numerical dissipation were previously proposed by Harten [9] to capture 1D contact discontinuities in compressible flows. The discretized form can only conserve a limited number of physical quantities of the pulse corresponding to the limited number of grid nodes across the pulse. Here, these are taken to include total amplitude at each grid point and the centroid speed. The modified form of equation (2.3) with the confinement term is then

$$\frac{\partial \phi}{\partial t} = -c \frac{\partial \phi}{\partial x} + E \quad 2.4$$

where, $E = \frac{\partial^2 F}{\partial x^2}$. The idea is that the second order form of F acts as a “pulse shaping” term and, as long as $F \rightarrow 0$ as $|x| \rightarrow \infty$, the quantities of interest are conserved. A simple form of F can be taken as

$$F = \mu' \phi - \varepsilon' \Phi, \quad 2.5$$

where μ' , ε' . Here, constants and Φ is the harmonic mean given as

$$\Phi(x) = \frac{3}{\frac{1}{\phi(x-h)} + \frac{1}{\phi(x)} + \frac{1}{\phi(x+h)}}, \quad 2.6$$

where h is the distance between two points, which tends to zero in the continuous limit. The confinement term is chosen such that it acts as an “expansion” or positive diffusion for shorter wavelengths and a “contraction” or negative diffusion for longer wavelengths. Also, the method must have a nonlinear term, otherwise some modes will decouple and diverge. The nonlinear term prevents the amplitude to escape to long wavelengths and can keep the pulse confined to a few grid cells.

2.1.2 WC as a Nonlinear Partial Differential Equation

To better understand the properties of the WC method, the harmonic mean, Φ , given in equation (2.6) is approximated as a partial differential equation. The Taylor expansions for $\phi(x+h)$ and $\phi(x-h)$ to second order are

$$\begin{aligned} \phi(x+h) &= \phi(x) + h\phi'(x) + \frac{h^2}{2!}\phi''(x), \\ \phi(x-h) &= \phi(x) - h\phi'(x) + \frac{h^2}{2!}\phi''(x), \end{aligned} \quad 2.7$$

where $\phi' = \partial_x \phi$, $\phi'' = \partial_x^2 \phi$. By substituting the Taylor expansions into equation (2.6), and neglecting the higher order terms, the nonlinear term, Φ , is then approximated as

$$\Phi = \phi - \frac{2h^2}{3} \left(\frac{\phi'^2}{\phi} \right) + \frac{h^2 \phi''}{3} + h^4 \left(\frac{\phi'^4}{3\phi^3} + \frac{\phi'^2 \phi''}{3\phi^2} \right), \quad 2.8$$

where $h \rightarrow 0$ in continuous limit and terms of the order h^4 can be neglected. Using equation (2.8), equation (2.4) becomes

$$\partial_t \phi = -c \partial_x \phi + \partial_x^2 \left(-\alpha \phi - \lambda \partial_x^2 \phi + 2\lambda \frac{(\partial_x \phi)^2}{\phi} \right), \quad 2.9$$

where $\alpha = \varepsilon' - \mu'$, $\lambda = \frac{\varepsilon' h^2}{3}$. All the parameters, μ' , ε' , α , and λ , are chosen to be positive. When $\phi = 0$, the above equation fails but, it is important to note that the discretized form of equation (2.9) used for computations is given in equation (2.19). The harmonic mean in equation (2.109) is still finite for $\phi = 0$.

Equation (2.9) describes the conservation of the following integral quantities: total amplitude,

$$A_T = \int \phi dx, \quad 2.10$$

and centroid speed,

$$\frac{d\hat{x}}{dt} = \frac{\int \phi c dx}{A_T}, \quad 2.11$$

where $\hat{x}$ is the centroid, given by

$$\hat{x} = \frac{\int x \phi dx}{A_T}. \quad 2.12$$

It is important to note that all the terms in equation (2.9) are homogenous of degree one unlike many non-linear equations, which use non-homogenous terms for the nonlinear term [10]. Thus, the confinement does not depend on the scale of the quantity to be convected (which is, of course, a property of the original linear equation (2.1)). Defining the confinement term in the equation (2.9) as the sum of three terms gives,

$$E = E_0 + E_1 + E_2, \quad 2.13$$

where $E_0 = -\alpha \partial_x^2 \phi$, $E_1 = -\lambda \partial_x^4 \phi$, and $E_2 = 2\lambda \partial_x^2 \left(\frac{(\partial_x \phi)^2}{\phi} \right)$. It is interesting that the role

of the second order term (E_0) in equation (2.13) is different from typical nonlinear pde's studied, such as KdV, that harbor solitary waves: In these, the linear second order term is the “expansion” term, and the “contraction”, or “steepener” term is the nonlinear Burgers-like convection: $(\partial_x \phi^2 / 2)$. In WC, the linear second order term, E_0 , *contracts* the pulse and the nonlinear term, E_2 , prevents ϕ from changing sign and transfers amplitude from long wavelength to short, and the fourth order term, E_1 , acts as a diffusion for short wavelengths and prevents the solution from diverging. The amplitude imparted to the field by the WC terms remains confined and propagated as stable wave packets without dispersion/diffusion. The appearance of ϕ in the denominator of equation (2.9) makes E_2 diverge as $\phi \rightarrow 0$. This prevents ϕ from changing sign. Since the total amplitude, A_T , is conserved, the integral of ϕ over any finite region cannot then diverge. In the discretized version defined in Section 2.2, none of the grid values can diverge in appropriate ranges of α and λ . This ensures realizability if ϕ is a physical quantity. Smolarkiewicz [11] also has rearranged the discretized convection equation so that there is such a term in the denominator for this reason.

By rearranging the terms, equation (2.9) may be expressed as

$$\partial_t \phi = -c \partial_x \phi + \partial_x^2 \left(\frac{1}{\psi^2} (-\alpha \psi + \lambda \partial_x^2 \psi) \right), \quad 2.14$$

where $\psi = 1/\phi$. In the convecting frame, $\xi = x - ct$, the above pde simply reduces to the heat equation,

$$\partial_t \phi = \partial_\xi^2 F \quad 2.15$$

where $F = \frac{1}{\psi^2} (-\alpha \psi + \lambda \partial_\xi^2 \psi)$. At convergence, $F \rightarrow 0$, and therefore reduces to

$$-\alpha\psi + \lambda\partial_\xi^2\psi = 0. \quad 2.16$$

It can be seen that the above equation is similar to the simplest form of Sturm-Liouville eigenvalue equation with $\beta = \sqrt{\alpha/\lambda}$. The solution to the equation (2.16) will then be of the form,

$$\psi = C_1(e^{\beta\xi} + e^{-\beta\xi}). \quad 2.17$$

Then the solution for ϕ is

$$\phi = A \operatorname{sech}[\beta(\xi)], \quad 2.18$$

where C_1 , C_2 and A are arbitrary constants.

2.2 Discretization

Using first order upwind in time and second order centered in space, the discretized form of equation (2.4) is written as

$$\phi_j^{n+1} = \phi_j^n - \frac{\nu}{2}(\phi_{j+1}^n - \phi_{j-1}^n) + \delta_j^2 F_j^n \quad 2.19$$

where $\delta_j^2 F_j^n = F_{j+1}^n - 2F_j^n + F_{j-1}^n$, $\nu = \frac{c\Delta t}{h}$, Δt is the time step and h is the grid cell size. Even on the discrete domain, the confinement term, F , conserves the essential quantities such as total amplitude,

$$A_T = \sum_{j=1}^J \phi_j = \text{const}, \quad 2.20$$

and centroid speed,

$$\nu = \langle j \rangle^n - \langle j \rangle^{n-1} = \frac{\sum_{j=1}^J j \phi_j^n - \sum_{j=1}^J j \phi_j^{n-1}}{A_T}, \quad 2.21$$

where J is the total number of grid points. The confinement term F is defined as

$$F_j^n = \mu \phi_j^n - \varepsilon \Phi_j^n, \quad 2.22$$

where $\mu = \frac{\mu' \Delta t}{h^2}$, $\varepsilon = \frac{\varepsilon' \Delta t}{h^2}$, μ' is a diffusion coefficient and ε' is the confinement coefficient. ε' along with μ' , control the width and rate of convergence of the pulse, and Φ is the nonlinear harmonic mean of N (including $N-1$ neighboring nodes) grid points. For 1D, $N=3$, and,

$$\Phi_j^n = \left[\frac{\sum_{l=-1}^{+1} (\phi_{j+l}^n)^{-1}}{3} \right]^{-1}. \quad 2.23$$

Equation (2.19) would simply reduce to the Euler explicit form, which is unconditionally unstable when $\varepsilon = 0$ and $\mu = 0$. The two (positive) parameters, ε and μ , are determined by the requirements for convergence, such that the small features relax to their solitary wave shape in a small number of time steps and can have an effective support (to have a significant magnitude) of a small number of grid cells. In the semi-discrete limit, the scalar field ϕ relaxes to a definite shape as derived in Section 2.2.1. In the moving frame ($\nu = 0$), equation (2.19) reduces to a discrete heat equation,

$$\phi_j^{n+1} = \phi_j^n + \delta_j^2 F_j^n \quad 2.24$$

that simulates the stationary pulse. At convergence, the positive and negative artificial dissipation terms are balanced with the nonlinearity to produce stable pulses that can stay confined and structurally stable to numerical perturbations (caused, for example, by the discrete convection term or another wave passing through).

2.2.1 Convergence

At convergence, $\delta_j^2 F \rightarrow 0$, $F \rightarrow 0$ and $(\mu\phi - \varepsilon\Phi) \rightarrow 0$. If $N = 3$ in the harmonic mean, the equation for F at a point is

$$\mu \left(\frac{1}{\phi_{j-1}} + \frac{1}{\phi_j} + \frac{1}{\phi_{j+1}} \right) - \frac{3\varepsilon}{\phi_j} = 0. \quad 2.25$$

Solutions to equation (2.25) that satisfy the boundary conditions: $F \rightarrow 0$, $|j| \rightarrow \infty$ are of the form,

$$\phi \rightarrow A \operatorname{sech}[\gamma(j - j_0)], \quad 2.26$$

where j_0 is the approximate position of the centroid. Here, γ is a pulse width coefficient and is determined by substituting equation (2.26) into (2.25):

$$\cosh \gamma = \left[\frac{1}{2} \left(\frac{3\varepsilon}{\mu} - 1 \right) \right]. \quad 2.27$$

Furthermore, the above equation can be reduced to a quadratic form in e^γ ,

$$e^{2\gamma} - \left(\frac{3\varepsilon}{\mu} - 1 \right) e^\gamma + 1 = 0, \quad 2.28$$

for which the solution is

$$e^\gamma = \frac{b \pm \sqrt{b^2 - 4}}{2} \quad 2.29$$

or

$$\gamma = \ln \left(\frac{b \pm \sqrt{b^2 - 4}}{2} \right), \quad 2.30$$

where $b = \left(\frac{3\varepsilon}{\mu} - 1 \right)$. The solution is real if $b^2 - 4 > 0$ and this requires that $\varepsilon/\mu \geq 1$.

If $\varepsilon = 0$, F acts as a positive diffusion for all wavelengths and the solution decays. When $\mu = 0$, the solution becomes unstable even if $\varepsilon = 0$. So, a minimal amount of diffusion is essential for the stability purpose. If $\mu \leq \varepsilon$, F acts as negative diffusion for longer wavelengths so that the features remain confined and stable to perturbations against spreading.

2.3 Results

2.3.1 Stationary Pulse

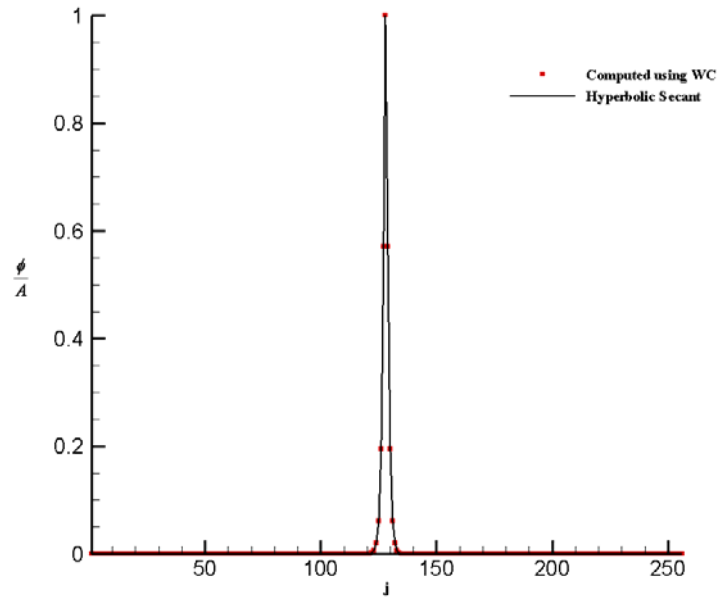
For the evaluation of the scheme and to demonstrate the efficient properties of the WC method, a number of tests are performed and presented. To show convergence, a single point pulse of arbitrary amplitude chosen to be 2,

$$\phi_j = \begin{cases} 2, & j = j_0 \\ 0, & j \neq j_0 \end{cases} \text{ at } n = 0, \quad 2.31$$

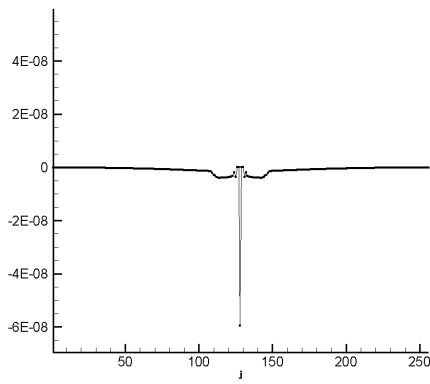
with parameters $\nu = 0$, $\varepsilon = 0.3$, $\mu = 0.2$ and $j_0 = 128$ is simulated on a grid of 256 points using periodic boundary conditions. The discretized equation used for this computation is

$$\phi_j^{n+1} = \phi_j^n + F_{j+1}^n - 2F_j^n + F_{j-1}^n. \quad 2.32$$

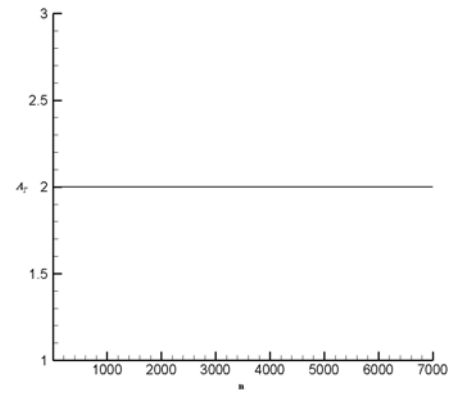
When the solution reaches convergence, the pulse is relaxed to a hyperbolic secant given by equation (2.26). In Figure 2.1a, the solid line is the algebraically calculated hyperbolic secant with width given by equation (2.30), and the points are computed using equation (2.32). For better comparison, the normalized function, ϕ/A is plotted against the hyperbolic secant. It can be clearly seen that the solution matches quite well with the analytically calculated hyperbolic secant. The difference between analytical and converged WC solutions is shown in Figure 2.1b, which appears to be quite low. The integral quantity, or sum, is conserved as shown in Figure 2.1c. When



(a)



(b)



(c)

Figure 2.1: Pulse simulation using equation (2.32). The initial conditions for this computation are defined in equation (2.31).

- a) Comparison to exact
- b) Error of converged solution
- b) The total sum plotted for 7000 time steps.

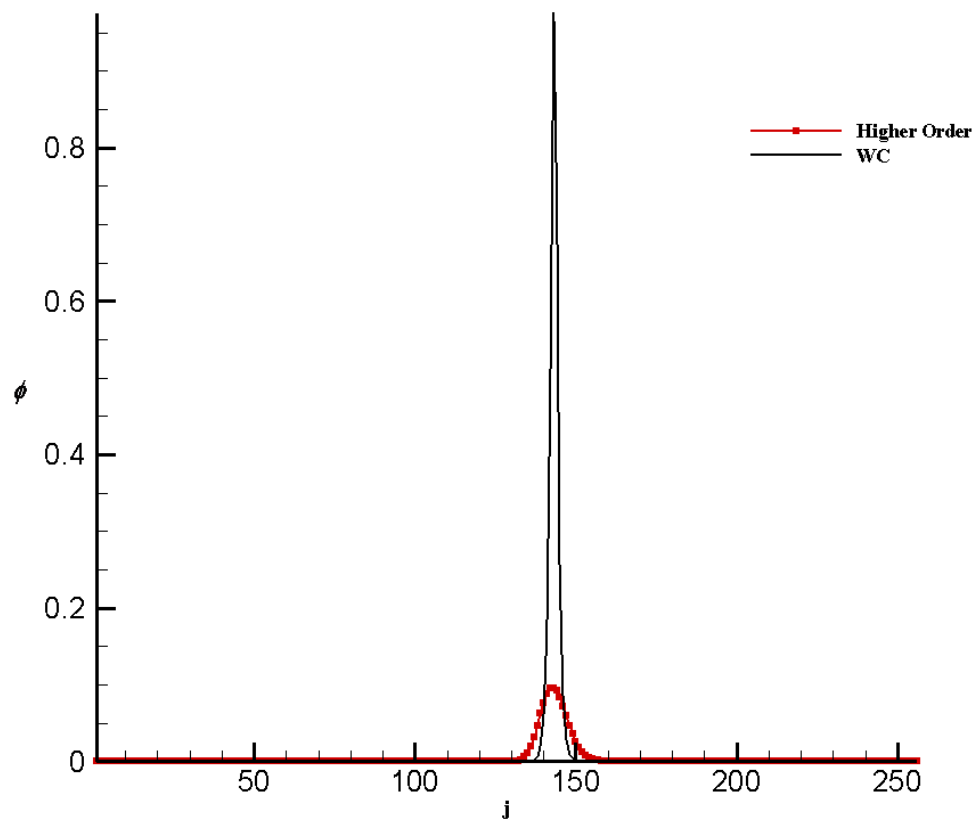


Figure 2.2: Comparison to higher order method. The pulse is propagated for 100 time steps using both WC and a higher order method.

discretized form of equation (2.3) is solved using a higher order method, the solution quickly spreads to a large number of grid points. It can be seen in Figure 2.2 that the solution from a higher order method spreads by large amount after only 100 times steps, but the confinement method keeps the pulse compact.

The width of the converged solution is independent of the initial width (although restricted to delta function-like forms). For example, for a constant μ and ε , the solution will converge to equation (2.26) with a final pulse width coefficient, γ_f (which is only dependent on the parameters, μ and ε). The above statement is demonstrated using the initial condition,

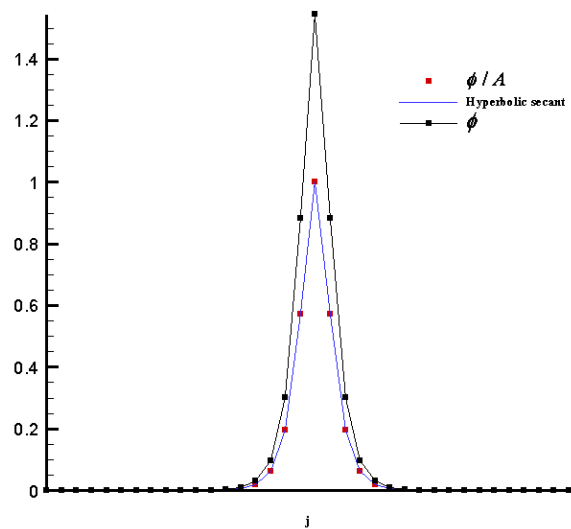
$$\phi_j = A_0 \text{sech}[\gamma_0(j - j_0)] \quad \text{at } n = 0, \quad 2.33$$

for different values of γ_0 , which is a measure of initial width, $\mu = 0.2$ and $\varepsilon = 0.3$, and therefore $\gamma_f = 1.1588$ from equation (2.30). In Figures 2.3a, 2.3b, and 2.4a, $\gamma_0 = 0.75, 1.5, 2.5$ respectively. The computed solutions from equation (2.32), ϕ and $\frac{\phi}{A}$, are plotted against an exact hyperbolic secant of final width coefficient, γ_f . It can be seen that WC solution converges to a hyperbolic secant of the same coefficient γ_f for all initial γ_0 values tested. The final solution converges to

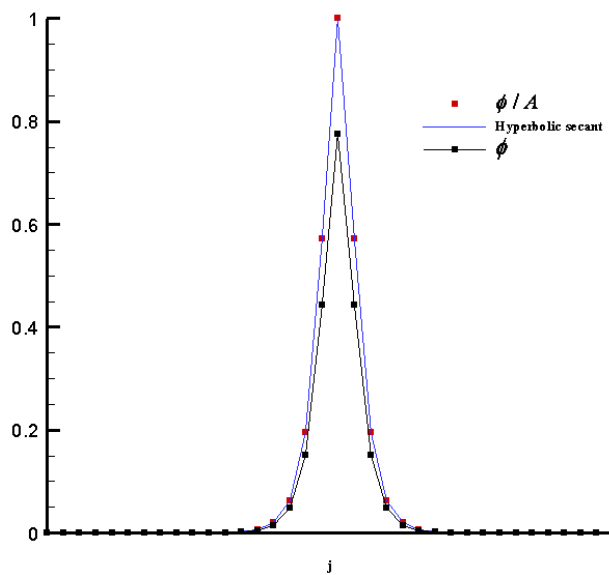
$$\phi_j = A_f \text{sech}[\gamma_f(j - j_0)] , \quad 2.34$$

where the converged amplitude factor, A_f , in equation (2.34) is dependent on the initial conditions. It is already known that the total amplitude, A_T , stays constant throughout the entire computation as can be seen in Figure (2.4b). In the continuous limit, for a hyperbolic secant (started initially as in equation (2.33)), the total amplitude,

$$A_c = \int_{-\infty}^{+\infty} \phi dx = \frac{A_0 \pi}{\gamma_0} = \text{const} . \quad 2.35$$



(a)

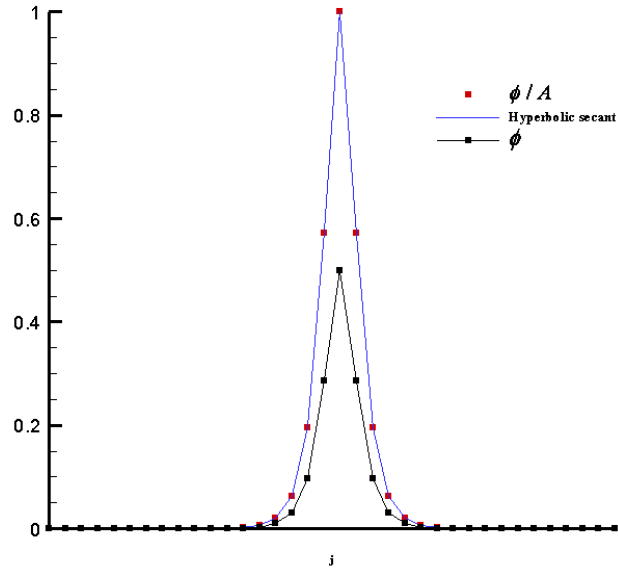


(b)

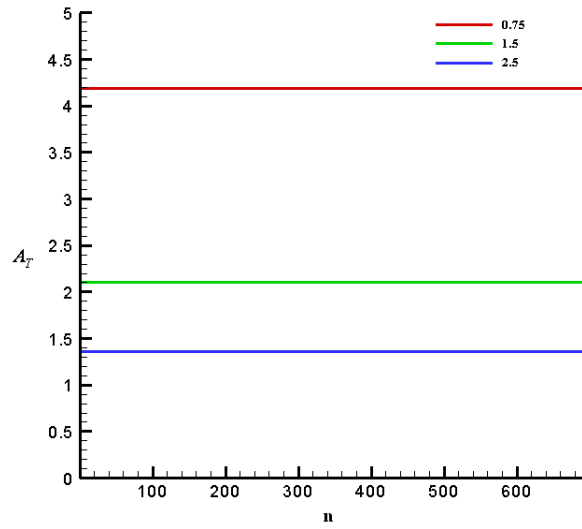
Figure 2.3: Comparison to analytically computed hyperbolic secant.

a) $\gamma_0 = 0.75, \varepsilon = 0.3, \mu = 0.2$

b) $\gamma_0 = 1.50, \varepsilon = 0.3, \mu = 0.2$



(a)



(b)

Figure 2.4: WC computation for different pulse width coefficients.

(a) $\gamma_0 = 2.5, \varepsilon = 0.3, \mu = 0.2$.

(a) Conserved sum for $\gamma_0 = 0.75, \gamma_0 = 1.5$ and $\gamma_0 = 2.5$

The same is approximately valid on a discrete domain

$$\frac{A_0\pi}{\gamma_0} = \frac{A_f\pi}{\gamma_f} = A_T = \text{const} . \quad 2.36$$

Therefore, the converged amplitude factor, A_f , is

$$A_f \approx \frac{A_T\gamma_f}{\pi} . \quad 2.37$$

Equation (2.37) is validated for different pulse width coefficients and presented in Table 2.1. A_f is computed using the above equation (2.37) and A_{WC} is computed using WC. The error of A_{WC} compared to A_f can be seen to be quite low and computed values from WC match quite well with values calculated using equation (2.37).

The convergence of ϕ can be shown in a simpler way when F is further reduced to,

$$F = \mu\phi - \varepsilon\Phi = \alpha(\gamma)\phi \quad 2.38$$

where

$$\alpha(\gamma) = \mu - \frac{3\varepsilon}{2\cosh(\gamma) + 1} . \quad 2.39$$

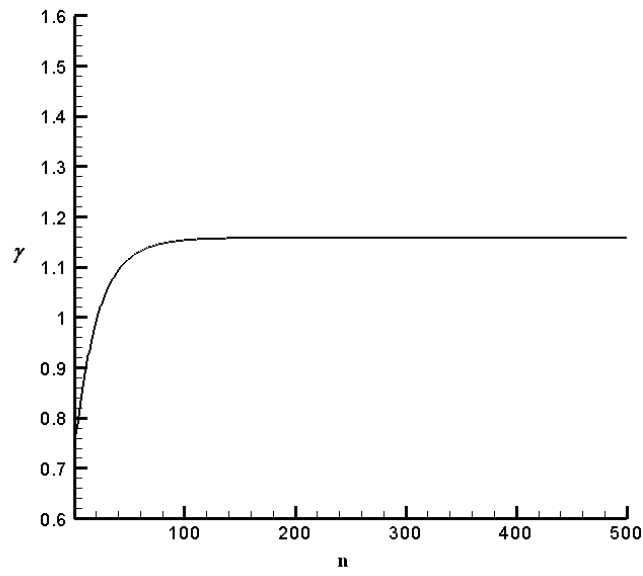
Therefore the discretized heat equation in simpler form is given as,

$$\phi_j^{n+1} = \phi_j^n + \delta_j^2 \alpha(\gamma) \phi_j^n . \quad 2.40$$

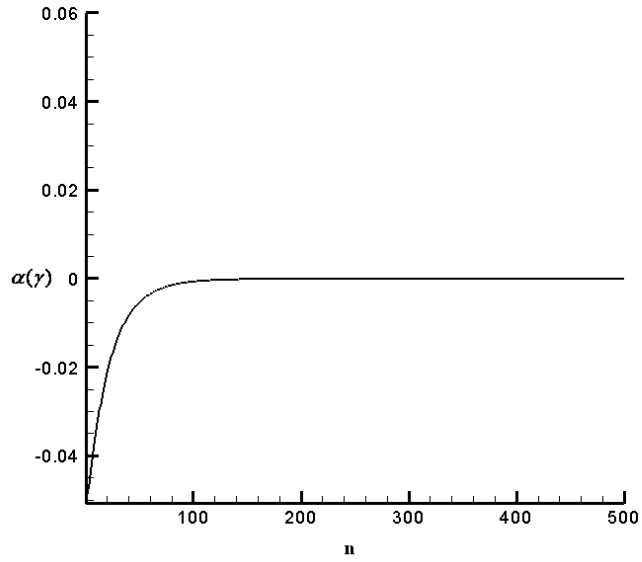
At convergence, $\alpha(\gamma) \approx 0$. For an initial width coefficient, $\gamma_0 = 0.75$, the pulse in equation (2.33) is computed and the value of γ is plotted for each time step in Figure 2.5a. It is obvious that the width coefficient converges to γ_f as defined in equation (2.30). The value $\alpha(\gamma)$ defined in equation (2.39) is shown in Figure 2.5b. When the pulse reaches convergence, $\alpha(\gamma)$ tends to zero as mentioned above.

γ_0	A_c	A_{WC}	A_f	$ A_f - A_{WC} $
0.25	12.566	4.6346	4.5906	0.044
0.5	6.2831	2.3237	2.3157	0.008
0.75	4.188	1.5455	1.5447	0.0008
1.0	3.1422	1.1581	1.1590	0.0009
1.5	2.1060	0.77686	0.7768	0.00006
2	1.6163	0.59614	0.5961	0.00004
2.5	1.3555	0.49998	0.49999	0.00001

Table 2.1: Error estimate for A_{WC} for different values of γ_0 .



(a)



(b)

Figure 2.5: Convergence

(a) Pulse width converges approximately to γ given by equation (2.30)

(b) $\alpha(\gamma)$ in equation (2.40) tends to zero as the pulse converges

2.3.1 Convecting Pulse

When $\nu \neq 0$, the discrete form of the convection is given as,

$$\phi_j^{n+1} = \phi_j^n - \frac{\nu}{2}(\phi_{j+1}^n - \phi_{j-1}^n) + \delta_j^2 F_j^n. \quad 2.41$$

The solution of the above equation is approximately a hyperbolic secant defined as,

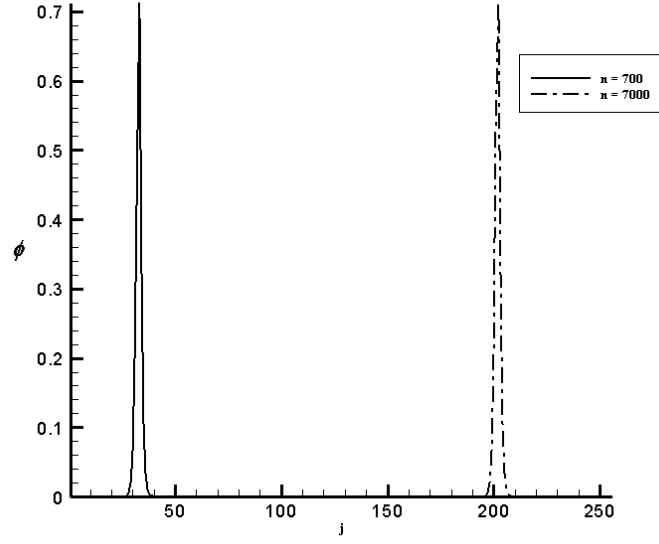
$$\phi_j^n \rightarrow A \text{sech}[\gamma(j - j_0 - \nu n)] \quad 2.42$$

where νn is the phase due to convection and j_0 is the initial position of the centroid. It is shown in Figure 2.6a that the pulse propagates on the lattice for long distances accurately with no systematic increase in numerical errors. Initial conditions used for this computation are the same as equation (2.31). The computation is done on 128 grid points with periodic boundary conditions to approximate an infinite domain. The pulse is plotted for two different times to show its accuracy (no increase in numerical errors). For this test case, the coefficients used are $\varepsilon = 0.3$, $\mu = 0.2$ and $\nu = 0.23$. It can be seen that the pulse “lives” on the grid even after many time steps. Further, the solution remains stable for a wide range of ε . The conservation of total amplitude and centroid speed are shown in Figures 2.6b and 2.7a respectively.

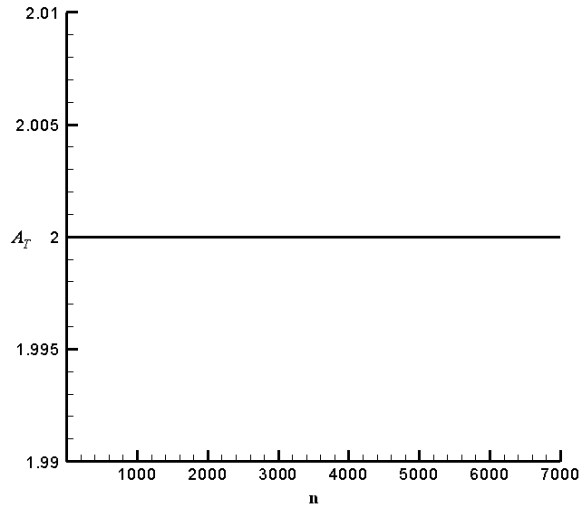
The pulse from equation (2.41) is plotted in a co-moving frame and compared to the solution from equation (2.26). As can be seen in Figure 2.7b, the converged solution does not exactly satisfy equation (2.26). This is due to the dispersion that is produced from the centered difference scheme used for convection term in equation (2.41). For a smooth solution, the artificial diffusion in the confinement term counteracts the effect of dispersion, resulting in the reduction of the effective diffusion coefficient,

$\tilde{\mu}$. Therefore, $\tilde{\mu} < \mu$ and in turn, $\frac{\varepsilon}{\tilde{\mu}} > \frac{\varepsilon}{\mu}$. The same kind of behavior is observed

when the upwind scheme, which is highly diffusive, is used for convection. More artificial diffusion is then added into the system increasing $\tilde{\mu}$ and therefore decreasing $\frac{\varepsilon}{\tilde{\mu}}$. This problem can easily be overcome by increasing the amount of confinement.



(a)

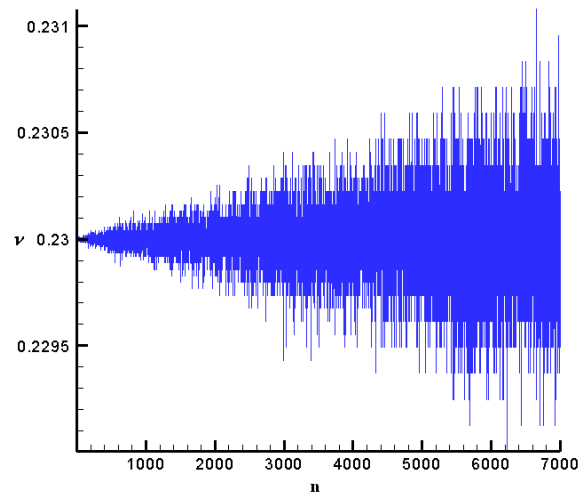


(b)

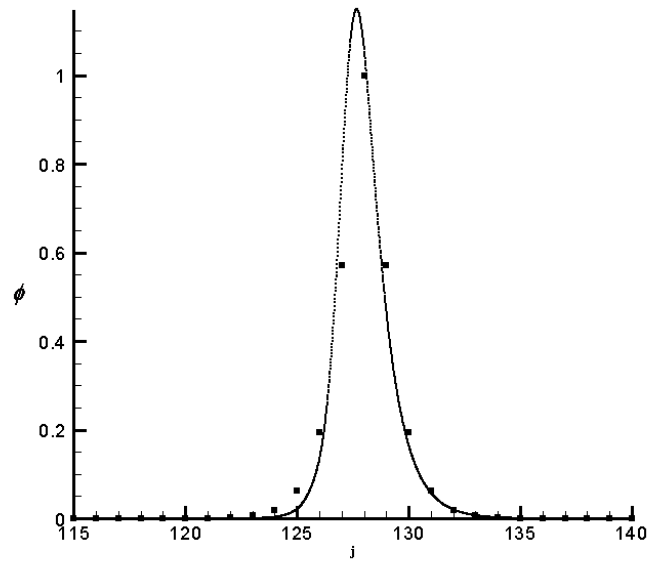
Figure 2.6: The pulse is simulated using equation (2.41). The initial conditions are defined in equation (2.31) and $\nu = 0.23$, $\varepsilon = 0.3$, $\mu = 0.2$.

a) The solid line is the pulse at time $(n) = 700$ and the dashed line is at $n = 7000$

b) The total sum of convecting pulse plotted for 7000 time steps



(a)



(b)

Fig.2.7: Convergence of a convecting pulse

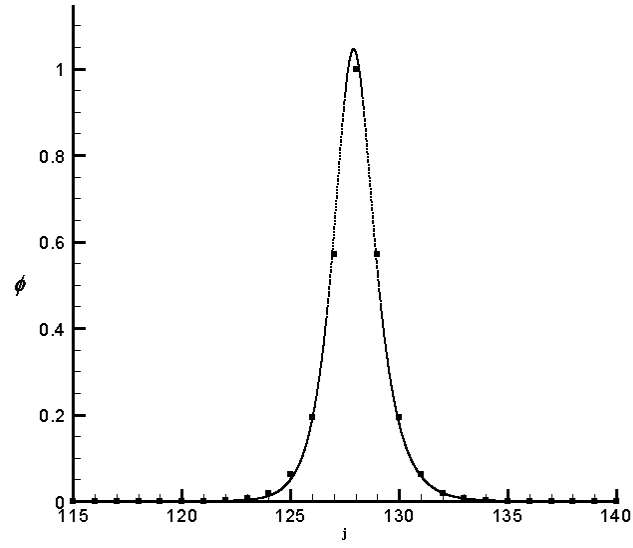
a) Centroid speed, ν , is plotted for 7000 time steps

b) Converged solution in co-moving frame and dots are the analytically computed hyperbolic secant

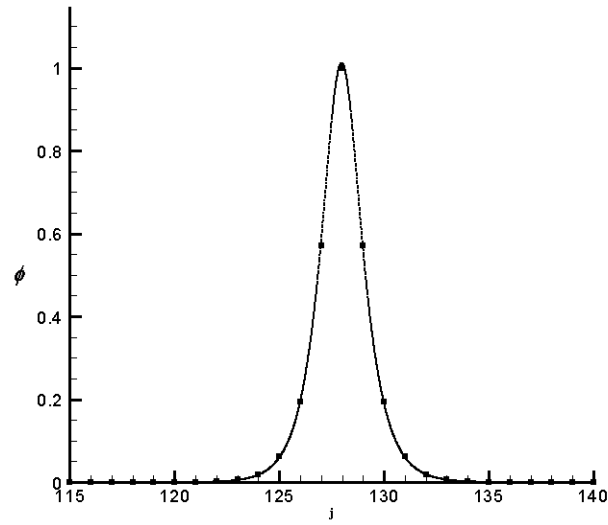
The amount of confinement is defined in 2 steps as below:

$$\begin{aligned} \text{Convection Step: } \phi_j^* &= \phi_j^n - \frac{\nu}{2} (\phi_{j+1}^n - \phi_{j-1}^n) \\ \text{Confinement Step: } \phi_j^{n+1} &= \phi_j^* + \delta_j^2 F(\phi_j^*). \end{aligned} \tag{2.43}$$

The confinement step is repeated depending on the convergence that has to be achieved. It is important to not forget that the repetition of the confinement step does not affect the essential integral quantities. However, it plays an important role in the convergence and is shown in Figures 2.8a and 2.8b. In Figure 2.8a, the confinement step is repeated 2 times every time step and in Figure 2.8b, it is repeated 4 times. It can be observed that in Figure 2.8b, the converged solution matches with the hyperbolic secant of width coefficient given by equation (2.30) to plottable accuracy. The amplitude ϕ varies periodically as it passes through each grid cell. This is demonstrated in Figure 2.9 for $\varepsilon = 0.3$, $\mu = 0.2$ and $\nu = 0.23$. As the centroid travels from j_1 to j_2 , (as shown in Figure 2.9) the amplitude on j_1 decreases and the amplitude on j_2 increases. This is often called “wobble” [12].



(a)



b)

Figure 2.8: The converged pulse computed using equation (2.43)

- a) Converged solution in co-moving frame with confinement step repeated 2 times every time step. Dots are the analytically computed hyperbolic secant
- b) Converged solution in co-moving frame with confinement step repeated 4 times every time step.

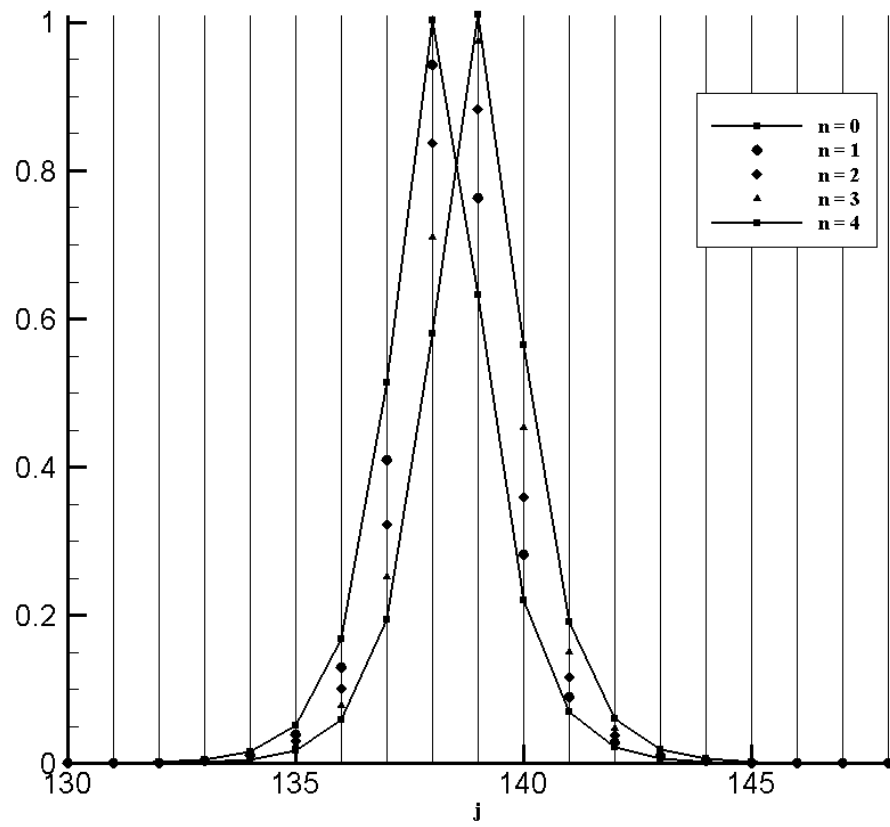


Figure 2.9: Propagation of pulse through a grid cell. The computed quantity, ϕ , plotted for 6 consecutive time steps as the centroid passes a grid cell

Chapter 3

FORMULATION

In the previous chapter, it was established that the W-C method works quite well to solve the advection equation in 1D. To further continue the study, W-C is incorporated in the full wave equation and analyzed. First, a discrete formulation to solve the wave equation is presented. The basic idea of wave propagation is that the important features of a disturbance propagate with a given velocity. The features have delta function-like forms and the present work is focused on propagating these thin pulses on a discrete grid with no systematic discretization errors. In the current chapter, a novel numerical scheme for solving the scalar linear wave equation is presented. In this chapter, formulation, implementation, and numerical testing of the method is described. The wave equation describes many physical phenomena, a notable example being the propagation of acoustic and electromagnetic waves. The partial differential equation to describe full wave propagation (equation (2.1)) is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi \quad 3.1$$

with

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad 3.2$$

in 3D, where ϕ is a scalar field and c is the speed of propagation, which depends on the medium properties. In this chapter, c is assumed to be constant for simplicity, although varying speed conditions are studied in Chapter 4. The solution of the wave equation can then describe the time-dependent propagation of an initial impulse in the environment. When the above equation is solved using W-C, no significant numerical dispersion and dissipation occur. The formulation in 1D is first presented and discussed before studying higher dimensions.

3.1 WC for 1D Wave Equation

As in convection, a nonlinear term, E is added to the one dimensional full wave equation. The wave equation with confinement terms, that control the shape of the pulse, is

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi + E \quad 3.3$$

where $E = \partial_t \partial_x^2 F$. A simple discretization for the above pde will be

$$\phi_j^{n+1} = 2\phi_j^n - \phi_j^{n-1} + \nu^2 \delta_j^2 \phi + \delta_n^- \delta_j^2 F \quad 3.4$$

where $\delta_n^- f = f^n - f^{n-1}$, $\delta_j^2 f_j = f_{j-1} - 2f_j + f_{j+1}$, $\nu = \frac{c\Delta t}{h}$, Δt is the time step and h is the grid cell size, and F is defined as in the advection case.

$$F_j^n = \mu \phi_j^n - \varepsilon \Phi_j^n \quad 3.5$$

where $\mu = \frac{\mu' \Delta t}{h^2}$, $\varepsilon = \frac{\varepsilon' \Delta t}{h^2}$. At convergence (when the propagating pulse shape converges), the solution to the equation (3.4) is

$$\phi \rightarrow A_1 \text{sech}(\gamma(j - j_0 + m)) + A_2 \text{sech}(\gamma(j - j_0 - m)) \quad 3.6$$

where A_1 and A_2 are determined as explained in Section 2.3.1, j_0 is the initial position of the centroid and the pulse width coefficient, γ , is defined as in equation (2.30). Equation (3.6) describes two pulses moving forward and backward. It is shown in Chapter 2 that the addition of WC terms in the form of second derivatives of a function that has short range do not change the propagation speed or the total amplitude. The same is true for the wave equation, if an additional time derivative is applied. The main constraint on the confinement term, F , as in advection is that it forces an initial isolated, propagating short range pulse with single maxima to remain short range and also not develop any additional maxima. An important property of the wave equation is that it is linear. Therefore, properties of the computed solution

should not depend on the amplitude. The behavior of solution computed using equation (3.3) is shown in Figure 3.1. When $\mu = 0$ and $\varepsilon = 0$, the solution becomes unstable. A little amount of diffusion ($\mu = 0.2$) will prevent the unstable behavior but the solution suffers large amount of spreading. Now, when the confinement term ($\varepsilon = 0.3$) is added, the pulse becomes stable and at the same time, remains thin. The dissipative/dispersive effects are balanced with those of the nonlinearity to produce stable localized structures [13].

3.2 Results

In this section, numerical examples are provided to confirm that no discretization errors are developed in the quantities of interest and that there is no significant spreading of the pulses during propagation. The formulation given by equation (3.4) is used for the numerical experiments. While analyzing the formal accuracy of the present scheme, a theoretical error estimate is beyond the scope, because the equation is nonlinear. However, evaluation of numerical error is done experimentally, and demonstrated that the scheme possesses the convergence that was discussed in the previous chapter.

3.2.1 Long-time propagation

To begin with, the propagation of pulses for long distances is observed. This is the idealization of waves without dissipation. Convergence analysis is illustrated by examining the behavior of a delta function as an initial pulse. Let the initial pulse be

$$\phi_j^n = \begin{cases} 2, & j = j_0 \\ 0, & j \neq j_0 \end{cases} \text{ at } n = 0 \text{ and } n = 1 \quad 3.7$$

where $j_0 = 128$ and the computation is done on 256 grid points. The pulse from WC is compared to the solution from a conventional higher order method and is shown in Figure 3.2. The two pulses in the plot are the forward and backward propagating scalar functions. It is obvious that WC preserves the thickness and total amplitude of the pulse with no significant numerical dissipation. Higher order methods do not help because increasing the order of the numerical scheme is only efficient when there are a large number of points across the propagating function.

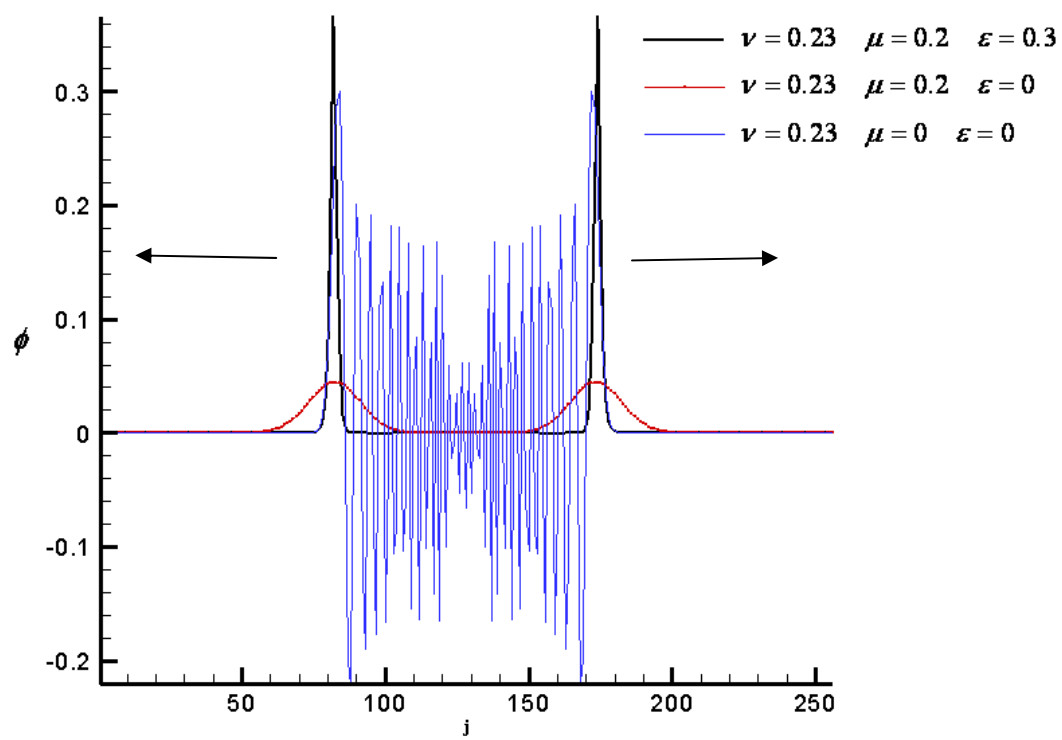


Figure 3.1: Behavior of 1D wave equation solution

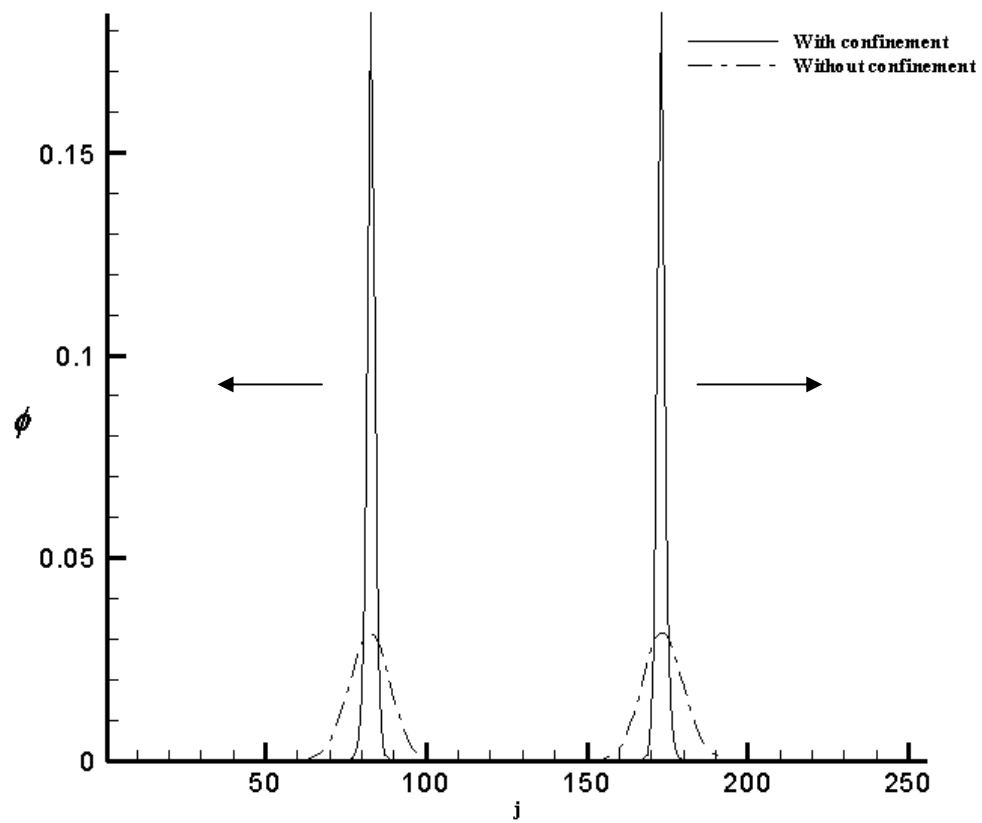


Figure 3.2: Computed pulses with and without confinement. The initial conditions are given in equation (3.7).

The solution is simulated for a much longer time to demonstrate the efficiency of WC in capturing the thin features accurately, as shown in Figure 3.3a. The parameters used for this computation are $\nu = 0.23$, $\mu = 0.2$ and $\varepsilon = 0.3$. With confinement, the pulses tend to retain their shape during propagation without dissipation. The conservation of total amplitude is shown in Figure 3.3b. Using the new numerical scheme, it has been successfully verified that there are no significant numerical errors in long distance propagation.

Using the initial conditions as given below,

$$\phi_j^n = \begin{cases} A_1 \operatorname{sech}(\gamma_0(j - j_0)) & n = 0 \\ \frac{A_1}{A_2} \operatorname{sech}(\gamma_0(j - j_0 - \nu)) & n = 1 \end{cases} \quad 3.8$$

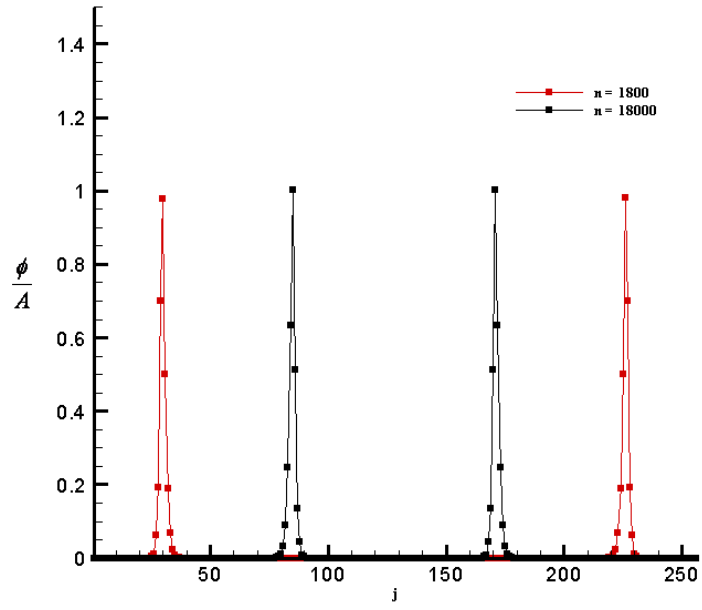
propagation of a single pulse can be observed where $j_0 = 32$ and $\nu = 0.23$ on a grid of 128 points with periodic boundary conditions. In Figure 3.4, the pulse is plotted every 5 time steps on $j - \phi - n$ plane. The pulse converges to a shape close to a

hyperbolic secant with a final pulse width coefficient dependent on the ratio, $\frac{\varepsilon}{\mu}$. It

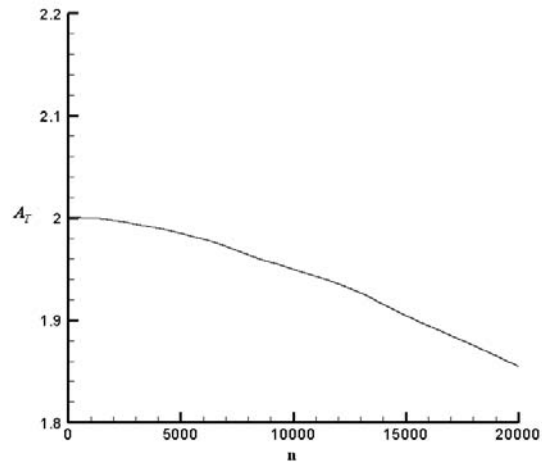
can also be observed that there are no anomalous reflections in the backward direction in spite of the equation being two-way as is demonstrated in Figure 3.4. In Figure 3.5a, the pulse is compared to the exact hyperbolic secant for two different times. It can be seen that the pulse approximately retains its shape even after many time steps. The total amplitude is shown in Figure 3.5b and is seen to decay by only 2.2% in 10000 time steps.

3.2.2 Wave Interaction

Before moving to higher dimensions, the pulse interaction phenomenon is discussed to understand the performance of WC during multiple collisions of pulses. To represent interacting wave equation solutions, when pulses pass through each other, there must be no amplitude exchange or phase shift. Otherwise, the actual wave equation being studied could not be accurately simulated, since it is linear. However, a nonlinear term is required in the WC equation in order to create a solitary wave representation of the pulse, which will be non-diffusing when discretized.



(a)



(b)

Figure 3.3: Forward and backward moving pulses generated by wave equation.

(a) The two pulses for $n = 1800$ (blue) and $n = 18000$ (red)

b) Total Amplitude $A_T^n = \sum \phi_{ij}^n$

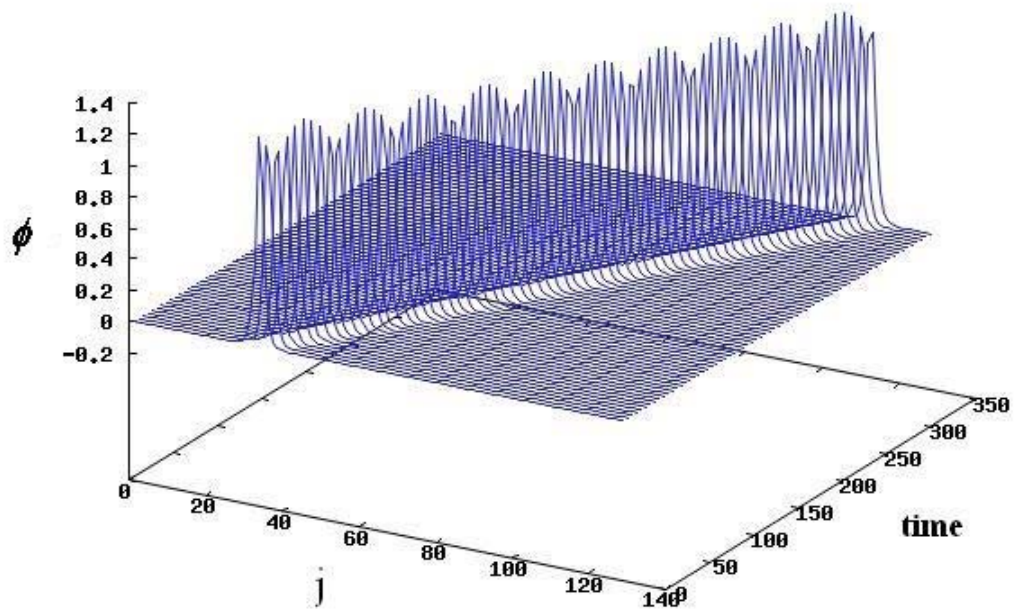
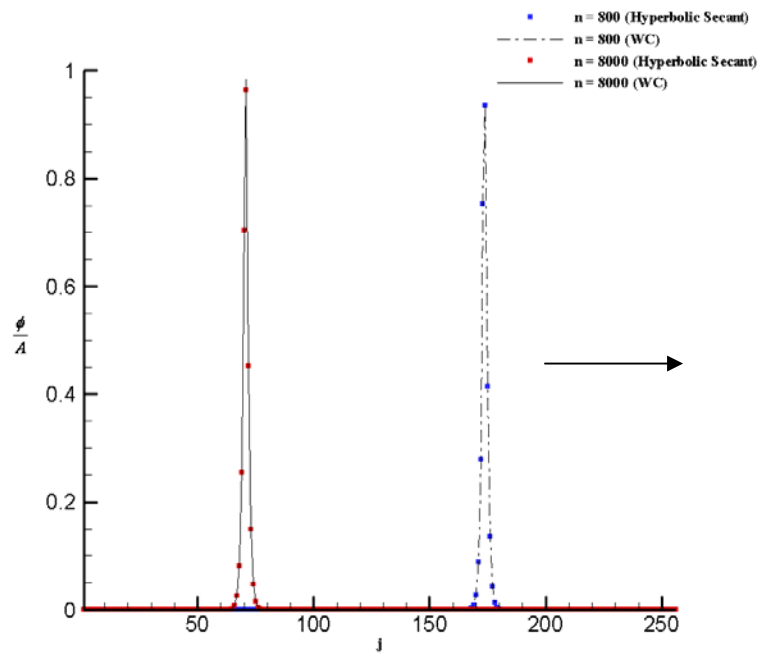
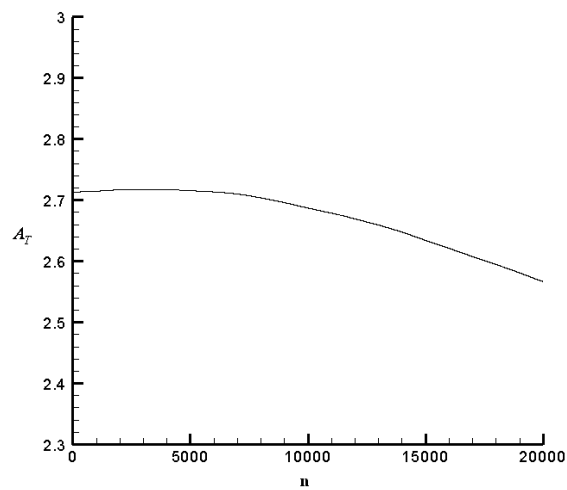


Figure 3.4: Forward propagating pulse with no reflections. The initial conditions for this computation are defined in equation (3.8).



(a)



(b)

Figure 3.5: Long distance propagation of forward propagating pulse.
(a) The pulse is shown for $n = 700$ and $n = 7000$ and compared to the analytically calculated hyperbolic secant.
b) Total Amplitude

Interaction between soliton-like pulses is an important property, which has been studied for a long time using nonlinear equations [14]. It is more appropriate to only re-introduce the nonlinear pde for WC, which was derived in Chapter 2, to establish a relation to other existing nonlinear pde's. The nonlinear pde for WC is

$$\partial_t \phi = \partial_x^2 \left(-\alpha \phi - \lambda \partial_x^2 \phi + 2\lambda \frac{(\partial_x \phi)^2}{\phi} \right). \quad 3.9$$

The above equation has a similarity to the Cahn-Hilliard (CH) equation, which describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. A simple form of the CH equation is

$$\partial_t \phi = \partial_x^2 \left(-\alpha_1 \phi - \alpha_2 \partial_x^2 \phi + \alpha_3 \phi^3 \right) \quad 3.10$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary, positive constants. The commonality includes the same linear terms and the appearance of a Laplacian in front of the non-linear term. This commonality is not surprising since the CH equation was one of the leading models proposed for a phenomenological description for a fluid interface and phase separation (decomposition into pure phases) [15]. The CH equation describes the evolution of concentrated fields (like thin pulses evolved from WC). Though the equations are similar, the resulting non-linear term is not exactly the same as in the commonly used form of the CH equation.

An important point is that other nonlinear pde's like Kdv, which can successfully propagate localized structures, suffer phase shift after collision [16]. This is not present in WC. For the new formulation used, this interaction effect vanishes. This vanishing is due to the fact that both the Laplacian and the time derivative operator operate on the nonlinear term.

The preservation of centroid speed during interaction is shown in Figure 3.6 for an initial condition given by equation (3.7). The centroid positions of forward and backward propagating pulses are plotted. Periodic boundary conditions are used for this simulation over a grid of 256 grid points. It can be noticed that even after many collisions, no jump in the centroid position is noticed after interaction.

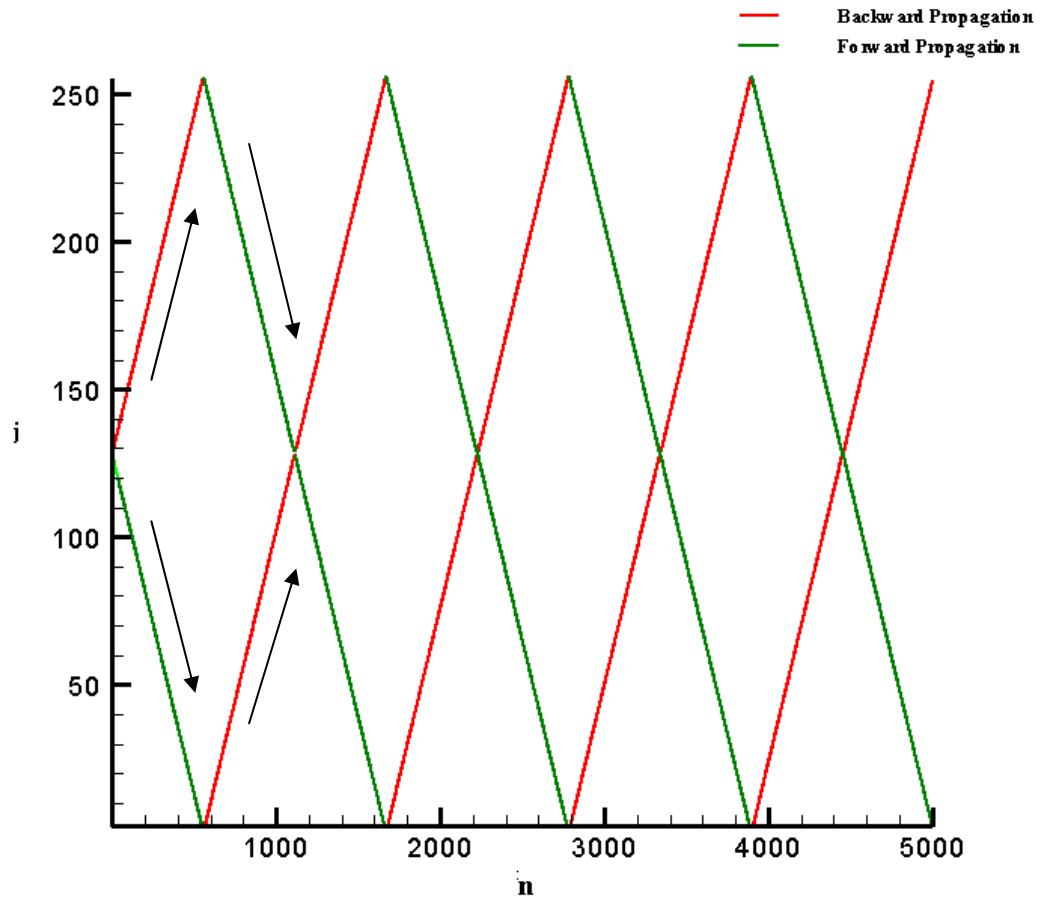


Figure 3.6: Centroid position.Collision of forward ($j + vn$) and backward ($j - vn$) propagating pulses with each other during propagation

The importance of preserving the right speed plays a dominant role in generating constant phase surfaces, which will be discussed in Chapter 5. The same is true for preservation of total amplitude. In Figure 3.7, it can be seen that two pulses of different amplitudes are effectively transparent (after a short relaxation time) to one another and do not lose their amplitudes during collisions. For this computation, the initial conditions used are,

$$\phi_j^n = \begin{cases} 2, & j = j_{01} \\ 1, & j = j_{02} \\ 0, & j \neq j_{01} \text{ and } j \neq j_{02} \end{cases}. \quad 3.11$$

The two initial pulses are first relaxed to hyperbolic secants of respective amplitudes using heat equation ($c=0$). So, the two initial conditions of the two pulses are then

$$\phi_j^n = \begin{cases} A_1 \operatorname{sech}[\gamma(j - j_{01})] + A_2 \operatorname{sech}[\gamma(j - j_{02})], & n = 0 \\ A_1 \operatorname{sech}[\gamma(j - j_{01} - \nu)] + A_2 \operatorname{sech}[\gamma(j - j_{02} + \nu)], & n = 1 \end{cases} \quad 3.12$$

where $A_2 = \frac{A_1}{2}$, $j_{01} = 128$ and $j_{02} = 140$. The pulses are then propagated towards each other to observe the interaction between the pulses.

3.2.3 Reflection from boundaries

When a wave encounters a boundary, (under ideal conditions of perfect reflection and no absorption) it should reflect from the boundary and also undergo a 180 degree phase change. This is an important phenomenon in simulating scattering from obstacles. WC treats the thin pulses irrespective of the sign of the amplitude. Reflection of a pulse from the boundary is shown in Figure 3.8a and the phase change is shown in the total amplitude plot in Figure 3.8b. It can be seen that the reflection is as expected.

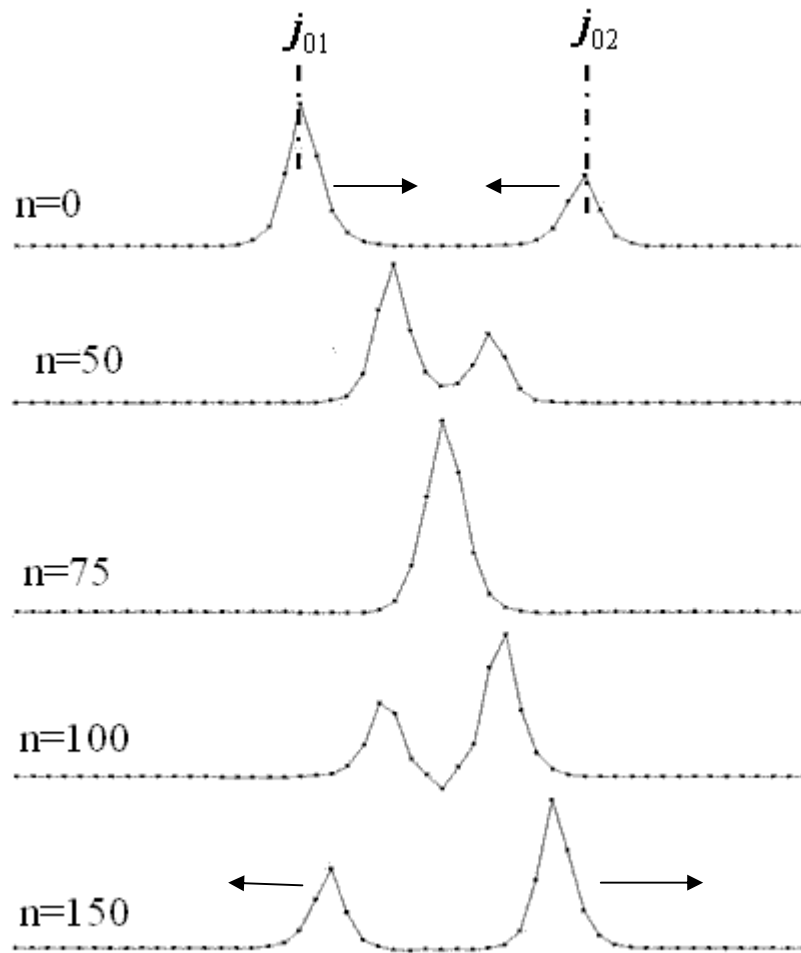
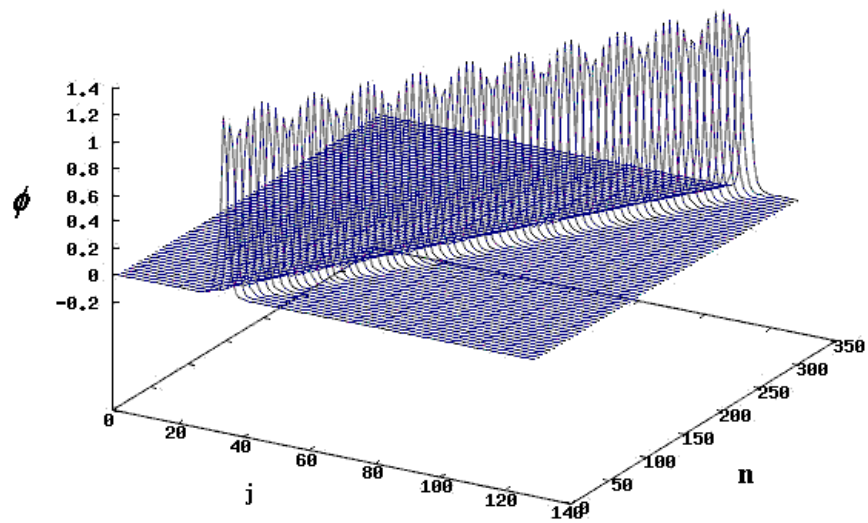
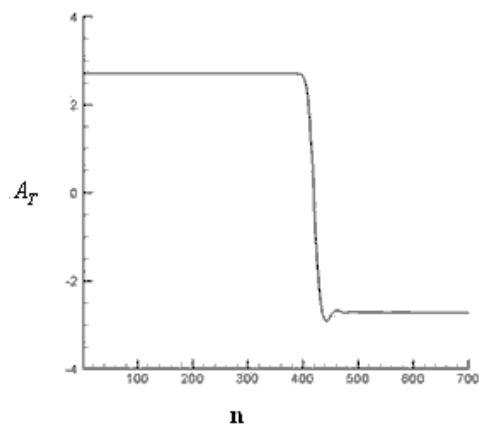


Figure 3.7: Interaction between two pulses of different amplitudes. The pulses undergo head-on collision during propagation for the initial conditions given in equation (3.12).



(a)



(b)

Figure 3.8 : Reflection from boundaries.

a) Change of sign at reflective boundaries (demonstrates 180 degrees phase change).

WC does not have restriction on the sign of the propagating function

b) The total amplitude becomes negative after reflection.

3.3 WC for higher dimensions

3.3.1 2D

In higher dimensions, WC behave s the same way as in 1D. The essential physics is accurately preserved as analyzed in Chapter 2. The discretized equation in 2D is

$$\phi_{i,j}^{n+1} = 2\phi_{i,j}^n - \phi_{i,j}^{n-1} + \nu^2 \nabla^2 \phi + \delta_n^- (\nabla^2 F) \quad 3.13$$

where i is the grid index in x-direction, j is the grid index in y-direction and ∇^2 is the 2D extension of the discrete Laplacian used in 1D . For convenience the aspect ratio of the grid is 1. However, behavior of WC with different aspect ratios is studied in Chapter 4. The nonlinear factor, $\Phi_{i,j}^n$, in the confinement term F is

$$\Phi_{i,j}^n = \left[\frac{(\phi_{i\pm 1,j}^n)^{-1} + (\phi_{i,j\pm 1}^n)^{-1} + (\phi_{i,j}^n)^{-1}}{N} \right]^{-1} \quad 3.14$$

where the number of terms in the sum, $N = 5$. Here, it is assumed that $\phi_{i,j}^n \geq 0$. Negative values can also be accommodated with a small extension. Both μ and ε are positive. Assuming convergence as $n \rightarrow \infty$ (for $\nu = 0$),

$$\nabla^2 (\mu \phi - \varepsilon \Phi) = 0 \quad 3.15$$

for which the solution at convergence is

$$\phi_{i,j} = A \sec h[\gamma(r_{i,j} - r_0)] \quad 3.16$$

where

$$r_{i,j} = x_i \cos \theta + y_j \sin \theta \quad 3.17$$

A , z_0 , θ are constants, $x_i = ih$, $y_j = jh$, h is the grid cell size, and γ is the inverse pulse width. The solution converges to a straight pulse (in 2-D) concentrated about a line at angle, θ and at any position. By substituting the solution in equation (3.15), it is easy to see that γ satisfies

$$\frac{\varepsilon}{\mu} = [1 + 2ch(\gamma h \cos \theta) + 2ch(\gamma h \sin \theta)] / 5. \quad 3.18$$

The solution has translational invariance and γ depends (to a small extent) on orientation, θ . However, the centroid speed and amplitude in the normal direction is independent of θ as can be seen from equation (3.18). To demonstrate long distance propagation with no increasing numerical errors, a diverging circular wave of speed, $\nu = 0.23$ is simulated on a $(128)^2$ cell grid with periodic boundary conditions. Confinement values used were $\mu = 0.2$, $\varepsilon = 0.3$. The initial conditions for this computation are

$$\begin{aligned} \phi_{i,j}^{n=0} &= A_0 \operatorname{sech}[\gamma r_{i,j}] \\ \phi_{i,j}^{n=1} &= \frac{A_0}{A_1} \operatorname{sech}[\gamma(r_{i,j} - \nu)] \end{aligned} \quad 3.19$$

where r is the radius of the wave. The actual wave equation exhibits a “tail” behind a pulse in 2-D, which can be seen to be suppressed by the Confinement, and, effectively, only the steep pulse front is accurately computed. In Figure 3.9, the circular wave for different time steps is presented. It can be seen that the wave persists on the grid for very long time. To provide a close look at the interaction, a part of the field from Figure 3.9 is magnified and shown in Figure 3.10. Also, as in 1-D, there is no discernable interaction between intersecting waves. The waves retain their form and orientation in spite of multiple head-on collisions.

3.3.2 3D

In 3D, the discretized form of the wave equation with the confinement term is

$$\phi_{i,j,k}^{n+1} = 2\phi_{i,j,k}^n - \phi_{i,j,k}^{n-1} + \nu^2 \nabla^2 \phi_{i,j,k}^n + \delta_n^-(\nabla^2 F_{i,j,k}^n) \quad 3.20$$

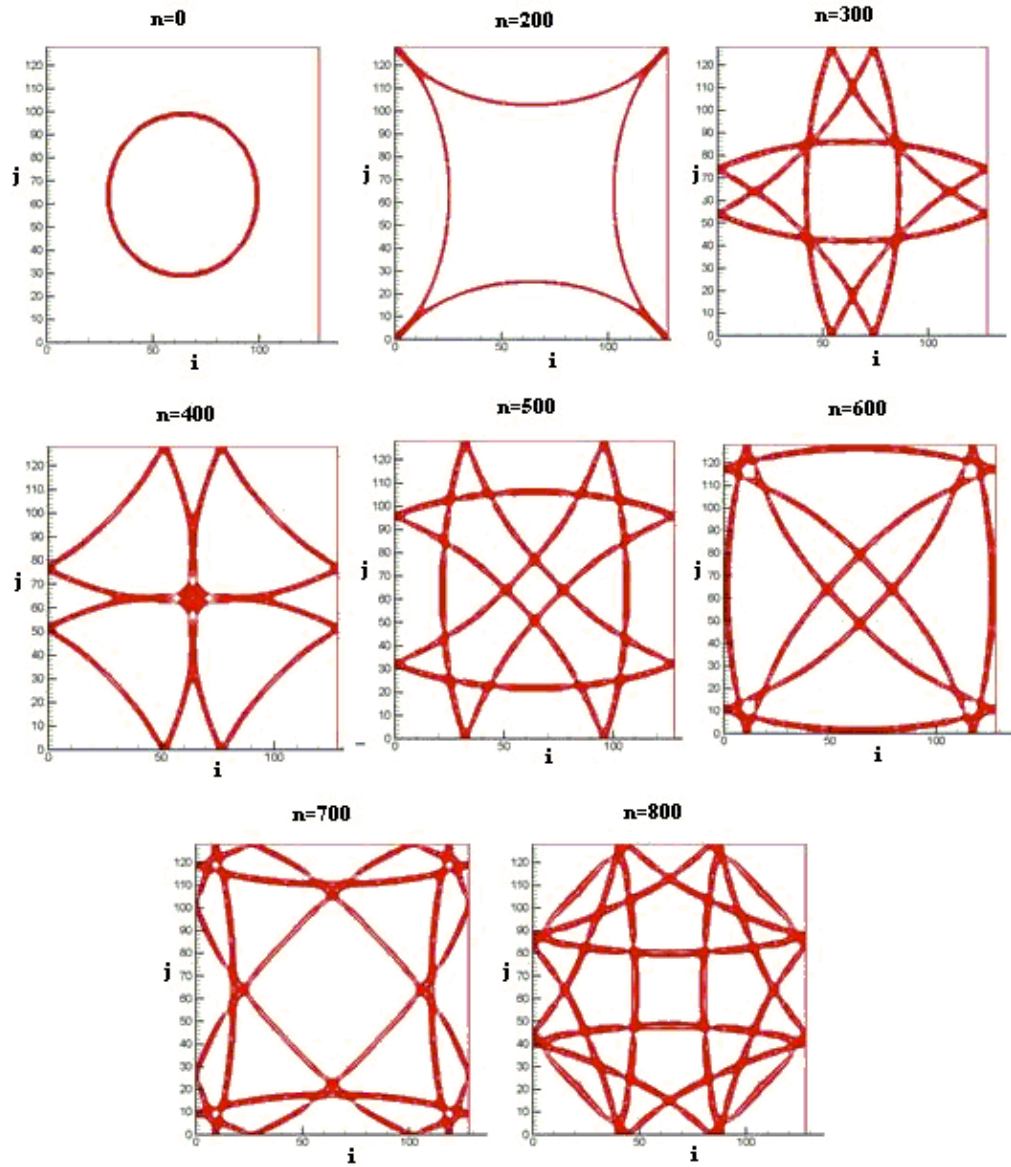


Figure 3.9 : Diverging circular wave front solved using equation (3.13)

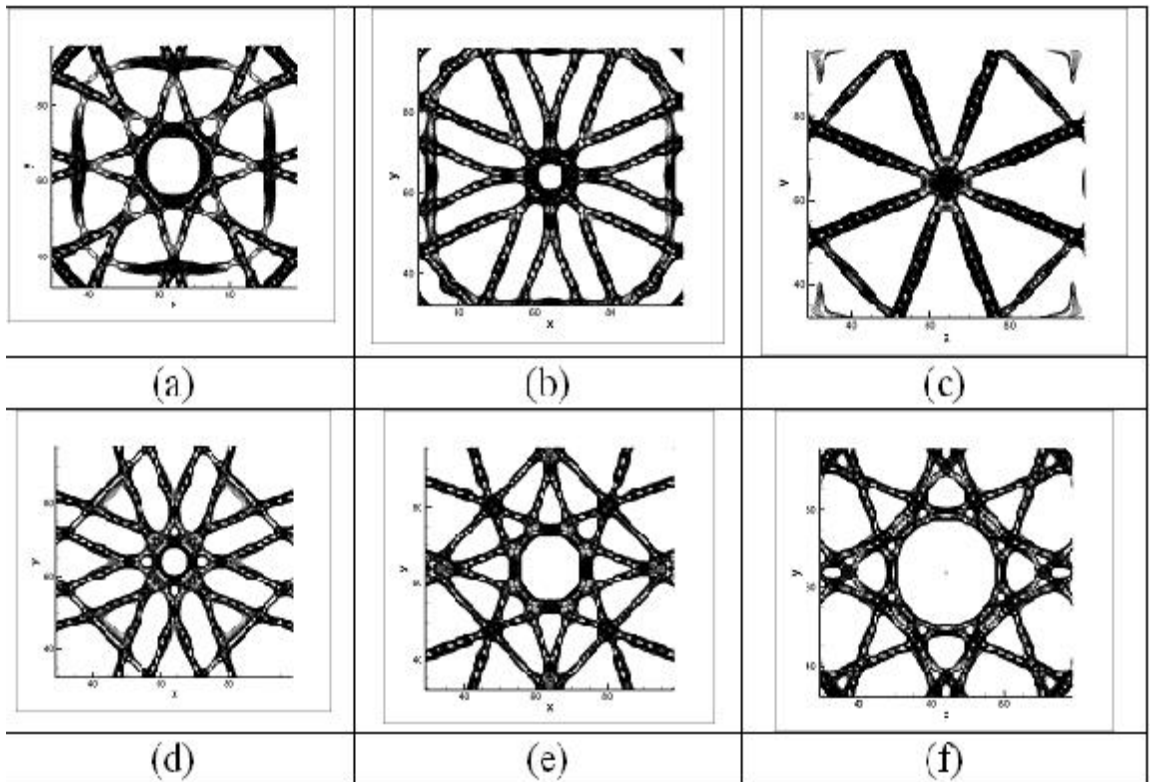


Figure 3.10: Magnified at a part of the field in Figure 3.9

where ∇^2 is the 3D extension of the discrete Laplacian used in 1D, and the nonlinear harmonic mean at a point, $\Phi_{i,j,k}^n$, is

$$\Phi_{i,j,k}^n = \left[\frac{\sum_{\alpha=-1}^{+1} \sum_{\beta=-1}^{+1} \sum_{\chi=-1}^{+1} (\phi_{i+\alpha, j+\beta, k+\chi}^n)^{-1}}{27} \right]^{-1}. \quad 3.21$$

Long distance propagation in 3D is simulated for an expanding, initially spherical wave and is shown in Figure 3.11. The computation is done on a coarse, $(64)^3$ cell grid with periodic boundary conditions. The initial diameter for this computation is 32 grid cells. Confinement values used were $\mu = 0.2$ and $\varepsilon = 0.3$. As in the 2-D case, the wave remains completely confined. Unlike in 2D, the actual solution does not exhibit a tail behind the front and the solution corresponds to the full physical solution rather than just the wave front. The solution at a given point $P(x, y, z)$, depends only on the information on the sphere and not on the interior of the sphere. Thus the interior of the sphere is a lacuna for the solution. It is true for odd numbers of space dimension.

3.5 Conclusion

A new Eulerian technique is introduced for solving the wave equation for short pulses. The method, “Wave Confinement”, reduces to standard Eulerian schemes for smooth, long wavelength pulses. However, unlike conventional schemes, it does not diffuse short pulses. Instead, they are “Confined” and propagate as nonlinear solitary waves that “live” indefinitely on the computational lattice. As such, they can be propagated over indefinitely long distances, while remaining only a few cells thick. These pulses *represent* the short physical pulses and accurately propagate integral quantities at each point along the pulse surface, such as total amplitude, centroid position, pulse width and other desired moments. It is argued that, for thin pulses, the method can easily be implemented in existing codes, allowing them to be extended to treat much shorter wavelengths/shorter pulses, without extensive reformulation.

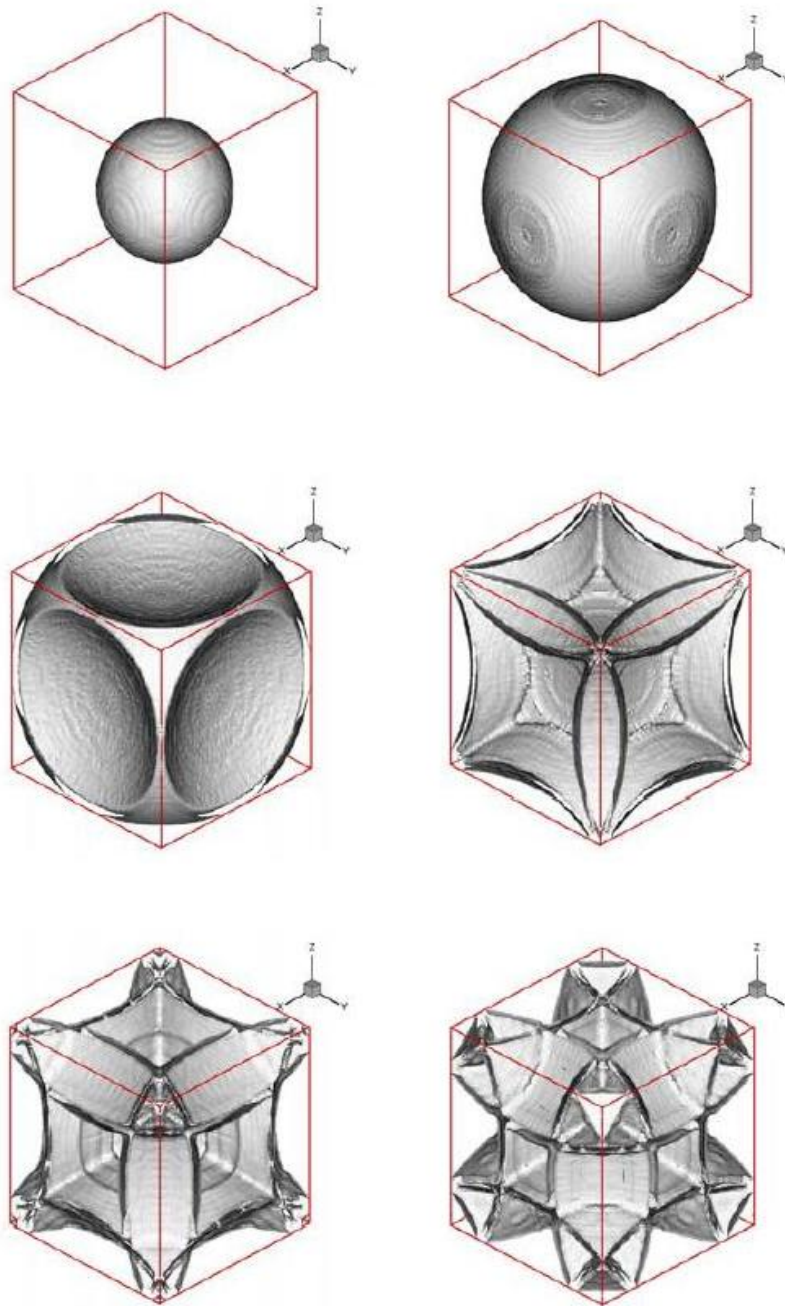


Figure 3.11: Spherical wave front propagation solved using equation (3.20)

Chapter 4

WAVE FRONT PROPAGATION

Everyday concepts of visible light lead naturally to the concept of rays of light, propagating in straight lines, reflected and refracted according to geometrical laws. The concept of a light wave and its reconciliation with ray theory was first introduced by Huygens in the 1600s. He postulated that light propagates as a wave front, and that at any instant every point on the wave front is the origin of a secondary wave, which propagates outward as a spherical wave. The secondary waves then combine to form a new wave front, so that the successive positions of the wave front can be calculated by a step-by-step process. The time dependent wave equation or time independent Eikonal equation generates these wave fronts, which are constant phase surfaces. In this chapter, WC is demonstrated for wave front propagation. Validation studies have been performed using Lagrangian markers, or “rays” for accuracy check.

4.1 Focusing

Focusing is the phenomenon of the waves converging on a point or line. This is important in studying caustic regions. A common situation where caustics are visible is when light shines on a drinking glass. Focusing phenomenon has been extensively applied in a variety of engineering, optics, computer graphics and medical applications. In caustic regions, many waves intersect. Fixed grid Eulerian methods have limits on the effective resolution due to the limited number of grid points in a caustic unlike Ray Tracing schemes [4] [17]. To overcome this problem, complex schemes have been considered such as Osher’s higher dimensional Liouville equation representation [18] and localization of caustics [19]. However, these seem to be complex and computationally costly.

It was first thought that, since during focusing, the number of grid points on the wave decreases, WC is not going to help to retain the information at focusing regions. It

was later shown that WC accurately computes waves through caustic regions and is stable to discretization effects produced on the grid. Since the interest is in the long distance propagation, the detailed resolution at the focusing itself was not an issue and that focusing in intermediate regions does not interfere with the long distance wave propagation. Focusing is computed using two different methods, which will be shown in Sections 4.1.1 and 4.1.2 respectively. Even though WC alone can efficiently capture a caustic, a new approximation called Eulerian/Lagrangian Transition is used at first, which will be discussed in detail in Section 4.1.1 and smooth transition is also demonstrated.

4.1.1 Eulerian Lagrangian Transition

This “hybrid” method involves an elegant algorithm to convert the Eulerian representation of the propagating waves to Lagrangian markers in focusing regions. In the concave (focusing) region, Lagrangian markers are used as they can maintain the continuous surface due to the relatively large number of markers in the small region. When the distance between the markers is large (diverge), WC is turned on as shown in Figure 4.1a. Simple interpolation techniques are used to interpolate values on the grid points using the markers, which might not be on the grid points. For better understanding of this hybrid method, a flow chart to describe the logical sequence of the method is provided in Figure 4.1b. Even though it is complicated than just using WC, it might have some applications in the future. This extension is similar to the improvement to a scheme originally used by Ruuth [20].

A simple test case is used to demonstrate the ability of WC to smoothly coordinate with Lagrangian markers and is discussed below. An initially isolated elliptical wave,

$$\phi_{i,j} = \begin{cases} 1, & 0.98 \leq \left(\frac{i-i_0}{a} \right)^2 + \left(\frac{j-j_0}{b} \right)^2 \leq 1.1 \\ 0, & \text{every where else} \end{cases} \quad n=0 \text{ and } n=1, \quad 4.1$$

is propagated inward with parameters $\nu = 0.23$, $\mu = 0.2$, $\varepsilon = 0.3$, where (i_0, j_0) is the center of the domain, $a = 40$ and $b = 32$. The outgoing front is not plotted for better understanding. The grid cells used for the conversion computation are in a region where the wave had not focused yet and still had adequate resolution.

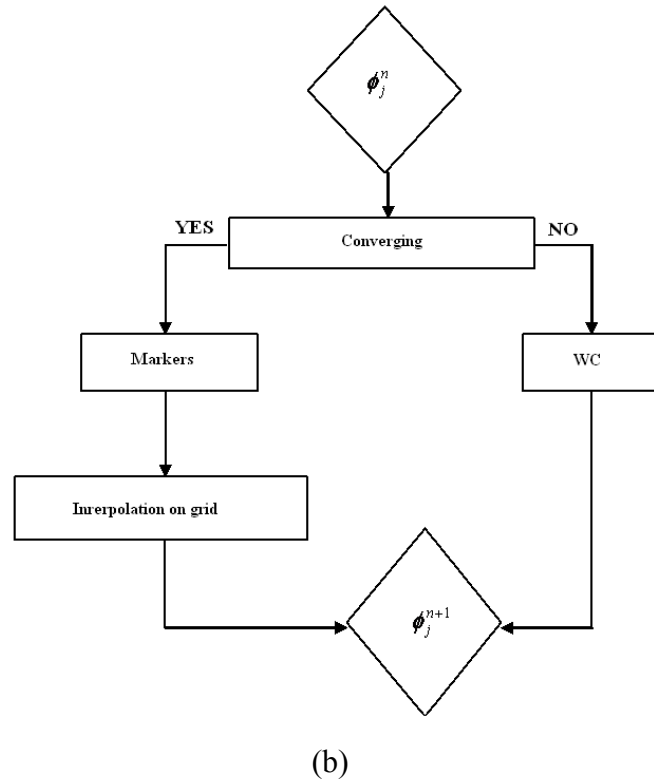
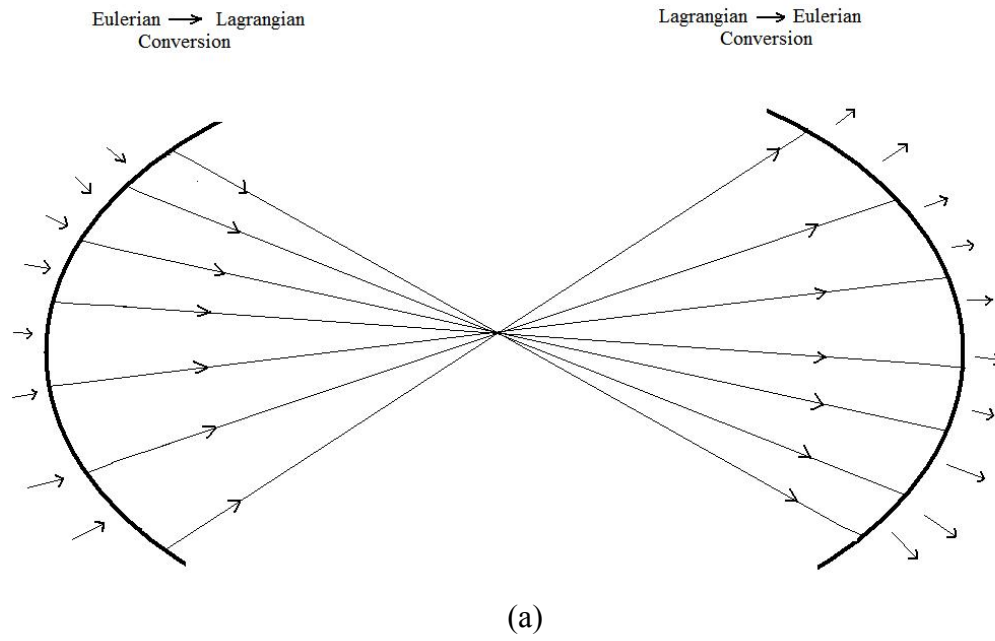


Figure 4.1: Eulerian/Lagrangian transition

(a) Representation of conversion logic

(b) Flow chart for logical sequence of the hybrid method

When the wave is converging, Ray Tracing is turned on and the grid points on the pulse are initiated as markers. The new positions of the markers are computed using the equations,

$$\begin{aligned} x_{new} &= x_{old} + \nu s_x \\ y_{new} &= y_{old} + \nu s_y \end{aligned} \quad 4.2$$

where s_x , s_y are the components of direction of rays and ν is the speed of propagation. The amplitudes of the markers are interpolated onto the grid for a smooth continuous wave. Until the waves pass through the focus and diverged to an extent where the Lagrangian components are not adequate to describe the smooth surface of the propagating quantity, Ray Tracing continues to be adequate. Results of this 'hybrid' computation for the closed surface given in equation (4.1) are shown in Figure 4.2. The computation is performed on a 128×128 grid with $i_0 = 64$ and $j_0 = 64$. The elliptical wave front for 6 different time steps during propagation is plotted and it can be obviously seen that the transition from and to the markers is smooth with no anomalous reflections.

Another test case with more complex boundary conditions is simulated to study the accuracy of this hybrid method. A plane wave propagated towards a boundary of shape,

$$f = j - 0.2 * \sin(\pi i) = 0 \quad 4.3$$

is reflected and converged/diverged according to the curvature on the point of reflection. The scattered field from the concave part converges and finally focuses at a point. The wave diverges when it reflects from the convex part. Initially, the plane wave is defined only on the grid, however, it is converted to a group of Lagrangian rays in the part of the field where the wave reflects from the concave surface. These rays are converted back to an Eulerian wave representation on the grid after passing through the focus, which then is used to continue propagation. The wave front for 6 different time steps is plotted and shown in Figure 4.3. As mentioned above, the transition during the propagation appears seamless.

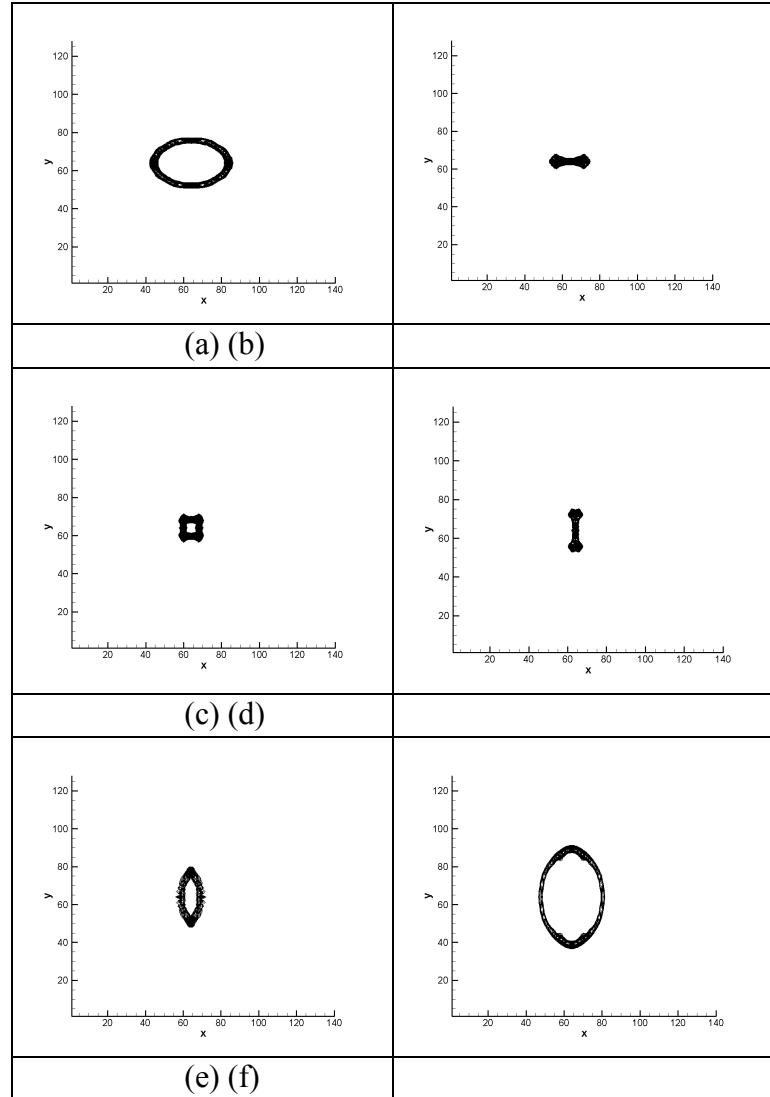


Figure 4.2: Inward propagating wave front computed using hybrid method
(a) $n = 0$, (b) $n = 50$, (c) $n = 70$, (d) $n = 90$, (e) $n = 110$, (f) $n = 160$

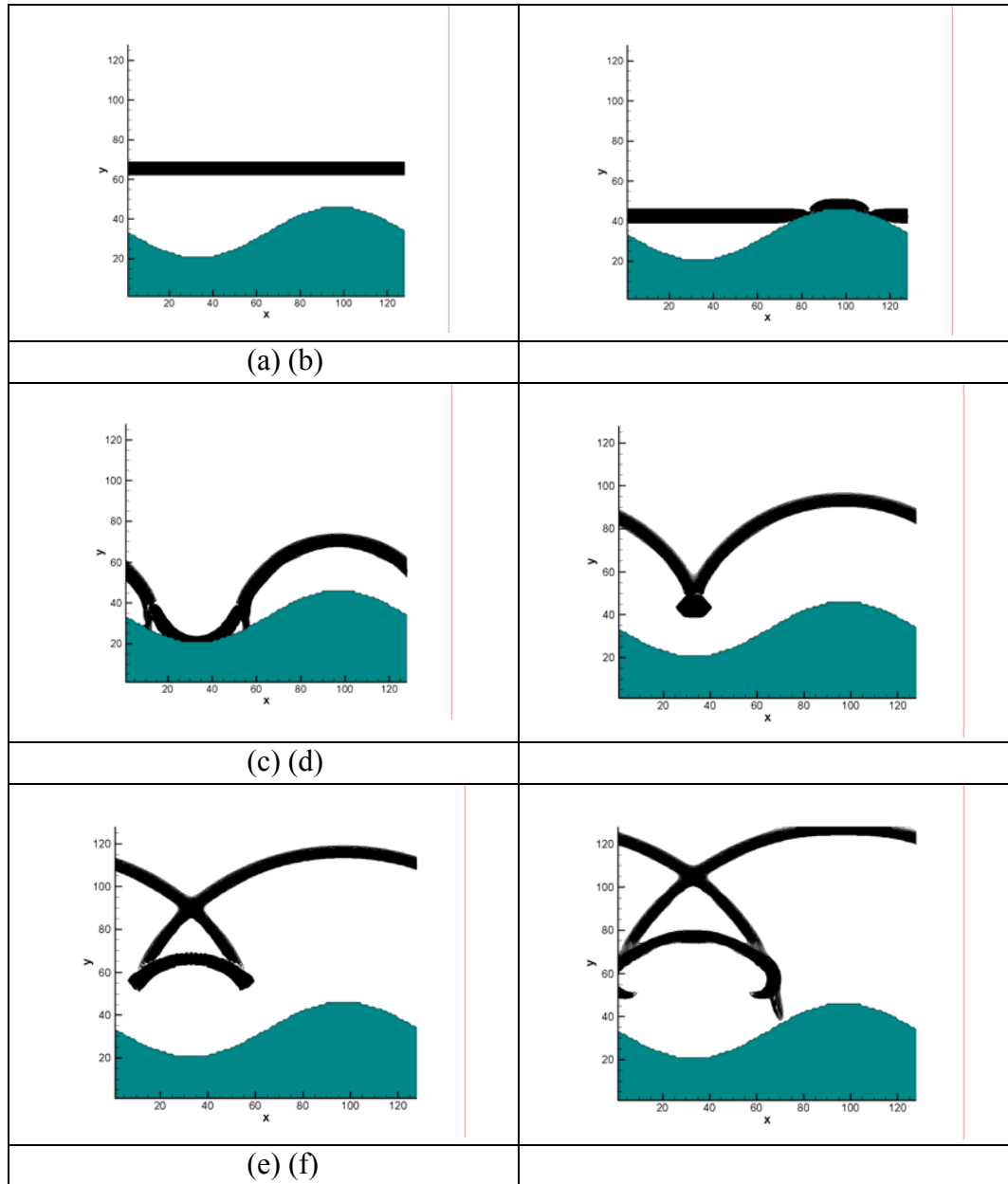


Figure 4.3: Reflection of plane wave from an irregular wall. The computation is carried on using Eulerian/Lagrangian Transition method.

(a) $n = 0$, (b) $n = 100$, (c) $n = 200$, (d) $n = 300$, (e) $n = 400$, (f) $n = 500$

4.1.2 WC

After the “conversion” method was developed, it was realized that there may be a simpler way. The original method by itself proved to be sufficient in caustic regions. An inward going elliptical wave given in equation (4.1) is computed to demonstrate the ability of WC to retain the form and speed of the propagating quantity. There appear to be a number of conserved variables that allow propagation of the wave through the focusing region with the original simple pulse capturing algorithm, with no use of a Lagrangian component, no additional logic, and with no grid refinement. Using this idea, the wave would then not need to be resolved in the focusing region, but the conserved variables in the field, acting as constraints, would allow the automatic reconstruction after the focusing. In Figure 4.4, amplitude contours are compared to Lagrangian markers (exact solution in high frequency approximation). It can be seen that the wave contours match quite well with markers. Also, the basic information defined by the initial conditions is not lost, even though only the simple Eulerian algorithm was used, with no additional logic. Recent developments in level set methods can capture focusing regions [21]. However, these are computationally complex unlike WC.

The existence of conserved variables had been already conjectured based on the multiple interactions of an expanding circular wave on a grid with periodic boundary conditions in Chapter 3. These demonstrate that the waves are extremely robust, even when over seven meet at the same point. It appeared then that the field outside the focusing region was influencing the waves and allowing them to be automatically reconstructed after the interaction. Constraints that appear to be obeyed for a focusing, closed surface can be shown from the equations (4.4) and (4.5). In 2D (for simplicity),

$$\begin{aligned}\delta_n^2 \langle i^2 \rangle^n &= 2\nu^2 \\ \delta_n^2 \langle j^2 \rangle^n &= 2\nu^2 \\ \delta_n^2 \langle ij \rangle^n &= 0\end{aligned}\tag{4.4}$$

where

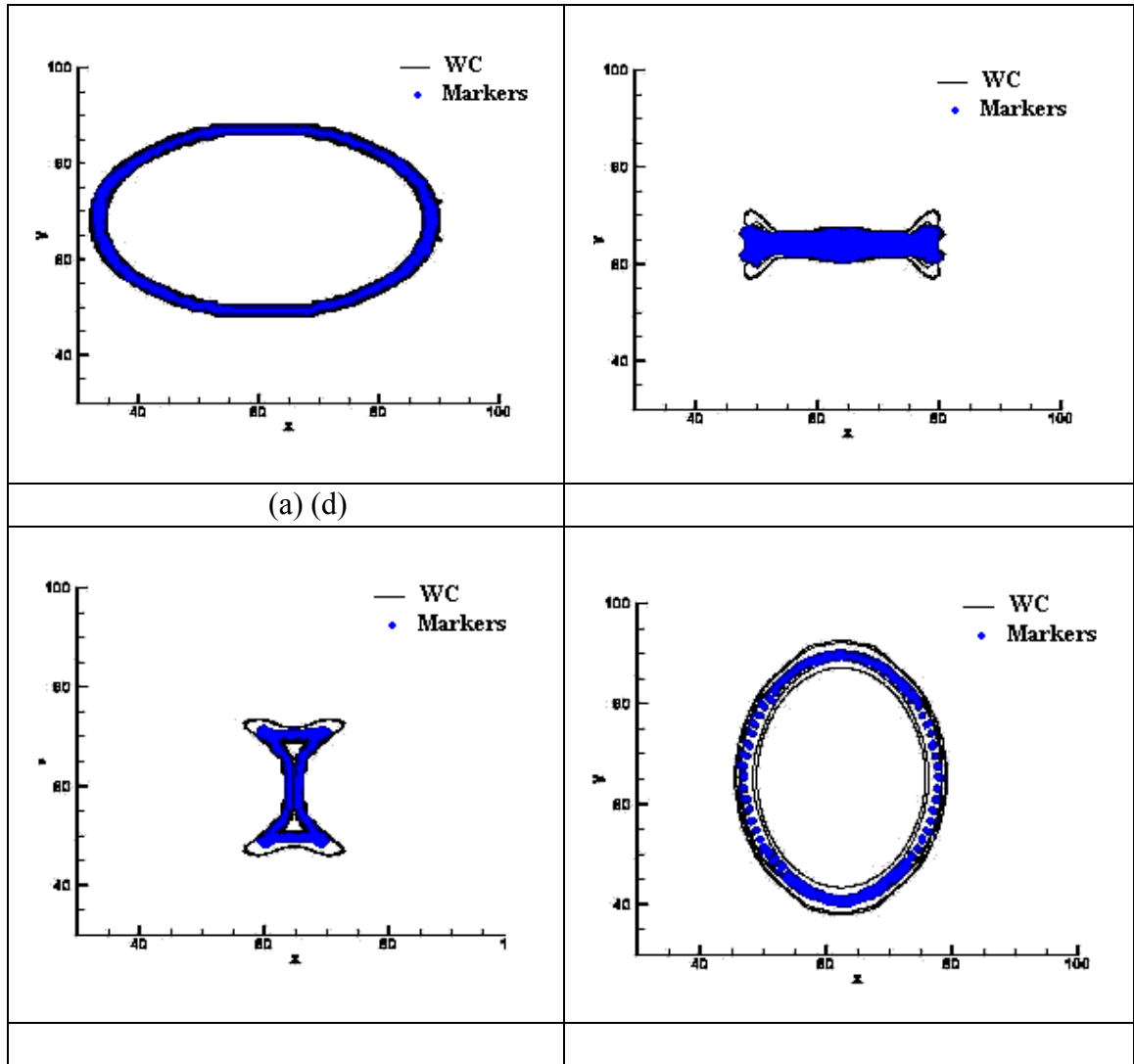


Figure 4.4: Inward propagating wave front computed using WC. The initial condition for this computation is given in equation 4.1.

(a) $n = 0$, (b) $n = 140$, (c) $n = 170$, (d) $n = 250$

$$\langle i^2 \rangle^n = \frac{\sum i^2 \phi_{i,j}^n}{\sum \phi_{i,j}^n}, \quad \langle j^2 \rangle^n = \frac{\sum j^2 \phi_{i,j}^n}{\sum \phi_{i,j}^n}, \quad \langle ij \rangle^n = \frac{\sum ij \phi_{i,j}^n}{\sum \phi_{i,j}^n}. \quad 4.5$$

4.2 Reflection from Complex Boundaries

The property of waves to reflect from boundaries is captured accurately using WC without having to use adaptive grids around the boundaries. This reflection of waves is responsible for echoes, interference, et c. The energy and momentum of the waves is reflected back with or without 180 degrees phase change depending on the properties of the boundary. The form of a reflected wave front is determined by those of the incident wave front and the surface. Boundaries may contain many irregularities and for short pulses (high frequency content), these irregularities can be larger than the wavelengths. Higher order discretization may be needed for conventional treatments of long distance propagation to reduce numerical diffusion in the propagation. Since WC does not have this diffusion problem, complex configurations can be accurately captured using very efficient, lower order discretizations with no loss of accuracy. A very effective method for treating reflections can then be implemented that does not require complex surface fitted grids, but allows the surface to be simply ‘immersed’ in a uniform Cartesian grid as shown in Figure 4.5. This method employs a “level set” representation of the surface and can easily accommodate very complex topography with little computational effort, unlike complicated conformal mapping methods [22] and adaptive grid methods [23]. For example, reflection from a boundary of function, $f(i, j) = 0$, is shown, where,

$$\phi_j^n = 0, \quad f \leq 0, \text{ at all } n. \quad 4.6$$

The effects of boundaries are smooth and no stair casing is observed. Reflection from complex terrains with many irregularities is important to capture interference effects in radio wave propagation. The irregularities can be very much larger than the wavelength of the short of waves that are propagated.

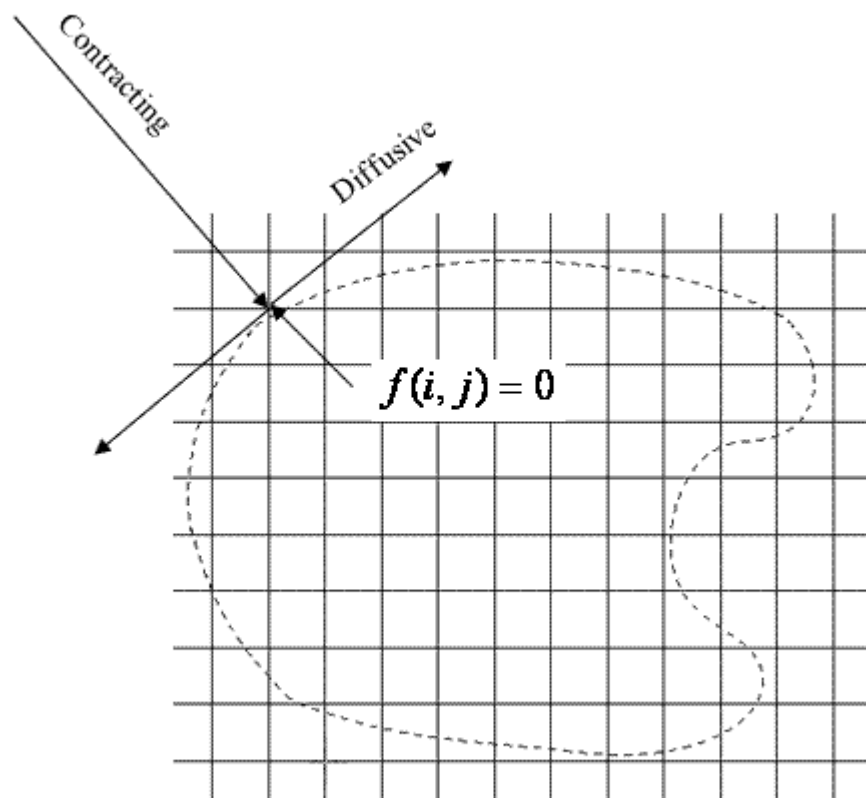


Figure 4.5: Immersed boundary to treat complex boundaries.

4.2.1 Scattering from Cylinder

Scattering is a physical process in which the direction of a wave deviates from its normal trajectory due to non-uniformities in the medium or in interaction with boundaries. A wave packet with a bundle of rays parallel to each other will reflect in different angles if they encounter a non-uniform obstacle (diffuse reflection or scattering). If such a bundle encounters a smooth surface such as mirror, the rays remain as bundle upon leaving the surface (specular reflection). The difference between specular and diffuse reflection is shown in Figure 4.6.

Scattering from the cylinder is diffuse reflection and is simulated accurately using WC. Consider the oblique incidence of a plane wave on a perfectly conducting infinite cylinder. The scalar field, ϕ , is set to zero inside the cylinder of radius 32 grid cells. The reflected surface is manipulated to re-direct or re-shape the plane wave front into a cylindrical scattered field. If the plane wave is polarized, the polarity is reversed when it encounters perfectly conducting obstacles or hard boundaries (This is not demonstrated here). The confinement term, F is diffusive in the tangential direction and contracting in the normal direction. This prevents stair-casing effects and generates a smooth scattered field, which can be seen in Figure 4.7. The scattered field does not quickly diffuse on the grid and can be detected near the receiver, which is located far away from the target. This is important in applications such as target detection. The scalar quantity, ϕ , is very sensitive and can be used to detect targets smaller than the grid cell size. When the field of computation is very large, the number of grid points across the target will be relatively small. The wave scattered by one localized scattering point is computed and is shown in Figure 4.8. The direct scattering problem involves determining the scattered field, which depends on the characteristics of the scatterer studied in this section for different cases. However, a lot of work is yet to be done in inverse scattering, which is considered for future research.

Before proceeding to the study of complex configurations of obstacles, a plane wave reflection from a sine wave boundary (equation (4.3)) is computed and shown. The scattered field from this configuration is already computed using the hybrid method in Section 4.1.1. The same can be effectively captured using WC and is shown in Figure 4.9. It can be seen that WC itself can capture the effects of the boundary.

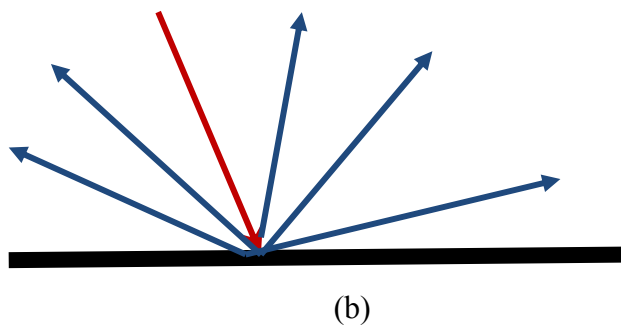
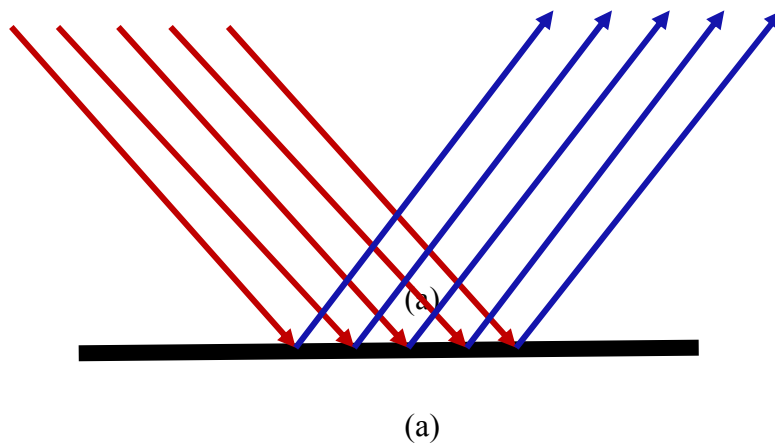


Figure 4.6: Reflection
(a) Specular reflection
(b) Diffuse reflection

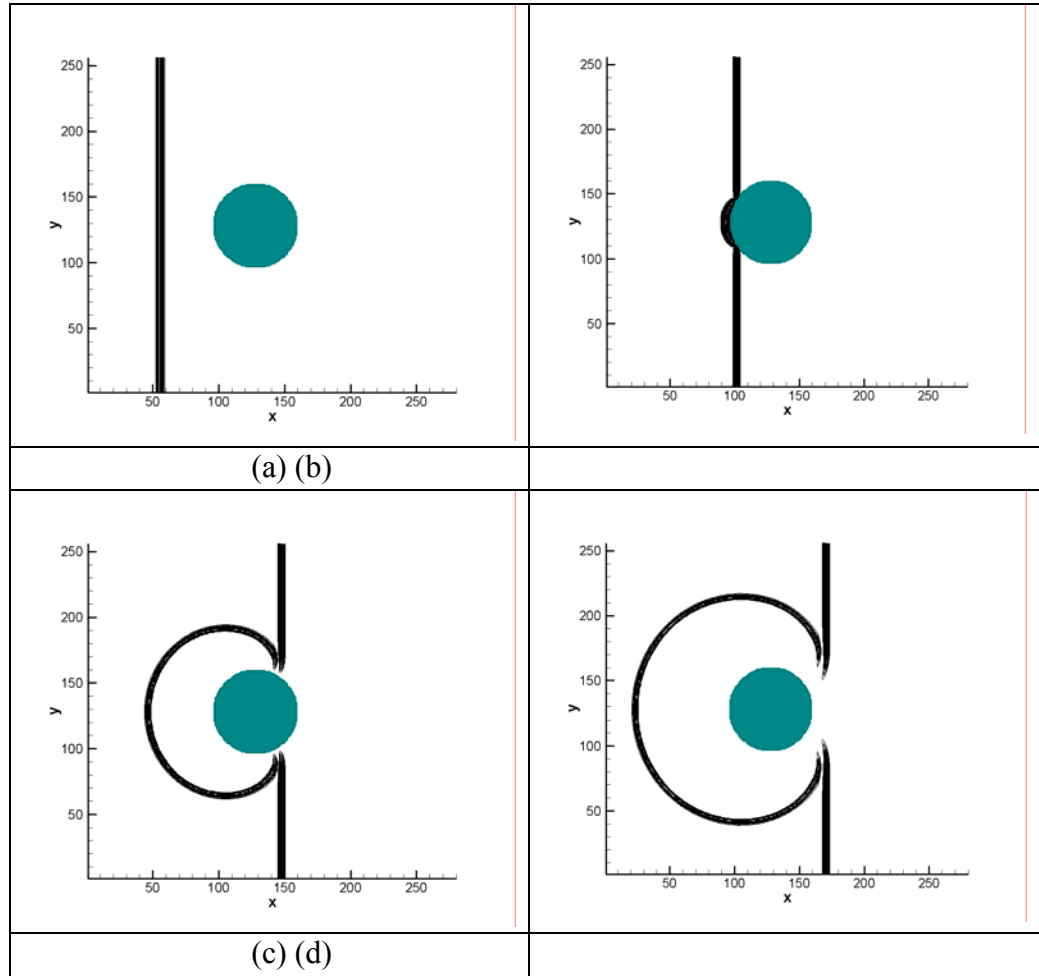


Figure 4.7: Scattering from Infinite Cylinder
(a) $n = 0$, (b) $n = 200$, (c) $n = 400$, (d) $n = 600$

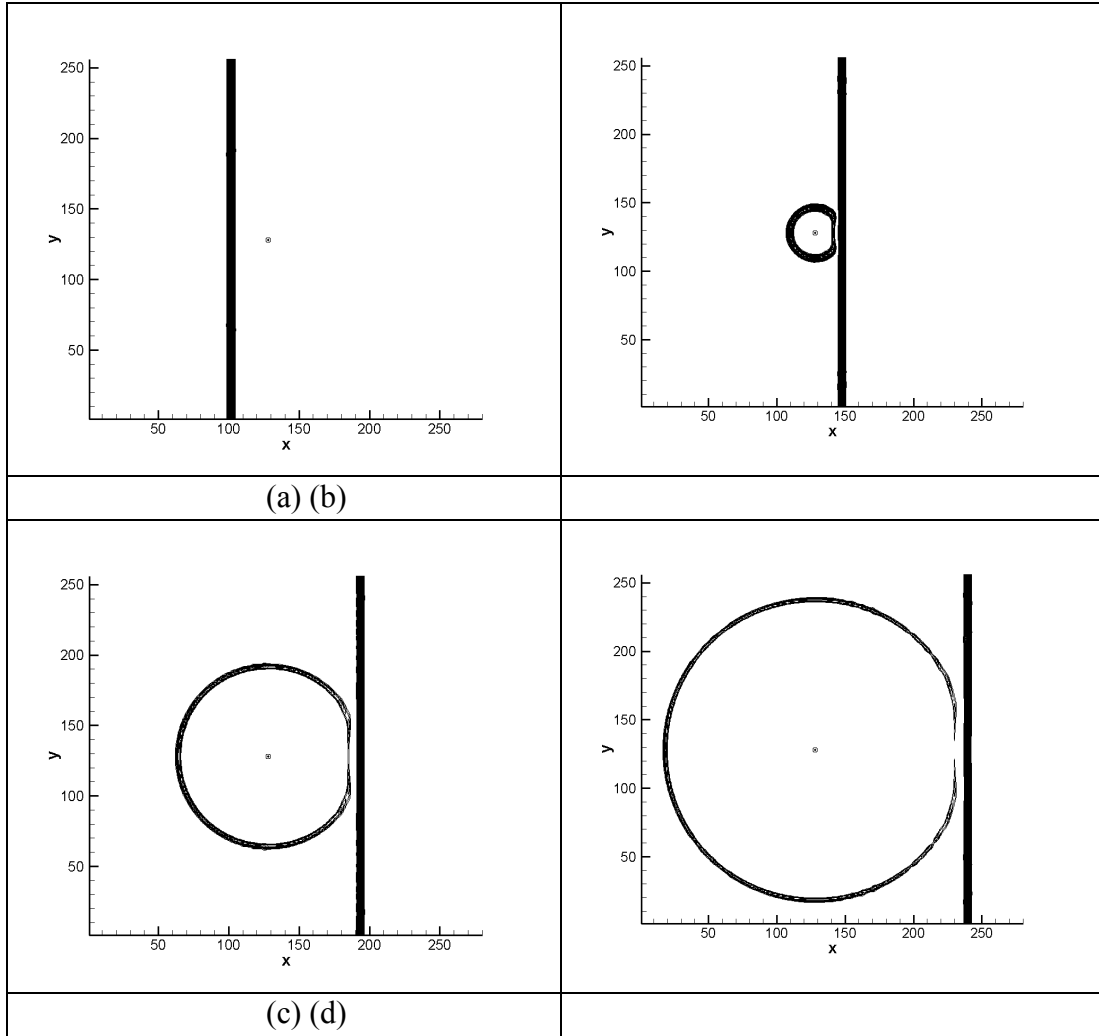


Figure 4.8: Scattering from a single scatterer using WC
(a) $n = 0$, (b) $n = 200$, (c) $n = 400$, (d) $n = 600$

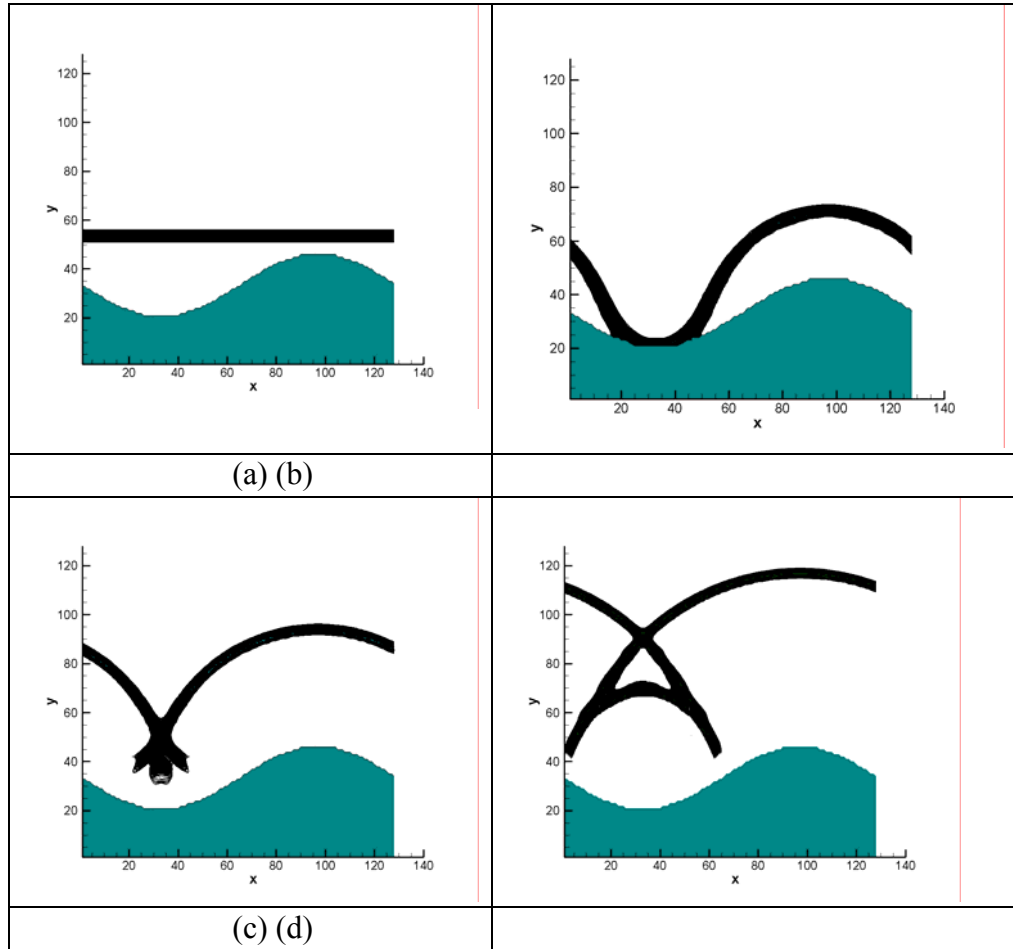


Figure 4.9: Reflection from the boundary with shape of a sine wave
(a) $n = 0$, (b) $n = 200$, (c) $n = 300$, (d) $n = 500$

4.2.2 Complex Object in 3D

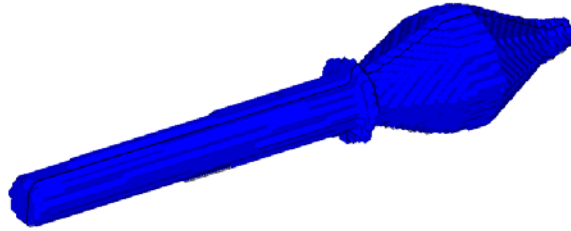
WC is employed in 3D to efficiently compute scattering from complex objects. A uniform Cartesian grid, as mentioned before, is used along with level set approximation described in Section 4.2. Results of confinement for scattering from a model RPG (Rocket Propelled Grenade) are discussed in this section. To describe in a few words, RPG is a hand held weapon equipped with an explosive warhead. The geometry of the RPG used for the computations is shown in Figure 4.10a. As can be seen in Figure 4.10b, the body is immersed in the Cartesian grid with no special logic. At very long distances, the wave fronts transmitted from the antenna are planar. So, a plane wave propagating towards the RPG is simulated. The contours of a scalar quantity, ϕ , representing the wave front of electromagnetic field are presented in Figure 4.11. The results are plotted in the cross sectional plane used in Figure 4.10b. It can be seen in Figure 4.11 that unlike geometrical optics, there is no sharp shadow and some part of the field is imparted to the non-illuminated region. Although the effect comes from the dissipative behavior of confinement term in the tangential direction, it can be modeled as diffraction. This is yet to be studied as part of future research.

4.3 Varying Index of Refraction

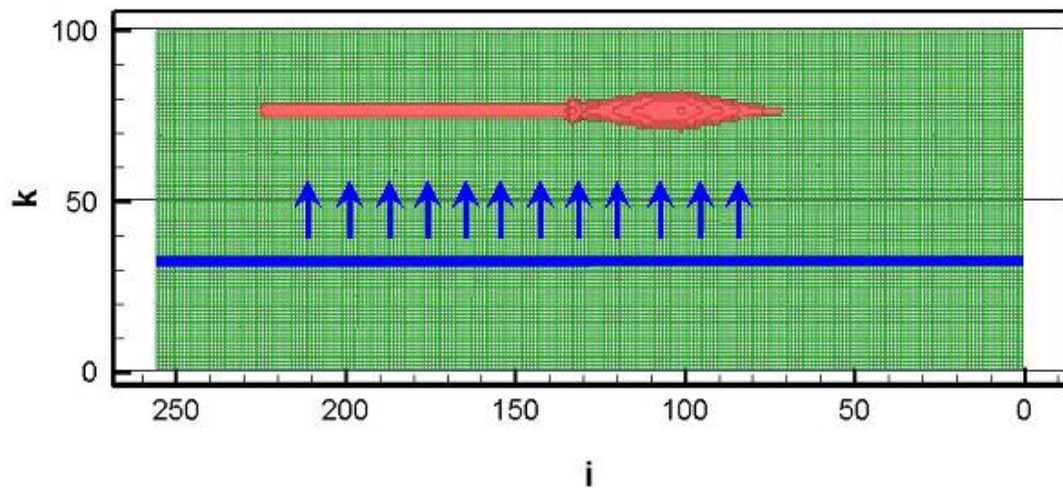
When a wave interacts with the medium of varying density, it refracts (scatters). Refraction can be described by Snell's law, which states that the angle of incidence is related to the angle of refraction.

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1} \quad 4.7$$

where n_1, n_2 are the indices of refraction of the two layers of medium, θ_1, θ_2 are the angles of incidence and refraction respectively and v_1, v_2 are the speeds of the pulse in the respective layers. One example of refraction in the atmosphere is mirage.



(a)



(b)

Figure 4.10: 3D Complex object

(a) Geometry

(b) Cross-sectional plane ($z=664$)

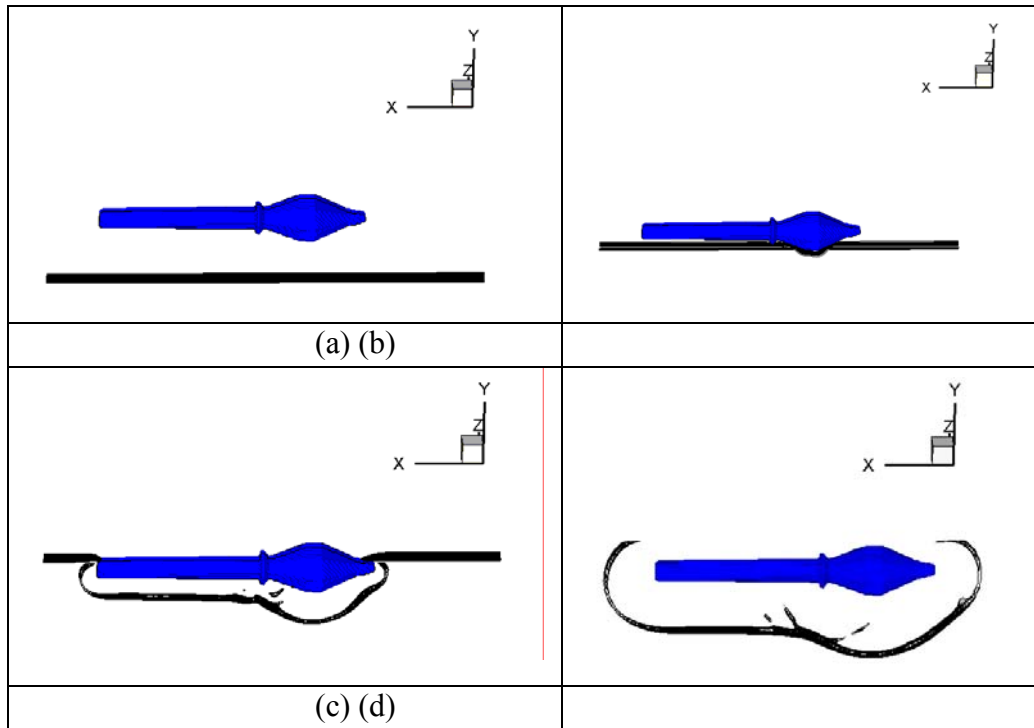


Figure 4.11: Amplitude contours of scattered field from RPG. Iso-surfaces are shown for RPG to show the 3D object.

(a)– (e) are wave plots for consecutive time steps

In acoustics, refraction is the bending or curving of a sound ray that results when the ray passes through a sound speed gradient from a region of one sound speed to a region of a different speed. The amount of ray bending is dependent upon the amount of difference between sound speeds. For water, sound speed depends on the variation in temperature, salinity, and pressure of the water. Similar acoustics effects are also found in the Earth's atmosphere. The phenomenon of refraction of sound in the atmosphere has been known for centuries; however, beginning in the early 1970s, widespread analysis of this effect came into existence through the designing of urban highways and noise barriers to address the meteorological effects of bending of sound rays in the lower atmosphere. Refraction of radio waves in evaporated ducts (formed due to variations in humidity near sea) and ionospheric ducts (formed due to variation in solar radiation) is important in radio wave propagation.

4.3.1 Profile of varying index of refraction: Converging waves

Consider a profile for varying speed,

$$v(j) = \text{const} * \exp(\alpha(j - j_0)^2), \quad 4.8$$

which has maximum speed (v) at $j = j_0$. An isolated plane wave propagated across the medium with the profile given in equation (4.8) is computed. The discretized equation used for the wave propagation is,

$$\phi_{i,j}^{n+1} = 2\phi_{i,j}^n - \phi_{i,j}^{n-1} + \delta_i^2 v_j^2 \phi_{i,j}^n + \delta_j^2 v_j^2 \phi_{i,j}^n + \delta_n (\delta_i^2 F_{i,j}^n + \delta_j^2 F_{i,j}^n). \quad 4.8$$

When the index of refraction is non-uniform, the waves tend to bend towards regions of higher index. It can be seen in Figure 4.12 that the computed pulses travel without diffusion and numerical errors in speed were insignificant to plottable accuracy. The computed pulse is compared to Langrangian markers (the blobs in Figure 4.11) for validation, which are calculated as,

$$\begin{aligned} x_{new} &= x_{old} + v S_{x,new} \\ y_{new} &= y_{old} + v S_{y,new} \end{aligned} \quad 4.10$$

where s_x and s_y are the components of the propagating direction. The direction is updated according to the gradient in index of refraction on that marker as,

$$\begin{aligned} s_{x,*} &= s_{x,old} \\ s_{y,*} &= s_{y,old} - \frac{dv}{dy} \end{aligned} \quad 4.11$$

and the normalized components of the direction are,

$$\begin{aligned} s_{x,new} &= \frac{s_{x,*}}{\sqrt{s_{x,*}^2 + s_{y,*}^2}} \\ s_{y,new} &= \frac{s_{y,*}}{\sqrt{s_{x,*}^2 + s_{y,*}^2}} . \end{aligned} \quad 4.12$$

When the index of refraction is randomly varying and does not have a smooth profile, curve fitting methods must be used to fit the random variations. This becomes very complex when the index of variation is a function of time. But, for a smooth profile, the computation of markers is exact and is used for validation.

As shown in Figure 4.12, the pulse does not lose information in spite of the limited number of grid points across the focusing regions. However, Lagrangian markers suffer inadequacy in number of grid points and fail to maintain continuity. The wave is propagated in a long duct (long distance in duct-like refractive index profile) and plotted as shown in Figure 4.13. The wave comes out of focus accurately even after many time steps. This demonstrates the preservation of the property, width, even after travelling through many focusing regions. This feature is important for many applications in computer graphics, optics etc.

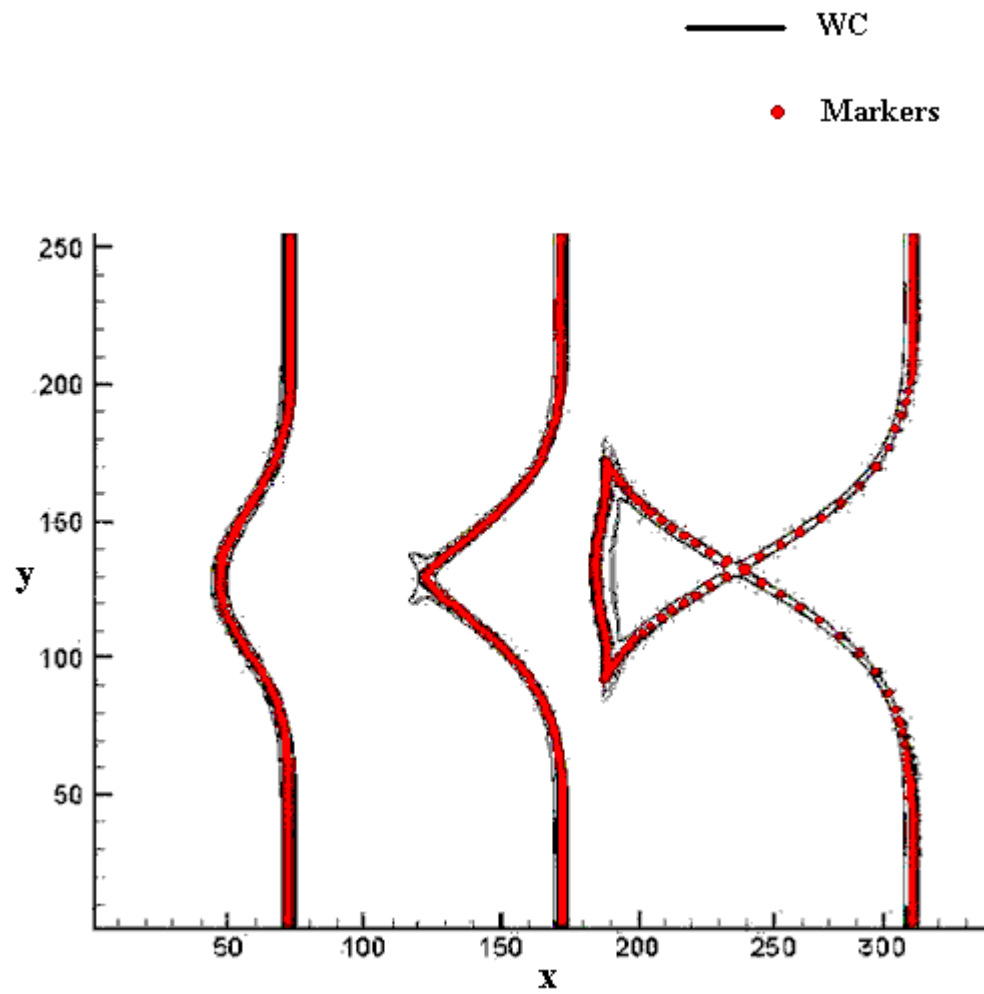


Figure 4.12: Varying index of refraction. The plane wave propagation across medium of profile given in equation 4.8

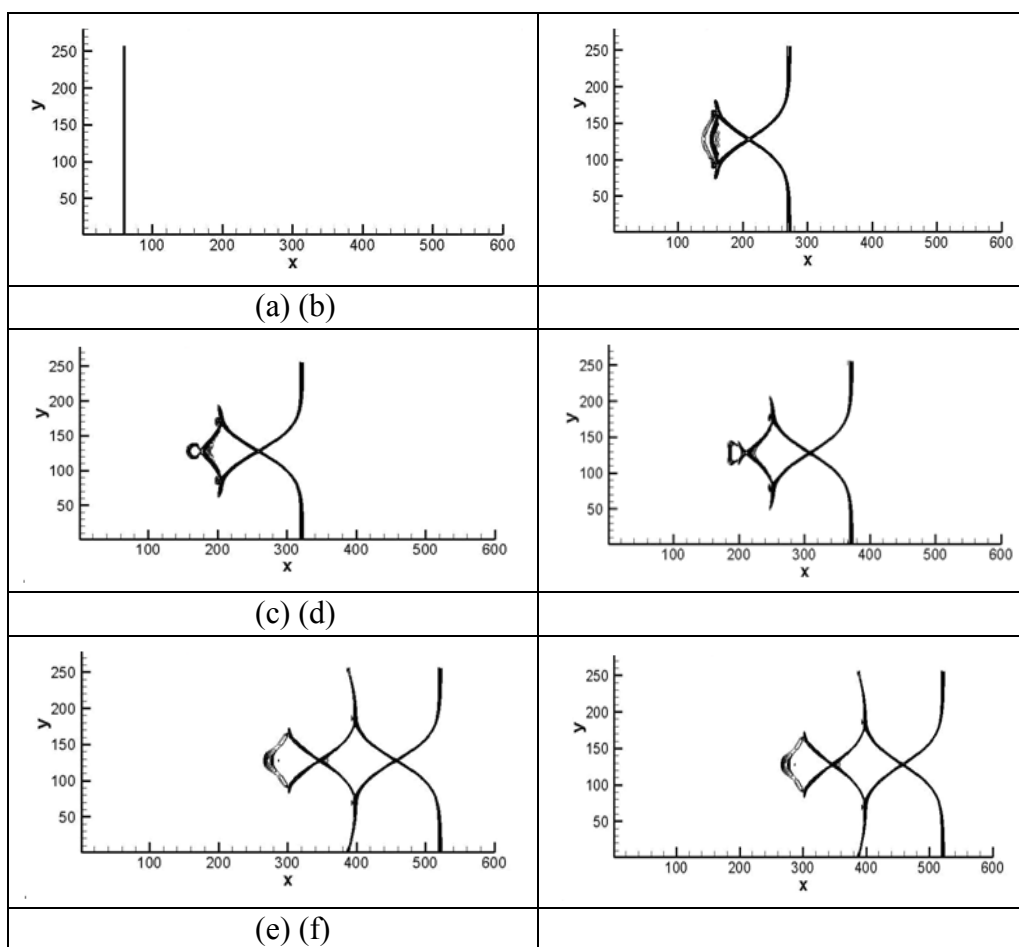


Figure 4.13: Propagation of plane wave in a long duct
(a) $n = 0$, (b) $n = 100$, (c) $n = 200$, (d) $n = 300$, (e) $n = 400$, (f) $n = 500$

4.3.2 Duct like profile used in EM propagation

Electromagnetic waves propagating in the atmosphere encounter regions of varying density, which cause bending of the waves. One example is the evaporation duct, which is associated with a sharp drop in moisture immediately above water bodies. The drop in moisture decreases the density and in turn increases the speed. A predetermined profile of varying index of refraction, which imitates such a duct is treated:

$$v(j) = 0.23 - 0.0023 * |j - j_0| \quad 4.13$$

where $j_0 = 30$. A curved surface configuration (for the ground) is employed, which is immersed in the uniform Cartesian grid. A plane wave propagating through the specified medium is shown in Figure 4.14. Lagrangian markers are also plotted for comparison purpose. The marker positions are computed as given in equations (4.10), (4.11), (4.12). It can be seen that the markers fail to maintain the continuous surface when the wave is diverging, but fall in the band of contours computed by WC.

4.4 Multiple grids

One other important use of WC for the wave equation involves cases with multiple grids with grid interfaces. This property plays an important role when a fine grid has to be used around the near field (source/antenna) and coarser grid in the far field. A fine grid is typically used near the antenna to resolve the high field gradients. To propagate the wave over long distances, a coarse grid scale must be used in the far field. The test case used only treats uniform grids, and not the antenna field or illustration. If only the discretized wave equation is used, with no Confinement, reflections are produced from the grid interfaces, unless special care is taken. Although several approaches have been devised over the years [24] [25] [26], a simple, stable and efficient approach is yet to be available. WC overcomes this problem and a pulse can propagate across the interface with no spurious reflections.

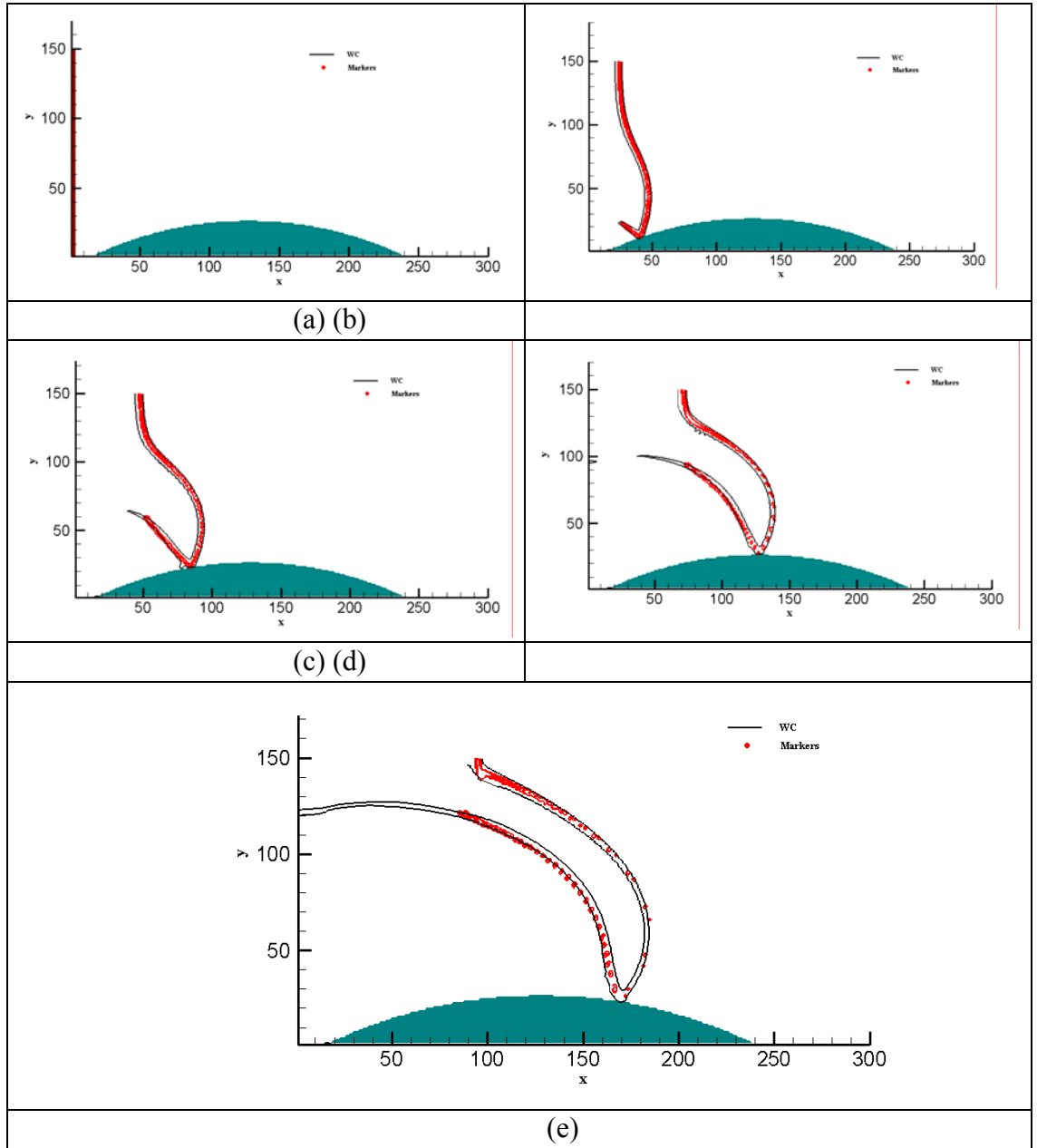


Figure 4.14: Plane wave propagation above curved surface, The medium has a diverging profile given equation (4.13) to imitate surface duct in atmosphere
(a) $n = 0$, (b) $n = 200$, (c) $n = 400$, (d) $n = 600$, (e) $n = 800$

This technique has many applications in astrophysics [27], electromagnetics [28], fluid dynamics [29], etc.

The discretized equation used to solve the wave for multiple grids is

$$\begin{aligned} \phi_{i,j}^{n+1} = & 2\phi_{i,j}^n - \phi_{i,j}^{n-1} + \nu_i^2 \delta_i^2 \phi_{i,j}^n + \nu_j^2 \delta_j^2 \phi_{i,j}^n + \delta_n^- \left[\mu_i \delta_i^2 \phi_{i,j}^n + \mu_j \delta_j^2 \phi_{i,j}^n \right] \\ & - \delta_n^- \left[\varepsilon_i \delta_i^2 \Phi_{i,j}^n + \varepsilon_j \delta_j^2 \Phi_{i,j}^n \right] \end{aligned} \quad 4.14$$

where $\delta_i^2 \phi_{i,j}^n = \phi_{i+1,j}^n - 2\phi_{i,j}^n + \phi_{i-1,j}^n$, $\delta_j^2 \phi_{i,j}^n = \phi_{i,j+1}^n - 2\phi_{i,j}^n + \phi_{i,j-1}^n$, $\delta_n^- = f^n - f^{n-1}$, $\nu_i = \frac{c\Delta t}{\Delta x_{i,j}}$,

$$\nu_j = \frac{c\Delta t}{\Delta y_{i,j}}, \quad \mu_i = \frac{\mu'\Delta t}{(\Delta x_{i,j})^2}, \quad \mu_j = \frac{\mu'\Delta t}{(\Delta y_{i,j})^2}, \quad \varepsilon_i = \frac{\varepsilon'\Delta t}{(\Delta x_{i,j})^2}, \quad \varepsilon_j = \frac{\varepsilon'\Delta t}{(\Delta y_{i,j})^2} \text{ and}$$

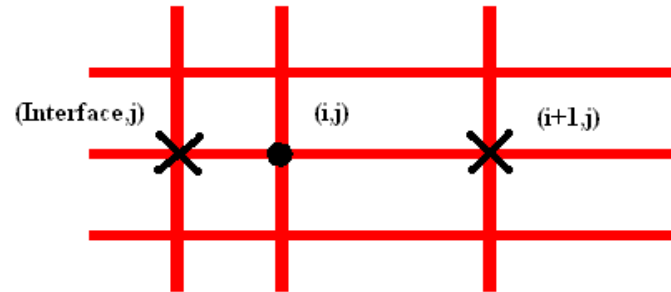
$\frac{\varepsilon_i}{\mu_i} = \frac{\varepsilon_j}{\mu_j} = \text{const.}$. It must be noted that the ratio $\frac{\varepsilon}{\mu}$ over the entire domain is constant

and therefore, the number of grid cells across the pulse remains constant.

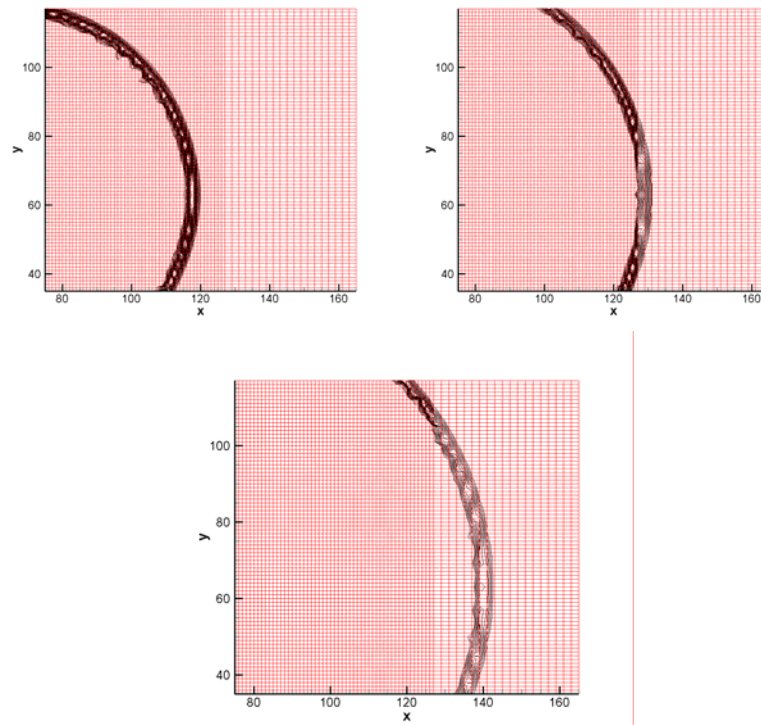
At the grid interface,

$$\begin{aligned} \delta_x^2 \phi_{i,j}^n &= \phi_{i+1,j}^n - 2\phi_{i,j}^n + \phi_{\text{interface},j}^n \\ \delta_y^2 \phi_{i,j}^n &= \phi_{i,j+1}^n - 2\phi_{i,j}^n + \phi_{i,\text{interface}}^n \end{aligned} \quad 4.15$$

where values at the interface are calculated as shown in Figure 4.15a. A cylindrical wave that passes through the variable grid interface is shown in Figure 4.15b. It can be seen that there is almost no production of reflected waves from the interface. The total amplitude as the wave is passing through fine grid / coarse grid interface is shown in Figure 4.16. This is checked for a range of grid ratios and presented in Figures 4.16a, 4.16b, 4.16c for grid ratios equal to 2, 4, 6 respectively. Anomalous numerical reflections during transition from the fine grid to the coarse grid can be suppressed by using variable confinement parameters, which are dependent on grid size and yet, the ratio $\frac{\varepsilon}{\mu}$ can be kept constant.



(a)



(b)

Figure 4.15: Multiple grids

(a) Computation of values at interface

(b) Circular wave front propagating across multiple grid interface

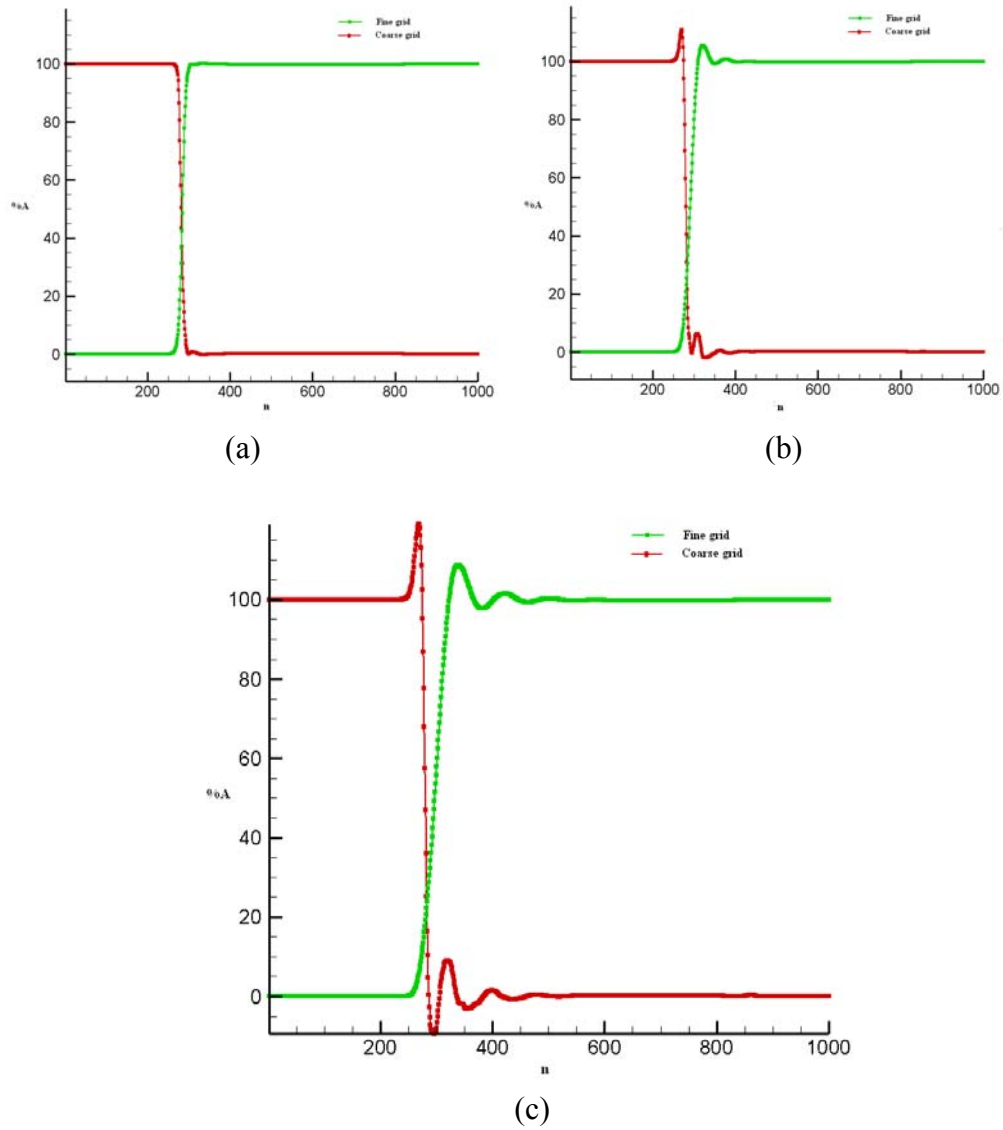


Figure 4.16: Amplitude transfer

- (a) Grid ratio = 2
- (b) Grid ratio = 4
- (c) Grid ratio = 6

4.5 Conclusion

WC appears to work quite well in capturing wave fronts in all realistic conditions. The centroid surfaces that propagate accurately on the grid can also be used in newly developed approximations to capture interference. Many conventional schemes that solve frequency dependent Helmholtz equation, suffer dissipation when computing rapidly varying field. Also, the time dependent wave equation can be solved using WC as the confinement prevents the unwanted dissipation/reflection/dispersion effects produced by discretization. The effects of complex boundaries are also efficiently captured without having to use fitted or adaptive grid. The ability to propagate other properties of the wave allowed the development of a new approximation, “Local Parabolic Method”, which will be discussed in Chapter 5.

Chapter 5

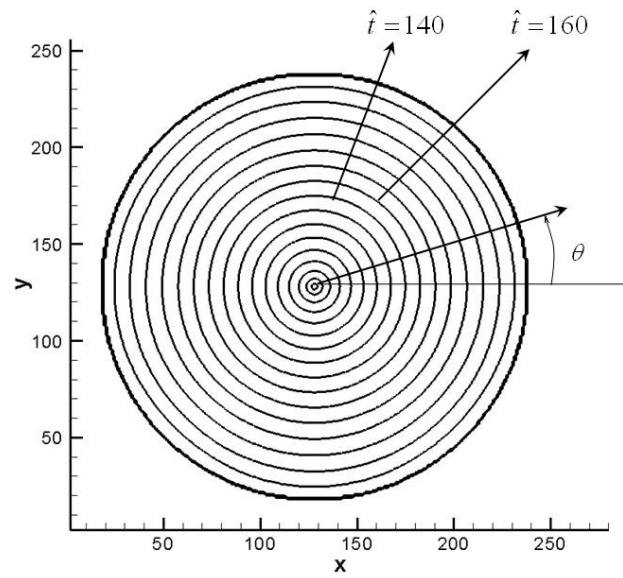
CARRIER FUNCTION

Developments that allow wave fronts to capture and propagate more details are described in this chapter. These details are required to compute interference. WC can be used to generate constant arrival time (Eikonal phase) surfaces accurately by storing the centroid arrival time at each grid point. Also, multiple arrival times are easily accommodated, which is complicated using Eikonal schemes. Some recent developments in Eikonal methods [2] can treat multiple arrival times but, these methods require extra independent variables and complex data management schemes are used to control memory requirements. To demonstrate the ability of WC to capture these surfaces, arrival times for the wave fronts generated from a point source located at (128,128) on a 256×256 grid. The parameters used for this computation are $\nu = 0.23$, $\mu = 0.2$, $\varepsilon = 0.3$. The arrival time, $\hat{t}$, is calculated for the passage of each wave as

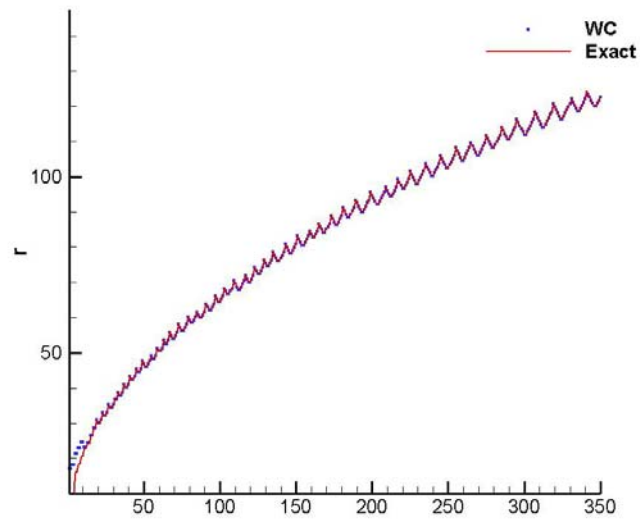
$$\hat{t}_{i,j} = \frac{\sum_n t \phi_{i,j}^n}{\sum_n \phi_{i,j}^n} \quad 5.1$$

It can be seen in Figure 5.1a that the contours of arrival times are smooth with no stair-casing effects, which shows that there are no plottable discretization errors. The arrival times at the grid points in a cone with a range of angles, $\theta = 29^\circ - 32^\circ$, are compared to the exact values for validation purpose. For each index, there are several grid points with varying j indices within the cone of $\theta = 29^\circ - 32^\circ$. The computed arrival times at each these grid points is plotted against exact values in Figure 5.1b. Assuming constant index of refraction, the exact arrival times are computed as

$$\hat{t}_{exact}(i, j) = \frac{r_{i,j}}{\nu} . \quad 5.2$$



(a)



(b)

Figure 5.1: Arrival times of cylindrical wave generated from point source

(a) Contours of arrival time

(b) WC Vs Exact for $\theta = 29^\circ - 32^\circ$

The direction of propagation, which is the gradient of the phase field, can also be accurately computed as

$$\vec{s}(s_x, s_y) = \frac{\bar{\nabla} \hat{t}_{i,j}}{|\bar{\nabla} \hat{t}_{i,j}|}, \quad 5.3$$

where s_x is the x-component of direction, and s_y is the y-component of direction. However, it is required that the wave front has entirely passed the grid point to accurately compute these properties. For very small wavelengths, the wave fronts and grid cells are hundreds of wavelengths wide and it becomes difficult to accommodate multiple arrival times when the waves are close to each other. To overcome that problem, the scalar field, ϕ , is used as a carrier function that can act as wave packet, which carries the required details of the propagating quantity. It is then not necessary for the wave front to entirely pass a grid point to capture the properties. The capability of WC is extended further using this “carrier” approach and numerous experiments are conducted.

5.1 Propagation Directions

The carrier approach is used to accurately propagate other properties of the wave with scalar function, ϕ . For example, to propagate the directions in 2D, the wave equation is solved three times for three quantities, ϕ , ϕs_x and ϕs_y as

$$\delta_n^2 \phi_l^n = \nu^2 \nabla^2 \phi_l^n + \mu \delta_n^- (\nabla^2 \phi_l^n) - \varepsilon \delta_n^- (\nabla^2 \Phi_l^n) \quad 5.4$$

where $l = 1, 2$ and 3 and ϕ_l is defined as,

$$\begin{aligned} \phi_1 &= \phi \\ \phi_2 &= \phi s_x \\ \phi_3 &= \phi s_y. \end{aligned} \quad 5.5$$

The updated directions are computed using the normalized relations,

$$s_x^{n+1} = \frac{\phi_2^{n+1}}{\sqrt{(\phi_2^{n+1})^2 + (\phi_3^{n+1})^2}} \quad 5.6$$

$$s_y^{n+1} = \frac{\phi_3^{n+1}}{\sqrt{(\phi_2^{n+1})^2 + (\phi_3^{n+1})^2}}.$$

In the test computation described, the initial conditions for ϕ are given as

$$\begin{aligned}\phi_{i,j}^{n=0} &= A_0 \sec h[\gamma r_{i,j}] \\ \phi_{i,j}^{n=1} &= \frac{A_0}{A_1} \sec h[\gamma(r_{i,j} - \nu)]\end{aligned}\tag{5.7}$$

with initial directions,

$$\begin{aligned}s_x &= \frac{i - i_0}{r_{i,j}} \\ s_y &= \frac{j - j_0}{r_{i,j}}\end{aligned}\tag{5.8}$$

where the radius of the wave, $r_{i,j} = \sqrt{(i - i_0)^2 + (j - j_0)^2}$ with origin at the center of the domain (i_0, j_0) . The parameters, ν , μ , and ε are the same for all three quantities and are $\nu = 0.23$, $\mu = 0.2$, and $\varepsilon = 0.3$. ϕ_1 and ϕ_2 behave as the scalar function itself during wave collision. In Figure 5.2, directions for the thin wave are shown at 4 different time steps. During interaction, directions temporarily are the mean values of the waves that are together. Then, they take their original forms after a short relaxation time.

5.2 Edge Diffraction

According to geometric optics, light travels in the form of rays and can bend in regions of non-uniformities in the medium. In isotropic media, the rays are normal to the primary wave fronts and fail to accommodate optical effects like diffraction.

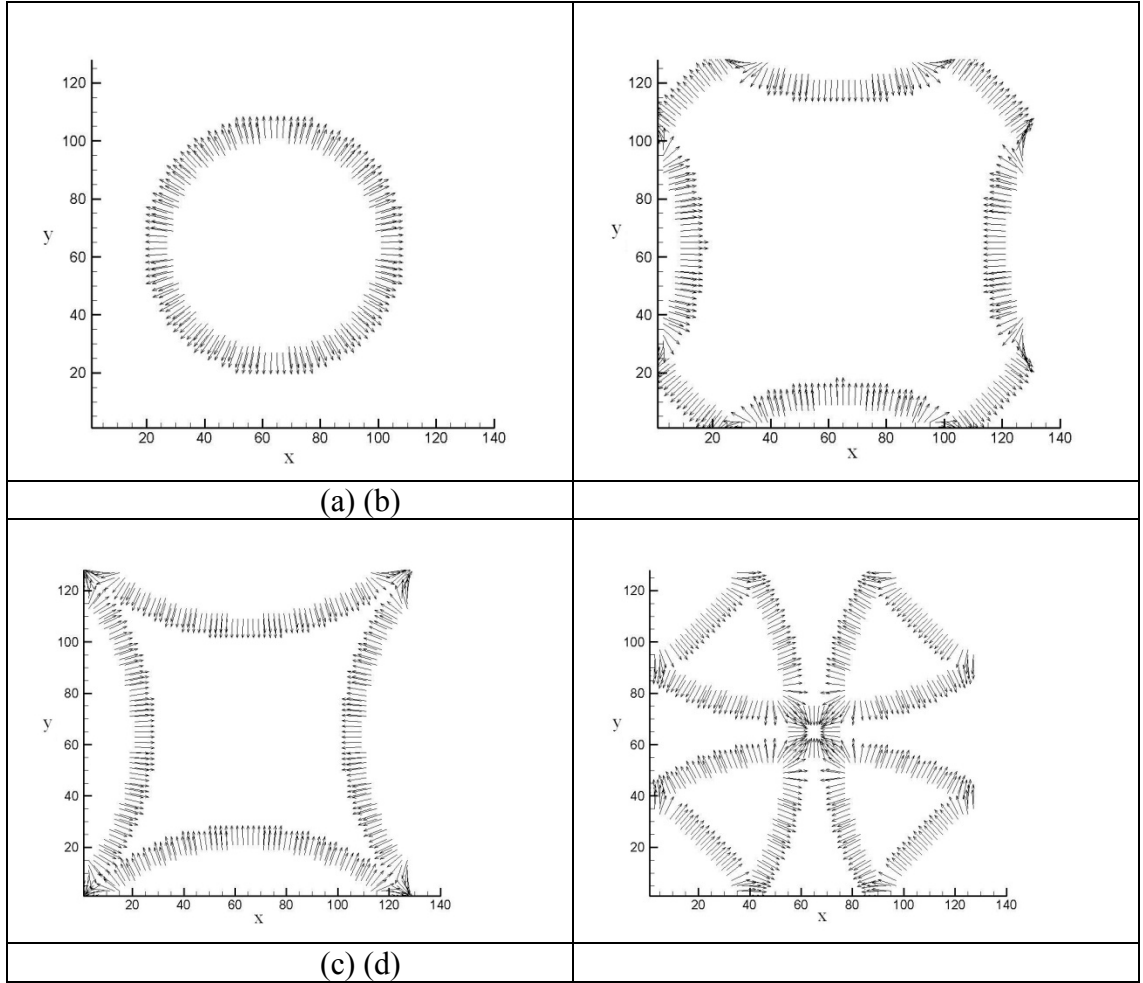


Figure 5.2: Propagation of directions

Since, diffraction was not described by the classical ray theory of light [30], it has been separately modeled using the Geometric Theory of Diffraction. On the other hand, Huygens principle (wave optics) postulates that every point on the wave front act as a point source for further propagation and can accurately account for

diffraction. One such example is diffraction of a plane wave from a semi-infinite screen. The total field obtained by Sommerfeld [31] consists of incident, reflected and a cylindrical diffracted wave. According to Huygens' principle, a secondary source is created at the edge and a part of the field from that source is transmitted into the non-illuminated (shadow) region as shown in Figure 5.3a. The incident field interferes with the diffracted field and interference fringes are observed in the illuminated region. At high frequency, the waves do not bend and maintain sharp shadows.

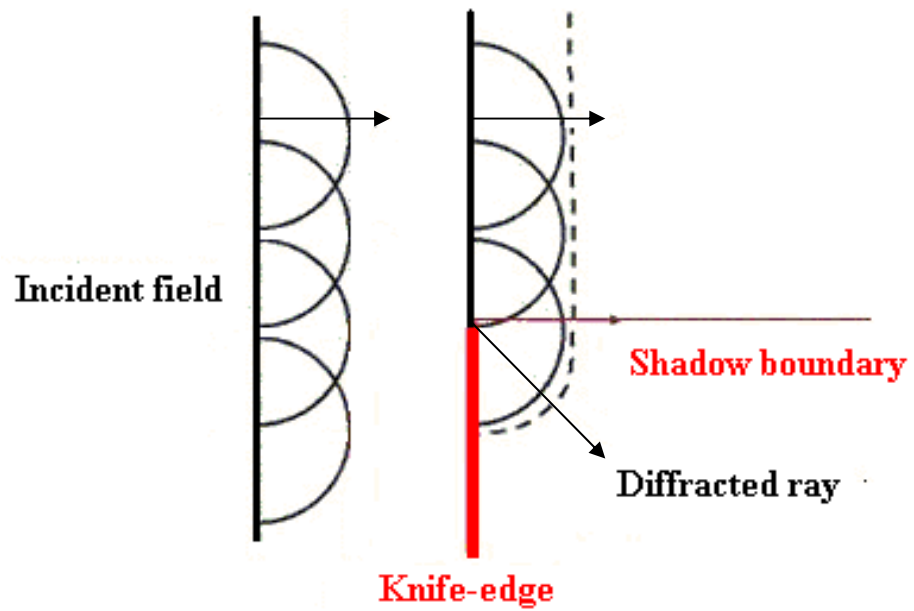
5.2.1 Keller's Approximation

The diffraction from a knife-edge can be computed using the Geometric Theory of Diffraction (GTD) developed by Joseph Keller [32] in the 1960s. It is an extension of geometric theory of optics. This approximation will incorporate diffraction effects into the ray theory of light. Also, it introduces secondary sources that generate diffracted rays from the edges and corners of boundaries. GTD has been successfully applied in acoustics and Optics. Let us consider an incident field propagating horizontally that encounters a knife-edge, as shown in Figure 5.3b. The fundamental principle of GTD is that ray propagation is a local phenomenon. Furthermore, all fields of any origin obey the laws of geometric optics locally. The same is also true for diffracted rays. When the incident field is propagated in the direction normal to the edge, the diffracted field is cylindrical with the edge as its axis. Now, the total field at a point is the sum of all fields that pass through a point. An interference pattern is observed due to superposition of the geometric incident wave (u^i) and diffracted edge wave (u^D).

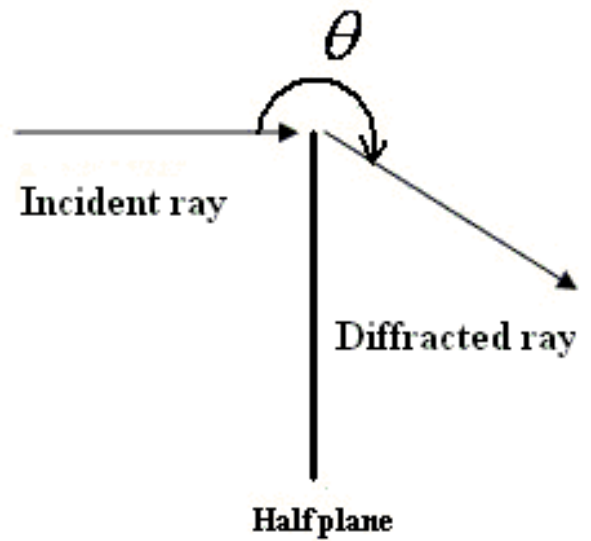
$$u = u^i + u^D \quad 5.9$$

where

$$\begin{aligned} u^i &= A^i e^{i\omega t_i} \\ u^D &= D \frac{A^i}{r^{1/2}} e^{i\omega t_D} \end{aligned} \quad 5.10$$



(a)



(b)

Figure 5.3: Knife edge diffraction

(a) Huygens principle

(b) Keller's geometric theory of diffraction

where ω is the frequency of the wave, r is the radius of the diffracted wave and D is the diffraction coefficient for the knife edge. The diffraction coefficient, D , developed by Keller is,

$$D = \frac{e^{i\pi/4}}{(2\pi k)^{1/2} \sin \theta} \left[\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) \right] \quad 5.11$$

where θ is the angle between incident and diffracted rays, as shown in Figure 5.3b.

5.2.2 WC with Keller's Approximation

The high frequency approximation is achieved by WC (as a pde) by restricting the dissipation and contraction to the tangential and normal directions respectively. The incident wave propagation with a knife edge in the path has a sideways dissipation when computed using the equation,

$$\begin{aligned} \phi_{i,j}^{n+1} = & 2\phi_{i,j}^n - \phi_{i,j}^{n-1} + \nu^2 \left[\delta_x^2 \phi_{i,j}^n + \delta_y^2 \phi_{i,j}^n \right] + \mu \delta_n^- \left[\delta_x^2 \phi_{i,j}^n + \delta_y^2 \phi_{i,j}^n \right] \\ & - \varepsilon \delta_n^- \left[\delta_x^2 \Phi_{i,j}^n + \delta_y^2 \Phi_{i,j}^n \right] \end{aligned} \quad 5.12$$

As explained in Section 4.2, the nonlinear term acts as dissipation in the tangential direction to generate smooth wave fronts. Due to this effect, the arrival times start to bend as the wave encounters the edge, as shown in Figure 5.4. Even though this looks like diffraction, it actually is produced by dissipation from the confinement term to maintain the smooth field for ϕ .

The scalar function can be treated as a wave packet, which can be used to propagate other variables such as direction, as demonstrated in Section 5.1. This allows WC to capture the resulting incident wave produced by a knife edge without numerical spreading. The equation used for the high frequency approximation is

$$\begin{aligned} \phi_{i,j}^{n+1} = & 2\phi_{i,j}^n - \phi_{i,j}^{n-1} + \nu^2 \left[\delta_x^2 \phi_{i,j}^n + \delta_y^2 \phi_{i,j}^n \right] + \mu \delta_n^- \left[s_x \delta_x^2 \phi_{i,j}^n + s_y \delta_y^2 \phi_{i,j}^n \right] \\ & - \varepsilon \delta_n^- \left[n_x \delta_x^2 \Phi_{i,j}^n + n_y \delta_y^2 \Phi_{i,j}^n \right] \end{aligned}, \quad 5.13$$

where s_x and s_y are propagated as described in Section 5.1, the components of the normal direction are $n_x = -s_y$ and $n_y = s_x$. These waves obey the principles of geometric optics and are called geometric waves.

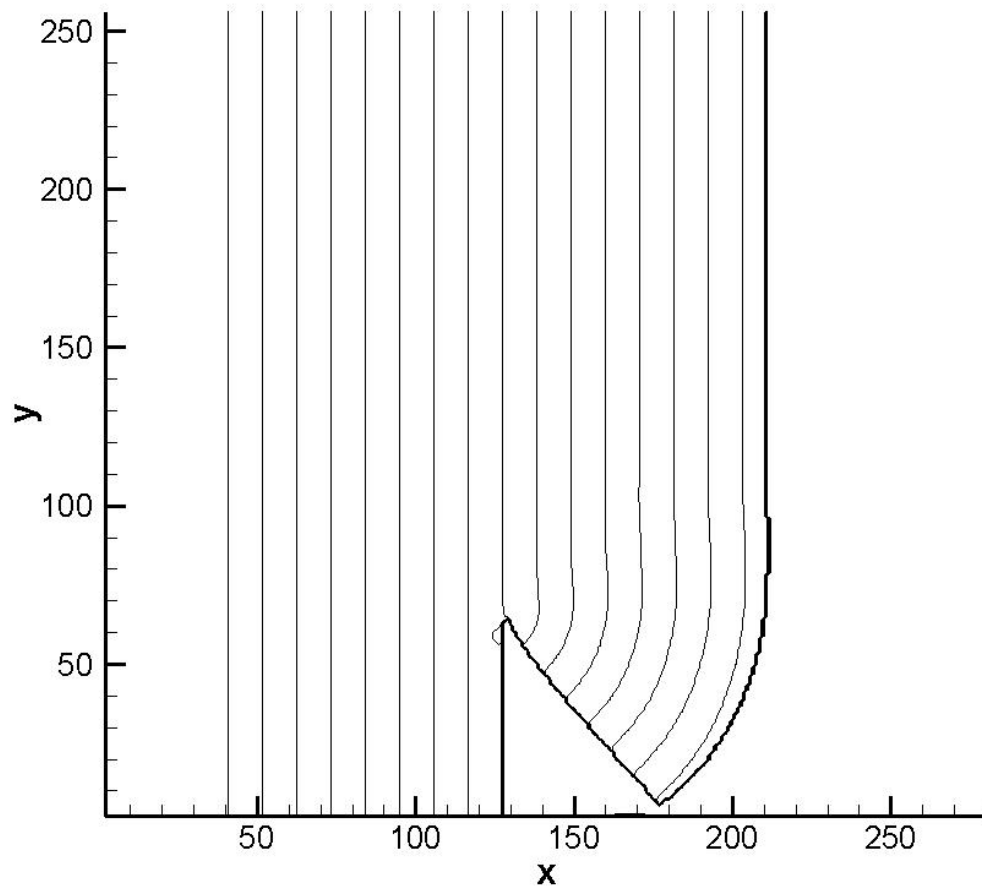
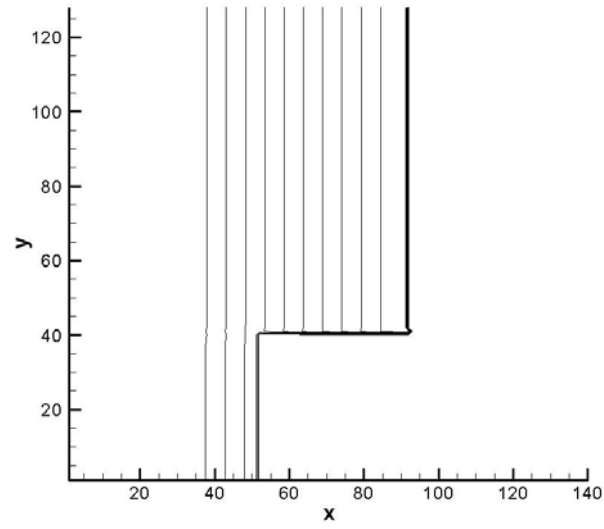
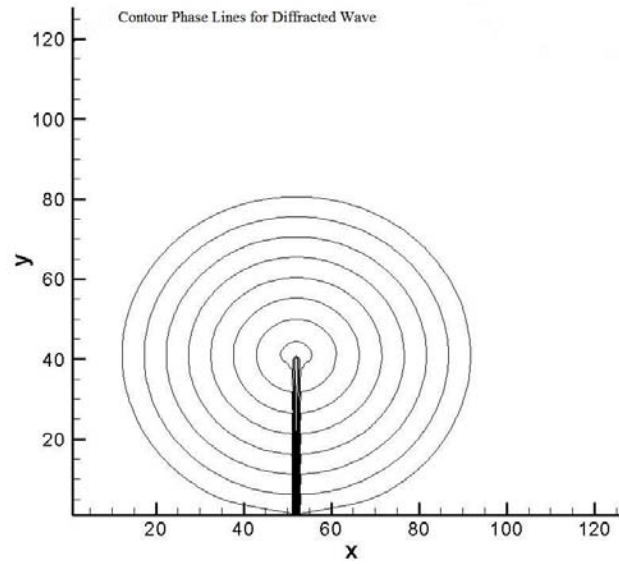


Figure 5.4 : Sideward bending of arrival time contours.

Constant Phase Contours(Geometric Wave)



(a)



(b)

Figure 5.5: Arrival times for knife edge.

(a) Incident field t^i

(b) Diffracted field t^D

The arrival times of incident field, t^i , are computed using the relation given in equation (5.1). It can be seen in Figure 5.5a that a sharp shadow boundary is generated.

The edge wave is propagated from a point source placed on the edge with the amplitude given by the incident field, A^i . The arrival times of the diffracted wave, t^D , are shown in Figure 5.5b. The total field is now calculated as

$$A = A^i \left(e^{i\omega t_1} + \frac{D}{\sqrt{r}} e^{i\omega t_2} \right). \quad 5.14$$

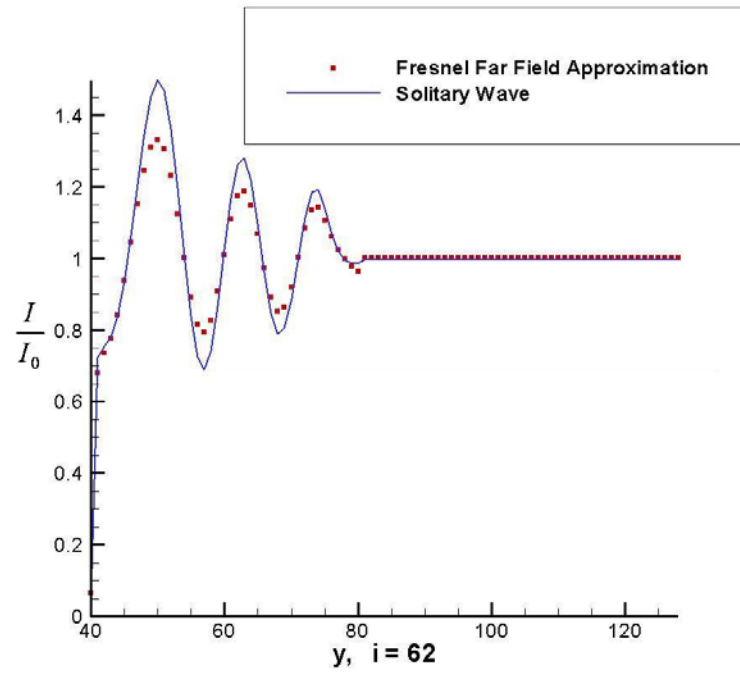
The interference pattern is computed using the equation 5.11 and compared to the Fresnel far field approximation [30]. The intensity ratio, $\frac{I}{I_0}$, for planes, $i = 62$ and $i = 68$ is shown in Figure 5.6a and Figure 5.6b respectively. It can be seen that the results agree closely with the Fresnel far field approximation.

5.3 Paraxial Approximation

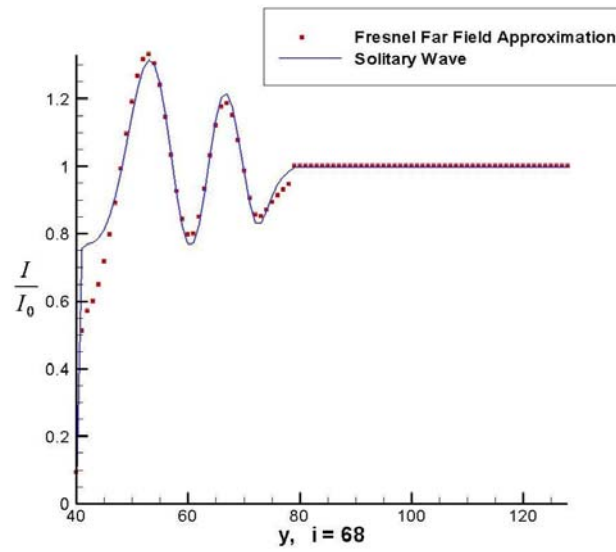
Paraxial approximation is an approximation to the multi-directional Helmholtz equation for long range propagation, which is valid in the limit of small angles with the horizontal. At very long distances, the field of importance propagates at angles close to a single preferred/paraxial direction. This approximation was introduced by Leontovich and Fock [33] in the 1940s to treat diffraction effects. It plays a very important role in radio wave propagation in atmosphere and has been extensively used in capturing interference. In spite of being advantageous in long range propagation problems, it fails to capture the effects of back scattering. To solve near-field problems, complicated transforms are applied to capture higher angles [6].

5.3.1 Standard Parabolic Equation

For problems where the waves propagate in a narrow angle the standard parabolic equation (SPE) is a good frequency domain approximation [34]. It involves factoring out the rapidly varying part of the wave. Taking the main direction as x , it involves solving for the slowly varying function, $u(x, y)$, which is related to the complex wave amplitude, $\psi(x, y)$, by



(a)



$y, i = 68$

Figure 5.6: Interference pattern in the illuminated region

(a) $i = 68$

(b) $i = 62$

(c)

$$u = e^{-\hat{i}kx}\psi \quad 5.15$$

where $\hat{i} = \sqrt{-1}$, k is the wave number and ψ satisfies the Helmholtz equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 n^2 \psi = 0, \quad 5.16$$

where n is the local index of refraction. For narrow propagation angles close to x the above equation operator can be approximated for propagation in x . The equation to be solved is then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\hat{i}k \frac{\partial u}{\partial x} + (n^2 - 1)k^2 u = 0, \quad 5.17$$

which is factored as

$$\left[\frac{\partial}{\partial x} + \hat{i}k \left(1 - \sqrt{\frac{1}{k^2} \frac{\partial^2}{\partial y^2} + n^2} \right) \right] \left[\frac{\partial}{\partial x} + \hat{i}k \left(1 + \sqrt{\frac{1}{k^2} \frac{\partial^2}{\partial y^2} + n^2} \right) \right] u = 0. \quad 5.18$$

Using first order Taylor expansions for the square root, the above equation for forward propagation is further reduced to standard parabolic equation as

$$\frac{\partial^2 u}{\partial y^2} + 2\hat{i}k \frac{\partial u}{\partial x} + (n^2 - 1)k^2 u = 0. \quad 5.19$$

The SPE solves the field by marching plane after plane in x direction. Various schemes have been developed to extend the accuracy to propagation at greater angles [35] [36]. These either involve higher order expansions about horizontal, or breaking the problem into distinct pieces, each of which involves a relatively narrow angle about a particular direction.

These extensions apparently do not result in a true multidirectional method. The difficulty is increased for higher frequencies. If θ is the angle of propagation and is not small, then the wave number will vary rapidly in y . In that case, $e^{\hat{i}kx}$ is not factored out accurately and the paraxial approximation fails to work. Other problems are that multiple arrival times (from very different directions) cannot be treated and,

of course, there must be a dominant propagation direction (or relatively small number of directions, one for each sector).

5.3.2 Local Parabolic Equation (LPM)

The newly developed LPM approximation involves the generalization of the above SPE method to general, multi-directions. The LPM will not be a higher order expansion about a single direction or a break-up of space into fixed, separate sectors to accommodate different directions: It will be radically different from other attempts to extend the SPE. In each region that the pulse passes through, amplitude and phase variables are updated as in the SPE, except that the direction is normal to the local constant phase surface. The directions used in this approximation are computed using WC and are explained in Section 5.1. Unlike the conventional SPE method, a local coordinate system (s, n) where s is the variable in propagation direction and n is the variable in the normal direction, is used for this solution. To start with, equation (5.1) is transformed to (s, n) coordinates as

$$A \frac{\partial \psi}{\partial s} + B \frac{\partial \psi}{\partial n} + C \frac{\partial^2 \psi}{\partial s^2} + D \frac{\partial^2 \psi}{\partial n^2} + 2E \frac{\partial^2 \psi}{\partial s \partial n} + k^2 n^2 \psi = 0 \quad 5.20$$

where $A = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}$, $B = \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}$, $C = \left(\frac{\partial s}{\partial x} \right)^2 + \left(\frac{\partial s}{\partial y} \right)^2$, $D = \left(\frac{\partial n}{\partial x} \right)^2 + \left(\frac{\partial n}{\partial y} \right)^2$,

$E = \frac{\partial s}{\partial x} \frac{\partial n}{\partial x} + \frac{\partial s}{\partial y} \frac{\partial n}{\partial y}$, $s = (xs_x + ys_y)$ and $n = (xn_x + yn_y)$. Now, the slowly varying

function, u , is defined as

$$u = e^{-iks} \psi. \quad 5.21$$

Equation (5.15) is now reduced to

$$\begin{aligned} A \frac{\partial u}{\partial s} + B \frac{\partial u}{\partial n} + C \frac{\partial^2 u}{\partial s^2} + D \frac{\partial^2 u}{\partial n^2} + 2E \frac{\partial^2 u}{\partial s \partial n} + ik \left(2C \frac{\partial u}{\partial s} + E \frac{\partial u}{\partial n} \right) \\ + \left[ikA + k^2 (n^2 - C) \right] u = 0 \end{aligned} \quad 5.22$$

where the transformation equation for the Laplacian is

$$A \frac{\partial u}{\partial s} + B \frac{\partial u}{\partial n} + C \frac{\partial^2 u}{\partial s^2} + D \frac{\partial^2 u}{\partial n^2} + 2E \frac{\partial^2 u}{\partial s \partial n} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} . \quad 5.23$$

Equation (5.22) can then be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \hat{k} \left(2C \frac{\partial u}{\partial s} + E \frac{\partial u}{\partial n} \right) + \left[\hat{k}A + k^2 (n^2 - C) \right] u = 0 , \quad 5.19$$

where $\frac{\partial u}{\partial s} = s_x \frac{\partial u}{\partial x} + s_y \frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial n} = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$.

5.4 Propagation results for LPM

As a first step, the LPM is used to compute plane wave propagation at high angles. For a single propagation direction (but not restricted to horizontal direction), $A = 0$, $E = 0$, $C = 1$ and equation 5.19 becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \hat{k} \left(2 \frac{\partial u}{\partial s} \right) + k^2 (n^2 - 1) u = 0 . \quad 5.20$$

The above equation can be solved using implicit or explicit methods. It is checked that an iterated explicit method converges to the same solution as the implicit method. The implicit form of the discretized equation used for the first computation is

$$\begin{aligned} & \frac{2\hat{k}s_x}{\Delta x} (u_{i+1,j} - u_{i,j}) + \frac{2\hat{k}s_y}{\Delta y} (u_{i+1,j} - u_{i+1,j-1}) + \frac{1}{\Delta x^2} (u_{i+1,j} + u_{i-1,j} - 2u_{i,j}) \\ & + \frac{1}{\Delta y^2} (u_{i+1,j+1} + u_{i-1,j-1} - 2u_{i+1,j}) + k^2 (n^2 - 1) u_{i+1,j} = 0 \end{aligned} \quad 5.21$$

where $u_{i,j}$ at planes $i=1$ and $i=2$ are known (provided as initial conditions) and is further reduced to tri-diagonal form,

$$\begin{aligned}
& u_{i+1,j-1} \left(-\frac{2\hat{i}ks_y}{\Delta y} + \frac{1}{\Delta y^2} \right) + u_{i+1,j} \left(\frac{2\hat{i}ks_x}{\Delta x} + \frac{2\hat{i}ks_y}{\Delta y} + \frac{1}{\Delta x^2} - \frac{2}{\Delta y^2} + k^2(n^2 - 1) \right) \\
& + u_{i+1,j+1} \left(\frac{1}{\Delta y^2} \right) = u_{i-1,j} \left(-\frac{1}{\Delta x^2} \right) + u_{i,j} \left(\frac{2}{\Delta x^2} + \frac{2\hat{i}ks_x}{\Delta x} \right)
\end{aligned} \tag{5.21}$$

The components of direction s (s_x, s_y) are pre-computed using WC as explained in Section 5.1. Finally, the Helmholtz solution is computed as,

$$\psi_{i,j} = e^{\hat{i}ks} u_{i,j} . \tag{5.22}$$

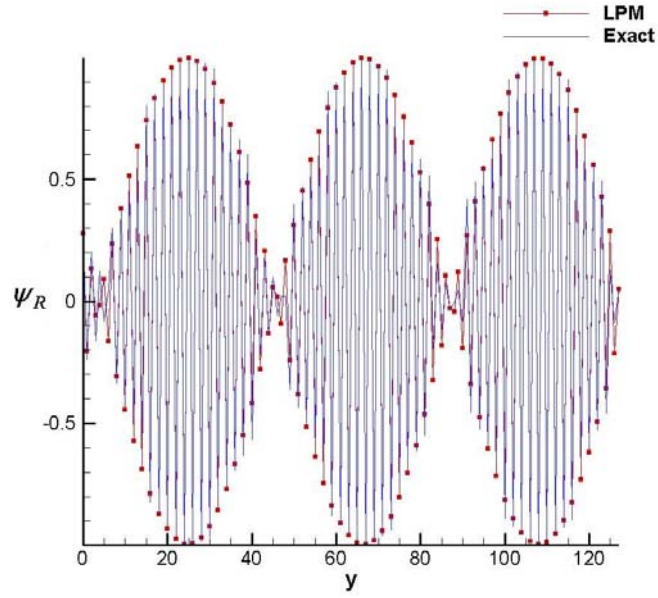
To validate the high angle propagation, a plane wave propagating at angle, $\theta = 30$ is computed and compared to the exact solution. Computations are performed on a 128×128 grid. The real part of the complex amplitude, ψ , is plotted at a distance of 6000λ and shown in Figure 5.7a. The wavelength used for this computation is $\lambda = \frac{\Delta x}{25}$. It can be seen that the LPM results match well with the exact solution.

Equation 5.12 is also used to solve the interference between two plane waves at angles $\theta = 60$ and $\theta = 60.06$ respectively. The pre-computed directions used for this purpose are the mean directions computed using WC. For example, the mean angle is 60.03 degrees for the present case, which is close to both angles. The reason for concentrating on waves close to a dominant direction at a point is that at long ranges, the interference is only significant when the rays are very close to each other. The interference pattern is computed for $\lambda = \Delta x$ and plotted for a plane 6000λ from the initial plane. The results are compared to the exact results as shown in Figure 5.7b. When equation (5.21) is used to solve for propagation in negative angles, the solution diverges, which can be shown in Figure 5.8a. This is because of the use of downwind value in $(u_{i+1,j} - u_{i+1,j-1})$. To overcome this problem, the difference operator is made dependent on the direction of propagation such that only upwind values are used for the computation (which has a similar approach as Jamson's rotated difference scheme for transonic flow [37]). This is even more important when there are varying propagating directions as in the cylindrical wave shown in Figure 4.2. For variable directions,

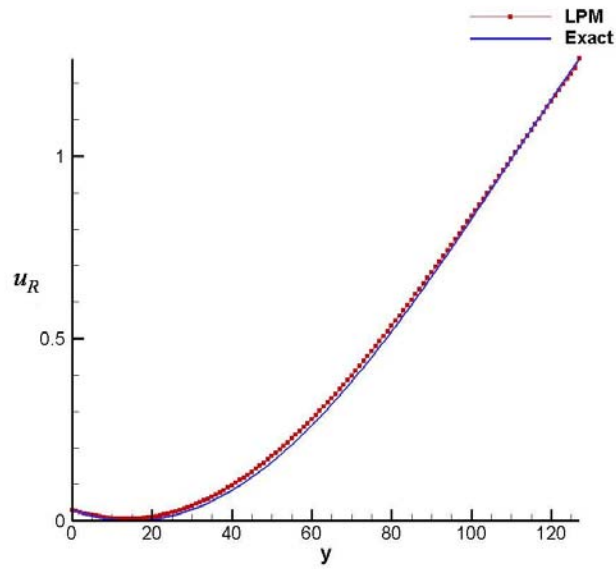
$$\text{If } s_y > 0, \quad \delta_j(u_{i+1,j}) = u_{i+1,j} - u_{i+1,j-1}$$

$$\text{If } s_y < 0, \quad \delta_j(u_{i+1,j}) = u_{i+1,j+1} - u_{i+1,j}$$

5.23



(a)



(b)

Figure 5.7 : Plane wave propagation at higher angles

- (a) $\theta = 30$, $\lambda = 0.04dx$, $x = 6000\lambda$
(b) $\theta_1 = 60$, $\theta_2 = 60.06$, $\lambda = \Delta x$, $x = 6000\lambda$

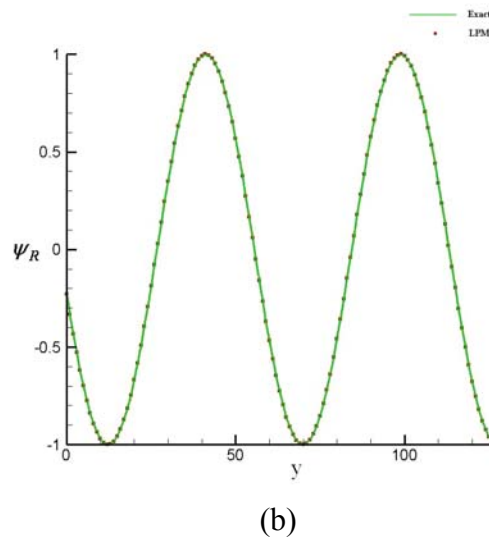
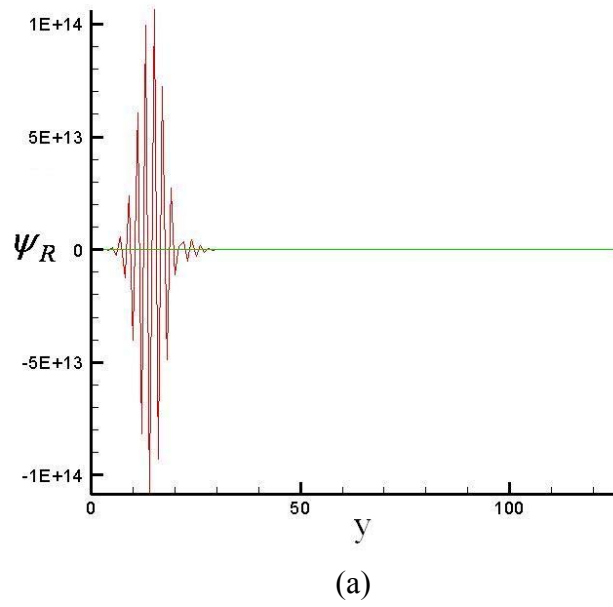


Figure 5.8: Propagation in negative angles
(a) Using equation 5.21
(b) Using equation 5.21 along with equation 5.23

Using equation (5.23), any angle is accommodated with no unstable behavior. In Figure 5.8b, reduced function, u for plane waves propagating in angles $\theta = -60$ and $\theta = -60.06$ is shown. It can be seen that the solution is stable and matches well with the exact solution.

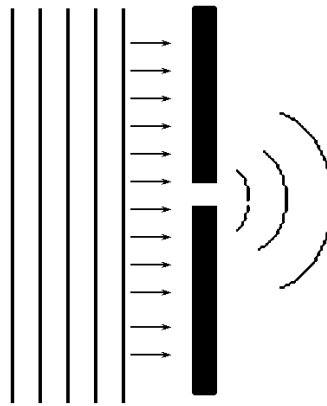
5.4.1 Single slit

When a plane wave front passes through a narrow slit, sideward propagation into the shadow region is observed as shown in Figure 5.9a. This occurs due to the diffraction, which is explained in Section 5.2. LPM is used to capture the effects of diffraction from a single slit. According to Huygens principle, each point on the aperture acts as a source for a secondary wave front. The interference between all of these sources produce the diffraction pattern. For a very narrow slit of width $b \ll \lambda$, the slit itself acts as one point source and produces a cylindrical wave. If $b \gg \lambda$, a bright spot as wide as slit is produced and grows as it propagates away from slit. When the slit size is comparable to the wavelength, an interference pattern is produced due to diffraction. In this section, diffraction computation using LPM are quantitatively investigated.

The equation used to solve the diffraction is given in equation 5.19. The discretized equation is

$$\begin{aligned} & \frac{2\hat{ik}_x C}{\Delta x} \delta_i u_{i+1,j} + \frac{2\hat{ik}_y C}{\Delta y} \delta_j u_{i+1,j} + \frac{\hat{ik}_x E}{\Delta x} \delta_i u_{i+1,j} + \frac{\hat{ik}_y E}{\Delta y} \delta_j u_{i+1,j} \\ & + \frac{\delta_i^2 u_{i+1,j}}{\Delta x^2} + \frac{\delta_j^2 u_{i+1,j}}{\Delta y^2} + \hat{ik} A u_{i+1,j} + k^2 (n^2 - C) u_{i+1,j} = 0 \end{aligned} \quad 5.24$$

where $\delta_i u_{i+1,j}$ and $\delta_j u_{i+1,j}$ are computed according to the special logic described in Section 5.4. In equation (5.24), A , C and E are computed as given below equation (5.15). Interference pattern for a single slit of width, $b = 30\lambda$ is computed and plotted for the plane, $x = 120\lambda$. The solution from LPM matches well with the exact solution as shown in Figure 5.9b.



(a)

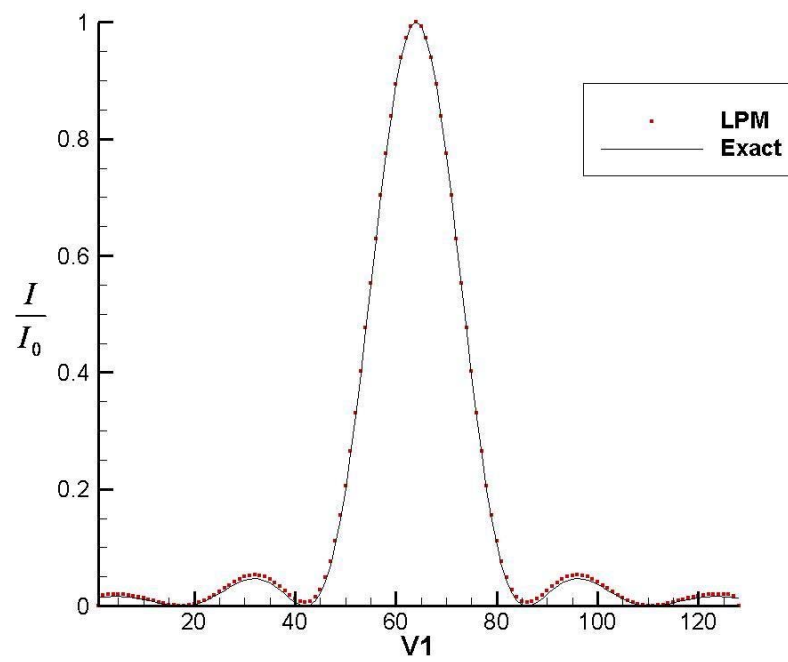


Figure 5.9: Single slit diffraction
 (a) Demonstration of diffraction phenomena
 (b) Interference pattern at $I = 120$

Chapter 6

CONCLUSION

The wave equation is accurately solved using the newly developed, Wave Confinement method, which includes the realistic effects such as varying index of refraction, scattering from complex objects and interference. Unlike conventional Eulerian schemes, WC does not suffer from numerical dissipation and the properties such as total amplitude and centroid are conserved. It generates wave equation solutions as thin solitary waves that can persist on the grid indefinitely. WC involves a nonlinear term in the propagation equation, unlike the original linear wave equation. However, this nonlinear term in the WC does not produce amplitude exchange or phase shift during wave collisions. After collision, the waves take their correct speed and form after a short relaxation time. Also, the interactions behave in the same way in any dimension. This is an important property in capturing the effects of focusing, multiple scattering and all the cases where many waves simultaneously meet at a point.

WC has been shown to capture caustic regions with no additional steps in the computation process. It is also observed that WC in combination with Ray Tracing is accurate. This led to a new approximation called Eulerian/Lagrangian Transition. The smooth transition is observed even when there are irregular boundaries. In spite of having a nonlinear term in WC, the waves are observed to focus with no shift in centroid positions. WC is also shown to generate smooth scattered surfaces during reflection from complex objects, preventing the need to use complicated adaptive grids. These objects are immersed in a Cartesian grid with no grid refinement and complex logic. This is due to the confinement term being diffusive in the tangential direction, keeping the scattered field smooth with no stair-casing effects. WC can also accommodate very small objects compared to the physical domain, and does not require fine grids, unlike conventional Eulerian schemes. This is an important feature in electromagnetic wave propagation in the atmosphere when complex surfaces such as terrain, buildings, etc., have to be accommodated.

Another important property of WC is its capability to capture the effects of varying index of refraction efficiently. This is an important feature in studying the effects of varying medium properties. Examples include evaporation or surface ducts (formed in atmosphere due to variations in density, temperature, etc), ionospheric ducts (formed in the ionosphere due to variations in electron density). These ducts can form caustic regions, multiple scattering etc. The confinement term is shown to treat such effects accurately and it is validated using Lagrangian method, which is an exact solution in high frequency approximation. Wave propagation across multiple grid interfaces is also proved to be accurate with no anomalous reflections. This will allow WC to use fine grids around an antenna and coarser grids in the far field. For the tested grid ratios, the transfer of amplitude across the interface is shown to be accurate.

WC is shown to accurately generate contours of constant arrival times as the wave passes a grid point. The scalar function propagated using WC can also be used as a carrier wave to propagate directions along with the wave. This allows efficient computation of the propagation directions. The properties propagated with the scalar function behave the same way as the scalar function itself during multiple interactions. Using the propagated directions, the wave solution in the high frequency approximation can be computed. When the wave encounters an edge, this will prevent sideward bending of the waves into the shadow region due to numerical diffusion. Using Keller's GTD along with WC, the diffraction pattern from a knife edge is computed and is shown to match quite well with the Fresnel far field approximation.

The local parabolic method is developed, which can compute propagation at any angle. It is an extension to the existing parabolic equation, which is confined to one preferred direction. Computations to simulate propagation at higher angles and single slit diffractions are done and validated using the exact analytical relations. Many computations are performed for a number of test cases and validated to check accuracy. The numerical properties of WC have to be well understood before using it for engineering applications. All the required validations have been performed for this new method to check the accuracy in actual complex conditions. WC still needs to be validated for propagation and interference effects for overlapping waves. As the pulsewidth will be relatively large compared to the wavelength in high frequency propagation, work has to be done to compute such effects within the wave. A more difficult, but widely studied challenge is the inverse scattering phenomena. This will be studied in the future to extend the applications of WC further.

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VITA

I was born in India in 1980. I graduated from Jawaharlal Nehru Technological University, India, with a Bachelor of Technology degree in Mechanical engineering in 2001. I received a Masters Degree in Mechanical engineering from Louisiana Technological University, Ruston, in 2004. I joined UTSI to pursue Ph.D. in Aerospace engineering and this dissertation is submitted to fulfill the degree requirements.