To the Graduate Council:

I am submitting herewith a dissertation written by Marek Alfred Galecki entitled "Enhanced cohomology and obstruction theory". I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy with a major in Mathematics.

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ENHANCED COHOMOLOGY AND OBSTRUCTION THEORY

A Dissertation

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Doctor of Philosophy

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Abstract

For any $n \geq 4$ and a finite regular CW-complex K we define an abelian group called the enhanced n-th cohomology group of K, denoted $EH^n(K)$. These groups have (contravariant) functorial properties; in fact, if $f \simeq g: K \to L$, then $f^* = g^*: EH^n(L) \to EH^n(K)$. Thus they are invariant on the homotopy type. There is an exact sequence

$$\ldots \to H^{n-1}(K;\mathbb{Z}) \xrightarrow{\mathrm{Sq}^2} H^{n+1}(K;\mathbb{Z}_2) \xrightarrow{I} EH^n(K) \xrightarrow{F} H^n(K;\mathbb{Z}) \xrightarrow{\mathrm{Sq}^2} H^{n+1}(K;\mathbb{Z}_2) \to \ldots$$

The enhanced cohomology groups give a well-defined obstruction to extend maps to some n-2-connected spaces from a subcomplex A of K to $K^{[n+1]} \cup A$. We thus have the primary and secondary obstructions encompassed in a single, well-defined step. We also have a well-defined obstruction to embedding an n-dimensional simplicial complex in $\mathbb{R}^{2n-1}$.

The classical Steenrod paper (Ann. of Math., **48**, 290-320) that introduced the squares produces a 2-step approach to solving the extension problem mentioned above, without realizing the existence of EH^n.

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CHAPTER 1

INTRODUCTION

The classical obstruction theory of Eilenberg contains the so-called "primary obstruction", which can be described as follows.

Let $n \geq 3$ and let Y be an n-2-connected topological space. Let (K,A) be a finite, regular CW-pair, and let $f{:}A \to Y$ be a map. Then f extends to $f^{\,n-1}{:}K^{[n-1]} \cup A \to Y$ and we have a cellular cochain $g \in C^{\,n}(K,A;\pi_{n-1}(Y))$, whose value on an n-cell σ^n is $[f^{\,n-1}|_{\mathrm{bd}\sigma}] \in \pi_{n-1}(Y)$. $[g] \in H^n(K,A;\pi_{n-1}(Y))$ is the primary obstruction: it does not depend on $f^{\,n-1}$, only on f, and $[g]=0$ if and only if f extends to $K^{[n]} \cup A$.

Suppose that $n \geq 4$ and $Y=S^{n-1}$. Then the primary obstruction can be translated geometrically as follows. Extend $f^{\,n-1}$ to $F^{\,n}{:}K^{[n]} \cup A \to D^n \supseteq \mathrm{bd}D^n = S^{n-1}$ in general position with respect to the center 0 of D^n. Then $F^{\,-1}(0)$ is a finite set; if this is $\emptyset$, we have extended f to $f^{\,n}{:}K^{[n]} \cup A \to S^{n-1}$. Around any point of this set take a small regular neighborhood in $K^{[n]}$, consider its boundary - an n-1-sphere, and look at the degree of $F^{\,n}$ restricted to that sphere (it will be +1 or -1). Summing the degrees up in a given n-cell will give us the value of g on that cell. In this way a configuration of points with numbers +1 or -1 attached to them gave rise to a cocycle. The primary obstruction works because cohomology can be interpreted in this way; n-cochains are finite sets of points in $K^{[n]}\setminus K^{[n-1]}$, with each point equipped with $+1$ or -1. Adding coboundaries corresponds to certain operations on such sets of points. These operations reflect what can happen to $(F^{n})^{-1}(0)$ during a homotopy of F^{n}; any two F^{n}'s are homotopic rel. A.

The attempt to extend f^{n-1} over two skeleta, to $f^{n+1}:K^{[n+1]}\cup A\to S^{n-1}$, starts with an extension $F^{n+1}:K^{[n+1]}\cup A\to D^{n}$, again in general position with respect to $0\in D^{n}$. Here $(F^{n+1})^{-1}(0)$ is a graph, which ordinary cohomology cannot handle. This, in short, is the reason why the secondary obstruction in classical theory is not well-defined. There is a need for a theory that handles such graphs in a way similar to the way ordinary cohomology handles points, explained above (and in detail in Chapter 5). This is called enhanced cohomology. This is a family of functors, for $n\geq 4$, from regular finite CW-pairs and homotopy classes of continuous maps to abelian groups and homomorphisms. The n-th enhanced cohomology group of (K,A) is denoted $\mathrm{E}H^{n}(K,A)$.

The enhanced cohomology gives a single, necessary and sufficient obstruction for extending f to $f^{n+1}:K^{[n+1]}\cup A\to Y$ for certain spaces Y. The obstruction, denoted μ_{f}, is in $\mathrm{E}H^{n}(K,A)$, and Y can be any Hausdorff space that is a subspace of an $n+1$-connected Hausdorff space C with a point $0\in C\setminus Y$ that has an n-dimensional Euclidean neighborhood in C, and such that $C\setminus 0$ retracts to Y. It is then easy to see that Y is n-2-connected. Thus the classical obstruction theory gives a well-defined obstruction for extending f to $K^{[n]} \cup A$.

In case $Y = S^{n-1}$, two obstructions were found in [St], whose vanishing means that f is extendable to $f^{n+1}: K^{[n+1]} \cup A \to Y$. That work can be easily modified to give two obstructions for our general Y. Using our terminology, these obstructions translate as follows.

There is an exact sequence

$$\ldots \to H^{n-1}(K,A;\mathbb{Z}) \xrightarrow{\mathrm{Sq}^2} H^{n+1}(K,A;\mathbb{Z}_2) \xrightarrow{I} EH^n(K,A) \xrightarrow{F} H^n(K,A;\mathbb{Z}) \xrightarrow{\mathrm{Sq}^2}$$
$$\xrightarrow{\mathrm{Sq}^2} H^{n+2}(K,A;\mathbb{Z}_2) \to \ldots,$$

look at $F([\mu_f])$ - this is the first obstruction; if it is 0, look at $I^{-1}([\mu_f])$ and see if it belongs to $\mathrm{Im}\,\mathrm{Sq}^2$.

Nevertheless, the group $EH^n(K,A)$ seems to be the most appropriate place in which the single obstruction lies.

There is an equivariant version of enhanced cohomology, obtained by considering cochains preserved, in a sense, by an involution $T: K \to K$. This version yields a necessary and sufficient obstruction to embed an n-dimensional simplicial complex K^n in $\mathbb{R}^{2n-1}$, $n \geq 5$. This is because the existence of such an embedding is equivalent to the existence of an equivariant map $F: K \times K \setminus \Delta \to S^{2n-2}$ (where Δ is the diagonal); such a map is obtained for free on codimension 2 skeleton of $K \times K \setminus \Delta$ and we can apply our obstruction theory (modified to equivariant setting) to extend the map over two next skeleta. See Chapter 19 for details.

The theory also yields a necessary and sufficient obstruction to embed K^n in $\mathbb{R}^{2n}$, $n \geq 3$, that coincides with the Shapiro-Wü obstruction ([Sh]), and a necessary obstruction to embed K^n in $\mathbb{R}^{2n-k}$, $k \geq 2$, that is not weaker than Shapiro-Wü's.

It should be theoretically possible to build functors which would give obstructions to extend maps as above, but to $K^{[n+k]}$ for $k > 1$. This would use k-dimensional objects in a way similar to our use of graphs. We would then have *enhanced cohomology of the k-th level for $k=0,1,2,\ldots$*. For $k=0$ this is ordinary cohomology, and for $k=1$ this is our enhanced cohomology; but even for $k=2$ the technical difficulties seem formidable.

The enhanced cohomology groups can be computed by a finite algorithm, although we do not do that here (more about this in Remark 20.1).

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First and foremost, I would like to thank my advisor Professor Lawrence S. Husch. Without his help, support, and criticism this work could not be completed. He answered countless questions and suggested ways to shorten several proofs.

I also want to thank others who have helped me. Professor R. Daverman helped shorten one of the proofs, suggested the name "enhanced cohomology" and answered numerous questions. Professors J. Dydak and M. Thistlethwaite also answered many questions. Finally, M. Bestvina and J. Stasheff suggested the connection with

Steenrod squares.

We will now briefly discuss the contents and main results.

Chapter 2 contains several preliminary Lemmas, whose proofs we could not find in the literature.

In Chapter 3, a certain operation $J:[S^n \times S^1, S^n] \to \pi_{n+1}(S^n)$ is studied. In Chapter 4, an operation on graphs and their neighborhoods, the saddle move, is studied. They both are crucial to the development of enhanced cohomology.

In Chapter 5, we formulate ordinary cohomology in a geometric fashion.

In Chapter 6, we geometrically define enhanced n-cochains in a finite regular CW-complex K and define the six Operations on the cochains. An enhanced coboundary is the difference between the output and the input of an Operation. Enhanced cochains form an abelian semigroup. All of that is supposed to resemble Chapter 5.

In Chapter 7, we develop an alternative approach that uses only particularly nice cochains, called non-singular. They are then used to specify enhanced cocycles. Enhanced cocycles and coboundaries are defined in completely different ways: the enhanced cohomology does not have a coboundary operator which could give rise to them both.

In Chapter 8, we prove that the coboundaries are cocycles, and define the enhanced cohomology semigroups $EH^n(K)$ as the quotient of cocycles over coboundaries. We prove that they are, in fact, groups.

The next few Chapters develop the functorial properties. Given a map $f:K\to L$ and an enhanced cochain S in L, we define $f^*(S)$ in Chapter 10. Then a lot of work is needed to see that the cohomology class of $f^*(S)$ depends only on the cohomology class of S and the homotopy class of f (we will usually say "cohomology class", "cohomologous" suppressing the word "enhanced"). Proposition 11.1 is the single most important step in the development of the theory. It roughly establishes that $f^*(S)$ is cohomologous to $g^*(S)$ as long as $f\simeq g$. Chapter 12 transfers f^* from the cochain level to the cohomology level. The main theorem 12.6 states that, for any $n\geq 4$, we have a contravariant functor from the category of regular finite CW-complexes and homotopy classes of continuous maps to the category of abelian groups. As a corollary, if K, L are homotopy equivalent finite regular CW-complexes, then $\mathrm{E}H^n(K)=\mathrm{E}H^n(L)$. In particular, $\mathrm{E}H^n(K)$ does not depend on the choice of a regular CW-structure.

In Theorem 15.1, we find a "classifying space" for the enhanced cohomology. In Chapter 16, we develop a connection between the enhanced cohomology, ordinary cohomology, and Steenrod squares. This is the exact sequence of 16.4. We then give calculations for several examples (16.5 - 16.7).

In Chapter 17, the relative theory is developed for use in the obstructions of Chapter 18. Corollary 18.4 describes out the obstruction for extending a map over two non-trivial skeleta. The crucial step is being able to realize local changes (Operations) on a cochain - "preimage" $(F^{n+1})^{-1}(0)$ - by a global change of the map F^{n+1}. This is accomplished, for various Operations, by Theorems 7.8, 4.2, and 6.7.

Chapter 19 develops the equivariant theory and the obstruction to embeddings (19.5, 19.6).

Notation and terminology.

We follow the convention, according to which an oriented manifold M induces the orientation for its boundary bd$M$ as in Figure 1.

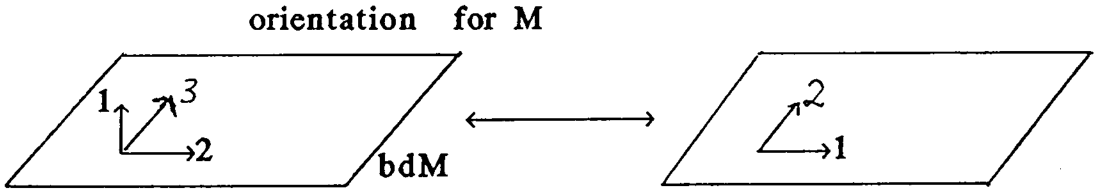
An oriented ball induces orientations on its subballs and then their boundaries - according to our convention.

According to [L-W], a CW-complex is called regular if each (closed) cell is homeomorphic to a ball. A regular finite CW-complex has a subdivision which is a finite simplicial complex. We fix this subdivision for each finite regular CW-complex.

If T is a CW-structure on a space, then $T^{[n]}$ denotes the n-skeleton.

The interior of a cell in a CW-complex is understood as the "open cell", not as the interior with respect to the whole complex.

B.H.E.T stands for the Borsuk Homotopy Extension Theorem for ANR's (or its PL-version for polyhedra).



"drop the (pointing inside) first vector".

Figure 1. How the manifold orients its boundary

The repetition set theory.

Suppose $S=\{x_i:i\in I\}$, where x_i's are (not necessarily pairwise different) elements of some fixed set and one wants to distinguish among the repetitions of the same element. One can form a set of pairs

$$S'=\{(x_i,d_i):i\in I\},$$

where d_i's are pairwise different elements of some set. A set with repetitions is a set S treated in this way. The assignment of d_i's is always arbitrary, and is subject to change without notice. If A has three repetitions of x, and B - one, then $A\setminus B$ has two repetitions, $B\setminus A$ - zero, $A\cap B$ - one, and $A\cup B$ - four. The assignment of d_i's to the resulting repetitions of x has nothing to do with the assignments before the operation was performed (not even the sets that contain d_i's are related).

"Elements of a set with repetitions are pairwise disjoint" means in particular that all non-empty elements have one repetition.

aff$A$ for $A\subseteq\mathbb{R}^n$ denotes the affine span of A.

Each non-zero vector in E^n has a direction which is a point in $S^{n-1}=\mathrm{bd}D^n$.

Δ^n is the standard n-simplex. o is the center of Δ^n. $k\Delta^n$, $k\in\mathbb{R}$, is $k(\Delta^n-o)+o$ (using the operations in the linear space $\mathbb{R}^{n+1}$).

D^n is the standard n-ball in $\mathbb{R}^n$. S^{n-1}, however, is $\mathrm{bd}D^n$ or $\mathrm{bd}\Delta^n$, the latter being the case for example when there is a PL-map that involves S^{n-1}. We fix an orientation for D^n, which induces an orientation for S^{n-1}.

A face of an n-ball is any standardly embedded n-1-ball in its boundary. A face of a simplex, however, has its standard meaning here.

The ball $B(x,r)$ is closed.

If $A \subseteq X$, $B \subseteq Y$, and $f:(\mathrm{cl}A;\mathrm{bd}A,\mathrm{int}A) \to (\mathrm{cl}B;\mathrm{bd}B,\mathrm{int}B)$ then bd$f$ means $f|_{\mathrm{bd}A}$.

cX is the cone over a compact metric space X.

$[X \to Y]$ is the set of homotopy classes from X to Y (we strongly dislike $[X,Y]$).

$A \subseteq B$ is PL means the inclusion $i:A \to B$ is PL.

By a simplicial approximation we mean a simplicial map homotopic to the original.

We take the notion of a graph from [B-S]. We only consider finite graphs. $G^{[0]}$ is the set of vertices of a graph G. We treat a graph as a "simplicial complex" with each edge being a simplex, even though many edges can have the same two vertices, or an edge can have one vertex. Then we can talk about subdivisions and certain approximation theorems hold. For example, any map between graphs has a "simplicial approximation", with the domain graph suitably subdivided, whereas the range keeps its original "triangulation" (the proof of this is left to the careful reader).

We will be frequently and tacitly (or writing "by codimension 3") using codimension-3-isotopy-extension theorems, namely [R-S]7.3 and, for the fixed-on-the-boundary case, [H1] Theorem 1.

In some easy parts of proofs we omit some conditions for the constructions performed. The careful reader should feel free to supplement the proofs as needed. In particular, words like "nice" or "standard" call for this kind of effort.

CHAPTER 2

PRELIMINARY LEMMAS

Lemma 2.1.

Let $B_3 \subseteq B_2 \subseteq B_1$ be PL, where B_1, B_2, B_3 are PL-balls. Suppose that B_2 is a topological neighborhood of B_3 in B_1, B_3 is proper and unknotted in B_2, and $B_2 \cap \mathrm{bd}B_1$ is a regular neighborhood of $B_3 \cap \mathrm{bd}B_1$ in $\mathrm{bd}B_1$. Then B_2 is a regular neighborhood of B_3 in B_1.

Proof.

Use [C]9.1(c).

Lemma 2.2.

Every triangulation T of the boundary of a PL-manifold M can be extended to the whole manifold.

Proof.

Let c be a PL-collar on the boundary. We can triangulate the exterior of c ($=M\setminus\mathrm{int}c$), and then subdivide so that the triangulation on $\mathrm{fr}_M c$ is a subdivision of T, modulo the PL-homeomorphism between the two boundaries of c. Now it is easy to extend the triangulation from the two boundaries to the whole collar.

Lemma 2.3. (Recognition Lemma)

Let S_1^1, S_2^{n-1} be disjoint PL-spheres in $\mathrm{int}\Delta^{n+1}$, $n \geq 3$. Suppose that there is a PL $n+1$-ball $B \subseteq \mathrm{int}\Delta^{n+1}$ and a homeomorphism $h: B \to \Delta^n \times I$ such that

$$h(S_1^1 \cap B) = o \times I, \quad S_2^{n-1} \subseteq \mathrm{bd}B,$$
$$h(S_2^{n-1}) = \Delta^n \times *, \ * \in I.$$

Then the linking number of S_1^1 and S_2^{n-1} is ± 1.

Proof.

Extend h to a PL-homeomorphism $\bar{h}: \Delta^{n+1} \to (2\Delta^n) \times [-1,2]$. Then, using codimension 3, PL-isotope $(2\Delta^n) \times [-1,2]$ relative $\Delta^n \times I$ to put $\bar{h}(S_1^1)$ in a standard position.

Lemma 2.4.

Let X be a compact metric space, and let $f: X \times I \to \mathbb{R}$ be a continuous map such that $f(X \times 0) \subseteq (-\infty, 0]$, $f(X \times 1) \subseteq [1, \infty)$, and, for any $x \in X$,

$$f|_{x \times I}$$

is increasing. Then the function $g: X \times I \to I$ defined by:

$$g(x,s) = t \text{ if } f(x,t) = s,$$

is continuous.

Proof.

$\bar{f}: X \times I \to X \times \mathbb{R}$ defined by $\bar{f}(x,t) = (x, f(x,t))$ is an embedding, and the inverse is $\bar{g}(x,s) = (x,t)$, where $s = f(x,t)$.

Lemma 2.5.

Let $n \geq 4$, and let T be a space PL-homeomorphic to $\Delta^n \times S^1$ that is embedded in the interior of a PL $n+1$-manifold M. Let s be a simple closed curve PL-embedded in int$T$. Suppose that s generates $\pi_1(T)$. Then T is a regular neighborhood of s in M.

Proposition 2.6.

Let N be a PL n-submanifold of a PL n-manifold M such that $N \cap \mathrm{bd}M$ is a PL n-1-manifold. Let $H:(N \cup \mathrm{bd}M) \times I \to M$ be a PL-isotopy that moves points from int$M$ within int$M$, and points from bd$M$ within bd$M$. Then H can be covered by an ambient isotopy of M.

Lemma 2.7.

Let $h:(D^n \times I, D^n \times \mathrm{bd}I) \to (M^{n+1}, \mathrm{bd}M^{n+1})$ be an embedding in an orientable manifold M with connected boundary and suppose the levels of $D^n \times I$ are continuously oriented. Then the top and bottom levels orient (via h) bdM^{n+1} in two different ways.

Proof.

The two orientations of bd$M$ are induced by different orientations of M.

Lemma 2.8.

Let $n \neq 4,5$. Let $B \subseteq C$ be a PL-inclusion of n-balls such that $B \cap \mathrm{bd}C = \mathrm{bd}B \setminus (\mathrm{int}A_0 \cup \mathrm{int}A_1)$, where A_0, A_1 are disjoint PL n-1-faces of B. Then the pair (C,B) is standard, or PL-homeomorphic to $(\Delta^{n-1} \times [0,3], \Delta^{n-1} \times [1,2])$.

Proof.

The triad $(B;A_0,A_1)$ is standard, or PL-homeomorphic to $(\Delta^{n-1}\times I;\Delta^{n-1}\times 0,\Delta^{n-1}\times 1)$. Let $A\subseteq B$ be (the inverse image of) the middle ball $\Delta^{n-1}\times\frac{1}{2}$. By [C], B is a regular neighborhood of A in C. We note that a proper locally flat PL-ball pair (B^n,B^{n-1}), $n\neq 4,5$, is PL-standard (this follows from the discussion of sphere pairs and Schönflies Conjecture in [R-S]). Therefore, A is standard in C. Thus (C,B,A) is PL-homeomorphic to $\Delta^{n-1}\times\left([0,3],[1,2],\frac{3}{2}\right)$.

CHAPTER 3

THE MAP "J"

We shall construct a map J from $[S^n \times S^1 \to S^n]$ to $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$ for $n=3,4,\ldots$. If we restrict to classes of maps $f: S^n \times S^1 \to S^n$ with $\deg f|_{S^n \times *} = 1$, this will turn out to be a bijection, and the two sets of homotopy classes will be interchanged throughout our work.

Embed $D^{n+1} \times S^1$ in $\mathrm{int} D^{n+2}$ in a standard way. Any map $f: S^n \times S^1 = \mathrm{bd} D^{n+1} \times S^1 \to S^n$ easily extends to the complement of $\mathrm{int} D^{n+1} \times S^1$, and we land in $\pi_{n+1}(S^n)$ by restricting the extension to $\mathrm{bd} D^{n+2}$. For the proof of the next theorem, we organize this description as shown below.

Assume that $\mathbb{R}^n$, $\mathbb{R}^{n+1}$ are contained in $\mathbb{R}^{n+2}$ by using first n, $n+1$ coordinates, respectively. Regard $D^{n+1} \times S^1$ as lying in $\mathbb{R}^{n+2}$ by rotating $B((0,\ldots,0,2,0),1) \cap \mathbb{R}^{n+1}$ around $\mathbb{R}^n$ (namely, the identification is

$$((x_1,\ldots,x_{n+1}),(y_1,y_2)) \leftrightarrow (x_1,\ldots,x_n,y_1(x_{n+1}+2),y_2(x_{n+1}+2)))).$$

Denote the rotation by $r_t: \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$, $t \in [0,1]$:

$$r_t(x_1,\ldots,x_n,x_{n+1}) = (x_1,\ldots,x_n,x_{n+1}\cos 2\pi t, x_{n+1}\sin 2\pi t).$$

Regard, for the time of this discussion of J, D^{n+2} as $B(0,4) \subseteq \mathbb{R}^{n+2}$, and S^{n+1} as $\mathrm{bd} B(0,4)$.

We can extend f to $\overline{\overline{f}}: E \to S^n$, where $E = D^{n+2} \setminus \mathrm{int}(D^{n+1} \times S^1)$ by first extending over a nice neighborhood D of a 2-disc $\check{D}$ spanning $* \times S^1$.

The above description is illustrated in Figure 2.

Let us define $J([f])=\left[\left.\bar{\bar{f}}\right|_{S^{n+1}}\right]$.

J is well-defined because E deformation retracts to $(S^n\times S^1)\cup\check{D}$ and if $\bar{f}\simeq\bar{f}':S^n\times S^1\to S^n$ and $\bar{f},\bar{f}':(S^n\times S^1)\cup\check{D}\to S^n$ are extensions, then $\bar{f}\simeq\bar{f}'$.

An orientation on S^n induces, by translating, an orientation on $S^n+(0,\dots,0,2,0)\subseteq S^n\times S^1$, which, in turn, induces orientations for all the spheres $S^n\times *\subseteq S^n\times S^1$.

Theorem 3.1.

Assume $n\geq 3$. There are two homotopy classes of maps $f:S^n\times S^1\to S^n$ such that $\deg f|_{S^n\times *}=1$: $[\rho]$, where $\rho:S^n\times S^1\to S^n$ is $\rho(x,y)=x$, and $[\gamma]$, where γ on $S^n\times\{e^{2\pi it}\}$ is induced by the element

$$\begin{bmatrix} \cos 2\pi t & \sin 2\pi t & 0 & \dots & 0 \\ -\sin 2\pi t & \cos 2\pi t & 0 & \dots & 0 \\ 0 & 0 & 1\ 0 & \dots & 0 \\ 0 & 0 & 0\ 1\ 0 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 0\ 1 \end{bmatrix}$$

of $SO(n+1)$. Moreover, $J([\rho])=0$, $J([\gamma])=1$.

An analogous statement holds for maps of degree -1.

Furthermore, representatives of $[\rho]$ and $[\gamma]$ can be chosen to be PL-homeomorphisms on each $S^n\times *$.

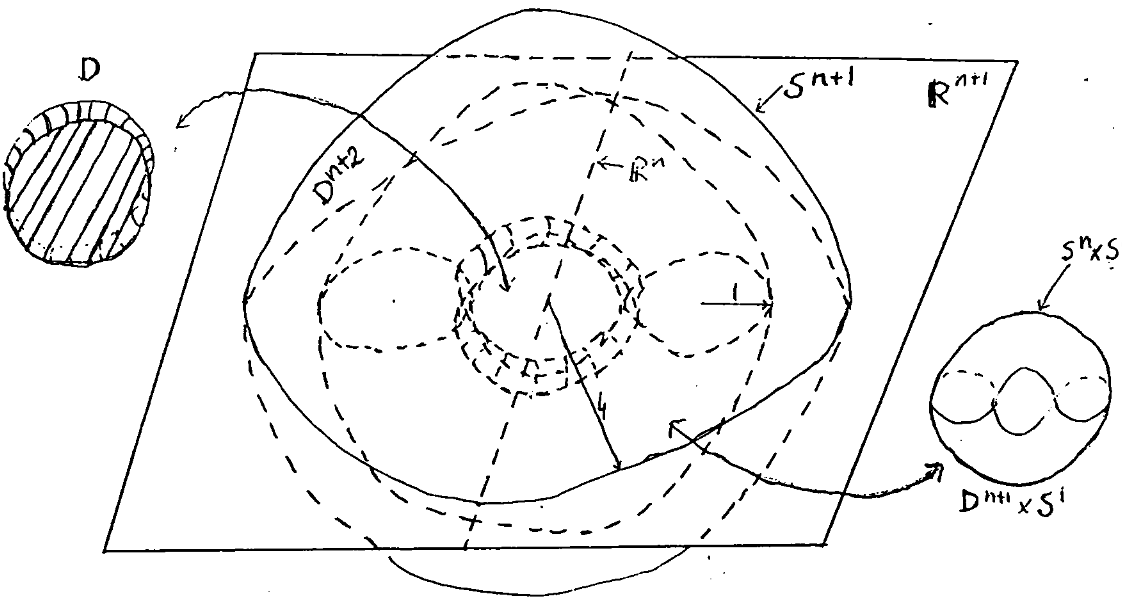


Figure 2. Objects needed to define J

Proof.

After we pick base points $*\in S^n, S^1$, we can identify, by homotoping, $[S^n\times S^1\to S^n]$ with $\pi_n(F\Omega S^n)$, where $F\Omega S^n$ is the space of all maps of S^1 to S^n.

$F\Omega S^n$ fibers over S^n by sending a map to its evaluation at $*$; the fiber is ΩS^n (the loops beginning at $*$). Consider the exact homotopy sequence of this fibration. The connecting homomorphisms $\pi_m(S^n)\to\pi_{m-1}(\Omega S^n)$ are trivial: lift the map $g:(D^m,\mathrm{bd}D^m)\to(S^n,*)$ to $F\Omega S^n$ using constant loops.

$$\begin{array}{ccccccccc}
\dots\to\pi_{n+1}(S^n) & \xrightarrow{0} & \pi_n(\Omega S^n) & \to & \pi_n(F\Omega S^n) & \to & \pi_n(S^n) & \xrightarrow{0} & \pi_{n-1}(\Omega S^n)\to\dots \\
 & & \wr\Vert & & & & \Vert & & \Vert \\
 & & \pi_{n+1}(S^n) & & & & \mathbb{Z} & & \mathbb{Z} \\
 & & \Vert & & & & & & \\
 & & \mathbb{Z}_2 & & & & & &
\end{array}$$

Therefore, the sequence splits so that $\pi_n(F\Omega S^n)\cong\mathbb{Z}_2\oplus\mathbb{Z}$ in such a way that the coordinate of $[f]$ in $\mathbb{Z}$ is determined by the degree of $f\big|_{S^n\times *}$. Consequently, there are two classes in $[S^n\times S^1\to S^n]$ with this degree equal to 1.

It is easy to see $J([\rho])=0$ by suitably extending ρ to E.

Now we prove that $J([\gamma])=1\in\pi_{n+1}(S^n)$. Let $\mathbb{R}^{n+1}_+$ be the subspace of $\mathbb{R}^{n+1}$ consisting of points with non-negative last coordinate. Rotate $\mathbb{R}^{n+1}_+$ around $\mathbb{R}^n$ in $\mathbb{R}^{n+2}$ obtaining at any moment $t\in[0,1]$ a half-hyperplane $H_t=r_t(\mathbb{R}^{n+1}_+)$ $(H_0=H_1=\mathbb{R}^{n+1}_+)$. Let $t\in[0,1]$ and let $\gamma_t:S^n+(0,\dots,0,2,0)\to S^n$ be the composition of the translation τ by $(0,\dots,0,-2,0)$ and the map $\gamma(t):S^n\to S^n$ determined by the element $\gamma(t)$

of $SO(n+1)$. $H_t \cap (S^n \times S^1)$ is an n-sphere in $S^n \times S^1$ which is mapped by γ to S^n by the composition of the rotation r_t^{-1} (that takes $H_t \cap (S^n \times S^1)$ to

$$H_0 \cap (S^n \times S^1) = S^n + (0,\dots,0,2,0))$$

and γ_t.

We are ready to build a certain extension $\Psi: E \to S^n$. Let $C = B(0;4) \cap \mathbb{R}^{n+1}_+$, where $B(0;4)$ is in $\mathbb{R}^{n+1}$. Consider a "radial projection" p in $\mathbb{R}^{n+1}$ with center at $(0,\dots,0,1) \in \mathbb{R}^{n+1}$, from bd$C$ to $S^n + (0,\dots,0,2)$. There is a map

$$\Psi_0 : C \setminus \mathrm{int}(D^{n+1} + (0,\dots,0,2,0)) \to S^n$$

that restricts to $\tau \circ p : \mathrm{bd}C \to S^n$, and $\tau : S^n + (0,\dots,0,2,0) \to S^n$. Denote $C_t = r_t(C \setminus \mathrm{int}(S^n + (0,\dots,0,2,0)))$ and let $\Psi_t : C_t \to S^n$ be the composition $\gamma(t)\Psi_0 r_t^{-1}$. Note that $B(0;4) \subseteq \mathbb{R}^n$ is the intersection of any two different C_t's and that the restriction of Ψ_t to this $B(0;4)$ is always the same; note also that $\Psi_0 = \Psi_1$. Therefore, one can define $\Psi: E \to S^n$ by gluing Ψ_t's together. This map is continuous.

To prove $J([\gamma]) = 1$ we need to see that $h = \Psi\big|_{\mathrm{bd}D^{n+2}}$ is essential. We shall do that by "desuspending" (up to homotopy) this map to a map $g: S^3 \to S^2$. We will then see that g determines an odd number in $\pi_3(S^2) \cong \mathbb{Z}$ and so, by the Freudenthal suspension theorem ([Wh]7.7.13), h is essential. We now provide some details.

Observe that h maps the equator

$$Q = \{(x_1,\dots,x_{n+2}) \in S^{n+1} : x_n = 0\}$$

of S^{n+1} to the equator $\{(x_1,\dots,x_{n+1}) \in S^n : x_n = 0\}$ of S^n, and the upper (lower) hemisphere $x_n \geq 0$ ($x_n \leq 0$) to the upper (lower) hemisphere. Therefore $h \simeq \Sigma(h|_Q)$ and h can be desuspended (up to homotopy) to $h|_Q$. Continue in this fashion (replace n-th coordinate by n-1-st, n-2-nd, and so on) until the equator has dimension 3. We are now faced with the map

$$g = h\Big|_{\{(x_1,\ldots,x_{n+2}) \in S^{n+1}:\ x_3 = \ldots = x_n = 0\}} : S^3 \to S^2,$$

which can be described as follows.

In this paragraph, we will only work with points $(x_k) \in \mathbb{R}^{n+2}$ such that $x_3 = \ldots = x_n = 0$. Express S^3 as $D^2 \times S^1/_\sim$, where all the circles $(\mathrm{bd}D^2) \times *$ are identified to become one ($D^2 \times e^{2\pi it}$ corresponds to $r_t(\{(x_k) \in S^{n+1}: x_{n+2} = 0,\ x_{n+1} \geq 0\})$; $\mathrm{bd}D^2 \times *$ corresponds to $\{(x_k) \in S^{n+1}: x_{n+1} = x_{n+2} = 0\}$), and let D_ε be a concentric, smaller, disc in D^2 (D_ε, when considered as a subset of $D^2 \times \{(1,0)\}$, corresponds to $\{(x_k) \in S^{n+1}: x_{n+1} \geq 1,\ x_{n+2} = 0\}$).

Let $\gamma_t: S^2 \to S^2$, $t \in [0,1]$, be induced by the element

$$\begin{bmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ -\sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of $SO(3)$, and let $\gamma: S^2 \times S^1 \to S^2$ be defined on $S^2 \times e^{2\pi it}$ by γ_t. Then there is a map $k: D_\varepsilon \to S^2$ collapsing $\mathrm{bd}D_\varepsilon$ to $* = (0,0,1) \in S^2 \subseteq \mathbb{R}^3$ and being 1-1 otherwise, such that $g([(x,t)]) = \gamma_t k(x)$ for $x \in D_\varepsilon$ and $g([(x,t)]) = *$ otherwise (the last property comes from the fact that p sends

$$\{(x_1,\ldots,x_{n+1}) \in \mathrm{bd}C:\ x_{n+1} \leq 1\}$$

to $(0,\ldots,0,1)$).

Consider two different points p_1, p_2 on the equator $\{(x_1,x_2,x_3)\in S^2\subseteq\mathbb{R}^3\colon x_3=0\}$ of S^2. It is easy to see that $g^{-1}(p_1)$, $g^{-1}(p_2)$ are parallel (1,1)-curves on some "standard" torus in S^3, so their linking number is ±1. As it is well-known, this means that $[g]=\pm1\in\pi_3(S^2)$ (see [P]§§10, 13, for example).

Because $J([\gamma])\neq J([\rho])$, we have $[\gamma]\neq[\rho]$, but there are only two homotopy classes in $[S^n\times S^1\to S^n]$ with the suitable degree, so they have to be $[\gamma]$, $[\rho]$. ∎

Corollary 3.2.

Let $n\geq3$. Let $S^n\xrightarrow{1}S^n$ be the space of maps from S^n to itself of degree 1 with base point Id_{S^n}. Then

$$\pi_1(S^n\xrightarrow{1}S^n)=\mathbb{Z}_2.$$

Moreover, $[\rho]=0\in\pi_1(S^n\xrightarrow{1}S^n)=\mathbb{Z}_2$ and $[\gamma]=1$. We identify ρ and γ here with their suitable adjoint maps.

Analogous result holds for degree -1.

Proof.

3.1 states that there are two homotopy classes of loops in $S^n\xrightarrow{1}S^n$. This means $\pi_1(S^n\xrightarrow{1}S^n)$ has two conjugacy classes. It suffices to show that this group is abelian.

Consider $J\pi\colon\pi_1(S^n\xrightarrow{1}S^n)\to\pi_{n+1}(S^n)$, where π is a map that takes an element of $\pi_1(S^n\xrightarrow{1}S^n)$ to its conjugacy class in $[S^n\times S^1\to S^n]$.

Claim. $J\pi$ is a group homomorphism.

The claim implies that, for $x,y\in\pi_1(S^n \xrightarrow{1} S^n)$,

$$J\pi(y^{-1}x^{-1}yx)=J\pi(1)=1,$$

so $\pi(y^{-1}x^{-1}yx)=\pi(1)$ and thus $y^{-1}x^{-1}yx=1$.

Proof of the claim.

If f, g represent elements of $\pi_1(S^n \xrightarrow{1} S^n)$, then we can assume that

$$f(x)=\mathrm{Id}_{S^n} \text{ for } x\in\left[0,\tfrac{1}{2}\right],$$

$$g(x)=\mathrm{Id}_{S^n} \text{ for } x\in\left[\tfrac{1}{2},1\right].$$

It is then possible to extend f to $\bar{\bar{f}}$ and g to $\bar{\bar{g}}$ on E so that $\bar{\bar{f}}\Big|_{\mathbb{R}_-^{n+2}\cap E}$ is the same as $\bar{\bar{g}}\Big|_{\mathbb{R}_+^{n+2}\cap E}$, modulo the reflection. $\bar{\bar{f}}\Big|_{\mathbb{R}_+^{n+2}\cap E}\cup\ \bar{\bar{g}}\Big|_{\mathbb{R}_-^{n+2}\cap E}$ represents the element gf.

Corollary 3.3.

(a) Let $h_0,h_1:S^n\to S^n$ be maps of degree 1. There are two homotopy classes of maps $h:S^n\times I\to S^n\times I$ such that

$$h\big|_{S^n\times\{i\}}=h_i:S^n\times\{i\}\to S^n\times\{i\},\quad i=0,1,$$

rel $S^n\times\{0,1\}$.

(b) If h_0, h_1 are PL-homeomorphisms, then both classes have representatives that are homeomorphisms.

(c) (a) and (b) remain true with S^n replaced by the pair (Δ^{n+1},S^n).

Corollary 3.4.

Let $f:S^n \times S^1 \to S^n$, $n \geq 3$, be a map such that $\deg f|_{S^n \times *} = \pm 1$. Let $g,d:S^n \to S^n$ be maps of degree -1. Then $f \circ (g \times \mathrm{Id}_{S^1}):S^n \times S^1 \to S^n$ is homotopic to df.

Proof.

Assume $\deg f|_{S^n \times *} = 1$. It is enough to prove that, for any maps $\alpha, \beta: S^n \to S^n$ of degree -1,

$$\beta \circ f \circ (\alpha \times \mathrm{Id}_{S^1}) \simeq f: S^n \times S^1 \to S^n.$$

Because there are only two homotopy classes possible, it is enough to prove

$$\beta \circ f \circ (\alpha \times \mathrm{Id}_{S^1}) \simeq p \Leftrightarrow f \simeq p.$$

The implication "$\Leftarrow$" holds because, as it is easy to see, $\beta \circ p \circ (\alpha \times \mathrm{Id}_{S^1}) \simeq p$. Now assume $\beta \circ f \circ (\alpha \times \mathrm{Id}_{S^1}) \simeq p$. Then

$$f \simeq \beta \circ \beta \circ f \circ (\alpha \times \mathrm{Id}_{S^1}) \circ (\alpha \times \mathrm{Id}_{S^1}) \simeq \beta \circ p \circ (\alpha \times \mathrm{Id}_{S^1}) \simeq p.$$

The case of $\deg f|_{S^n \times *} = -1$ is analogous.

Remark 3.5.

Assume $n \geq 5$. The two homotopy classes of maps $f:S^n \times S^1 \to S^n$ such that $\deg f|_{S^n \times *} = 1$ can be also recognized by means of the "framed bordism".

We now generalize the definition of J.

Observation 3.6.

Let s be a PL-simple closed curve in the interior of a PL-ball

B^{n+1}, where $n \geq 4$. Let N be a regular neighborhood of s in int$B$ and let h:fr$N \to S^{n-1}$ be a map. Then because the pair (B,N) is homeomorphic to $(D^{n+1}, D^n \times S^1)$ h determines a map $\bar{\bar{h}}$ from bd$B$ to S^{n-1} as in the definition of J, whose homotopy class depends only on the homotopy class of h. Thus we have a function from [fr$N \to S^{n-1}$] to [bd$B \to S^{n-1}$], which we shall also denote by J. ■

We can extend the definition of J to several simple closed curves, basically by adding the values for each of the curves. Our application of this requires that we use a sphere as the ambient space.

Observation 3.7.

Let $s_1,\ldots,s_k$ be disjoint PL simple closed curves in a PL $n+1$-sphere S, where $n \geq 4$. Let N be their regular neighborhood, consisting of regular neighborhoods N_i for s_i, $i=1,\ldots,k$. Let h:fr$N \to S^{n-1}$ be a map. We choose PL $n+1$-ball $B \subseteq S$ that contains all the curves in its interior. We can find a "chain" of PL $n+1$-balls $B_1,\ldots,B_k$, such that:

- $N_i \subseteq \text{int}B_i$ for $i=1,\ldots,k$,
- $\bigcup_i B_i = B$,
- $B_i \cap B_j = \emptyset$ unless $|i-j|=1$, in which case B_i and B_j meet at a face of both, or $i=j$.

Let $h_i = h|_{\text{fr}N_i}$ for $i=1,\ldots,k$. One can extend each h_i to

$\overline{h}_i: B_i \setminus \mathrm{int} N_i \to S^{n-1}$. Then one can modify those extensions to get an extension of h to $\overline{h}: B \setminus \mathrm{int} N \to S^{n-1}$. Let $\overline{\overline{h}} = \overline{h}\big|_{\mathrm{bd}B}$; let $\overline{\overline{h}}_i = \overline{h}_i\big|_{\mathrm{bd}B_i}$. Now $[\overline{\overline{h}}] = \sum_i [\overline{\overline{h}}_i]$ in $\pi_n(S^{n-1})$, and so $[\overline{\overline{h}}] = \sum_i J([h_i])$. In particular, $[\overline{\overline{h}}]$ does not depend on the choice of the balls B_i, extensions, or modifications. In this way we get a function from $[\mathrm{fr}N \to S^{n-1}]$ to $[\mathrm{bd}B \to S^{n-1}]$. Moreover, as the next proposition implies, $[\overline{\overline{h}}] \in \pi_n(S^{n-1})$, does not depend on the choice of B. Therefore, we have a well-defined function $J: [\mathrm{fr}N \to S^{n-1}] \to \pi_n(S^{n-1})$. Note that we can write $J([h]) = \sum_{i=1}^{k} J([h_i])$.

We single out the following obvious

Proposition 3.8.

$[\overline{\overline{h}}] = 0 \in \pi_n(S^{n-1})$ iff there is an extension of h to $S \setminus \mathrm{int} N$.

$[\overline{\overline{h}}] = 1$ iff there is no such extension.

Corollary 3.9.

Let $\alpha: S \to S$ be a homeomorphism of the $n+1$-sphere. Then $J\left(\left[h\alpha^{-1}\big|_{\mathrm{fr}(\alpha(N))}\right]\right) = J([h])$.

Proposition 3.8 implies

Lemma 3.10.

Let $f: S^{n-1} \to S^{n-1}$ be a map of degree ± 1. Then $J([h]) = J([fh])$.

CHAPTER 4

THE SADDLE MOVE

Let $n \geq 1$ and, if $x \in \mathbb{R}^{n+1}$, then let x_k be its k-th coordinate. Consider the following PL-ball:

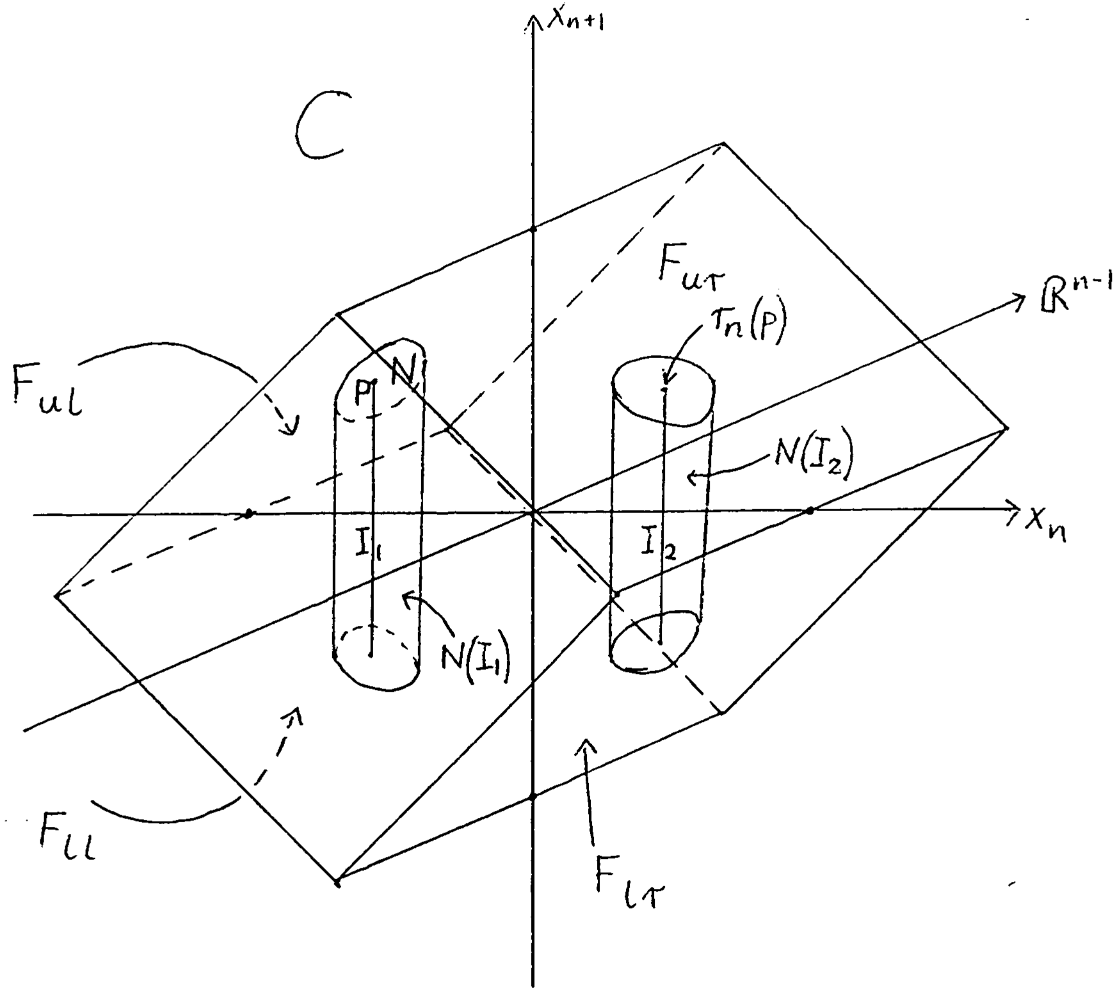
$$C = \{x \in \mathbb{R}^{n+1}\colon\ |x_n| + |x_{n+1}| \leq 1,\ \forall_i\ |x_i| \leq 1\},$$

and consider its faces as pictured in Figure 3. Let $\pi_v : C \to F_{ul} \cup F_{ur}$ be the ("vertical") projection changing only $n+1$-st coordinate, and $\pi_h : C \to F_{ul} \cup F_{ll}$ - the projection changing only n-th coordinate. Let r_k be the reflection with respect to $\{x\colon x_k = 0\}$, for any k.

Let $p \in \mathrm{int}F_{ul}$ (interior in $\mathrm{aff}F_{ul}$), and let N be a regular neighborhood of p in $\mathrm{int}F_{ul}$. Let $h : \mathrm{fr}N \to S^{n-1}$ be a map of degree ± 1. Denote $I_1 = \pi_v^{-1}(p)$, $\mathrm{N}(I_1) = \pi_v^{-1}(N)$, $I_2 = \pi_v^{-1}(r_n(p))$, $N(I_2) = \pi_v^{-1}(r_n(N))$. Then $N(I_i)$ is a regular neighborhood of I_i in C such that $r_n(N(I_1)) = N(I_2)$. Let $h_{I_1} : \mathrm{fr}N(I_1) \to S^{n-1}$ be the composition $h\pi_v|_{\mathrm{fr}N(I_1)}$. Let $h_{I_2} = h_{I_1} r_n : \mathrm{fr}N(I_2) \to S^{n-1}$. Denote further $J_1 = \pi_h^{-1}(p)$, $N(J_1) = \pi_h^{-1}(N)$, $J_2 = r_{n+1}(J_1)$, $N(J_2) = r_{n+1}(N(J_1))$. Let $h_{J_1} : \mathrm{fr}N(J_1) \to S^{n-1}$ be $h\pi_h$; let $h_{J_2} : \mathrm{fr}N(J_2) \to S^{n-1}$ be $h_{J_1} r_{n+1}$. The pass from I_1, I_2, h_{I_1}, h_{I_2} to J_1, J_2, h_{J_1}, h_{J_2} will be called a *saddle move*. The end result is shown in Figure 4.

Theorem 4.1 (realization of the saddle move).

Let $n \neq 2$. Let D^n be embedded in a space T. Let $f : C \to T$ be a map



$$F_{ul}=\{x\in C\colon x_{n+1}=x_n+1\} \text{ (upper left)}$$

$$F_{ur}=\{x\in C\colon x_{n+1}+x_n=1\}$$

$$F_{ll}=\{x\in C\colon x_{n+1}+x_n=-1\}$$

$$F_{lr}=\{x\in C\colon x_{n+1}=x_n-1\}$$

Figure 3. The situation before the saddle move

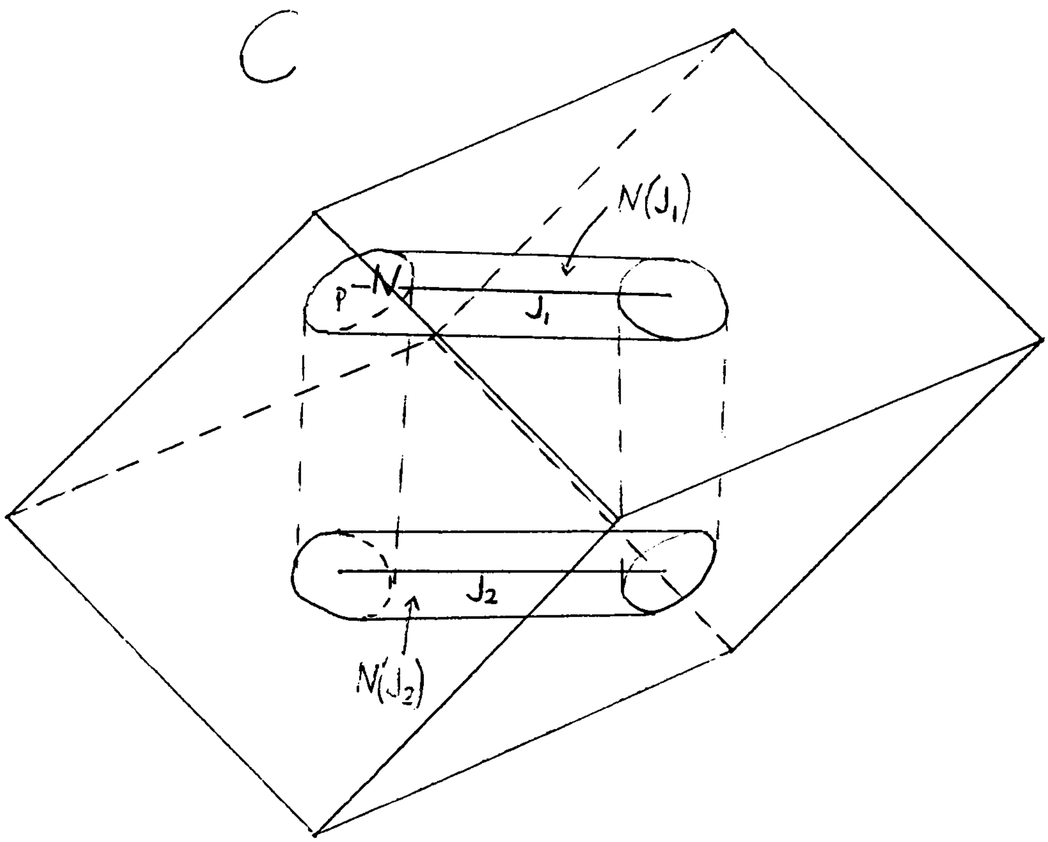


Figure 4. The situation after the saddle move

such that $f^{-1}(0)=I_1 \cup I_2$ $(0 \in D^n)$, $f(N(I_i)) \subseteq D^n$, $i=1,2$, and $f|_{\mathrm{fr}N(I_i)}=h_{I_i}$, $i=1,2$. Then there is a map $f':C \to T$ such that $f'|_{\mathrm{bd}C}=f|_{\mathrm{bd}C}$, $f'^{-1}(0)=J_1 \cup J_2$, $f'(N(J_i)) \subseteq D^n$, $f'|_{\mathrm{fr}N(J_i)}=h_{J_i}$, $i=1,2$.

Proof.

f' is already defined on $\mathrm{bd}C \cup \mathrm{fr}N(J_1) \cup \mathrm{fr}N(J_2)$. To prove the theorem, we need to extend it to the rest of C. The extension over $N(J_1) \cup N(J_2)$ is easy to produce. The extension over $C \setminus (N(J_1) \cup N(J_2))$ is more difficult. In order to construct it, consider $C \times [0,1]$ and regard I_1, I_2 as lying on the 0-level, and J_1, J_2 - on the 1-level.

Let s be the simple closed curve in $\mathrm{bd}(C \times [0,1])$ obtained by adding I_1, I_2, J_1, J_2 and the ends of those arcs multiplied by I. There is a regular neighborhood $N(s)$ of s in $\mathrm{bd}(C \times [0,1])$ obtained by adding $N(I_1)$, $N(I_2)$, $N(J_1)$, $N(J_2)$ and their "ends" multiplied by I. Then there is a map $g:\mathrm{fr}N(s) \to S^{n-1}$ obtained by using h_{I_1}, h_{I_2}, h_{J_1}, h_{J_2} and by defining $g(x,t)=h(x)$ for $x \in \mathrm{bd}C$, where h is (the applicable) one of h_{I_i}'s.

We can now construct:

- a 2-dimensional proper unknotted PL-disc D in $C \times I$, whose boundary is s,
- a regular neighborhood $N(D)$ of D such that

$$N(D) \cap \mathrm{bd}(C \times I) = N(s),$$

- a map $\overline{g}:\mathrm{fr}N(D) \to S^{n-1}$ that extends g

In order to do that, one PL-isotopes s and $N(s)$ in a controlled manner into a standard position. D is then a standard spanning disc, and $N(D)$ is a standard neighborhood. Finding $\bar{g}$ amounts to extending $g: S^{n-1} \times \mathrm{bd}D^2 \to S^{n-1}$ to $\bar{g}: S^{n-1} \times D^2 \to S^{n-1}$. This can be done iff $J([g])=0$ (see chapter 3), or $g \simeq \rho$, using the notation of Chapter 3. g was constructed to have this property.

Define a map

$$k:(\mathrm{bd}C \setminus (\mathrm{int}N(I_1) \cup \mathrm{int}N(I_2))) \times [0,1] \to S^{n-1}$$

by $k(x,t)=f(x)$. Let F denote the union of f, $\bar{g}$, and k. We wish to extend F to $F: C \times [0,1] \setminus \mathrm{int}N(\mathrm{D}) \to \mathbb{R}^n \setminus \{0\}$. Once F is extended, its restriction, f', to $C \setminus (\mathrm{int}N(J_1) \cup \mathrm{int}N(J_2))$ on the 1-level will be our desired map.

Choose a point in each of $\mathrm{fr}N(I_i)$ and run a proper PL-arc in $(C \setminus \cup \mathrm{int}N(I_i)) \times \{0\}$ whose boundary are these points. Also, choose another PL-arc joining these points in $\mathrm{fr}N(D)$, and lying in $C \times (0,1)$ (except for the ends). It is easy to see that the circle σ thus formed spans a PL 2-disc W outside $N(D)$. Let $N(W)$ be a regular neighborhood of W in $C \times I \setminus \mathrm{int}N(D)$. $N(W)$ can be naturally identified with $D^n \times D^2$, and we note that F is already defined on a part of $N(W)$ identified with $D^n \times S^1$. Obviously, we can extend F to $N(W)$.

We will now collapse the rest of the domain of the proposed extension of F to the parts on which F is already defined, and use the resulting retraction to extend F. The details follow.

Since $N(D) \cap N(W)$ is a ball, $A=N(D) \cup N(W)$ is a PL $n+2$-ball. Consider a collar structure on $\mathrm{bd}(C \times I)$. We can assume that, in the collar area, A occupies the collar lines above $A\cap(\mathrm{bd}C\times I)$. Let $Z=C\times\{0\}\cup\mathrm{bd}C\times[0,1]$. F is defined on the collar lines over $\mathrm{bd}_Z(A\cap Z)$, and also on $Z\setminus A$, and we can extend it to the collar lines over the latter by collapsing.

As A meets Z in an $n+1$-face, A together with the collar lines over Z form an $n+2$-ball Y. F is now defined on $\mathrm{bd}_{C\times I}Y$, and all we need to do is to retract $C\times I\setminus\mathrm{int}_{C\times I}Y$ to $\mathrm{bd}_{C\times I}Y$.

It is easy to see what $Y'=Y\cap\mathrm{bd}(C\times I)$ is: it is a regular neighborhood of the space θ that looks like this letter, in the plane, multiplied by an n-1-disc. Let $c\theta$ be a PL-cone over θ, built inside Y. Attach a collar over $\mathrm{bd}(C\times I)\setminus\mathrm{int}Y'$, pinched at $\mathrm{bd}Y'$, to the outside of $C\times I$, to get a bigger PL-ball X.

Claim.

Both Y and $C\times I$ are relative regular neighborhoods of $c\theta\cup Y'$, relative $\mathrm{bd}Y'$, in X. In such a situation, the space "in between" the two regular neighborhoods has a product structure, which enables one to find the deformation retraction as required.

Proof of the Claim.

The proof for Y and $C\times I$ is the same; all we use is that we have a ball that meets $\mathrm{bd}X$ in Y'. We use [C]9.1, as usual. The only problem is to find a collapsible retraction, or, by [C]8.1, a collapse of Y to $c\theta\cup Y'$ relative $\mathrm{bd}Y'$. To do this, we view Y as $I^3\times\Delta^{n-1}$, and place $c\theta$ in $I^3\times 0$ with θ lying on $\mathrm{int}I^2\times 0\times o$. We now collapse in I^3 and "multiply the process" by Δ^{n-1}. We eat away all of I^3 except $c\theta$ and a regular neighborhood of θ in $I^2 \times 0$, which (multiplied by Δ^{n-1}) can be regarded as Y'. We then only have to collapse $(c\theta \setminus \theta) \times \Delta^{n-1}$ to $(c\theta \setminus \theta) \times o$.

CHAPTER 5

A GEOMETRIC INTERPRETATION OF ORDINARY COHOMOLOGY

We will provide a geometric interpretation of (cell) cohomology with integral coefficients for a regular CW-complex K. This is done to motivate our definition of enhanced cohomology and thus we will only sketch the arguments.

We will look at the n-th cohomology group for some $n>1$.

Consider a finite set of pairs of the form (p,h), where $p\in\mathrm{int}\sigma^n$ (open cell) for some cell σ^n of K, N is a regular neighborhood of p in $\mathrm{int}\sigma^n$ and $h:\mathrm{fr}N\to S^{n-1}$ is a map of degree ±1. The finite sets we consider may contain repetitions. Consider the family $\mathscr{C}^n(K)$ of all such sets. Consider the addition operation on $\mathscr{C}^n(K)$: $S_1+S_2=S_1\cup S_2$ (sum of sets with repetitions). $\mathscr{C}^n(K)$ is an abelian semigroup under $+$.

Consider the following Operations on an element S of $\mathscr{C}^n(K)$.

Operation 1: Take a pair (p,h) from S, where $p\in\mathrm{int}\sigma$, $h:\mathrm{fr}N\to S^{n-1}$, and take a PL-isotopy $F:N\times I\to\mathrm{int}\sigma$; then exchange (p,h) for

$$(F_1(p),hF_1^{-1}|_{F_1(\mathrm{fr}N)}:F_1(\mathrm{fr}N)\to S^{n-1}).$$

Operation 2: Take a pair $(p,h:\mathrm{fr}N\to S^{n-1})$ from S and homotope $h:\mathrm{fr}N\to S^{n-1}$ to a map h'; exchange (p,h) for (p,h').

Operation 3: Add $\{(p,h),(p,h')\}$ to S, where h, h' have the same $\mathrm{fr}N$ as domain and $\deg h=1$, $\deg h'=-1$ (according to some orientation of fr$N$).

Operation 4: Orient an n-1-cell α^{n-1} and orient all the n-cells $\sigma_1,\ldots,\sigma_m$ having α^{n-1} as a face in such a way that they induce the given orientation on α^{n-1}. Add to S

$$\{(p_1,h_1),\ldots,(p_m,h_m)\},$$

where $p_i \in \sigma_i$ for each i and all h_i's are of the same degree (1 or -1).

The inverse of Operation r will be also called Operation r.

Consider the following equivalence relation $\sim$ on $\mathscr{C}^n(K)$: $S_1 \sim S_2$ iff S_1 can be obtained from S_2 by finitely many Operations 1-4.

Define a sub-semigroup $\mathscr{Z}^n(K)$ of $\mathscr{C}^n(K)$ as follows. Consider an $n+1$-cell τ^{n+1} of K. Consider

$$\{(p_1,h_1),\ldots,(p_m,h_m)\} \in \mathscr{C}^n(K), \quad h_i\colon \mathrm{fr}N_i \to S^{n-1},$$

and consider those p_i's in the interiors of faces of τ^{n+1}. Fix an orientation for bdτ. This induces an orientation on frN_i so that we can calculate the sum of degrees, $\sum \deg h_i$ (for those i's specified above). Let $\{(p_1,h_1),\ldots,(p_m,h_m)\} \in \mathscr{Z}^n(K)$ iff this sum is 0 for each $n+1$-dimensional cell τ^{n+1} of K. Easily $\mathscr{Z}^n(K)$ is a subsemigroup of $\mathscr{C}^n(K)$.

We notice that the Operations 1-4 preserve $\mathscr{Z}^n(K)$ and that + induces an abelian group structure on $\mathscr{Z}^n(K)/_\sim$.

Orient all the cells of K. Define a homomorphism $g\colon \mathscr{C}^n(K) \to C^n(K,\mathbb{Z})$ (cell cochains) as follows. Suppose $\{(p_1,h_1),\ldots,(p_m,h_m)\}$ is an element of $\mathscr{C}^n(K)$, so that $p_i \in \mathrm{int}\sigma_i^n$, h_i is a map from frN_i to S^{n-1}. Each N_i induces an orientation on frN_i. Define the value of $g(\{(p_i,h_i)\})$ on σ_i to be $\deg h_i$, and on other cells to be 0. Define

$$g(\{(p_1,h_1),\ldots,(p_m,h_m)\})=\sum_{i=0}^{m} g(\{(p_i,h_i)\}).$$

It is easy to see that g is a homomorphism of semigroups, that $g(\mathcal{G}^n(K))\subseteq Z^n(K,\mathbb{Z})$ (cocycles), and that g induces a well-defined homomorphism

$$g:\mathcal{G}^n(K)/_{\sim}\to H^n(K,\mathbb{Z})$$

(Operations 1-4 do not change a cohomology class in $C^n(K,\mathbb{Z})$ via our homomorphism). It is then easy to manufacture g^{-1} to see that g is an isomorphism. Therefore $\mathcal{G}^n(K)/_{\sim}$ furnishes an alternative, geometric description of $H^n(K,\mathbb{Z})$ in terms of sets of points and maps from the boundaries of their regular neighborhoods.

Let $f:K\to L$ be a continuous map between regular finite CW-complexes, and assume that K, L are triangulated. One can homotope f to a cellular map which is simplicial between some subdivisions $f':K'\to L'$. Take $[\{(p_1,h_1),\ldots,(p_m,h_m)\}]\in\mathcal{G}^n(L)$. One can assume that each p_i is in the interior of some n-simplex of L' in an n-simplex of L; one can also assume that $\mathrm{fr}N_i=\mathrm{dom}h_i$ is in the same interior (use Operation 1 if needed). Then $f'^{-1}(p_i)\cap K^{[n]}$ is a finite set of points: $\{r_{i,1},\ldots,r_{i,k_i}\}$. Furthermore, $f'^{-1}(N_i)\cap K^{[n]}$ consists of regular neighborhoods $V_{i,1},\ldots,V_{i,k_i}$ of $r_{i,1},\ldots,r_{i,k_i}$ respectively. Let

$f^*([\{p_1,h_1),\ldots,(p_m,h_m)\}])=$

$$=[\bigcup_{i=1}^{m}\{(r_{i,1},h_i f|\mathrm{fr}V_{i,1}),\ldots,(r_{i,k_i},h_i f|\mathrm{fr}V_{i,k_i})\}].$$

One can see that this f^* corresponds to $f^*:H^n(K,\mathbb{Z})\to H^n(L,\mathbb{Z})$ via g. Roughly speaking, in our interpretation of cohomology, f^* is obtained by taking preimages of points and regular neighborhoods, under f.

CHAPTER 6

ENHANCED COCHAINS, ENHANCED COBOUNDARIES, AND THE SEMIGROUP STRUCTURE

Let $n>0$, $n\neq 2$. We shall now describe several types of "admissible objects". They will be the building blocks for the "geometric enhanced cochains" $\mathscr{C}^n(K)$.

Type 1.

Consider two different n-cells in K: σ_0, σ_1, which are faces of some $n+1$-cell τ. Consider a proper PL-arc l in τ running from $\mathrm{int}\sigma_0$ to $\mathrm{int}\sigma_1$. Consider a regular neighborhood N of l in τ such that $N\cap \mathrm{bd}\tau\subseteq \mathrm{int}\sigma_0\cup \mathrm{int}\sigma_1$. Note that there is a PL-homeomorphism between $(\Delta^n\times I,\ \Delta^n\times\{0\},\ \Delta^n\times\{1\},\ o\times I)$ and (N,N_0,N_1,l), where $N_i=N\cap \mathrm{int}\sigma_i$. Consider a map $h{:}\mathrm{fr}N\to S^{n-1}$ with the property $|\deg h|_{\mathrm{bd}\Delta^n\times *}|=1$, for any $*\in I$, where $\Delta^n\times *$ is treated as a subset of N via the above homeomorphism (this does not depend on the choice of the homeomorphism).

An *admissible object of type 1* is the pair (l,h).

Type 2.

Consider an n-cell σ and an $n+1$-cell τ, such that σ is a face of τ. Regard σ as $\sigma=\sigma_0=\sigma_1$, and repeat the above definition. An *admissible object of type 2* is the pair (l,h).

Type 3.

Consider an $n+1$-cell τ. Consider a PL simple closed curve $s \subseteq \mathrm{int}\tau$, and a regular neighborhood N of s in $\mathrm{int}\tau$. Let $h:\mathrm{fr}N \to S^{n-1}$ be a map. We require that, for some (embedded) n-1-sphere S in fr$N$ with linking number ± 1 with s

$$(*) \qquad \deg h|_S = \pm 1.$$

This requirement does not depend on the choice of S. We will usually check $(*)$ in a tacit manner. Lemma 2.3 can be used to recognize a suitable n-1-sphere.

We will need another condition for Type 3 which is equivalent to $(*)$. We identify fr$N$ with $S^{n-1} \times S^1$ by any PL-homeomorphism, and require that

$$(**) \qquad \deg h|_{S^{n-1} \times *} = \pm 1.$$

An *admissible object of type 3* is the pair (s,h).

The following Lemma follows from a standard position argument and Theorem 3.1, and it asserts that there exist, basically, two kinds of objects of Type 3.

Lemma 6.1.

Let τ be as above. There exist admissible objects (s,h) and (s,h') of Type 3 in τ with $J([h])=0$ and $J([h'])=1$. ■

Type 4.

Consider a free n-cell τ. Then an *admissible object of type 4* is a pair of the form (p,h), where $p \in \operatorname{int}\tau$, N is a regular neighborhood of p in $\operatorname{int}\tau$ and $h:\operatorname{fr}N \to S^{n-1}$ is a map of degree ± 1.

▪ (Types)

A cochain in our theory will be a set of objects glued together by their ends without leaving any "free" ends, as pictured in Figure 5.

More precisely, the intersection of objects of Type 1 or 2 with boundaries of $n+1$-cells in which they lie will be called ends (an end involves a point and a map). Each such object has two ends. Imagine an n-cell that is a face of a number of $n+1$-cells. Suppose that in each of these $n+1$-cells there exists an object such that all these objects have one common end in the n-cell (in particular the maps from the end to S^{n-1} coming from all these objects are the same). These objects and the end will then be called a *glued family.*

Let S be a set of admissible objects (possibly with repetitions). There are (possibly) objects of Type 1 or 2 there, and let us consider all the pairs (A,a), where A is such an object and a is an end of A. S will then be called a *geometric enhanced cochain of dimension* (or *level*) n *in* K. Because of possible repetitions of objects, it is important to remember which (repetitions of) objects are glued. This "gluing pattern" is a part of the definition of a cochain. The set of all the cochains will be denoted $E\mathscr{C}^n(K)$. We make all these definitions for $n=1,3,4,5,\ldots$.

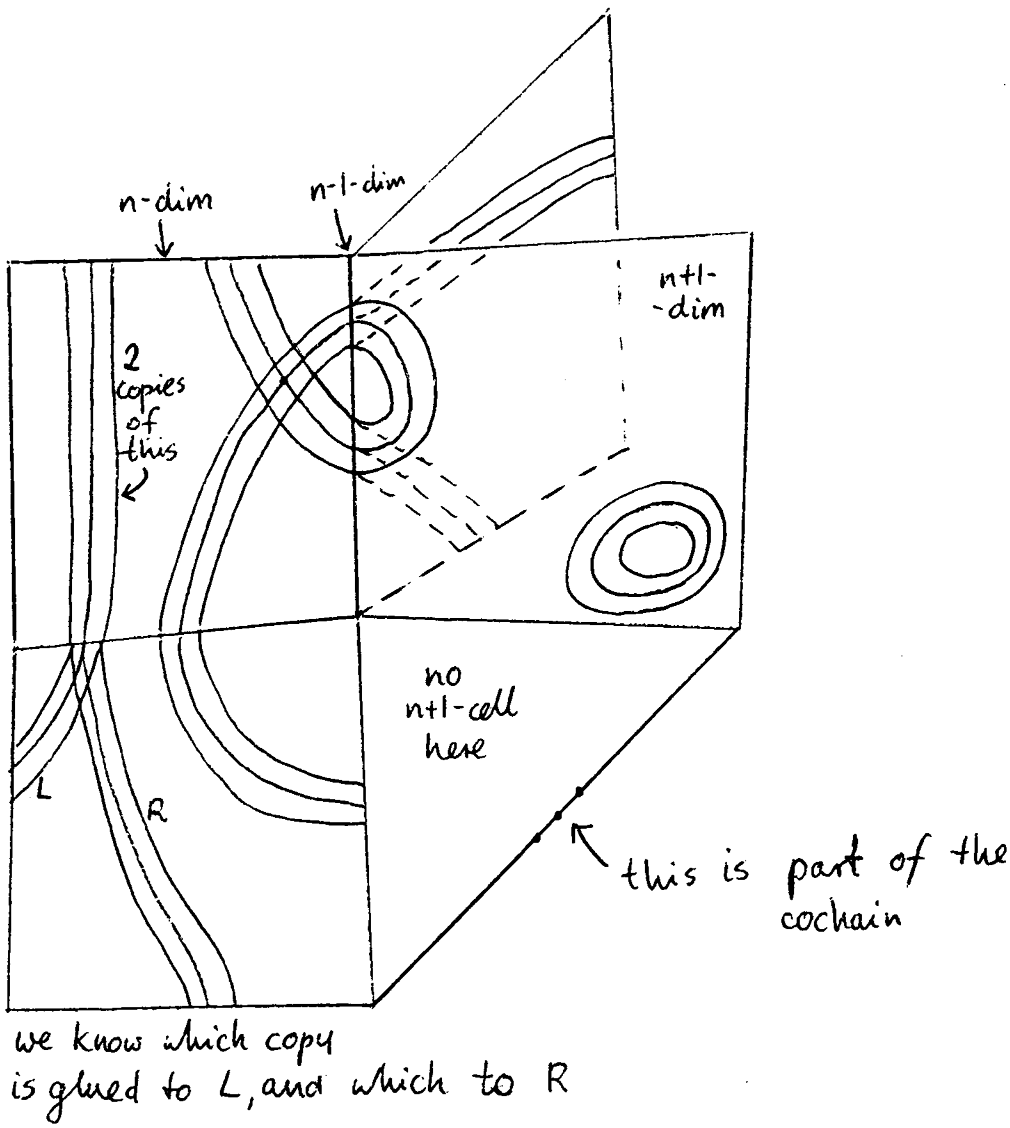


Figure 5. An enhanced cochain

By the set-theoretic addition of objects and gluing patterns one obtains an abelian semigroup structure on $E\mathscr{C}^n(K)$.

Consider a geometric enhanced n-cochain S in K, with the property that in every $n+1$- or free n-cell σ of K, the regular neighborhoods involved in objects from S are pairwise disjoint (in particular, there are no repetitions). Such a cochain will be called *non-singular.*

Remark 6.2.

By adding the curves and points involved in a non-singular cochain S one obtains a 0- or 1-dimensional subpolyhedron G of K, which we will call *the core* of S.

Since S is non-singular, by adding all of the regular neighborhoods together one obtains a regular neighborhood N of G in $K^{[n+1]}$, which will be called *the neighborhood induced by S.* Finally, by gluing the maps involved in S one gets a map $h:\mathrm{fr}N \to S^{n-1}$ with certain spheres in fr$N$ going to S^{n-1} with degree ± 1. We will call it *the map given by S.*

Remark 6.3.

If L is a subcomplex of K, then an enhanced cochain S in K can be restricted to one in L, written $S|_L$, by considering only objects in L.

We will now describe certain Operations on a geometric enhanced

n-cochain S in K that produce another enhanced cochain. The Operations will be used to define "cohomologous" cochains.

Operation 1 - the isotopy move.

Let $(l,h:\mathrm{fr}N \to S^{n-1})$ be an object from S of Type i, $i=1,2,3,4$, using the notation from the definition of Type i. Let $H_t:N \to \tau$, $t \in [0,1]$, be a PL-isotopy moving, for Type 1 or 2, $N \setminus (N_0 \cup N_1)$, N_0, N_1 in $\mathrm{int}\tau$, $\mathrm{int}\sigma_0$, $\mathrm{int}\sigma_1$, respectively, or for Type 3 or 4, N in $\mathrm{int}\tau$. The operation changes (l,h) to $H_1(l,h)$, where $H_1(l,h)$ means $(H_1(l), hH_1^{-1}:H_1(\mathrm{fr}N) \to S^{n-1})$. This is also an admissible object of Type i, by Lemma 2.1 or condition (**) from the definition of Type 3.

If some objects in the cochain are glued, then each of them is moved by an isotopy, and we require that the isotopies preserve the gluing, that is, agree on the ends, at any time.

Operation 2 - the homotopy move.

This operation permits the substitution of the map h by a homotopic map, for every object. Again, the homotopies should agree on the glued ends at any time.

Operation 3.

Operation 3 requires that S be non-singular. We will use the setting from the description of the saddle move in the ball C. Take an $n+1$-cell τ of K and assume that there is a PL $n+1$-ball $B \subseteq \mathrm{int}\tau$ and a PL-homeomorphism $h:C \to B$, such that, via h, the picture we see in B is the picture we saw in C before the saddle move. Operation 3 replaces that picture in B with the picture after the saddle move.

There is another requirement for the saddle move to be possible. In B, we see parts of the neighborhoods of (one or two) objects. Let M be the sum of those neighborhoods. We require that the components of $\overline{M \setminus B}$ be PL $n+1$-balls.

Let us draw one possible situation (Figure 6).

Double the picture and use the generalized Alexander Duality [Sp]6.2.17 to see that the quantity of the components is as shown. Lemma 2.8 can sometimes be used, to verify that a component is a ball. In particular, if $n \geq 5$, all the components marked by stars are automatically balls as required.

It is easy to see that after the saddle move we get (one or two) admissible objects. That the neighborhoods we get are indeed regular follows from the above requirement about the $n+1$-balls, 2.1, and 2.5. ■ (Operation 3)

We now observe that the saddle move does not change the value of J.

Observation 6.4.

Suppose that we have a number of pairwise disjoint PL simple closed curves in the interior of a PL $n+1$-ball, their regular neighborhood N and a map from its boundary to S^{n-1}, where $n \geq 3$. Suppose further that we are in a position to perform a saddle move in some PL $n+1$ subball B (this is analogous to the description of Operation 3). Let us do the saddle move and observe that, after, we again have disjoint curves, a regular neighborhood (2.5) and a map.

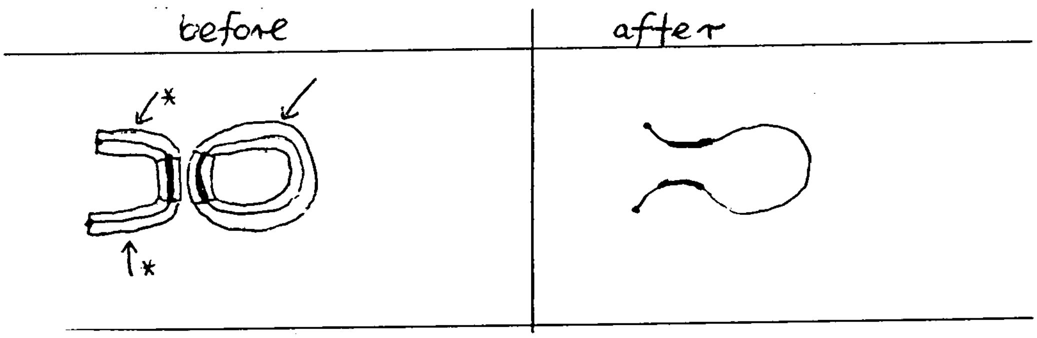


Figure 6. A saddle move

We can compute the value of J before and after the move and a natural question to ask is if the values are the same. This is indeed the case, by Proposition 3.8 and Theorem 4.1.

Operations 4-6 add some elements to the cochain S. Roughly speaking, we triangulate $K^{[n+1]}$, take an n-1-simplex $\overline{\alpha}$, and look at the 1-dimensional link around it. This is the core of what we add. We consider a nice regular neighborhood and a natural map to S^{n-1}. The whole thing is roughly equivalent to Operation 4 for ordinary cohomology. Operation $4+i$ here concerns with the case when $\overline{\alpha}$ is in the interior of an n-$1+i$-cell.

During the description of the Operations 4-6, we will be triangulating a certain set $U \subseteq K^{[n+1]}$. We require that this be PL-compatible with the original triangulation.

Operation 4 - the link move.

Consider an n-1-cell α and a point $p \in \mathrm{int}\alpha$. Triangulate a neighborhood U of p in $K^{[n+1]}$ as follows. p should be in the interior of an n-1-simplex $\overline{\alpha} \subseteq \mathrm{int}\alpha$. For each n-cell σ_i, containing the face α, there is an n-simplex $\overline{\sigma_i} \subseteq \sigma_i$ with $\overline{\alpha}$ as a face. For each $n+1$ cell τ_{ij}, two of whose faces are σ_i, σ_j, we have:

- an n-simplex $\overline{\sigma_{ij}}$ that lies in $\mathrm{int}\tau_{ij}$ except for its face $\overline{\alpha}$,
- an $n+1$-simplex that has $\overline{\sigma_i}$, $\overline{\sigma_{ij}}$ as faces and that lies in $\mathrm{int}\tau_{ij}$ except for $\overline{\sigma_i}$,
- an $n+1$-simplex that has $\overline{\sigma_j}$, $\overline{\sigma_{ij}}$ as faces and that lies in $\mathrm{int}\tau_{ij}$

except for $\overline{\sigma}_j$.

All of these simplices (together with their faces) sum up to U, and constitute a triangulation of U (this new triangulation of U is PL with respect to the original triangulation of K). Note that the mentioned $n+1$-simplices do not have any vertices other than those picked from the mentioned n-simplices.

Consider a simplicial map $\lambda: U \to 2\Delta^n$, which maps each of the above-mentioned n-simplices isomorphically onto $2\Delta^n$. In a sense, λ "folds" U onto $2\Delta^n$.

$\lambda^{-1}(\Delta^n)$ is a regular neighborhood of $\lambda^{-1}(o)$ in K, whose boundary $\lambda^{-1}(S^{n-1})$ gets mapped to S^{n-1} by λ. This data determines in the obvious way a certain set of admissible objects of Type 1, one for each τ_{ij}, and of Type 4, one for each free σ_i, and a gluing pattern for them. This constitutes an enhanced n-cochain which we denote by L_p. The link move adds L_p to S.

Operation 5.

Consider an n-cell σ and $p \in \mathrm{int}\sigma$. Assume first that σ is not free. Triangulate a neighborhood U of p in K as follows. p should be in the interior of an n-1-simplex $\overline{\alpha} \subseteq \mathrm{int}\sigma$. There are two n-simplices $\overline{\sigma}$, $\overline{\sigma}'$ in $\mathrm{int}\sigma$, which meet in the common face $\overline{\alpha}$. In each $n+1$-cell τ_i which contains σ as a face, U is the cone over $\overline{\sigma} \cup \overline{\sigma}'$ from a point in the interior of τ_i. Now we proceed similarly to the case of Operation 4. Map $\lambda: U \to 2\Delta^n$ by folding as before, obtain L_p, and add it to S.

If σ is free, then the Operation 5 is: add $L_p=\{(p,h),(p,h')\}$ to S, where $p\in\text{int}\sigma$, h, h' have the same domain and $\deg h=1$, $\deg h'=-1$.

Operation 6.

Let τ be an $n+1$-cell and $p\in\text{int}\tau$. Triangulate a neighborhood U of p in $\text{int}\tau$ as follows. p is to be in the interior of an n-1-simplex α. There are then 3 n-simplices: σ_1, σ_2, σ_3 having α as a face, and there are 3 n+1-simplices τ_{12}, τ_{13}, τ_{23}, where the vertices of τ_{ij} are taken from σ_i and σ_j. Let $\lambda:U\to 2\Delta^n$ be the folding map. Operation 6 is: add to S

$$L_p=\{(\lambda^{-1}(o),\lambda\big|_{\lambda^{-1}(S^{n-1})}:\lambda^{-1}(S^{n-1})\to S^{n-1})\}.$$

■ (Operation 6)

The neighborhood U used in Operations 4-6 will be called the *folding neighborhood.*

Remark 6.5.

We want to consider the inverses of the Operations as well. Therefore, Operation n, $n=1,\ldots,6$, shall mean: the Operation n as described above, or its inverse (for Operations 1, 2, 3 this makes no difference).

Proposition 6.6. (geometric realization of the link move)

a. Let Δ^n be embedded in a path-connected space T. Let $p\in U\subseteq K^{[n+1]}$ be as above, for any of the Operations 4-6. We adopt other notation from the definitions of these Operations, as well.

Let $f:U\to T$ be a map. Then there exists a map $f':U\to T$ such that $f'=f$ on bd$U$, and

(∗) $\quad f'^{-1}(o)=\lambda^{-1}(o)$, $f'(\lambda^{-1}(\Delta^n))\subseteq\Delta^n$, and $f'=\lambda$ on $\lambda^{-1}(\Delta^n)$.

In other words, we can always introduce L_p to the preimage.

b. We can cancel L_p, as well: Let $f':U\to T$ be a map such that (∗) holds. Then there exists a map $f:U\to T$ such that $f=f'$ on bd$U$ and $f(U)$ misses o.

Proof.

To prove a., use contractibility of U and a collar on bd$U$. b. for Operations 5 and 6 is easy to see. To prove b. for Operation 4, first add a copy of L_p, with an opposite orientation, parallel to L_p, then realize saddle moves and delete using b. for Operations 5 and 6.

In Operation 4, one might argue that it is equally natural to use more than one such an n-simplex for each τ_{ij} and proceed in the obvious fashion as (schematically) pictured in Figure 7.

In the same spirit, one might use more than one n-simplex inside each τ_i in Operation 5, and more than 3 n-simplices in Operation 6.

Lemma 6.7.

The definitions of Operations 4-6 using more simplices yield the same Operations as the original definitions.

Proof.

This is an easy exercise in elementary PL-techniques.

One needs the right way to reduce the number of n- and

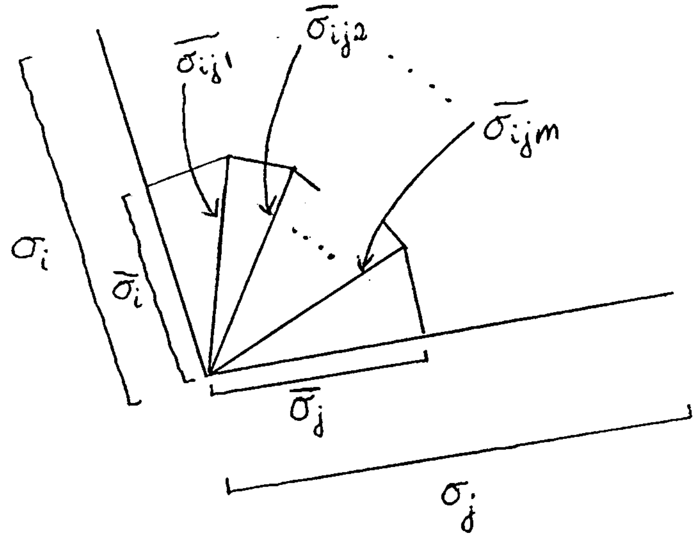


Figure 7. The alternative picture of Operation 4

$n+1$-simplices. We want to reduce two adjacent $n+1$-simplices and the n-simplex between them to one $n+1$-simplex. This reduction amounts to the following obvious fact.

Call the vertices of Δ^{n+1} $v_0,v_1,\ldots,v_{n+1}$, and the vertices of Δ^n - $w_0,w_2,w_3,\ldots,w_{n+1}$. Consider the simplicial map $f:\Delta^{n+1}\to\Delta^n$ determined by $f(v_0)=f(v_1)=w_0$, $f(v_i)=w_i$ for $i>1$. Let $v_{-1}\in\mathrm{int}\overline{v_0v_1}$ and consider the subdivision Δ of Δ^{n+1} with two $n+1$-simplices spanned by $\{v_{-1},v_0,v_2,\ldots,v_{n+1}\}$ and by $\{v_{-1},v_1,v_2,\ldots,v_{n+1}\}$. Let $g:\Delta\to\Delta^n$ be the simplicial map determined by $g(v_{-1})=g(v_0)=g(v_1)=w_0$, $g(v_i)=w_i$ for $i>1$. Then $f=g$. ■

We now can introduce an equivalence relation $\sim$ between enhanced n-cochains; namely, $C_0\sim C_1$ iff C_1 can be obtained from C_0 by finitely many Operations 1-6. We will also say that C_0 and C_1 are cohomologous.

Standard PL arguments can be used to show the following.

Lemma 6.8.

Every geometric enhanced n-cochain, $n\geq 3$, can be changed to a non-singular one by Operation 1.

Lemma 6.9.

The addition is well-defined on $\mathrm{E}\mathscr{C}^n(K)/_\sim$, for $n\geq 3$.

Proof.

We need to prove that $C_0+C_2 \sim C_1+C_2$ if C_1 is obtained from C_0 by Operation r, $r=1,\ldots,6$. For $r\neq 3$, this is obvious. For $r=3$, use Lemma 6.8, slightly modified to move only one of the terms and to avoid a pre-determined PL-ball in the interior of an $n+1$-cell.

CHAPTER 7

NON-SINGULAR COCHAINS AND ENHANCED COCYCLES

In this Chapter we study non-singular cochains, and use them to define cocycles. We assume throughout that K is a finite, regular CW-complex.

Many times we will work inside a neighborhood of a non-singular cochain. The following Proposition will help us find our way around.

Proposition 7.1.

Let S be a non-singular enhanced n-cochain in K with core c, neighborhood N, and map h. Then there is a graph (possibly without edges) G and a PL-homeomorphism $f{:}(G\times\Delta^n, G\times o)\to(N,c)$, such that:

- $G=G_0\oplus G_1$ (disjoint union),
- $f(G_0\times\Delta^n)$ is the neighborhood of the objects of Types 1, 2, 4,
- $f^{-1}(N\cap K^{[n]})=G_0^{[0]}\times\Delta^n$,
- G_1 is a simplicial complex, whose components are circles, and $f(G_1\times\Delta^n)$ is the neighborhood of the objects of Type 3,
- the composition

$$\mathrm{bd}\Delta^n \hookrightarrow g\times \mathrm{bd}\Delta^n \xrightarrow{f} \mathrm{fr}N \xrightarrow{h} S^{n-1}$$

has degree 1, for any $g\in G$.

Proof.

The major obstacle here would be if the mapping torus of an orientation-reversing homeomorphism of Δ^n were contained in N, but the existence of h prevents that.

Proposition 7.2.

The PL-homeomorphism f from Proposition 7.1 can be improved so that the composition

$$G\times \mathrm{bd}\Delta^n \xrightarrow{f} \mathrm{fr}N \xrightarrow{h} S^{n-1}=\mathrm{bd}\Delta^n$$

is homotopic to the projection $\pi: G\times \mathrm{bd}\Delta^n \to \mathrm{bd}\Delta^n$.

Proof.

Take $f':(G\times\Delta^n, G\times o)\to(N,c)$ from Proposition 7.1. We will modify h by homotopies. First, we can assume that h is PL and that $hf'|_{x\times \mathrm{bd}\Delta^n}$, for $x\in G^{[0]}$, is the identity. Next, using the PL-version of Corollary 3.2, we can assume that, for any $g\in G$, the map

$$h|_{f'(g\times \mathrm{bd}\Delta^n)}: f'(g\times \mathrm{bd}\Delta^n)\to \mathrm{bd}\Delta^n$$

is a PL-homeomorphism. Then define f on $g\times \mathrm{bd}\Delta^n$ to be the composition

(*) $$g\times \mathrm{bd}\Delta^n \xrightarrow[\mathrm{Id}]{} \mathrm{bd}\Delta^n \xrightarrow[h|_{f'(g\times \mathrm{bd}\Delta^n)}^{-1}]{} f'(g\times \mathrm{bd}\Delta^n) \hookrightarrow N$$

and extend by coning.

Remark 7.3.

We will sometimes identify c with $G\times o$, or G, using Proposition 7.1.

We now present a number of tricks we can play with non-singular cochains (7.4-7.9).

Let us consider an isotopy move on a non-singular cochain S_0, using the isotopy H_t. For $t>0$, $H_t(S_0)$ is not necessarily a non-singular cochain; if it is, we say that H_t determines a *non-singular isotopy move* from S_0 to another cochain. By general position and [H]8.13, we have

Proposition 7.4.

Let a non-singular n-cochain S_1 in K be obtained from another non-singular cochain S_0 by an isotopy move. Then there is a non-singular isotopy move from S_0 to S_1.

Non-singular isotopy moves are ambient in the following sense.

Proposition 7.5.

Given a non-singular isotopy move in K, there exists a PL-isotopy of K that moves each cell within itself and covers the move.

Proof.

We wish to extend $F_t:N_0 \to K^{[n+1]}$ to certain isotopy $H_t:K \to K$. We first set $H_t=\mathrm{Id}$ on $K^{[n-1]}$. We then define H_t on $K^{[n]}$, and on $K^{[n+1]}$ using 2.6. We finally cone to extend H_t over any cell of dimension more than $n+1$.

As a corollary of 3.9, 7.4, and 7.5, we have

Proposition 7.6.

Let τ be an $n+2$-cell in a regular CW-complex. Consider two non-singular n-cochains (with maps h_0, h_1) in bdτ, related by an isotopy move. Then $J([h_0])=J([h_1])$.

Lemma 7.7.

Consider an admissible object (s,h: fr$N \to S^{n-1}$) of Type 3, in an $n+1$-cell τ, with $n \geq 4$. If $J([h])=0$, then the enhanced n-cochain consisting of (s,h) is cohomologous to $\emptyset$.

Proof.

Use a non-singular isotopy move (with isotopy H_t) to move s and N to the "core" of the neighborhood N' induced by an object (s',h') arising in Operation 6. Using 7.5 to cover H_t by an ambient isotopy of τ, we get $J([hH_1^{-1}])=0$.

Because $J([h'])=0$ as well, we can assume $hH_1^{-1} \underset{PL}{\simeq} h'$ (essentially, use Theorem 3.1 here). Thus a homotopy move and Operation 6 finish off the job.

Whenever one thinks one can do a saddle move, one can!

Lemma 7.8 (can-do saddle move).

Let S be a non-singular cochain in K, let τ^{n+1} be a cell in K, and let A_1, A_2 be objects of Types 1-3 in τ from S. Then one can do a homotopy move that changes things only in the parts of A_1 and A_2 in intτ, and then a saddle move in τ that involves A_1 and A_2.

Proof.

This is a relatively easy exercise in PL-topology and we just sketch the solution. We will be using the notation from the definition of a saddle move.

Using a non-singular isotopy move, we isotopy move A_1 and its core, rel. bdτ, so that their intersection with B' is $N(I_1)$ and I_1, resp. We do the analogous thing to A_2, making sure it does not intersect A_1 after the move. We desire that the maps from fr$N$ (on bd$C$) and fr$r_n(N)$ to S^{n-1} we get be related by r_n. If that is not the case, we re-do one of the above isotopy moves to map a part of the core of, say, A_1 onto I_1, with opposite orientation.

Let $H_1:\tau\to\tau$ be the end of a covering isotopy for the two isotopy moves above.

We "almost" can do the saddle move in $H_1^{-1}(B)$. The problem is that the maps from fr$N(I_1)$, fr$N(I_2)$ to S^{n-1} which we have are not necessarily related by r_n. The degrees, however, match up, and we can use a homotopy move on A_1 and A_2 to achieve the happy end.

Lemma 7.9.

Let τ^{n+1} be a cell from K, and $n\geq 4$. Let S_0, S_1 be enhanced cochains that consist only of objects of Type 3 in τ. Suppose that the sum of the values of J for the objects is the same for S_0 and S_1. Then $S_0 \sim S_1$ by isotopy, homotopy and saddle moves.

Proof.

First, we prove the Lemma for S_i's that consist of 1 object each. By an ambient isotopy, we can move one core to the other, and the neighborhood of S_0 to the one of S_1. We want to apply 3.1 to homotopy move one object to the other. If the degrees of the maps (restricted to a sphere with linking number ± 1) are not compatible, prior to doing that we rotate one of the objects by an isotopy.

Now for the general case. Roughly, make S_i's non-singular, saddle and homotope them to make them consist of 1 object, move the two objects together. Here is the list of what one needs (in this order): 6.8, 7.5, 3.9, 7.8, 6.4, the special case.

■

Our goal now is to define a sub-semigroup of $E\mathscr{C}^n(K)$, denoted by $E\mathscr{G}^n(K)$, for $n \geq 4$. Its elements will be called *the geometric enhanced n-cocycles in K.*

Consider a geometric enhanced n-cochain S in K. We will formulate conditions under which S is a cocycle. Consider an $n+2$-cell τ of K. By Lemma 6.8, $S|_{\mathrm{bd}\tau}$ can be, by Operation 1, changed to a non-singular cochain. We now have (by 6.2) a number of simple closed curves in bdτ, their regular neighborhood N and a map $h{:}\mathrm{fr}N \to S^{n-1}$, and we can compute $J([h]) \in \pi_n(S^{n-1})$ (Observation 3.7). We define that $S \in E\mathscr{G}^n(K)$ iff $J([h])=0$ for any n+2-cell τ. Note by Proposition 7.6 that this does not depend on the choice of the non-singular cochain. We leave to the reader to check that $E\mathcal{Z}^n(K)$ is closed with respect to the addition.

CHAPTER 8

THE DEFINITION OF ENHANCED COHOMOLOGY

Proposition 8.1.

Let $S_0 \sim S_1$ be two non-singular n-cochains. Let τ be any $n+2$-cell in K. Let $h_i:\mathrm{fr}N_i \to S^{n-1}$ be the maps of the frontiers of the neighborhoods associated with $S_i|_{\mathrm{bd}\tau}$. Then (computed in bdτ) $J([h_1])=J([h_2])$. In particular, Operations 1 - 6 preserve $E\mathcal{Z}^n(K)$ $(n\geq 4)$.

Proof.

We assume that S_0 and S_1 are related by a single Operation. We also drop the assumption that they be non-singular.

Operation 1.

Use Proposition 7.6 here.

Operation 2.

This case is obvious.

Operation 3.

If the saddle move takes place in a ball $B\subseteq \mathrm{bd}\tau$, the values of J in bdτ computed from C_0 and C_1 coincide because of Proposition 3.8 and Theorem 4.1.

Operation 4.

If the link move takes place around an n-1-cell α, and $\alpha\subseteq \mathrm{bd}\tau$, then $S_1|_{\mathrm{bd}\tau}$ is obtained from $S_0|_{\mathrm{bd}\tau}$ by a link move in bdτ. In fact, the move takes place in a PL $n+1$-ball $U \subseteq \mathrm{bd}\tau$ from the definition of the link move. One can assume that $S_0|_{\mathrm{bd}\tau}$ and $S_1|_{\mathrm{bd}\tau}$ miss U, except for L_p. The core of L_p in $\mathrm{bd}\tau$ is just one simple closed curve, and, if h is the map from the boundary of its regular neighborhood, we only need to see $J([h])=0$ (computed in $\mathrm{bd}\tau$). In principle, this follows from the fact that h is "homotopic" to ρ; see theorem 3.1. But one can see this directly from the definition of J; use Lemma 6.9.

Operations 5, 6 - the proof here is similar to the case of Operation 4.

In view of Proposition 8.1, one can form a set of equivalence classes $E\mathcal{J}^n(K)_{/\sim}$, where two cocycles are related by $\sim$ iff one can be obtained from the other by finitely many Operations 1 - 6. By 6.9, this set becomes a semigroup.

Proposition 8.2.

$E\mathcal{J}^n(K)_{/\sim}$ is an abelian group, $[\emptyset]_\sim$ being 0.

Proof.

Let S be an enhanced n-cocycle. We need to find a representative C of -[S]. First, one can assume that S is non-singular. Let N be the regular neighborhood induced by S. Let $f: G\times I^n \to N$ be the PL-homeomorphism from Proposition 7.1.

We will work entirely in N, or, via f, in $G\times I^n$.

We can change S by an isotopy move to a cocycle S' with the induced neighborhood $N'=G\times A$, where

$$A=\left[\frac{1}{6},\frac{1}{3}\right]\times\left[\frac{1}{3},\frac{2}{3}\right]^{n-1}\subseteq I\times I^{n-1}=I^{n}.$$

Moreover, we can assume that the core of S' is

$$l=G\times\frac{1}{4}\times\frac{1}{2}\times\ldots\times\frac{1}{2}\times\frac{1}{2}.$$

Let $s:G\times I^{n}\to G\times I^{n}$ be defined by

$$s(y,x_1,\ldots,x_n)=(y,1-x_1,x_2,\ldots,x_n).$$

Then our candidate for $-[S]$ will be $[C]$, where the core of C is $s(l)$, the regular neighborhood induced by C is $s(N')$, and the map determined by C is $h's:\mathrm{fr}\,s(N')\to S^{n-1}$ (easily C is a cocycle).

In order to change $S+C$, by Operations, to $\emptyset$, we need the following main steps:

a. Perform saddle moves as in Figure 8 along the length of N; more precisely, we need a move on each side of any part of the form $v\times\Delta^{n}$, where v is a vertex of G. After that, the situation in $G\times\Delta^{n}$ is as pictured in Figure 9.

b. Erase the rectangles between the saddle moves using Operation 6, as pictured in Figure 10.

c. Erase the parts like the one in Figure 11, near n-cells, using Operation 5.

Comments:

In a., homotopy moves are needed to prepare for the saddle moves,

In b., we can do this because $J([h])=0$ (and we use 7.7), where h is the map from the frontier of our "piece". This is because h is symmetric and can be homotoped to ρ from the definition of J.

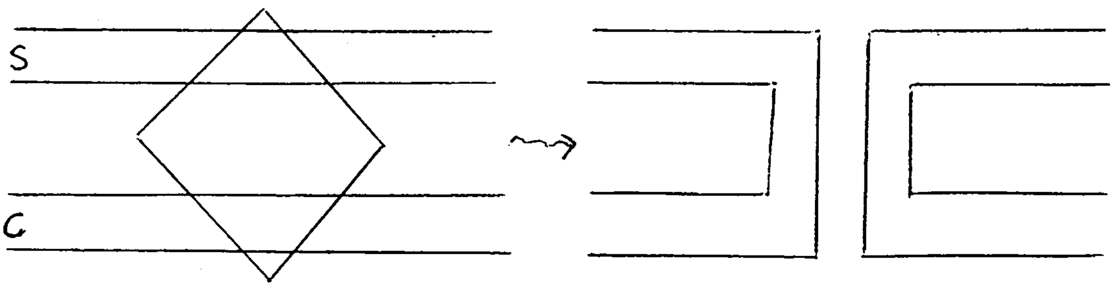


Figure 8. One of the saddle moves to start the process of changing $S+C$ to $\emptyset$

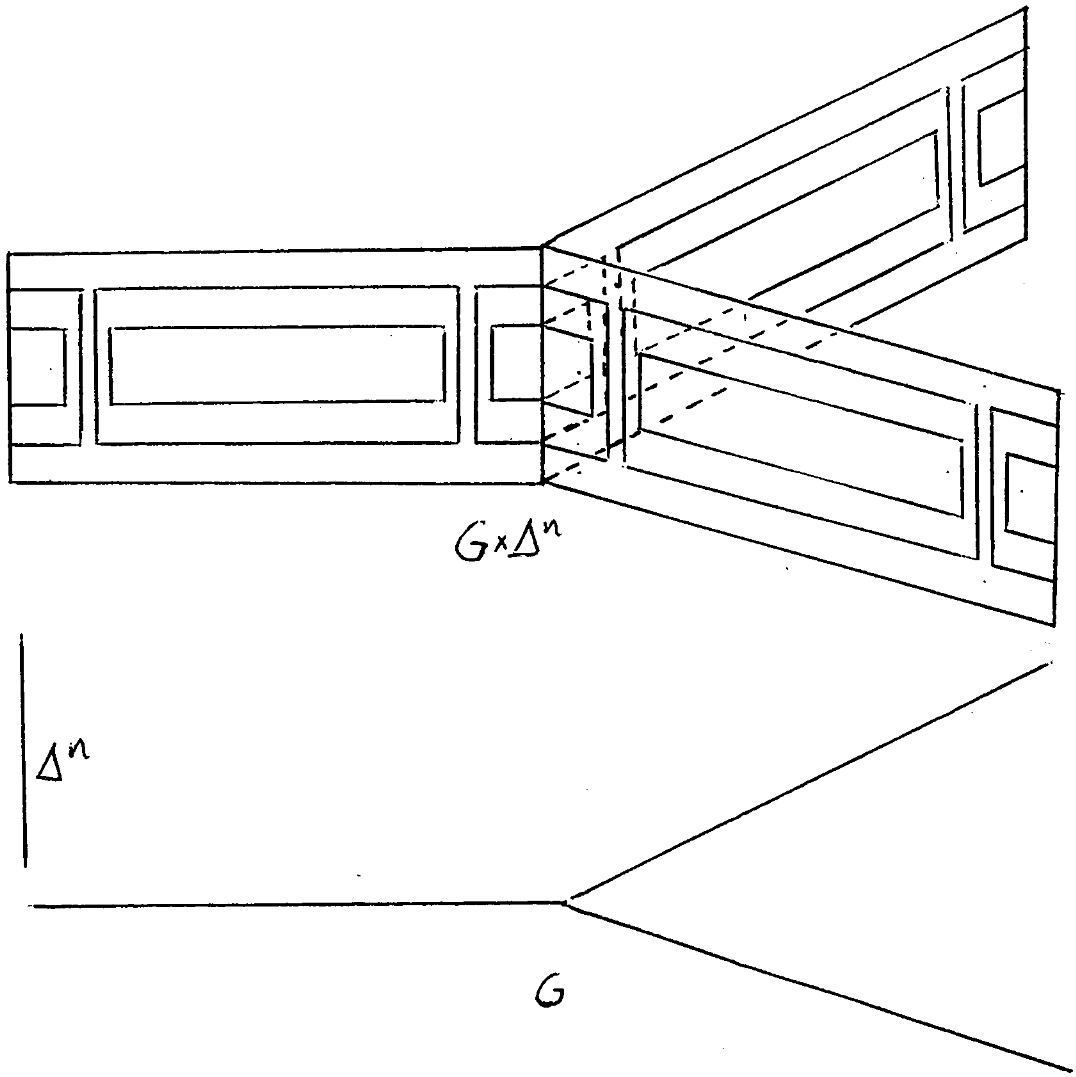


Figure 9. The situation after the saddle moves

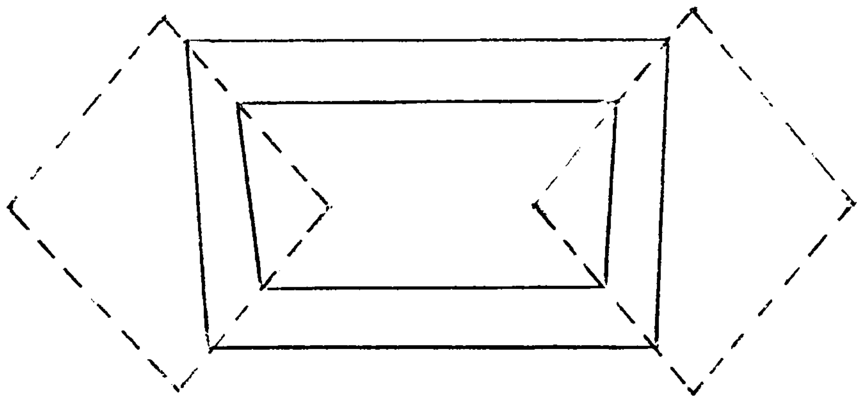


Figure 10. The rectangles to be erased after saddle moves

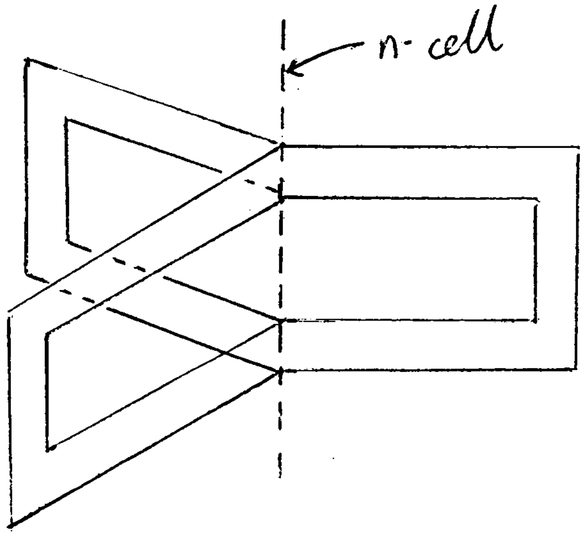


Figure 11. The other parts to be erased

In c., before we do the saddle moves of b., we adjust $S+C$ by homotopy move so that near n-cells the map given by $S+C$ is nice. Then the map from the frontier of the piece to be erased in c. is nice, and we can erase the piece. In the case of a free n-cell, we have pairs of Objects of Type 4 in it. Each pair is to be erased by Operation 5.

We have erased everything, and the proof of 8.2 is complete.

Remark 8.3.

One can find a nicer representative of -[S]. It is obtained by composing all maps from S with a fixed orientation-changing homeomorphism of S^{n-1} the range. The proof of this essentially follows the lines of the proof performed above, but is more complicated.

Definition 8.4.

$E\mathcal{Z}^n(K)_{/\sim}$ will be called the *n-th enhanced cohomology group of K*, and will be denoted by $EH^n(K)$.

Observation 8.5: The non-singular theory.

Using only non-singular cochains, equivalent enhanced cohomology groups can be built as follows.

Non-singular n-cochains, $n\geq 4$, in K form a set. We define an equivalence relation $\sim$ on this set using the following non-singular counterparts of the Operations. We use non-singular isotopy moves as described before. The homotopy move replaces the map given by the cochain by a homotopic one. The saddle move needs no change. When adding (or subtracting) L_p in Operations 4-6, we assume the rest has no contact with the folding neighborhood U.

Cocycles are defined as before.

The cohomology classes can be added. Having $[C_1]$ and $[C_2]$, we define $[C_1]+[C_2]$ by moving C_1 and C_2 apart by an isotopy move (6.8) and then adding. This is well-defined by 7.4.

The isomorphism from the "ordinary" to the non-singular group is defined by $[C]\to[C']$, where C' is obtained from C by an isotopy move. The inverse isomorphism is [Id].

The only thing that needs to be checked is

Lemma 8.6.

Suppose one gets from C_0 to C_1 by an isotopy move, Operation r ($r=1,\ldots,6$), and another isotopy move. Suppose that C_0 and C_1 happen to be non-singular. Then $C_0 \sim C_1$ in a non-singular fashion.

Proof.

For case $r=1$ use 7.4. For case $r=2$, use 6.6. In case $r=3$, there is nothing to do.

Suppose $r=4,5,6$. After the first isotopy move, note the folding neighborhood U in which you are about to work. It is easy to isotopy-move everything off U. Then add L_p and reverse the isotopy move to put everything back where it belongs. Now use 7.4 twice.

CHAPTER 9

CENTRAL SUBSETS OF SIMPLICIAL COMPLEXES

To simplify the rather technical proofs of the functoriality properties we would like that the cochains, cores, and certain related 2-dimensional objects in simplices look nice: are "straight" and in some sort of central position. In this Chapter we develop some information about such "central" objects.

Let K be a simplicial complex, $n \geq 0$, $\varepsilon \in (0,1)$, and let $A \subseteq M \subseteq K$ be subsets such that for any simplex Δ of K there is a simplicial map $\alpha_{\Delta}: \Delta \to \Delta^n$ such that $M \cap \Delta = \alpha_{\Delta}^{-1}(\varepsilon \Delta^n)$ and $A \cap \Delta = \alpha_{\Delta}^{-1}(o)$. Then we call A an *n-central subset of K, and M - the special ε-neighborhood of A.* Observe that A and ε determine M, and thus we will be saying "the special ε-neighborhood". A will be called the *core* of M.

Observe that, for any simplex Δ,

(*) $$\Delta \cap \mathrm{bd}_K M = \alpha_{\Delta}^{-1}(\mathrm{bd}\varepsilon\Delta^n).$$

Example.

There is a triangulation of the Möbius Band such that the equator is a 1-central set. This shows that there may not be a global simplicial map such that A is the preimage of o.

Observation 9.1.

Let $M \subseteq K$ and $\{\alpha_\Delta\}$ be as above, and let $N = \bigcup_\Delta \alpha_\Delta^{-1}(\delta\Delta^n)$, where $\delta \in (0,1)$. It is easy to see that N is the special δ-neighborhood of the core of M. ∎

The following is a consequence of the definitions and (*):

Proposition 9.2.

Let K, L be simplicial complexes, and $f: K \to L$ - a simplicial map. Let $n \geq 0$, $\varepsilon \in [0,1)$, and let M be a special ε-neighborhood of an n-central set c in L. Then:

(i) $f^{-1}(M)$ is the special ε-neighborhood of an n-central set $f^{-1}(c)$ in K,

(ii) $\mathrm{bd}_K f^{-1}(M) = f^{-1}(\mathrm{bd}_L M)$.

Proposition 9.3.

Let K be a simplicial complex, $n \geq 0$, $\varepsilon, \delta \in (0,1)$. Let M, N be the special neighborhoods of the same n-central set c in L. Then there is a PL-isotopy $H: M \times I \to K$ such that:

- $H_1(M, \mathrm{bd}M) = (N, \mathrm{bd}N)$,
- for any t, $H_t|_c = \mathrm{Id}_c$,
- for any simplex Δ of K, $H_t(M \cap \Delta) \subseteq \Delta$, and $H_t(M \cap \mathrm{int}\Delta) \subseteq \mathrm{int}\Delta$.

Sketch of Proof.

Observe that $M \cap \Delta$ is a regular neighborhood of $(c \cap \Delta) \cup (M \cap \mathrm{bd}\Delta)$, relative $\mathrm{bd}M \cap \mathrm{bd}\Delta$, in Δ. Define H inductively over a sequence of simplices of K of non-decreasing dimension.

Let K be a simplicial complex, $n \geq 4$, and C - a non-singular enhanced n-cochain in K such that the core of C is an n-central subset of K, and the neighborhood N induced by C is the special ε-neighborhood of c. Then C will be called *a central n-cochain of size* ε.

CHAPTER 10

PRELIMINARIES TO THE FUNCTORIALITY OF ENHANCED COHOMOLOGY

Consider a cellular n-cocycle in a regular finite CW-complex K as discussed in Chapter 5. It involves some structure inside $K^{[n]}$: points, regular neighborhoods, maps. It is not hard to see that one can extend that structure to the inside of $n+1$-cells. That is, one can pair points on the boundaries of $n+1$-cells using arcs running through their interiors, extend the regular neighborhoods of the points to regular neighborhoods of the arcs, and do it all in such a way that the maps from the frontiers to S^{n-1} can be extended to the frontiers of the new regular neighborhoods. An analog for enhanced theory is given below. We are also going to provide a triangulation for the complex, under which the enhanced cocycle and its "extension" to inside of $n+2$-cells are "central" (see Chapter 9).

Lemma 10.2.

Let L be a finite regular CW-complex, $n\geq 4$, and S - a non-singular enhanced n-cocycle in L with core s, neighborhood N and map h. Then there is a 2-dimensional subpolyhedron M of $L^{[n+2]}$, its (closed) PL-neighborhood N_M in $L^{[n+2]}$, a map $h_M\colon \mathrm{fr}_{L^{[n+2]}} N_M \to S^{n-1}$, and a triangulation T of $L^{[n+2]}$, which have the following properties.

Let $\tau_1,\ldots,\tau_k$ be the $n+2$-cells in L. We have

$$M=s\cup M_1\cup M_2\cup\ldots\cup M_k,$$

where M_i, $i=1,\ldots,k$, is a proper PL 2-manifold in τ_i with $\mathrm{bd}M_i=s\cap\mathrm{bd}\tau_i$. We have $N_M\cap L^{[n+1]}=N$. We require that h_M extend the map h. T subdivides the cell structure on $L^{[n+2]}$. M is an n-central set in $L^{[n+2]}$ triangulated by T, and N_M is the special $\frac{1}{2}$-neighborhood of this n-central set in $L^{[n+2]}$ (this, in particular, means $M\cap T^{[n-1]}=\emptyset$). If Δ is any n-simplex from T such that $M\cap\Delta\neq\emptyset$ then

$$h_M:\mathrm{fr}_\Delta(N_M\cap\Delta)\to S^{n-1}$$

is defined. We require that the degree of this map be ±1 for any Δ.

Proof.

We will construct M, N_M, and T first.

We first triangulate a neighborhood of N in $L^{[n+1]}$. Let $f:(G\times\Delta^n,G\times o)\to(N,s)$ be the PL-homeomorphism obtained from Proposition 7.2, and let $t=f(\times2):G\times\frac{1}{2}\Delta^n\to N$, where $\times2:\frac{1}{2}\Delta^n\to\Delta^n$ is a map defined by $\times2(x)=2(x-o)+o$. Using collars, there is a PL-embedding $T:G\times\Delta^n\to L^{[n+1]}$ that extends t and such that, if $t(\{x\}\times\frac{1}{2}\Delta^n)\subseteq\mathrm{int}\sigma$ for a cell σ, then $T(\{x\}\times\Delta^n)\subseteq\mathrm{int}\sigma$. We subdivide G to make it a simplicial complex and then we subdivide $G\times\Delta^n$ to a simplicial complex in a standard way. In this way, T is a triangulation for $T(G\times\Delta^n)$ such that $T(G\times\Delta^n)\cap L^{[n]}$ is a subcomplex.

Notice that we have a canonical simplicial map

$$\mathscr{T}:(T(G\times\Delta^n),T(G\times\mathrm{bd}\Delta^n))\to(\Delta^n,\mathrm{bd}\Delta^n)$$

(take T^{-1} and project), such that $s=\mathscr{T}^{-1}(o)$ and $N=\mathscr{T}^{-1}\left[\frac{1}{2}\Delta^n\right]$.

We will now work in one $n+2$-cell τ_i. $s\cap\mathrm{bd}\tau_i$ is a union of pairwise disjoint PL simple closed curves and it is easy to see that $2N=T(G\times\Delta^n)\cap \mathrm{bd}\tau_i$ is a regular neighborhood of this union, and is a subcomplex of $T(G\times\Delta^n)$.

Claim.

$$J([\mathcal{T}\!:\mathrm{fr}2N=T(G\times \mathrm{bd}\Delta^n)\cap \mathrm{bd}\tau_i\to \mathrm{bd}\Delta^n])=0$$

(computed in $\mathrm{bd}\tau_i$).

Proof.

This is because S is a cocycle. The details follow.

We have $J\left(\left[h\Big|_{\mathrm{fr}N\cap \mathrm{bd}\tau_i}\right]\right)=0$. By 7.2, $h\simeq\times 2\pi t^{-1}\!:\mathrm{fr}N\cap \mathrm{bd}\tau_i\to \mathrm{bd}\Delta^n$, where $\pi\!:G\times \mathrm{bd}\tfrac{1}{2}\Delta^n\to \mathrm{bd}\tfrac{1}{2}\Delta^n$ is the projection.

Let $H=t^{-1}(s\cap \mathrm{bd}\tau_i)\subseteq G=G\times o$ and extend T to a PL-embedding $\overline{T}\!:H\times 2\Delta^n \hookrightarrow \mathrm{bd}\tau_i$. We now have a nested sequence

$$(\mathrm{Im}\overline{T},2N,N\cap \mathrm{bd}\tau_i)=\overline{T}(H\times 2\Delta^n,H\times\Delta^n,H\times\tfrac{1}{2}\Delta^n).$$

There exists a PL-homeomorphism $\alpha\!:\mathrm{bd}\tau_i\to \mathrm{bd}\tau_i$, that moves $N\cap \mathrm{bd}\tau_i$ onto $2N$ by $\overline{T}(\mathrm{Id},\times 2)\overline{T}^{\,-1}$.

Using Corollary 3.9, we see

$$J([\times 2\pi t^{-1}\alpha^{-1}\!:\mathrm{fr}2N\to \mathrm{bd}\Delta^n])=0.$$

But $\times 2\pi t^{-1}\alpha^{-1}\big|_{\mathrm{fr}2N}=\mathcal{T}\big|_{\mathrm{fr}2N}$. ∎ (claim)

The Claim allows us to extend $\mathcal{T}$ to a map $\mathcal{T}_i''\!:\mathrm{bd}\tau_i\setminus \mathrm{int}2N\to \mathrm{bd}\Delta^n$.

We will now extend T to a triangulation T_i of τ_i, and obtain a simplicial map $\mathcal{T}_i\!:\tau_i\to\Delta^n$ that extends $\mathcal{T}$ and sends $\mathrm{bd}\tau_i\setminus \mathrm{int}2N$ to $\mathrm{bd}\Delta^n$.

Let T_i'' be a triangulation of $\mathrm{bd}\tau_i$ that extends T. Apply [Z] to

$\mathcal{T}_i''$ and T_i'' to get a triangulation T_i' of $\mathrm{bd}\tau_i$ that extends T and a simplicial (with respect to T_i') map $\mathcal{T}_i':\mathrm{bd}\tau_i\to\Delta^n$ that extends $\mathcal{T}$ and maps $\mathrm{bd}\tau_i\setminus\mathrm{int}2N$ to $\mathrm{bd}\Delta^n$. Extend T_i' and $\mathcal{T}_i'$ to a PL-collar C_i on $\mathrm{bd}\tau_i$ in τ_i (the need for a collar will become apparent later). Finally, cone to obtain T_i and $\mathcal{T}_i$.

We now define T on the whole $L^{[n+2]}$. In each $n+2$-cell τ_i, we consider T_i on $\mathrm{cl}(\tau_i\setminus C_i)\cup(2N\times I)$ (using the collar structure) and we set $T=T_i$ there. We let T on $L^{[n-1]}$ be an arbitrary triangulation that subdivides the cell structure. Each n-cell is now triangulated on the boundary and on some balls inside; extend using 2.2. Each $n+1$-cell τ is now triangulated on the boundary and on the "thickened" N (like $2N\cap\tau$, if $\tau\subseteq\mathrm{bd}\tau_i$). We know (by Lemma 2.1) that the thickened N in τ is a regular neighborhood of $s\cap\tau$ in τ. Use a standard position/codimension 3 argument to make the picture of τ standard and see that we can extend T to the whole τ using 2.2.

Finally, each $n+2$-cell τ_i is now triangulated on the boundary, on $2N\times I$, and on $\mathrm{cl}(\tau_i\setminus C)$.

$$\mathrm{cl}(\tau_i\setminus((2N\times I)\cup\mathrm{cl}(\tau_i\setminus C))).$$

is a PL-manifold and we can use 2.2 to finish T off.

We define $M_i=\mathcal{T}_i^{-1}(o)$ and $N_M\cap\tau_i=\mathcal{T}_i^{-1}\left[\frac{1}{2}\Delta^n\right]$; we only need to construct h_M by extending h from $\mathrm{fr}N$ to $\mathrm{fr}N_M$. But, for any i, we have the map $\times 2\mathcal{T}_i:\mathrm{fr}N_M\to\mathrm{bd}\Delta^n$, and

$$\times 2\mathcal{T}_i\simeq h:\mathrm{fr}N_M\cap\mathrm{bd}\tau_i\to\mathrm{bd}\Delta^n,$$

because we used 7.2 to manufacture $\mathcal{T}_i$. Use B.H.E.T. ∎

Let L be a finite regular CW-complex, $n \geq 4$, and S - a non-singular enhanced n-cochain in L. Let T be a new, PL, regular CW-structure on $L^{[n+1]}$ that subdivides the old one. Let c be the core of S, and N - the neighborhood induced by S. Suppose that there is a non-singular n-cochain C in $L^{[n+1]}$, built with respect to T, with the property that the core of C is c, and the neighborhood induced by C is N. Then we say that S and T are *compatible*, or that one is compatible with the other. Actually, it is easy to see that if there is a cochain C as above, then there is one whose map is f.

Suppose further that C has only objects of Types 1 and 4, and that each $n+1$-cell or free n-cell from T supports at most 1 object. Then we say that S and T are *strongly compatible*.

By following the proof of 10.1, paying attention only to the cellular $n+1$-skeleton of L, one can prove

Lemma 10.2.

Let L be a finite regular CW-complex, $n \geq 4$, and S - a non-singular enhanced n-cochain in L. Then there is a triangulation T of $L^{[n+1]}$ that is strongly compatible with S.

Let K, L be finite regular CW-complexes, $n \geq 4$, and S - a non-singular enhanced n-cochain in L, with core s, neighborhood N, and map $h{:}\mathrm{fr}N \to S^{n-1}$. Let $f{:}K \to L$ be a map, and C - a non-singular enhanced n-cochain in K, with core c and neighborhood M, such that:

$f(K^{[n+1]}) \subseteq L^{[n+1]}$ (the cell skeleta),

$f^{-1}(s) \cap K^{[n+1]} = c$,

$f(M) \subseteq N$, $f(\mathrm{fr}M) \subseteq \mathrm{fr}N$,

$hf|_{\mathrm{fr}M}{:}\mathrm{fr}M \to S^{n-1}$ is the map given by C.

We will then say that C is a *pull-back of S by f*, and that *S can be pulled-back by f*. The pull-back may not be unique.

The definition immediately yields the following results.

Proposition 10.3.

A pull-back of a pull-back is a pull-back by the composition. More precisely, let S pull back to C by f, and C pull back to A by g. Then S pulls back to A by fg.

Proposition 10.4.

If $i{:}K \to L$ is an inclusion of a subcomplex, then $S|_K$ is a pull-back of S by i.

Proposition 10.5.

Any pull-back of a non-singular sum (that is, a non-singular cochain that is the sum of non-singular cochains) is the sum of some pull-backs of the terms; all the pull-backs here are done with respect to the same map.

Let K, L, n, S, s, N, h be as above. Let $f{:}K \to L$ be a map and C -

a non-singular enhanced n-cochain in K with the core c and the induced neighborhood M such that:

$f(K^{[n+1]}) \subseteq L^{[n+1]}$ (the cell skeleta),

$f^{-1}(s) \cap K^{[n+1]} = c$,

$f(\mathrm{fr}M) \subseteq \mathrm{fr}N$,

$hf|_{\mathrm{fr}M}: \mathrm{fr}M \to S^{n-1}$ is the map given by C,

there is a map $f': K \to L$, homotopic to f, with the first 3 properties above, and such that $f'(M) \subseteq N$, $f|_{\mathrm{fr}M} \simeq f'|_{\mathrm{fr}M}: \mathrm{fr}M \to \mathrm{fr}N$.

We will then say that *C is a weak pull-back of S by f.* In all the development that follows, we will only use pull-backs for the brevity of the definition. Everything would go through with "weak pull-backs", though. This is the most general definition possible.

Proposition 10.6.

Let K, L, S, n be as above. Let $f: K \to L$ be a map. Then there is a map $f' \simeq f$ such that S can be pulled-back by f'.

Proof.

Consider a triangulation T_L of $L^{[n+1]}$, that is compatible with S. Consider also a triangulation T_K of K that subdivides the cell structure.

Homotope f to a map $g: K \to L$ such that $g(K^{[n+1]}) \subseteq L^{[n+1]}$. Then we can take f' as a simplicial approximation of $g|_{K^{[n+1]}}$, using T_L and some repeated barycentric subdivision of T_K, and extend it to K by the B.H.E.T.

We will be interested in the kind of pull-back that occurs in the last proof. Namely, let K, L be finite regular CW-complexes, $n \geq 4$, and S - a non-singular enhanced n-cochain in L. Let T_L be a triangulation of $L^{[n+1]}$ that is compatible with S. Let T_K be some triangulation of K that subdivides the cell structure. Let $f:K \to L$ be a map such that:

- $f(K^{[n+1]}) \subseteq L^{[n+1]}$ (the original cell skeleta),
- $f|_{K^{[n+1]}}$ is simplicial with respect to T_K, T_L.

Then S can be pulled-back by f to a non-singular n-cochain C (cochain with respect to the original cell structure in K). Such C is unique. We will then say that C is a *simplicial pull-back of S by f*, and that *S is simplicially pulled-back by f*. We can add that the pull-back is done *with respect to T_K and/or T_L*.

The definition of a pull-back is flexible because it is based on any kind of map under rather weak assumptions, and is thus nice for the purpose of applications. To prove things, however, we would prefer to use simplicial pull-back since it provides more structure. We therefore need a "bridge" between the two definitions. This is very technical

Proposition 10.7.

Let K, L, S, n be as above. Let T_L be a triangulation of $L^{[n+1]}$ that is strongly compatible with S. Let C be a pull-back of S by a map $f{:}K{\to}L$. Then there is a simplicial map $g{:}K{\to}L$ such that:

- $g{\simeq}f$,
- S is simplicially pulled-back by g, with respect to T_L, to obtain a cochain B in K,
- $B \sim C$.

Proof.

Throughout the proof, T_L will be the only cell structure on $L^{[n+1]}$ used; we forget about the original one ($L^{[n+1]}$ will, however, denote the original cell skeleton). We can actually assume that S is defined with respect to T_L.

Let N be the neighborhood induced by S, and s - the core of S. Using Proposition 7.1, we identify N with $H \times \Delta^n$, where H is a graph and $s = H \times 0$. Any cell of the form $b \times \Delta^n$, where b is an edge, is now lying in an $n+1$-simplex of T_L, and the cell's ends are contained in faces of the simplex.

We will proceed in eight Steps. In each Step, we change f by a homotopy to obtain a new f and a new pull-back (new C) of S by new f, related to the old pull-back (by old f) by $\sim$. At the end of the last Step, we will get g and B. In the last paragraph of Step i, called the Output, we gather almost all the information needed from then on (the one exception is that we do not mention that we have a pull-back). We sometimes do constructions in between Steps; they may be needed throughout.

We will usually homotope f only on a crucial part of the domain.

If all that is needed after is the (possibly multiple) use of B.H.E.T, then we leave this to the reader.

Step 1.

Let M be the neighborhood of C and let c be the core of C. Using Proposition 7.1, identify M with $G' \times \Delta^n$, where G' is a graph Now c is $G' \times 0$. In Step 1, we want to make f map "slices" $x \times \Delta^n$, $x \in G'$, to "slices" $y \times \Delta^n$, $y \in H$. Define a homotopy $F_t : M \to N$ by

$$F_t(x,y) = (\pi_1 f(x,(1\text{-}t)y), \pi_2 f(x,y)),$$

where $x \in G'$, $y \in \Delta^n$, and $\pi_1 : N \to H$, $\pi_2 : N \to \Delta^n$ are projections. We have $F_0 = f|_M$, and $F_t(\mathrm{fr}M) \subseteq \mathrm{fr}N$. Observe that $F((\mathrm{fr}M) \times I)$ misses s. $f(K^{[n+1]} \setminus \mathrm{int}M)$ misses s, as well. Thus we can use B.H.E.T, with $L^{[n+1]} \setminus s$ as the range, to extend F_t to a homotopy $\overline{F}_t : K^{[n+1]} \to L^{[n+1]}$. $\overline{F}_1$ is our new f. The core of C and the neighborhood induced by C do not change. The new map given by the new C is $hF_1|_{\mathrm{fr}M}$.

Observe that, indeed, the new f maps slices to slices.

∎ (Step 1)

Step 2.

Step 2 will improve f on c to a "simplicial" map.

Let $F_t : c = G' \times o \to s = H \times o$ be a homotopy such that $F_0 = f|_c$ and F_1 is a "simplicial approximation" of $f|_c$ with respect to some subdivision c' of c. The triangulation of s does not change.

We extend F_t to $\overline{F}_t : M \to N$ by

$$\overline{F}_t(x,y) = (F_t(x), \pi_2 f(x,y)).$$

The new f and C are obtained as in Step 1.

f is now simplicial on c'.

■ (Step 2)

Before going on to the next step, we manufacture a triangulation T_1 of K such that:

- T_1 subdivides the cell structure,
- if Δ is an $n+1$- (n-, less than n-dimensional) simplex in T_1 in $K^{[n+1]}$, then $\Delta \cap c$ is either $\emptyset$ or an edge of c' (either $\emptyset$ or a vertex of c', $\emptyset$).

Observe that any edge of c' must then join different n-1-faces of an n-simplex. We will refer to this situation by saying: "c' is c chopped by T_1".

T_1 is first defined on $K^{[n-1]}$ arbitrarily. Then M is suitably triangulated (in order to accomplish this, c' must be a very fine subdivision of c; we can certainly assume that). Finally, we use 2.2.

Step 3.

M contains certain simplices of T_1, and so $M \cap \Delta$ is not a regular neighborhood of $c \cap \Delta$, in Δ, for certain simplices Δ of T_1. But there is a thinner regular neighborhood M' of c in $K^{[n+1]}$ (original cell skeleton), that possesses this property. Step 3 changes M to M' by means of a nice ambient isotopy H_t of $K^{[n+1]}$: H_t is supposed to move each cell within itself and keep c fixed. M' is our "new M"; the "new f" is fH_1^{-1} (extended to K by B.H.E.T). c does not change.

The new c, M, and f determine the new C, which is obtained from the old one by an isotopy move.

f is simplicial on c'; c' is c chopped by the triangulation T_1; c' is a simplicial complex. $M \cap \Delta$ is a regular neighborhood of $c \cap \Delta$, for any $\Delta \in T_1$.

Step 4.

Step 4 is very similar to Step 1. Recall how M and N were "sliced" in Step 1 with respect to the original cell structure of K and T_L. In Step 4 we use the same slicing for N, but M is now sliced using our brand new triangulation T_1. Exactly as in Step 1, we make f map slices to slices.

The output from Step 4 is the output from Step 3 plus the following. If $\Delta \in T_1$, $\Delta' \in T_L$ are simplices such that $f(c \cap \Delta) \subseteq s \cap \Delta'$, then $f(M \cap \Delta) \subseteq N \cap \Delta'$.

Step 5.

We now "thin M 2 times". Using T_1 and 7.1, we find a PL-homeomorphism $j: G \times \Delta^n \to M$, where G is a simplicial complex with $\dim G \leq 1$. We want our "new M" and c to be as stated below. Step 5 is an easier version of Step 3.

The output of Step 5 is the output of Step 4 plus the construction of a subpolyhedrom $j(G \times \Delta^n)$ in $K^{[n+1]}$ with M as $j(G \times \frac{1}{2}\Delta^n)$ and c as $j(G \times o)$.

■ (Step 5)

We will now build a special triangulation T_2 of $j(G \times \Delta^n)$. To do that, we subdivide $G\times\Delta^n$ to a simplicial complex in such a way that any $v\times\Delta^n$, $v\in G^{[0]}$, is a simplex, and then we need a PL-homeomorphism $T_2\colon G\times\Delta^n\to j(G\times\Delta^n)$ (j is not suitable as T_2).

The conditions we want T_2 to satisfy are:

(i) $T_2(w\times\Delta^n)=j(w\times\Delta^n)$ for any vertex or edge w of G.

Consider $v\in G^{[0]}$ and let σ^n be the simplex of T_L with $f(v)\in\operatorname{int}\sigma^n$.

(ii) Because of (i), $j(v\times\Delta^n)$ is a simplex of T_2. We want that there be a simplicial (with respect to T_2 and T_L) isomorphism $f_v\colon j(v\times\Delta^n)\to\sigma^n$, such that

$$f_v(j(v\times\tfrac{1}{2}\Delta^n),j(v\times o))=(N\cap\sigma^n,s\cap\sigma^n)$$

(this is supposed to hold for any $v\in G^{[0]}$).

These are possible to achieve because, for any v, there is a PL-homeomorphism

$$f_v\colon(j(v\times\Delta^n),j(v\times\tfrac{1}{2}\Delta^n),j(v\times o))\to(\sigma^n,N\cap\sigma^n,s\cap\sigma^n).$$

Furthermore, one can insist that

$$(*)\qquad f_v\simeq f\colon j(v\times\operatorname{bd}\tfrac{1}{2}\Delta^n)\to\operatorname{fr}N\cap\sigma^n.$$

When $G\times\Delta^n$ is identified with $j(G\times\Delta^n)$ via j, $c=G\times o$, and $M=G\times\frac{1}{2}\Delta^n$. From now on we identify $G\times\Delta^n$ with $j(G\times\Delta^n)$ via T_2, and we forget about j. Because of the change, the above equalities no longer hold.

We are now ready to make f simplicial on some parts of the domain in

Step 6.

c and M will not change in Step 6. f will be changed by a homotopy F_t, $F_0=f$, such that $F_t(\operatorname{fr}M)\subseteq\operatorname{fr}N$ and $F_1|_{v\times\Delta^n}=f_v$.

Let $v \in G^{[0]}$ and let p be the point $c \cap (v \times \Delta^n)$. $M \cap (v \times \Delta^n)$ is a regular neighborhood of p in $v \times \Delta^n$. Let $f(p) \in \sigma^n$, where σ^n is an n-simplex of T_L. Because of (∗), there is a homotopy

$$F_t:(v \times \Delta^n, M \cap (v \times \Delta^n), p) \to (L^{[n+1]}, N \cap \sigma^n, s \cap \sigma^n = f(p)),$$

such that:

- $F_0 = f$,
- $F_t(\mathrm{fr} M \cap (v \times \Delta^n)) \subseteq \mathrm{fr} N \cap \sigma^n$,
- $F_t^{-1}(s \cap \sigma^n) = p$,
- $F_1 = f_v$.

We will now (nicely) extend F_t to K.

First, set $F_t = f$ on c.

Next, let τ be an $n+1$-simplex in T_1 in $K^{[n+1]}$ such that $\tau \cap c$ is an edge of c'. Let $\tau \cap c$ join the faces σ_0^n, σ_1^n of τ. We will now define F_t on $M \cap \tau$.

Case 1:

$f(c \cap \tau)$ is an arc in s in an $n+1$-simplex β of T_L.

Let $f(M \cap \sigma_i^n) \subseteq \alpha_i$, where α_i is an n-face of β. We have already defined F_t on $M \cap \sigma_i^n$ and $c \cap \tau$, and we have $F_t(M \cap \sigma_i^n) \subseteq N \cap \alpha_i$ (in fact, there is equality there). All we require from F_t on $M \cap \tau$ is that $F_t(M \cap \tau) \subseteq N \cap \beta$, $F_t(\mathrm{fr} M \cap \tau) \subseteq \mathrm{fr} N \cap \beta$, and $F_t^{-1}(s \cap \beta) = c \cap \tau$ (we really only need $F_1^{-1}(s \cap \beta) = c \cap \tau$, but this is no easier). To that end, we identify $M \cap \tau$ with $\Delta^n \times I$ and $c \cap \tau$ with $o \times I$. We use a standard construction: F_t is determined by f on $\Delta^n \times \left[\frac{t}{3}, 1-\frac{t}{3}\right]$, by $F_t|_{\Delta^n \times 0}$ on $\Delta^n \times \left[0, \frac{t}{3}\right]$, and by $F_t|_{\Delta^n \times 1}$

on $\Delta^n \times \left[1-\frac{t}{3},1\right]$.

▪ (Case 1)

Case 2:

$f(c\cap\tau)$ is a point p in an n-simplex α of T_L.

Take $X=N\cap\alpha$, $A=\mathrm{fr}N\cap\alpha$, $B=s\cap\alpha=p$, and proceed as in Case 1.

▪ (Case 2)

We can now extend F_t to $K^{[n+1]}$ (original cell skeleton) using B.H.E.T. Our goal here (that is, in $K^{[n+1]}\setminus(G\times\Delta^n)$) is to avoid s, so the range in B.H.E.T will be $L^{[n+1]}\setminus s$.

▪ (Step 6)

The output of Steps 1-6 is as follows.

PL-embedded in $K^{[n+1]}$ is a copy of $G\times\Delta^n$, where G is a simplicial complex of dimension ≤ 1. $G\times\Delta^n$ has a triangulation T_2. Every $v\times\Delta^n$, $v\in G^{[0]}$, is a simplex of T_2, and f is a simplicial isomorphism from $v\times\Delta^n$ to an n-dimensional simplex of T_L.

We have $M\subseteq G\times\mathrm{int}\Delta^n$. For every $v\times\Delta^n$, $v\in G^{[0]}$, $p_v=c\cap(v\times\Delta^n)$ is a point, $M\cap(v\times\Delta^n)$ is a regular neighborhood of p_v in $v\times\Delta^n$, and $f(M\cap(v\times\Delta^n))\subseteq\mathrm{int}\alpha_v$ for some n-simplex α_v of T_L.

Let b be any edge of G, and let v_0, v_1 be its ends. $c\cap(b\times\Delta^n)$ is a proper PL-arc in $b\times\Delta^n$ between $p_0=p_{v_0}$ and $p_1=p_{v_1}$, and $M\cap(b\times\Delta^n)$ is a regular neighborhood of that arc in $b\times\Delta^n$. If $f(p_0)\neq f(p_1)$ then $f(p_0)$ and $f(p_1)$ are in two different n-faces of an $n+1$-simplex β of T_L, and $f(M\cap(b\times\Delta^n))\subseteq\beta$. If $f(p_0)=f(p_1)$, then $f(M\cap(b\times\Delta^n))\subseteq\alpha$ for some n-simplex

α of T_L.

■ (output)

We have, essentially, localized the problem to $b\times\Delta^n$.

In Step 7, we treat $I\times\Delta^n$ as a simplicial complex. To that end, we fix a standard subdivision (with $n+1$ $n+1$-simplices) with $0\times\Delta^n$ and $1\times\Delta^n$ as simplices. Also, f is already equal to g on $G^{[0]}\times\Delta^n$; although we will construct many maps with domains that contain $G^{[0]}\times\Delta^n$, they will all equal f there.

Step 7.

The ultimate goal of Step 7 is to make f simplicial with respect to some triangulation of $G\times\Delta^n$. The first part of Step 7 concerns with the situation inside $b\times\Delta^n$ for an edge b of G.

Case 1:

$f(p_0)\neq f(p_1)$ (recall that v_0, v_1 are the ends of b, and $p_i=c\cap(v_i\times\Delta^n)$).

f on $v_i\times\Delta^n$, $i=0,1$, is a simplicial isomorphism onto an n-simplex α_i of T_L. α_0 and α_1 are different faces of the $n+1$-simplex β. The situation in $b\times\Delta^n$ is pictured in Figure 12. f maps $b\times\Delta^n$ to $L^{[n+1]}$, $M\cap(b\times\Delta^n)$ to $N\cap\beta$, $\mathrm{fr}M\cap(b\times\Delta^n)$ to $\mathrm{fr}N\cap\beta$, and $M\cap(v_i\times\Delta^n)$ to $N\cap\alpha_i$, $i=0,1$. Moreover, in $b\times\Delta^n$, M is a regular neighborhood of c.

On the other hand, if we denote the points $s\cap\alpha_i$, $i=0,1$, by s_i, then $(N\cap\beta,N\cap\alpha_i)$ is a regular neighborhood of $(s\cap\beta,s_i)$ in (β,α_i). $s\cap\beta$ is a proper PL-arc in β.

Our goals in Case 1 are as follows. We wish to triangulate $b\times\Delta^n$

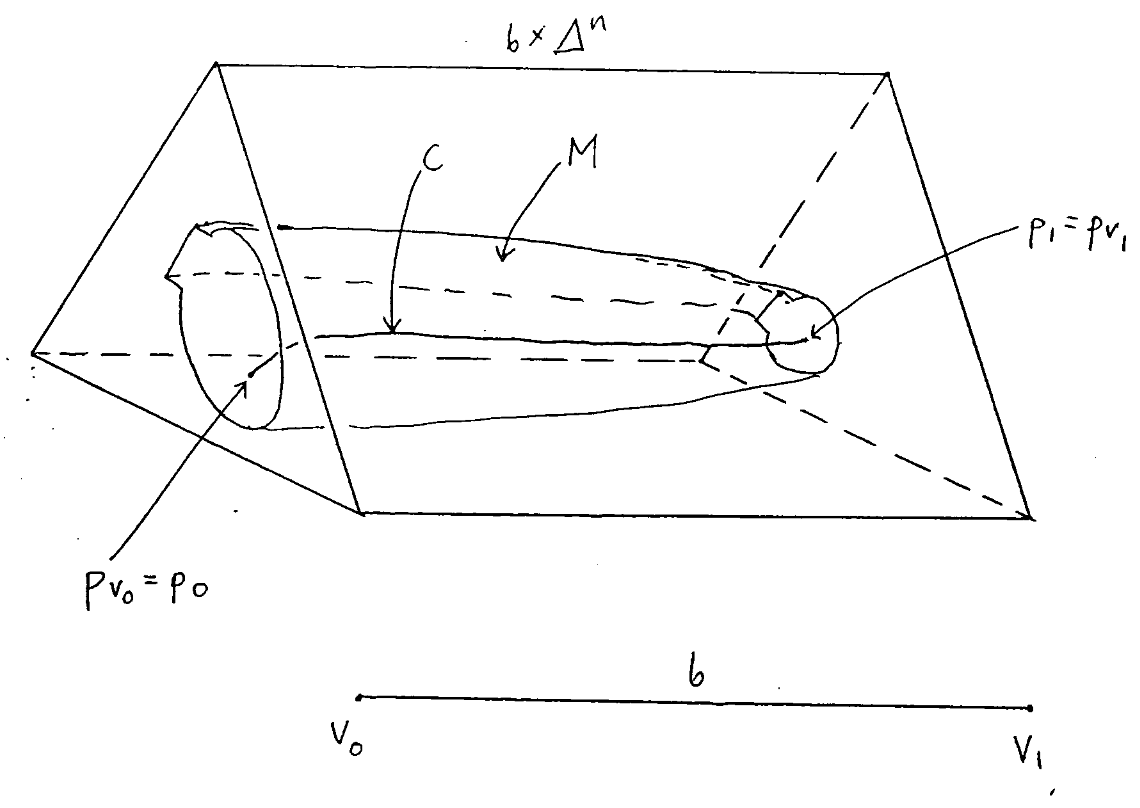


Figure 12. The situation in $b \times \Delta^n$

by a PL-homeomorphism $T_b:I\times\Delta^n\to b\times\Delta^n$ such that $T_b(i\times\Delta^n)=v_i\times\Delta^n$, and $T_b|_{i\times\Delta^n}:i\times\Delta^n\to v_i\times\Delta^n$ is a simplicial isomorphism for $i=0,1$. $I\times\Delta^n$ is considered a simplicial complex in a standard way, subdividing the cell structure, and using only existing vertices. We will also exhibit a simplicial (with respect to T_b and T_L) map $f_b:b\times\Delta^n\to\beta$ (this will be a part of our new f after Step 7), such that:

(1) $$f\simeq f_b:(b\times\Delta^n,\mathrm{fr}M\cap(b\times\Delta^n))\to(L,\mathrm{fr}N\cap\beta)\text{ rel. }\mathrm{bd}b\times\Delta^n,$$

(2) $$f_b(M\cap(b\times\Delta^n))\subseteq N\cap\beta,$$

(3) $$f_b^{-1}(s\cap\beta)=c\cap(b\times\Delta^n).$$

Our first move is to construct a map $f':b\times\Delta^n\to\beta$ such that $f'=f$ on $(\mathrm{bd}b\times\Delta^n)\cup(M\cap(b\times\Delta^n))$ and

$$f'((b\times\Delta^n)\setminus\mathrm{int}M,b\times\mathrm{bd}\Delta^n)\subseteq(\beta\setminus\mathrm{int}N,\tilde{\beta}),$$

where $\tilde{\beta}=\mathrm{bd}\beta\setminus(\mathrm{int}\alpha_0\cup\mathrm{int}\alpha_1)$. We need to realize that, by codimension 3 and [C], the situations within $b\times\Delta^n$ and β are standard. It is then easy to manufacture f'.

f' is a prototype for T_b^{-1}. The problems are two: that these two maps have different ranges, and that f' need not be a PL-homeomorphism. Therefore, some reconciliation work is needed.

Let $\pi:I\times\Delta^n\to\beta$ be the simplicial map determined as follows. Let a_i, $i=0,1$, be the only vertex of α_i that is not in α_{1-i}. $\pi|_{0\times\Delta^n}$ is any simplicial isomorphism onto α_0; let w_0 satisfy $\pi(0,w_0)=a_0$. Then set $\pi(1,w)=\pi(0,w)$ for any vertex $w\neq w_0$, and $\pi(1,w_0)=a_1$. Now we define a triangulation T'_b of $b\times\Delta^n$ to be a PL-homeomorphism that makes the diagram

$$\begin{tikzcd}
((b\times\Delta^n, b\times \mathrm{bd}\Delta^n); v_0\times\Delta^n, v_1\times\Delta^n) \arrow[dr, "f'"'] & & ((I\times\Delta^n, I\times\mathrm{bd}\Delta^n); 0\times\Delta^n, 1\times\Delta^n) \arrow[ll, "T'_b"'] \arrow[dl, "\pi"] \\
& \left(\left[\beta,\tilde{\beta}\right]; \alpha_0, \alpha_1\right) &
\end{tikzcd}$$

commute up to homotopy at the first place (the pair), and commute exactly at the other two (the homotopy should take place relative the second and third places). The existence of T'_b follows, essentially, from Corollary 3.3(c). Let $f'_b\colon b\times\Delta^n\to\beta$ be the simplicial, with respect to T'_b and T_L, map determined by

$$f'_b\big|\mathrm{bd}b\times\Delta^n = f'\big|\mathrm{bd}b\times\Delta^n = f\big|\mathrm{bd}b\times\Delta^n$$

(all the vertices of T'_b lie in $\mathrm{bd}b\times\Delta^n$). What the last diagram really says is that, upon identifying $b\times\Delta^n$ with $I\times\Delta^n$ via T'_b, π is f'_b, and

(4) $$f'\simeq f'_b\colon(b\times\Delta^n, b\times\mathrm{bd}\Delta^n)\to\left[\beta,\tilde{\beta}\right]$$

relative the "ends".

The difficulty with T'_b and f'_b is that $f'^{-1}_b(N\cap\beta)$ is not necessarily $M\cap(b\times\Delta^n)$, although this is so on $\mathrm{bd}b\times\Delta^n$. Yet $f'^{-1}_b(N\cap\beta)$ is nice enough to apply codimension 3 unknotting and [C] to find an isotopy $W_t\colon b\times\Delta^n\to b\times\Delta^n$, relative $\mathrm{bd}(b\times\Delta^n)$, that moves

$$(f'^{-1}_b(N\cap\beta),\ f'^{-1}_b(s\cap\beta))$$

onto $(M\cap(b\times\Delta^n), c\cap(b\times\Delta^n))$.

We finally let $T_b=W_1\circ T'_b$, and, as before, we let $f_b\colon b\times\Delta^n\to\beta$ be the simplicial, with respect to T_b and T_L, map determined by $f_b=f$ on $\mathrm{bd}b\times\Delta^n$. Observe that f_b satisfies (2) and (3). We leave it to the reader to see that (4), considered on $b\times \mathrm{bd}\Delta^n$, implies (1).

We have $f_b=f$ on $\mathrm{bd}b\times\Delta^n$, so we can extend the homotopy to be constant there; then we extend it to give $f_b \simeq f: M\cap(b\times\Delta^n)\to N\cap\beta$ by coning (the domain and the range here are balls). Finally, to extend to the rest of $b\times\Delta^n$, we use a suitable strong deformation retraction (from the rest of $b\times\Delta^n$). ▪ (case 1)

Case 2: $f(p_0)=f(p_1)$, is easier!

The goals of Case 2 are obtained from the goals of Case 1 by the substitution $\beta=\alpha$. To achieve the goals, proceed similarly to Case 1.

▪ (Case 2).

On $G\times\Delta^n$, our "new" f (output of Step 7) will be all the f_b's glued, and the new f is the old f on $v\times\Delta^n$, for any free vertex v of G.

We extend the new f from $G\times \mathrm{bd}\Delta^n$ to $K^{[n+1]}$ (the original cell skeleton) using B.H.E.T as usual, so as to avoid s.

Because of (1), the new and old f's are homotopic.

The new M and c are the same as the old ones. The new M, c, and f determine the new C. By (1) again, the new C is obtained from the old one by a homotopy move.

▪ (Step 7)

We now have a triangulated polyhedron W ($=G\times\Delta^n$ triangulated by T_3) of $K^{[n+1]}$, such that:

- $W\cap K^{[n-1]}=\emptyset$,
- $\mathrm{fr}_{K^{[n+1]}}W$ and $W\cap K^{[n]}$ are subcomplexes of W,

- $f|_W$ is simplicial,
- $M \subseteq \mathrm{int}_{K^{[n+1]}} W$

(we use the original cell skeleta here). C will not change anymore, so B is the present C.

Step 8.

Consider the *simplicial complement* of s in $L^{[n+1]}$:

$$C(s,L^{[n+1]}) = \{\Delta \in T_L : \Delta \cap s = \emptyset\}.$$

It is easy to find a homotopy $f \simeq f' : K^{[n+1]} \to L^{[n+1]}$, relative W, such that

$$(**) \qquad f'(K^{[n+1]} \setminus \mathrm{int}_{K^{[n+1]}} W) \subseteq C(s,L^{[n+1]}).$$

We now only need to use relative simplicial approximation to make f' simplicial on $K^{[n+1]} \setminus \mathrm{int} W$ with respect to some triangulation of $K^{[n+1]}$ that extends the triangulation of W.

CHAPTER 11

THE MAIN STEP TO FUNCTORIALITY

Proposition 11.1.

Let K, L be finite regular CW-complexes, $n \geq 4$, and S - a non-singular enhanced n-cocycle in L. Then there exists a triangulation T_L of $L^{[n+1]}$ with the following two properties.

T_L subdivides the cell structure and is strongly compatible with S.

If C_0, C_1 are simplicial pull-backs of S with respect to T_L, by homotopic maps $f_0, f_1 : K \to L$ respectively, then $C_0 \sim C_1$.

Proof.

We first outline Part 1 of the proof.

T_L will be the restriction of a triangulation of $L^{[n+2]}$ obtained in Lemma 10.1. S is extended to $L^{[n+2]}$ as in that Lemma. We can assume that $f: K^{[n+1]} \times I \to L^{[n+2]}$ is a simplicial homotopy between f_0 and f_1 (under some triangulation of $K \times I$) and consider the preimage of the extension of S under f.

We embed $K \times I$ in $\mathbb{R}^m \times I$ for some m so that $K \times \{i\}$ is in $\mathbb{R}^m \times \{i\}$, $i=0,1$. $\mathbb{R}^m \times \{t\}$, any t, is called the t-level. We make sure that all the points of $f^{-1}(s) \cap (K \times I)^{[n]}$ lie on different levels, except all the points on 0 and 1-level. We also arrange that the intersection of $K \times I$ with a t-level, for any t, is naturally homeomorphic to K

(however, this intersection need not be $K\times\{t\}$).

Part 1: Preparation.

We adopt M, N_M and the map $h_M:\mathrm{fr}_{L^{[n+2]}}N_M\to S^{n-1}$ from the statement of Lemma 10.1. Throughout the proof, K will be regarded as a subpolyhedron of some $\mathbb{R}^q$. Let $f:K\times I\to L$ be the homotopy from f_0 to f_1. One of the primary goals of Part 1 is to build a very special copy of the domain of f. It will be contained in some $\mathbb{R}^{m+1}$, which is where most of our activity will take place. If Y is a subset of $\mathbb{R}^{m+1}$ or $K\times I$, then, unless stated otherwise, Y_t is the intersection of Y with the t-level.

Let $T_0:W_0^{\,p}\to K\times\left\{\frac{1}{3}\right\}$ be the triangulation with respect to which f_0 is simplicial. We can assume that $W_0^{\,p}$ is a simplicial complex in $\mathbb{R}^m\times\left\{\frac{1}{3}\right\}\subseteq\mathbb{R}^m\times\mathbb{R}=\mathbb{R}^{m+1}$, where m is very large. Recall that T_0 subdivides the cell structure on K, so $T_0^{-1}(\sigma)\subseteq W_0^{\,p}$ is a subcomplex for any cell σ in K. Similarly, define $W_1^{\,p}\subseteq\mathbb{R}^m\times\left\{\frac{2}{3}\right\}$ and $T_1:W_1^{\,p}\to\mathbb{R}^m\times\left\{\frac{2}{3}\right\}$.

We will construct a simplicial complex $W^{\,p}\subseteq\mathbb{R}^m\times\left[\frac{1}{3},\frac{2}{3}\right]$ which has $W_0^{\,p}=W^{\,p}\cap\left(\mathbb{R}^m\times\frac{1}{3}\right)$, and $W_1^{\,p}=W^{\,p}\cap\left(\mathbb{R}^m\times\frac{2}{3}\right)$ as subcomplexes, and a PL-homeomorphism $T^{\,p}:W^{\,p}\to K\times\left[\frac{1}{3},\frac{2}{3}\right]$, such that $T^{\,p}\big|_{W_0^{\,p}}=T_0$, $T^{\,p}\big|_{W_1^{\,p}}=T_1$. The superscript p in most of the gadgets here stands for "prototype". These gadgets will be later modified and will lose the superscript. We arrange all the cells of K in an order $\sigma_1,\dots,\sigma_z$ of non-decreasing dimension, and we define T^{P} and W^{P} simultaneously by the usual inductive argument. Namely, suppose that we have a simplicial complex $W_{i-1} \subseteq \mathbb{R}^m \times \left[\frac{1}{3},\frac{2}{3}\right]$ and a PL-homeomorphism

$$T^{\mathrm{P}}: W_{i-1} \to \left(K \times \left\{\frac{1}{3}\right\}\right) \cup \left(\left(\sigma_1 \cup \ldots \cup \sigma_{i-1}\right) \times \left[\frac{1}{3},\frac{2}{3}\right]\right) \cup \left(K \times \left\{\frac{2}{3}\right\}\right).$$

We treat $\sigma_i \times \left[\frac{1}{3},\frac{2}{3}\right]$ as a PL-cone with the vertex

$$(\xi_i, z_i) \in \operatorname{int}\sigma_i \times \left(\frac{1}{3},\frac{2}{3}\right) \subseteq \sigma_i \times \left[\frac{1}{3},\frac{2}{3}\right].$$

We can require that the projection function $\Pi: \sigma_i \times \left[\frac{1}{3},\frac{2}{3}\right] \to \left[\frac{1}{3},\frac{2}{3}\right]$ maps each segment from the base to the vertex linearly. Because m is large, we can find $(\zeta_i, z_i) \in \mathbb{R}^m \times \left(\frac{1}{3},\frac{2}{3}\right)$ such that the cone Z with (ζ_i, z_i) as vertex and the simplicial complex $(T^{\mathrm{P}})^{-1}\left(\mathrm{bd}\left(\sigma_i \times \left[\frac{1}{3},\frac{2}{3}\right]\right)\right)$ as the base, is embedded and Z intersects W_{i-1} only at its base. We take $W_i = W_{i-1} \cup Z$ (Z has the obvious simplicial structure). We extend T^{P} (in Step i) by coning. Note that T^{P} is level-preserving:

$$(*) \qquad \forall t \in \left[\frac{1}{3},\frac{2}{3}\right] \quad T^{\mathrm{P}}(W^{\mathrm{P}} \cap (\mathbb{R}^m \times \{t\})) = K \times \{t\}.$$

The homotopy f can now be viewed as $f: W^{\mathrm{P}} \to L$, with $f|_{W_0^{\mathrm{P}}} = f_0$, $f|_{W_1^{\mathrm{P}}} = f_1$. The cell structure on $K \times \left[\frac{1}{3},\frac{2}{3}\right]$ transports, via $(T^{\mathrm{P}})^{-1}$, to W, and by the relative cellular approximation and B.H.E.T we can assume that

$$f\left((T^{\mathrm{P}})^{-1}\left(K^{[n+1]} \times \left[\frac{1}{3},\frac{2}{3}\right]\right)\right) \subseteq L^{[n+2]}$$

(original cell skeleta). By [Z] and B.H.E.T we can perform a repeated barycentric subdivision on W^{P}, relative $W_0^{\mathrm{P}} \cup W_1^{\mathrm{P}}$, and assume that $f{:}W^{\mathrm{P}} \to L$ is simplicial on $(T^{\mathrm{P}})^{-1}\left[K^{[n+1]} \times \left[\frac{1}{3},\frac{2}{3}\right]\right]$, with respect to the new subdivision of W^{P} (which we shall now call W^{P}), and T_{L}.

Let $W=\left[W_0^{\mathrm{P}} \times \left[0,\frac{1}{3}\right]\right] \cup W^{\mathrm{P}} \cup \left[W_1^{\mathrm{P}} \times \left[\frac{2}{3},1\right]\right]$, where W_i^{P} is treated as a subset of $\mathbb{R}^m$. We subdivide W to a simplicial complex in $W \setminus W^{\mathrm{P}}$, coning either between the $\frac{2}{3}$- and 1-level, or the 0- and $\frac{1}{3}$-level.

T^{P} is now extended to $T{:}W \to K \times I$ in the obvious fashion.

Consider the subcomplex $X=T^{-1}(K^{[n+1]} \times I)$ of W. We extend f to a simplicial map $f{:}X \to L^{[n+2]}$ as follows. We require $f\Big|W_0^{\mathrm{P}} \times \{0\} = f\Big|W_0^{\mathrm{P}} \times \left\{\frac{1}{3}\right\}$, and $f\Big|W_1^{\mathrm{P}} \times \{1\} = f\Big|W_1^{\mathrm{P}} \times \left\{\frac{2}{3}\right\}$. Let Γ be any simplex of $X \cap (W_0^{\mathrm{P}} \cup W_1^{\mathrm{P}})$. $\Gamma \times \left[\frac{2}{3},1\right]$ or $\Gamma \times \left[0,\frac{1}{3}\right]$ was subdivided above by coning, and let p_Γ be the vertex used. If $\dim\Gamma \leq n$, we let $f(p_\Gamma)$ be any vertex of $f(\Gamma)$. If $\dim\Gamma = n+1$, then $f^{-1}(s) \cap \Gamma$ is either $\emptyset$ or a proper PL-arc that joins two n-faces Δ_0 and Δ_1 of Γ (recall that s is the core of S, and S is strongly compatible with T_{L}). If $f^{-1}(s) \cap \Gamma = \emptyset$, we let $f(p_\Gamma)$ to be any vertex of $f(\Gamma)$. In the other case, we let $f(p_\Gamma)$ to be $f(v_0)$, where v_0 is the vertex of Γ that is in Δ_0 but not in Δ_1.

Claim.

Let Γ_1, Γ_2 be different n-simplices of X such that $f^{-1}(s) \cap \Gamma_1 \neq \emptyset \neq f^{-1}(s) \cap \Gamma_2$. Then precisely one of the following holds:

- each of Γ_1 and Γ_2 is on 0- or 1-level,
- there is a point v that does not lie on the 0- or 1-level and

that is a vertex of precisely one of Γ_1, Γ_2.

Proof.

We leave the proof to the reader. The Claim follows from the way we extended f. ■ (Claim)

We went through the special effort of defining f on $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$-levels because we want the above claim.

We now slightly adjust W to make it satisfy certain "general position" requirements that involve the last coordinate of $\mathbb{R}^{m+1}$. Namely, let $\{(q_i',u_i')\}$ be the set of all the vertices of W not on 0- or 1-level, and let $\delta>0$. Then we move (q_i',u_i') to a δ-close point (q_i,u_i), and we extend the move to W linearly (keeping the 0- or 1-level vertices fixed). We have an embedding α of W in $\mathbb{R}^m\times[0,1]$, provided δ is small enough. Following are additional conditions that we want the move to satisfy.

Condition 1.

$u_1,\ldots,u_z$ are algebraically independent over $\mathbb{Q}$. Therefore, no (≥ 1 dimensional) simplex of $\alpha(W)$ sits on a level, unless it is a 0- or 1-level.

Condition 2.

For each $x\in K$, the function $\Pi\alpha T^{-1}|_{x\times I}:x\times I\to\mathbb{R}$ is increasing, where $\Pi:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$ is the projection. Why can we demand this?

Recall how $K\times I$ lives in $\mathbb{R}^{q+1}$, and consider simplicial subdivisions $(K\times I)'$ of $K\times I$, and W' of W, with respect to which T is a simplicial isomorphism. Consider any simplex Δ of $(K\times I)'$ such that, for some $x \in K$, $\dim \Delta \cap (x \times I) = 1$. Let $\Gamma' = T^{-1}(\Delta)$.

$$\mathscr{F}_{\Delta} = \{\Delta \cap (x \times I) \colon x \in K,\ \dim \Delta \cap (x \times I) = 1\}$$

is a family of closed segments in Δ. Equip each segment with an arrow that points down. $\mathscr{F}_{\Delta}$, by T^{-1}, gets mapped to a likewise family (of parallel oriented segments) $\mathscr{F}_{\Gamma'}$ in the simplex Γ'. Because T^{-1} preserved levels, the intervals of $\mathscr{F}_{\Gamma'}$ point down, and have a non-zero angle to $\mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R}$. If b is an oriented interval in a simplex Γ in a Euclidean space, a very small affine movement of Γ will change the direction of b very little, and the same change will occur for any interval in Γ parallel to b. Therefore, if Γ' (as above) is contained in a simplex Γ of W, a very small affine movement of Γ changes the direction of the parallel family $\mathscr{F}_{\Gamma'}$ very little. Because there is only finitely many simplices like Γ' (and families like $\mathscr{F}_{\Gamma'}$), we see that, after a small move, in $\alpha(W)$, all the families still point down. This is what we want. ∎ (Condition 2)

Since α does not preserve levels, a level of $\alpha(W)$ need not be homeomorphic to K. To recover this property, we define another PL-homeomorphism $T' \colon \alpha(W) \to K \times I$.

We set $T'(\alpha(y,t)) = (x,t)$ if $T(y,t) = (x,t')$ for some t'. This is 1-1 by Condition 2, and, trivially, onto. Lemma 2.4 easily implies that T' is continuous. Observe that T' is the αT is T on 0- and 1-levels. T' is PL since it is the product of PL-maps.

We now forget about the original T and W, and T' and $\alpha(W)$ become our new T and W.

Recall the subsets M, N_M of $L^{[n+2]}$. With respect to the triangulation T_L of $L^{[n+2]}$, N_M is a special $\frac{1}{2}$-neighborhood of an n-central set M in $L^{[n+2]}$. Our simplicial map $f{:}W \to L$ takes X to $L^{[n+2]}$. We "pull N_M and M back", using f, as in Proposition 9.2. Let $c=f^{-1}(M)\cap X$ and $U=f^{-1}(N_M)\cap X$. Let $h_U=h_M f{:}\mathrm{fr}_X U \to S^{n-1}$.

c is an n-central set in X and $c\cap X^{[n]}$ is finite. By the Claim and Condition 1, all the points of $(c\cap X^{[n]})\cap(\mathbb{R}^m\times(0,1))$ lie on different levels. This means c is, in a sense, "skewed".

■ (Preparation)

Let us summarize the parts of the construction above which will be used in the rest of the proof.

We have a simplicial complex $W\subseteq\mathbb{R}^m\times I$ with a level-preserving triangulation $T{:}W\to K\times I$. For any cell σ of K, $T^{-1}(\sigma\times I)$ is a subcomplex of W. X denotes the subcomplex $T^{-1}(K^{[n+1]}\times I)$. U is the special $\frac{1}{2}$-neighborhood U of an n-central set c in X. We have a map $h_U{:}\mathrm{fr}_X U\to S^{n-1}$ with the property

(1) $$\deg h_U\big|\mathrm{fr}_\Delta U\cap\Delta=\pm1,$$

for any n-simplex Δ of X. c is skewed in the following sense. All points of $(c\cap X^{[n]})\cap(\mathbb{R}^m\times(0,1))$ lie on different levels. In particular, no interval $c\cap\Gamma$, where Γ is an $n+1$-simplex of X, lies in a level, unless it is the 0- or 1-level. Also, no linear 2-disc $c\cap\Delta$, where Δ is an $n+2$-simplex of X, lies in a level. The triangulation is skewed: no (≥ 1 dimensional) simplex of W lies in a level, unless it is the 0- or 1-level.

Because of 9.2(ii), $\mathrm{fr}_{X_i} U_i = (\mathrm{fr}_X U) \cap X_i$. For $i=0$ or 1, U_i, c_i, and $h_U : \mathrm{fr}_{X_i} U_i \to S^{n-1}$ determine, via T, the n-cochain (with respect to the original cell structure) C_i in K. C_i can be also regarded as a cochain with respect to the triangulation $T: W_i \to K$ (see Chapter 10); then, C_i is an n-central cochain of size $\frac{1}{2}$. Whenever we speak about size, we will use the second meaning; otherwise we use the first.

Our ultimate goal is to prove $C_0 \sim C_1$.

Part 2: Attack.

Let us outline our strategy.

We reduce U (9.3) to a very thin size. Let

$$c \cap X^{[n]} = \{(a_1,1),\ldots,(a_a,1),(b_1,0),\ldots,(b_b,0),(d_1,t_1),\ldots,(d_k,t_k)\},$$

where $1 > t_1 > \ldots > t_k > 0$. On any level t, except close to t_e, $e=1,\ldots,k$, we see an n-cochain C_t in K. We will observe that $C_t \sim C_{t'}$ if $[t,t']$ is not too close to any t_e. The final step is to show that $C_{t_e - \iota} \sim C_{t_e + \iota}$, where $\iota > 0$ is a suitably small number. This is the most difficult part of the proof.

■ (outline)

First, two general comments.

Let us apply Proposition 9.3 to our core and size $\frac{1}{2}$ neighborhood U. This allows us to arbitrarily change the size of U to say, ε-size, by a nice isotopy. h_U follows along, and is defined on the frontier of the adjusted U. Now we look what happened at the top

1-level. A routine comparison between the conclusion of 9.3 and the requirements of the isotopy move shows that we have changed the central n-cochain C_1 from size $\frac{1}{2}$ to size ε by an isotopy move. This is all done with respect to the triangulation $T{:}W_1 \to K$. But an isotopy move with respect to a fine CW-structure is an isotopy move with respect to a rougher one. Thus C_i, with original meaning, is isotopy-moved. The same applies to the bottom and the conclusion is: we are allowed to place conditions on the size of U as we please (as long as there is a size that satisfies them all!), and not have to worry about anything. In practice, we will finitely many times insist that ε be small enough.

For the second comment, recall the non-singular homotopy move. What it means, is that we are allowed to freely maneuver, by homotopy, the map h_U. If we see a cochain at a level (that is: after passing through

$$W_t \xrightarrow{T} K \times I \xrightarrow{\Pi} K,$$

we see an n-cochain in K) it will be changed by a homotopy move.

We start from the last step of the outline. We fix $e=1,\dots,$k. We will need to determine ι, which will depend on c and e, and not on U or the map. After ι is chosen, ε will have to be adjusted. This is done k times.

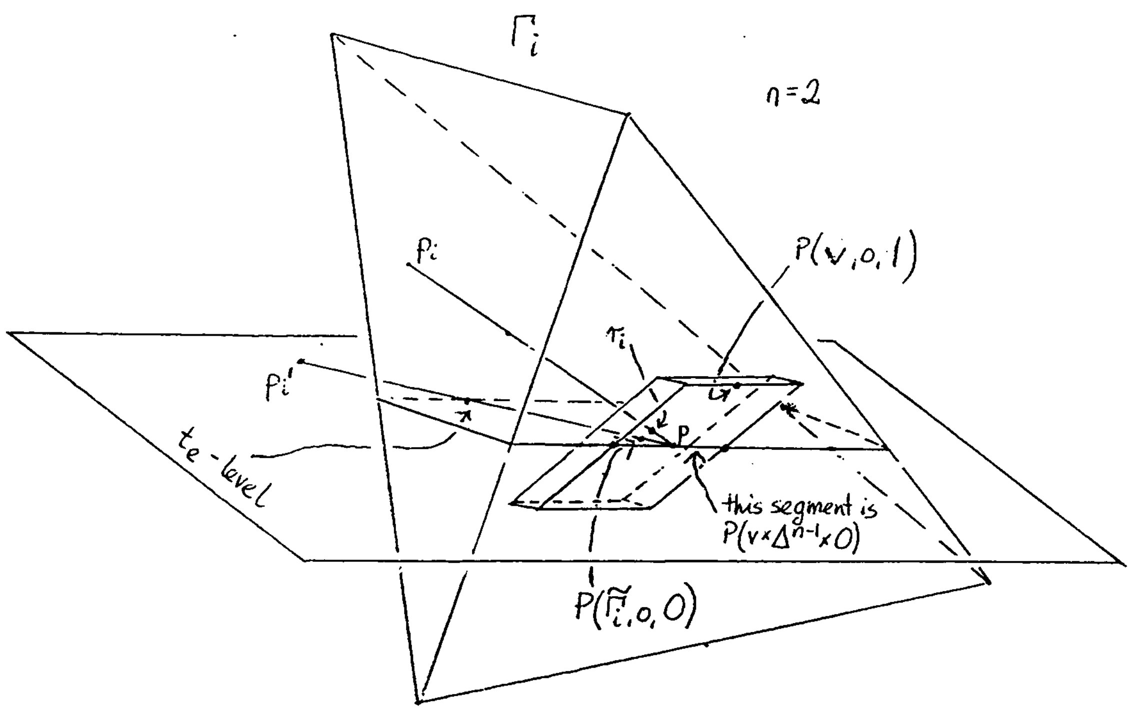
"What happens around (d_e,t_e)"

Let us look at a neighborhood of $p=(d_e,t_e)$. Let Δ be the n-simplex of X that contains p in its interior. Let $\Gamma_1,\dots,\Gamma_w$ be all the $n+1$-simplices of X one of whose faces is Δ. If Γ_i and Γ_j, $i \neq j$, are faces of an $n+2$-simplex of X, then this simplex will be denoted Γ_{ij}. Let G denote the link of Δ in X, that is, the graph whose vertices $\tilde{\Gamma}_i$ correspond to Γ_i's and whose edges $\tilde{\Gamma}_{ij}$ correspond to Γ_{ij}'s.

Let $\mathrm{bd}\Gamma_i \cap c = \{p, p_i\}$. We will now define an affine homeomorphism P from $E = \mathrm{c}G \times \Delta^{n-1} \times [-1,1]$, where $\mathrm{c}G$ is the cone over G, onto a neighborhood V of p in X (here "affine" means: affine on each $\mathrm{c}b \times \Delta^{n-1} \times [-1,1]$, where b is a vertex or an edge of G) as follows.

Let v be the vertex of $\mathrm{c}G$. We first set $P(v,o,0)=p$. Then let $P|_{v \times \Delta^{n-1} \times 0}$ be an affine homeomorphism onto a neighborhood of p in $\mathrm{int}\Delta$ on the t_e-level. Let $(v,o,1)$ go to a point close to p and lying on $t_e+\upsilon$-level, where $\upsilon > 0$. How close? So that $P|_{v \times \Delta^{n-1} \times [-1,1]}$ is determined as an affine homeomorphism onto a neighborhood of p in $\mathrm{int}\Delta$.

Let p_i' be the projection of p_i, parallel to the vector $\overrightarrow{pP(v,o,1)}$ to the t_e-level. The vector $\overrightarrow{pp_i}$ is contained in $\mathrm{aff}\Gamma_i$, and we set $P(\tilde{\Gamma}_i,o,0)$ to be the point on $\overrightarrow{pp_i'}$ close to p. How close? First, so that $P|_{\mathrm{c}\tilde{\Gamma}_i \times \Delta^{n-1} \times [-1,1]}$ is determined as an affine homeomorphism onto a neighborhood of p in $\mathrm{int}\Gamma_i \cup \mathrm{int}\Delta$. The second condition will be given later. The behaviour of P on $\mathrm{c}\tilde{\Gamma}_i \times \Delta^{n-1} \times [-1,1]$ is shown in Figure 13. Finally, $P|_{\mathrm{c}\tilde{\Gamma}_i \times \Delta^{n-1} \times [-1,1]}$ and $P|_{\mathrm{c}\tilde{\Gamma}_j \times \Delta^{n-1} \times [-1,1]}$



the front parallelogram is $P(v\times\Delta^{n-1}\times[-1,1])$

the whole parallelepiped is $P(c\tilde{\Gamma}_i\times\Delta^{n-1}\times[-1,1])$

p_i should be the center of the back triangle

Figure 13. The behaviour of P on $c\tilde{\Gamma}_i\times\Delta^{n-1}\times[-1,1]$

determine $P\big|_{c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times[-1,1]}$ as an affine homeomorphism onto a neighborhood of p in $\mathrm{int}\Gamma_{ij}\cup\mathrm{int}\Gamma_i\cup\mathrm{int}\Gamma_j\cup\mathrm{int}\Delta$. The behaviour of P on $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times[-1,1]$ is shown in Figure 14.

Observe that $P{:}E\to V$ behaves well with respect to levels:

$$P(cG\times\Delta^{n-1}\times\{t\})=V_{t_e+t\upsilon},\ t\in[-1,1].$$

Consider the convex set

$$V_i=P(c\tilde{\Gamma}_i\times\Delta^{n-1}\times[-1,1])\subseteq\mathrm{aff}\Gamma_i.$$

The vector $\overrightarrow{pp_i}\subseteq c$ starts at p, goes through $\mathrm{int}_{\mathrm{aff}\Gamma_i}V_i$ and then hits $\mathrm{bd}V_i$ again at a point r_i. We insist that r_i be inside the "back face" of V_i, that is,

$$r_i\in P(\tilde{\Gamma}_i\times\mathrm{int}(\Delta^{n-1}\times[-1,1])).$$

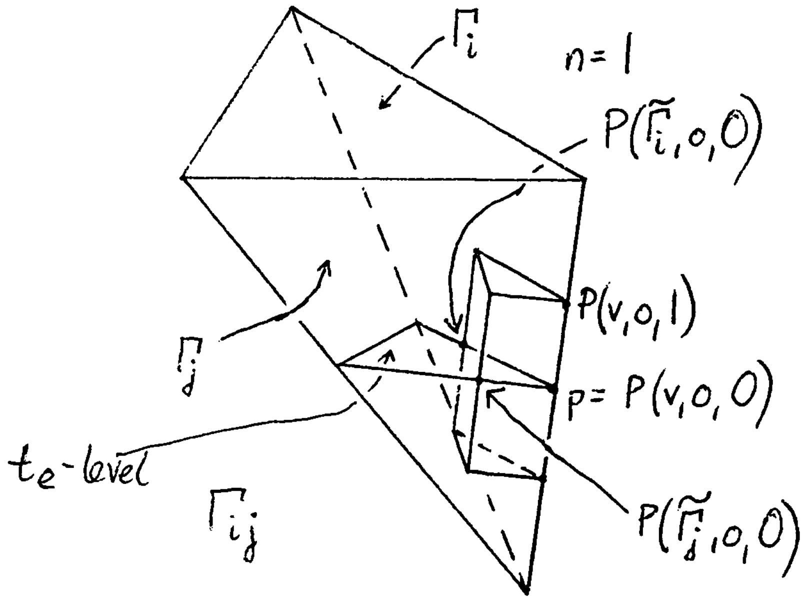
This is achieved by putting $P(\tilde{\Gamma}_i,o,0)$ sufficiently close to p.

P is finally constructed, and we use it to identify V with E.

Let s_i be the last coordinate of $r_i\in\mathbb{R}^{m+1}$. Let $\tilde{s}_i\in(-1,1)$ be the level (namely, $cG\times\Delta^{n-1}\times\{\tilde{s}_i\}$) that P takes to the s_i-level. We fix $\iota>0$ to be less than half of the distances from t_e to any of the following: 1, $t_1,\dots,t_k$ (except t_e), 0, $s_1,\dots,s_w$.

▪ "What happens around (d_e,t_e)"

We now take a look at U around p (U is the special $\frac{1}{2}$-neighborhood of c in X). $c\cap(c\tilde{\Gamma}_i\times\Delta^{n-1}\times[-1,1])$ is the closed segment from $(v,o,0)$ $(=p)$ to $r_i=(\tilde{\Gamma}_i,o,\tilde{s}_i)$; we will call it the distinguished segment.



the wedge is $P(c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times[-1,1])$

Figure 14. The behaviour of P on $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times[-1,1]$

The entire $c\cap(c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times[-1,1])$ lies in $c\tilde{\Gamma}_{ij}\times o\times[-1,1]$ and is the triangle as pictured (we will call it the distinguished triangle) in Figure 15. Consider the space $cG\times\Delta^{n-1}\times\mathbb{R}\supseteq E$ and the map $\gamma:cG\times\Delta^{n-1}\times\mathbb{R}\to v\times\Delta^{n-1}\times\mathbb{R}$ whose restriction on each $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times\mathbb{R}$ $(c\tilde{\Gamma}_{i}\times\Delta^{n-1}\times\mathbb{R})$ is the projection parallel to the distinguished triangle (segment) contained in that part. The map is pictured in Figure 16. Let us denote $U_0=U\cap(v\times\Delta^{n-1}\times\mathbb{R})$. We have $U\cap E=\gamma^{-1}(U_0)\cap E$. We make the size ($\varepsilon$) of U small enough so that $\gamma^{-1}(U_0)\subseteq cG\times\mathrm{int}(\Delta^{n-1}\times[-1,1])\subseteq E$. In addition, ε should be so small that $U_0\subseteq v\times\mathrm{int}\Delta^{n-1}\times(-\tilde{\iota},\tilde{\iota})$, where $\tilde{\iota}$-level corresponds to ι-level via P.

"The two crucial cochains"

We will look at $cG\times\Delta^{n-1}\times\{\tilde{\iota}\}$ and $cG\times\Delta^{n-1}\times\{-\tilde{\iota}\}$. Note that $0<\tilde{\iota}<\frac{1}{2}\min_{i=1,\ldots,w}|\tilde{s}_i|$. On each of these two levels we will see (using c, U, and h_U) an n-cochain, and we will investigate how the two cochains are related. cG and Δ^{n-1} have natural CW-structures, and each level has a natural CW-structure as the product $cG\times\Delta^{n-1}$. This is the CW-structure with respect to which the two cochains will be considered and maneuvered.

We will first take a close look at $\gamma^{-1}(U_0)$ on $\tilde{\iota}$- and $-\tilde{\iota}$-level. Since the Δ^{n-1}-coordinate is preserved by γ, we will simply picture the action of

$$\gamma:c\tilde{\Gamma}_i\times\{x\}\times\{-\tilde{\iota},\tilde{\iota}\}\to v\times\{x\}\times\mathbb{R}$$

and

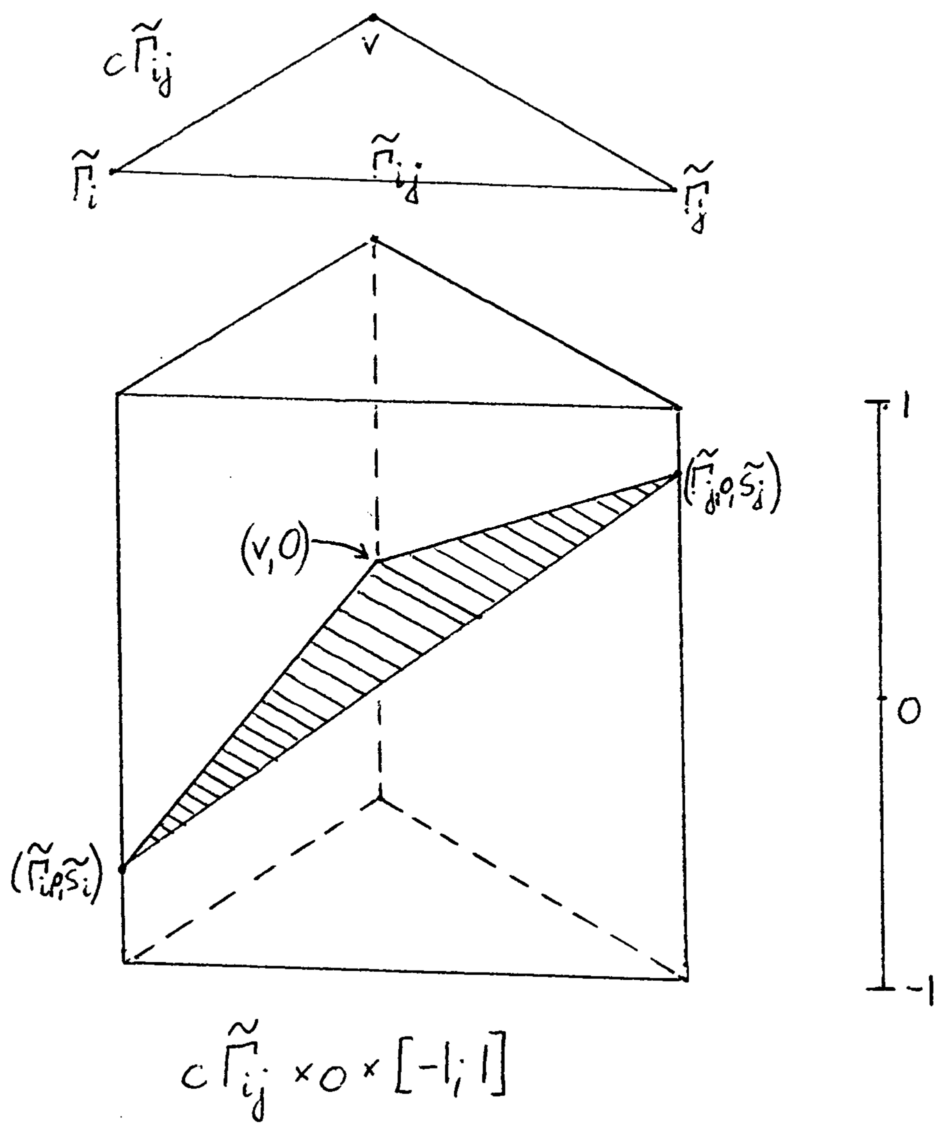


Figure 15. The distinguished triangle

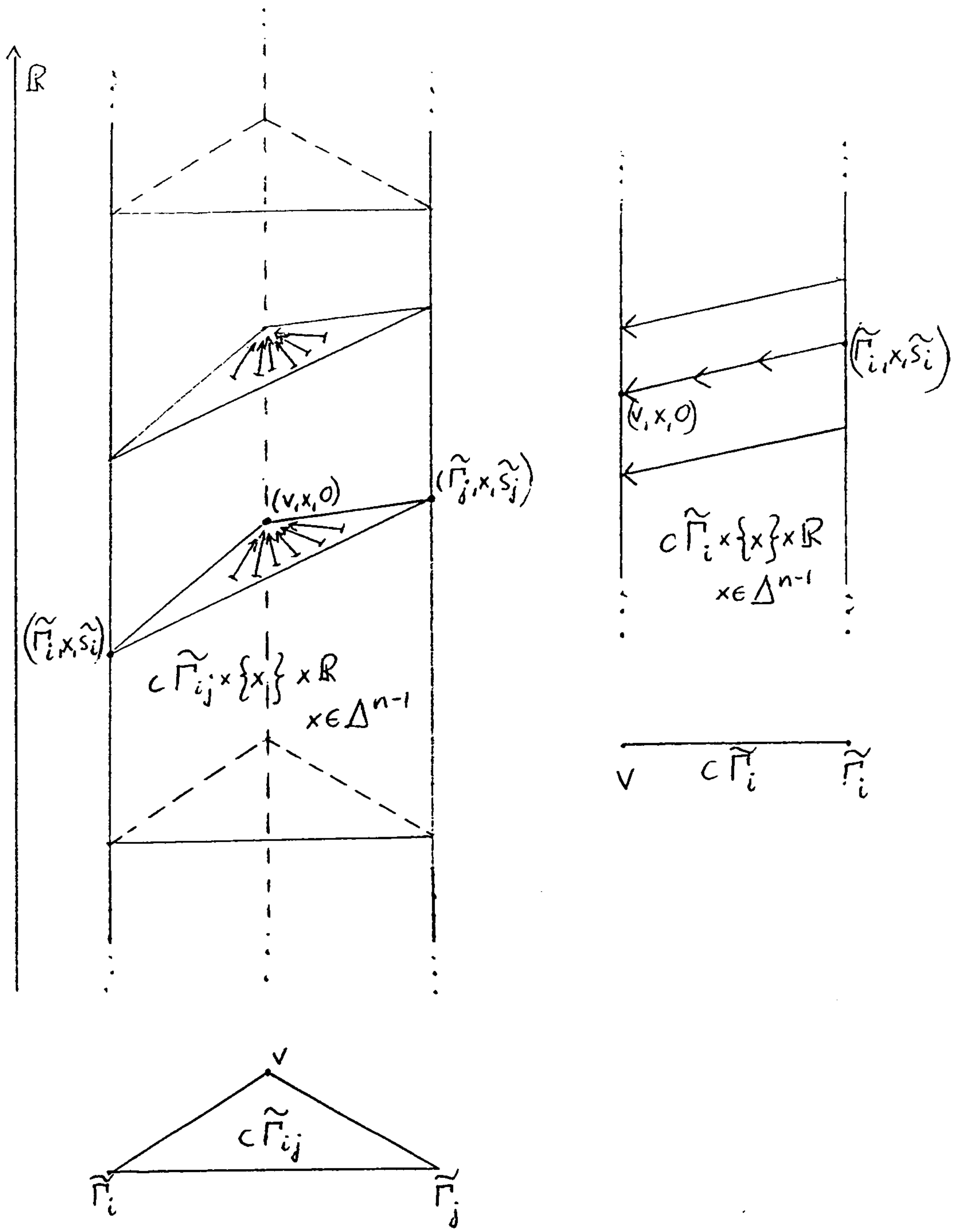


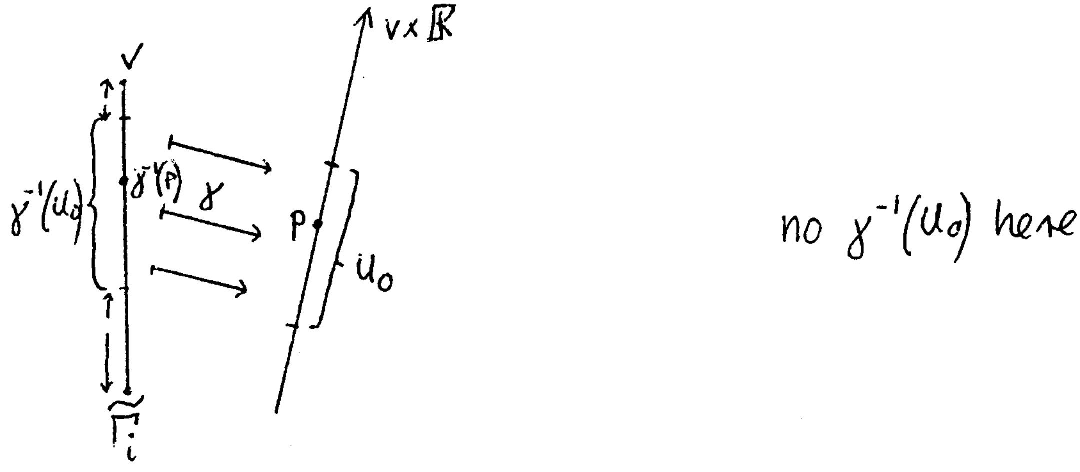
Figure 16. The action of γ

$$\gamma : c\tilde{\Gamma}_{ij} \times \{x\} \times \{-\tilde{\imath}, \tilde{\imath}\} \to v \times \{x\} \times \mathbb{R},$$

for some x, and suppress x. The whole picture of γ is then obtained by taking products with Δ^{n-1}. Recall that U_0 is an n-simplex in $v \times \Delta^{n-1} \times [-1,1]$, so that $U_0 \cap (v \times \{x\} \times [-1,1])$ is $\emptyset$, a point, or an interval, depending on x. The last case is reflected in our Figures 17 - 21. On these Figures γ is the projection parallel to the vectors drawn as $\longmapsto$. The sign $\longleftarrow\!\!\longrightarrow$ stresses that a non-zero distance as shown is present. This simply indicates that $\gamma^{-1}(U_0)$ is separated from $v \times \Delta^{n-1}$ and from $G \times \Delta^{n-1}$. The "cores" $\gamma^{-1}(v,o,0)$ are shown in those pictures. If $x \neq o$, simply disregard the cores. One last comment: note the changing position of the arrow on $v \times \mathbb{R}$. The reason is: the orientation of the parallel projection, pictured in Figure 22, of the segment $[v, \tilde{\Gamma}_i]$ onto the line $v \times \mathbb{R}$ changes depending on whether $[v, \tilde{\Gamma}_i]$ is above p or below. From Figures 17 - 21, we can develop a complete understanding of how U and $c = \gamma^{-1}(p)$ looks on levels $\tilde{\imath}$ and $-\tilde{\imath}$. By Lemma 2.1, we will see n-cochains in $cG \times \Delta^{n-1}$, if we can verify the degree requirements for the restriction of the map h_U to the frontier $\mathrm{fr}U_{\tilde{\imath}}$, $\mathrm{fr}U_{-\tilde{\imath}}$ of U on $\tilde{\imath}$-, $-\tilde{\imath}$-level, respectively (that is, $\mathrm{fr}U_{\pm\tilde{\imath}} \subseteq cG \times \Delta^{n-1} \times \{\pm\tilde{\imath}\}$ is the intersection of $\mathrm{fr}U$ with the $\pm\tilde{\imath}$-level). We first homotope h_U.

Claim.

One can homotope h_U so that the diagram

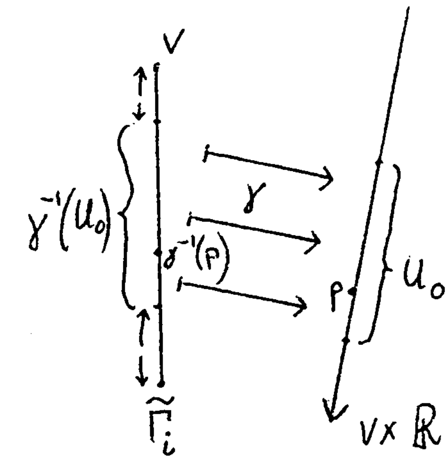


level $\tilde{\imath}$ level $-\tilde{\imath}$

$c\tilde{\Gamma}_i$, $\tilde{s}_i > 0$

Figure 17. The action of $\gamma: c\tilde{\Gamma}_i \times \{x\} \times \{-\tilde{\imath}, \tilde{\imath}\} \to v \times \{x\} \times \mathbb{R}$ in case when $\tilde{s}_i > 0$

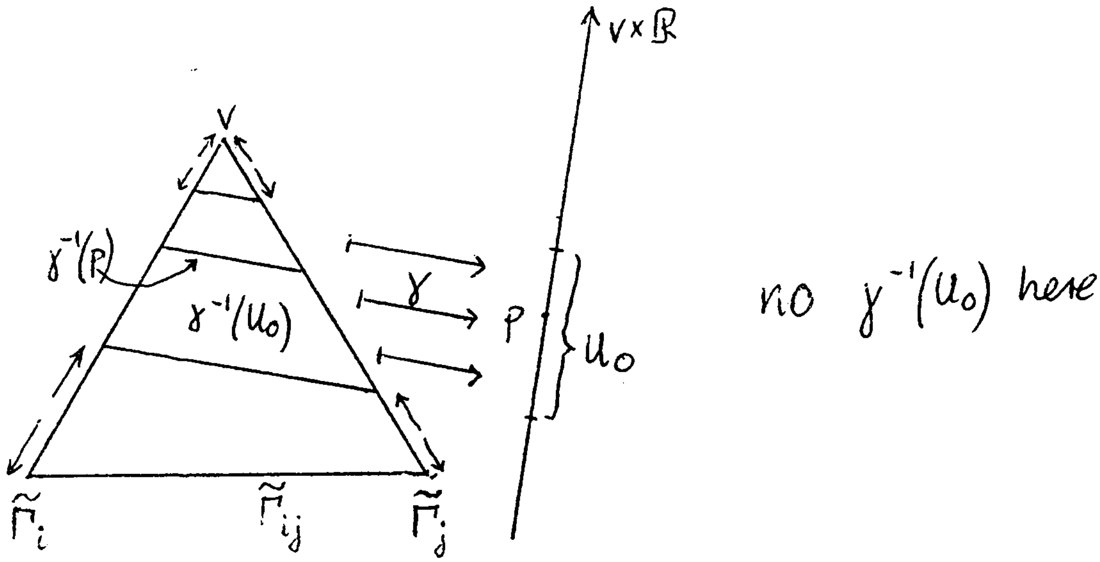
no $\gamma^{-1}(U_0)$ here



level $\tilde{\iota}$ level $-\tilde{\iota}$

$c\tilde{\Gamma}_i$, $\tilde{s}_i < 0$

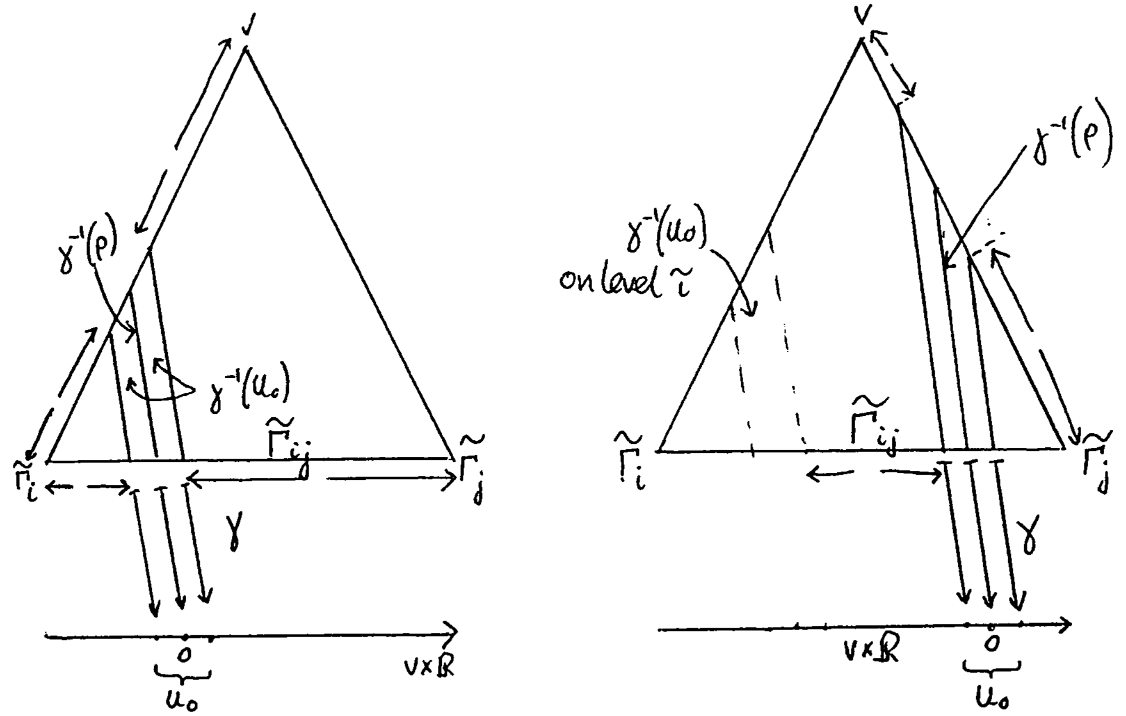
Figure 18. The action of $\gamma: c\tilde{\Gamma}_i \times \{x\} \times \{-\tilde{\iota}, \tilde{\iota}\} \to v \times \{x\} \times \mathbb{R}$ in case when $\tilde{s}_i < 0$



level $\tilde{\imath}$ level $-\tilde{\imath}$

$c\tilde{\Gamma}_{ij}$, $\tilde{s}_i>0$, $\tilde{s}_j>0$

Figure 19. The action of γ:$c\tilde{\Gamma}_{ij}\times\{x\}\times\{-\tilde{\imath}\,,\tilde{\imath}\,\}\to v\times\{x\}\times\mathbb{R}$ in case when $\tilde{s}_i>0$, $\tilde{s}_j>0$

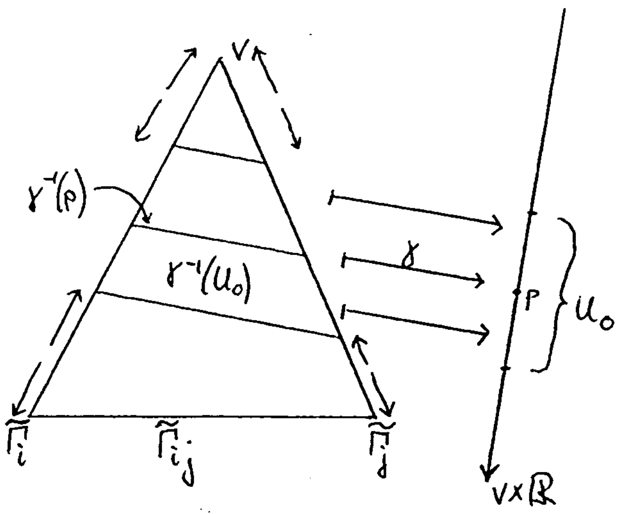


level $\tilde{\imath}$ level $-\tilde{\imath}$

$c\tilde{\Gamma}_{ij}$, $\tilde{s}_i>0$, $\tilde{s}_j<0$

Figure 20. The action of $\gamma: c\tilde{\Gamma}_{ij}\times\{x\}\times\{-\tilde{\imath},\tilde{\imath}\}\to v\times\{x\}\times\mathbb{R}$ in case when $\tilde{s}_i>0$, $\tilde{s}_j<0$

No $\gamma^{-1}(U_0)$ here



level $\tilde{\imath}$ level $-\tilde{\imath}$

$c\tilde{\Gamma}_{ij}$, $\tilde{s}_i<0$, $\tilde{s}_j<0$

Figure 21. The action of $\gamma: c\tilde{\Gamma}_{ij} \times \{x\} \times \{-\tilde{\imath}, \tilde{\imath}\} \to v \times \{x\} \times \mathbb{R}$ in case when $\tilde{s}_i<0$, $\tilde{s}_j<0$

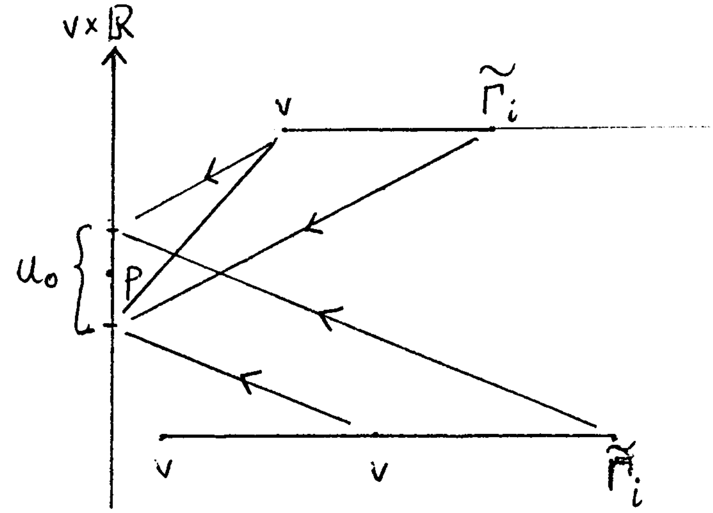


Figure 22. Why the position of the arrow on $v \times \mathbb{R}$ changes

$$(4) \qquad \begin{tikzcd} \mathrm{fr}U_{\pm\tilde{\imath}} \arrow[rr, "h_U|\mathrm{fr}U_{\pm\tilde{\imath}}"] \arrow[dr, "\gamma"'] & & S^{n-1} \\ & \mathrm{fr}U_0 \arrow[ur, "h_U|\mathrm{fr}U_0"'] & \end{tikzcd}$$

commutes (in both "+" and "-" cases).

■ (the two crucial cochains)

Proof of the Claim.

The Claim is a close relative of the following:

Let X be a metric space and let $A\subseteq X$ be a closed subset which is contractible in X. Let Y be a metric ANR. Then any map $h:X\to Y$ is homotopic to a map which is constant on A. ■ (the relative)

To prove the Claim, it is enough to find a compact, contractible metric space A, a point $x\in A$, two subsets $A_+,A_-\subseteq A$, and an embedding $u:A\times\mathrm{fr}U_0\hookrightarrow\mathrm{fr}U$, such that

$$u(x\times\mathrm{fr}U_0,A_+\times\mathrm{fr}U_0,A_-\times\mathrm{fr}U_0)=(\mathrm{fr}U_0,\mathrm{fr}U_{\tilde{\imath}},\mathrm{fr}U_{-\tilde{\imath}}),$$

and such that the diagram

$$\begin{tikzcd} \mathrm{fr}U_{\pm\tilde{\imath}} \arrow[d, "\gamma"] & A_\pm\times\mathrm{fr}U_0 \arrow[l, hook', "u"'] \arrow[d, "*\times\mathrm{Id}"] \\ \mathrm{fr}U_0 & \{x\}\times\mathrm{fr}U_0 \arrow[l, hook', "u"'] \end{tikzcd}$$

commutes.

Towards that end, we will find A and construct certain embedding (which will be an extension of u)

$$\bar{u}:A\times(v\times\Delta^{n-1}\times(-\tilde{\imath},\tilde{\imath}))\to\gamma^{-1}(v\times\Delta^{n-1}\times(-\tilde{\imath},\tilde{\imath})).$$

$\bar{u}$, first of all, preserves the Δ^{n-1}-coordinate; let us then drop v and $x\in\Delta^{n-1}$, consider $\gamma{:}cG\times\mathbb{R}\to\mathbb{R}$, and construct $\bar{u}{:}A\times(-\tilde{\imath},\tilde{\imath})\to\gamma^{-1}(-\tilde{\imath},\tilde{\imath})$.

We work according to the following scheme. We will have $A\subseteq cG$. As cG is the sum of the cones over vertices and edges of G, let us consider such a vertex of edge b, and construct $A\cap cb$. $A\cap cb$ will be a sum of straight, closed intervals (they will be exhibited in a moment). Working with such an interval $J\subseteq cb$, we define

$$\bar{u}{:}J\times(-\tilde{\imath},\tilde{\imath})\to\gamma^{-1}(-\tilde{\imath},\tilde{\imath})\cap(J\times\mathbb{R})\subseteq\gamma^{-1}(-\tilde{\imath},\tilde{\imath})$$

as in one of the Figures 23, 24. Not all closed intervals that are subsets of cb fit into one of them, but all we will use will. For the pictures in Figures 23 and 24 to exist, it is important that $\tilde{\imath}\leq\frac{1}{2}|\tilde{s}_i|$. $\bar{u}$ is linear on each of the vertical intervals. The distinguished object in $cb\times\{x\}\times\mathbb{R}$ is the translate of the one previously defined in $cb\times\{o\}\times\mathbb{R}$.

We now consider different cases for b; in each of them, we explain what $A\cap cb$ is and what the intervals J are.

Case 1. $b=\tilde{\Gamma}_i$, $\tilde{s}_i>0$.

Then $A\cap c\tilde{\Gamma}_i$ is (one) interval J with $J_0=v$, and use Figure 23. $J_1\in c\tilde{\Gamma}_i$ is the point with $\gamma(J_1,\tilde{\imath})=-\tilde{\imath}$.

Case 2. $b=\tilde{\Gamma}_i$, $\tilde{s}_i<0$

Proceed as above, but use Figure 24. We have $\gamma(J_1,-\tilde{\imath})=\tilde{\imath}$.

■ (Case 2)

In the Figures 25, 26 for subsequent cases, $A\cap cb$ is the

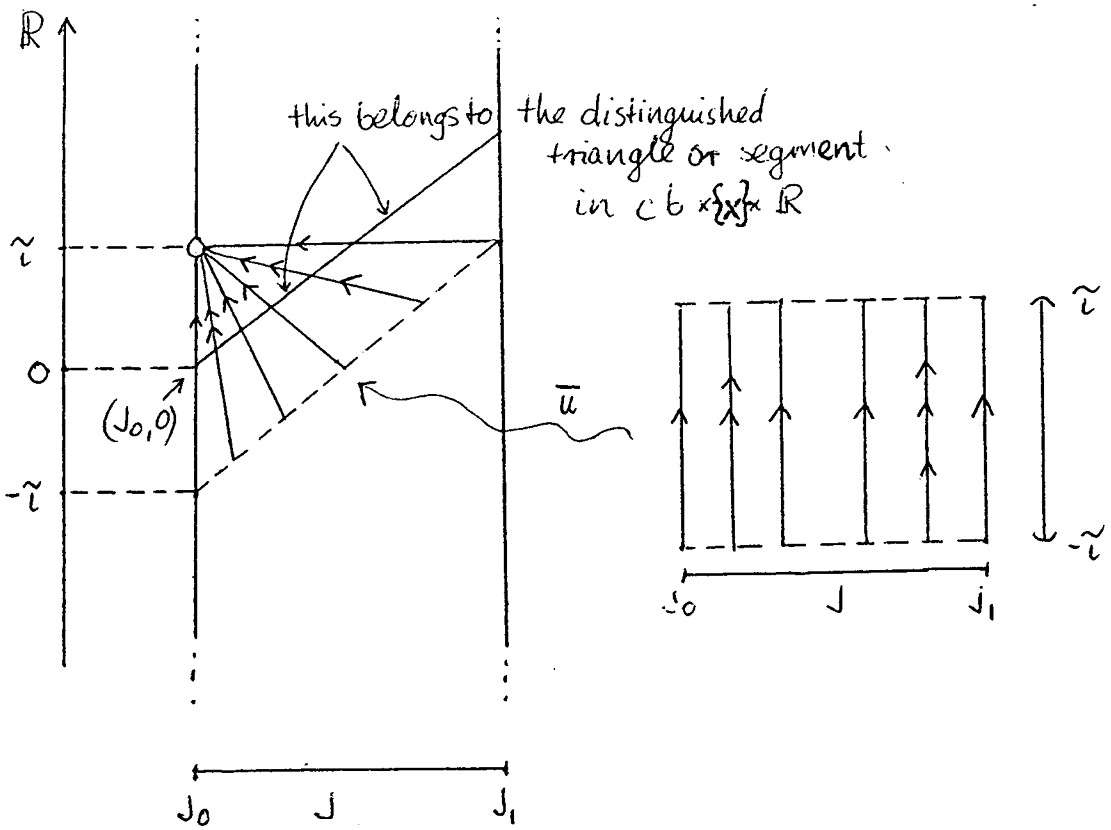


Figure 23. One way $\bar{u}:J\times(-\tilde{\imath},\tilde{\imath})\to\gamma^{-1}(-\tilde{\imath},\tilde{\imath})\cap(J\times\mathbb{R})\subseteq\gamma^{-1}(-\tilde{\imath},\tilde{\imath})$ can behave

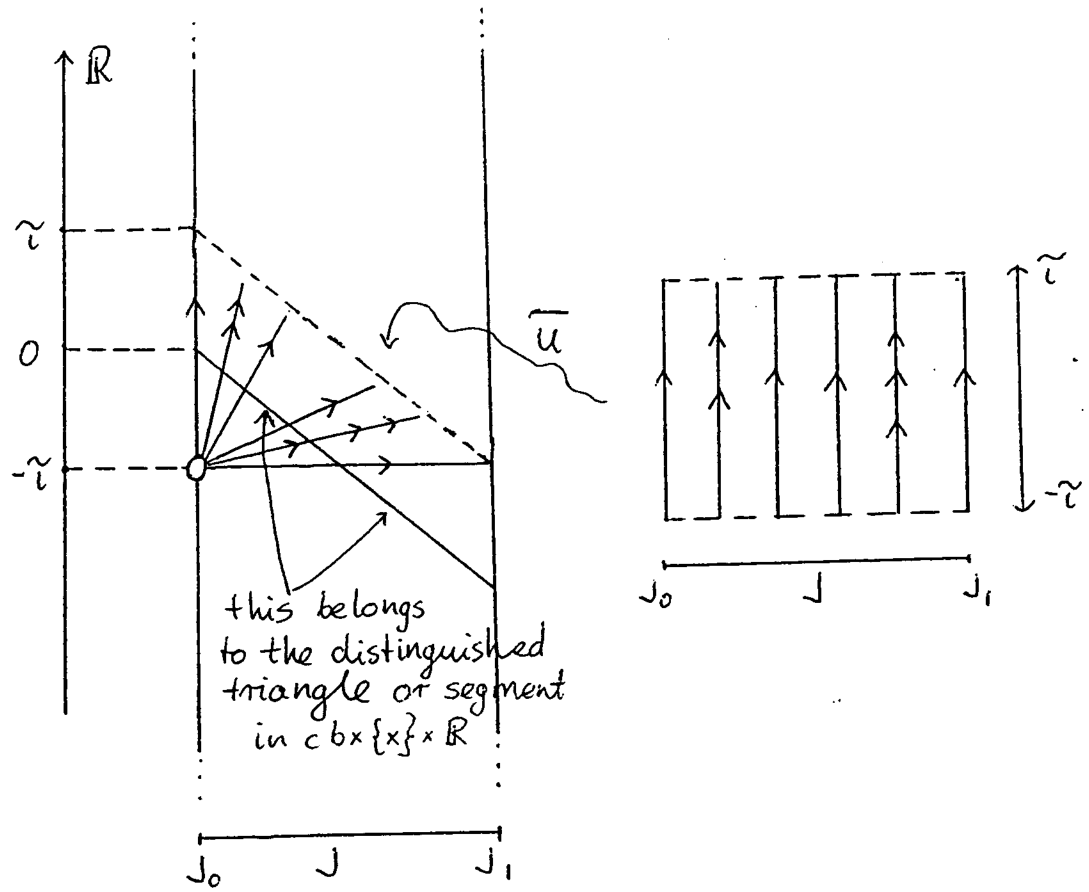


Figure 24. **Another way** $\overline{u}:J\times(-\tilde{\imath},\tilde{\imath})\to\gamma^{-1}(-\tilde{\imath},\tilde{\imath})\cap(J\times\mathbb{R})\subseteq\gamma^{-1}(-\tilde{\imath},\tilde{\imath})$ **can behave**

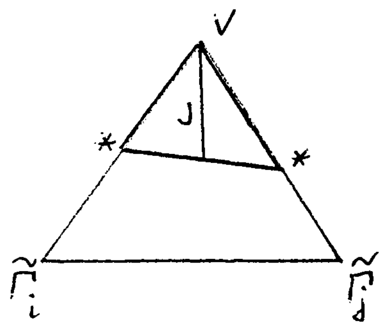
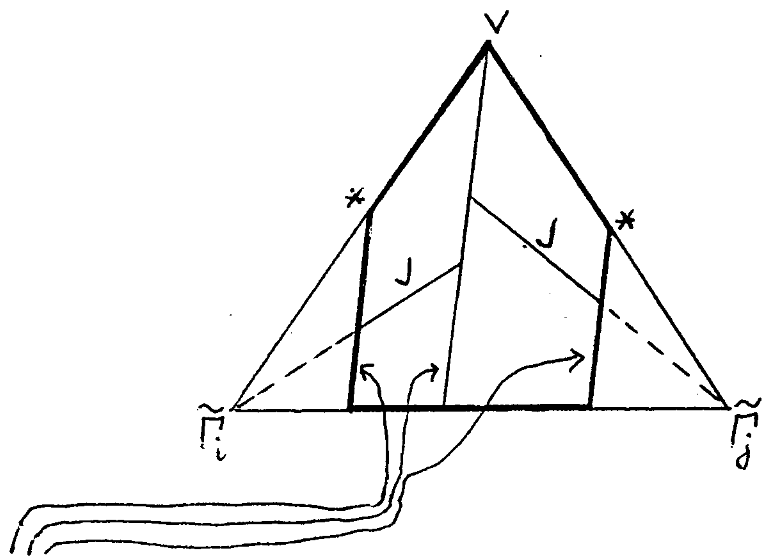


Figure 25. The typical interval J in case $b=\tilde{\Gamma}_{ij}$, $\tilde{s}_i>0$, $\tilde{s}_j>0$



these parallel lines are the projections of level lines of the distinguished triangle.

Figure 26. The typical interval J in case $b=\tilde{\Gamma}_{ij}$, $\tilde{s}_i>0$, $\tilde{s}_j<0$

2-dimensional polygon whose boundary is drawn bald.

Case 3. $b=\tilde{\Gamma}_{ij}$, $\tilde{s}_i>0$, $\tilde{s}_j>0$.

The J's start at v and end at the opposite edge. Use Figure 25. Note that where the vertices denoted by $*$ are is already determined.

Case 4. $b=\tilde{\Gamma}_{ij}$, $\tilde{s}_i>0$, $\tilde{s}_j<0$.

There are two kinds of J's here: the left ones are subintervals of intervals that start at $\tilde{\Gamma}_i$ and end at the segment that starts at v and is parallel to the projection of a level line of the distinguished triangle. The right ones are analogous. Use Figure 23 for the left J's, and Figure 24 for the right ones.

Case 5. $b=\tilde{\Gamma}_{ij}$, $\tilde{s}_i<0$, $\tilde{s}_j<0$.

Proceed as in Case 3 but use Figure 24. ∎ $(\overline{u})$

To see that (4) commutes, observe that γ can be thought of as the composition of: projection to $(c\tilde{\Gamma}_i \cup c\tilde{\Gamma}_j)\times\mathbb{R}$ parallel to the level lines of the distinguished triangle, and $\gamma:(c\tilde{\Gamma}_i \cup c\tilde{\Gamma}_j)\times\mathbb{R}\to v\times\mathbb{R}$.

∎ (Claim)

By Claim and (1) the degree requirements are satisfied. We change h_U as in the Claim. We now have a complete understanding of the two n-cochains C_+, C_- in $cG\times\Delta^{n-1}$, that come from the $\tilde{\imath}$-, $-\tilde{\imath}$-levels, respectively.

We are now prepared to start the transformation of $C_{t_e-\imath}$ to $C_{t_e+\imath}$ in K. The process can be divided into two major stages, which we first outline.

Stage 1.

Note that $C_{t_e+\iota}=C_+\cup\overline{C_{t_e+\iota}\setminus C_+}$ (under the obvious meaning of "closure" and sum). We perform Operations on C_+ keeping $(\overline{C_{t_e+\iota}\setminus C_+})\cap C_+$ fixed to get $\bar{C}_-$. $\bar{C}_-$ is really C_-, except that $\bar{C}_-=C_+$ on $G\times\Delta^{n-1}$. We extend the process to the outside of $P(cG\times\Delta^{n-1}\times\{\tilde{\iota}\})$ by not changing anything.

Stage 2.

We first concentrate on the outside of $V=P(cG\times\Delta^{n-1}\times[-1,1])$. Between $t_e+\iota$-level and t_e-ι-level we see an isotopy of $\overline{C_{t_e+\iota}\setminus C_+}$ and we move this to $\overline{C_{t_e-\iota}\setminus C_-}$ by a combination isotopy/homotopy move. We keep track of what happens on the "border" $P(G\times\Delta^{n-1}\times[-1,1])$. We then move to the inside and perform an isotopy move from $\bar{C}_-$ to C_- that reflects what was happening on the border (so that the outside and inside processes can be glued).

Stage 1.

The change of C_+ to $\bar{C}_-$ is done in $cG\times\Delta^{n-1}$ relative "the base" $G\times\Delta^{n-1}$. First, we sketch the main ideas.

We first use Operation 4 to add a link cochain around the simplex $v\times\Delta^{n-1}$ of cG; the map of that cochain has degree opposite to that of C_+. Figures 27, 28, 29, 30, and 31 show the changes in the level $\tilde{\imath}$ parts of Figures 17, 18, 19, 20, and 21, respectively. The folding neighborhood in which the link cochain is constructed is chosen sufficiently small so that (after the addition) we have a non-singular cochain. We now use saddle moves in each of the cases represented by the Figures 29 and 30. The balls in which the saddle moves are performed are drawn in Figures 32 and 33, respectively, using dashed lines, and the results of the saddle moves are shown. Next, we cancel the torus which is shown in the middle part of Figure 32 using Operation 6. We then cancel the contents of Figure 34 using Operation 5. What is left is $\overline{C}_-$. $\overline{C}_-$ consists of what is pictured in Figures 28, 31, and the right part of Figure 33.

Throughout the process of changing C_+ to $\overline{C}_-$ we freely use homotopy and isotopy moves (fixed on $G\times\Delta^{n-1}$) to adjust and prepare the objects for the next step.

In order to achieve precision, we reduce the picture to a 2-dimensional one multiplied trivially, in a sense, by Δ^{n-1}.

Consider an isotopy $H_t:U_0\times I\to U_0$ that keeps p fixed and moves U_0 to $H_1(U_0)=l\times\alpha\subseteq\mathbb{R}\times\Delta^{n-1}=v\times\mathbb{R}\times\Delta^{n-1}$, where l is a short closed interval in $\mathbb{R}$ around 0 (l will be drawn with the orientation it inherits from $\mathbb{R}$), and α is a small n-1-simplex in Δ^{n-1} around o.

We will change $\gamma^{-1}(U_0)$ on $\tilde{\imath}$-level so as to reflect the isotopy.

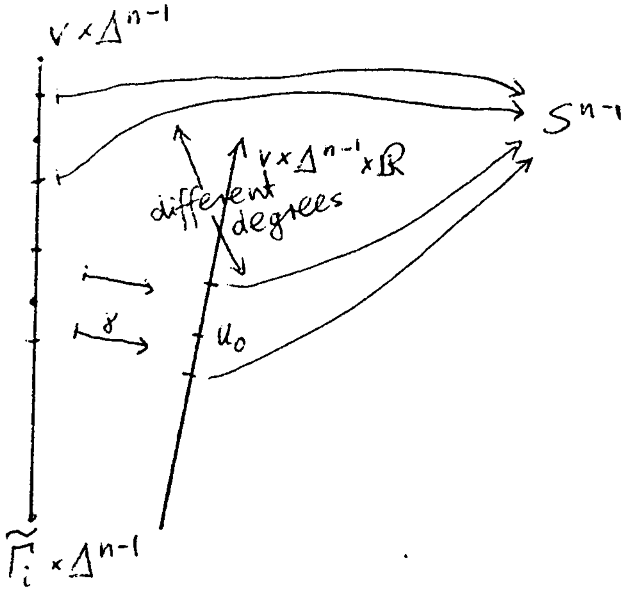


Figure 27. The situation in $c\tilde{\Gamma}_i \times \Delta^{n-1}$, in case $\tilde{s}_i > 0$, after adding the link cochain

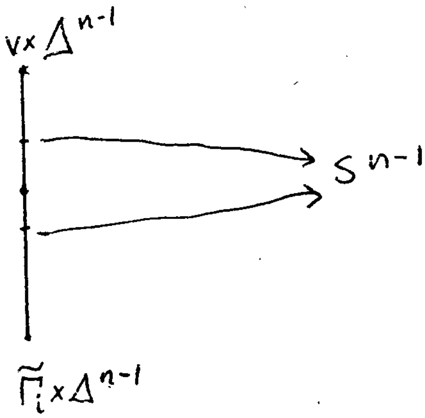


Figure 28. The situation in $c\tilde{\Gamma}_i \times \Delta^{n-1}$, in case $\tilde{s}_i < 0$, after adding the link cochain

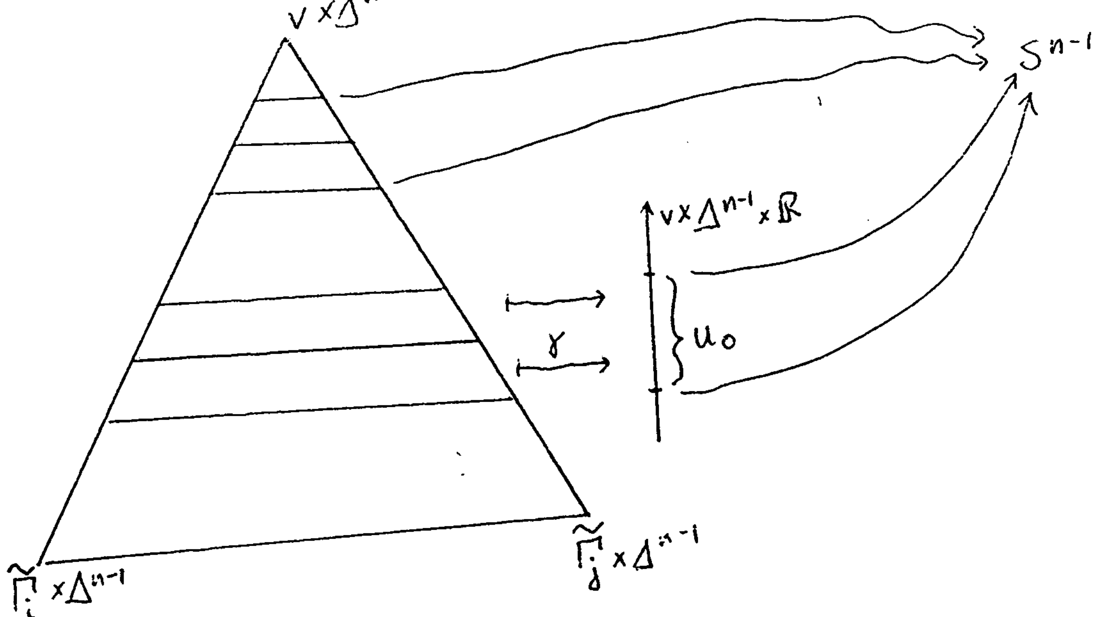


Figure 29. The situation in $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}$, in case $\tilde{s}_i>0$, $\tilde{s}_j>0$, after adding the link cochain

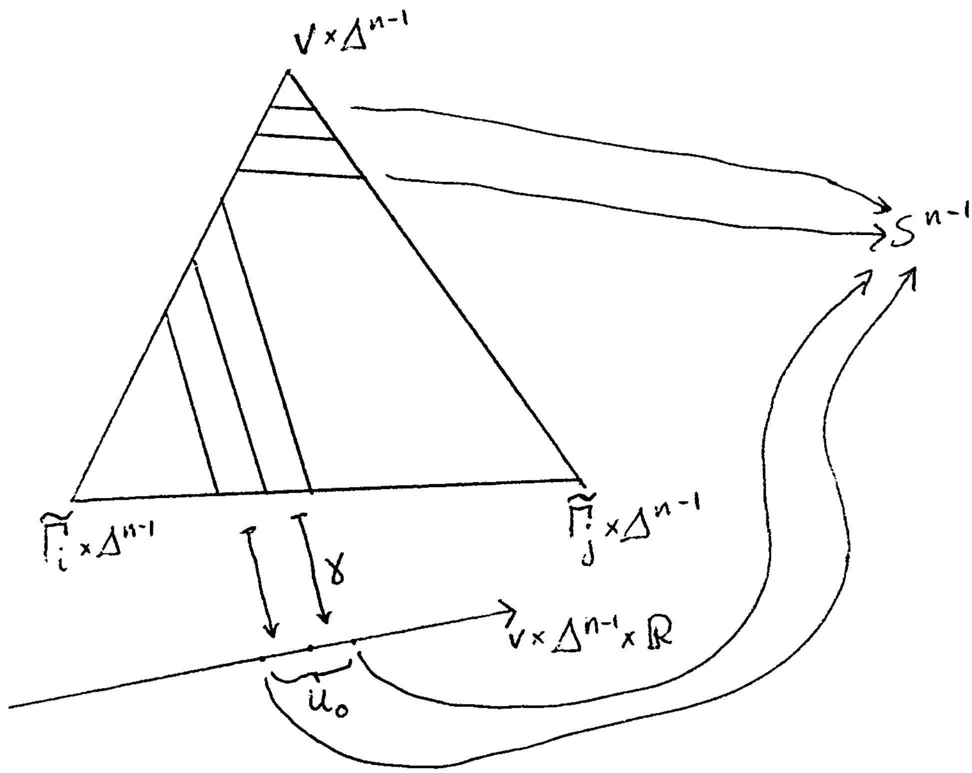


Figure 30. The situation in $c\tilde{\Gamma}_{ij} \times \Delta^{n-1}$, in case $\tilde{s}_i > 0$, $\tilde{s}_j < 0$, after adding the link cochain

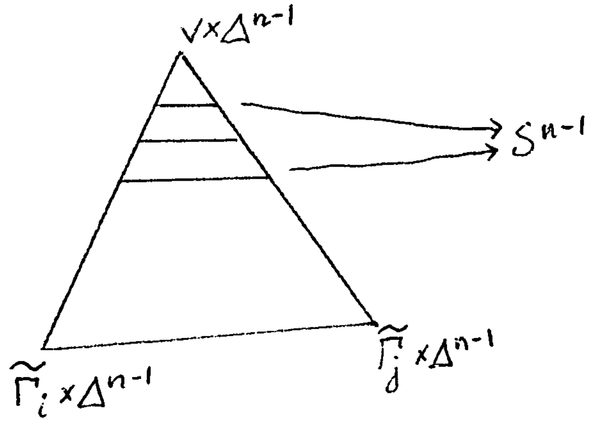


Figure 31. The situation in $c\tilde{\Gamma}_{ij} \times \Delta^{n-1}$, in case $\tilde{s}_i < 0$, $\tilde{s}_j < 0$, after adding the link cochain

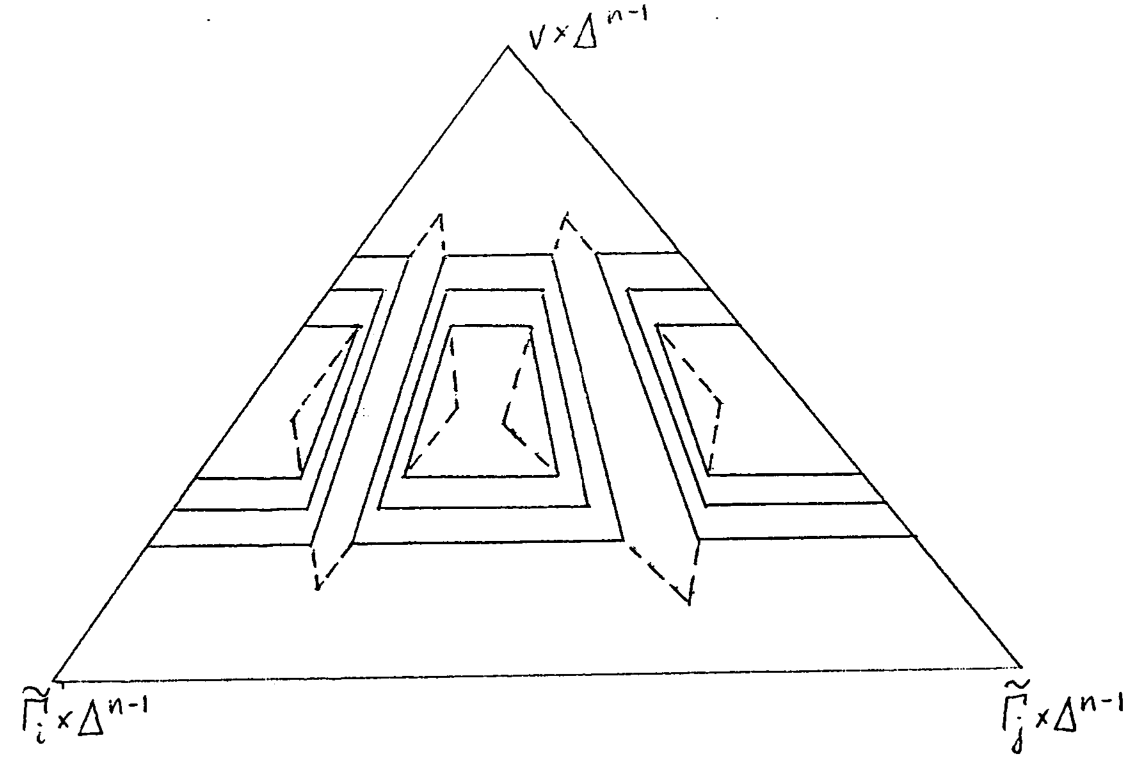


Figure 32. After the saddle moves in $c\tilde{\Gamma}_{ij} \times \Delta^{n-1}$, $\tilde{s}_i > 0$, $\tilde{s}_j > 0$

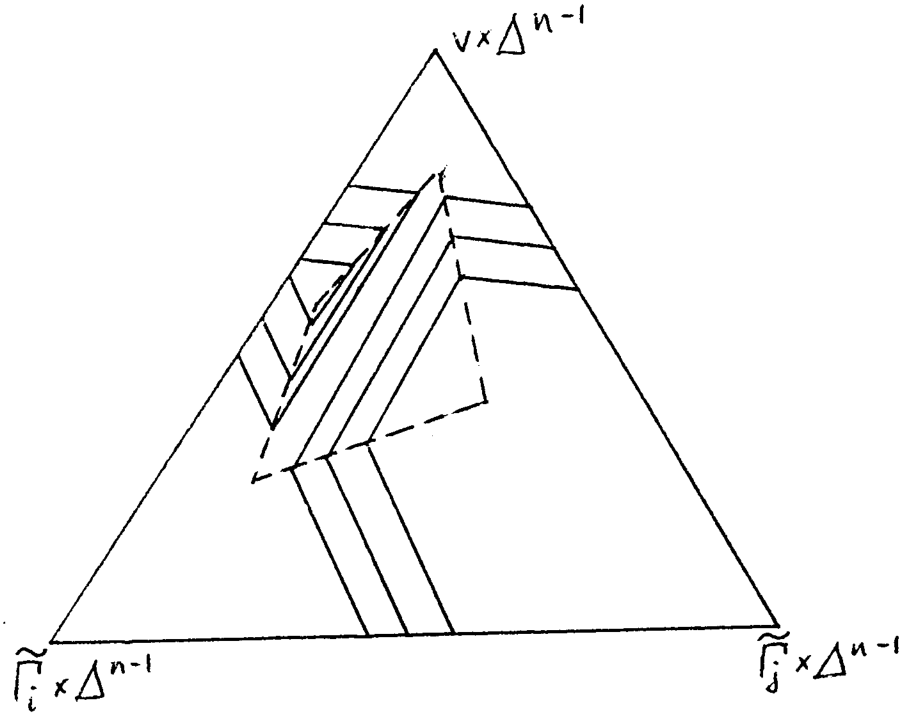


Figure 33. After the saddle moves in $c\widetilde{\Gamma}_{ij} \times \Delta^{n-1}$, $\tilde{s}_i > 0$, $\tilde{s}_j < 0$

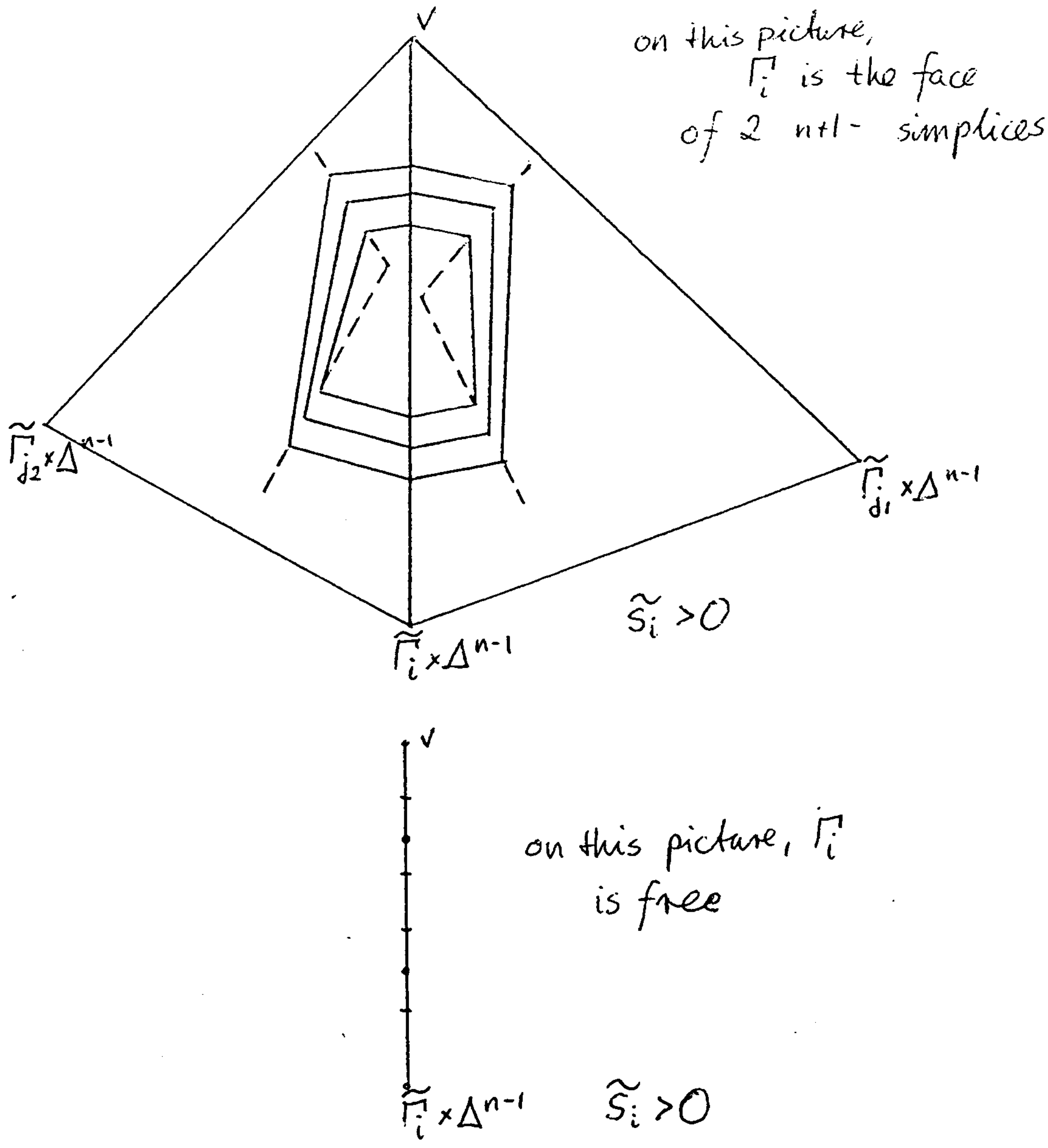


Figure 34. Link cancellations after the saddle moves

The work inside $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}$ for $\tilde{s}_i>0$, $\tilde{s}_j>0$ is easier. We want to PL-identify U on $\tilde{\gamma}$-level inside $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}$ with $U_0\times I$, so that the projection $\Pi:U_0\times I\to U_0$ corresponds to $\gamma:U\cap(c\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times\{\tilde{\gamma}\})\to U_0$. This is easy to do, only the standard mistake [R-S]Ch1 has to be avoided. Now the isotopy H_t transports to the $\tilde{\gamma}$-level as $H_t\times\mathrm{Id}:U_0\times I\to U_0\times I$ and gives an isotopy move for the object of C_t that lies in $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}$.

The work inside $c\tilde{\Gamma}_{ij}\times\Delta^{n-1}$ for $\tilde{s}_i>0$, $\tilde{s}_j<0$ is a bit more difficult, for we have to keep the picture on $G\times\Delta^{n-1}$ constant, and a "buffer" is needed. The buffer will be $Y_{ij}\times\Delta^{n-1}\subseteq c\tilde{\Gamma}_{ij}\times\Delta^{n-1}$, where Y_{ij} is a trapezoid as pictured in Figure 35. Y_{ij} is so thin that, for any $x\in\Delta^{n-1}$, $\gamma^{-1}(U_0)$ is split in two parts as shown in Figure 36 (or $\gamma^{-1}(U_0)\cap(c\tilde{\Gamma}_{ij}\times\{x\}\times\mathbb{R}=\emptyset)$. The upper part is dealt with similarly as above for the $\tilde{s}_i>0$, $\tilde{s}_j>0$ case. The lower part can be naturally PL-identified with $Y\times I$, where $Y=Y\times 0=\gamma^{-1}(U_0)\cap(\tilde{\Gamma}_{ij}\times\mathbb{R})$. This enables one to taper the isotopy off to Id on $Y=Y\times 0$ using H_t in the usual way, as shown in Figure 37. What is more, this Figure and $Y\times I$ are related by a PL-isotopy (of $Y\times I$ in itself) in the standard way: any level $t_0\in[0,1]$ stays in itself, between times 0 and t_0 following H_t, then not moving at all.

The work inside $c\tilde{\Gamma}_i\times\Delta^{n-1}$, for a free Γ_i is done by restricting the work in any of the two cases above.

We have changed C_+, by isotopy moves, to another non-singular

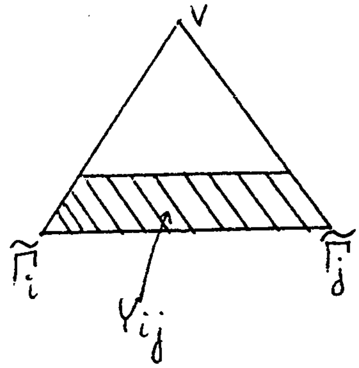


Figure 35. The trapezoid Y_{ij}

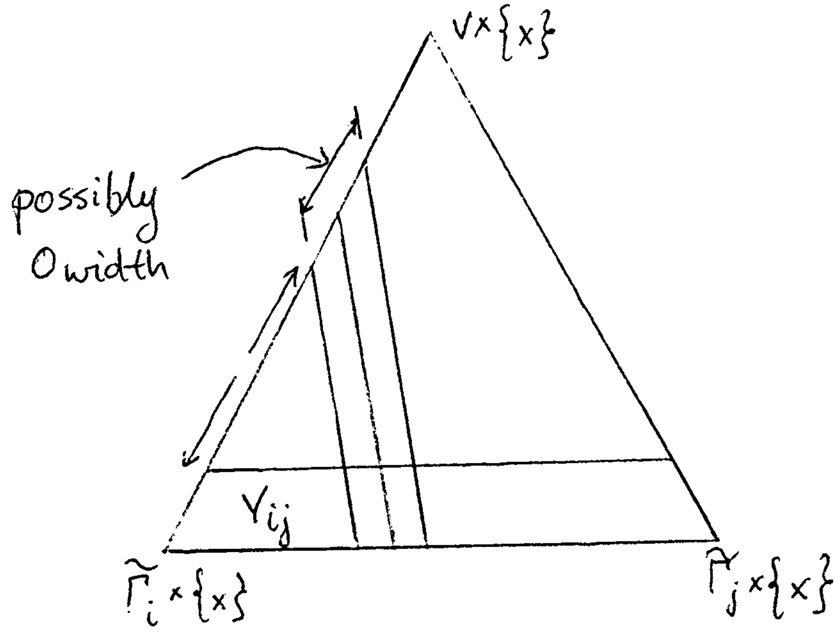


Figure 36. How $\gamma^{-1}(U_0)$ is split

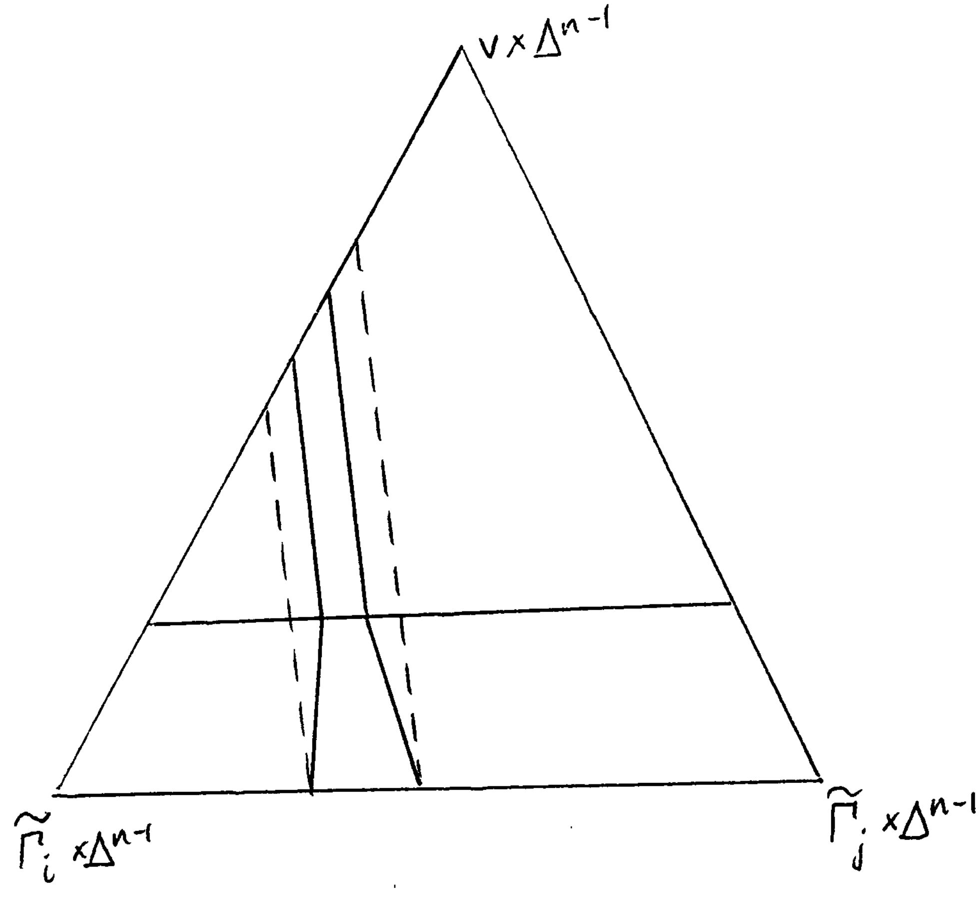


Figure 37. How the isotopy $H_t \times \mathrm{Id}$ is tapered off

n-cochain C'_+ in $cG\times\Delta^{n-1}$. The isotopy moves did not change anything on $G\times\Delta^{n-1}$. C'_+ has, outside the buffer zones, particularly nice induced neighborhood N'. N' is as pictured in Figures 38 and 39, multiplied by $\alpha\subseteq\Delta^{n-1}$. Every piece of the neighborhood maps by projection γ onto $l\times\alpha$ as pictured. Then the composition

$$\mathrm{fr}N' \xrightarrow{\gamma} \mathrm{bd}(l\times\alpha) \xrightarrow{h_U H_1^{-1}} S^{n-1}$$

is the map given by C'_+ (restricted to $\mathrm{fr}N'$). $h_U H_1^{-1}$ will be denoted by h_0.

We treat C_- in the analogous fashion to produce C'_-. We make sure that essentially the same isotopy H_t and essentially the same trapezoid Y_{ij} (for each $\tilde{\Gamma}_{ij}$ with $\tilde{s}_i>0$, $\tilde{s}_j<0$) are used (this might require shrinking the above trapezoid). To understand the phrase "essentially the same isotopy", we note that Y on $-\tilde{\imath}$-level is a translate of the Y on $\tilde{\imath}$-level within $\tilde{\Gamma}_{ij}\times\Delta^{n-1}$. This follows from the definition of γ: the lines along which γ projects from $\tilde{\imath}$- and $-\tilde{\imath}$-levels are parallel. Similarly, we speak of both lower parts of $\gamma^{-1}(U_0)$ being essentially the same. Now "essentially the same" means "same modulo the translation". C'_- will be used later.

To precisely describe how to get from C'_+ to $\overline{C}_-$ we go back to the Figures 27-33 and treat them as 2-dimensional: change the factor Δ^{n-1} to o in all of them. They now represent Operations on 1-cochains and the program outlined in them acquires precision (we do not want to touch anything inside the buffer zones; this will be

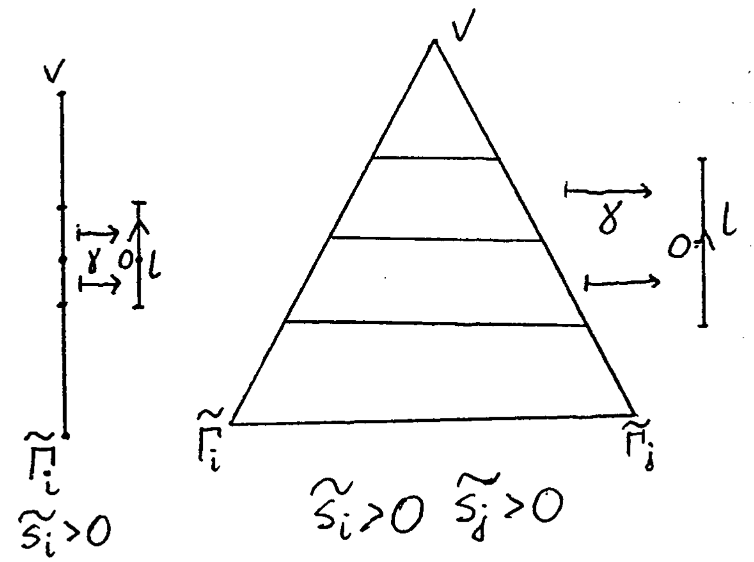


Figure 38. How N' looks in $\tilde{\Gamma}_i \times \Delta^{n-1}$, in case $\tilde{s}_i > 0$, and in $\tilde{\Gamma}_{ij} \times \Delta^{n-1}$, in case $\tilde{s}_i > 0$, $\tilde{s}_j > 0$

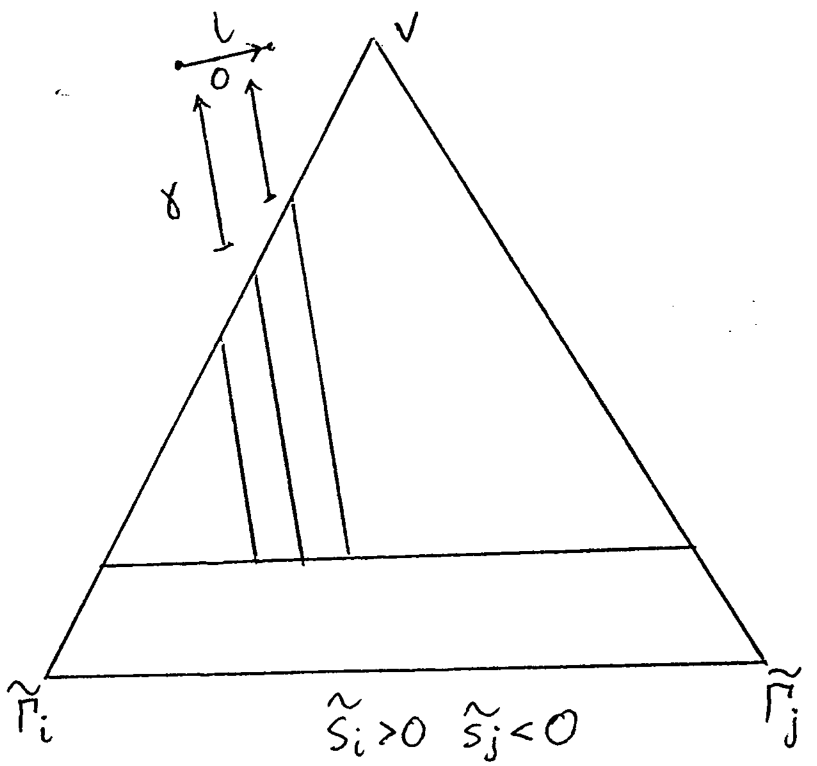


Figure 39. How N' looks in $\tilde{\Gamma}_{ij} \times \Delta^{n-1}$, in case $\tilde{s}_i > 0$, $\tilde{s}_j < 0$

further clarified). We have a passage between certain 1-dimensional cochains. C'_+ and $\overline{C}_-$ are obtained from them by "multiplication by Δ^{n-1}". We need to transform the Operations between 1-cochains to certain Operations between suitable n-cochains.

We discuss this "stability" issue in more detail.

In one step, we have two 1-cochains D_0, D_1; D_1 is obtained from D_0 by Operation r. Let N_i be the neighborhood induced by D_i. D_i gives the map g_i:bd$N_i \to$bd$I$, if we agree that bd$I = S_0$. But, in fact, we have extensions $g_i:N_i \to I$. Suppose we multiply all the cells in sight by Δ^{n-1}, the neighborhoods N_i by a small n-1-simplex $\alpha \subseteq \text{int}\Delta^{n-1}$, and the cores by a point $p \in \text{int}\alpha$. We also consider the compositions

$$\text{bd}(N_i \times \alpha) \xrightarrow{\text{bd}(g \times \text{Id})} \text{bd}(I \times \alpha) \xrightarrow{h_0} S^{n-1}.$$

It is easy to see that the process yields two n-cochains, denoted by $D_i \times \alpha$, and we wish to see that they are related by a suitable Operation(s). This is an exercise in elementary PL-techniques and we leave it to the reader. If the two 1-cochains are related by an Operation r, $r \geq 4$, then the two n-cochains are related by an Operation r and a homotopy move. In all other cases, no additional Operation is needed.

We now state what we expect to get through the process.

We have added buffer zones and using Operations 1-6, we change C'_+ to a non-singular n-cochain $\overline{C}_-$ outside of these buffer zones. $\overline{C}_-$ is obtained from the following 2-dimensional pictures in Figure 40 by trivial multiplications by $\alpha \subseteq \Delta^{n-1}$, in exactly the same way that C'_+

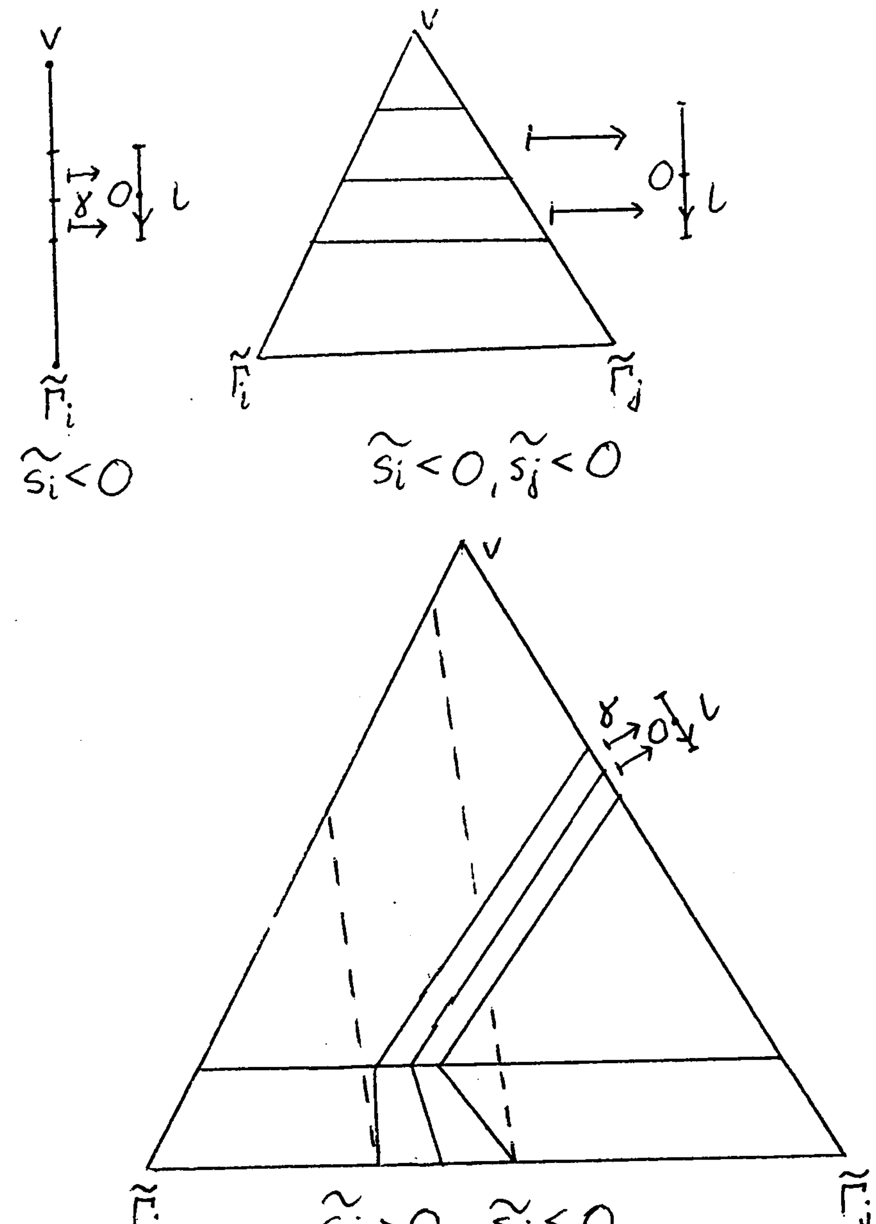


Figure 40. How $\overline{C}_-$ looks

was obtained. The intersections with the buffer zones are exactly as for C'_+.

It is easy to see that C_+ glues with the rest of the $t_e+\iota$-level of X to produce the cochain $C_{t_e+\iota}$ in K, via πT. This might require adjusting ε further. Moreover, it should be obvious that we can, instead of C_+, glue $\overline{C_-}$ to the rest of $C_{t_e+\iota}$ and obtain a non-singular n-cochain in K, which we shall denote by C'. Then, when we recall the Operations used to get from C_+ to $\overline{C_-}$, we see that we transform $C_{t_e+\iota}$ to C' using the Operations. Only few comments are needed here.

First, all the Operations we will do will be "supported" by $j(cG)$; that is, in particular, the balls and neighborhoods for Operations 3-6 are in there.

Second, recall that we first added a "link cochain"; Operation 4 was used. This now translates to Operation $3+r$, where r is the number such that $\pi TP(v\times\Delta^{n-1}\times\tilde{\iota})\subseteq \mathrm{int}\alpha^{n-2+r}$, α^{n-2+r} - a cell in K. This is rather straightforward; the only non-trivial ingredient is Lemma 6.7. In particular, observe that if Γ_i, for some $i\leq w$, is free, then $j(c\tilde{\Gamma}_i\setminus\{\tilde{\Gamma}_i\})$ can not lie in the interior of an $n+1$-cell; what is more, it has to lie in the interior of a free n-cell.

■ (work on the level $t_e+\iota$ and stage 1)

Stage 2.

Recall the outline of Stage 2. We want to use a combination isotopy/homotopy move to make the outside move. To prescribe the isotopy, we need a suitable isotopy $F:(\overline{U_{t_e+\iota}\setminus V})\times I\to\overline{U\setminus V}$ such that

$$F((U_{t_e+\iota}\setminus V)\times t)=U_{t_e+\iota-2t\iota}\setminus V.$$

Our isotopy move will then follow F_t.

In order to construct F, we need to analyze c and U outside V between $t_e+\iota$ and t_e-ι-level. c on $t_e+\iota$-level contains points in the interiors of some $n+1$-simplices of X, and points inside "back faces" of V: $P(G\times\Delta^{n-1}\times[-1,1])$ (one point in each $P(\tilde{\Gamma}_{ij}\times\Delta^{n-1}\times[-1,1])$ with $\tilde{s}_i>0$, $\tilde{s}_j<0$). These points are joined by straight segments inside $n+2$-simplices of X whenever two such points lie in such a simplex. While moving down to t_e-ι-level we see that this picture is essentially isotoped, or multiplied by I; the points and segments mentioned above stay within their simplices or back faces. In each of these simplices and on each level from $t_e+\iota$ to t_e-ι, U is a nice regular neighborhood of c in that simplex, possibly with the exclusion of V whenever a back face of V is involved. In this way the whole picture of U between $t_e+\iota$ and t_e-ι-levels outside V is the picture of U on $t_e+\iota$-level PL-isotoped downwards. Moreover, this isotopy F can be arranged so as to preserve the inclusions of parts of its domain in any $n+1$-simplex of X, any $n+2$-simplex of X, or any back face of V.

Now for the "inside" part. We are going to produce a certain homotopy/isotopy move from $\bar{C}_-$ to C'_-; we then go to C_- by the move already described (rel. G). We start at time $t=0$, end at $t=1$, and at any time $t\in I$ we will have a certain enhanced cochain $\bar{C}_t$ in $cG\times\Delta^{n-1}$. We want to glue that cochain, by $P|_{cG\times\Delta^{n-1}\times s}$, for a suitable $s\in[-\tilde{\imath},\tilde{\imath}]$, to $U_{t_e+l-2tl}\setminus V$ to obtain a cochain in K (via T). In this way the inside and outside processes will be glued and we see that the cochain at the end of Stage 1 ($\bar{C}_-$ glued to outside of $P(cG\times\Delta^{n-1}\times\tilde{\imath})$) is cohomologous to C_{t_e-l}. This proves $C_{t_e-l}\sim C_{t_e+l}$.

The outside move within $P(G\times\Delta^{n-1}\times[-1,1])$ can be, via P^{-1}, translated to $G\times\Delta^{n-1}$ and studied there. We will produce a move from $\bar{C}_-$ to C'_- in $cG\times\Delta^{n-1}$, respecting the outside effect on $G\times\Delta^{n-1}$, and then use P as an isotopy (from $P|_{cG\times\Delta^{n-1}\times\tilde{\imath}}$ to $P|_{cG\times\Delta^{n-1}\times(-\tilde{\imath})}$) to put everything back in X, and, via T^{-1}, in K.

We now analyze the outside move within $G\times\Delta^{n-1}$. We first need a close look at the proof of the Claim before Stage 1. The original conclusion of this Claim was used before; we now need that the diagram

$$\begin{tikzcd}
\mathrm{fr}\,U_t\cap G \arrow[rr, "h_U"] \arrow[dr, "\gamma"'] & & S^{n-1} \\
& \mathrm{fr}\,U_0 \arrow[ur, "h_U"'] &
\end{tikzcd}$$

for any t between $\tilde{\imath}$ and $-\tilde{\imath}$, commutes. This actually follows from the construction (in the proof of the Claim); restrict your attention to a point in Δ^{n-1} and follow the pictures.

Recall the construction of C'_- and the observation we made there that $\mathrm{fr}U_{-\tilde{l}} \cap G$ is a translate of $\mathrm{fr}U_{\tilde{l}} \cap G$. The same look at γ tells us that actually any $\mathrm{fr}U_t \cap G$ is obtained by translation. The above diagram certifies that any two of

$$h_U\big|_{\mathrm{fr}U_t \cap G} : \mathrm{fr}U_t \cap G \to S^{n-1}$$

are the same modulo the translation. The outside move, as happening on $G \times \Delta^{n-1}$, is really a simple translation.

We are home now. Above any trapezoid (in any $\tilde{\Gamma}_{ij} \times \Delta^{n-1}$ with $\tilde{s}_i > 0$, $\tilde{s}_j < 0$), and in any $\tilde{\Gamma}_{ij}$ without a trapezoid, we move $\bar{C}_-$ to C'_- using the 2-dimensional picture and multiplying by $\alpha \subseteq \Delta^{n-1}$. We make sure that we do the translation on the top of each trapezoid. Now all that is left is to do is a translation within each trapezoid.

■ (Stage 2)

■ $(C_{t_e - l} \sim C_{t_e + l})$

By our experience in the outside part of Stage 2 above, that $C_t \sim C_{t'}$ if $[t,t']$ is away from any t_e, 0, and 1, is trivial. Use a combination homotopy/isotopy move.

We only need to see that $C_1 \sim C_{1-\xi}$ for very small $\xi > 0$. Essentially, they are close, that is, the cores are close, no new objects appear, and the neighborhoods are "similar"; we then do a short isotopy move, and the homotopy move is automatic, as the two maps are close.

The biggest problem happens around an n-simplex Δ of X_1 that contains a point of c. One needs to analyze the neighborhood of that simplex and how c and U behave there. If Δ is from the interior of an $n+1$-simplex of K, we have a simple "link" of Δ in X, as shown in Figure 41.

If Δ is from the interior of an n-simplex Γ of K, then first observe that there must be an $n+1$-simplex $\bar{\Delta}$ going down inside $\Gamma \times I$, having Δ as a face. This is illustrated in Figure 42. Then below each $n+1$-simplex Δ' on 1-level of X, whose face is Γ, we see the link as in Figure 43.

This analysis of links yields the following information about the relationship of c_1 with $c_{1-\xi}$. We view $c_1 \subseteq K$ as built out of segments (intersections with $n+1$-simplices of X_1) and "special" points (intersections with n-simplices of X_1). Similarly, $c_{1-\xi}$ is built of segments (intersections with $n+2$-simplices of X on 1-ξ-level) and points (intersections with $n+1$-simplices).

Each segment of c_1 gives rise to a segment of $c_{1-\xi}$ by taking the $n+1$-simplex in X_1 and looking beneath to the $n+2$-simplex of X glued to it, as the dashed lines show. Every special point from c_1 that lands in an n-cell of K gives rise to a corresponding point in $c_{1-\xi}$ in a similar way (dotted arrow). However, a special point of c_1 that lands in an $n+1$-cell of K gives rise to a chain of segments and special points in $C_{1-\xi}$ nearby, as labeled by "addition".

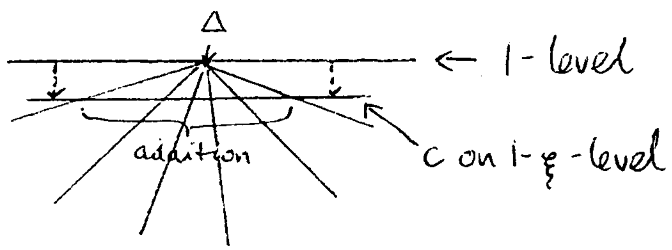


Figure 41. The link of a simplex Δ from the interior of an $n+1$-simplex of K, in X

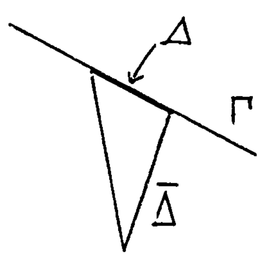


Figure 42. How a simplex Δ from the interior of an n-simplex of K extends downwards

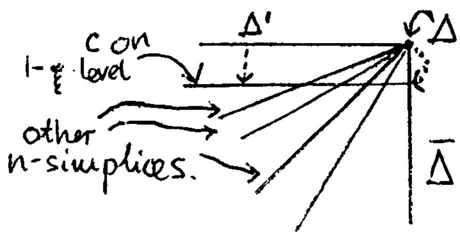


Figure 43. The link of a simplex Δ from the interior of an n-simplex of K, in X

Let us try to illustrate how c_1 and $c_{1-\xi}$ might look in K. This is Figure 44.

We also observe that $\mathrm{fr}U_1$ and $\mathrm{fr}U_{1-\xi}$ are Hausdorff close. This is because a part of $\mathrm{fr}U_{1-\xi}$ that lies in some $n+2$-simplex ($n+1$-simplex) of X is close to a part of $\mathrm{fr}U_1$ that lies in the 1-level $n+1$-face (n-face) of that simplex; these parts are then close in K because T is uniformly continuous. Similarly we find a point from $\mathrm{fr}U_{1-\xi}$ close to any point of $\mathrm{fr}U_1$; any simplex from X_1 "extends" downwards. We now see that the cores and neighborhoods of C_1 and $C_{1-\xi}$ satisfy the assumptions of the observation below (C_1 is C and $C_{1-\xi}$ is C'). Because $\mathrm{fr}U$ exists between $\mathrm{fr}U_1$ and $\mathrm{fr}U_{1-\xi}$ and h is uniformly continuous, the maps given by C_1 and $C_{1-\xi}$ also satisfy the assumptions.

"Close" cochains are related by homotopy and isotopy moves:

Observation.

Let $n \geq 3$. Let K be a regular finite CW-complex and let C be an enhanced n-cochain in K, with core c, neighborhood N, and map $h:\mathrm{fr}N \to S^{n-1}$. Let $\varepsilon > 0$ and let C' be another enhanced n-cochain in K, with core c', neighborhood N', and map $h':\mathrm{fr}N' \to S^{n-1}$ such that there is a PL-homeomorphism $f:c \to c'$ that moves points less than ε, $\mathrm{fr}N$ and $\mathrm{fr}N'$ are ε-close using Hausdorff distance, and

(*) $$\forall_{x' \in \mathrm{fr}N'} \exists_{x \in \mathrm{fr}N}\ \rho(x,x') < \varepsilon,\ \rho(h(x),h'(x')) < \varepsilon.$$

Then, provided ε is sufficiently small, C and C' are cohomologous.

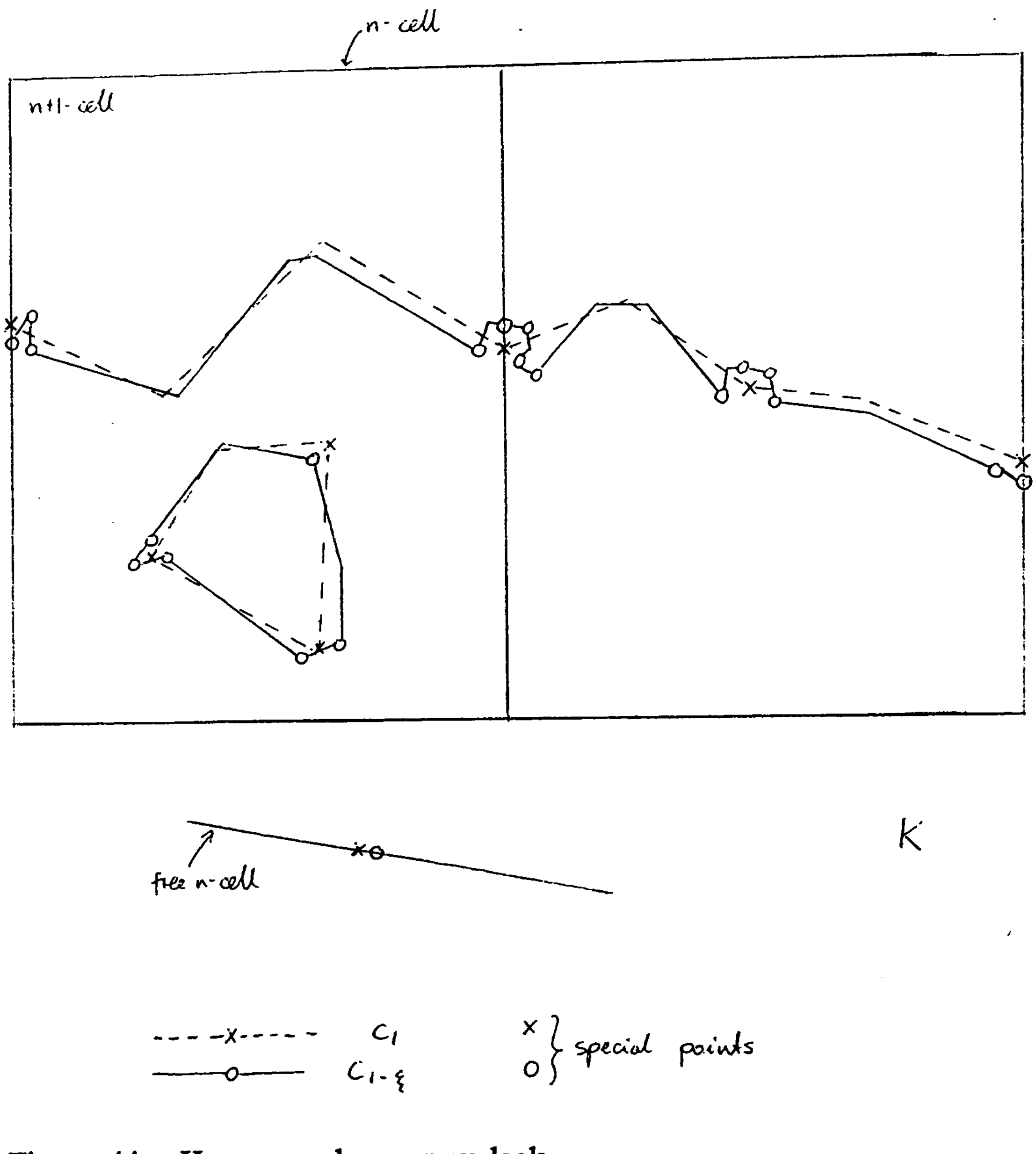


Figure 44. How c_1 and $c_{1-\xi}$ may look

Proof.

We isotope c' onto c keeping the intersection with the interior of any cell in that interior. We cover the isotopy by an ambient isotopy of K that moves only points that are very close to c and moves each cell in itself. Codimension 3 is used here, and we first build the covering isotopy on $K^{[n]}$.

We now can assume $c=c'$, and nothing else has changed. Recall the "coordinate system" in N that is given by 7.1. By radial projection r, each point of $N \setminus c$ can be mapped to a point on fr$N$. By isotoping N' inside itself (using 7.1) we can assume that $N' \subseteq N$ (but fr$N$ and frN' are still close).

We now homotope h' so that the diagram

$$\begin{tikzcd} \mathrm{fr}N' \arrow[rr, "r"'] \arrow[dr, "h'"'] & & \mathrm{fr}N \arrow[dl, "h"] \\ & S^{n-1} & \end{tikzcd}$$

commutes. This can be done because the radial projection on frN' moves points very little and because of (*).

We enlarge N to $2N$ as in 10.1, by using collars. r extends to the radial projection $r: 2N \setminus c \to \mathrm{fr}N$. We move N' inside $2N$ onto N by an isotopy H_t keeping $c'=c$ fixed. The intersections with any cell are moved within that cell. C and C' now have the same neighborhood and we define a homotopy move

$$h_t(x) = rH_t(H_1^{-1}(x)) \text{ for } x \in \mathrm{fr}N$$

from C' to C.

CHAPTER 12

THE FUNCTORIALITY

The following consequence of 10.7 and 11.1 is an intermediate step to prove that the pull-back depends only on homotopy class of the map and cohomology class of the cocycle.

Proposition 12.1.

Let K, L be finite, regular CW-complexes, $n \geq 4$, and S_0, S_1 - non-singular enhanced n-cocycles in L such that S_1 is obtained from S_0 by an isotopy move. Let $f_0 \simeq f_1 : K \to L$ and suppose that S_i pulls back by f_i to form C_i. Then $C_0 \sim C_1$.

Proof.

The special case when $S_0 = S_1$ follows from Propositions 10.7 and 11.1.

To prove the general version, let N_i be the neighborhood induced by S_i and let $h_i : \mathrm{fr} N_i \to S^{n-1}$ be the map given by S_i. Use Propositions 7.4 and 7.5 to obtain an isotopy H_t of L such that $H_1(\mathrm{fr} N_0) = \mathrm{fr} N_1$ and $h_1 = h_0 H_1^{-1} \big|_{\mathrm{bd} N_1}$. Let $f_1' = H_1 \circ f_1$ and observe that S_1 pulls back to C_0 by f_1'. But $f_1 \simeq f_1' : K \to L$, and the special case implies $C_0 \sim C_1$.

Remark.

As we know, there may be more than 1 pull-back of a cochain by a map. We now see that they are all cohomologous.

Let K, L be regular, finite CW-complexes, $n \geq 4$, S - an enhanced n-cocycle in L, and $f:K \to L$ - a map. We define $f^*(S) \in E\mathscr{C}^n(K)/_\sim$ as follows. Change S to a non-singular n-cochain S' by an isotopy move. S' is a cocycle by 8.1. Let $f \approx f'$ so that S' can be pulled back by f' (10.6) to C. Then we define $f^*(S) = [C]$. This is well-defined by Proposition 12.1.

Theorem 12.2.

A pull-back of a (non-singular) n-cocycle, $n \geq 4$, is an n-cocycle.

Proof.

Let K, L, n be as above, and let S be a non-singular enhanced n-cocycle in L. Let $f:K \to L$ pull S back to C. Let τ be an $n+2$-cell in K. $C \cap \mathrm{bd}\tau$ is a collection of circles in $\mathrm{bd}\tau$; let $h:\mathrm{fr}N \to S^{n-1}$ be the induced map of the frontier of its regular neighborhood. The n-cochain $C|_{\mathrm{bd}\tau}$ in τ is a pull-back of S by $f|_\tau$. $f|_\tau:\tau \to L$ is homotopic to the constant map $*$ to a point outside the core of S. Of course S pulls back by $*$ to $\emptyset$. Therefore, by Propositions 8.1 and 12.1, $J([h]) = 0$.

10.5 now immediately yields

Proposition 12.3.

$f^*:\mathrm{E}\mathscr{G}^n(L) \to \mathrm{E}H^n(K)$ is a semigroup homomorphism.

■

What is left to define f^* on cohomology level is to see that f^* takes coboundaries to $0 \in EH^n(K)$.

Let K be a simplicial complex, and let Δ be an m-simplex of K. For use in the next Lemma, we introduce the notion of the partial star of Δ in K, denoted $\frac{1}{m+2}\mathrm{St}\Delta$, as follows.

Let the vertices of Δ be $v_0,\ldots,v_m$; let p_v be the barycentric coordinate, with respect to a vertex v of K, of a point $p\in K$. Then

$$\frac{1}{m+2}\mathrm{St}\Delta=\left\{p\in K:\forall_{i=0,1,\ldots,m}\ \sum_{v\notin\Delta^m} p_v\leq p_{v_i}\right\}$$

(where $\sum\emptyset=0$). This is pictured in Figure 45. If K is a simplicial complex in some $\mathbb{R}^k$, then $\frac{1}{m+2}\mathrm{St}\Delta$ is also a simplicial complex in $\mathbb{R}^k$, naturally isomorphic with the star $\mathrm{st}_K\Delta$. The interior of $\frac{1}{m+2}\mathrm{St}\Delta$ in K is

$$\left\{p\in K:\ \forall_{i=0,1,\ldots,m}\ \sum_{v\notin\Delta^m} p_v< p_{v_i}\right\}.$$

The partial stars of different simplices of the same dimension can intersect only at (parts of) their frontiers in K.

Recall the n-cochains L_p that occur in Operations 4-6.

Lemma 12.4.

A pull-back of L_p is cohomologous to 0.

Proof.

Let $f:K\to L$ be a map between finite regular CW-complexes and let L_p be obtained in L. Suppose L_p pulls back by f and we are interested

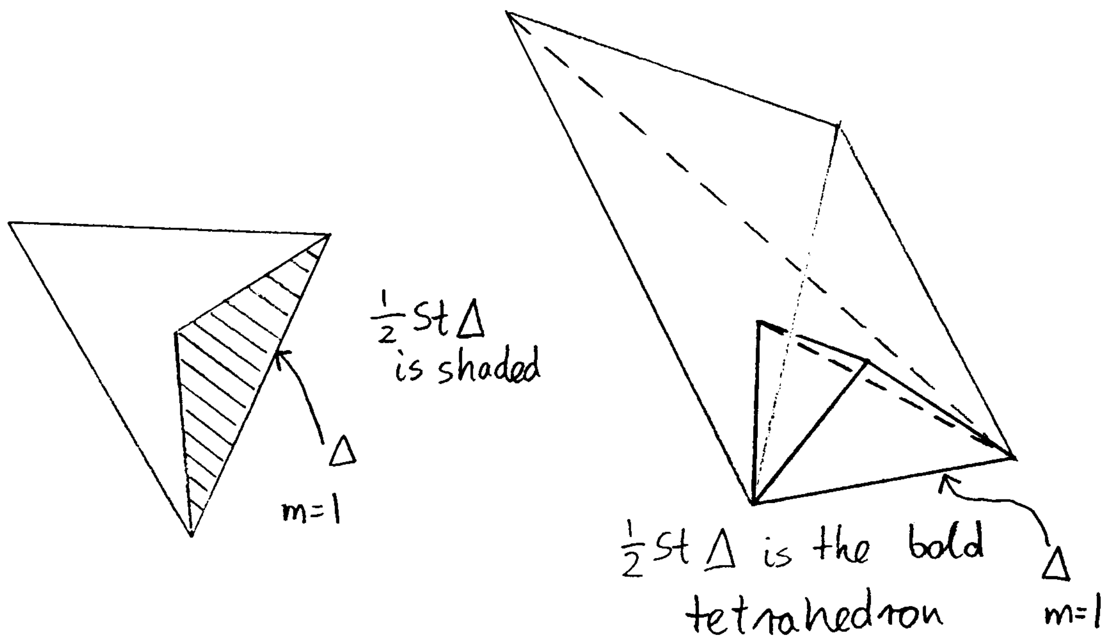


Figure 45. The partial star

in that pull-back. To obtain L_p (in any of the Operations 4-6) we triangulated a PL-neighborhood U of a certain point $p \in L^{[n+1]} \setminus L^{[n-2]}$. We first extend that triangulation to a triangulation T_L of L that subdivides the cell structure. We also triangulate K so as to subdivide the cell structure and so that $K^{[n]}$ is a full subcomplex. 12.1 lets us assume that $f:K \to L$ is simplicial with respect to some subdivision T_K of that triangulation, and T_L. $K^{[n]}$ is still full with respect to T_K. Throughout the rest of the proof, T_K is the only triangulation of K used. Recall the "central" simplex α and the folding map $\lambda: U \to 2\Delta^n$ that produces L_p. We start the sequence of Operations that produces $f^*(L_p)$ from $\oslash$.

Consider the set Z of all n-1-simplices of T_K from $K^{[n+1]}$ such that $f(\Delta)=\alpha$. For each such Δ, we add a cochain, denoted L_Δ, which is contained in the partial star $\frac{1}{n+1}\mathrm{St}\Delta$ of Δ in $K^{[n+1]}$. We first define a "folding" map $\lambda_\Delta: \frac{1}{n+1}\mathrm{St}\Delta \to 2\Delta^n$ that is a simplicial map that equals λf on Δ, and takes all the vertices of $\frac{1}{n+1}\mathrm{St}\Delta$ not in Δ to the "remaining" vertex of $2\Delta^n$ (the one not in $\lambda f(\Delta)$). λ_Δ produces L_Δ exactly like λ produced L_p. By 2.1 and 2.3 we get an enhanced n-cochain, with respect to the original cell structure of K. Note, by Lemma 6.7 and the fullness of $K^{[n]}$, that this is a cochain of the form L_q, for any point $q \in \mathrm{int}\Delta$, thus the addition of L_Δ does not change the enhanced cohomology class of $\oslash$.

Let $\Gamma \in T_K$ from $K^{[n+1]}$ be of dimension $\geq n$, and we want to investigate $\left[\sum_{\Delta \in Z} L_\Delta\right] \cap \Gamma$. In our Figures 46-51, n=2, but the necessary

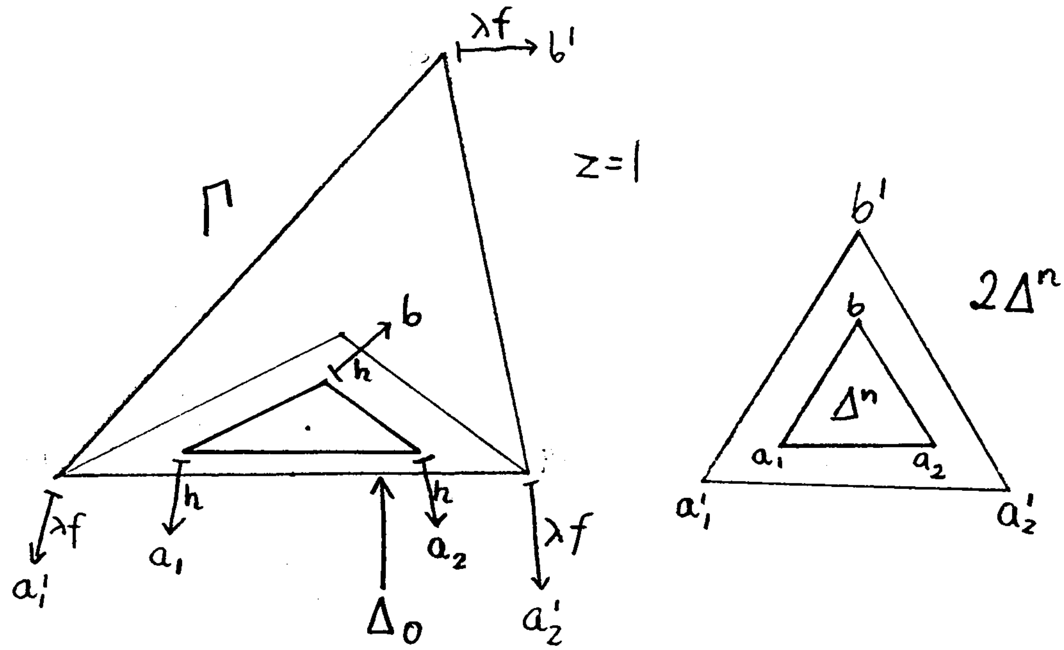


Figure 46. How ΣL_{Δ} looks in Γ in Case 2

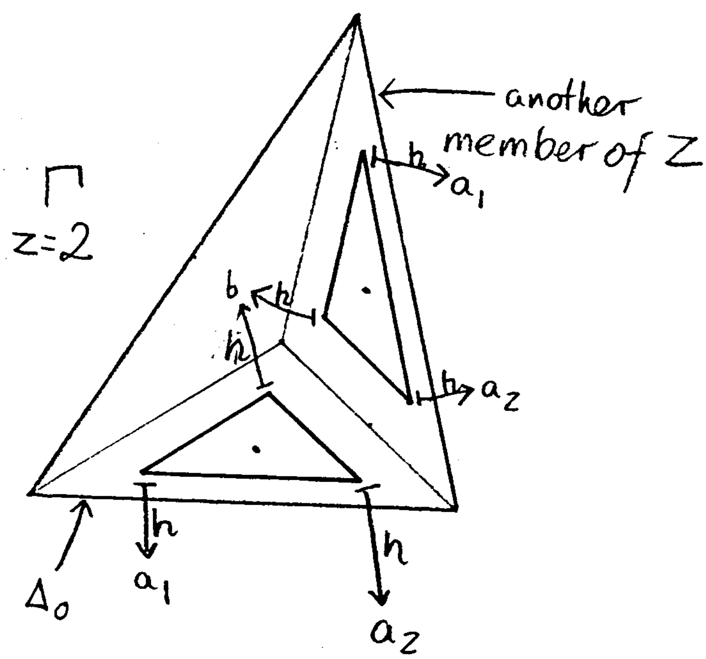


Figure 47. How ΣL_{Δ} looks in Γ in Case 3

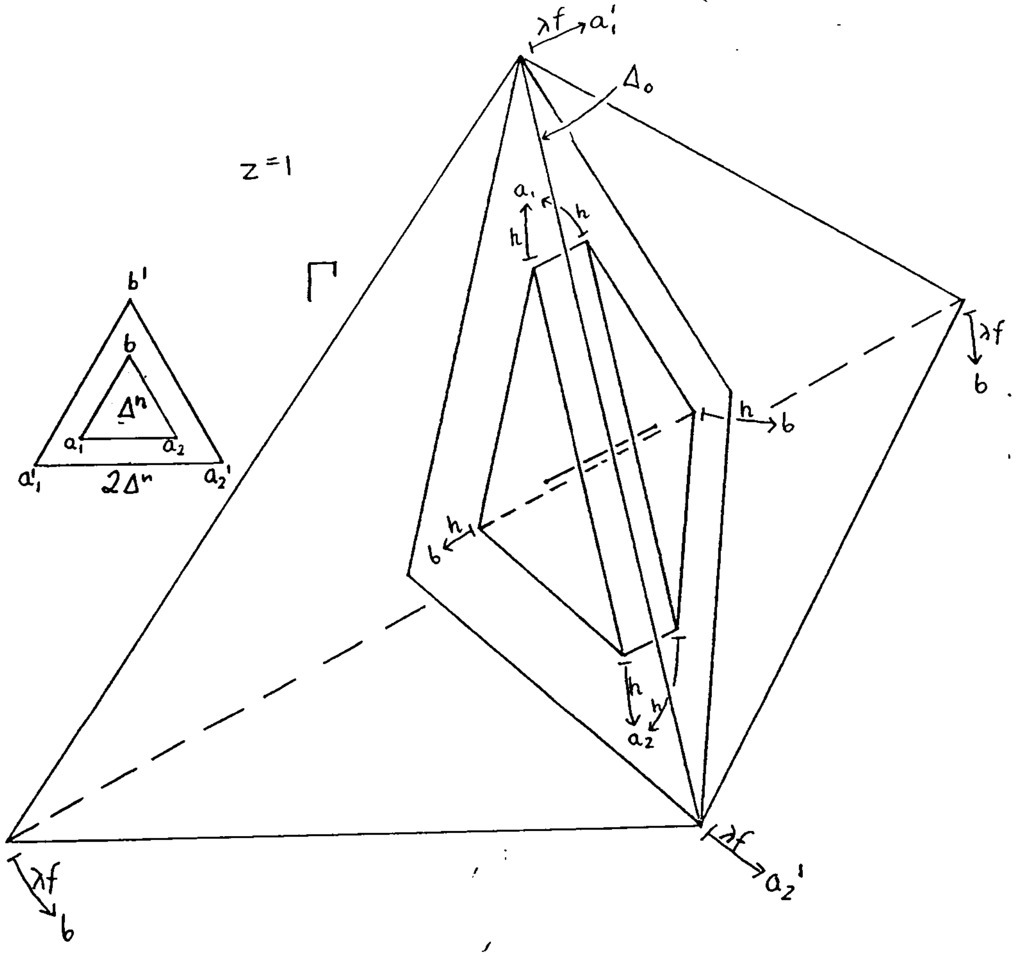


Figure 48. How $\sum L_{\Delta}$ looks in Γ in Case 4

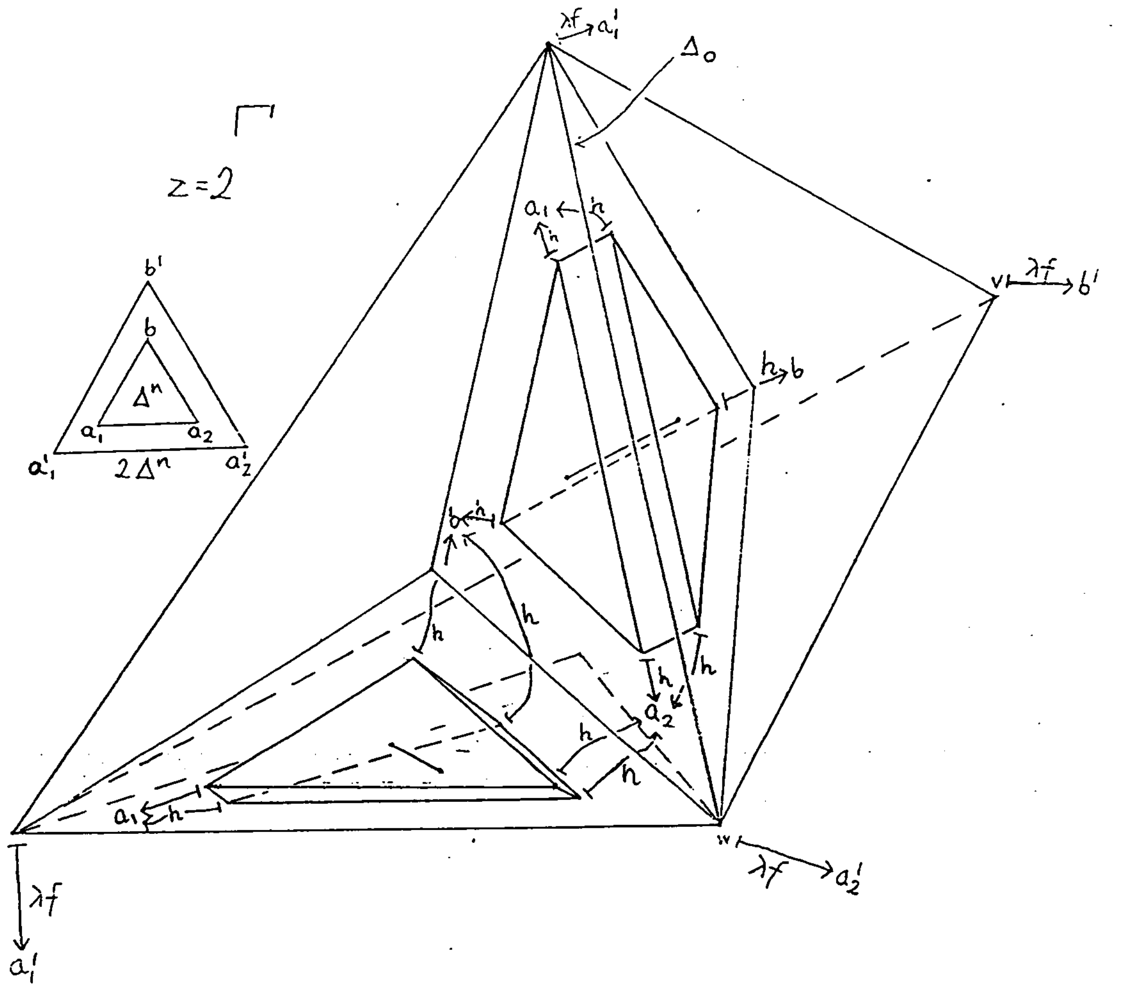


Figure 49. How ΣL_Δ looks in Γ in Case 5

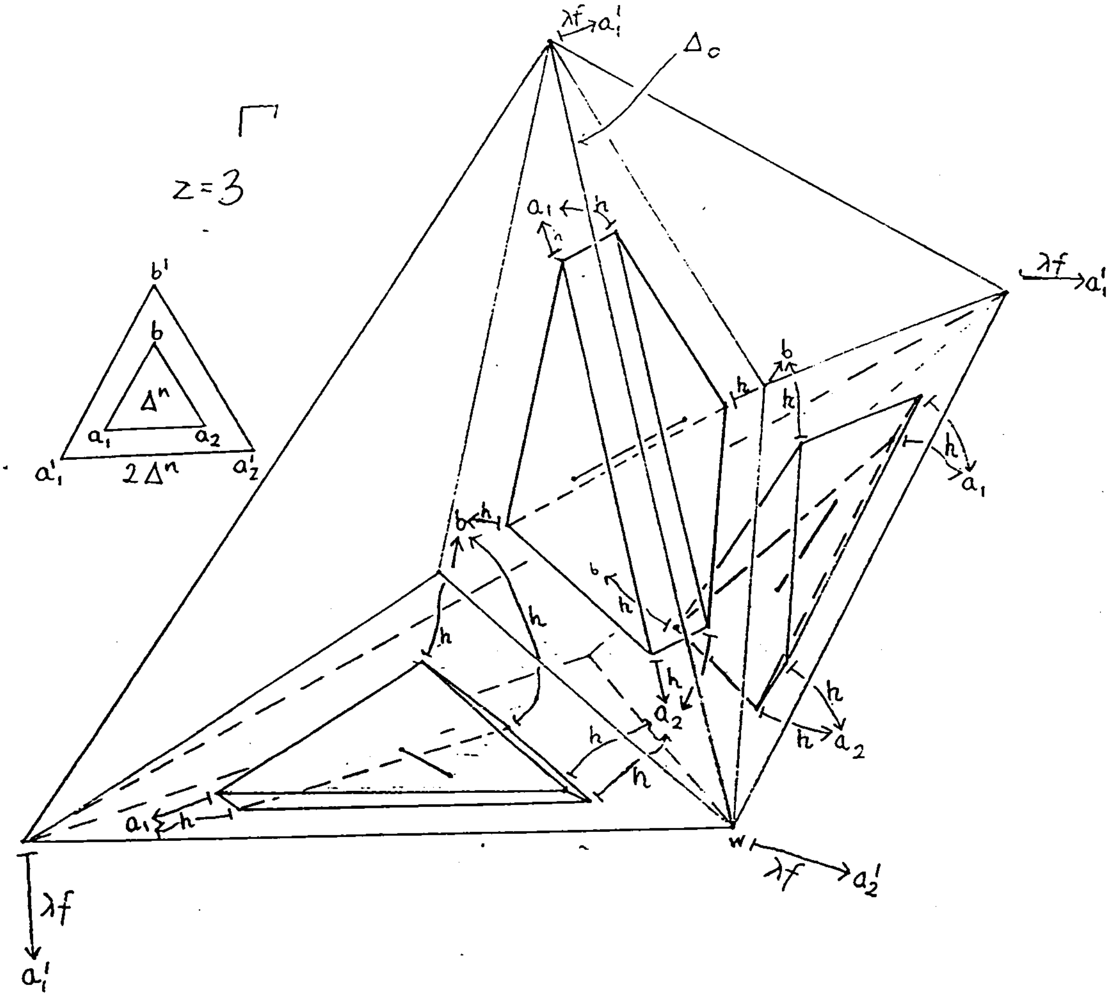


Figure 50. How $\sum L_\Delta$ looks in Γ in Case 6

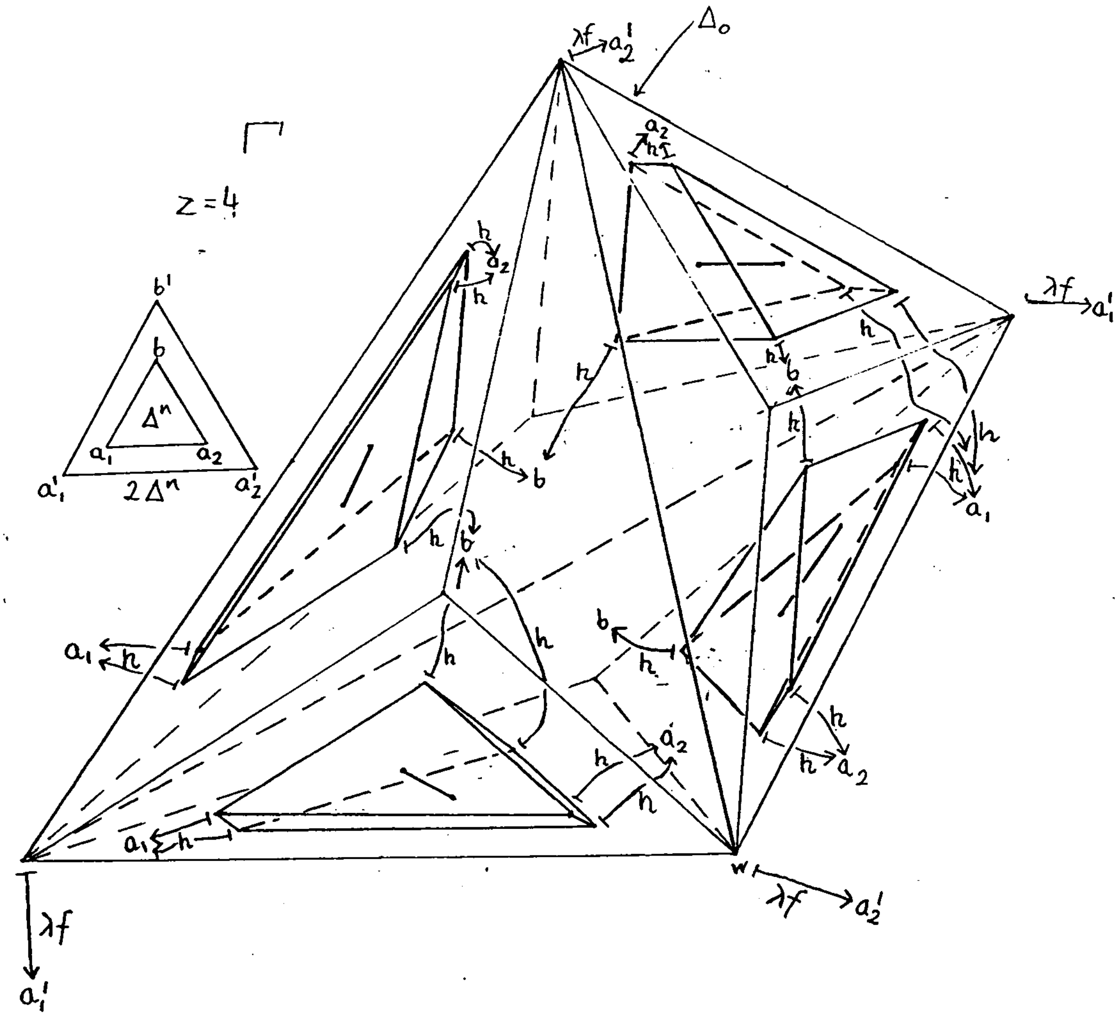


Figure 51. How ΣL_{Δ} looks in Γ in Case 7

adjustments for the general n will be discussed. The frontier of the neighborhood N induced by $\sum L_\Delta$ is shown using bold lines. The core is either some bold points or some bold straight segments. The map h given by $\sum L_\Delta$ is simplicial on the bold triangles. a_i''s are the vertices of $2\Delta^n$ that span $\lambda(\alpha)$; a_i's are the corresponding vertices of Δ^n. Let z be the number of faces of Γ that belong to Z. If $z>0$, we pick a face $\Delta_0 \in Z$ of Γ, and use the adjective "remaining" for the vertices of Γ not in Δ_0. There are seven Cases to consider; we first discuss briefly the geometry of the Cases which are illustrated in the accompanying Figures 46-51.

Before we start, we discuss some basic adjustments for general n. In our Figures 46-51, Δ_0 has two vertices that go by λf to a_1' and a_2'. In general, Δ_0 has n vertices, and they go to $a_1',a_2',\dots,a_n'$ (all the vertices of $2\Delta^n$). As for the map h, any bold segment in fr$N$ whose ends are mapped by h to a_1 and a_2, has to be, for general n, replaced by an n-1-simplex, whose vertices are mapped to $a_1,a_2,\dots,a_n$.

Case 1: $z=0$.

In this case, $\sum L_\Delta$ misses Γ entirely.

Case 2: $\dim\Gamma=n$, Γ is free, $z>0$, and the remaining vertex of Γ is mapped by f outside α.

Case 3: $\dim\Gamma=n$, Γ is free, and the remaining vertex is mapped by f to a vertex of α. z is then 2.

Case 4: $\dim\Gamma=n+1$, $z>0$, and both remaining vertices are mapped by f outside α. Note that the images of these two vertices are the same. Note that z is 1.

$N \cap \Gamma$ is the space between the two bold triangles (which are contained in faces of Γ). The simplicial map h on the boundaries of these triangles is determined by Figure 48. h on the whole fr$N$ is then determined in the obvious manner. This comment will apply to all the remaining cases.

Case 5: $\dim\Gamma = n+1$, $z>0$, of the two remaining vertices only one, call it v, is mapped by f outside α.

Suppose that the other remaining vertex is mapped by λf to a'_1. What in our Figure 49 is a single vertex w, is really n-1-vertices that are mapped by λf to $a'_2, \ldots, a'_n$.

Case 6: $\dim\Gamma = n+1$, $z>0$, the two remaining vertices of Γ are mapped by f to the same vertex of α.

The comment about w (from Case 5) still applies.

Case 7: $\dim\Gamma = n+1$, $z>0$, and the two remaining vertices of Γ are mapped by f to different vertices of α.

What Figure 51 does not show is n-2 vertices added to Δ_0 and mapped by λf to $a'_3, \ldots, a'_n$.

We now consider $f^*(L_p)$ for each of the above cases. The map given by $f^*(L_p)$ is depicted exactly like h was before.

Cases 1, 3, 6, 7.

No $f^*(L_p)$ here since in order for $f^*(L_p)$ to show in Γ, an n-1-face of Γ, but not the whole Γ, has to be mapped onto α. Case 1 fails the first test, Cases 3, 6, 7 pass the first, but fail the second.

Cases 2, 4, 5.

See Figures 52-54.

■ (pictures of $f^*(L_p)$)

We will briefly describe how to use Operations 1-6 to move from ΣL_Δ to $f^*(L_p)$. We first do a finite number of saddle moves in Cases 5, 6, and 7.

In any simplex satisfying Case 5, we consider the "front face" on Figure 49, that is, the n-face spanned by everything but v. Two objects start inside that face and it is easy to see from Figure 49 that the two ends have maps (h) to $S^{n-1}=\mathrm{bd}\Delta^n$ of opposite degrees (under the same orientation for both ends). We can then perform a saddle move in an elongated ball near the front face, as suggested in the following simplified picture in Figure 55, which drops a dimension by suppressing w.

From now until the end of the proof, we will sometimes speak of symmetry in Γ with respect to certain n-planes. What we mean is that the symmetry occurs after Γ is affinely transformed into the standard $n+1$-simplex Δ^{n+1}.

The saddle move should be symmetric to the n-plane spanned by the midpoint between the two vertices of Γ mapped by λf to a_1', and all the other vertices of Γ. The effect on the core is shown in Figures 56 and 57. To show which faces the ends of the arcs belong to, we surround each end with a miniature of "its" face.

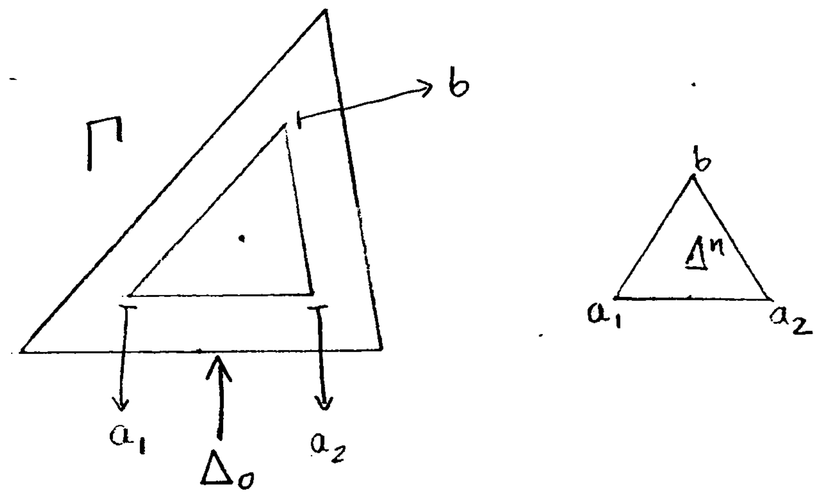


Figure 52. How $f^*(L_p)$ looks in Γ in Case 2

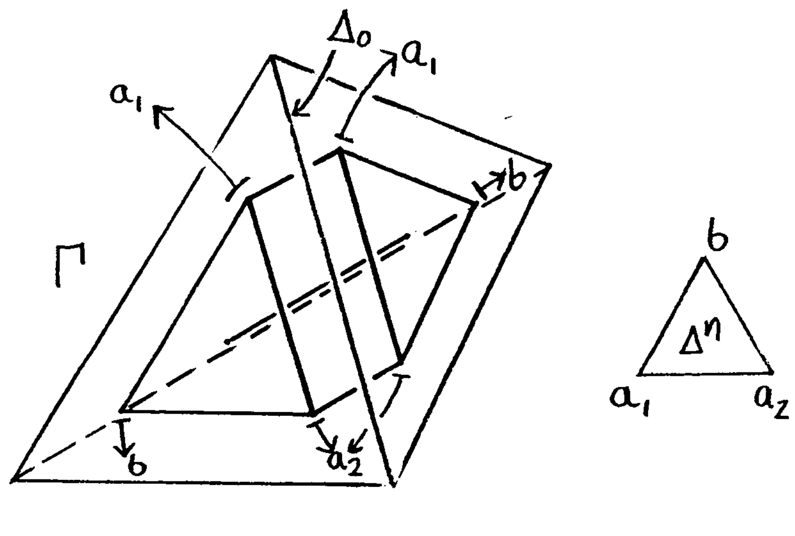


Figure 53. How $f^*(L_p)$ looks in Γ in Case 4

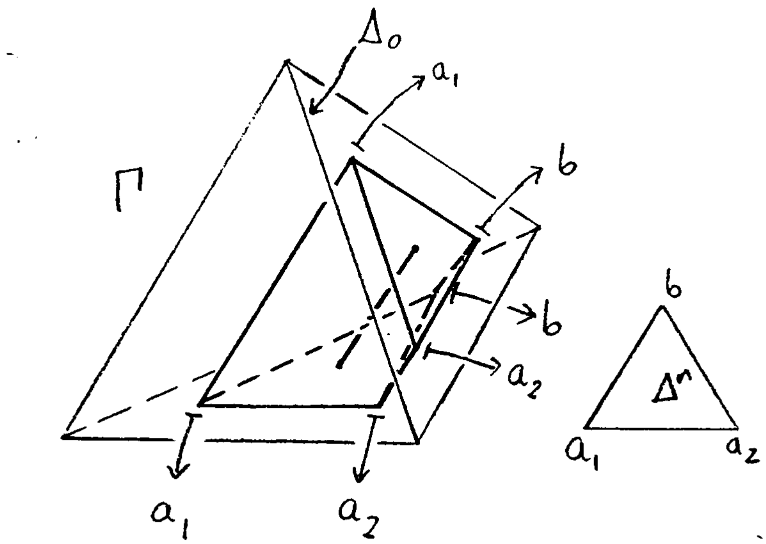


Figure 54. How $f^*(L_p)$ looks in Γ in Case 5

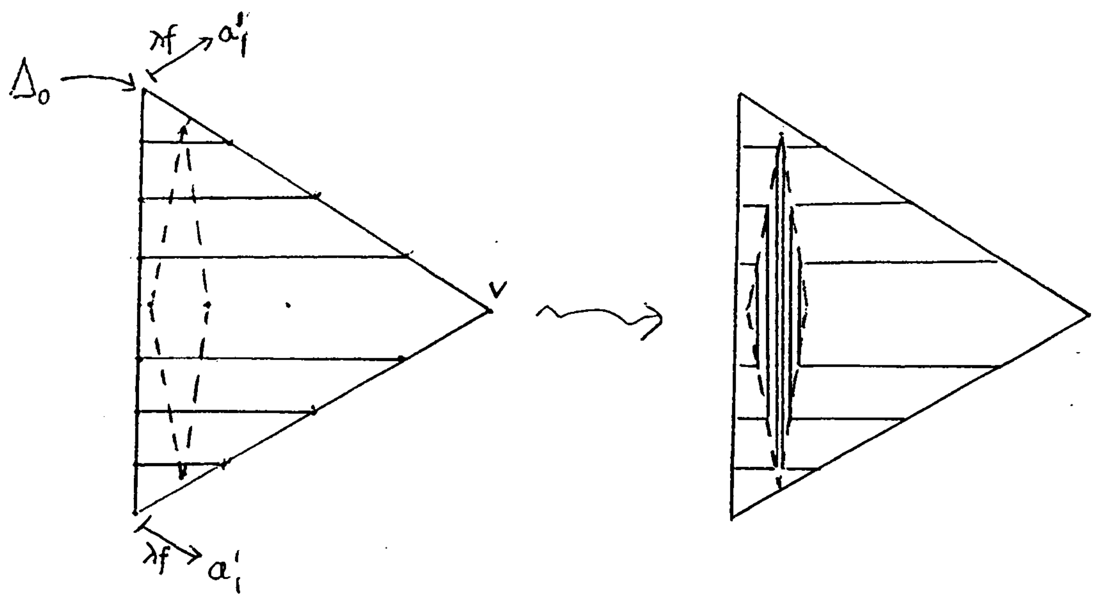


Figure 55. How to perform the saddle move in Case 5

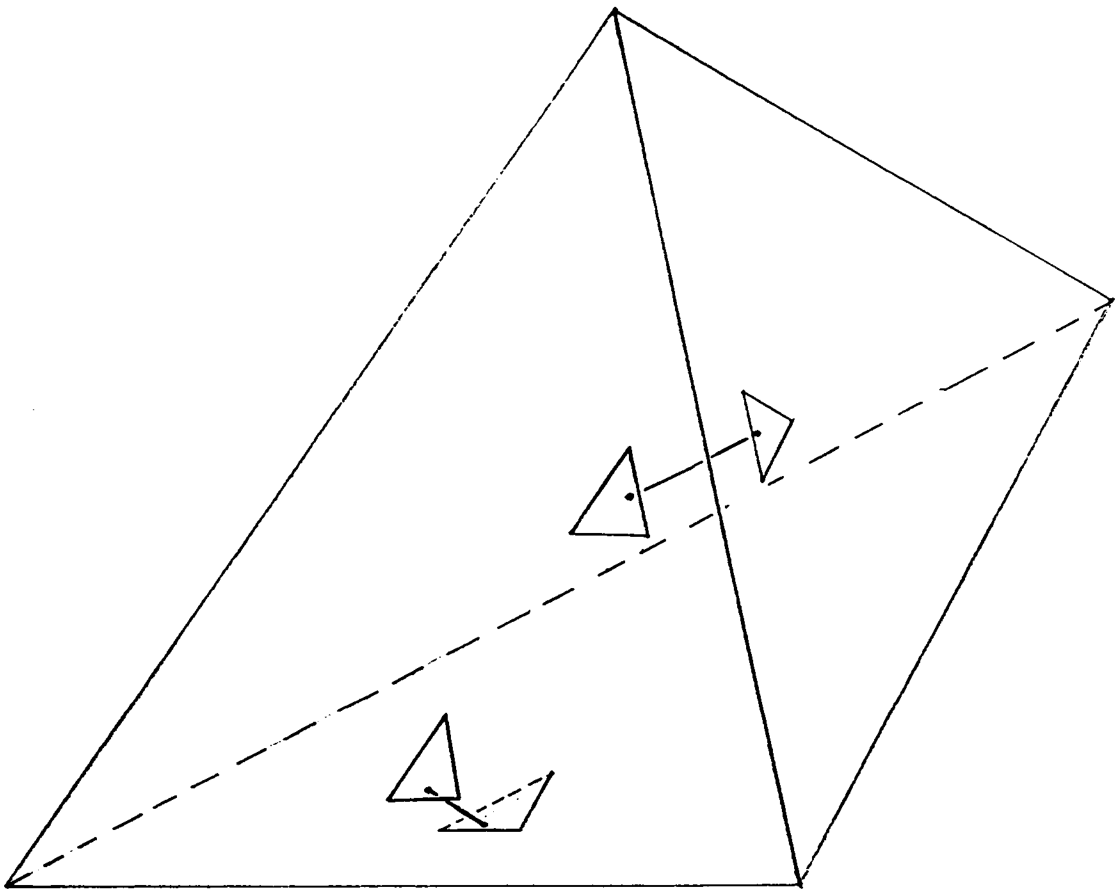


Figure 56. Before the saddle move in Γ in Case 5

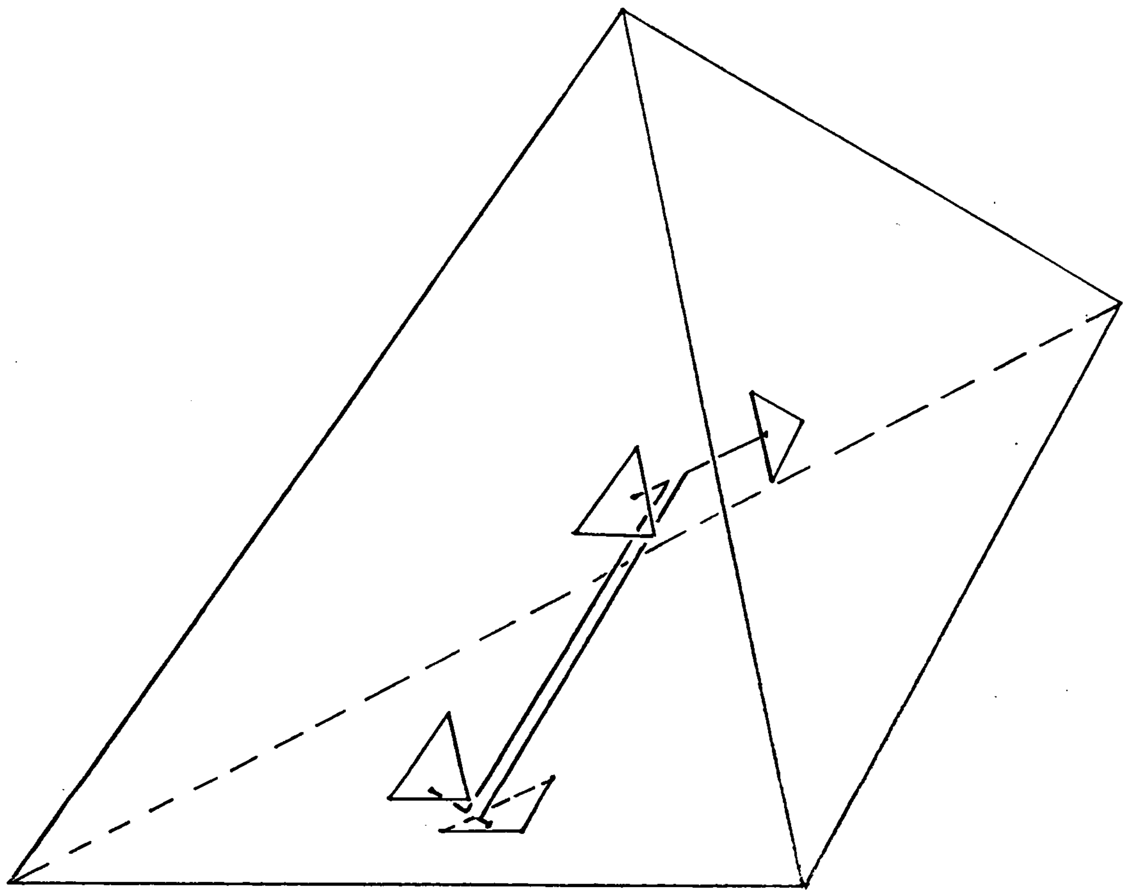


Figure 57. After the saddle move in Γ in Case 5

The move should be as simple as possible.

In Case 6, we do three saddle moves: near each of the three n-faces of Γ that are mapped onto α by f we do a saddle move analogous to the one in Case 5. This is pictured in Figures 58 and 59.

In Case 7, we similarly do 4 saddle moves, as depicted in Figures 60 and 61.

After the saddle moves, we cancel the simple closed curves from the interiors of simplices of Cases 6 and 7 using Operations 6. Lemma 7.7 is used. That the values of J encountered are 0 essentially follows by requiring symmetry of the constructions. The components of $\sum L_{\Delta} \cap \Gamma$ are symmetric with respect to an n-plane in Γ. We can do all the saddle moves in Γ in the "same" fashion, modulo the symmetry.

There are still some unwanted parts remaining around any non-free n-simplex Y that is mapped onto α by f. In each $n+1$-simplex whose face is Y, we did a saddle move close to Y. These moves result in a cluster of objects of Type 2 glued to Y, or a lookalike L'_{p} of a link cochain around a point $p \in \operatorname{int} Y$. This should be eliminated by Operation $5+r$, where Y is in the interior of an $n+r$-cell of K. It is easy to produce the suitable triangulated neighborhood U around p. That the result of each of the saddle moves close to Y was symmetric ensures that we can homotopy move L'_{p} to a link cochain L_{p}.

We also remove the stuff in any free n-simplex Y that is mapped

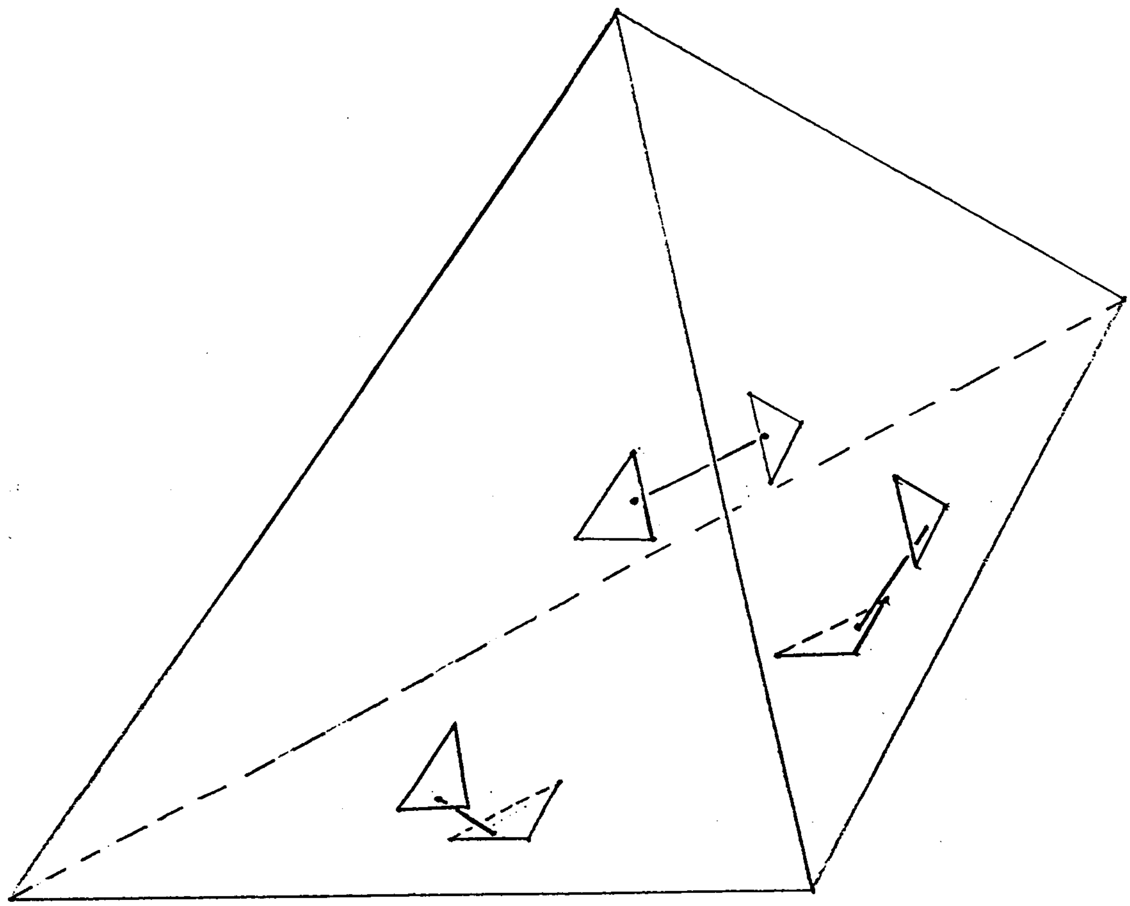


Figure 58. Before the saddle move in Γ in Case 6

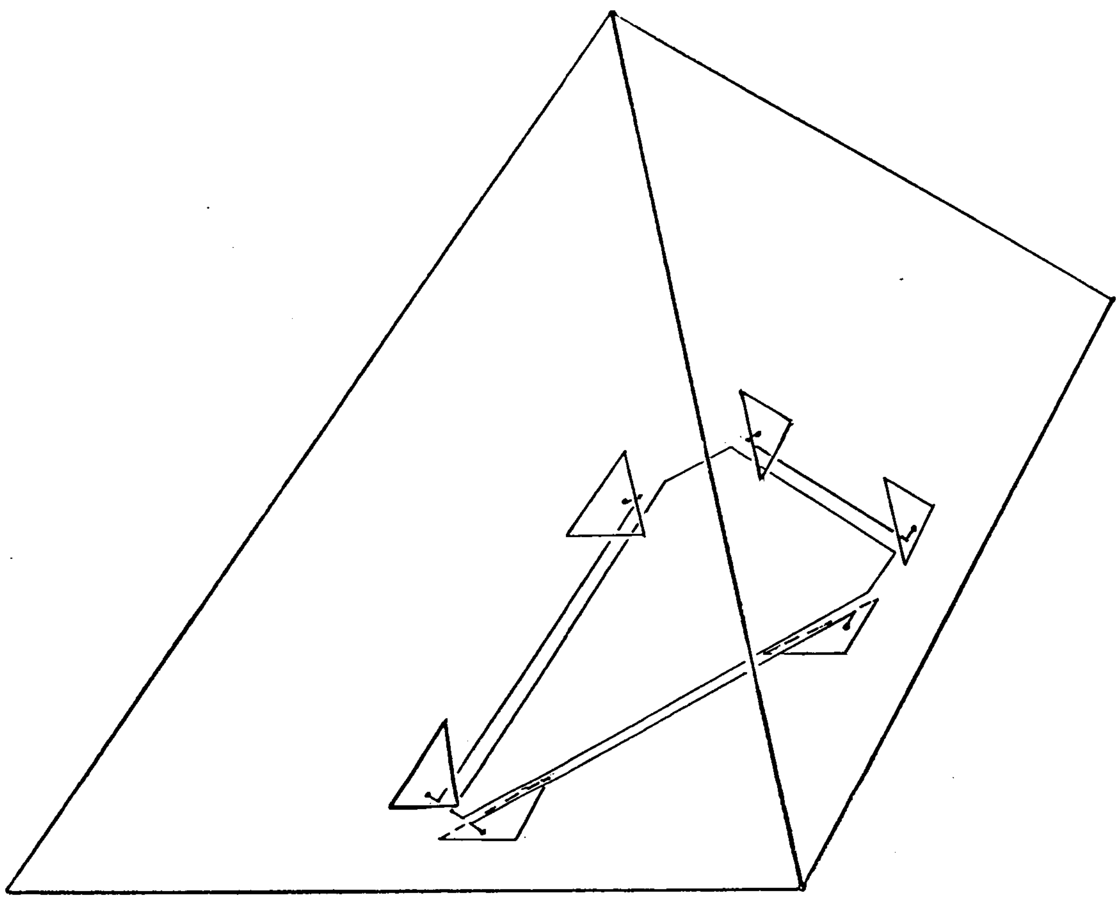


Figure 59. After the saddle move in Γ in Case 6

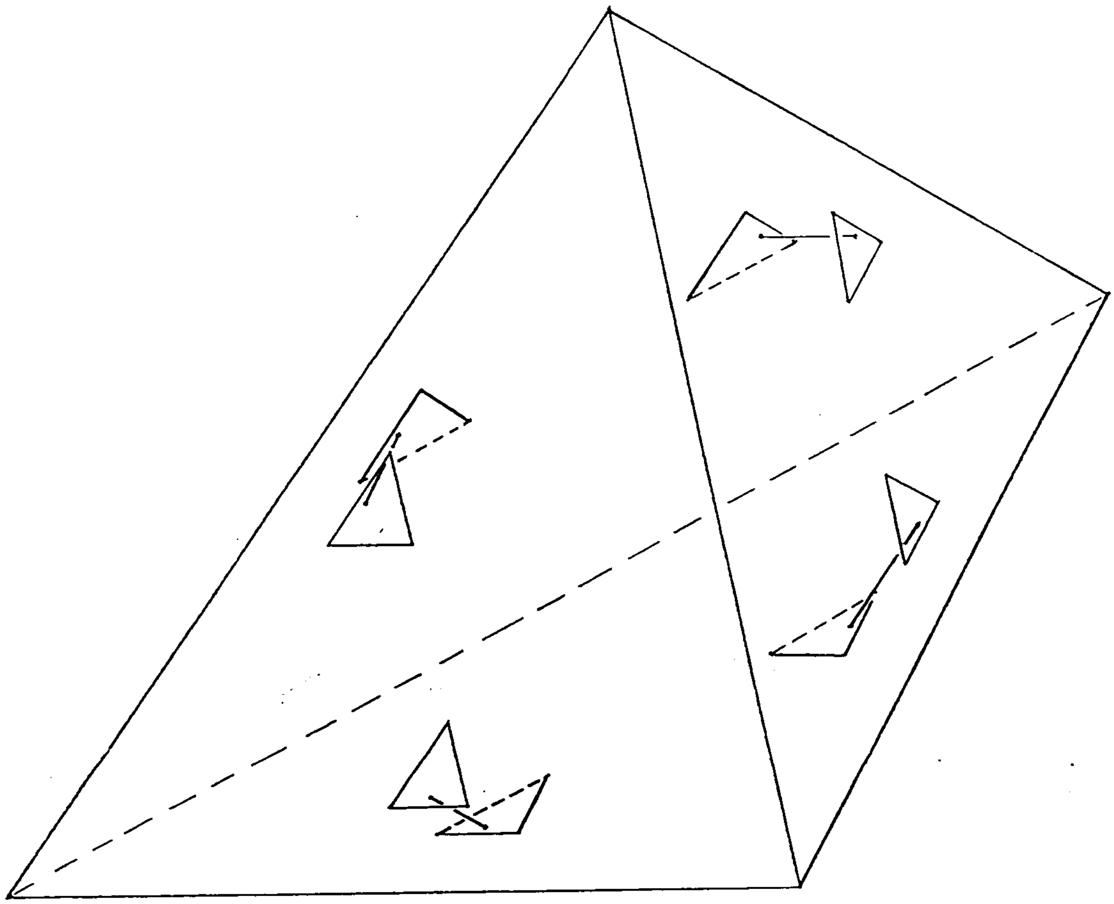


Figure 60. Before the saddle move in Γ in Case 7

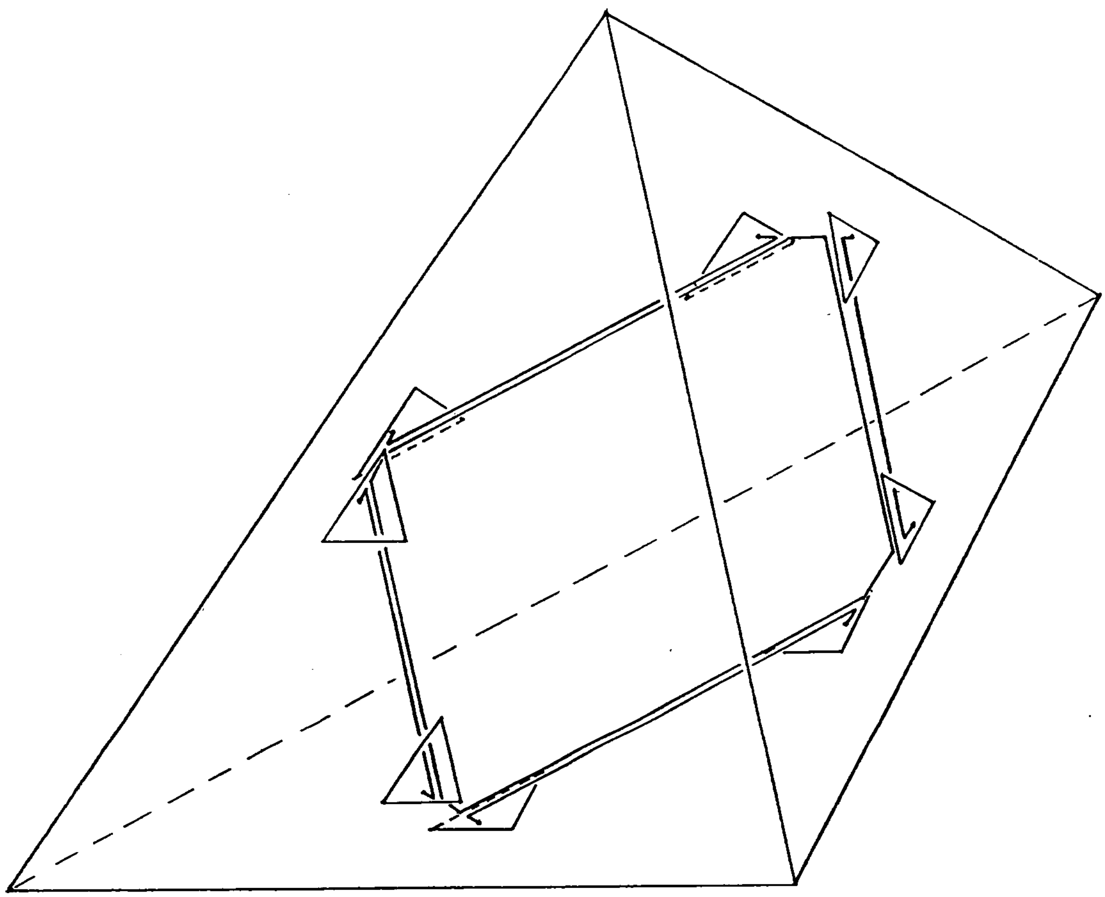


Figure 61. After the saddle move in Γ in Case 7

onto α by f. The two objects of Type 4 that occur in Case 3 have maps of different degrees, as Figure 47 shows. Use Operation 5.

All is left to do is an isotopy move.

■

We can now strengthen Proposition 12.1 to

Theorem 12.5.

Let K, L be finite regular CW-complexes, $n\geq 4$, and $S_0 \sim S_1$ - non-singular enhanced n-cocycles in L. Suppose that S_i pulls back by $f_i:K\to L$ to C_i, and $f_0\simeq f_1:K\to L$. Then $C_0 \sim C_1$.

Proof.

By 12.1, we can freely maneuver f_i's within their homotopy class, and thus we let f stand for any map in that class.

S_0 and S_1 are connected by a finite number of intermediate Stages (enhanced n-cocycles; two consecutive ones related by an Operation) and each of them can be made non-singular by an isotopy move (6.8). Therefore we can, without a loss, assume that one gets from S_0 to S_1 by an isotopy move, an Operation r, and another isotopy move.

Case $r=1$.

Use Proposition 12.1.

Case $r=2$.

The homotopy and isotopy moves commute in such a way that we can trade

$$S_1 = (\text{isotopy move}) \circ (\text{homotopy move}) \circ (\text{isotopy move}) S_0$$

for

$$S_1 = (\text{homotopy move}) \circ (\text{isotopy move}) \circ (\text{isotopy move}) S_0.$$

12.1 takes care of the last two steps; the first step is obvious.

Case $r=3$.

We can assume that S_1 is obtained from S_0 by a saddle move in a PL-ball $B \subseteq \operatorname{int}\tau$, where τ is some $n+1$-cell. There exists a triangulation T of $L^{[n+1]}$ that subdivides the cell structure, is compatible with S_0, and has B in the interior of some $n+1$-simplex.

We now homotope f like in the proof to Proposition 10.6, so that the resulting map allows for a simplicial pull-back of S_0. But observe that it then also allows for a simplicial pull-back of S_1. The two pull-backs differ by a number of saddle moves: $f^{-1}(B)$ is a disjoint union of PL $n+1$-balls, each mapped homeomorphically onto B, and we need one saddle move for each component.

Cases $r=4$, 5, 6.

These cases follow from 12.4. We add here certain non-singular n-cochain L_p in L (see the description of Operation r). The bottom line of the diagram in Figure 62 is what we have. But we can construct a non-singular cochain S_1' and an isotopy move M as in the right vertical arrow by Lemma 6.8. We then subtract L_p from S_1' to get S_0'. 12.1 implies that any pull-backs of S_0 and S_0' are cohomologous; the same holds for S_1' and S_1. To see that any pull-backs of S_0' and S_1' are cohomologous we use 10.5 and 12.4.

We now have a well-defined homomorphism

$$
\begin{tikzcd}
& \substack{S_0' \\ \text{non-singular}} \arrow[rr, "\text{add } L_p"] \arrow[d, "\substack{\text{isotopy} \\ \text{move} \\ - \\ \text{restriction} \\ \text{of } M}"'] & & \substack{S_1' \\ \text{non-singular}} \arrow[d, "\substack{\text{isotopy move} \\ M \\ \text{that fixes } L_p}"] & \\
S_0 \arrow[r, "\text{isotopy}", "\text{move}"'] & {} \arrow[rr, "\text{add } L_p"] & & {} \arrow[r, "\text{isotopy}", "\text{move}"'] & S_1
\end{tikzcd}
$$

Figure 62. Adding L_p as a "non-singular" proposition

$$f^*: \mathrm{E}H^n(L) \to \mathrm{E}H^n(K),$$

for any $n \geq 4$, associated with the given map $f: K \to L$.

10.3 and 12.5 yield

Theorem 12.6.

(a) If $f \simeq g$ then $f^* = g^*$,

(b) $(fg)^* = g^* f^*$,

(c) $(\mathrm{Id})^* = \mathrm{Id}$.

CHAPTER 13

THE HOMOTOPIC CLASSIFYING SPACE FOR ENHANCED COHOMOLOGY

Let Ω_n, $n \geq 4$, be a CW-complex such that the homotopy groups $\pi_n(\Omega_n)=\mathbb{Z}$, $\pi_{n+1}(\Omega_n)=\mathbb{Z}_2$, and all the other homotopy groups vanish; Ω_n is obtained by adding cells to S^n, starting with dimension $n+3$, to successively kill the homotopy groups starting with $n+2$-nd (consult [Wh]5.2). We will now show that Ω_n is a classifying space for enhanced cohomology.

Identify a PL n-ball Δ^n in $S^n \subseteq \Omega_n$, let o be the barycenter of Δ^n, and consider a map $f: K \to \Omega_n$. One can homotope f so that the preimage of (Δ^n, o) in $K^{[n+1]}$ gives a non-singular enhanced n-cochain in K (that is, $f^{-1}(\Delta^n)$ is the neighborhood, $f^{-1}(\mathrm{bd}\Delta^n)$ is its frontier, f is the map).

Theorem 13.1.

This correspondence induces a bijection α from $[K \to \Omega_n]$ to $EH^n(K)$.

Proof.

We first prove that homotopic maps give cohomologous cocycles. One can push the homotopy on $K^{[n+1]}$ into S^n. Consider the enhanced n-cocycle on S^n, $(o, \mathrm{Id}: \mathrm{bd}\Delta^n \to \mathrm{bd}\Delta^n)$; then the two cochains obtained as preimages are pull-backs of this cocycle (and therefore cocycles) by homotopic maps from $K^{[n+1]}$ to S^n. By 12.6, these cocycles are cohomologous, and we have a well-defined map $\alpha:[K\to\Omega_n]\to EH^n(K)$.

The proof that α is "onto" is easy. Take a class $[C]\in EH^n(K)$ and "extend" to $K^{[n+2]}$ using 10.1. This involves a map $h_M:\mathrm{fr}_{K^{[n+2]}}M\to S^{n-1}$, using the notation of 10.1. We can further h_M to the desired map from K to Ω_n.

Now we show that α is 1-1. We use the non-singular theory here. We have two maps, $f_0:K\times 0\to\Omega_n$ and $f_1:K\times 1\to\Omega_n$, that give cohomologous preimages of Δ^n; we would like to extend f_0 and f_1 to a homotopy $F:K\times I\to\Omega_n$. It suffices to define F on $K^{[n+1]}\times I$. Because $\Omega_n^{[n+1]}=S^n$, we can assume that the f_i's take $K^{[n+1]}$ to S^n, and we will define $F:K^{[n+1]}\times I\to S^n$. Denote by N_i the neighborhood given by the cocycle on the i-level, $i=0,1$.

The goal is to find a subset W of $K^{[n+1]}\times I$, and a map $F:W\to \mathrm{bd}\Delta^n$ such that:

$\mathrm{fr}N_0,\mathrm{fr}N_1\subseteq W$,

F extends $f_i:\mathrm{fr}N_i\to\mathrm{bd}\Delta^n$, $i=0,1$,

W separates $K^{[n+1]}\times I$ into two open sets, one of which contains $\mathrm{int}N_0$ and $\mathrm{int}N_1$, and the other - $K^{[n+1]}\times 0\setminus N_0$ and $K^{[n+1]}\times 1\setminus N_1$.

After that is done, an extension of F to $K^{[n+1]}\times I$ to either side of W is easily obtained, as the preferred ranges are cells for each side of W.

The two cocycles are cohomologous; it suffices to consider the case when they are related by a single Operation r.

Case $r=1$.

The isotopy move easily gives W and F.

Case $r=2$.

$W=\mathrm{bd}N_0\times I$.

Case $r=3$

follows from the proof of 4.1: for example we can choose W as $\mathrm{fr}N(D)$, and then F is $\bar{g}$.

Case $r=4$.

We can assume that $N_1\subseteq N_0$. We will use the notation from the definition of Operation 4. We assume that neighborhood $U\subseteq K\times 0$. Outside of $U\times I$, W will be $\mathrm{fr}N_0\times I$ and F is $f_0\times I$.

We will triangulate a neighborhood of p in $K^{[n+1]}\times I$ as a cone using the given triangulation of U. The cone point v should be in $\alpha\times I$, and if a simplex of U is in a cell σ, then the cone over that simplex should be in $\sigma\times I$. We want to extend λ to a simplicial map from this neighborhood to $2\Delta^n$. λ maps all the vertices of $U\setminus\bar{\alpha}$ to one vertex of $2\Delta^n$; This is where we define $\lambda(v)$. Now W in $U\times I$ is $\lambda^{-1}(\mathrm{bd}\Delta^n)$ and F is this extension of λ.

The proofs in cases $r=5,6$ are similar to the proof in Case $r=4$.

CHAPTER 14

THE CONNECTION WITH STEENROD SQUARES

Using the geometric interpretation of cell cohomology in Chapter 5, we can define a map

$$F{:}E\mathscr{C}^{n}(K)\to\mathscr{C}^{n}(K)$$

by restricting our attention to the n-cells of K, or forgetting the interiors of $n+1$-cells.

Theorem 14.1.

F gives a well-defined homomorphism

$$F{:}EH^{n}(K)\to H^{n}(K),$$

which will be called the canonical forgetting homomorphism. If $n\geq\dim K$, then F is an isomorphism.

Proof.

This is easy to see. By 2.7, F sends any cochain to a cocycle. ∎

We now define a map

$$I{:}C^{n+1}(K,\mathbb{Z}_2)\to E\mathscr{C}^{n}(K)/_{\sim},$$

where $C^{n+1}(K,\mathbb{Z}_2)$ are the cell cochains. Let $f\in C^{n+1}(K,\mathbb{Z}_2)$. Define $I(f)$ as the class of a cochain that consists of objects of Type 3, one in each $n+1$-cell of K. The map h induced by the object in τ^{n+1}, is defined so that $J([h])=f(\tau^{n+1})$. This is allowed by Lemma 6.1, and

I is well-defined by 7.9.

Proposition 14.2.

I is a homomorphism

$$I:H^{n+1}(K,\mathbb{Z}_2)\to EH^n(K).$$

Proof.

That the cocycles go to cocycles follows from the additivity of J (end of 3.7). Hence, we have $I:Z^{n+1}(K,\mathbb{Z}_2)\to EH^n(K)$; that this respects addition follows from 7.9.

Now, we only need to see $I(\delta f)=0\in EH^n(K)$, where $f(\sigma)=1$ for some n-cell σ, and $f(\alpha)=0$ for any other n-cell α. All the objects of $I(\partial f)$ with the value of J equal to 0 can be eliminated by using 7.7.

We are left with one object in each $n+1$-cell that has σ as a face. We add a "link cochain" around σ using Operation 5, use a saddle move in each $n+1$-cell, homotope, and cancel the whole thing using Operation 5. A symmetrical arrangement is helpful. The process is illustrated in Figure 63. The details are left to the reader.

It is easy to see

Lemma 14.3.

The image of $I:H^{n+1}(K,\mathbb{Z}_2)\to EH^n(K)$ consists of enhanced cohomology classes that have a representative that does not intersect $K^{[n]}$.

Let K be a finite regular CW-complex, and consider the second

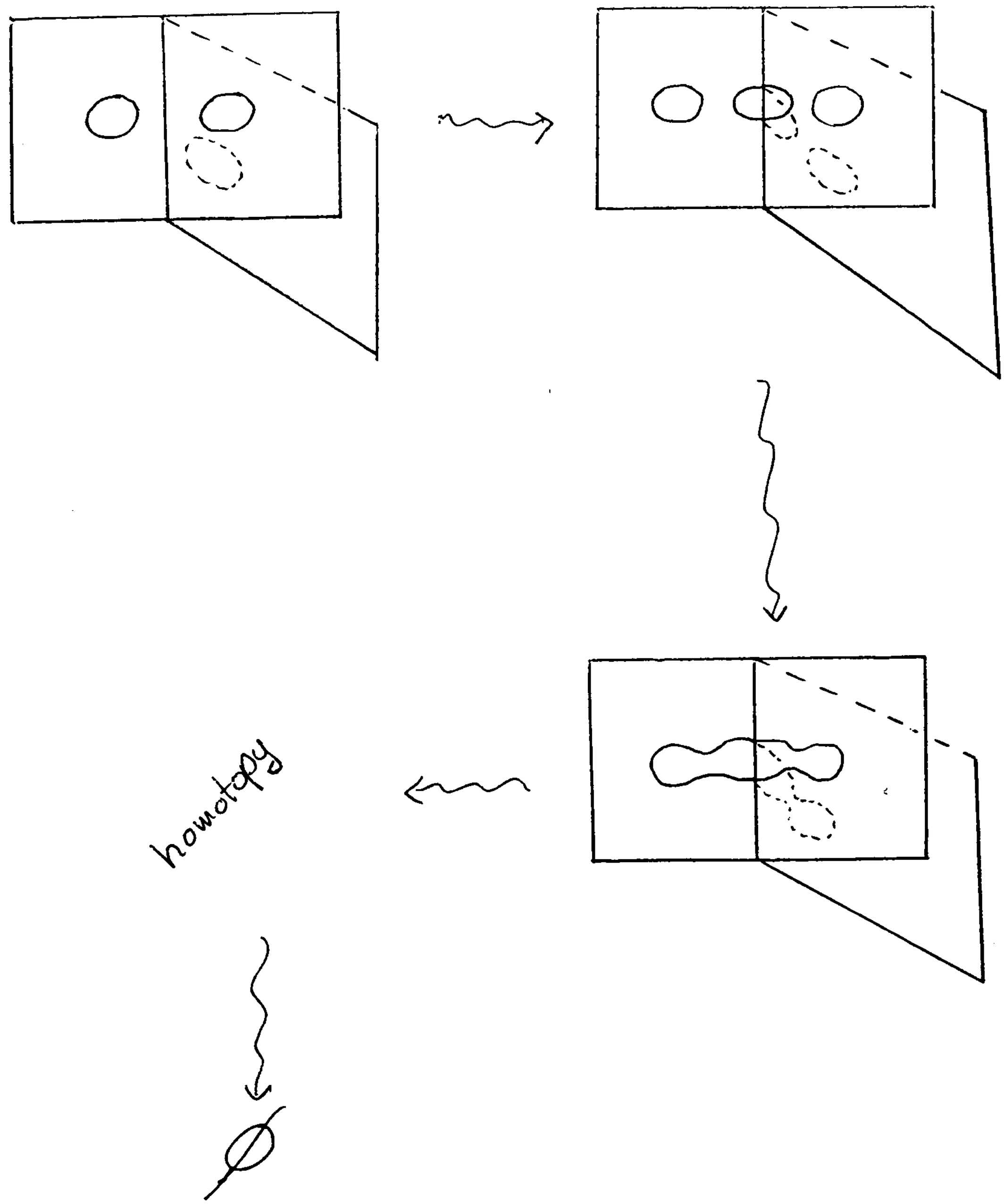


Figure 63. Why I preserves coboundaries

Steenrod square $\mathrm{Sq}^2:H^{n-1}(K,\mathbb{Z})\to H^{n+1}(K,\mathbb{Z}_2)$. We will use the following definition of this Operation for $n\geq 4$ (see [St]21.1).

We use cellular cohomology. Let g be an n-1-cocycle (coefficients in $\mathbb{Z}$) in K. Choose a base point $*$ and an orientation for S^{n-1}, and consider the constant map $f:K^{[n-2]}\to *$. Let σ^{n-1} be any n-1-cell of K. Extend f to $f:\sigma^{n-1}\to S^{n-1}$ with the degree equal to $g(\sigma^{n-1})$. Since g is a cocycle, there is an extension of f to $K^{[n]}$. Let τ^{n+1} be any $n+1$-cell of K. The class of $f:\mathrm{bd}\tau\to S^{n-1}$ in $\pi_n(S^{n-1})=\mathbb{Z}_2$ is then the value of $\mathrm{Sq}^2(g)$ on τ. [St]21.1 implies that this is a definition of Sq^2.

Theorem 14.4.

The sequence

$$H^3(K,\mathbb{Z})\xrightarrow{\mathrm{Sq}^2}\dots$$
$$\to H^{n-1}(K,\mathbb{Z})\xrightarrow{\mathrm{Sq}^2}H^{n+1}(K,\mathbb{Z}_2)\xrightarrow{I}EH^n(K)\xrightarrow{F}H^n(K,\mathbb{Z})\xrightarrow{\mathrm{Sq}^2}H^{n+2}(K,\mathbb{Z}_2)\to\dots$$

is exact.

Proof.

Let us see $I\mathrm{Sq}^2=0$ first. Consider $[c]\in\mathrm{Im}(I\mathrm{Sq}^2)$. We have $f:K^{[n]}\to S^{n-1}$ and, whenever for an $n+1$-cell τ $f|_{\mathrm{bd}\tau}$ is essential, we see a simple closed curve $s\subseteq\mathrm{int}\tau$, a regular neighborhood N of s in τ and a map $h:\mathrm{fr}N\to S^{n-1}$ with $J([h])=1$. We can thus extend f and h to $\tau\setminus\mathrm{int}N$. We can also define $\bar{h}:N\to\Delta^n\supseteq\mathrm{bd}\Delta^n=S^{n-1}$ so that $\bar{h}^{-1}(o)=s$ and $\bar{h}|_{\mathrm{fr}N}=h$, and extend f to any $n+1$-cell with $f|_{\mathrm{bd}\tau}$ inessential.

We now see that c, as a cocycle in $K^{[n+1]}$, is a pull-back of the "identity" enhanced n-cocycle (Δ^n, o) in the regular CW-complex $2\Delta^n$. Since $EH^n(2\Delta^n)=0$, c is cohomologous to 0 in $K^{[n+1]}$; it is then cohomologous to 0 in K.

To prove $\mathrm{Ker} I \subseteq \mathrm{ImSq}^2$, take $[x] \in H^{n+1}(K, \mathbb{Z}_2)$ that dies as an element $[c]$ of $EH^n(K^{[n+1]})$.

Claim.

c is a pull-back of the above identity cocycle by some map $F: K^{[n+1]} \to 2\Delta^n$.

Proof.

It suffices to show that if two cochains are related by an Operation r and one is a pull-back, then the other is one as well (everything here is within non-singular theory). For $r=1$ this follows from 7.5, for $r=2$ - from homotopy extension and the collar structure on the "boundary" of a cochain, for $r=3$ - from 4.1, and for $r \geq 4$ - from 6.6.

■ (Claim)

Now, homotope

$$F:(K^{[n+1]}, K^{[n]}) \to (2\Delta^n, 2\Delta^n \setminus o)$$

to map $K^{[n]}$ to $S^{n-1} = \mathrm{bd}\Delta^n$ and $K^{[n-2]}$ to $* \in S^{n-1}$, and call the new map F'. If τ^{n+1} is a cell, then

$$\left[F' \middle|_{\mathrm{bd}\tau}\right] = \left[F \middle|_{\mathrm{bd}\tau}\right] = x(\tau) \in \pi_n(S^{n-1}) = \mathbb{Z}_2.$$

Using our definition of Sq^2, we now easily see $x \in \mathrm{ImSq}^2$.

By Lemma 14.3, the composition FI is 0.

To see $\mathrm{Ker}F \subseteq \mathrm{Im}I$, consider $F: E\mathscr{C}^n(K) \to \mathscr{C}^n(K)$. If $F(\bar{c}) = c$ and c is cohomologous to 0, we will show that $\bar{c}$ is cohomologous to a cochain living off $K^{[n]}$; this suffices by 14.3.

Using Chapter 5, if c is cohomologous to 0 then we can use Operation 4 adding sets of points (c is a sum of elements of the form $\delta\bar{\sigma}^n$, where $\bar{\sigma}^n$ is a cochain that takes a cell σ^n to 1 and the other cells to 0; we add sets of points that correspond to the cells σ^n in the sum) repeatedly to transform c to a cochain with the sum of the degrees equal to 0, in every n-cell. It is easy to "realize" each addition on the enhanced level by Operations 4 and 2 from Chapter 6.

We now have an enhanced cochain d, cohomologous to $\bar{c}$, that, when restricted to any n-cell, produces maps with sum of degrees 0. We will inductively reduce the number of glued families in d (not changing the cohomology class) attaining 0 of them at the end.

At each step, we reduce the number by two. First, apply an isotopy move to make d non-singular. Let h be the map of d and consider the two ends of objects of d in the same n-cell σ and with opposite degrees of h. We can isotopy move d to make it look as in Figure 64 (τ_i's are all the $n+1$-cells that have σ as a face). We then follow the process outlined in Figures 64-66. The details are easy to furnish.

We want to identify a copy of $\Delta^{n-1} \times I \times I$ in each τ_i; the copies should by glued by $\Delta^{n-1} \times I \times 0$ in σ. We then want d to occupy $D \times \left(\left[\frac{1}{6}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{5}{6}\right]\right) \times I$, where D is a PL-ball in $\mathrm{int}\Delta^{n-1}$. We then homotopy

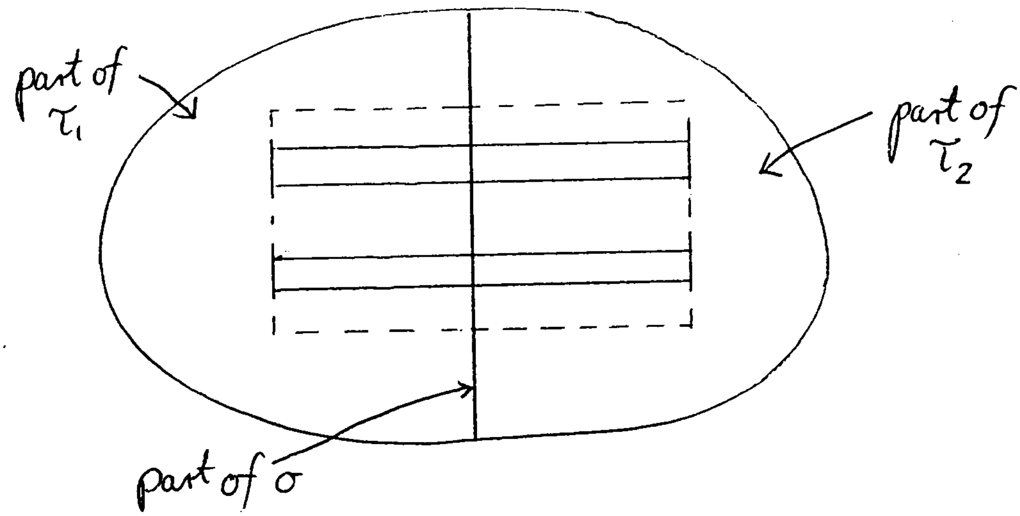


Figure 64. Part of what happens near an n-cell in proving $\mathrm{Ker}F \subseteq \mathrm{Im}I$

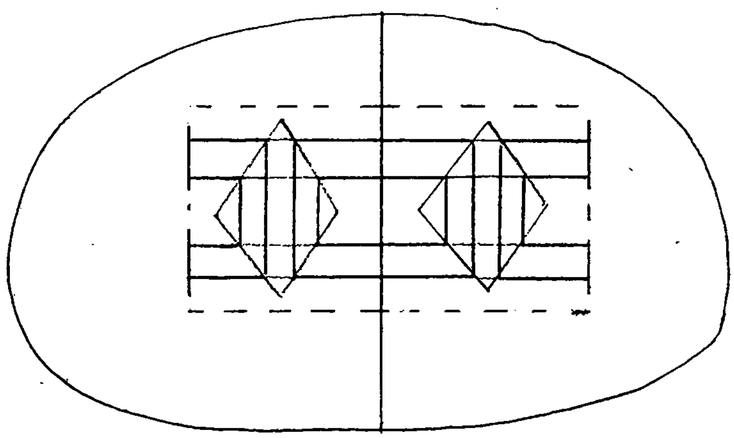


Figure 65. After the saddle moves near σ

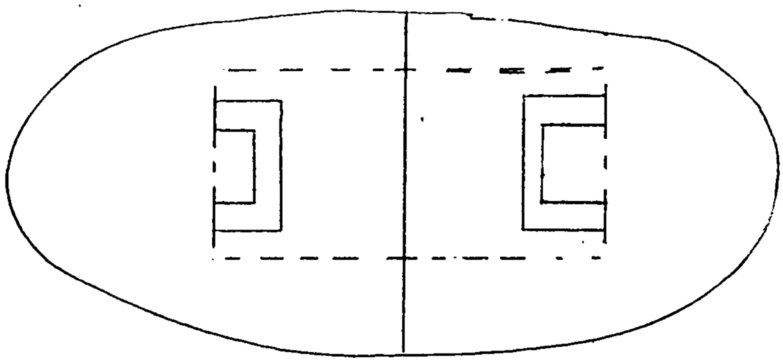


Figure 66. The situation near σ after the link cochain is removed

and saddle move to obtain situation as in Figure 65. The homotopy move should make the map on $\mathrm{bd}\left[D\times\left[\frac{1}{6},\frac{1}{3}\right]\right]\times 0$ the same as the map on $\mathrm{bd}\left[D\times\left[\frac{2}{3},\frac{5}{6}\right]\right]\times 0$, modulo the symmetry with respect to $\frac{1}{2}\in I$. Moreover, it should make the map on $\mathrm{bd}(D\times\left[\frac{1}{6},\frac{1}{3}\right]\Big]\times t$ the same for all $t\in I$; and similarly for $\mathrm{bd}\left[D\times\left[\frac{2}{3},\frac{5}{6}\right]\right]\times t$. We can then easily erase the central part by Operation 5.

We will now prove $\mathrm{Sq}^2F=0$. We consider an enhanced n-cochain C with map $h{:}\mathrm{fr}N\to S^{n-1}$. To see $\mathrm{Sq}^2F(C)$, we shall, using the definitions, produce a map $f{:}K^{[n+1]}\to S^n$, where S^n is regarded as two n-discs D_1, D_2 sewn along S^{n-1}, with the base point $*$ of S^n lying in D_1. We extend h to $f{:}N\to D_2$, set $f(K^{[n-1]})=*$, and extend f so that $f(K^{[n+1]}\setminus N)\subseteq D_1$. to the rest arbitrarily, but with values in D_1. We only need to see that, for any $n+2$-cell τ, $f|_{\mathrm{bd}\tau}$ is inessential. Since the value of J in $\mathrm{bd}\tau$ is 0, we can produce $f'{:}\mathrm{bd}\tau\to S^n$ that agrees with f on N, but takes the rest to S^{n-1}. But $f|_{\mathrm{bd}\tau}\simeq f'\simeq *$.

Finally, if $[c^n]\in\mathrm{Ker}\,\mathrm{Sq}^2$, we shall see that c extends to an enhanced n-cocycle C.

We first interpret c as in Chapter 5; we assume that the points and their neighborhoods are pairwise disjoint. So we have a finite collection of points $z\subseteq K^{[n]}\setminus K^{[n-1]}$, their regular neighborhood N in $K^{[n]}$, and a map $h{:}\mathrm{fr}N\to S^{n-1}$.

When we use the above sewing, h extends to a map $f{:}K^{[n+1]}\to S^n$, such that $f(K^{[n-1]})=*$, $f^{-1}(D_2)\cap K^{[n]}=N$, $f^{-1}(p)\cap K=z$, where p is the center of D_2. Using relative simplicial approximation, we can assume that $f^{-1}(p)$, $M=f^{-1}(D_2)$, and $f:f^{-1}(S^{n-1})\to S^{n-1}$ give an enhanced cochain C'. We of course have $F(C')=c$, but C' need not be an enhanced cocycle.

Claim.

Let τ^{n+2} be a cell in K. The value of J computed using C' on bdτ is $\left[f\,|\,\mathrm{bd}\tau\right]$.

Proof.

Let B be a small $n+1$-ball in bdτ^{n+2} disjoint from C'. Consider a map $f':\mathrm{bd}\tau\to S^n$ such that $f'=f$ on M, $f'(B)\subseteq D_1$, $f'(\mathrm{bd}\tau\setminus(B\cup M))\subseteq S^{n-1}$ (for existence see 3.6). Now

$$\left[f\,|\,\mathrm{bd}\tau\right]=[f']=\left[\Sigma\left(f'\,|\,\mathrm{bd}B\right)\right]=\Sigma\left[f'\,|\,\mathrm{bd}B\right]=\left[f'\,|\,\mathrm{bd}B\right].$$

but the last thing is the value of J we need (see 3.6). ■

Consider a cochain $g\in C^{n+2}(K,\mathbb{Z}_2)$, whose value on a cell τ^{n+2} is $\left[f\,|\,\mathrm{bd}\tau\right]$. We have $[g]=\mathrm{Sq}^2([c])$, so g is a coboundary, or a sum of elements of the form $\delta\sigma^{n+1}$, where $\sigma^{n+1}\in C^{n+1}(K,\mathbb{Z}_2)$ sends a cell σ^{n+1} to 1 and the rest to 0. For each term $\delta\sigma^{n+1}$, add an object of Type 4, with J value 1, to C'. The total is C. C is an enhanced cocycle because of additivity of J (end of 3.7) and the last Claim.

■

The above theorem yields

Example 14.5.

If M is a compact, triangulated, topological $n+1$-manifold, $n\geq 4$, with non-empty boundary in each component, then $F:EH^n(M)\to H^n(M)$ is an isomorphism. ∎

We now give an example of I being one-to-one. This shows that the enhanced cohomology groups differ from the ordinary ones (with any group of coefficients) on finite regular CW-complexes.

Proposition 14.6.

If M is a closed 2-connected triangulated n-manifold, $n \geq 5$, then $I: \mathbb{Z}_2 \cong H^n(M,\mathbb{Z}_2) \to EH^{n-1}(M)$ is an isomorphism.

Proof.

By Poincaré Duality, $H^{n-2}(M,\mathbb{Z})=0$ and $H^{n-1}(M,\mathbb{Z})=0$ and from the exact sequence I is an isomorphism.

On the other end of the spectrum is

Example 14.7.

Let $n \geq 4$ and let $f: S^n \to S^{n-1}$ be a PL essential map. Let C_f be the mapping cone of f. Recall the sequence

$$\begin{array}{ccccccc} H^{n-1}(C_f,\mathbb{Z}) & \xrightarrow{\mathrm{Sq}^2} & H^{n+1}(C_f,\mathbb{Z}_2) & \xrightarrow{I} & EH^n(C_f) & \xrightarrow{F} & H^n(C_f). \\ \| & & \| & & & & \| \\ \mathbb{Z} & & \mathbb{Z}_2 & & & & 0 \end{array}$$

By [St]20.2, Sq^2 is onto here. Thus $EH^n(C_f)=0$ and I fails to be one-to-one.

∎

It is easy to see

Proposition 14.8.

The sequence from 14.4 is natural, that is, any diagram

$$\begin{tikzcd}
\to H^{n-1}(K,\mathbb{Z}) \arrow[r, "Sq^2"] & H^{n+1}(K,\mathbb{Z}_2) \arrow[r, "I"] & EH^n(K) \arrow[r, "F"] & H^n(K,\mathbb{Z}) \arrow[r, "Sq^2"] & H^{n+2}(K,\mathbb{Z}_2) \to \dots \\
\to H^{n-1}(L,\mathbb{Z}) \arrow[r, "Sq^2"] \arrow[u, "f^*"] & H^{n+1}(L,\mathbb{Z}_2) \arrow[r, "I"] \arrow[u, "f^*"] & EH^n(L) \arrow[r, "F"] \arrow[u, "f^*"] & H^n(L,\mathbb{Z}) \arrow[r, "Sq^2"] \arrow[u, "f^*"] & H^{n+2}(L,\mathbb{Z}_2) \to \dots \arrow[u, "f^*"]
\end{tikzcd}$$

commutes.

CHAPTER 15

THE RELATIVE THEORY

The enhanced cohomology theory can be relativized as follows. The analogue of anything that we do not discuss is easily obtained by mimicking the absolute case. We will frequently need a combination of [Z] and B.H.E.T. We consider a CW-pair (K,A) of finite, regular CW-complexes.

In Operation 4 of Chapter 5, we require that α be disjoint from A.

The admissible objects in (K,A) are defined as the admissible objects in K that miss A. Lemma 6.1 holds for τ not in A. The relative cochains are denoted as $E\mathscr{C}^n(K,A)$. Remark 6.3 restricts a cochain in (K,A) to one in $(L,A\cap L)$. The cells α, σ, τ in Operations 4, 5, 6, respectively, should not be from A. The relative cocycles are denoted $E\mathscr{Z}^n(K,A)$. In Proposition 7.5, we can assume that the covering isotopy is fixed on A.

The enhanced cohomology groups are denoted $EH^n(K,A)$.

Proposition 10.4 reads: if $i{:}(K,K\cap B)\longrightarrow(L,B)$ is an inclusion of subcomplexes, then $S|_K$ is a pull-back of S by i.

In the statements of 10.7 to 14.8 we only use maps and homotopies of pairs. In any of the Steps 1, 2, 4, 6, 7 of the Proof of Proposition 10.7: we do our thing as in the absolute Proof, set the homotopy considered to be constant on A, extend to $K^{[n+1]}\setminus A$ by B.H.E.T, extend to K by B.H.E.T. In Steps 3 and 5, we insist that H_t be Id on A. In Step 8, we first consider a simplicial approximation of $f|_{A\cap K^{[n+1]}}:A\cap K^{[n+1]}\to B\cap L^{[n+1]}$. We use B.H.E.T several times, and after a while we are ready to use [Z] to approximate a map defined on $K^{[n+1]}\setminus\operatorname{int}W$ relative $\operatorname{fr}W\cup(A\cap K^{[n+1]})$.

In the Proof of Proposition 11.1, we build the simplicial complex $W^{\,p}\subseteq\mathbb{R}^m\times\left[\frac{1}{3},\frac{2}{3}\right]$, and the triangulation $T^{\,p}:W^{\,p}\to K\times\left[\frac{1}{3},\frac{2}{3}\right]$ as in the absolute case. Using [Z] and B.H.E.T repeatedly we can insist that $f:W^{\,p}\to L$ take $(T^{\,p})^{-1}\left[A\times\left[\frac{1}{3},\frac{2}{3}\right]\right]$ to B. We then see that $f:X\to L^{[n+2]}$ takes $T^{-1}(A\times I)$ to B, so c (and U) does not touch $T^{-1}(A\times I)$. Therefore, the interior of the simplex Δ (defined, for any e, in the "Landscape" part) is mapped, by T, outside $A\times I$.

To define $f^{\,*}$ we use a homotopy of pairs.

In our definition of relative Steenrod squares, we require that f send A to $*$. This is used in the proof of 14.4; $I\mathrm{Sq}^2=0$ because c is a pull-back of a cocycle in $(2\Delta^n,*)$.

14.5, 14.6, 14.7 are not relativized.

Remark 15.1.

$EH^{\,n}(X)$ and $EH^{\,n}(X,\emptyset)$ are canonically isomorphic.

Proposition 15.2.

Let (K,A,B) be a CW, finite, regular triple. The "short triple sequence"

$$EH^n(K,A) \to EH^n(K,B) \to EH^n(A,B),$$

with homomorphisms induced by inclusions, is exact.

Proof.

The composition is obviously 0.

The argument for the other inclusion mimics the part of the proof of the zig-zag lemma that corresponds to our situation, except that we do not have the boundary homomorphism. All we need is the following.

Let $i:(A,B) \to (K,B)$ be the inclusion, and let $i^*(d)=c$, where $d \in E\mathcal{Z}^n(K,B)$, $c \in E\mathcal{Z}^n(A,B)$. Suppose that c is changed by an Operation r (in A relative B) to c'. Then there is $d' \in E\mathcal{Z}^n(K,B)$, cohomologous (in K relative B) to d with $i^*(d')=c'$.

This is trivial for all the cases except the subtraction versions of Operations 4, 5.

CHAPTER 16

THE APPLICATION TO OBSTRUCTION THEORY

Let $A \subseteq K$ be a regular finite CW-pair and $n \geq 4$. Let Y be a Hausdorff space. Suppose Y is a subspace of a space C such that

a. C is Hausdorff,

b. C is $n+1$-connected,

c. there is an open subset U of C, not contained in Y, homeomorphic to $\mathbb{R}^n$; equivalently, there is a homeomorphism $\xi: U \to \mathbb{R}^n$, where $U \subseteq C$ is open and $\xi^{-1}(0) \notin Y$.

Conditions a-c will be called the *extension design.* We will usually identify U and $\mathbb{R}^n$ by ξ and suppress ξ.

Suppose we have a map $f: A \to Y$ and the question is: can one extend f to $\bar{f}: K^{[n+1]} \cup A \to Y$? We will first be contented with an easier task of extending f to $\bar{\bar{f}}: K^{[n+1]} \cup A \to C \setminus 0$.

Let $r: \mathbb{R}^n \setminus 0 \to S^{n-1}$ be the radial projection. A map $g: K^{[n+1]} \cup A \to C$ will be called *pretty* if there exists a non-singular relative enhanced n-cochain in (K,A) with the core s, the neighborhood N, and the map $h: \mathrm{fr}N \to S^{n-1}$ such that

$$s = g^{-1}(0), \quad g(N) \subseteq \mathbb{R}^n, \quad h = rg: \mathrm{fr}N \to S^{n-1}.$$

Lemma 16.1.

Let $Y \subseteq C$ satisfy an extension design. Any map $f: A \to Y \subseteq C$ has a pretty extension.

Proof.

Roughly, take any extension $\bar{f}$ and use general position about 0. That C is Hausdorff is used in the following way. We first modify $\bar{f}$ to make $0 \notin \bar{f}(A \cup K^{[n-1]})$. Because C is Hausdorff, $\bar{f}(A)$ and 0 can be separated, and, if U is a neighborhood of 0 in $\mathbb{R}^n$ with $U \cap \bar{f}(A \cup K^{[n-1]}) = \emptyset$, we do general position in $\bar{f}^{-1}(U)$.

Theorem 16.2.

For any pretty extension g of f, an enhanced cochain S it gives is a relative cocycle in (K,A). The class $[S] \in \mathrm{E}H^n(K,A)$ is independent of g and S.

Proof.

Take two pretty extensions $g_0, g_1 : K^{[n+1]} \cup A \to C$. and suppose they give enhances cochains S_0, S_1, respectively. We find a homotopy $g_t : K^{[n+1]} \cup A \to C$ that is constant on A. Roughly, this homotopy will allow us to write S_0, S_1 as pull-backs of a cocycle in C by homotopic maps. We can also extend g_1 to $g_1 : K^{[n+2]} \cup A \to C$.

Let N_i be the neighborhood given by S_i. Let $0 \in \Delta \subseteq \mathbb{R}^n \subseteq C$ be a PL n-ball such that $g_i(N_i) \subseteq \Delta$ for i=0,1. Let Δ' be a copy of Δ and let P be Δ and Δ' glued along the boundary. Because $C \setminus \mathrm{int}\Delta$ is Hausdorff, there is a continuous map $\varphi : C \to P$ with $\varphi|_\Delta = \mathrm{Id}$, $0 \notin \varphi(C \setminus \Delta)$. Let $h_t = \varphi g_t : K^{[n+1]} \cup A \to P$; we also have $h_1 : K^{[n+2]} \cup A \to P$.

Let $a > 0$ be so small that $h_t(A)$ stays outside $2a\Delta$. The space P has a regular CW-structure with $2a\Delta$ as an n-cell. Let $B = P \setminus \mathrm{int}2a\Delta$.

Let $r{:}\mathbb{R}^n\setminus 0\to S^{n-1}$ be the radial projection and consider a relative enhanced n-cocycle

$$\Lambda=\{(0,r{:}\mathrm{bd}a\Delta\to S^{n-1})\}$$

in (P,B). Using 7.1, it is easy to homotope $h_1{:}(K^{[n+2]}\cup A,A)\to(P,B)$ to a map h_2 (homotopy between maps of pairs) so that S_1 is a pull-back of Λ by h_2; we only need to push N_1 to $a\Delta$. This, in particular, shows that S_1 is a cocycle in $(K^{[n+2]}\cup A,A)$, which means it is a cocycle in (K,A). Do the same with S_0 and h_0 to see that S_0 and S_1 are pull-backs of Λ by homotopic maps and therefore are cohomologous in $(K^{[n+1]}\cup A,A)$. This implies that they are cohomologous in (K,A).

Definition.

Let (K,A) be a regular finite CW-pair and $n\geq 4$. Let Y be a Hausdorff space and let (C,ξ) be an extension design for Y. Let $f{:}A\to Y$ be a map. Then we define

$$\mu_{f,C,\xi}\in \mathrm{E}H^n(K,A)$$

to be the class of an enhanced cocycle arising from any pretty extension of f.

Theorem 16.3.

$\mu_{f,C,\xi}=0$ is a necessary and sufficient condition for f to be extendable to $\overline{\overline{f}}{:}K^{[n+1]}\cup A\to C\setminus 0$.

Proof.

Necessity. If f extends to $\overline{\overline{f}}$, then this is a pretty extension that gives the class $[\emptyset]$.

Sufficiency. Consider a pretty extension $g:K^{[n+1]}\cup A\to C$ of f that yields a cocycle $S\in E\mathcal{J}^{n}(K,A)$ and suppose that S' is obtained from S by a (relative) Operation r. We need to see that there is a pretty extension g' that yields S'. We use non-singular theory here.

Case $r=1$.

Use the relative version of 7.5.

Case $r=2$.

Use the homotopy extension and a collar structure on the boundary of S (7.1).

Case $r=3$. Use 4.1.

Case $r\geq 4$. Use 6.6. Note that $U\cap A=\emptyset$.

Corollary 16.4.

Suppose that $C\setminus 0$ retracts to Y. Then $\mu_{f,C,\underline{\xi}}=0$ is a necessary and sufficient condition for $f:A\to Y$ to be extendable to $f:K^{[n+1]}\cup A\to Y$.

CHAPTER 17

THE EQUIVARIANT THEORY AND APPLICATION TO EMBEDDING THEORY

We now discuss the "equivariant" enhanced cohomology. The development essentially parallels the ordinary case and we will only discuss nontrivial points.

Let K be a finite, regular CW-complex. Let $E:K\to K$ be a PL, cellular, fixed point free involution ($E^2=\mathrm{Id}$). It then follows that E maps each cell homeomorphically onto some other cell.

Consider two admissible objects in K: C_i, $i=1,2$, with cores l_i, neighborhoods N_i, and maps $h_i:\mathrm{fr}N_i\to S^{n-1}$ ($n>0$, $n\neq 2$). We will say that C_1, C_2 form an equivariant pair if $E(N_1,l_1)=(N_2,l_2)$, and the diagram

$$\begin{tikzcd} \mathrm{bd}\, N_1 \arrow[r, "E"'] \arrow[d] & \mathrm{bd}\, N_2 \arrow[d] \\ S^{n-1} \arrow[r, "\text{antipodal}", "\text{map}"'] & S^{n-1} \end{tikzcd}$$

commutes.

A cochain from $\mathscr{C}^n(K)$ is called equivariant if each of its objects forms an equivariant pair with exactly one other object. The subgroup of equivariant enhanced cochains is denoted $\mathscr{C}^n(K;E)$. The equivariant cocycles and Operations are defined and the equivariant cohomology groups $\mathrm{E}H^n(K;E)$, $n\geq 4$, arise.

A new CW-structure with respect to which E remains cellular will be called *equivariant*. We observe that there is a triangulation of K under which E is a simplicial isomorphism. This is used in the proof to define $\overline{T}$ on $K^{[n-1]}$. The covering isotopy of 7.5 should commute with E. 7.6 is not changed.

Throughout Chapter 9, E is assumed to be a simplicial isomorphism and n-central sets should be preserved under E. What follows is that special neighborhoods are preserved under E. In 9.2, both K and L have the simplicial involutions, and f should commute with them. In 9.3, H should commute with E. An equivariant central n-cochain is one whose neighborhood and core is preserved under E.

M, N_M, h_M in 10.1 are "equivariant". 10.2 gives an equivariant triangulation.

Throughout the discussion of pull-backs and functoriality, the maps between CW-complexes should commute with their involutions, or "be equivariant"; likewise, the homotopies should be equivariant. We note that the equivariant versions of absolute and relative simplicial and cellular approximation theorems (for example [Z]) are true. The proofs use the ordinary versions and are inductive on the dimension of a skeleton.

In the proof of 10.7, we used B.H.E.T. many times to extend homotopies. In the equivariant version, we always extend in steps, inductively on the skeleta of K. At each step (involving a pair of cells) we use B.H.E.T. The general comment on the proof of 10.7 is that it's "local"; this is the primary reason why the equivariant version goes through.

During Part 1 of the proof of 11.1, all the activity in $\mathbb{R}^{m+1}$ is equivariant, that is, as we build and modify W, a suitable level-preserving involution of W is carried along. Then T^p and T are equivariant. The small moves of vertices to force Conditions 1 and 2 deal with a pair of vertices at a time, and move them to the same level.

The additions to the summary of part 1 are as follows: W is equipped with a level-preserving fixed point free simplicial involution. T, c, U, h_U are equivariant. The points of $(c \cap X^{[n]}) \cap (\mathbb{R}^m \times (0,1))$ are paired; each pair lies on one level, all the pairs lie on different levels.

After this, part 2 translates easily to the equivariant setting. Most of part 2 is "local"; we now work in two places at once.

With an equivariant map $f:K \to L$, we associate a well-defined homomorphism $f^*:\mathrm{E}H^n(L;E_L) \to \mathrm{E}H^n(K;E_K)$, $n \geq 4$.

If $f \simeq g$ through equivariant maps, then $f^* = g^*$.

We stop the development of the equivariant theory after Chapter 12.

The equivariant enhanced cohomology theory gives rise to an obstruction to embedding an m-dimensional simplicial complex in $\mathbb{R}^{2m-1}$, $m \geq 5$, as follows.

Let K be a finite, regular CW-complex. Let K^* be the subcomplex of $K \times K$ composed of the products of disjoint cells, and consider the involution $T: K^* \to K^*$: $T(x,y)=(y,x)$. We will consider maps $F: K^* \to \mathbb{R}^n$ that commute with T and the map: $x \mapsto -x$ of $\mathbb{R}^n$. They will be called equivariant. A *pretty equivariant map* $F: K^* \to \mathbb{R}^n$, $n \geq 4$, is one for which there exists an equivariant enhanced n-cochain C in K^* with the core c, the neighborhood N, and the map $h: \mathrm{fr}N \to S^{n-1}$, such that $c = F^{-1}(0)$ and $h = rF|_{\mathrm{fr}N}$, where $r: \mathbb{R}^n \setminus 0 \to S^{n-1}$ is the radial retraction.

We observe that pretty equivariant maps exist and that C has to be a cocycle (the proof of this last observation is similar to a part of the proof of 16.2).

Theorem 17.1.

The class $\mu^n_K = [C] \in EH^n(K^*;T)$ is independent on the choice of a pretty equivariant map F and the cochain C in the definition of a pretty equivariant map.

Proof.

In principle, this is similar to the proof of 16.2. The details are left to the reader.

Corollary 17.2.

If K is embeddable in $\mathbb{R}^n$, then $\mu^n_K = 0$.

Proof.

Let f be an embedding and consider the pretty equivariant map $F(x,y) = f(x) - f(y)$.

Theorem 17.3.

Let $m \geq 5$ and suppose that K is an m-dimensional simplicial complex. Then K embeds in $\mathbb{R}^{2m-1}$ if and only if

$$\mu_K^{2m-1} = 0 \in EH^{2m-1}(K^*; T).$$

Proof.

"Only if" is proven above.

To prove "if", it is enough to find an equivariant map

$$F: K \times K \setminus \Delta \to S^{n-1}$$

(see [We]), but since $K \times K \setminus \Delta$ equivariant retracts to K^*, we only need to find an equivariant map

$$F: K^* \to \mathbb{R}^{2m-1} \setminus 0.$$

We start from a pretty equivariant map and proceed as in the proof of 16.3. We use equivariant versions of 7.5, 7.1, 4.1, and 6.6.

Remark.

We also have that K embeds in $\mathbb{R}^{2m}$ if and only if $\mu_K^{2m} = 0$; but $EH^{2m}(K^*; T) \cong \overline{H}^{2m}(K^*)$, using the notation of [Sh], and our obstruction coincides with the Shapiro obstruction.

As for the lower dimensions ($n \leq 2m-1$), our obstruction is not weaker that Shapiro's. For if (our) $\mu_K^n = 0$ then it is easy to see that the Shapiro obstruction is 0.

CHAPTER 18

THE ALGORITHM TO COMPUTE ENHANCED COHOMOLOGY

The enhanced cohomology groups can be computed by a finite algorithm. The key to this lies in the fact that the semigroup of cochains is, up to isotopy and homotopy moves, finitely generated, and the generators can be readily identified.

Let us for example look at objects of Type 1 in an $n+1$-cell between given two faces. They can be all isotopy-moved to one position, and then all that matters is the homotopy type of the map on the frontier of the regular neighborhood, relative "the ends". There are two classes here for degree 1 (and two for degree -1): take anything as one and, roughly, follow it by γ for the second (3.3). We can then see that every pair of different faces of one $n+1$-cell contributes $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to the cochains; similarly a face of an $n+1$-cell contributes $\mathbb{Z}_2$ (Type 2), any $n+1$-cell contributes $\mathbb{Z}_2$ (Type 3), and any free n-cell contributes $\mathbb{Z}$. Then one only needs to identify the effect of link moves (in terms of the generators) and the effect of the saddle move for each suitable pair of generators.

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APPENDIX: INDEX

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VITA

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