To the Graduate Council:

I am submitting herewith a thesis written by Ruth Ann Lewis entitled "Introduction to Nonlinear Differential Equations." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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INTRODUCTION TO NONLINEAR DIFFERENTIAL EQUATIONS

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Degree

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Ruth Ann Lewis

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ABSTRACT

Frequently students are exposed to the general theory of nonlinear differential equations only at the graduate level. The purpose of this thesis is to provide a text on this topic to be used on the undergraduate level. It presupposes a knowledge of elementary calculus and differential equations, and a rudimentary knowledge of Fortran is needed for Chapter VII. A few important theorems are stated without proof because it is felt that the student should become aware of such important results, even though the proofs are beyond the scope of this text.

The first chapter defines nonlinear differential equations and presents several physical problems modeled by such equations. Various graphical methods of solution are outlined in Chapter II. Chapter III is devoted to phase plane analysis for first-order systems. A thorough treatment of the concepts of critical points and stability, both for linear and nonlinear autonomous systems, is to be found in Chapter IV.

The existence of periodic solutions is discussed in Chapter V. A graphical procedure for plotting the trajectories of a particular first-order system is given. The Liénard equation and the notion of limit cycles are also discussed. Chapter VI discusses both perturbation and averaging methods for finding approximate solutions of a particular second-order equation. Chapter VII is intended only as a brief introduction to numerical methods of solution of initial-value problems. A digital computer is used both to find numerical solutions and to graphically output this data.

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CHAPTER I

INTRODUCTION

1.1. Nonlinear Differential Equations Defined. A common occurrence in nature is that a certain desired function x (either real- or vector-valued) is known to satisfy a differential equation

$$x'(t) = f[t,x(t)] \quad . \tag{1.1}$$

The function f is determined by the physical laws connected with the function x . Equation (1.1) states that the rate of change of x at time t depends only upon x and the time t . It is desired to determine x for all time t by knowing the initial value x_0 at some initial time t_0 .

A differential operator L (such as $L(x) = x'(t) - 3x(t)$ or $L(x) = 2e^{4x(t)} + x''(t)$) is said to be linear if

$$L(c_1x_1 + c_2x_2) = c_1L(x_1) + c_2L(x_2)$$

for each pair of constants c_1 and c_2 . Clearly if L is linear and x_1 and x_2 are solutions of $L(x) = 0$, then $c_1x_1 + c_2x_2$ is also a solution. This property is usually referred to as the principle of superposition.

If L is linear we refer to the equation $L(x) = 0$ as a linear homogeneous equation and to the equation $L(x) = g(t)$, where g is a given nonzero function, as a linear nonhomogeneous equation. If L is not linear, we refer to the equation $L(x) = 0$ as a nonlinear equation, e.g., $\ddot{x}'(t) + [x(t)]^2 = 0$.

Nonlinear differential equations arise frequently when physical problems are described by mathematical models. Nonlinearities occur, for example, in the study of astronomy, electricity, sound, hydrodynamics, aerodynamics, biology, and ecology. Despite the widespread applicability, the general theory of nonlinear differential equations is not nearly so highly developed as that of linear equations. In some problems a nonlinear system can be approximated by a corresponding linear one using a process called quasilinearization; the behavior of solutions of the nonlinear system can then be inferred from those of the linear system. In other problems the behavior of the system is an intrinsic property of the nonlinearity and is lost in a linear approximation.

To illustrate how nonlinearization may arise in physical problems and how such problems differ from linear ones, several mathematical models of physical problems are considered in the remaining sections of this chapter.

1.2. Vibration of a Simple Pendulum. As a simple example, consider the angular displacement of the vertical pendulum shown in Figure 1. This pendulum consists of a bob of mass m suspended from

a fixed point O by a straight wire of length ℓ and negligible mass.

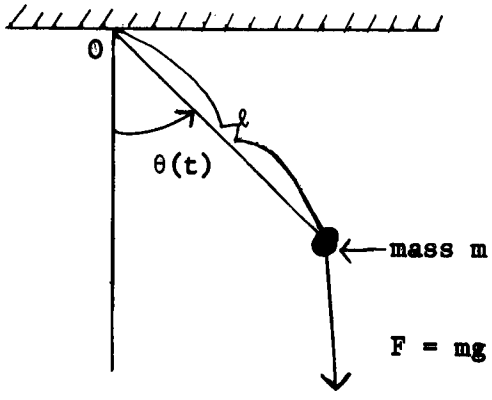


Figure 1. Simple pendulum.

It can be shown that the angular displacement θ of the pendulum from its position of equilibrium satisfies the second-order nonlinear differential equation

$$\theta'' + g/\ell \sin \theta = 0 , \quad (' = d/dt) , \tag{1.2}$$

where g is the acceleration due to gravity and θ is positive when measured counterclockwise ([16], p. 143). The general solution of Equation (1.2) in the form of θ as a function of t cannot be obtained in terms of the usual elementary functions; it is obtainable

only by using elliptic functions.

Recalling that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

for small angular displacements θ, $\sin\theta$ can be approximated by θ. Thus for θ sufficiently small, consider the linear differential equation

(1.3) $$\theta'' + g/\ell\,\theta = 0 \ , \ (' = d/dt) \ ,$$

whose characteristic equation

$$\lambda^2 + g/\ell = 0$$

has roots $\lambda = \pm \sqrt{g/\ell}\, i$. The general solution of Equation (1.3) can be written

(1.4) $$\theta(t) = c_1 \sin \sqrt{g/\ell}\, t + c_2 \cos \sqrt{g/\ell}\, t \ ,$$

where c_1 and c_2 are arbitrary constants. Equation (1.4) exemplifies the fact that the linear equation (1.3) satisfies the principle of superposition. Notice also that Equation (1.4) can be rewritten in the form

(1.5) $$\theta(t) = c \sin (\sqrt{g/\ell}\, t + \phi) ,$$

where $c = \sqrt{c_1^2 + c_2^2}$ and $\tan \phi = c_2/c_1$. From Equation (1.5) it is easily seen that the differential equation (1.3) describes simple harmonic motion.

A. PROBLEM LIST 1

1. For the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$, give the general solution of Equation (1.3) and find the amplitude and period of this solution. Note that the period of oscillation is independent of the initial angular displacement.

1.3. Falling Body Problems. Newton's second law states that the time rate of change of momentum of a body is proportional to the resultant force acting on the body. Thus the resultant force F on a body of mass m with velocity v is given by

(1.6) $$\frac{d}{dt}(mv) = k F .$$

There are several systems of units in which $k = 1$. Two commonly used ones are given below along with the units of measurement used by each.

TABLE I

CONSISTENT FORCE UNITS

	Force	Mass	Distance	Time	Acceleration
British	pounds	slugs	feet	seconds	ft/sec^2
cgs	dynes	grams	centimeters	seconds	cm/sec^2

As an application of Equation (1.6), consider an object falling in a vacuum near the surface of the earth. Thus the acceleration due to gravity may be considered constant. It follows from Equation (1.6) that

$$m \frac{dv}{dt} = F \quad \text{or} \quad mg = F ,$$

where g is the gravitational acceleration and F is the weight of the object. Now g is approximately equal to 32 ft/sec^2 in the British system and 980 cm/sec^2 in the cgs system. Hence to compute the mass of an object in slugs, divide its weight in pounds by 32.

As a second application of Equation (1.6), consider an object of mass m falling through the atmosphere near the surface of the earth with air resistance considered. This resistive force is commonly considered to be either

(1) proportional to the velocity, or

(2) proportional to the square of the velocity.

Suppose that the object starts its fall from a height h and that $x(t)$ is the distance fallen in t seconds. Since $v(t) = x'(t)$, Equation (1.6) becomes

$$mv' = gm - cv \ , \quad (' = d/dt) \ , \tag{1.7}$$

for Assumption (1) and

$$mv' = gm - dv^2 \ , \quad (' = d/dt) \ , \tag{1.8}$$

for Assumption (2). The constants c and d would be calculated by physical measurements. Note that Equation (1.7) is linear, while Equation (1.8) is nonlinear. The equation which more accurately represents the actual situation can only be determined by experiment.

Now consider Equations (1.7) and (1.8) in the British system. Dividing both sides of Equation (1.7) by m, we obtain the equation

$$v' = 32 - \alpha v \ , \quad \alpha = c/m \ . \tag{1.9}$$

Solving Equation (1.9) with the initial condition $v(0) = 0$ gives

$$v(t) = 32[1 - e^{-\alpha t}]/\alpha \ . \tag{1.10}$$

Similarly we obtain the equation

$$v' = 32 - \beta v^2 \ , \ \beta = d/m \ , \tag{1.11}$$

from Equation (1.8) for Assumption (2). To solve this equation, it

is convenient to use the method of partial fractions. A calculation shows that

$$(1.12)\quad 1 = \frac{v'}{32 - \beta v^2} = \frac{1}{2\sqrt{32\beta}}\left[\frac{1}{\sqrt{32/\beta} - v} + \frac{1}{\sqrt{32/\beta} + v}\right]v' .$$

Integration of Equation (1.12) yields

$$(1.13)\quad t + K = \frac{1}{2\sqrt{32\beta}} \ln\left|\frac{\sqrt{32/\beta} + v(t)}{\sqrt{32/\beta} - v(t)}\right| ,$$

where K is a constant of integration. The initial condition $v(0) = 0$ implies $K = 0$, and solving Equation (1.13) for $v(t)$ yields

$$(1.14)\quad v(t) = \sqrt{32/\beta} \cdot \frac{1 - e^{-2\sqrt{32\beta}\, t}}{1 + e^{-2\sqrt{32\beta}\, t}} .$$

One can observe from both Equation (1.10) and (1.14) that as t increases a terminal velocity is approached. The problems below give a further comparison of the two equations. We shall see in other physical problems that the linear and nonlinear models give very different results.

B. PROBLEM LIST 2

1. Solve Equations (1.9) and (1.11) given $\alpha = 1/2$, $\beta = 1/128$, and $v(0) = 0$. Sketch the graph of the solution in both cases

within the same rectangular coordinate system by plotting $v(t)$ for $t = 0, 1, 2, 4, 6$.

2. Find $x(t) = \int_0^t v(s)ds$ in each of the two cases in Problem 1. Sketch the graphs of the two functions obtained in the same rectangular coordinate system by plotting $x(t)$ for $t = 0, 2, 5, 10$.

3. A man weighing 160 pounds starts free fall at 8000 feet. The air resistance (in pounds) to his fall is $1.2v$ where v is his velocity (ft/sec). Find his velocity and height at time t. Estimate (to within 1/2 second) the time required for him to fall to a height of 1000 feet.

4. A sled weighing 196 pounds is pushed across a level field of snow by a rocket with a thrust of 800 pounds. If the resistive force acting on the sled is $v^4/2$ pounds, where v is the velocity in ft/sec, then find the differential equation which v satisfies. What is the limiting velocity of the sled?

1.4. A Predator-Prey Model. As was mentioned earlier, nonlinear equations often arise in biological and ecological problems. The Lotka-Volterra model of the growth of two conflicting populations may be studied either in the predator-prey form or the host-parasite form; since the analyses are similar, only the predator-prey model is considered here.

Suppose x and y measure the number of individuals in a prey population X and a corresponding predator population Y, respectively. If x is large, then in the presence of so much prey, the predator population will flourish and y will increase. But as y increases

the number of prey will decrease; that is, x will decrease. Starvation of the predator population will then ensue, causing y to decrease. However, as the predator population declines the prey population begins to increase. Thus a cyclic pattern emerges.

Before a mathematical model of this problem can be constructed, certain basic assumptions need to be made. We will assume:

1. The mass of the food supply of X is so large during the period of time under consideration that X is never limited by its food supply.
2. In the absence of X, Y becomes extinct.
3. The rate of consumption of X is proportional to the probability that the members of the two populations encounter. Moreover, this probability depends on the product of x and y.
4. The rate of increase in y due to x, the amount of the food supply, is proportional to x.

The mathematical model which describes this problem and satisfies the four assumptions is the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy = x(a - by) \\ \frac{dy}{dt} &= -cy + dxy = y(-c + dx) \end{aligned} \tag{1.15}$$

where a, b, c, and d are positive proportionality constants of the system. Observe that $dx/dt = 0$ when and only when $x = 0$ or $y = a/b$, and $dy/dt = 0$ when and only when $y = 0$ or $x = c/d$.

Since x and y are always nonnegative, the solution curves of System (1.15) will always lie in the first quadrant of the x,y-plane. The lines $x = c/d$ and $y = a/b$ divide this quadrant into four regions; the direction of the solution curves in each region may be obtained by observing the values of dx/dt and dy/dt in that region. Such an analysis has been made in Figure 2.

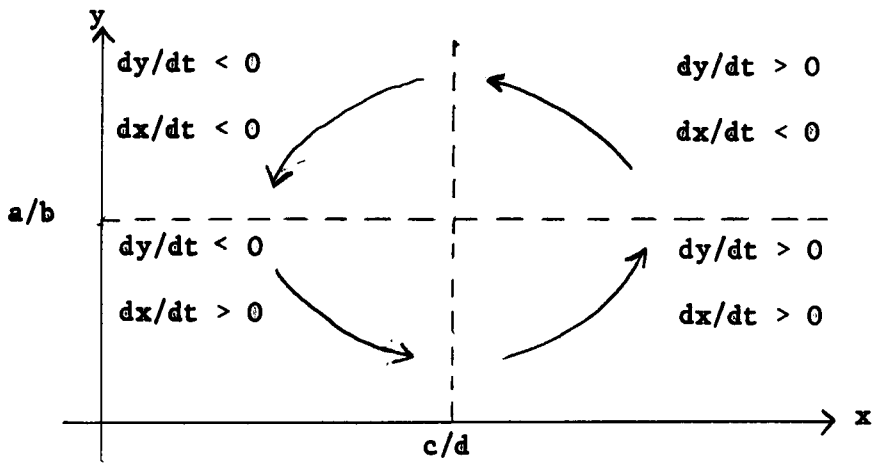


Figure 2. Direction of the solution curves of the Lotka-Volterra equations in each region.

Since the right-hand sides of System (1.15) are independent of t, we may write

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)} \quad , \tag{1.16}$$

from which the equation

(1.17) $dy(a/y - b) + dx(c/x - d) = 0$

may be obtained. Integration of Equation (1.17) gives

(1.18) $\ell n(y^a x^c K) = by + dx$,

where ℓnK is a constant of integration determined by the initial conditions. Notice that Equation (1.18) can be rewritten in the form

(1.19) $$\frac{1/K\, e^{dx}}{x^c} = e^{-by}\, y^a ,$$

which will be useful in Section 2.4, where a graphical procedure for plotting y as a function of x will be given.

C. PROBLEM LIST 3

1. Verify that each of the four assumptions are satisfied by System (1.15).
2. Algebraically derive Equations (1.16) - (1.19).

1.5. Neutron Transport and the Riccati Equation. In Section 1.3 a particular solution of the nonlinear differential equation

(1.11) $v'(t) = 32 - \beta v^2$

was found. Equation (1.11) belongs to the class of differential

equations of the form

$$(1.20)\quad y'(x) + q(x)y(x) + r(x)[y(x)]^2 = p(x)$$

known as Riccati equations. If y is a solution of Equation (1.20) and $u(x) = C \exp\left[\int r(x)y(x)dx\right]$ for some nonzero constant C, then u is a solution of the second-order linear homogeneous equation

$$(1.21)\quad ru'' - (r' - qr)u' - pr^2u = 0\ ,\ (' = d/dx)\ .$$

(See Problem 1.)

On the other hand, suppose u is a nonzero solution on an interval I of the second-order linear homogeneous equation

$$(1.22)\quad a(x)u''(x) + b(x)u'(x) + c(x)u(x) = 0\ .$$

Let $y(x) = \dfrac{u'(x)}{r(x)u(x)}$ for some nonzero function r. You are asked to verify in Problem 2 that y satisfies the Riccati equation

$$(1.23)\quad y' + \left[\frac{r'}{r} + \frac{b}{a}\right]y + ry^2 = -\frac{c}{ar}\ ,\ (' = d/dx)\ .$$

Riccati equations are important for reasons other than their intimate relationships with second-order linear equations. They arise in numerous applications and have been studied extensively. Although no method exists which always produces the general solution

of a Riccati equation, the solution can be obtained if any particular solution is known.

Suppose y_1 is a particular solution of Equation (1.20). The substitution of $y = y_1 + 1/v$ into Equation (1.20) yields

$$(1.24) \quad v' - (q + 2ry_1)v - r = 0 ,$$

since y_1 satisfies Equation (1.20). Equation (1.24) can then be solved using elementary methods.

EXAMPLE 1.1. Solve the Riccati equation

$$(1.25) \quad y' - 2xy - y^2 = x^2 - 2 , \quad (' = d/dx) ,$$

given that a particular solution is $y_1(x) = 1 - x$.

SOLUTION. In this case $q(x) = -2x$, $r(x) = -1$, and $p(x) = x^2 - 2$. Substituting into Equation (1.24), we obtain the first-order linear nonhomogeneous equation

$$(1.26) \quad v' + 2v = -1 .$$

Multiplying both sides of Equation (1.26) by the integrating factor e^{2x} yields $(e^{2x}v)' = -e^{2x}$; after integrating both sides, the general solution of Equation (1.26) is found to be

(1.27) $v(x) = ce^{-2x} - 1/2$

for an arbitrary constant c . Hence the general solution of Equation (1.25) is

(1.28) $y(x) = y_1(x) + \frac{1}{v(x)} = 1 - x + \frac{2e^{2x}}{c - e^{2x}}$.

The Riccati equation arises in the study of neutron transport. Think of a closed interval [0,x] as a material rod capable of transporting particles. These particles may collide with the fixed nucleii of the medium, the probabilities of such collisions and their expected results being specified. At present we make the following two assumptions:

1. Each time a particle suffers a collision, two particles are produced - one moving in the same direction as the original particle and another moving in the opposite direction. (The original particle may be one of these.)
2. While traversing a short rod of length Δ , a particle has a probability of $\sigma\Delta + \varepsilon(\Delta)$ of suffering a collision, where σ is a constant and ε is a function with the property that $\frac{\varepsilon(\Delta)}{\Delta} \to 0$ as $\Delta \to 0$.

Thus if the right-hand end of the rod is bombarded with a left-moving flux of c particles/sec , then a flux, say d particles/sec , will be observed to exit from the right-hand end moving toward the right. This flux is a result of collisions in the interior of

the rod. We denote the fraction d/c by R and write R(x) to show dependence on the length of the rod. R is called the reflection function and d = c R(x) , the reflected flux. It is the function R that we wish to determine.

Consider a rod of length $x + \Delta$ for Δ "small" and an incident, left-moving flux of 1 particle/sec.

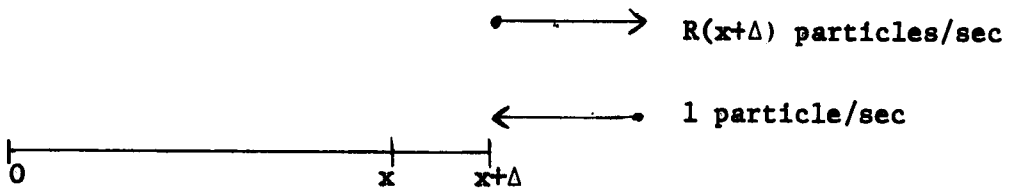


Figure 3. The reflected flux when c = 1 particle/sec.

Of the 1 particle/sec entering at $x + \Delta$, the expected number having collisions in the subrod $[x, x + \Delta]$ is $\sigma\Delta + \varepsilon(\Delta)$. Each collision produces a left-moving particle and a right-moving particle. Thus of the initial 1 particle/sec , we have from collisions in the subrod, $[x, x + \Delta]$, $\sigma\Delta + \varepsilon(\Delta)$ particles/sec moving to the right and $1 = [1 - (\sigma\Delta + \varepsilon(\Delta))] + (\sigma\Delta + \varepsilon(\Delta))$ particles/sec arriving at x and moving to the left. Of these left-moving particles we have R(x) reflected back to the right. Of these right-moving particles, $R(x)[\sigma\Delta + \varepsilon(\Delta)]$ is the expected number having collisions. Thus the R(x) right-moving particles produce $R(x)[\sigma\Delta + \varepsilon(\Delta)]$ left-moving particles and $R(x) = R(x)[1 - (\sigma\Delta + \varepsilon(\Delta))] + R(x)(\sigma\Delta + \varepsilon(\Delta))$

right-moving particles. Of the $R(x)[\sigma\Delta + \varepsilon(\Delta)]$ left-moving particles arriving at x, $[R(x)]^2[\sigma\Delta + \varepsilon(\Delta)]$ are reflected back to the right. (We neglect the possibility of two collisions in a rod of length Δ.) Continuing this argument gives

$$R(x + \Delta) = [\sigma\Delta + \varepsilon(\Delta)] + R(x) \tag{1.29}$$
$$+ [R(x)]^2 [\sigma\Delta + \varepsilon(\Delta)] + [R(x)]^3 [\sigma\Delta + \varepsilon(\Delta)]^2 + \ldots .$$

Hence

$$\frac{R(x + \Delta) - R(x)}{\Delta} = \sigma + \frac{\varepsilon(\Delta)}{\Delta} + [R(x)]^2 [\sigma + \frac{\varepsilon(\Delta)}{\Delta}] \tag{1.30}$$
$$+ \Delta[R(x)]^3 [\sigma + \frac{\varepsilon(\Delta)}{\Delta}]^2 + \ldots .$$

Letting $\Delta \to 0$ yields

$$R'(x) = \sigma + [R(x)]^2\sigma = \sigma[1 + [R(x)]^2] . \tag{1.31}$$

$R(0) = 0$ since a rod of length 0 reflects no flux. Equation (1.31) with this initial condition has solution

$$R(x) = \tan \sigma x . \tag{1.32}$$

Thus as $x \to \frac{\pi}{2\sigma}$, the expected number of right-moving particles (from an incident flux of only 1 particle/sec) increases without

bound. For this reason the length $\frac{\pi}{2\sigma}$ is called the critical length.

A more complicated model may be considered by changing the two assumptions to:

1'. If a particle suffers a collision at x, then $F(x)$ particles are produced moving in the same direction as the original particle, and $B(x)$ particles are produced moving in the opposite direction.

2'. While traversing a short rod of length Δ centered at x, a particle has a probability of $\sigma(x)\Delta + \varepsilon(\Delta)$ of suffering a collision where $\frac{\varepsilon(\Delta)}{\Delta} \to 0$ as $\Delta \to 0$.

Reasoning as above leads to the equation

$$(1.33)\quad R'(x) = B(x)\sigma(x)+2\sigma(x)[F(x)-1]R(x)+\sigma(x)B(x)[R(x)]^2,\ R(0) = 0 .$$

D. PROBLEM LIST 4

1. Show that if y is a solution of Equation (1.20) and $u(x) = C \exp [\int r(x)y(x)dx]$ for some nonzero constant C, then u is a solution of Equation (1.21).

2. If u is a nonzero solution of Equation (1.22) on some interval I, show that for any nonzero function r on I $y(x) = \frac{u'(x)}{r(x)u(x)}$ satisfies Equation (1.23).

3. Show that if y is a solution of Equation (1.20), then $z = 1/y$ is a solution of $z' + p(x)z^2 - q(x)z = r(x)$.

4. Transform each of the following equations into a Riccati equation:

   a. $xu'' + u' + xu = 0$;

   b. $xu'' + (3x^2 - 1)u' + 2x^3u = 0$;

   c. $(1 - x^2)u'' - 2xu' + 6u = 0$;

   d. $u'' - u' - e^{2x}u = 0$;

   e. $x^2u'' + xu' - u = 0$.

5. Transform the following Riccati equations into second-order linear equations:

   a. $y' - y^2 = 1 - x^2$;

   b. $y' = x^2y - xy^2 + 1$;

   c. $xy' = 1 + y + xy^2$;

   d. $x^2y' + 1 = y^2$.

6. Use the particular solution $y_1(x) = 1$ to solve the equation $y' = x - 1 + y - xy^2$.

7. Solve the equation $y' - x^2y' = 1 - y^2$ using the particular solution:

   a. $y_1 = 1$;

   b. $y_1 = -1$;

   c. $y_1(x) = x$.

8. Solve Equation (1.33) and find the critical length for each of the following cases:

   a. $B(x) = F(x) = 1$, $\sigma(x) = 2x$;

   b. $B(x) = 2$, $F(x) = 1$, $\sigma = 0.1$;

   c. $B(x) = 1$, $F(x) = 2$, $\sigma = 0.1$;

   d. $B(x) = F(x) = 2$, $\sigma = 0.1$.

9. Consider the subclass of Riccati equations of the form (1.20) where $r(x) = 0$; that is, suppose

(1.34) $y'(x) + q(x)y(x) = p(x)$.

a. Show that the general solution of Equation (1.34) is
$y(x) = y(0) \exp\left[-\int_0^x q(\tau)d\tau\right] + \int_0^x \exp\left[-\int_s^x q(\tau)d\tau\right]p(s)ds$.

b. If $p(x) = 0$ and $\int_0^\infty |q(x)|dx < \infty$, show that the general solution of Equation (1.34) is bounded on $0 \leq x < \infty$.

c. If $|p(x)| \leq M$ for $0 \leq x < \infty$ and $q(x) \geq C > 0$ for some constants C and M , show that the general solution of Equation (1.34) is bounded.

d. If $p(x) = 0$ and $\int_0^x q(\tau)d\tau \to 0$ as $x \to \infty$, show that the general solution y of Equation (1.34) approaches 0 as $x \to \infty$.

Several examples of nonlinear differential equations are given in Spain [18] and Aggarwal [1]. For more detailed discussions of the predator-prey model, see Lotka [11] or Davis [7]. Also see Davis [7] for a more extensive study of Riccati equations. Further discussion of the neutron transport problem is to be found in Wing [19].

CHAPTER II

GRAPHICAL METHODS OF SOLUTION

2.1. Line Elements and Direction Fields. An initial-value problem for a first-order differential equation is usually formulated as

$$(2.1)\quad y'(x) = f(x,y(x)),\ y(x_0) = y_0\ ,$$

where x_0 and y_0 are given real numbers and f is a given function which may be linear or nonlinear. It can be proved, under very general conditions on f , that on some interval I containing x_0 Equation (2.1) possesses a unique solution (15, pp. 375-376). This is illustrated in Figure 4.

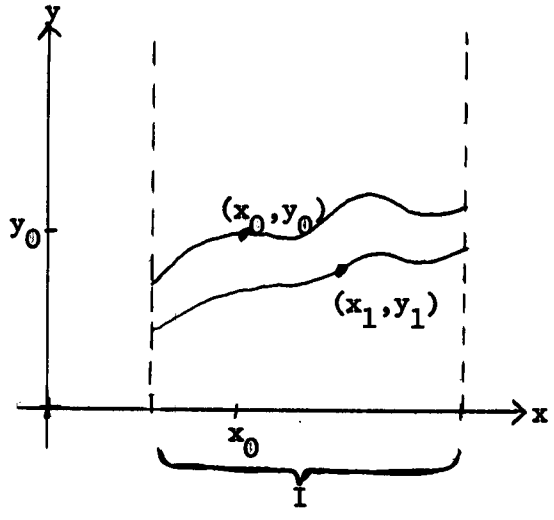


Figure 4. A solution of the initial-value problem (2.1) exists and is unique.

If we choose a point (x_1,y_1) with x_1 in I and (x_1,y_1) not on the graph through (x_0,y_0), then we have a different solution of

$$(2.2) \qquad y'(x) = f(x,y(x))$$

satisfying $y(x_1) = y_1$. In the case of the linear equation

$$(2.3) \qquad y'(x) + p(x)y(x) = F(x) ,$$

I can be taken as any interval on which both p and F are defined. (This may be the whole real line.) In the nonlinear case, the interval I must sometimes be restricted. For example, the initial-value problem

$$(2.4) \qquad y'(x) = 1 + [y(x)]^2 , \; y(0) = 0$$

has the solution $y(x) = \tan x$; the graph of this function has a vertical asymptote at $x = \pi/2$.

The graphs of the solutions of Equation (2.2) in the x,y-plane are called the integral curves of the differential equation. At each point $(x,y(x))$ for which f is defined, Equation (2.2) gives the slope $f(x,y(x))$ of the integral curve passing through that point.

An explicit calculation for the function y in terms of elementary functions is generally not possible. However, in many physical problems a rough sketch of the graph of a solution may be all that

is needed. Since Equation (2.2) defines the slope at a given point $(x,y(x))$ of the integral curve passing through this point, we can construct the tangent to the integral curve at the given point without actually knowing the solution function of which this integral curve is the graph. A short line segment drawn through the point $(x,y(x))$ with slope, or direction, $f(x,y(x))$ is called a <u>line element</u>. The function f in Equation (2.2) is called the <u>direction field</u> associated with the differential equation.

The graphical method of solution of Equation (2.2) which involves line elements and direction fields may be outlined as follows. First choose a grid of points $P_1, P_2, \ldots, P_n$ at which f is defined and compute $f(P_i)$ for $i = 1, 2, \ldots, n$. Construct the line element for each point; that is, draw a short line segment of slope $f(P_i)$ with midpoint P_i for $i = 1, 2, \ldots, n$. This process is called obtaining a field of line elements. The solution of Equation (2.2) which passes through the point P_i will be tangent to its line element at that point. When a sufficient number of line elements have been constructed so that the integral curves may be approximated, draw smooth curves tangent to the line elements to produce approximate graphical solutions.

EXAMPLE 2.1. Construct a field of line elements for the first-order equation

$$y'(x) = 1/2\ y(x) \ . \tag{2.5}$$

SOLUTION. In this case, the direction field depends only upon $y(x)$. If a solution of Equation (2.5) crosses the horizontal line $y = 1$, then it must cross with slope 1/2; if it crosses the line $y = 2$, then it must do so with slope 1, etc. Several approximate graphical solutions of Equation (2.5) are sketched in Figure 5.

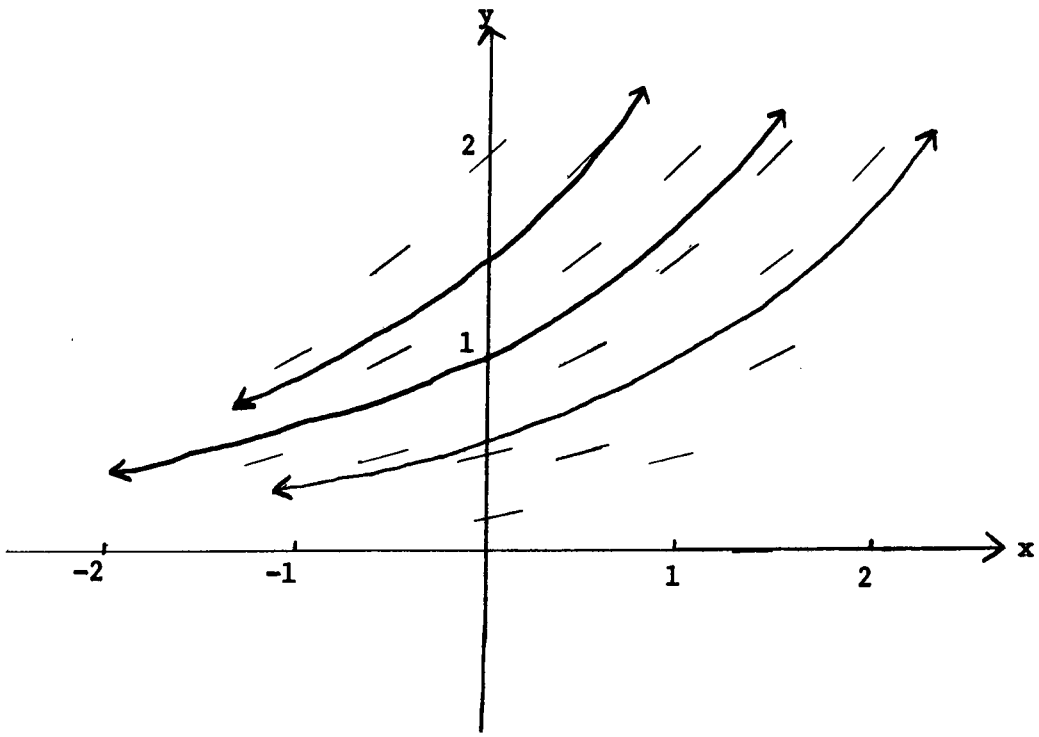


Figure 5. Approximate solutions of Equation (2.5) using a field of line elements.

While this graphical method of solution can be applied to any first-order equation of the form (2.2), it has several obvious disadvantages. The construction of line elements can be a tedious and time-consuming task. Furthermore, the integral curves produced are only approximations to the graphs of the exact solutions, and the accuracy of these approximate graphs is not known. Of course, a more detailed field of line elements will yield a better approximation, but this requires more time and labor. Although the speed at which approximate graphical solutions may be obtained by this method depends considerably upon the skill of the investigator, the process may sometimes be speeded up significantly using the method of isoclines, which will be considered in the next section.

A. PROBLEM LIST 5

1. Construct a field of line elements and sketch three approximate solutions of each of the following first-order equations:

   a. $y'(x) = 4x$;

   b. $y'(x) = 3x - y(x)$;

   c. $y'(x) = 2x + y(x)$;

   d. $y'(x) = 1/x$.

2. For the linear equation (2.3), the following simple technique can be used to sketch the tangent segments along the line $x = x_0$. Given x_0 such that $p(x_0) \neq 0$, let

$$P_0 = \left(x_0 + \frac{1}{p(x_0)} , \frac{F(x_0)}{p(x_0)}\right) .$$

For a given point (x_0,y_0) , use a straightedge to sketch through (x_0,y_0) a segment of the line joining (x_0,y_0) and P_0 .

a. Show that this is the line element through the point (x_0,y_0).

b. Using this procedure, construct the graph of the solution of the initial-value problem $y'(x) + x^2y(x) = 1$, $y(0) = 1$.

2.2. The Method of Isoclines. Again consider the first-order differential equation

$$y'(x) = f(x,y(x)) \quad (2.2)$$

An isocline of Equation (2.2) is a curve along which its direction field f is constant; that is, the isoclines of Equation (2.2) are the family of curves $f(x,y) = c$, where c is a parameter. This definition suggests another graphical method of solution of Equation (2.2). The procedure which will be explained below is usually referred to as the method of isoclines, but is sometimes called the method of equal slopes. First determine the family of isoclines, $f(x,y) = c$, and draw the graphs of several of these curves. When any integral curve crosses the isocline $f(x,y) = c$, it must have slope c at that point. Thus at all points (x,y) of this particular isocline, the line elements have slope c . At this stage, construct the field of line elements for each isocline already drawn. Draw smooth integral curves tangent to the line elements, thereby producing approximate graphical solutions of Equation (2.2). For the initial-value problem (2.1), it suffices to draw only the particular integral curve passing through the point (x_0,y_0) .

EXAMPLE 2.2. Use the method of isoclines to sketch several approximate integral curves for the equation

$$y'(x) = [y(x)]^2 - x \,. \tag{2.6}$$

SOLUTION. The family of isoclines are the curves $[y(x)]^2 - x = c$. In Figure 6 the isoclines for $c = 1, 0, -1, -2$, and -3 have been constructed. Then several line elements with the appropriate slope have been drawn on each isocline. Four approximate solutions of Equation (2.6) are sketched in Figure 6. Notice that y_2 increases without bound as x increases and that y_1 is asymptotic to the parabola $y^2 = x$.

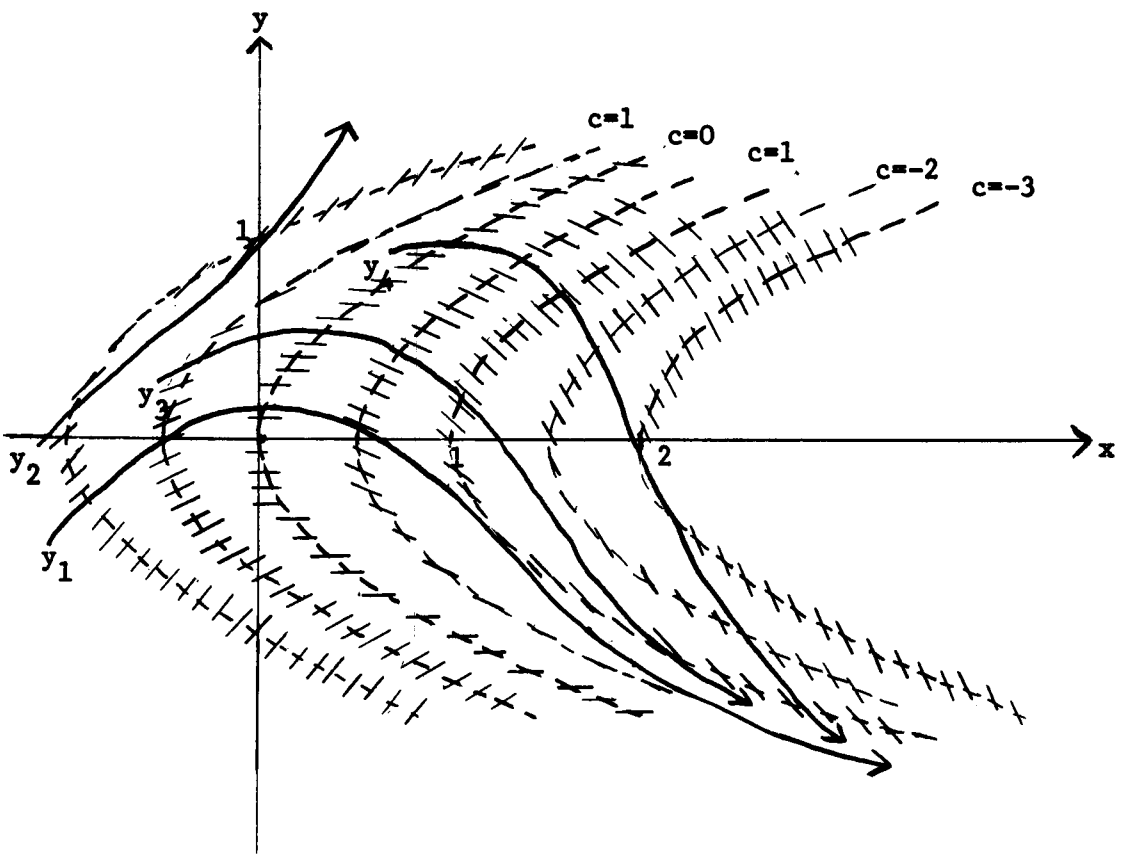


Figure 6. Approximate solutions of Equation (2.6) using the method of isoclines.

It must be stressed that, in general, the isoclines are not the solutions of Equation (2.2), although occasionally an isocline will also be an integral curve. In such a case, however, the isocline must be a straight line of slope c .

B. PROBLEM LIST 6

1. Use the method of isoclines to construct the integral curves of the initial-value problems given below:

   a. $y'(x) = x$ with $y(0) = 0$, $y(0) = 1$;

   b. $y'(x) = y(x) + x$ with $y(0) = 0$;

   c. $y'(x) = x^2 + [y(x)]^2$ with $y(0) = 0$, $y(0) = 1$, $y(0) = 2$;

   d. $y'(x) = x^2 - [y(x)]^2$ with $y(0) = 0$, $y(0) = -1$;

   e. $y'(x) = 1 - \sin [y(x)]$ with $y(0) = 0$.

2. Show that no solution of Problem 1.e has a vertical asymptote.

2.3. Single-Population Growth. In the remaining three sections of this chapter, several graphical techniques will be employed in the analyzation of applied problems in ecology. As an introduction to the more complex problems of growth of two conflicting populations, we first consider the problem of single-population growth.

Human populations, as well as those of crystals, certain plants and animals, the number of bacteria in a culture, the number of cells in the human body, and many other natural phenomena have been observed to grow in a characteristic pattern. The simplest pattern is that of Quetelet's uninhibited growth (named for

L. A. J. Quetelet (1796-1874), a famous Belgian statistician and astronomer): The rate of increase is proportional to the size of the population. If $N(t)$ denotes the size of the population at time t and k is the proportionality constant, then Quetelet's uninhibited growth equation is

$$N'(t) = k\,N(t)\ , \tag{2.7}$$

whose solution is easily seen to be

$$N(t) = N(0)e^{kt}\ . \tag{2.8}$$

Making the transformation $T = \frac{\ell n\ 2}{k}$, Equation (2.8) becomes

$$N(t) = N(0)2^{\frac{t}{T}}\ . \tag{2.9}$$

Equations (2.8) and (2.9) are frequently called the law of growth. Although this law may hold in the initial stages of population growth, it is obvious that it cannot hold indefinitely.

Ecologists frequently replace Quetelet's model by the assumption that the actual rate of increase equals the potential rate of increase times the degree of realization of potential. Since the degree of realization of potential is a function of the size of the population, this mathematical model may be written as

$$(2.10) \qquad N'(t) = k\,N(t)f(N(t)) \ .$$

For example, it might be observed that

$$(2.11) \qquad f(N(t)) = \frac{L - N(t)}{L} \ ,$$

where L is the maximum size of the population that can be supported. Substitution of Equation (2.11) into Equation (2.10) yields the Riccati equation

$$(2.12) \qquad N'(t) = kN(t) \cdot \frac{L - N(t)}{L} \ .$$

By the separation of variables and the use of partial fractions, the solution of Equation (2.12) can be shown to be

$$(2.13) \qquad N(t) = L\,\frac{Ce^{kt}}{1 + Ce^{kt}} \ ,$$

where C is a constant. The initial population size at time $t = 0$ is then $N(0) = L \cdot \frac{C}{1+C}$. In Figure 7 the solutions of Equation (2.12) are sketched for several values of C and k.

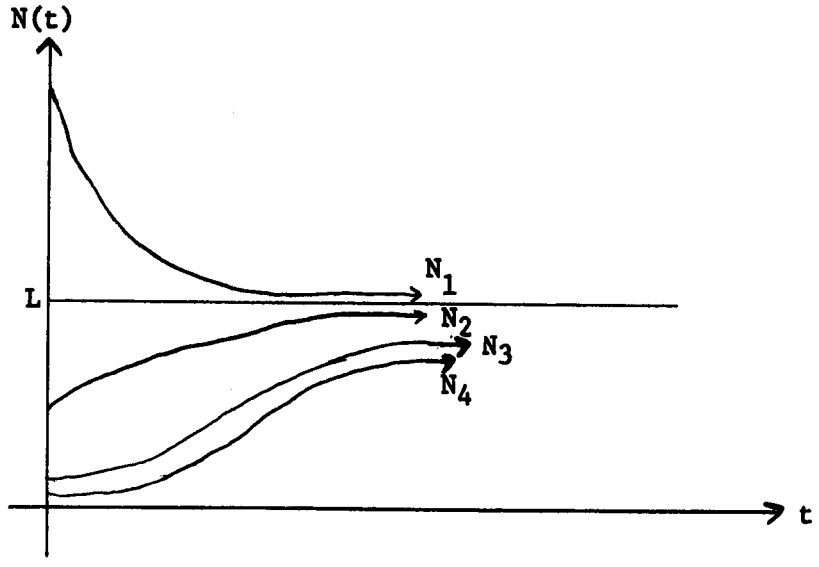


Figure 7. Approximate solutions of Equation (2.12).

From Equations (2.10) through (2.14) and Figure 7, the following noncomputational observations can be made:

1. $f(L) = 0$ implies that $N(t) = L$ is a constant solution of Equation (2.12);
2. $N(t) < L$, or $f(N(t)) > 0$, implies that N is increasing;
3. $N(t) > L$, or $f(N(t)) < 0$, implies that N is decreasing;
4. A solution curve $N(t) \neq L$ will never intersect the curve $N(t) = L$ by the uniqueness of solutions for such initial-value problems. (See Section 2.1)
5. Regardless of the value of $N(0)$, $N(t) \to L$ as $t \to \infty$. This is easily seen since Equation (2.13) can be rewritten as $N(t) = L \cdot \frac{C}{C + e^{-kt}}$ and $e^{-kt} \to 0$ as $t \to \infty$.

C. PROBLEM LIST 7

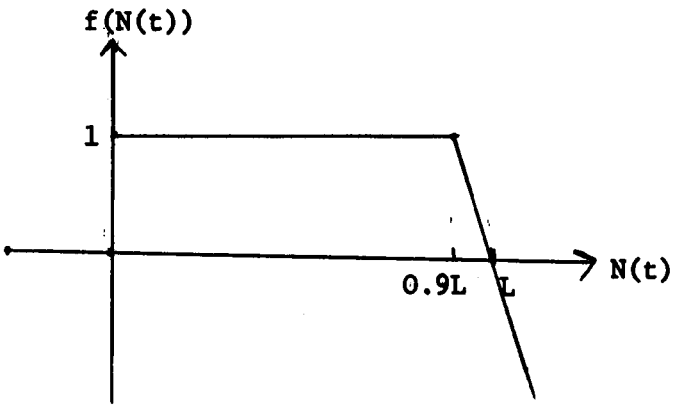
1. Using the transformation $T = \frac{\ell n\ 2}{k}$, show that Equations (2.8) and (2.9) are equivalent.

2. Using the methods of separation of variables and partial fractions, show that Equation (2.13) is the general solution of the Riccati equation (2.12).

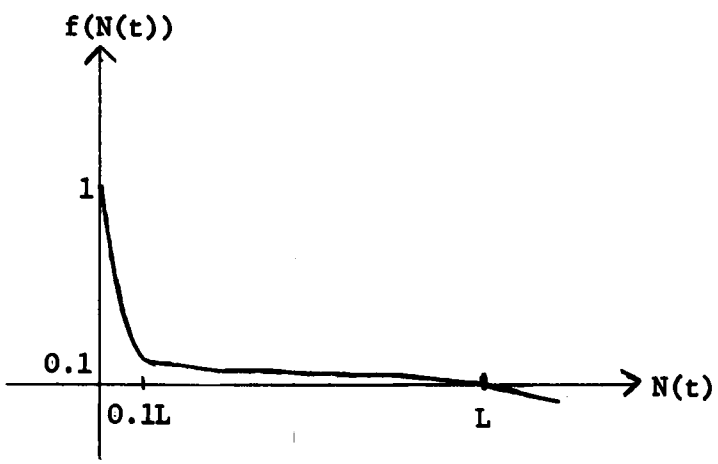
3. Find a number η_0 so that if $N(0) < \eta_0$ the graph of N has an inflection point, but $\eta_0 \leq N(0) < L$ does not give an inflection point.

4. Roughly sketch several solutions of Equation (2.10) for $k = 1$ and the following choices of the function f . Comment upon the effect of crowding in each case.

a.



b.



5. Show that if $C \geq 0$ then $N(0) < L$, whereas $C < -1$ implies $N(0) > L$. Why does Equation (2.13) not make sense for $-1 \leq C < 0$?

6. If a species grows at a rate proportional to the number present, it obeys a growth equation of the form (2.7) where $k = b - d$, the difference between the birth and death rates. Suppose that the death rate d is a function of time; in particular, assume $d = cN(t)$. We now have the growth equation

$$N'(t) = [b - cN(t)]N(t) .$$

Solve this equation and sketch N when $N(0) < b/c$, $N(0) = b/c$, and $N(0) > b/c$.

7. Consider the first-order equation

$$N'(t) = q(t)f(N(t))$$

with $q(t) \geq 0$ for all t and $f(u) > 0$ for all u.

a. Suppose $\int_0^\infty q(t)dt = \infty$ and $\int_0^\infty \frac{du}{f(u)} < \infty$. Show that if N is a solution, there is a time $T > 0$ such that $N(t) \to \infty$ as $t \to T$. (For example, consider $N'(t) = 1 + [N(t)]^2$.)

b. Suppose $\int_0^\infty q(t)dt < \infty$ and $\int_0^\infty \frac{du}{f(u)} = \infty$. Show that if N is a solution, then N does not become unbounded at some finite time, and, in fact, there is a constant L such that $N(t) \to L$ as $t \to \infty$.

2.4. The Lotka-Volterra Model. In Section 1.4 the Lotka-Volterra model of the growth of two conflicting populations was discussed in the predator-prey form. If the size of the prey and predator

populations are represented by x and y, respectively, the rates of growth of the two populations are given by the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cx + dxy \end{aligned} \quad , \tag{2.14}$$

where a, b, c, and d are positive constants. The general solution of this equation was found to be

$$\frac{1/K\, e^{dx}}{x^c} = e^{-by}y^a \quad . \tag{2.15}$$

Because the relationship between x and y is a complicated transcendental equation, the solution curves are most easily constructed using a graphical procedure such as the following:

Step 1. Determine $\frac{1}{K}$ using the initial conditions.

Step 2. Define the functions g and h as follows:

$$\begin{aligned} g(y) &= e^{-by}y^a \\ h(x) &= \frac{1/K\, e^{dx}}{x^c} \end{aligned} \tag{2.16}$$

Step 3. Plot the graph of g versus y.

Step 4. Assuming several values of x, compute $h(x)$ and then read the corresponding values of y from the graph of g.

Step 5. When a sufficient number of points (x,y) have been calculated to construct the solution curve, plot the points and connect them with a smooth curve.

EXAMPLE 2.3. Use this graphical method to solve System (2.14) given $a = 1$, $b = \frac{1}{100}$, $c = 2$, and $d = \frac{1}{1000}$ for the initial values $(x(0), y(0)) = (2000, 150)$.

SOLUTION. Substituting these values into Equation (2.15) yields $1/K = 1.81182 \cdot 10^7$. Thus $g(y) = e^{-y/100}y$ and $h(x) = \frac{1.81182 \cdot 10^7 \, e^{x/1000}}{x^2}$. The graph of the function g is plotted in Figure 8. Since $g(y) = h(x)$, we need only consider those values of x for which $h(x)$ is in the range of g. A few calculations show that x must be approximately between 1400 and 2600. In Table II $h(x)$ is given for several such values of x. Estimates from Figure 8 of those values of y for which $g(y) = h(x)$ are also shown in Table II. In Figure 9 these ordered pairs (x,y) are plotted and connected with a smooth curve.

Although in Step 4 the investigator must find approximate y values by inspection of the graph of g, this graphical method of solution is sufficiently accurate for many physical applications.

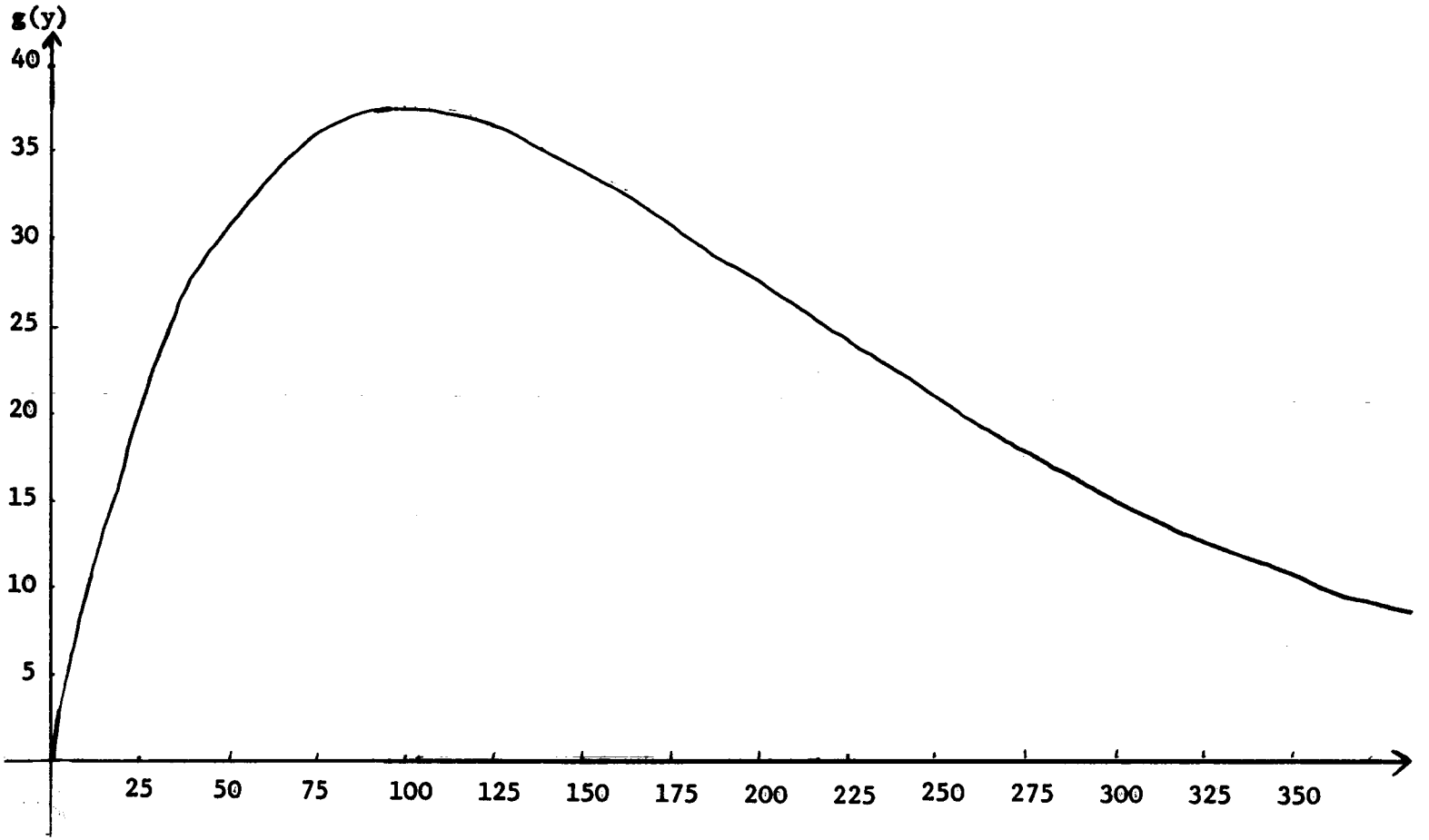


Figure 8. The graph of $g(y) = e^{-y/100}y$.

TABLE II

APPROXIMATE y VALUES OBTAINED FROM THE RELATIONSHIP g(y) = h(x)

x	h(x)	Approximate y Values	
1000	49.25	-	-
1300	39.34	-	-
1400	37.49	105	115
1500	36.09	83	129
1700	34.32	70	141
2000	33.47	65	150
2300	34.16	66	146
2500	35.32	75	135
2600	36.09	83	129
2700	36.98	-	-

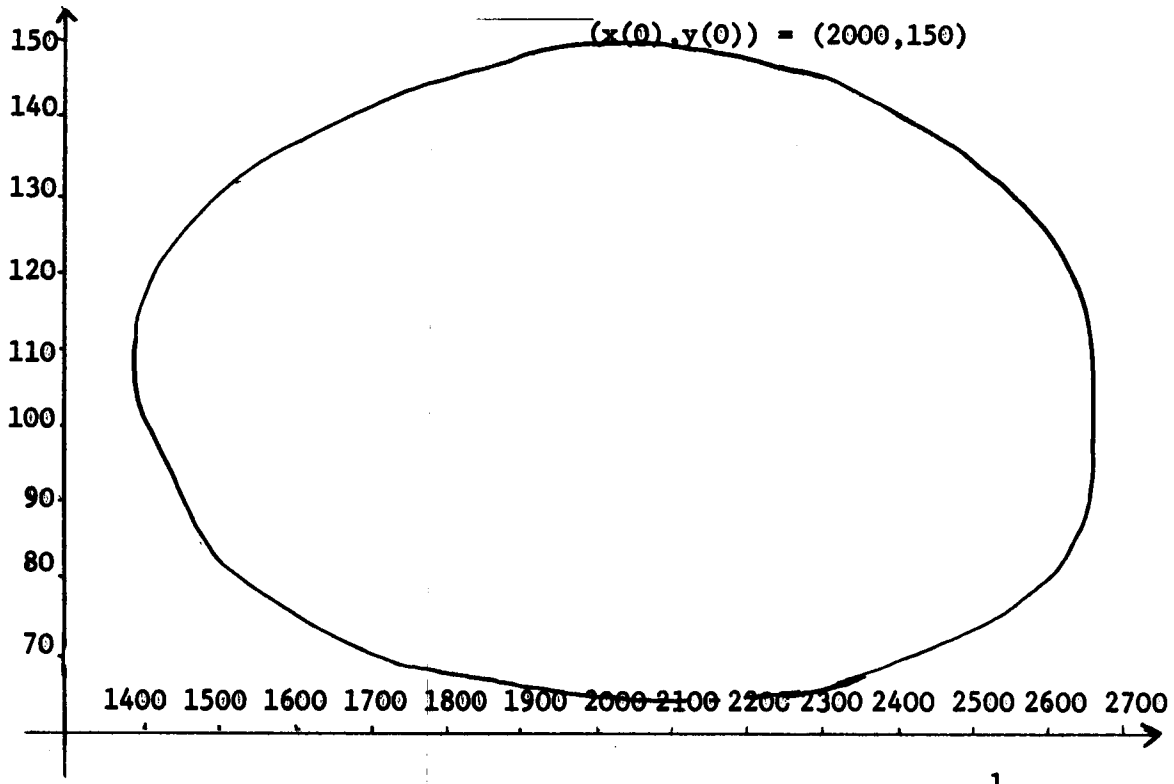


Figure 9. Integral curve for System (2.14) with a = 1, b = $\frac{1}{100}$, c = 2, d = $\frac{1}{1000}$, and (x(0),y(0)) = (2000, 150).

D. PROBLEM LIST 8

1. Use the graphical procedure described in this section to sketch the solution curve for System (2.14) given $a = 1$, $b = \frac{1}{100}$, $c = 2$, and $d = \frac{1}{1000}$ for each of the following initial data:

   a. $(x(0), y(0)) = (1000, 100)$;

   b. $(x(0), y(0)) = (2000, 300)$;

   c. $(x(0), y(0)) = (2000, 400)$.

2.5. Model of Two Species Competing for the Same Food Supply

We have already seen how certain ecological situations can be studied using mathematical models. However, because of the complex interrelationships between the living organisms and their nonliving environment which exist in any ecosystem, the formulation of any mathematical model presupposes certain simplifying assumptions. The desirability of a given model depends upon the closeness with which it follows and predicts the behavior of the ecological system.

Suppose x and y measure the number of individuals in a given ecosystem of the species X and Y, respectively. Furthermore, assume that species X and Y are competing for much the same food supply. Before constructing a mathematical model, several basic simplifying assumptions must be made. Assume that only two species with a constant, but limited, food supply are involved. (Note that in practice the food supply would usually vary due to seasonal fluctuations.) The rate of reproduction of each species is assumed to be constant; furthermore, there are no changes in population

size due to migration or predation. Assume that the maximum population of species X which the ecosystem will support in the absence of species Y is the constant x^* ; similarly, suppose the saturation population of species Y is y^* . Let the constant α denote the degree to which species Y interferes with species X's pursuit of food. Similarly, let β measure the ability of X to interfere with Y . A reasonable mathematical model which satisfies these simplifying assumptions is

$$\frac{dx}{dt} = kx \frac{[x^* - (x+\alpha y)]}{x^*}$$

(2.17)

$$\frac{dy}{dt} = cy \frac{[y^* - (y+\beta x)]}{y^*} \quad .$$

Given the initial population size of species X and Y and the values of the six constants involved, the ecologist is interested in whether one species will eventually become extinct, allowing the other to reach its maximum population size, or whether the two species can coexist.

Since k and x^* are positive, $\frac{dx}{dt} = 0$ when and only when $x = 0$ or $x^* = x + \alpha y$. The latter is the equation of a straight line in the x,y-plane with y-intercept $\frac{x^*}{\alpha}$ and x-intercept x^* . (See Figure 10, part a.) Above this line $\frac{dx}{dt} < 0$ (indicated by an arrow directed toward the left), and below this line $\frac{dx}{dt} > 0$ (indicated by an arrow directed toward the right). A similar construction for $\frac{dy}{dt}$ is shown in part b of Figure 10.

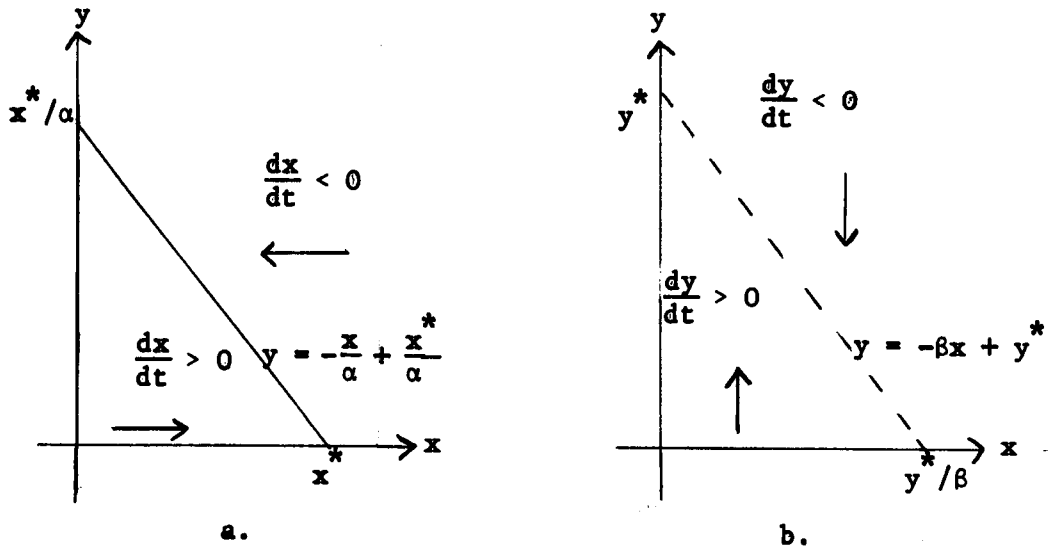


Figure 10. Analysis of the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in each region.

The four important cases to consider for the positions of these two lines on which $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ are shown in Figure 11. The direction of the arrows in each region is also indicated.

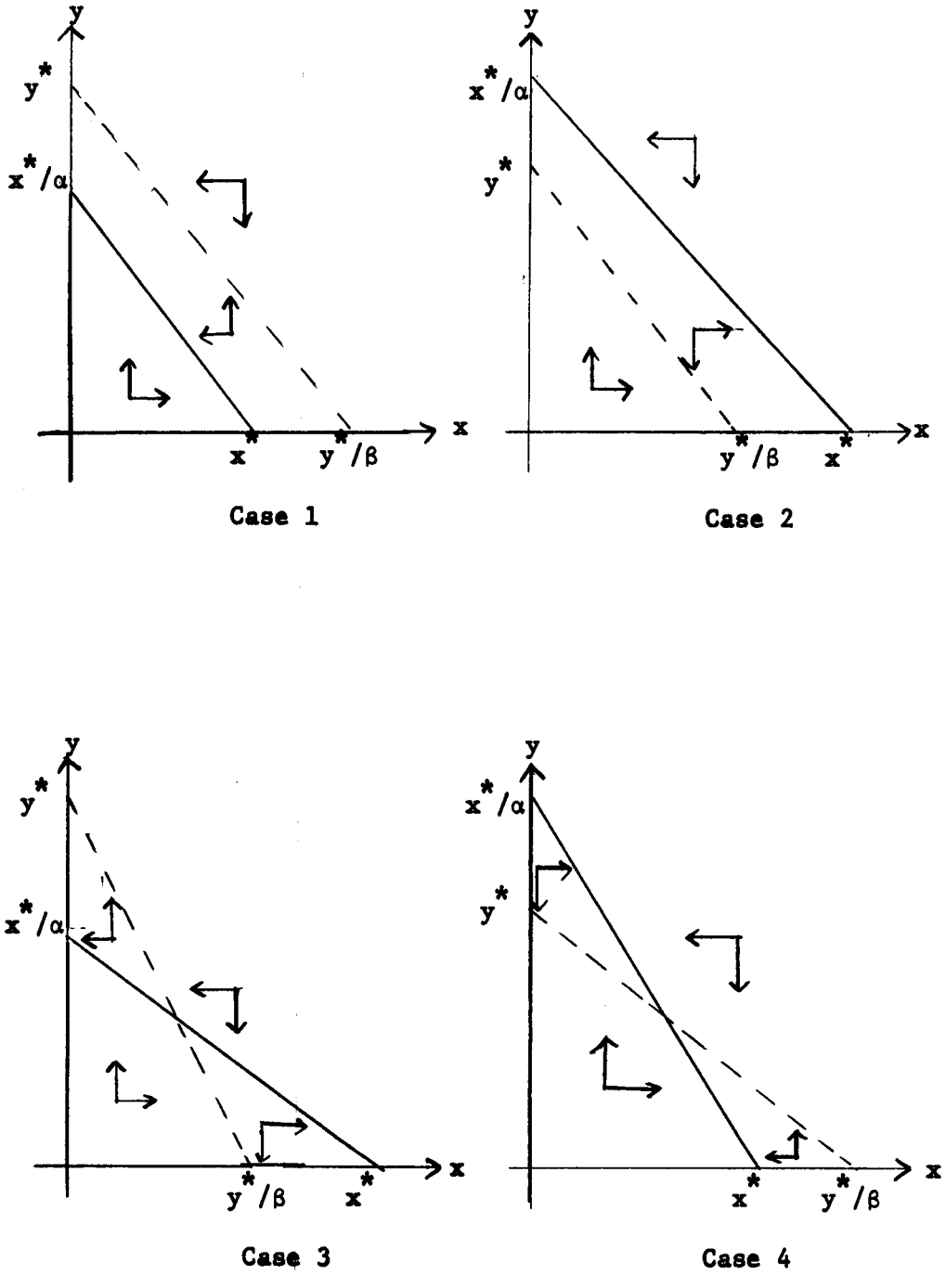


Figure 11. The four most important relative positions of $y = -\frac{1}{\alpha}(x-x^*)$ and $y = -\beta x+y^*$.

In Figure 12 several solution curves are sketched for System (2.17), Case 1, corresponding to various initial values. The solution curves coincident with the axes exemplify the fact that if there are no individuals of one species at time $t = 0$, then there can never be any members of this species in the ecosystem; furthermore, the other species eventually reaches its maximum population size. All other solution curves tend toward the limiting value $(0,y^*)$ as time increases. In this case, which species becomes dominant?

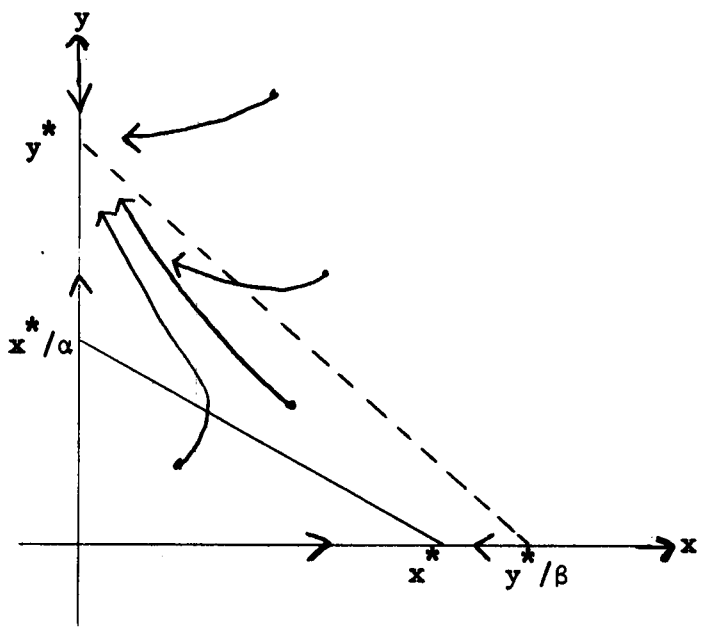


Figure 12. Several solution curves for System (2.17), Case 1.

E. PROBLEM LIST 9

1. A point (x_0,y_0) is a <u>critical point</u> for System (2.17) provided $\frac{dx}{dt} = \frac{dy}{dt} = 0$ at that point. Find all three critical points for Case 1.

2. Why is it impractical to consider those cases for which the two lines coincide or intersect on one of the coordinate axes?

3. If the two lines did coincide, how many critical points would there be?

4. Make an analysis similar to the one above for Cases 2, 3, and 4. For each case sketch several solution curves and find all of the critical points.

For additional reading to supplement Sections 2.1 and 2.2, see Kaplan [9], Pollard and Tennenbaum [14], Rabenstein [15], and Ross [16], and for Sections 2.3 and 2.4, see Davis [7]. Section 2.5 closely parallels the discussion in Penney [13].

CHAPTER III

PHASE PLANE ANALYSIS

3.1. Introduction. The mathematical model of many physical problems involves two real-valued functions, x and y, which are coupled by a pair of first-order differential equations

$$\begin{aligned} x'(t) &= f[t, x(t), y(t)] \\ y'(t) &= g[t, x(t), y(t)] , \end{aligned} \tag{3.1}$$

where f and g are functions determined by the physical problem. The independent variable t usually represents time. A solution of System (3.1) is a pair of functions (x,y) which satisfies (3.1) on some interval of time, I. An important existence and uniqueness theorem for solutions of System (3.1) is now stated without proof.

THEOREM 3.1. Suppose that f and g are continuous real-valued functions with continuous partial derivatives for $a \leq t \leq b$ and (x,y) in some region U of the x, y-plane. Given t_0 with $a \leq t_0 \leq b$ and (x_0,y_0) in U, then on some subinterval I of $[a,b]$ containing t_0 there exists a unique solution (x,y) of System (3.1) which satisfies the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$.

The graphs of the solutions of System (3.1) in the t,x,y-plane are called integral curves, but the projection of an integral curve onto the x,y-plane is called a trajectory or orbit. Since a solution pair (x,y) is often called a phase of the physical system (3.1), the x,y-plane is referred to as the phase plane.

Define a function ϕ from an interval I into the phase plane by $\phi(t) = [x(t),y(t)]$, where the pair (x,y) is a solution of System (3.1). Then the pictorial representation of ϕ produced by sketching its range is just a trajectory of System (3.1). It is customary to indicate the direction of motion along a trajectory with increasing time by use of arrows. Furthermore, if System (3.1) is time - dependent, or nonautonomous, several points in the range of ϕ are usually labeled by their corresponding times. Such a picture is called a phase plane representation or phase portrait of ϕ. The phase portrait of the function $\phi(t) = (t^2, t)$ is sketched in Figure 13.

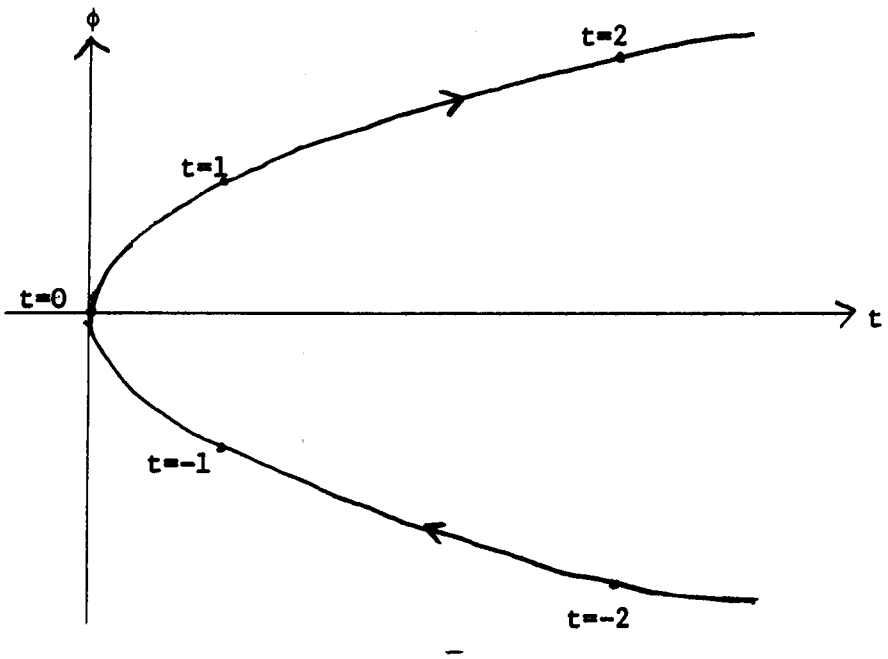


Figure 13. Phase portrait of $\phi(t) = (t^2, t)$.

Since the function $\psi(t) = (t^6, t^3)$ has the same trajectory as ϕ, it is necessary to distinguish between a solution of System (3.1) and the trajectory of the solution. A solution of System (3.1) is a pair of functions $[x(t), y(t)]$ which satisfies that system; a trajectory, however, may be defined parametrically by more than one solution.

In many applications, the functions f and g of System (3.1) are time - independent, or autonomous. In this case f and g are functions of x and y only, so System (3.1) can be written as

$$x'(t) = f[x(t),y(t)]$$

(3.2)

$$y'(t) = g[x(t),y(t)] \ .$$

In physical problems this means that the rate of change of the system is a function of the state of the system only and does not depend on the time at which it arrived in that state. For the autonomous system (3.2), the trajectories are often called <u>paths</u>.

If System (3.2) can be written in the form

$$x'(t) = a(t)x(t) + b(t)y(t) + e(t)$$

(3.3)

$$y'(t) = c(t)x(t) + d(t)y(t) + f(t) \ ,$$

then it is called <u>linear</u>, and <u>linear homogeneous</u> if $e(t) = f(t) = 0$. In the next section we discuss the solutions and phase portraits of autonomous, linear homogeneous systems.

A. PROBLEM LIST 10

1. Sketch the phase portraits of the following function pairs:
   a. $x(t) = 2e^{t}$, $y(t) = e^{t}$;
   b. $x(t) = e^{-t} + e^{-2t}$, $y(t) = 2e^{-t} + 3e^{-2t}$;
   c. $x(t) = e^{t} + e^{2t}$, $y(t) = 2e^{t} + 3e^{2t}$;
   d. $x(t) = 2 \cos 3t$, $y(t) = \sin 3t$;

e. $x(t) = 2e^{-t} \cos 3t$, $y(t) = e^{-t} \sin 3t$;

f. $x(t) = 2e^{t} \cos 3t$, $y(t) = e^{t} \sin 3t$.

2. Show that the second-order equation

$$\sigma''(t) + a(t)\sigma'(t) + b(t)\sigma(t) = c(t)$$

can be regarded as a system of the form (3.1) by choosing $x(t) = \sigma(t)$ and $y(t) = \sigma'(t)$. Under what conditions will the resulting system be autonomous?

3. Suppose (x,y) is a solution of the autonomous system (3.2) for t in some interval I . Let t_0 be a real number and define $(\hat{x},\hat{y})$ by $\hat{x}(t) = x(t - t_0)$ and $\hat{y}(t) = y(t - t_0)$ for all t such that $t - t_0$ is in I . Show that $(\hat{x},\hat{y})$ is a solution of System (3.2). What is the difference between the phase portraits of (x,y) and $(\hat{x},\hat{y})$?

4. Find the solutions of the linear autonomous system

$$x'(t) = x(t)$$

$$y'(t) = x(t) + y(t)$$

satisfying the initial conditions $x(0) = 1$, $y(0) = 3$, and $x(4) = e$, $y(4) = 4e$. Show that these two different solutions represent the same path. Also find the differential equation which gives the slope of the tangent to the paths, and solve it to find the one-parameter family of solutions representing each path.

3.2. Linear Homogeneous Systems. Consider the autonomous, linear homogeneous system

$$x' = ax + by$$
$$y' = cx + dy \quad , \tag{3.4}$$

where a, b, c, and d are real constants, x and y are functions of t, and $'$ denotes $\frac{d}{dt}$. We know by Theorem 3.1 that a solution of System (3.4) is uniquely determined when the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ are specified for some time t_0.

The most straightforward approach to finding solutions of System (3.4) is to solve for y in terms of x and x' in the first equation, differentiate this derived equation for y, and substitute these two expressions for y and y' into the second equation. The second-order linear differential equation in x thus produced can then be solved using elementary methods. This procedure is illustrated in the following example.

EXAMPLE 3.1. Find the general solution of the linear system

$$x' = x + 2y$$
$$y' = 3x + 2y \; . \tag{3.5}$$

SOLUTION. Solving the first equation for y gives

$$(3.6) \qquad y = \frac{x' - x}{2} \ ;$$

differentiating both sides of Equation (3.6) yields

$$(3.7) \qquad y' = \frac{x'' - x'}{2} \ .$$

If Equations (3.6) and (3.7) are substituted into the second equation, the single second-order linear equation in one unknown,

$$(3.8) \qquad x'' - 3x' - 4x = 0 \ ,$$

is obtained. The equation $\lambda^2 - 3\lambda - 4 = 0$ is called the <u>characteristic equation</u> (often abbreviated (C)) associated with System 3.5. The two roots of this quadratic equation, $\lambda_1 = 4$ and $\lambda_2 = -1$, are called the <u>characteristic roots</u> of (C). Thus the general solution of Equation (3.8) is

$$(3.9) \qquad x(t) = c_1 e^{4t} + c_2 e^{-t} \ .$$

If x and x' are substituted into Equation (3.6), we find that

$$(3.10) \qquad y(t) = \frac{3}{2} c_1 e^{4t} - c_2 e^{-t} \ .$$

It is left to the reader to verify that the pair (x,y) defined by Equations (3.9) and (3.10) satisfies System (3.5).

Although the method of solution outlined above works well for any linear system with constant coefficients in two unknowns, if three or more unknowns are involved, this procedure does not work. For this reason, in addition to the facilitation in proving the theorems concerning stability in the next chapter, we will consider an alternate method of solution of System (3.4).

In seeking solutions of System (3.4), recall from your experience with elementary differential equations that a single nth-order linear equation with constant coefficients has exponential solutions. Remembering the relationship which exists between linear systems and single higher-order linear equations, it is reasonable to assume that System (3.4) might have solutions of the form

$$x(t) = Ae^{\lambda t}$$
$$y(t) = Be^{\lambda t} \quad . \tag{3.11}$$

If these equations are substituted into System (3.4), we obtain

$$\lambda Ae^{\lambda t} = aAe^{\lambda t} + bBe^{\lambda t}$$
$$\lambda Be^{\lambda t} = cAe^{\lambda t} + dBe^{\lambda t} \quad ,$$

which is equivalent to the system

$$(a - \lambda)\, A + bB = 0$$

(3.12)

$$c\, A + (d - \lambda)B = 0 \quad .$$

Obviously System (3.12) has the trivial solution $A = B = 0$, but this gives only the trivial solution $x(t) = y(t) = 0$ of System (3.4). Therefore we seek nontrivial solutions of System (3.12). By a theorem of linear algebra, a nontrivial solution of the homogeneous system (3.12) exists when and only when

$$(3.13) \qquad 0 = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) \ .$$

Equation (3.13) is called the characteristic equation associated with System (3.4). In order that the pair (3.11) be a solution of System (3.4), λ must be one of the characteristic roots of Equation (3.13). In seeking solutions of System (3.4), we distinguish three cases:

1. λ_1 and λ_2 are real and distinct;
2. $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, where $\beta \neq 0$;
3. $\lambda_1 = \lambda_2$; that is, Equation (3.13) has one real root of multiplicity two.

Case 1. The Roots of (C) Are Real and Distinct.

In this case we can solve System (3.12) first with $\lambda = \lambda_1$ and then with $\lambda = \lambda_2$ to obtain two distinct solutions

$$x_1(t) = A_1 e^{\lambda_1 t} \qquad x_2(t) = A_2 e^{\lambda_2 t}$$

(3.14) and

$$y_1(t) = B_1 e^{\lambda_1 t} \qquad y_2(t) = B_2 e^{\lambda_2 t} \quad .$$

Because these solutions are nontrivial, we have that not both A_1 and B_1 or both A_2 and B_2 are zero. Since $\lambda_1 \neq \lambda_2$, these two solutions are clearly linearly independent; that is, the only constants, k_1 and k_2, for which $k_1[x_1(t),y_1(t)] + k_2[x_2(t),y_2(t)] = (0,0)$ for all t are $k_1 = k_2 = 0$. Thus the general solution of System (3.4) is of the form

$$(3.15) \qquad [x(t),y(t)] = c_1[x_1(t),y_1(t)] + c_2[x_2(t),y_2(t)] \ ,$$

where c_1 and c_2 are arbitrary constants.

Suppose $P = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$. You are asked to show in Problem 2 that the system (3.12) with solutions (3.14) is equivalent to the matrix equation

$$(3.16) \qquad P^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} .$$

EXAMPLE 3.2. Find the general solution of the system

$$x' = -4x - 3y$$

(3.17)

$$y' = 2x + y \quad .$$

SOLUTION. This system has characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ with characteristic roots $\lambda_1 = -1$ and $\lambda_2 = -2$. One solution of System (3.12) with $\lambda = -1$ is $A_1 = 1$, $B_1 = -1$, while $A_2 = 3$, $B_2 = -2$ is a solution for $\lambda = -2$. Thus the general solution of System (3.17) is

$$x(t) = c_1 e^{-t} + 3c_2 e^{-2t}$$

(3.18)

$$y(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \quad .$$

Several trajectories for System (3.17) are sketched in Figure 14 corresponding to various values of c_1 and c_2. Notice that

$$\frac{y(t)}{x(t)} = \frac{-c_1 - 2c_2 e^{-t}}{c_1 + 3c_2 e^{-t}} \to -1$$

as $t \to \infty$ if $c_1 \neq 0$ (since $e^{-t} = \frac{1}{e^t} \to 0$ as $t \to \infty$). Furthermore, if $c_1 = 0$ then $\frac{y(t)}{x(t)} = -\frac{2}{3}$. In this case y is a multiple of x and the trajectory is a straight line with slope $-\frac{2}{3}$.

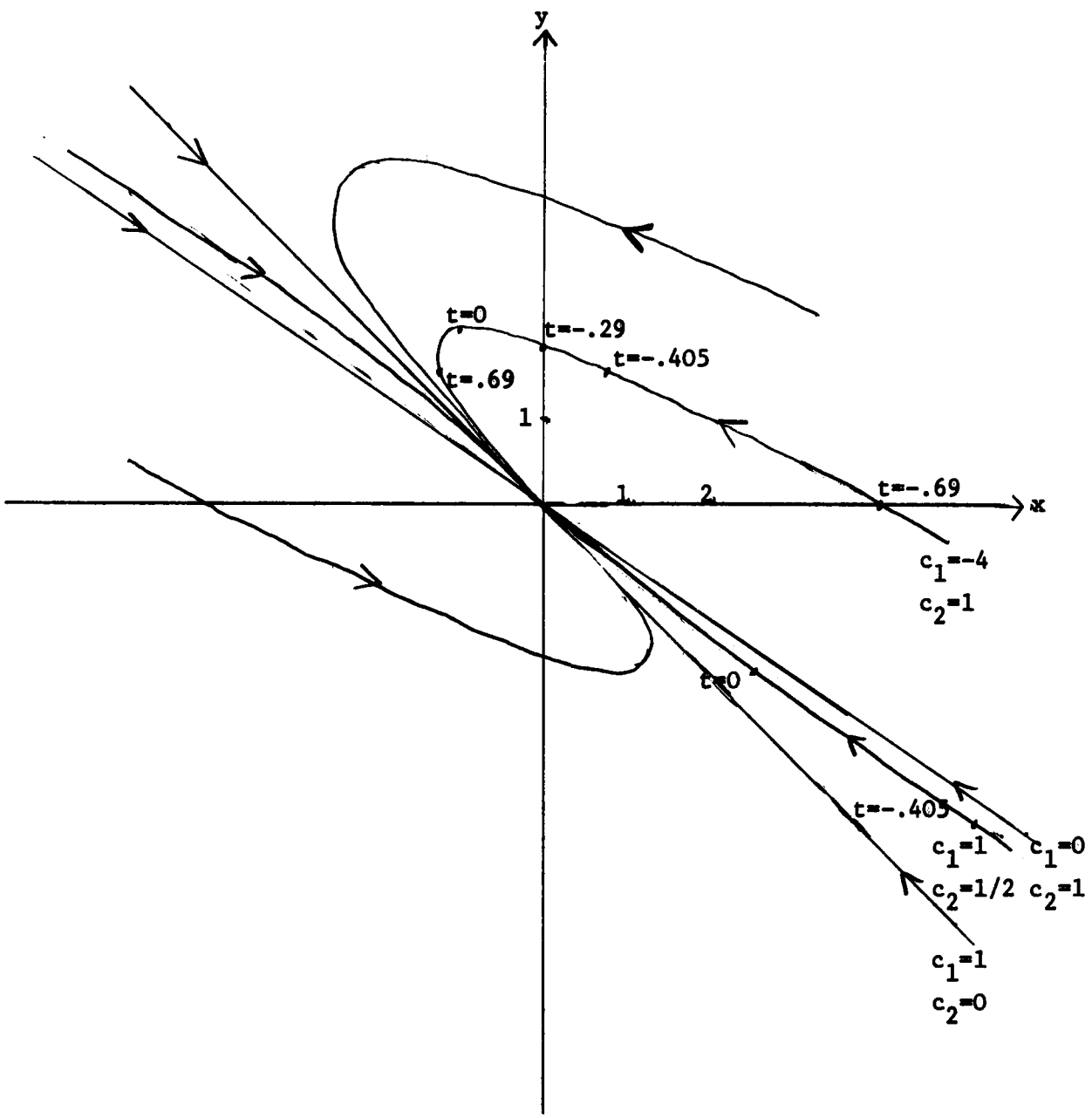


Figure 14. Phase portraits for solutions of System (3.17).

Case 2. The Roots of (C) Are Conjugate Complex.

Since imaginary roots of polynomial equations with real coefficients occur in conjugate pairs, the roots of (C) in this case can be written as $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \overline{\lambda}_1 = \alpha - \beta i$ for $\beta \neq 0$. (The bar denotes "complex conjugate".)

The procedure used in Case 1 may be followed since $\lambda_1 \neq \lambda_2$; however, in this case the solutions of System (3.12) will be complex. Furthermore if (A_1, B_1) is a solution of System (3.12) when $\lambda = \lambda_1$, then $(A_2, B_2) = (\overline{A}_1, \overline{B}_1)$ is a solution for $\lambda = \lambda_2 = \overline{\lambda}_1$. For since a, b, c, and d are assumed to be real, by taking conjugates in System (3.12) we obtain the system

$$\begin{aligned} (a - \overline{\lambda})\,\overline{A} + b\overline{B} &= 0 \\ c\,\overline{A} + (d - \overline{\lambda})\overline{B} &= 0 \,. \end{aligned} \tag{3.19}$$

Suppose $A_1 = C + Di$ and $B_1 = E + Fi$, where C, D, E, and F are real constants. Then we may choose $A_2 = C - Di$, $B_2 = E - Fi$. In Problem 3 you are asked to verify that with this choice the condition $A_1B_2 - A_2B_1 \neq 0$ reduces to $CF - ED \neq 0$. Thus if $Q = \begin{bmatrix} C & D \\ E & F \end{bmatrix}$, $\lambda = \alpha + Bi$, $A = A_1$, and $B = B_2$, the system (3.12) is easily shown to be equivalent to the matrix equation

$$Q^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} Q = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} . \tag{3.20}$$

(See Problem 4.)

Because $\lambda_2 = \overline{\lambda}_1$, it is only necessary to solve System (3.12) once to obtain the two distinct solutions of System (3.4)

$$x_1(t) = (C + Di)e^{(\alpha+\beta i)t}$$

(3.21)

$$y_1(t) = (E + Fi)e^{(\alpha+\beta i)t}$$

and

$$x_2(t) = (C - Di)e^{(\alpha-\beta i)t}$$

(3.22)

$$y_2(t) = (E - Fi)e^{(\alpha-\beta i)t} \quad .$$

However, to obtain <u>real</u> solutions of System (3.4), consider the solution (3.21). Applying Euler's formula, $e^{i\theta} = \cos\theta + i \sin\theta$, this solution can be rewritten as

$$x_1(t) = (C + Di)e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

$$= e^{\alpha t}[(C \cos \beta t - D \sin \beta t) + i(D \cos \beta t + C \sin \beta t)]$$

(3.23)

$$y_1(t) = (E + Fi)e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

$$= e^{\alpha t}[(E \cos \beta t - F \sin \beta t) + i(F \cos \beta t + E \sin \beta t)] \ .$$

It can be shown that a pair of complex functions $[p_1(t) + iq_1(t),\ p_2(t) + iq_2(t)]$ is a solution of System (3.4) if and only if both the pair of real parts $[p_1(t),\ p_2(t)]$ and the pair consisting of the imaginary parts $[q_1(t),\ q_2(t)]$ are solutions of System (3.4). Moreover, these two solutions are linearly independent. (See Problem 5.) Thus the general solution of System (3.4) for the case of complex conjugate roots may be written as

$$\begin{aligned} x(t) &= e^{\alpha t}[c_1(C \cos \beta t - D \sin \beta t) \\ &\quad + c_2(D \cos \beta t + C \sin \beta t)] \\ y(t) &= e^{\alpha t}[c_1(E \cos \beta t - F \sin \beta t) \\ &\quad + c_2(F \cos \beta t + E \sin \beta t)] . \end{aligned} \tag{3.24}$$

EXAMPLE 3.3. Find the general solution and sketch several trajectories for the system

$$\begin{aligned} x' &= -x + 2y \\ y' &= -2x - y . \end{aligned} \tag{3.25}$$

SOLUTION. The characteristic equation for this system, $\lambda^2 + 2\lambda + 5 = 0$, has roots $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$. A simple nontrivial solution of System (3.12) with $\lambda = -1 + 2i$

is $A_1 = 1$, $B_1 = i$ from which it follows that $C = 1$, $D = 0$, $E = 0$, and $F = 1$. From the equations (3.24), it follows that the general solution of System (3.25) is

$$x(t) = e^{-t}[c_1 \cos 2t + c_2 \sin 2t]$$

(3.26)

$$y(t) = e^{-t}[-c_1 \sin 2t + c_2 \cos 2t] .$$

The trajectories corresponding to $c_1 = 2$, $c_2 = 0$ and $c_1 = 4$, $c_2 = 0$ are sketched in Figure 15.

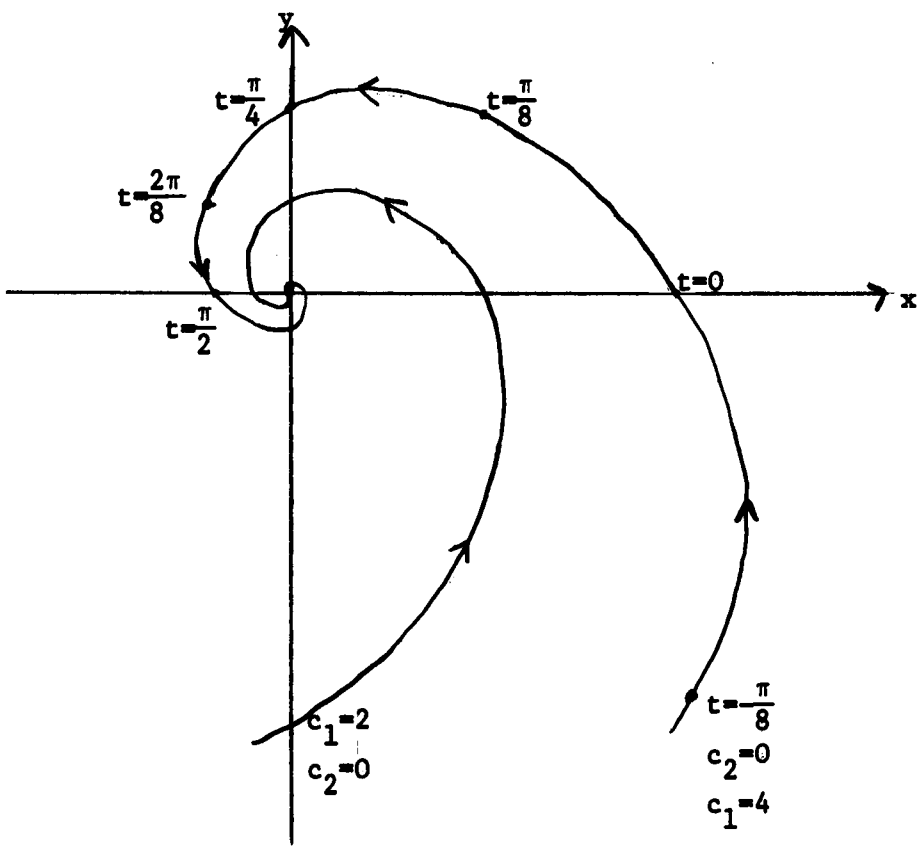


Figure 15. Phase portraits for solutions of System (3.25).

Case 3. The Roots of (C) Are Real and Equal.

If λ_0 denotes the root of (C) of multiplicity two, then by solving System (3.12) a solution of System (3.4) is obtained which is of the form

$$x(t) = A_1 e^{\lambda_0 t}$$

(3.27)

$$y(t) = B_1 e^{\lambda_0 t}$$

with not both A_1 and B_1 zero. Except in trivial cases a second, linearly independent solution of this form cannot be found. Recalling the analogous situation for a single nth-order linear equation, we know that if λ_0 is a root of (C) of multiplicity k , then the collection $c_1 e^{\lambda_0 t}$, $c_2 t e^{\lambda_0 t}$, $c_3 t^2 e^{\lambda_0 t}$, ..., $c_k t^{k-1} e^{\lambda_0 t}$ are all solutions of the linear equation. This would lead us to expect a second solution of System (3.12) of the form

$$x(t) = Ate^{\lambda_0 t}$$

$$y(t) = Bte^{\lambda_0 t} \quad .$$

However, the situation is actually not quite so simple. In Problem 6 you are asked to show that every solution of System (3.4) for Case 3 is of the form

$$x(t) = (A + Bt)e^{\lambda_0 t}$$

(3.28)

$$y(t) = (C + Dt)e^{\lambda_0 t} \ .$$

Notice that the solution (3.27) is of this form if $B = D = 0$. Thus the general solution of System (3.4) can be written as

$$x(t) = c_1 A_1 e^{\lambda_0 t} + c_2 (A + Bt)e^{\lambda_0 t}$$

(3.29)

$$y(t) = c_1 B_1 e^{\lambda_0 t} + c_2 (C + Dt)e^{\lambda_0 t} \ ,$$

where c_1 and c_2 are arbitrary constants. But to obtain a general solution of this form, the six constants A_1, A, B, B_1, C, and D must be evaluated. Thus we proceed to simplify the form of the general solution for Case 3. In the case of a double root λ_0, the characteristic equation must be

(3.30) $$0 = (\lambda - \lambda_0)^2 = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2 \ .$$

Comparing Equations (3.13) and (3.35) gives the relationships

(3.31) $$a + d = 2\lambda_0 \quad \text{and} \quad ad - bc = \lambda_0^2 \ .$$

We consider two subcases: $b = c = 0$ and not all of $a - \lambda_0$, b, c, and $d - \lambda_0$ are zero.

Subcase 1. $b = c = 0$.

The relationships (3.31) reduce to $ad = \lambda_0^2$ and $a + d = 2\lambda_0$. Thus $a = d = \lambda_0$. System (3.4) becomes

$$x' = \lambda_0 x$$

(3.32)

$$y' = \lambda_0 y \quad ,$$

whose general solution is

$$x(t) = c_1 e^{\lambda_0 t}$$

(3.33)

$$y(t) = c_2 e^{\lambda_0 t} \ .$$

Since in this subcase

(3.34) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix} \quad ,$$

the matrix is already in Jordon canonical form.

Subcase 2. Not all of $a - \lambda_0$, b, c, and $d - \lambda_0$ are zero.

Substitution of the equations (3.28) into System (3.4) yields the system

$$B = [(a - \lambda_0)B + bD]t + [(a - \lambda_0)A + bC]$$

(3.35)

$$D = [cB + (d - \lambda_0)D]t + [cA + (d - \lambda_0)C] \quad .$$

If we choose $B = A_1$ and $D = B_1$, where A_1 and B_1 are given by Equation (3.27), System (3.35) reduces to

$$A_1 = (a - \lambda_0)A + bC$$

(3.36)

$$B_1 = cA + (d - \lambda_0)C \ .$$

Suppose $R = \begin{bmatrix} A_1 & A \\ B_1 & C \end{bmatrix}$. We show that $\det R \neq 0$. Assume $\det R = A_1C - B_1A = 0$. Then from the systems (3.12) and (3.36), we find

$$A_1^2 = (a - \lambda_0)AA_1 + bCA_1$$

$$= (a - \lambda_0)AA_1 + bB_1A$$

$$= [(a - \lambda_0)A_1 + bB_1]A = 0$$

$$B_1^2 = cAB_1 + (d - \lambda_0)CB_1$$

$$= cA_1C + (d - \lambda_0)CB_1$$

$$= [cA_1 + (d - \lambda_0)B_1]C = 0 \ .$$

But this implies $A_1 = B_1 = 0$. This contradiction shows that the assumption det R = 0 must be false. Thus for Subcase 2, it may be shown that the canonical form is

$$R^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} R = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}. \tag{3.37}$$

(See Problem 7.) The general solution for this subcase can then be written as

$$\begin{aligned} x(t) &= c_1 A_1 e^{\lambda_0 t} + c_2 (A + A_1 t) e^{\lambda_0 t} \\ y(t) &= c_1 B_1 e^{\lambda_0 t} + c_2 (C + B_1 t) e^{\lambda_0 t} \ . \end{aligned} \tag{3.38}$$

The problem has thus been reduced for this subcase to the computation of only four constants.

EXAMPLE 3.4. Find the general solution and sketch phase portraits around the origin for the system

$$\begin{aligned} x' &= -x \\ y' &= -y \ . \end{aligned} \tag{3.39}$$

SOLUTION. The characteristic equation for this system, $\lambda^2 + 2\lambda + 1 = 0$, has the double root $\lambda_0 = -1$. Since $b = c = 0$,

this system is of Subcase 1. Thus the general solution is

$$x(t) = c_1 e^{-t}$$

(3.40)

$$y(t) = c_2 e^{-t} \quad .$$

Since $y(t) = \frac{c_2}{c_1} x(t)$, if $c_1 \neq 0$ the trajectories of the non-trivial solutions of System (3.39) are straight lines passing through the origin. Several of these paths are sketched in Figure 16. Notice that $e^{-t} \to 0$ as $t \to \infty$; hence all solutions approach the origin as time increases.

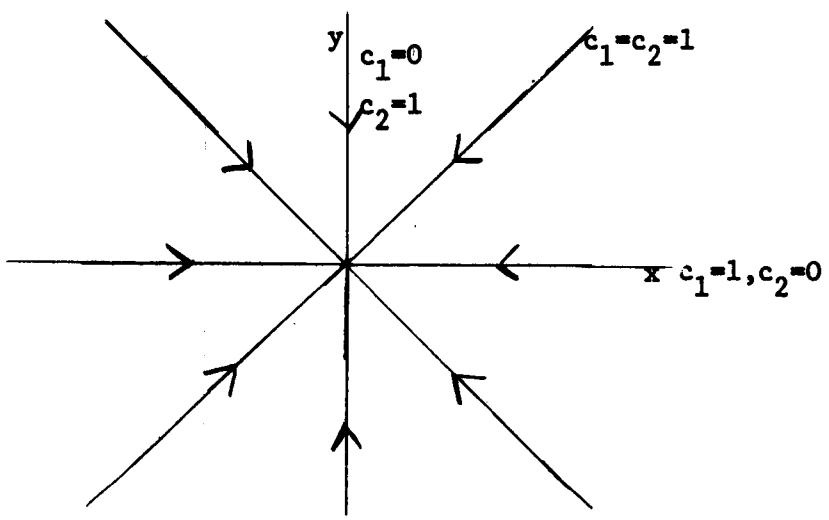


Figure 16. Phase portraits for solutions of System (3.39).

EXAMPLE 3.5. Find the general solution and sketch several trajectories of the system

$$x' = -3x - y$$
$$y' = 4x + y \tag{3.41}$$

SOLUTION. The characteristic equation for this system is the same as that of System (3.39); however, this system is of Subcase 2. One solution of System (3.12) with $\lambda = -1$ is $A_1 = 1$, $B_1 = -2$. Thus the pair

$$x_1(t) = e^{-t}$$
$$y_1(t) = -2e^{-t} \tag{3.42}$$

is a solution of System (3.41). To obtain a second, linearly independent solution, substitute these values into System (3.36) to obtain

$$1 = -2A - C$$
$$-2 = 4A + 2C \quad . \tag{3.43}$$

One solution of this system is $A = 1$, $C = -3$. Hence from the

equations (3.38) the general solution of System (3.41) is seen to be

$$x(t) = c_1 e^{-t} + c_2(1 + t)e^{-t}$$

(3.44)

$$y(t) = -2c_1 e^{-t} + c_2(-3-2t)e^{-t} \quad .$$

For the case $c_2 = 0$, $y(t) = -2x(t)$; thus the trajectory is a straight line with slope -2 passing through the origin. However, if $c_2 \neq 0$,

$$\frac{y(t)}{x(t)} = \frac{-2c_2 + \frac{1}{t}(-2c_1 - 3c_2)}{c_2 + 1/t(c_1 + c_2)} \to -2$$

as $t \to \infty$ since $1/t \to 0$ as $t \to \infty$. Thus for $c_2 \neq 0$ the trajectories of the solutions of System (3.41) approach the origin in such a way that the slope of a line drawn from the origin to a point on the curve approaches -2 as the point moves toward the origin. (See Figure 17.)

B. PROBLEM LIST 11

1. Find the general solution and sketch the phase portraits around the origin for each of the following:

a. $x' = 2x - 3y$

$y' = x - 2y$;

b. $x' = y$

$y' = -1/4x$;

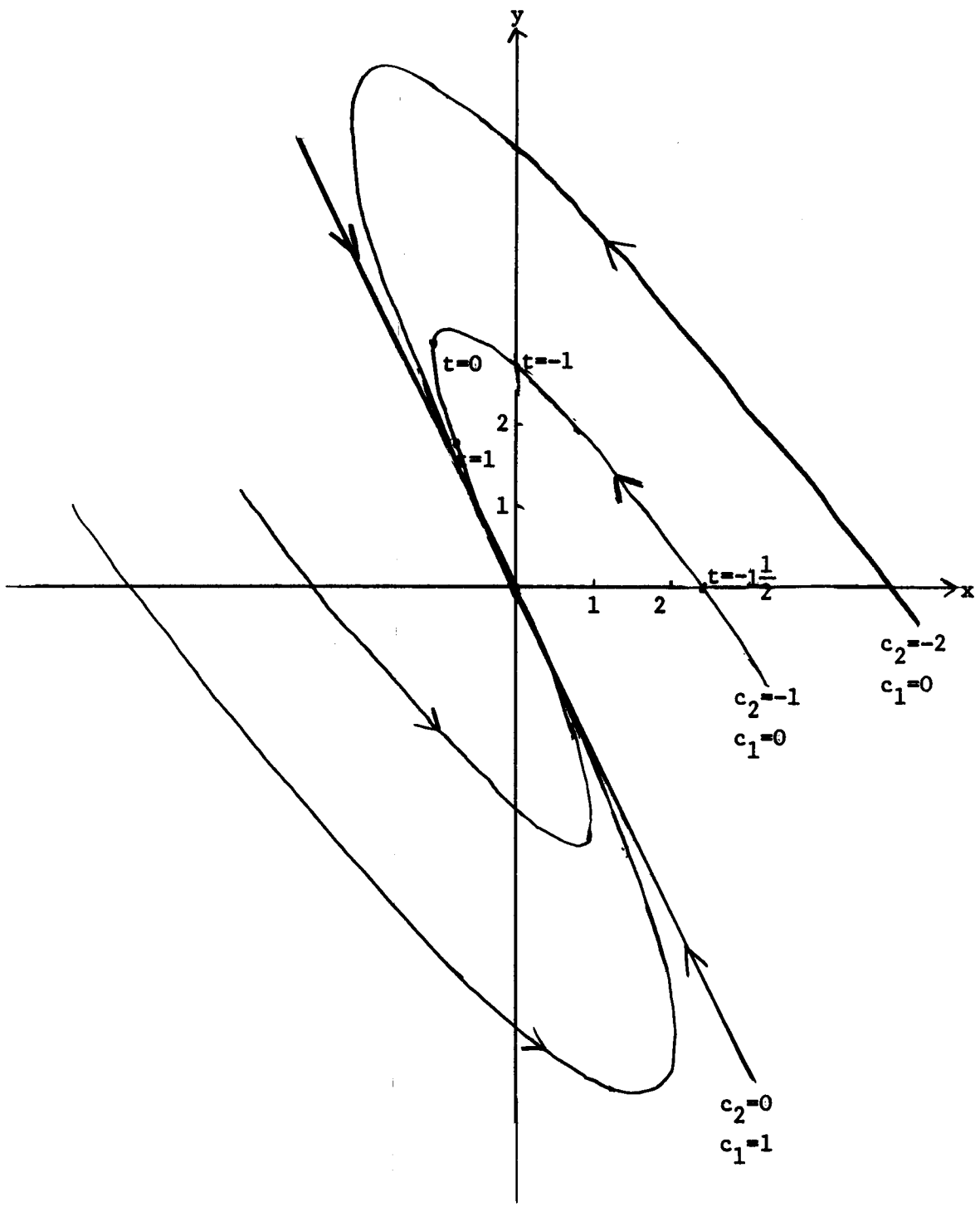


Figure 17. Phase portraits for solutions of System (3.41).

c. $x' = -2x - y$
$y' = -y$;

d. $x' = 2x + y$
$y' = 3x + 4$;

e. $x' = x - y$
$y' = -4x + y$;

f. $x' = -3x + 2y$
$y' = -2x + y$;

g. $x' = x - 3y$
$y' = 3x + y$;

h. $x' = x - y$
$y' = 2x - y$;

i. $x' = -2x + 2y$
$y' = -2x + 3y$;

j. $x' = 0$
$y' = x$;

k. $x' = -x$
$y' = -y$;

l. $x' = 4x - 3y$
$y' = 5x - 4y$.

2. Show that if the roots of (C) are real and distinct, then System (3.12) is equivalent to the matrix equation (3.16) in the following steps:

a. Verify that System (3.12) with the solutions (3.14) is equivalent to the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} P = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

b. Show that $\det P \neq 0$, from which it follows that P^{-1} exists. (Hint: Suppose $\det P = 0$. Then show $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = -\frac{c_2}{c_1} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$, $c_1 \neq 0$ implies $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is a contradiction.)

3. Verify that if $A_1 = \overline{A}_2 = C + Di$ and $B_1 = \overline{B}_2 = E + Fi$, then the condition $A_1B_2 - A_2B_1 \neq 0$ reduces to $CF - ED \neq 0$ for Case 2.

4. If the roots of (C) are conjugate complex, show that System (3.12) is equivalent to the matrix equation (3.20).

5. Write the equations (3.23) in the form

$$x(t) = x_1(t) + ix_2(t)$$

$$y(t) = y_1(t) + iy_2(t)$$

and then show that every solution of System (3.4) for Case 2 is of the form

$$x(t) = c_1x_1(t) + c_2x_2(t)$$

$$y(t) = c_1y_1(t) + c_2y_2(t) \ .$$

6. Consider System (3.4) when $a + d = 2\lambda_0$ and $ad - bc = \lambda_0^2$.

   a. Show that if $b = 0$, then $a = d = \lambda_0$ and all solutions are of the form

   $$x(t) = c_1e^{\lambda_0 t}$$

   $$y(t) = c_2e^{\lambda_0 t} + c_1cte^{\lambda_0 t} \ .$$

   b. If $b \neq 0$, then solve the first equation for y , substitute this into the second equation, and conclude that all solutions are of the form

   $$x(t) = (A + Bt)e^{\lambda_0 t}$$

   $$y(t) = (C + Dt)e^{\lambda_0 t} \ .$$

7. For Case 3, Subcase 2, verify that System (3.36) is equivalent to the matrix equation (3.37).

8. For Case 2 with $\alpha = 0$, show that every nontrivial solution has as its trajectory an ellipse centered at the origin.

9. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix with $\det A \neq 0$, then find a 2×2 matrix B such that $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (Call the unique such B the inverse of A and denote it by A^{-1} .)

10. Show that if (x,y) is a solution of System (3.4),

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $\det A \neq 0$, and (ξ,η) is defined by

$$\xi = a_{11}x + a_{12}y$$
$$\eta = a_{21}x + a_{22}y \quad ,$$

then

$$\xi' = b_{11}\xi + b_{12}\eta$$
$$\eta' = b_{21}\xi + b_{22}\eta$$

where $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = A \begin{bmatrix} a & b \\ c & d \end{bmatrix} A^{-1}$.

11. Show that the condition $b \cdot c > 0$ is sufficient, but not necessary, for System (3.4) to have two real linearly independent solutions of the form (3.11).

12. Extend the procedures developed in this chapter to solve the following linear systems of three equations in three unknowns:

a. $x' = x + y - z$
$y' = 2x + 3y - 4z$
$z' = 4x + y - 4z$;

b. $x' = x - y - z$
$y' = x + 3y + z$
$z' = -3x + y - z$.

For additional reading see Kaplan [9] and Ross [16].

CHAPTER IV

CRITICAL POINTS AND STABILITY

4.1. Introduction. Consider the autonomous system

$$x'(t) = f[x(t),y(t)]$$

(4.1)

$$y'(t) = g[x(t),y(t)] \ .$$

A point (x_0, y_0) is called a critical point of this system provided

$$f(x_0, y_0) = g(x_0, y_0) = 0 \ .$$

If (x_0, y_0) is a critical point, then $x(t) \equiv x_0$, $y(t) \equiv y_0$ is a constant solution of System (4.1); its trajectory is the single point (x_0, y_0) in the phase plane. A critical point (x_0, y_0) of System (4.1) is said to be isolated if there exists a circle,

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \ , \ r > 0 \ ,$$

inside which (x_0, y_0) is the only critical point of the system. In a physical problem a critical point corresponds to a stationary value of the system.

A critical point (x_0, y_0) of System (4.1) is defined to be stable provided for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such

that if $[x(t),y(t)]$ is a solution of System (4.1) satisfying

$$\{[x(0) - x_0]^2 + [y(0) - y_0]^2\}^{1/2} < \delta \quad ,$$

then

$$\{[x(t) - x_0]^2 + [y(t) - y_0]^2\}^{1/2} < \varepsilon$$

for all $t \geq 0$. Geometrically, this means that a critical point (x_0, y_0) is stable provided each trajectory which is inside a circle with radius δ at time $t = 0$ must remain inside the circle with radius ε for all $t \geq 0$. (See Figure 18.)

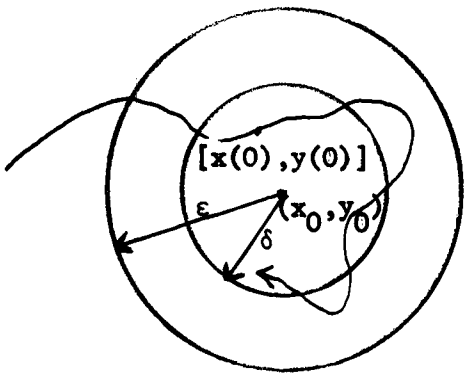


Figure 18. Stability of a critical point (x_0,y_0).

The critical point (x_0, y_0) is said to be <u>asymptotically stable</u> provided it is stable and there exists a number $\delta_0 > 0$ such that if $[x(t),y(t)]$ is a solution of System (4.1) which satisfies $\{[x(0) - x_0]^2 + [y(0) - y_0]^2\}^{1/2} < \delta_0$, then $[x(t),y(t)] \to (x_0, y_0)$ as $t \to \infty$. A critical point which is stable but not asymptotically stable is said to be <u>neutrally stable</u>. If a critical point (x_0, y_0) is not stable, it is called <u>unstable</u>.

The general solution of System (3.39) of Example 3.4 was found to be $x(t) = c_1e^{-t}$, $y(t) = c_2e^{-t}$. Clearly $(0,0)$ is the only critical point of this system. For a given $\varepsilon > 0$, let $\delta = \varepsilon$. Thus $(c_1^2 + c_2^2)^{1/2} < \delta$ implies $\{[x(t)]^2 + [y(t)]^2\}^{1/2} \leq (c_1^2 + c_2^2)^{1/2} < \varepsilon$ for all $t \geq 0$; so $(0,0)$ is a stable critical point of System (3.39). Furthermore, this critical point is asymptotically stable since $e^{-t} \to 0$ as $t \to \infty$. (See Figure 16, page 65.)

A. PROBLEM LIST 12

1. Geometrically, what does it mean for a critical point of System (4.1) to be unstable?

2. Show that the system

$$x' = y$$

$$y' = -x$$

is neutrally stable.

3. Show that the system

$$x' = 3x$$

$$y' = 2y$$

is unstable.

4. Show that a critical point of System (4.1) which is not isolated cannot be asymptotically stable.

4.2. Critical Points and Stability for Linear Autonomous Systems. The concept of stability will now be examined in relation to the linear system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \ , \quad ad - bc \neq 0 \ . \end{aligned} \tag{4.2}$$

The condition $ad - bc \neq 0$ ensures that $(0,0)$ is the only critical point of this system.

The solutions of System (4.2) were discussed extensively in Section 3.2. The critical point $(0,0)$ may be characterized according to the nature of the roots λ_1 and λ_2 of the characteristic equation (3.13). Although we considered only three cases in Section 3.2, the description of the nature of the critical point $(0,0)$ of System (4.2) is not quite so simple. Actually, we must consider six cases:

1. λ_1 , λ_2 conjugate complex but not pure imaginary;
2. λ_1 , λ_2 pure imaginary;
3. λ_1 , λ_2 real, unequal and of the same sign;
4. λ_1 , λ_2 real, unequal and of opposite sign;
5. Exceptional case: $a = d$, $b = c = 0 (\lambda_1 = \lambda_2 = a = d)$;
6. $\lambda_1 = \lambda_2$ but not the exceptional case.

Case 1. $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, $\alpha \neq 0$, $\beta \neq 0$.

The critical point (0,0) is called a <u>focus</u> (or sometimes a focal point or spiral point). If $\alpha < 0$ the critical point is asymptotically stable (Figure 19) and unstable if $\alpha > 0$ (Figure 20).

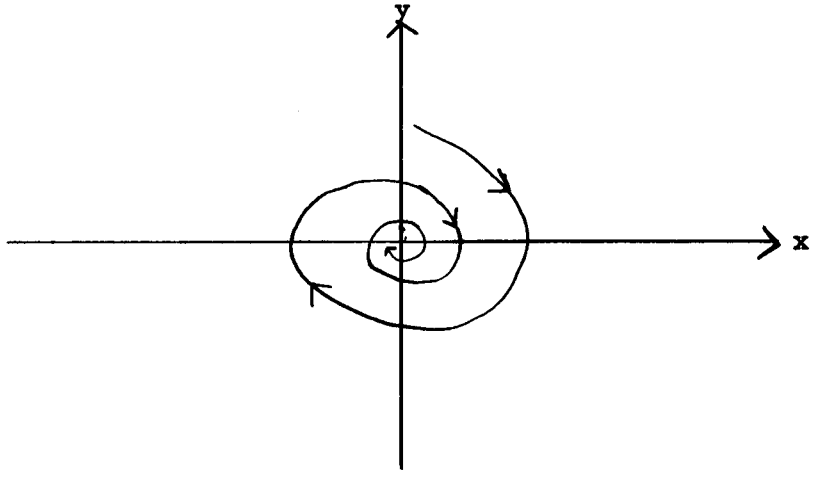


Figure 19. Asymptotically stable focus.

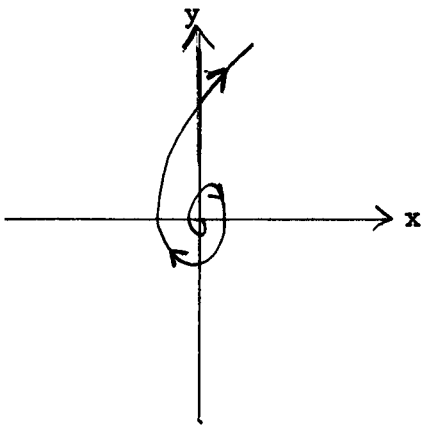


Figure 20. Unstable focus.

Case 2. $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$, $\beta \neq 0$.

The critical point (0,0) is called a <u>center</u> (or sometimes a vortex point). The trajectories are closed curves surrounding the origin. The critical point is stable even though the trajectories do not approach (0,0) ; that is, (0,0) is a neutrally stable critical point. (See Figure 21).

Case 3. λ_1 , λ_2 Real, Unequal and of the Same Sign.

In this case the critical point (0,0) is called a <u>node</u> (or sometimes a nodal point). If $\lambda_1 < \lambda_2 < 0$ the trajectories approach (0,0) as $t \to \infty$; thus (0,0) is an asymptotically stable critical point. (See Figure 22). If $0 < \lambda_1 < \lambda_2$ the solutions

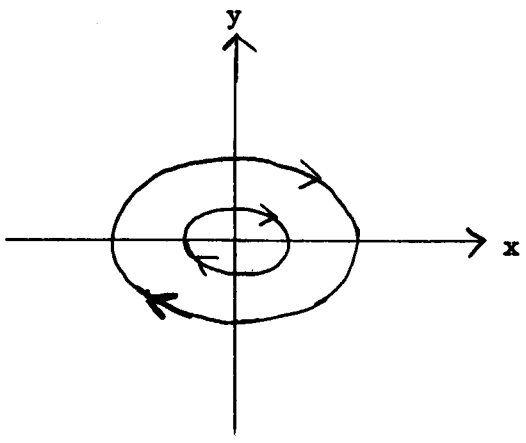


Figure 21. Stable center.

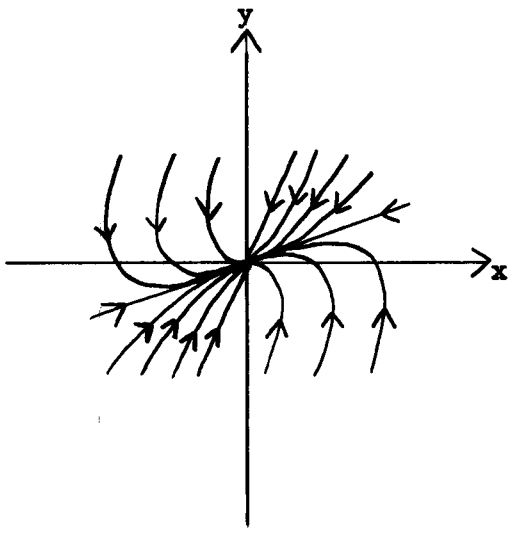


Figure 22. Asymptotically stable node.

$[x(t),y(t)] \to (0,0)$ as $t \to -\infty$; that is, (0,0) is an unstable critical point (Figure 22, page 78, with the arrows reversed).

Case 4. λ_1 , λ_2 Real, Unequal, and of Opposite Sign.

Suppose $\lambda_1 < 0 < \lambda_2$. The critical point (0,0) is a <u>saddle point</u> and is always unstable. (See Figure 23.)

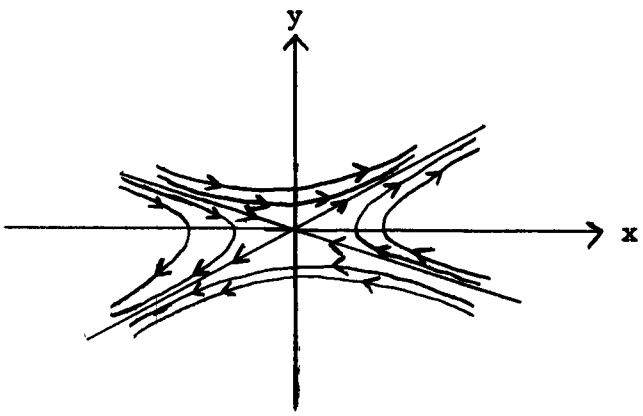


Figure 23. Unstable saddle point.

Case 5. $\lambda_1 = \lambda_2 = a = d$, $b = c = 0$.

In this case the trajectories are straight lines passing through the origin. (See Figure 24.) This configuration is known as a star and the critical point (0,0) is called a <u>star point</u> (or sometimes a node or nodal point). If $a < 0$ the star point is asymptotically

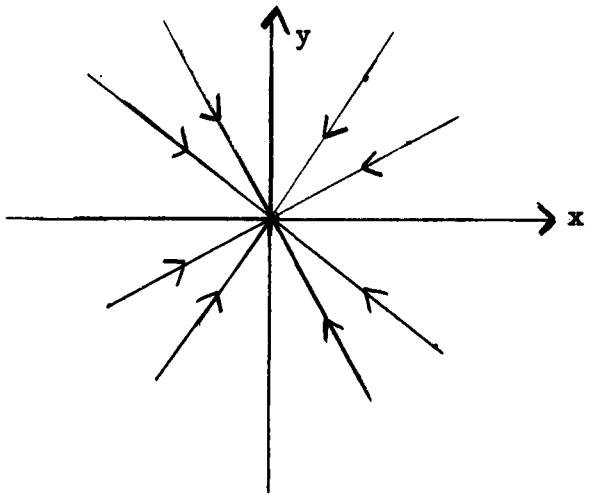


Figure 24. Asymptotically stable star point.

stable (Figure 24) and it is unstable if $a > 0$ (Figure 24 with the arrows reversed). (Note that $a \neq 0$ since we assumed $ad - bc \neq 0$.)

Case 6. $\lambda_1 = \lambda_2$ But Not Case 5.

The critical point (0,0) is called a <u>node</u> (or nodal point). If $\lambda_1 < 0$ the critical point is an asymptotically stable node; if $\lambda_1 > 0$, it is unstable, (Figure 25).

The results of this section are summarized in Table III. The conclusions concerning the stability of the critical point (0,0) for these six cases are verified by the following theorem, a proof of which can be found in Rabenstein [15].

THEOREM 4.1. The critical point (0,0) of the linear autonomous system (4.2) is stable if and only if both roots of the

characteristic equation (3.13) have nonpositive real parts. Furthermore, (0,0) is asymptotically stable if and only if both roots have negative real parts.

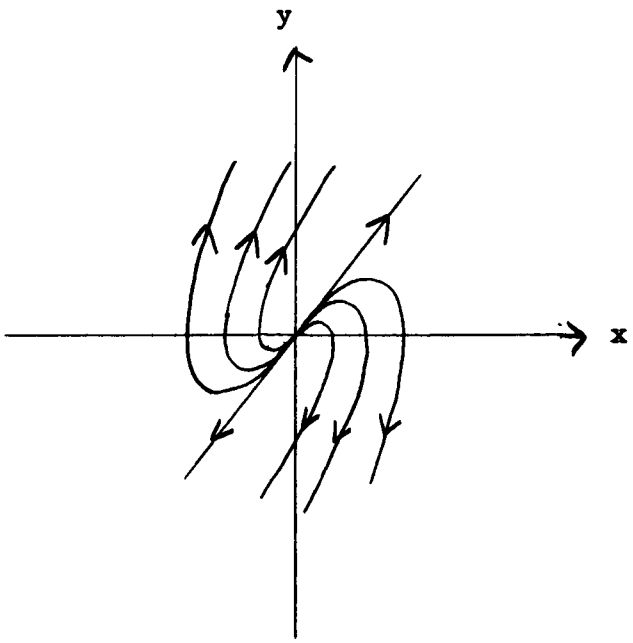


Figure 25. Unstable node.

TABLE III

STABILITY OF THE CRITICAL POINT (0,0) FOR THE LINEAR SYSTEM (4.2)

Nature of Roots	Nature of Critical Point	Stability
$\lambda_1 = \overline{\lambda}_2 = \alpha + \beta i$, $\alpha \neq 0$, $\beta \neq 0$	focus	asymptotically stable if $\alpha < 0$ unstable if $\alpha > 0$
$\lambda_1 = \overline{\lambda}_2 = \beta i$, $\beta \neq 0$	center	stable
$\lambda_1 < \lambda_2 < 0$ or	node	asymptotically stable
$0 < \lambda_1 < \lambda_2$	node	unstable
$\lambda_1 < 0 < \lambda_2$	saddle point	unstable
$\lambda_1 = \lambda_2 = a = d$, $b = c = 0$	star point	asymptotically stable if $a < 0$ unstable if $a > 0$
$\lambda_1 = \lambda_2$ but not the exceptional case	node	asymptotically stable if $\lambda_1 < 0$ unstable if $\lambda_1 > 0$

B. PROBLEM LIST 13

1. For each of the systems in Problem 1, Problem List 11, determine whether the origin is an asymptotically stable, a neutrally stable, or an unstable critical point.

2. Using the fact that the trajectories of solutions of System (4.2) are concentric ellipses when the roots of (C) are pure imaginary (see Problem 8, Problem List 11, give an argument to show that (0,0) is a stable critical point.

3. Consider the second-order equation with constant coefficients

$$\theta'' + b\theta' + a\theta = 0 \ , \ (' = d/dt) \ .$$

Discuss in detail the stability of the solution $\theta \equiv 0$ in relation to the roots of $\lambda^2 + b\lambda + a = 0$.

4. Show that if $ad - bc = 0$, then System (4.2) has infinitely many critical points, none of which is isolated.

5. Prove that the critical point (0,0) of System (4.2) is asymptotically stable if and only if $a + d < 0$ and $ad > bc$.

6. Determine relationships between the coefficients a, b, c, and d of System (4.2) which are necessary and sufficient for stability of the critical point (0,0).

7. For the linear autonomous system (4.2), show that

   a. if $a = d$ and b and c are of opposite sign, then (0,0) is a focus;

   b. if $a = -d$ and $ad > bc$, then (0,0) is a center;

   c. if $a = d$ and either $b = 0$ or $c = 0$ (but not both), then (0,0) is a node;

d. if $a = d$, b and c are of the same sign, and $0 < \sqrt{bc} > |a|$, then $(0,0)$ is a node;

e. if $a = d$, b and c are of the same sign, and $\sqrt{bc} > |a|$, then $(0,0)$ is a saddle point;

f. if $a = -d$ and b and c are of the same sign, then $(0,0)$ is a saddle point.

4.3. Critical Points and Stability for Nonlinear Autonomous Systems. In Section 2.5 the problem of two species competing for the same food supply was modeled by a nonlinear autonomous system. If $k = 0.1$, $c = 0.4$, $\alpha = 0.2$, $\beta = 0.5$, $x^* = 1000$, and $y^* = 2000$, then System (2.14) becomes

$$\frac{dx}{dt} = 0.1x\,\frac{[1000 - (x + 0.2y)]}{1000}$$

(4.3)

$$\frac{dy}{dt} = 0.4x\,\frac{[2000 - (y + 0.5x)]}{2000} \quad .$$

Since $\frac{x^*}{\alpha} > y^*$ and $x^* < \frac{y^*}{\beta}$, System (4.3) is seen to be of Case 4.

The point where the two lines of Figure 11, page 41, intersect is $(x_0, y_0) = (666\ 2/3, 1666\ 2/3)$. By making the change of variables $x_1 = x - x_0$, $y_1 = y - y_0$ this critical point can be translated to the origin. Then System (4.3) is equivalent to

$$\frac{dx_1}{dt} = 10^{-4}[-x_0(x_1 + 0.2y_1) - x_1(x_1 + 0.2y_1)]$$

(4.4)

$$\frac{dy_1}{dt} = 2 \cdot 10^{-4}[-y_0(0.5x_1 + y_1) - y_1(0.5x_1 + y_1)] \quad .$$

The linear approximation to System (4.4) formed by dropping the nonlinear terms is then

$$\frac{dx_2}{dt} = -x_0 \cdot 10^{-4}(x_2 + 0.2y_2)$$

(4.5)

$$\frac{dy_2}{dt} = -2y_0 \cdot 10^{-4}(0.5x_2 + y_2) \quad .$$

The characteristic equation of this linear autonomous system can be shown to be

(4.6) $$\lambda^2 + 0.366 \ldots \lambda + 0.02 = 0 \quad ,$$

and the characteristic roots are approximately $\lambda_1 = -0.34$ and $\lambda_2 = -0.06$. Thus by Case 3, Section 4.2, the origin is an asymptotically stable node for System (4.5). The question which naturally arises is: How well does the behavior of the linear system (4.5) approximate that of the nonlinear system (4.3)? That is, is (x_0, y_0) an asymptotically stable critical point for the nonlinear system? The rest of this chapter is devoted to seeking an answer to this question.

Consider the nonlinear autonomous system

$$x'(t) = f[x(t),y(t)]$$

(4.7)

$$y'(t) = g[x(t),y(t)] \ .$$

Suppose (x_0, y_0) is an isolated critical point of this system. Then by Taylor's theorem, if all of the nth partial derivatives of f and g are continuous in a closed region R of the x,y-plane containing (x_0, y_0), then f and g may be expanded around (x_0, y_0) using Taylor's formula. The Taylor series expansion for f would be

$$\begin{aligned} f(x,y) = {} & f(x_0, y_0) + D_x f(x_0, y_0)(x - x_0) \\ & + D_y f(x_0, y_0)(y - y_0) + D_{xx} f(x_0, y_0) \frac{(x-x_0)^2}{2} \\ & + D_{xy} f(x_0, y_0)(x - x_0)(y - y_0) \\ & + D_{yy} f(x_0, y_0) \frac{(y-y_0)^2}{2} + \ldots \ . \end{aligned} \tag{4.8}$$

Thus for a large class of functions, System (4.7) may be rewritten as

$$x'(t) = a(x - x_0) + b(y - y_0) + M(x - x_0, y - y_0)$$

(4.9)

$$y'(t) = c(x - x_0) + d(y - y_0) + N(x - x_0, y - y_0) ,$$

where $a = D_x f(x_0, y_0)$, $b = D_y f(x_0, y_0)$, $c = D_x g(x_0, y_0)$, $d = D_y g(x_0, y_0)$, $ad - bc \neq 0$, and the functions M and N satisfy the inequalities

$$|M(x - x_0, y - y_0)| \leq L \cdot [(x - x_0)^2 + (y - y_0)^2]$$

(4.10)

$$|N(x - x_0, y - y_0)| \leq L \cdot [(x - x_0)^2 + (y - y_0)^2]$$

for some nonnegative constant L and for all (x,y) within some circle centered at (x_0, y_0) . These inequalities guarantee that (x_0, y_0) is a critical point of System (4.9). Observe that the constant terms of the Taylor series expansion are missing in System (4.9) because $f(x_0, y_0) = g(x_0, y_0) = 0$. If $|x - x_0|$ and $|y - y_0|$ are sufficiently small, then $M(x - x_0, y - y_0)$ and $N(x - x_0, y - y_0)$ will be very small. The omission of these terms produces the linear autonomous system

$$x'(t) = a(x - x_0) + b(y - y_0)$$

(4.11)

$$y'(t) = c(x - x_0) + d(y - y_0)$$

where $ad - bc \neq 0$. Hence a system of the form (4.9) which satisfies the conditions (4.10) is often referred to as a perturbed linear system.

Most types of critical points are preserved by this quasilinearization process. Note that it is sufficient to suppose that $(x_0, y_0) = (0,0)$ since the substitution $[\hat{x}(t), \hat{y}(t)] = [x(t) - x_0, y(t) - y_0]$ reduces the problem to this case. With this assumption System (4.11) is equivalent to System (4.2). With reference to Section 4.2, it can be shown that the critical point (x_0, y_0) of System (4.9) is of the same type as for System (4.11) in Cases 1, 3, 4, and 6. Although (x_0, y_0) is a center for System (4.11) in Case 2, it may be either a center or a focus of System (4.9). Similarly, for Case 5 the critical point (x_0, y_0) may be either a star point or a focus for System (4.9).

It is also desirable to compare the stability of the critical points of Systems (4.9) and (4.11). The following theorem shows that the linear terms of System (4.9) are decisive in terms of asymptotic stability.

THEOREM 4.2. If the critical point (x_0, y_0) of the linear autonomous system (4.11) is asymptotically stable, then (x_0, y_0) is an asymptotically stable critical point of the perturbed linear system (4.9), where M and N satisfy the inequalities (4.10).

PROOF. Since the definition of stability is invariant under a linear transformation of coordinates, again it suffices to suppose that $(x_0, y_0) = (0,0)$. Recall from Theorem 4.1 that (0,0) is an asymptotically stable critical point of the linear system (4.11) if and only if both roots of the characteristic equation (3.13) have negative real parts. In order to use the canonical forms developed in Section 3.2, we distinguish three cases:

1. $\lambda_1 < \lambda_2 < 0$ or $\lambda_1 = \lambda_2 < 0$ when $b = c = 0$;
2. $\lambda_1 = \overline{\lambda_2} = \alpha + \beta i$ with $\alpha < 0$;
3. $\lambda_1 = \lambda_2 < 0$ when $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}$.

We consider in detail the proof for Case 1.

By Equations (3.16) and (3.34) there exists a nonsingular matrix $A = [a_{ij}]$ such that

$$A \begin{bmatrix} a & b \\ c & d \end{bmatrix} A^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} .$$

The change of variables $\begin{bmatrix} \xi \\ \eta \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$ leads to the matrix equation

$$\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} M^*(\xi,\eta) \\ N^*(\xi,\eta) \end{bmatrix} , \tag{4.12}$$

where $M^*(\xi,\eta) = a_{11}M(x,y) + a_{12}N(x,y)$ and $N^*(\xi,\eta) = a_{21}M(x,y) + a_{22}N(x,y)$. (Note that $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$.) By Problem 4 there exist constants k and K for which

$$(4.13) \qquad k(x^2 + y^2) \leq \xi^2 + \eta^2 \leq K(x^2 + y^2)$$

for all (x,y). Thus

$$\begin{aligned} |M^*(\xi,\eta)| &\leq |a_{11}|L(x^2 + y^2) + |a_{12}|L(x^2 + y^2) \\ &\leq (|a_{11}| + |a_{12}|)L(\xi^2 + \eta^2)/k \\ &\leq L_1(\xi^2 + \eta^2) \end{aligned} \qquad (4.14)$$

$$|N^*(\xi,\eta)| \leq L_1(\xi^2 + \eta^2) \ ,$$

where $L_1 = \max\{(|a_{11}| + |a_{12}|)L/k \ , \ (|a_{21}| + |a_{22}|)L/k\}$. Thus System (4.12) satisfies the same hypotheses as the original System (4.9). Since the definition of asymptotic stability is invariant under such a change of variables, it suffices to consider for Case 1 the matrix system (4.12) where M^* and N^* satisfy the inequalities (4.14). The other two cases can be treated similarly.

Suppose for the moment that (0,0) has been shown to be a stable critical point for System (4.12). Does this imply (0,0) is a stable critical point for System (4.9)? By Problem 4, if

$$\sqrt{[\xi(0)]^2 + [\eta(0)]^2} < \delta \Rightarrow \sqrt{[\xi(t)]^2 + [\eta(t)]^2} < \varepsilon \sqrt{k}$$

for all $t \geq 0$, then

$$\sqrt{[x(0)]^2 + [y(0)]^2} < \frac{\delta}{\sqrt{K}} \Rightarrow \sqrt{[\xi(0)]^2 + [\eta(0)]^2} < \delta$$

$$\Rightarrow \sqrt{[\xi(t)]^2 + [\eta(t)]^2} < \varepsilon\sqrt{k} \Rightarrow \sqrt{[x(t)]^2 + [y(t)]^2} < \varepsilon \ ;$$

hence stability of System (4.12) implies stability of System (4.9). The correspondence of asymptotic stability is left as an exercise.

We now show that (0,0) is a stable critical point for System (4.12). Let C be the circle centered at the origin of the ε,η-plane with radius $r = \frac{|\lambda_2|}{4L_1}$. Suppose (ε,η) is a solution of System (4.12) with $\{[\xi(0)]^2 + [\eta(0)]^2\}^{1/2} < r$. Define the function

$$E(t) = [\xi(t)]^2 + [\eta(t)]^2 \ . \tag{4.15}$$

E is called a Liapunov function. Then

$$\begin{aligned} E'(t) &= 2\xi(t)\xi'(t) + 2\eta(t)\eta'(t) \\ &= 2\lambda_1[\xi(t)]^2 + 2\lambda_2[\eta(t)]^2 \qquad (4.16) \\ &\quad + 2\xi(t)M^*[\xi(t),\eta(t)] + 2\eta(t)N^*[\xi(t),\eta(t)] \ . \end{aligned}$$

Thus if $[\xi(t),\eta(t)]$ is inside C, it follows from Equation (4.16), the inequalities (4.14), and $\lambda_1 \leq \lambda_2$ that

$$E'(t) \leq 2\lambda_2 \{[\xi(t)]^2 + [\eta(t)]^2\} + 2L_1[|\xi(t)| + |\eta(t)|]$$

$$\cdot \{[\xi(t)]^2 + [\eta(t)]^2\} \tag{4.17}$$

$$= 2(\lambda_2 + L_1|\xi(t)| + L_1|\eta(t)|)\{[\xi(t)]^2 + [\eta(t)]^2\} \ .$$

But since it is assumed that $[\xi(t)]^2 + [\eta(t)]^2 < r^2$, it follows that $|\xi(t)| < r = \frac{-\lambda_2}{4L_1}$ and $|\eta(t)| < \frac{-\lambda_2}{4L_1}$. Therefore

$$E'(t) \leq 2[\lambda_2 - \frac{\lambda_2}{2}]E(t) = \lambda_2 E(t) \ . \tag{4.18}$$

Several conclusions follow from (4.18). Notice first that as long as $[\xi(t), \eta(t)]$ is inside C , then $E'(t) < 0$; thus $[\xi(t), \eta(t)]$ moves closer to $(0,0)$ as t increases. So $[\xi(t), \eta(t)]$ remains inside C for all $t \geq 0$. The stability of the critical point $(0,0)$ of System (4.12) is now immediate. Given $\varepsilon > 0$, choose $\delta = \min \{r, \frac{\varepsilon}{2}\}$; then any solution of System (4.12) for which $\{[\xi(0)]^2 + [\eta(0)]^2\}^{1/2} < \delta$ remains within δ of $(0,0)$ for all $t \geq 0$. Moreover, from (4.15) and (4.18) it follows that $0 \leq E(t) \leq E(0)e^{\lambda_2 t}$. Taking the limit as $t \to \infty$ implies $E(t) \to 0$ since $e^{\lambda_2 t} \to 0$ for $\lambda_2 < 0$. Thus $[\xi(t), \eta(t)] \to (0,0)$ as $t \to \infty$; that is, $(0,0)$ is an asymptotically stable critical point for System (4.12).

The proofs for the two remaining cases are left to the reader. These cases should be treated similarly to Case 1; the one difference

is that for Case 3, define

$$E(t) = \lambda_0^2[\xi(t)]^2 + [\eta(t)]^2 \tag{4.19}$$

and use the inequality

$$\begin{aligned} &2\{\lambda_0^2\xi(t)[\lambda_0\xi(t) + \eta(t)] + \lambda_0[\eta(t)]^2\} \\ &= \lambda_0\{\lambda_0^2[\xi(t)]^2 + [\eta(t)]^2\}+ \qquad (4.20) \\ &\lambda_0[\lambda_0\xi(t) + \eta(t)]^2 \leq \lambda_0 E(t) \ . \end{aligned}$$

In addition we state the following theorem, a proof of which may be found in Coddington and Levinson [4].

THEOREM 4.3. Suppose the same conditions hold as in Theorem 4.2 except that at least one root of the characteristic equation (3.13) has positive real part. Then (x_0, y_0) is an unstable critical point of System (4.9).

Returning to the example of linear approximation with which we began this section, Theorem 4.2 guarantees that the critical point (666 2/3, 1666 2/3) of the nonlinear system (4.3) is asymptotically stable.

C. PROBLEM LIST 14

1. Making the change of variables $x_1 = x - x_0$, $y_1 = y - y_0$, show that System (4.3) is equivalent to System (4.4).

2. Show that Equation (4.6) is the characteristic equation of System (4.5) and find the approximate characteristic roots.

3. Verify that for Case 1 with $(x_0, y_0) = (0,0)$ the change of variables $\begin{bmatrix} \xi \\ \eta \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$ makes the systems (4.9) and (4.12) equivalent.

4. Define a transformation T of the x,y-plane by $T(x,y) = (\xi,\eta) = (a_{11}x + a_{12}y , a_{21}x + a_{22}y)$ with $a_{11}\, a_{22} - a_{21}\, a_{12} \neq 0$. Show that there exist numbers k and K such that for all (x,y)

$$k(x^2 + y^2) \leq \xi^2 + \eta^2 \leq K(x^2 + y^2) .$$

5. Show that if (0,0) is an asymptotically stable critical point for System (4.12), then (0,0) is asymptotically stable for System (4.9).

6. Use Taylor's formula to expand each of the following functions about the given point (s):

   a. $f(x,y) = 2x + 3y - 1$, (1,1), (2,0) ;

   b. $f(x,y) = x^2 + xy + y$, (1,2), (2,2) ;

   c. $f(x,y) = x^2 + x + xy^2 + 2y^2$, (1,1) ;

   d. $f(x,y) = \cos x \cos y$, (0,0) ;

   e. $f(x,y) = \sin x \sin y$, (0,0) .

7. For each system below, find all of the critical points and use the theorems of this section to determine their stability.

a. $x' = x + y + x^2 + y^2 = (x + 1/2)^2 + (y + 1/2)^2 - 1/2$

$y' = x - y - x^2 + y^2 = (y - x)(x + y - 1)$;

b. $x' = 1 + x - y^2$

$y' = 1 - x - y^2$;

c. $x' = 6y - 2y^2$

$y' = -10x - y + 4x^2 - 2xy + 4y^2$.

8. Use a graphical procedure (as in Chapter II) to sketch the trajectories of the system

$$x' = y$$

$$y' = -x + y(1 - x^2) .$$

Note: If a trajectory passes through (x_0, y_0) at time t_0 , it does so with slope $\dfrac{y'(t_0)}{x'(t_0)} = \dfrac{-x_0 + y_0(1 - x_0^2)}{y_0}$ if $y_0 \neq 0$.

9. A hydroplane operates according to the equation

$$m \frac{dv}{dt} = T(v) - D(v) ,$$

where m is the mass of the hydroplane, v is the velocity, $T(v)$ is the thrust of the engine, and $D(v)$ is the drag due to water and air resistance. Suppose T and D are as in the figure below. Sketch the shape of v versus t for values of $v(0) < v_0$, $v_1 < v(0) < v_2$, $v_2 < v(0) < v_3$, and $v_3 < v(0)$. Which modes of operation are unstable?

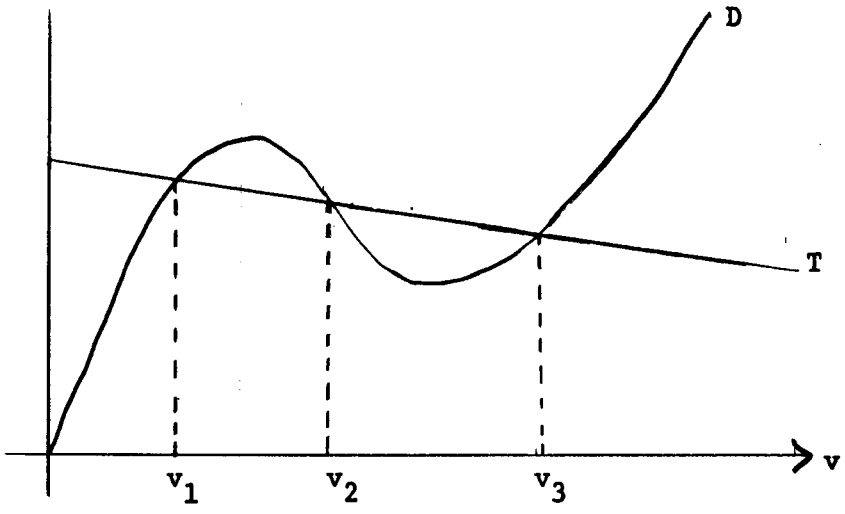


Figure 26. The thrust and drag of a hydroplane.

For a discussion of the nature of the trajectories of System (4.9) around a critical point, see Cesari [3]. The treatment here of critical points and stability for linear and nonlinear autonomous systems is based on Birkhoff and Rota [2] and Ross [16].

CHAPTER V

THE EXISTENCE OF PERIODIC SOLUTIONS

5.1. Nonlinear Conservative Systems. A conservative physical system is one in which the law of conservaton of energy is assumed to hold; that is, the sum of the kinetic energy and the potential energy is assumed to be constant in a conservative system.

Consider a particle of unit mass which is constrained to move along a straight line. Also assume that the particle is attracted toward a point 0 with a restoring force q that depends only on the displacement x , which is a function of time t . (See Figure 27.) Thus the velocity of this particle of unit mass at time t is $x'(t)$, while its kinetic energy is given by $1/2[x'(t)]^2$.

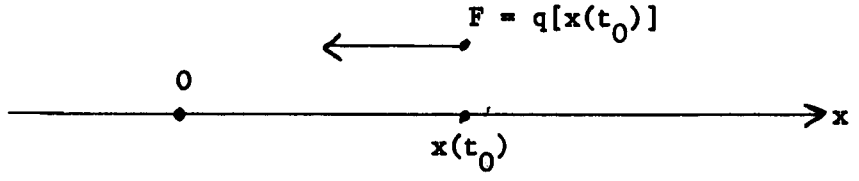


Figure 27. Rectilinear motion of a particle with unit mass.

Then by Newton's second law, the differential equation of motion is

$$x''(t) + q[x(t)] = 0 \ . \tag{5.1}$$

Some examples of physical systems which satisfy this equation are a weight attached to a spring $[q(x) = kx]$, the angular displacement of a vertical pendulum $[q(x) = k \sin x]$, and the deflection of a leaf spring $[q(x) = cx + dx^3]$. In general it will not be possible to solve Equation (5.1) explicitly for x . However, in many cases the phase portraits can be plotted, and thus a great deal of information concerning the qualitative behavior of the solutions can be obtained.

Suppose we restrict the class of admissible functions q to include only piecewise continuous functions. Then there exists a continuous function Q satisfying

$$Q'(x) = q(x) \tag{5.2}$$

for all x where q is continuous. By the fundamental theorem of calculus if q is continuous on $[x_0,x]$, then

$$\int_{x_0}^{x} q(u)du = Q(x) - Q(x_0) \ . \tag{5.3}$$

Multiplying Equation (5.1) by $x'(t)$ and then integrating yields the equation

$$1/2[x'(t)]^2 + Q[x(t)] = k \ , \tag{5.4}$$

where k is a constant which can be calculated from the initial values

of the displacement and the velocity. Notice that Equation (5.4) expresses the law of conservation of energy.

Since the second-order differential equation (5.1) is equivalent to the first-order autonomous system

$$(5.5) \qquad \begin{aligned} x'(t) &= y(t) \\ y'(t) &= -q[x(t)] \ , \end{aligned}$$

we can plot the trajectories in the x,y-phase plane. Then by Equation (5.4) the trajectories must satisfy the relation

$$1/2\ y^2 + Q(x) = k$$

or

$$(5.6) \qquad y = \pm \sqrt{2[k - Q(x)]} \ .$$

Hence for a given value of k, the trajectory defined by Equation (5.6) is a curve of constant energy in the x,y-phase plane with slope $\frac{dy}{dx} = -\frac{q(x)}{y}$. Notice also from Equation (5.6) that the trajectories are symmetrical with respect to the x-axis.

Recall from Section 4.1 that (x_0,y_0) is a critical point of System (5.5) provided $y(t) = y_0 = 0$ and $-q[x(t)] = -q(x_0) = 0$. Thus all critical points of System (5.5) are of the form $(x_0,0)$ where $q(x_0) = Q'(x_0) = 0$; that is, they all lie on the x-axis

and correspond to points on the graph of Q which have a horizontal tangent. Since at such a point both the velocity and the acceleration of the conservative physical system (5.5) are zero, the point $(x_0,0)$ is often called an equilibrium point of the system.

The following graphical procedure enables us to construct the trajectories of System (5.5) in the x,y-phase plane.

Step 1. In the x,z-plane, draw the graph of $z = Q(x)$ and the lines $z = k$ for various values of k.

Step 2. For each fixed value of k, compute the difference in ordinates $k - Q(x)$ for various values of x using the graph obtained in Step 1.

Step 3. For each k, use the relation (5.6) and the results of Step 2 to compute the corresponding y values for each x such that $k - Q(x)$ is nonnegative. Plot these points in the x,y-phase plane and connect them with a smooth curve.

Before applying this procedure to a specific example, we first consider three general cases.

Case 1. Q is monotonic.

Suppose Q is a monotone increasing function and consider the trajectory of constant energy $k = k_0$. Let x_0 be the point where $z = Q(x)$ and $z = k_0$ intersect; then $Q(x_0) = k_0$. Initially consider only the upper half of the trajectory where $y = \sqrt{2[k_0 - Q(x)]}$. Then as x approaches x_0 from the left, $k_0 - Q(x)$ decreases

monotonically to zero. But $\frac{dy}{dx}$ is infinite when $x = x_0$ since $y = 0$; thus the trajectory crosses the x-axis at a right angle. Since the trajectories are symmetric with respect to the x-axis, as k varies a family of parabola-like trajectories are produced which open to the left. (See Figure 29.) If Q is monotone decreasing, a similar analysis shows that a family of parabola trajectories which open to the right are produced.

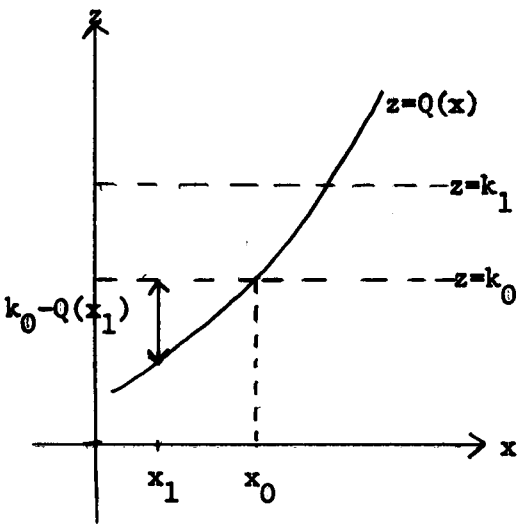


Figure 28. Q is monotonic.

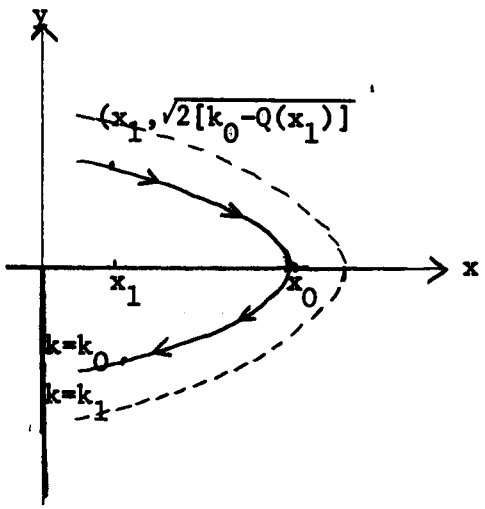


Figure 29. Family of parabola-like trajectories.

Case 2. Q has a local minimum at x_0.

To the left of x_0, $Q(x)$ is strictly decreasing, and to the right, strictly increasing. Thus from the analysis of Case 1, there must be a family of ellipse-like trajectories in the x,y-phase plane near the critical point $(x_0,0)$. (See Figure 31.) Thus the conservative system (5.5) has periodic solutions when the potential energy, $Q(x)$, is a minimum. As in Case 2, Section 4.2, this critical point is called a center and is obviously stable.

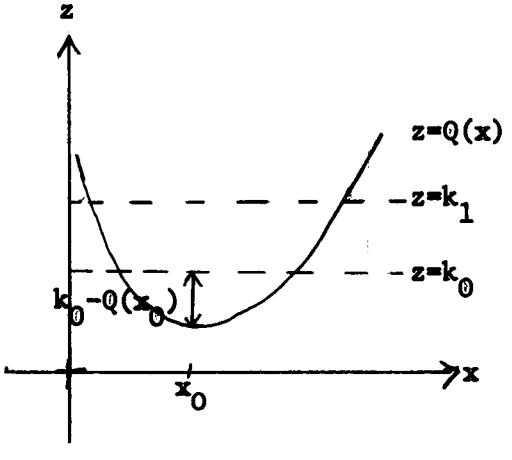


Figure 30. Q has a local minimum at x_0.

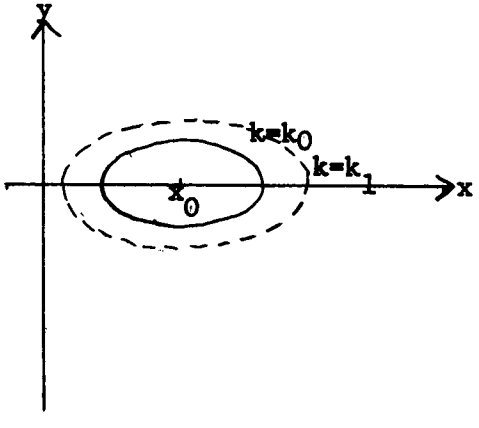


Figure 31. Family of ellipse-like trajectories.

Case 3. Q has a local maximum at x_0.

Suppose $k_0 = Q(x_0)$. Then since Q is monotone increasing to the left of x_0, we obtain a parabola-like curve which crosses the x-axis at $x = x_0$ and opens to the left. Similarly, the other half of this trajectory is a parabola-like curve opening to the right and passing through the point $(x_0,0)$. So for k near k_0 with $k < k_0$, the trajectories consist of pairs of parabola-like curves crossing the x-axis on opposite sides of x_0 and opening in opposite directions. For $k > k_0$, the trajectories will be pairs of curves which do not intersect the x-axis near x_0. (See Figure 33.) In this case the critical point $(x_0,0)$ is called a saddle point and is clearly unstable.

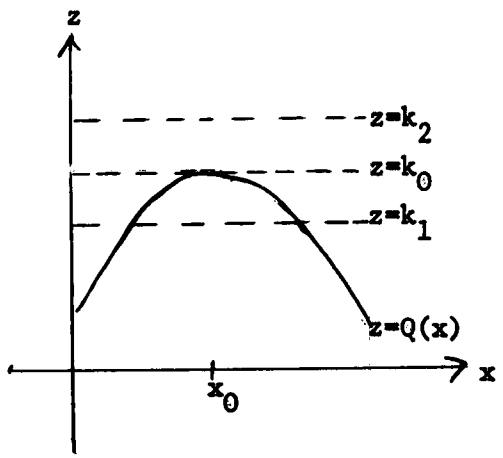


Figure 32. Q has a local maximum at x_0.

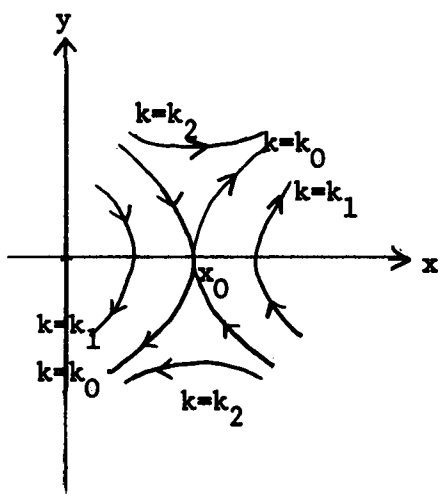


Figure 33. Family of parabola-like trajectories and directrixes.

Recall from Section 1.2 that the angular displacement of a simple vertical pendulum satisfies the equation

$$x''(t) + g/\ell \sin x(t) = 0 \,. \tag{5.7}$$

By quasilinearization approximate solutions were found for sufficiently small angular displacements. In this section, however, we use the procedure outlined above to determine if there are regions in the x,y-phase plane where this nonlinear conservative system has periodic solutions.

Consider the particular case $g/\ell = 1$ and $x_0 = 0$. Then $Q(x) = 1 - \cos x$ and Equation (5.6) becomes

$$y = \pm \sqrt{2(k - 1 + \cos x)} \,. \tag{5.8}$$

In Figure 34 several trajectories are sketched corresponding to various

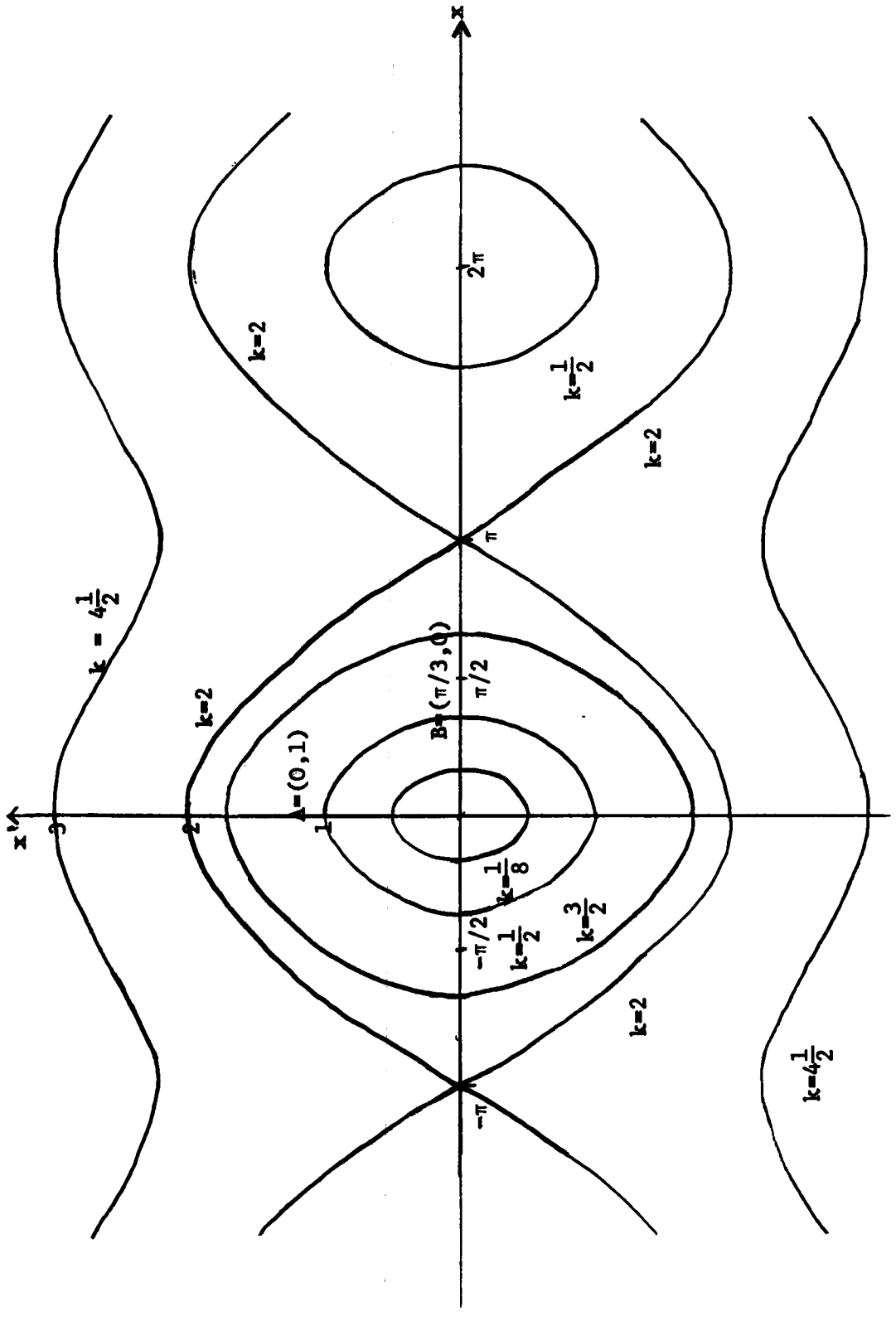


Figure 34. Phase portraits for solutions of $x'' + \sin x = 0$.

values of k. For $k < 2$, the trajectories are families of simple closed curves centered at the critical points $(0,0)$, $(\pm 2\pi,0)$, $(\pm 4\pi,0)$, ... These trajectories correspond to oscillations where y alternates in sign; that is, the pendulum oscillates back and forth. For $k > 2$, y is constant in sign. Referring to Figure 1, page 3, in this case the pendulum is rotating about the point 0; that is, it loops the loop. For $k = 2$, the pendulum approaches (but never reaches) an angular displacement of $x = \pm\pi$, $\pm 3\pi$, ... It is clear from Figure 34, page 105, that the critical points $[\pm(2n+1)\pi,0]$, $n = 0, 1, 2, \ldots$ are unstable.

Now that we have shown the existence of periodic solutions for $k < 2$ near the critical points $(\pm 2n\pi,0)$, $n = 0, 1, 2, \ldots$, we can calculate the period of the oscillations. In particular, we find the period T of the trajectory about the origin for $k = 1/2$. Notice from Figure 34, page 105, that the period of oscillation will be four times the time required to go from point A to point B on this trajectory, since this curve is symmetric with respect to both axes. But $x = 0$ at point A, and if B has coordinates $(b,0)$, then we must have $0 = 1/2 - 1 + \cos b = -1/2 + \cos b$ where $0 < b < \pi$. Thus $b = \pi/3$. In this case b is called the <u>amplitude</u> of the oscillation and the interval $[-b,b]$ is called the <u>span</u> of the oscillation.

When $y(t) > 0$ Equation (5.8) gives

$$x'(t) = y(t) = \sqrt{2 \cos [x(t)] - 1} \quad . \tag{5.9}$$

Hence the period of oscillation must be

$$T = 4\int_0^{T/4} \frac{x'(t)dt}{\sqrt{2\cos[(x(t)] - 1}} = 4\int_0^{\pi/3} \frac{dx}{\sqrt{2\cos x - 1}}$$

(5.10)

$$= 4\int_0^{\pi/3} \frac{dx}{\sqrt{1 - 4\sin^2(\frac{x}{2})}} ,$$

the last equality resulting from the trigonometric identity $\cos x = 1 - 2\sin^2(x/2)$. You are asked to show in Problem 6 that the substitution $2\sin(x/2) = \sin u$ reduces Equation (5.10) to

$$T = 4\int_0^{\pi/2} \frac{du}{\sqrt{1 - 1/4\sin^2 u}} .$$

(5.11)

This integral is an elliptic integral of the first kind whose value can be found from special tables, such as [17, p. 529]. It is found that $T = 4(1.6858) = 6.7432$.

In general, however, the trajectories corresponding to periodic solutions of System (5.5) will not be symmetric with respect to the y-axis. In this case the period is given by

$$T = 2\int_a^b \frac{dx}{\sqrt{2[k - Q(x)]}} ,$$

(5.12)

where $Q(a) = Q(b) = k$. Again the interval $[a,b]$ is called the span of the oscillation. This integral is an improper integral since $Q(a) = k = Q(b)$. It can be shown using advanced calculus

that this integral exists whenever both $Q'(a) = q(a) \neq 0$ and $Q'(b) = q(b) \neq 0$.

A. PROBLEM LIST 15

1. Sketch the phase portraits for solutions of System (5.5) around the critical point (0,0) for each of the following cases:

   a. $q(x) = 4x$;

   b. $q(x) = 4x^3$;

   c. $q(x) = x + x^3$;

   d. $$q(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} ;$$

   e. $$q(x) = \begin{cases} \frac{1}{x^2} & \text{if } x > 1 \\ x & \text{if } -1 \leq x \leq 1 \\ -\frac{1}{x^2} & \text{if } x < -1 \end{cases} ;$$

   (Use k = 1/4, 1/2, 1, 1 1/4.)

   f. $$q(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases} .$$

2. In parts (a) - (d) of Problem 1, express the period in terms of an integral. If possible, compute these integrals.

3. Show that the trajectories of

$$x'' - x + x^3 = 0$$

satisfy the relation

$$(x')^2 - x^2 + \frac{x^4}{2} = c \ .$$

Sketch several trajectories in the x,x'-phase plane and conclude that there are centers at $(\pm 1, 0)$ and a saddle point at $(0,0)$.

4. For the equation

$$x'' + qx + rx^3 = 0 \ ,$$

where $r > 0$, $q > 0$, show that the period T of an oscillation is

$$T = 4\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{[2q + rL^2(1 + \sin^2 \theta)]^{1/2}} \ ,$$

where L is the amplitude of an oscillation.

5. Consider the trajectory in Figure 34, page 105, which passes through point $(0,2)$. Show that the time τ_ε required to go from $(0,2)$ to $(\pi - \varepsilon, \sqrt{2[1 + \cos (\pi-\varepsilon)]})$ is

$$\tau_\varepsilon = \int_0^{\pi-\varepsilon} \frac{dx}{2(1+\cos x)} = \int_0^{\pi-\varepsilon} \frac{dx}{2 \cos (x/2)} \ .$$

Also show that $\tau_\varepsilon \to \infty$ as $\varepsilon \to 0$ and thus conclude that the state $(\pi,0)$ is never reached.

6. For the pendulum equation (5.7) with $g/\ell = 1$, the period of a trajectory about $(0,0)$ when $k < 2$ is

$$T = 4 \int_0^b \frac{dx}{\sqrt{2(k-1 + \cos x)}} \ .$$

Show that the substitution $\sqrt{2/k} \sin (x/2) = \sin u$ yields

$$T = 4 \int_0^{2\pi} \frac{du}{\sqrt{1 - k/2 \sin^2 u}} \ .$$

Evaluate T for $k = 0, 1/2, 1, 3/2$, and $7/4$, and sketch the graph of T versus k.

7. Suppose that x_1 and x_2 are two nonlinear oscillations with the same span $[a,b]$ satisfying $x'' + q_1(x) = 0$ and $x'' + q_2(x) = 0$, respectively. Show that if $\frac{q_2(x)}{x} > \frac{q_1(x)}{x}$ for $x \neq 0$, then the period T_2 of x_2 is less than the period T_1 of x_1.

8. Suppose that q in Equation (5.1) is an odd function (hence Q is even) and that x_1 and x_2 satisfy (5.1) and have spans $[-b_1,b_1]$ and $[-b_2,b_2]$ (with $b_2 > b_1$) and periods T_1 and T_2, respectively. Show that if $\frac{q(x)}{x}$, $x > 0$, is an increasing function, then $T_2 < T_1$, and if $\frac{q(x)}{x}$, $x > 0$, is a decreasing function, then $T_2 > T_1$. (Hint: Let $y(t) = cx_1(t)$ where $c = \frac{b_2}{b_1}$. Show that y has span $[-b_2, b_2]$ and satisfies

$$y''(t) + cq[y(t)/c] = y''(t) + Q[y(t)] = 0$$

where $Q(y) = cq(y/c)$. Then use Problem 7.

9. Determine whether $x = 0$ is a stable solution of each of the following:

a. $x'' + (1 - x^2)x' + x = 0$;

b. $x'' - (\cos x)x' + \sin x = 0$;

c. $x'' + \alpha x' + (x+2x^3) = 0$.

5.2. Limit Cycles and the Liénard Equation. A phenomenon of nonlinear oscillations that has no parallel in linear oscillations is the presence of what is called a limit cycle. When the linear oscillator described by the equation

(5.13) $\theta'' + b\theta' + a\theta = 0$

is negatively damped $(b > 0)$, the amplitude of oscillation always increases exponentially without bound. However, in certain negatively damped nonlinear oscillators, the amplitude of oscillation often approaches a finite limit. This was observed as early as 1883 by Lord Rayleigh in a discussion of the damping of a vibratory system under a viscosity factor in his paper, "On Maintained Vibrations." He showed how a steady state oscillation might occur in the equation

(5.14) $x'' + [-b + k(x')^2]x' + cx = 0$,

where b, k, and c are positive constants.

In 1922 in his study of the triode oscillator, van der Pol encountered the nonlinear equation

(5.15) $x'' + \mu(x^2 - 1)x' + x = 0$,

where μ is a positive constant. Using the method of isoclines van der Pol was able to demonstrate the existence of a limit cycle.

An explicit closed-form solution of the van der Pol equation (5.15) is not possible. However, the solution of the system

(5.16)
$$x' = y + x(1 - x^2 - y^2)$$
$$y' = -x + y(1 - x^2 - y^2)$$

can be obtained explicitly using polar coordinates (r,θ) . Substituting $x = r\cos\theta$, $y = r\sin\theta$ into System (5.16) gives the equivalent system

$$x' = r'\cos\theta - r\theta'\sin\theta = r\sin\theta + r\cos\theta\,(1 - r^2)$$

(5.17)

$$y' = r'\sin\theta + r\theta'\cos\theta = -r\cos\theta + r\sin\theta\,(1 - r^2) \quad .$$

Solving System (5.17) for r' and θ' gives the equivalent first-order system

$$r' = r(1 - r^2)$$

(5.18)

$$\theta' = -1 \ .$$

The first equation may be solved using partial fractions. After integration we find $r^2 = \dfrac{1}{1 + ce^{-2t}}$; hence the general solution of System (5.18) may be written as

$$r = \frac{1}{\sqrt{1 + ce^{-2t}}}$$

(5.19)

$$\theta = -t + t_0 \ .$$

If $c = 0$, then $r = 1$; so the trajectory is the circle $x^2 + y^2 = 1$, where the movement is clockwise as time increases. If $c \neq 0$, then $r \to 1$ as $t \to \infty$. If $c < 0$, the trajectory is outside the circle $x^2 + y^2 = 1$ but spirals toward it as $t \to \infty$;

for $c > 0$, the trajectory is inside the circle but spirals toward it as $t \to \infty$. Several trajectories are sketched in Figure 35.

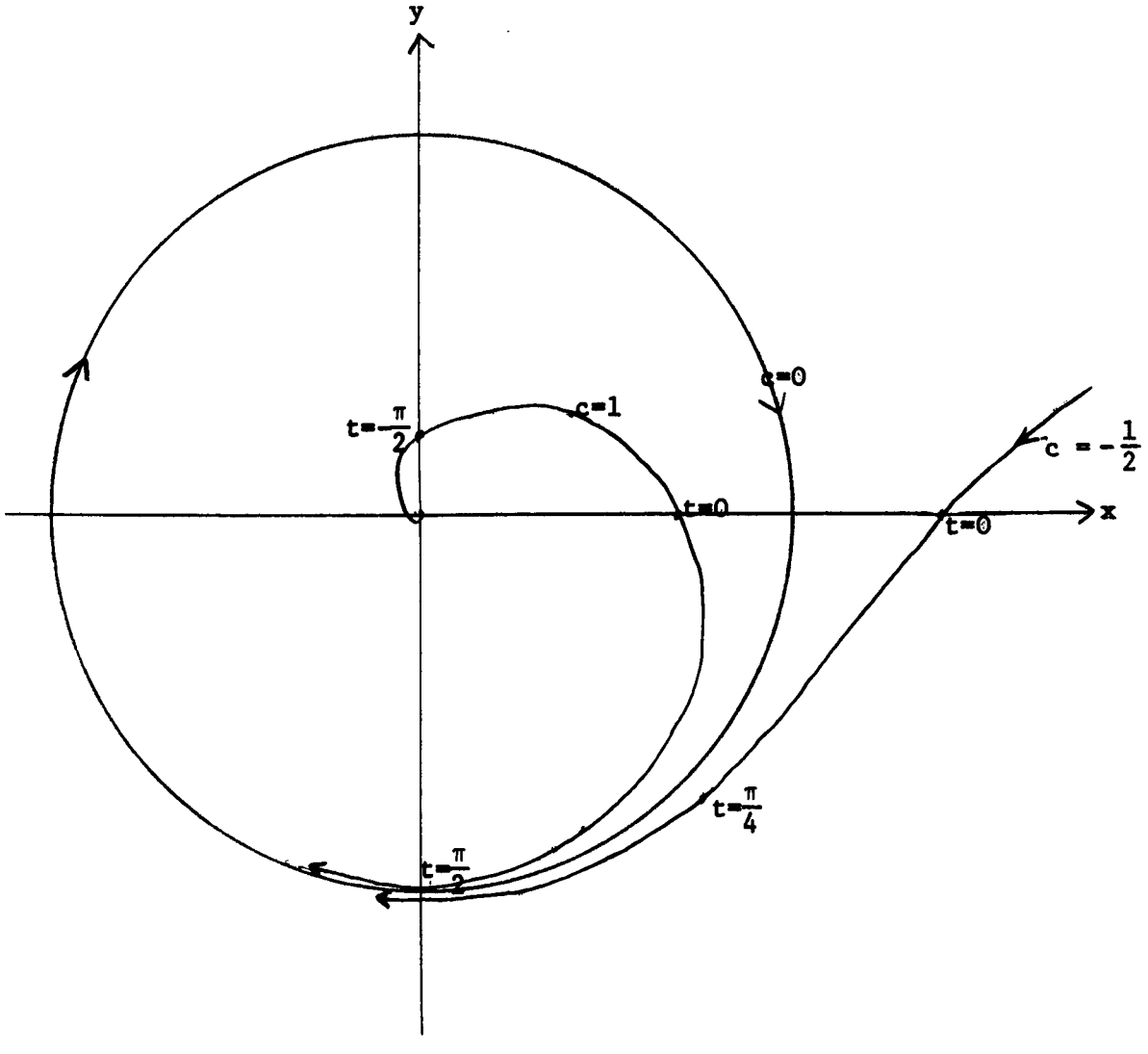


Figure 35. Phase portraits for solutions of System (5.16).

DEFINITION 5.1. If the trajectory of a nonconstant periodic solution of the system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{5.20}$$

is approached spirally from either the inside or the outside by the trajectory of a nonperiodic solution of System (5.20) either as $t \to \infty$ or as $t \to -\infty$, and if some neighborhood of it contains no other closed trajectory, then this closed trajectory is called a limit cycle.

We now discuss a general type of equation known as the Liénard equation:

$$x'' + f(x)x' + g(x) = 0 . \tag{5.21}$$

The following important theorem concerning the existence of periodic solutions of a specific class of second-order nonlinear differential equations is stated without proof.

THEOREM 5.1. Suppose the functions f and g are defined and continuous on $(-\infty,\infty)$ and satisfy the following conditions on $(-\infty,\infty)$:

1. f is even; i.e., $f(x) = f(-x)$;
2. g is odd; i.e., $g(x) = -g(-x)$;
3. $xg(x) > 0$ for $x \neq 0$;

4. There is a number $b > 0$ such that $\int_0^x f(u)du < 0$ for $0 < x < b$ and $\int_0^x f(u)du > 0$ for $x > b$; i.e., $\int_0^x f(u)du = 0$ has a unique positive root $x = b$;
5. $f(x) > 0$ for $x > b$ and $\int_0^x f(u)du \to \infty$ as $x \to \infty$;
6. $\int_0^x g(u)du \to \infty$ as $x \to \infty$.

Then the Liénard equation (5.21) has a unique limit cycle.

As an application of this theorem, we show the existence of a unique limit cycle for the van der Pol equation (5.15). In this case $f(x) = \mu(x^2 - 1)$ and $g(x) = x$. Thus f and g are continuous on $(-\infty,\infty)$, f is an even function and g , an odd function, and $xg(x) = x^2 > 0$ if $x \neq 0$. Define $F(x) = \int_0^x f(u)du = 1/3\ \mu x(x^2 - 3)$ and $G(x) = \int_0^x g(u)du = 1/2x^2$. Then $F(x) < 0$ for $0 < x < \sqrt{3}$ and $F(x) > 0$ for $x > \sqrt{3}$. Furthermore for $x > \sqrt{3}$, $f(x) = \mu(x^2 - 1) > 0$ and $F(x) \to \infty$ as $x \to \infty$. Finally $G(x) = 1/2x^2 \to \infty$ as $x \to \infty$. Thus the hypotheses of Theorem 5.1 are satisfied, and we can conclude that the van der Pol equation (5.15) has a unique limit cycle. Using the method of isoclines, Ross [16, p. 506] has constructed trajectories in the x,y-phase plane for $\mu = 0.1$, $\mu = 1$, and $\mu = 10$. These results are shown in Figure 36.

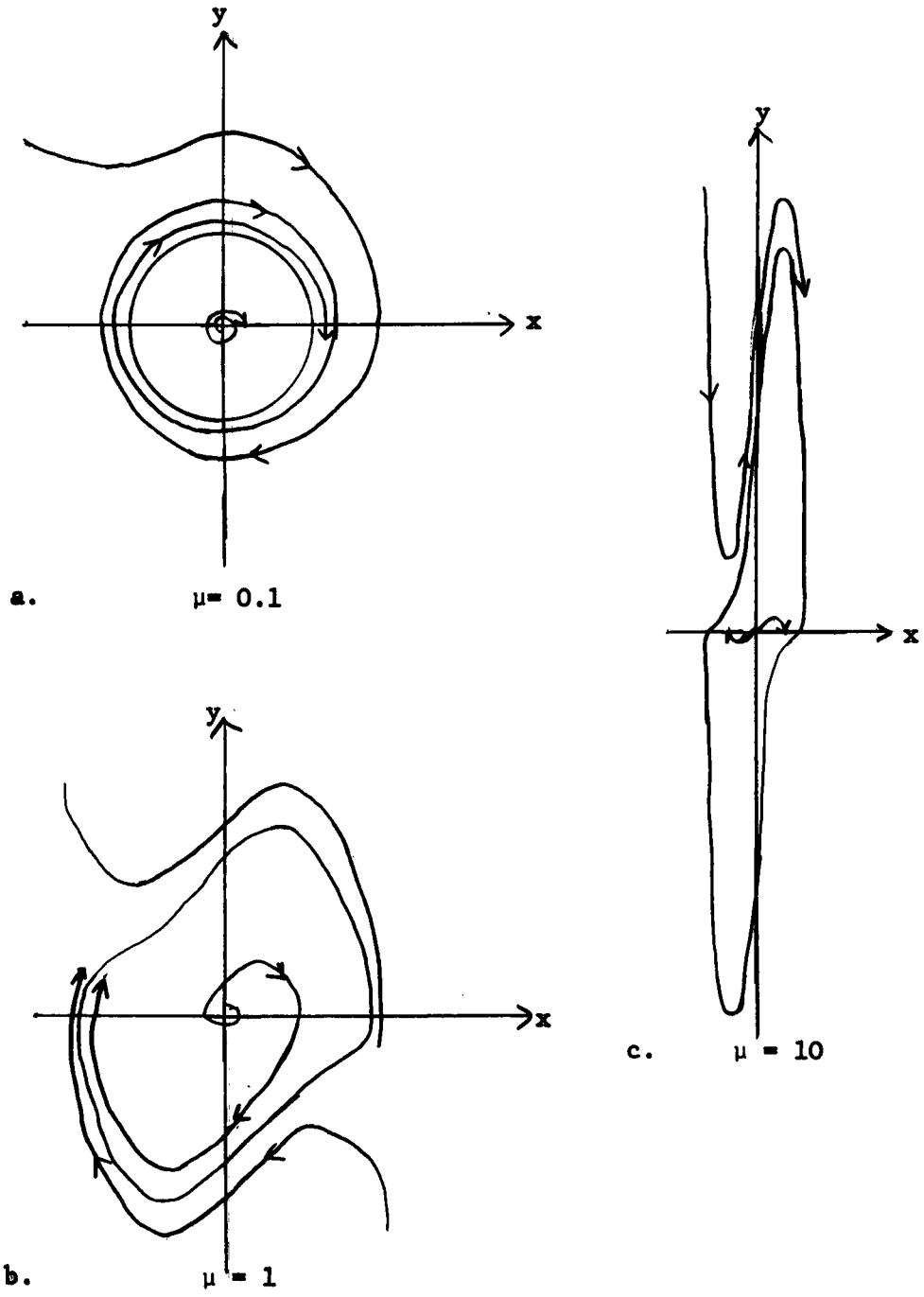


Figure 36. Unique limit cycle for the van der Pol equation.

B. PROBLEM LIST 16

1. For a solution x of Equation (5.14) define $y(u) = qx'(t)$, $t = pu$, where p and q are chosen so that $cp/q = 1$ and $q^2 = 3k/b$. Show that y satisfies

$$y''(u) + (pb)\{[y(a)]^2 - 1\}y'(u) + y(u) = 0 .$$

2. Solve (5.18) to obtain (5.19).

3. Show that the hypotheses of Theorem 5.1 are satisfied for each of the following equations:

   a. $x'' + (x^4 - 1)x' + x = 0$;

   b. $x'' + (3x^2 - 2)x' + x + x^3 = 0$;

   c. $x'' + (x^2 - \cos x)x' + \arctan x = 0$.

4. Prove the following negative criterion for the existence of a periodic solution of System (5.20): If f and g have continuous first partial derivatives inside a simple closed curve C in the x,y-plane and $f_x + g_y$ is of constant sign in the interior of C, then System (5.20) has no periodic solution whose trajectory lies in the interior of C. Hint: Use Green's theorem in the plane.

5. Show that the equation

$$x'' + f(x)x' + x = 0$$

is equivalent to the system

$$\begin{aligned} x' &= y - F(x) \\ y' &= -x , \end{aligned} \tag{5.22}$$

where $F(x) = \int_0^x f(u)du$.

6. If (x,y) is a solution of System (5.22), show that the line normal to the trajectory at the point $(x_0,y_0) = [x(t_0),y(t_0)]$ has slope $[y_0 - F(x_0)]/x_0 (x_0 \neq 0)$. Show that this normal goes through the point $[0,F(x_0)]$.

7. Justify the following procedure for drawing tangent segments for the trajectories of System (5.22):

Step 1. Construct the graph of F versus x.

Step 2. Given $P_0 = (x_0,y_0)$, draw a vertical line through P_0 and locate the intersection Q_0 with the graph of F.

Step 3. Draw the horizontal line through Q_0 and locate the intersection R_0 with the y-axis.

Step 4. Draw through P_0 a segment perpendicular to the line through R_0 and P_0.

8. Use the method of Problem 7 to sketch the phase portraits (for System (5.22)) of the following equations:

a. $x'' + (x^2 - 1)x' + x = 0$;

b. $x'' + (x^4 - x^2)x' + x = 0$;

c. $x'' + (\cos x)x' + x = 0$.

Section 5.1 closely parallels the discussion in Lakin and Sanchez [10]. A proof of Theorem 5.1 for the case $g(x) = x$ is to be found in Davies and James [6]. See also Rabenstein [15] and Ross [16].

CHAPTER VI

APPROXIMATION METHODS

6.1. Introduction. In the preceding chapter we discussed the existence of periodic solutions of certain types of nonlinear differential equations. Now we seek approximate explicit expressions for the periodic solutions of autonomous equations of the form

$$x'' + \omega^2 x = \varepsilon f(x,x') \ , \ (' = d/dt) \ , \tag{6.1}$$

where $\omega > 0$ and ε is a sufficiently small parameter so that the nonlinear term $\varepsilon f(x,x')$ is relatively small. If $\varepsilon = 0$ the solution of the resulting linear equation,

$$x'' + \omega^2 x = 0 \ , \tag{6.2}$$

may be written in the form

$$x(t) = a \cos (\omega t + \phi) \ , \tag{6.3}$$

where a and ϕ are constants which depend upon the initial conditions. In this case the solution is periodic with period $T = 2\pi/\omega$. Thus it seems reasonable to suspect that when ε is sufficiently small, periodic solutions of Equation (6.1) will lie

in a neighborhood of some (or perhaps all) of the solutions of the corresponding linear equation (6.2). Although various methods have been devised to obtain explicit closed forms of approximate solutions, only two of these will be discussed here - perturbation methods and averaging methods. Emphasis will be placed on the discussion of the technique of solution rather than on a mathematical justification of the method used.

6.2. Perturbation Methods. In seeking periodic solutions $x(t)$ of autonomous equations, there is no loss of generality in assuming $x'(0) = 0$, since this only effects a shift of the origin along the t-axis. Suppose x is the solution of Equation (6.2) satisfying the initial conditions $x(0) = A$, $x'(0) = 0$. Then it is desirable to know how much the presence of the disturbing function $\varepsilon f(x,x')$ alters or perturbs the solution of Equation (6.1) which satisfies the same initial conditions.

The perturbation method due to Poisson consists of the following steps:

Step 1. Assume that the periodic solutions can be expressed in a power series in the small parameter ε; that is,

$$x(t) = x_0(t) + x_1(t)\varepsilon + x_2(t)\varepsilon^2 + \dots \quad . \tag{6.4}$$

Step 2. Substitute the formal power series (6.4) into Equation (6.1). By equating coefficients of equal powers of ε,

determine successive linear differential equations for $x_0(t)$, $x_1(t)$, $x_2(t)$, ..., $x_k(t)$, where k is the desired order of approximation.

Step 3. Solve the differential equations for $x_i(t)$, $i = 0, 1, \ldots, k$ to determine the solution x to the kth order of approximation.

The major disadvantage of Poisson's method is the appearance of coefficients in the power series expansion (6.4), such as $t \cos t$, whose amplitude becomes unbounded as t approaches infinity. Such terms are called <u>secular terms</u>. To illustrate this concept, we consider the equation

$$x'' + x = -\varepsilon x$$

or

$$x'' + (1+\varepsilon)x = 0 \ , \ x(0) = A, \ x'(0) = 0 \ . \tag{6.5}$$

In this case the exact solution is easily found to be

$$x(t) = A \cos \sqrt{1+\varepsilon}\, t \ , \tag{6.6}$$

which is clearly periodic with period $T = 2\pi/\sqrt{1+\varepsilon}$. We now apply Poisson's method and assume that a periodic solution of Equation (6.5) is of the form (6.4) where $x_i'(0) = 0$ for $i = 0, 1, 2, \ldots$.

Since we know there is a parametric family of periodic solutions of Equation (6.5), we may assume that $x_0(0) = A$ and $x_i(0) = 0$ for $i \geq 1$. (In the case where only isolated periodic solutions exist and each $x_i(0)$ can assume only specific values, this assumption is not valid.) Then x' and x'' may be obtained by differentiating Equation (6.4) termwise. Substituting these results into Equation (6.5) yields

$$(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \ldots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)$$

(6.7)

$$+ (\varepsilon x_0 + \varepsilon^2 x_1 + \varepsilon^3 x_2 + \ldots) = 0 .$$

Suppose we desire a first order approximation. Equating the coefficients of ε^0 and ε^1 in Equation (6.7) results in the differential equations

(a) $x_0'' + x_0 = 0 , \; x_0(0) = A , \; x_0'(0) = 0 ,$

(6.8)

(b) $x_1'' + x_1 + x_0 = 0 , \; x_1(0) = x_1'(0) = 0 .$

The solution of Equation (6.8) (a) is easily found to be $x_0(t) = A \cos t$. Substituting this result into Equation (6.8)(b), one obtains the equation

(6.9) $\quad x_1'' + x_1 = -A \cos t , \; x_1(0) = x_1'(0) = 0 ,$

whose solution may be found by the method of undetermined coefficients to be

$$(6.10) \qquad x_1(t) = -1/2\, At \sin t \ .$$

Thus the solution of Equation (6.5) to first-order terms is

$$(6.11) \qquad x(t) = A \cos t - 1/2 \varepsilon At \sin t + \ldots \ .$$

The term $1/2\, At \sin t$ is a secular term since a poor approximation of the exact solution (6.6) is obtained for large values of t . Notice, however, that the Taylor's series expansion of the exact solution (6.6) about $\varepsilon = 0$ gives Equation (6.11) also. The exact solution is periodic, but this is not apparent from the series expansion (6.11). Hence the presence of secular terms does not imply that the solution diverges, but rather than an inappropriate choice of the form of the solutions has been made. The choice is inappropriate because, in general, the presence of secular terms prevents the determination of periodic solutions of the equation.

The reason for the occurrence of secular terms is that, in general, the frequency when $\varepsilon = 0$ is not the same as when $\varepsilon \neq 0$. A variation of Poisson's method, due to Lindstedt (see [3, pp. 116-118]) assumes that the frequency of the periodic solution may also be expressed as a power series in ε with constant coefficients. Lindstedt's method consists of removing the terms in the successive

linear differential equations, which lead to secular terms in the solution, as they arise. This procedure may be outlined as follows:

Step 1. Make the transformation $\tau = \Lambda t$ so that the solution, as a function of τ, is periodic with period $T = 2\pi$. Note: If x is periodic with period $2\pi/\Lambda$ in t, then x will be periodic in τ of period $T = 2\pi$ since

$$x(\tau) = x(\Lambda t) = x[\Lambda(t+2\pi/\Lambda)] = x(\Lambda t+2\pi) = x(\tau+2\pi) .$$

Step 2. To assure that $x(\tau) = x(\tau+2\pi)$, assume that $x_i(\tau) = x_i(\tau+2\pi)$ for $i = 0, 1, 2, \ldots$.

Step 3. Assume that the periodic solutions and their frequencies can be expressed in a power series in ε; that is,

$$x(\tau) = x_0(\tau) + x_1(\tau)\varepsilon + x_2(\tau)\varepsilon^2 + \ldots \tag{6.12}$$

and

$$\Lambda = \Lambda_0 + \Lambda_1\varepsilon + \Lambda_2\varepsilon^2 + \ldots \quad . \tag{6.13}$$

Step 4. Substitute the formal power series (6.12) and (6.13) into Equation (6.1) and equate the coefficients of equal powers of ε to obtain successive linear differential equations for $x_0(\tau)$, $x_1(\tau)$, $x_2(\tau)$, ... $x_k(\tau)$, where k is the desired order of approximation.

Step 5. Determine the periodic solution, $x(\tau)$, and its frequency, Λ , to the kth order of approximation by solving the successive differential equations for $x_i(\tau)$, $i = 0, 1, \ldots, k$, eliminating the secular terms as they arise.

EXAMPLE 6.1. Use Lindstedt's method to approximate the periodic solutions and their frequencies up to first-order terms for the equation

$$(6.14) \qquad x'' + x = \varepsilon x^3 \ , \ x(0) = A \ , \ x'(0) = 0 \ , \ (' = \frac{d}{dt}) \ .$$

SOLUTION. Since we do not know if a parametric family of periodic solutions exists, we can only assume

$$x(0) = A = A_0 + A_1\varepsilon + A_2\varepsilon^2 + \ldots$$

where $x_i(0) = A_i$ for $i = 0, 1, 2, \ldots$. The substitution $\tau = \Lambda t$, where Λ is given by Equation (6.13), transforms Equation (6.14) into

$$(6.15) \qquad \Lambda^2\ddot{x} + x = \varepsilon x^3 \ , \ x(0) = A \ , \ \dot{x}(0) = 0 \ , \ (\cdot = \frac{d}{d\tau}) \ .$$

We try to determine Λ so that Equation (6.15) will have a periodic solution in τ with period $T = 2\pi$.

Assume Equation (6.15) has a solution of the form (6.12) where $x_i(\tau + 2\pi) = x_i(\tau)$ and $\dot{x}_i(0) = 0$ for $i = 0, 1, 2, \ldots$. Step 4 of Lindstedt's method yields the equation

$$(\Lambda_0^2 + 2\varepsilon\Lambda_0\Lambda_1 + 2\varepsilon^2\Lambda_0\Lambda_2 + \varepsilon^2\Lambda_1^2 + \ldots)(\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \ldots)$$

$$(6.16) \quad + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)$$

$$= \varepsilon(x_0^3 + 3\varepsilon x_0^2 x_1 + 3\varepsilon^2 x_0^2 x_2 + 3\varepsilon^2 x_0 x_1^2 + \ldots) \ .$$

If Equation (6.16) is expanded and the coefficients of equal powers of ε are equated, one obtains the successive linear equations

$$(6.17) \quad \begin{aligned} &(a) \quad \Lambda_0^2\ddot{x}_0 + x_0 = 0 \\ &(b) \quad \Lambda_0^2\ddot{x}_1 + x_1 = x_0^3 - 2\Lambda_0\Lambda_1\ddot{x}_0 \end{aligned} \ .$$

The general solution of Equation (6.17) (a) is $x_0(\tau) = k_1 \cos(\tau/\Lambda_0) + k_2 \sin(\tau/\Lambda_0)$. However, the initial conditions $x_0(0) = A_0$, $\dot{x}_0(0) = 0$ imply $k_1 = A_0$, $k_2 = 0$, while the periodicity condition $x_0(\tau) = x_0(\tau + 2\pi)$ requires that $\Lambda_0 = 1$. If $x_0(\tau) = A_0 \cos\tau$, $\Lambda_0 = 1$, and the trigonometric identity $\cos^3\tau = 1/4(3\cos\tau + \cos 3\tau)$ are substituted into Equation (6.17)(b), one obtains

$$(6.18) \quad \begin{aligned} \ddot{x}_1 + x_1 &= A_0^3 \cos^2\tau + 2\Lambda_1 A_0 \cos\tau \\ &= A_0(3/4\ A_0^2 + 2\Lambda_1)\cos\tau + 1/4\ A_0^3 \cos 3\tau \ . \end{aligned}$$

Equation (6.18) can be solved by the method of undetermined coefficients, but the term $A_0(3/4\ A_0^2 + 2\Lambda_1)\cos\tau$ leads to a secular term in the solution of the form $k_1\ \tau\cos\tau + k_2\ \tau\sin\tau$, which is not periodic. Hence following Lindstedt's method we choose $\Lambda_1 = -3/8\ A_0^2$, and Equation (6.18) reduces to

$$\ddot{x}_1 + x_1 = 1/4\ A_0^3\cos 3\tau \quad , \tag{6.19}$$

whose solution is found to be

$$x_1(\tau) = A_1\cos\tau - 1/32\ A_0^3\cos 3\tau \quad . \tag{6.20}$$

Then to first-order terms the periodic solutions of Equation (6.14) are approximated by

$$x(\tau) = A_0\cos\tau + \varepsilon(A_1\cos\tau - 1/32\ A_0^3\cos 3\tau) + \ldots \tag{6.21}$$

where

$$\Lambda = 1 - 3/8\ A_0^2\varepsilon + \ldots$$

and

$$A = A_0 + A_1\varepsilon + A_2\varepsilon^2 + \ldots \ .$$

Although Lindstedt's method overcomes the problem of secular terms, it has a disadvantage of its own. Poincaré has shown that, in general, the series obtained by Lindstedt's method does not converge. Poincaré's variation of the perturbation method takes into

account that the solution x in Equation (6.4) may be periodic without all of the coefficients x_i , $i = 0, 1, 2, \ldots$ being periodic. Then the series (6.4) converges to a solution of Equation (6.1) and if certain periodicity conditions are satisfied, a periodic solution will exist. (For a discussion of Poincaré's method, see [3, pp. 118-120].) Poincaré's method is particularly suited to the motion of planets since the effect of secular terms is slight due to their relatively slow motion. However, Lindstedt's method is probably better suited to the study of nonlinear oscillations.

A. PROBLEM LIST 17

1. Use Lindstedt's method to approximate a periodic solution and its frequency, up to first-order terms, for each of the following equations satisfying the initial conditions $x(0) = A$, $x'(0) = 0$:

a. $x'' + \omega^2 x = \varepsilon x^3$;

b. $x'' + (1+\varepsilon)^2 x = 0$;

c. $x'' + \omega^2 x = \varepsilon x^5$;

d. $x'' + x = \varepsilon(x^3 + x')$;

e. $x'' + x = \varepsilon(x' - 1/3\ x'^3)$;

f. $x'' + x = \varepsilon x'$;

g. $x'' + x = -2\varepsilon(x')^2$;

h. $x'' + k_1 x' + k_2 x + k_3 x^3 = 0$;

i. $x'' + \omega^2 x = \varepsilon[(x')^3 + x^3]$;

j. $x'' + \omega^2 x = \varepsilon[1 + x^2]x'$;

k. $x'' + x = \varepsilon[8/9\ x^4 - 1]x'$.

2. Use the method of Lindstedt to show that an approximation to the limit cycle for the van der Pol equation,

$$x'' + x = \varepsilon(1 - x^2)x' ,$$

with small ε satisfying the initial conditions

$$x(0) = A_0 + A_1\varepsilon + A_2\varepsilon^2 + \ldots, \quad x'(0) = 0$$

is

$$x(\tau) = 2 \cos \tau + \varepsilon(3/4 \sin \tau - 1/4 \sin 3\tau) + \ldots$$

where the frequency is $\Lambda = 1 - 1/16\ \varepsilon^2 + \ldots$.

6.3. Averaging Methods. In this section we discuss an averaging method due to the Russian mathematicians Krylov and Bogolyubov (henceforth abbreviated "K & B method") for seeking approximate explicit expressions for the periodic solutions of Equation (6.1), where $\omega > 0$ and ε is sufficiently small, say $0 < |\varepsilon| << 1$. Recall from Section (6.1) that if $\varepsilon = 0$, the solution of the resulting linear equation (6.2) may be written in the form (6.3). The derivative of Equation (6.3) is

$$x'(t) = -a\omega \sin (\omega t + \phi) . \tag{6.22}$$

The K & B method assumes that the nonlinear equation (6.1) has solutions of the form (6.3) with derivatives of the form (6.22), provided that a and ϕ are now considered to be "slowly varying" functions of time. Notice that this is basically a method of variation of parameters analogous to the approach used to solve

nonhomogeneous linear equations when solutions of the corresponding homogeneous equation are known.

Thus the K & B method assumes that Equation (6.1) has a solution x of the form

$$x(t) = a(t) \cos [\omega t + \phi(t)] \tag{6.23}$$

with derivative x' of the form

$$x'(t) = -a(t)\omega \sin [\omega t + \phi(t)] \, . \tag{6.24}$$

Differentiation of Equation (6.23) yields

$$\begin{aligned} x'(t) = a'(t) \cos [\omega t + \phi(t)] - a(t)\phi'(t) \sin [\omega t + \phi(t)] \\ - a(t)\omega \sin [\omega t + \phi(t)] \, . \end{aligned} \tag{6.25}$$

Thus in order for x' to be of the form (6.24), we must require

$$a'(t) \cos [\omega t + \phi(t)] - a(t)\phi'(t) \sin [\omega t + \phi(t)] = 0 \, . \tag{6.26}$$

Differentiating the assumed derivatives (6.24), one obtains

$$\begin{aligned} x''(t) = -\omega a'(t) \sin [\omega t + \phi(t)] - \omega^2 a(t) \cos [\omega t + \phi(t)] \\ - \omega a(t)\phi'(t) \cos [\omega t + \phi(t)] \, . \end{aligned} \tag{6.27}$$

Substituting the assumed solution (6.23) and its assumed derivatives (6.24) and (6.27) into the differential equation (6.1) and simplifying yields

$$-\omega a'(t) \sin [\omega t + \phi(t)] - \omega a(t)\phi'(t) \cos [\omega t + \phi(t)]$$

$$= \varepsilon f\{a(t) \cos [\omega t + \phi(t)], -\omega a(t) \sin [\omega t + \phi(t)]\} \quad . \tag{6.28}$$

Solving Equations (6.26) and (6.28) simultaneously for $a'(t)$ and $\phi'(t)$, one obtains

$$\text{(a)} \quad a'(t) = -\frac{\varepsilon}{\omega} f\{a(t) \cos [\omega t + \phi(t)],$$

$$-a(t)\omega \sin [\omega t + \phi(t)]\}\sin [\omega t + \phi(t)] \tag{6.29}$$

$$\text{(b)} \quad \phi'(t) = -\frac{\varepsilon}{\omega a(t)} f\{a(t) \cos [\omega t + \phi(t)],$$

$$-a(t)\omega \sin [\omega t + \phi(t)]\} \cos [\omega t + \phi(t)] \quad .$$

These are the exact equations for finding the functions a and ϕ when Equation (6.1) is assumed to have a solution of the form (6.23) with derivative of the form (6.24). However, due to the complicated nature of these nonlinear autonomous equations, a solution cannot, in general, be obtained.

For ε sufficiently small, Equations (6.29)(a) and (b) show that a and ϕ are "slowly varying" functions of time, as was expected. Suppose we define $\psi(t) = \omega t + \phi(t)$. Then during a period of $T = 2\pi/\omega$ in t , $a(t)$ and $\phi(t)$ are almost constant, and $\psi(t)$ increases by approximately 2π . The K & B method assumes that $a(t)$ and $\phi(t)$ are constant during an interval of 2π in ψ . The first approximation consists of replacing the right-hand sides of Equations (6.35) by their mean values over an interval of 2π , that is, the first term of their Fourier series expansion. We thus obtain the equations

$$\text{(a)}\quad a'(t) = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi\ ,\ -a\omega\sin\psi)\sin\psi d\psi\ ;$$

(6.30)

$$\text{(b)}\quad \phi'(t) = -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi\ ,\ -a\omega\sin\psi)\cos\psi d\psi\ .$$

Thus a first approximation to the solution of Equation (6.1) is given by Equation (6.23), where $a(t)$ and $\phi(t)$ are determined by Equations (6.30). The refinement of the first approximation and the successive nth approximations for $n = 2, 3, 4, \ldots$ are beyond the scope of this text.

The method of averaging due to van der Pol assumes that Equation (6.1) is nonautonomous. For this reason the van der Pol equations, which are analogous to Equations (6.30), are nonautonomous and thus more complex. Since the two methods follow the same lines for the first approximation, the van der Pol method is not discussed here.

EXAMPLE 6.2. Use the K & B method of averaging to find the first approximation of a solution of the equation

$$x'' + \omega^2 x = \varepsilon x^3 \ , \ (' = d/dt) \tag{6.31}$$

satisfying the initial conditions $x(0) = A$, $x'(0) = 0$.

SOLUTION. Substituting $f(a\cos\psi \ , \ -a\omega \sin\psi) = a^3 \cos^3\psi$ into Equation (6.30)(a) and evaluating the integral, one obtains

$$\begin{aligned} a'(t) &= \frac{a^3\varepsilon}{2\pi\omega} \int_0^{2\pi} \cos^3\psi \ (-\sin\psi)d\psi \\ &= \frac{a^3\varepsilon}{2\pi\omega} \ \frac{\cos^4\psi}{4} \Big|_0^{2\pi} = 0 \ . \end{aligned} \tag{6.32}$$

Thus $a(t) = A_0$, a constant. Then from Equation (6.30)(b), we find

$$\begin{aligned} \phi'(t) &= -\frac{\varepsilon A_0^2}{2\pi\omega} \int_0^{2\pi} \cos^4\psi d\psi \\ &= -\frac{\varepsilon A_0^2}{2\pi\omega}\left\{\frac{3\psi}{8}\Big|_0^{2\pi} + \frac{\sin 2\psi}{4}\Big|_0^{2\pi} + \frac{\sin 4\psi}{32}\Big|_0^{2\pi}\right\} = -\frac{3\varepsilon A_0^2}{8\omega} \ . \end{aligned} \tag{6.33}$$

Hence

$$\phi(t) = -\frac{3\varepsilon A_0^2}{8\omega} \ t + \phi_0 \ , \tag{6.34}$$

where ϕ_0 is a constant of integration. Thus the first approximation of Krylov and Bogolyubov is

$$(6.35) \qquad x(t) \tilde{\sim} A_0 \cos\left[\left(\omega - \frac{3\varepsilon A_0^2}{8\omega}\right)t + \phi_0\right] .$$

If we choose $A_0 = A$ and $\phi_0 = 0$, then the initial conditions $x(0) = A$, $x'(0) = 0$ are satisfied.

B. PROBLEM LIST 18

1. Use the K & B method to find a first approximation to the solution of each of the differential equations in Problem 1, Problem List 17. In each case, assume $0 < \varepsilon << 1$ and $a(0) = a_0 > 0$.

2. Suppose that the function f in Equation (6.1) is independent of x'; that is, $x'' + \omega^2 x = \varepsilon f(x)$. Show that the amplitude $a(t)$ is a constant a_0 and that $\phi(t) = K(a_0)t + \phi_0$, where $K(a_0) = -\frac{\varepsilon}{2\pi\omega a_0}\int_0^{2\pi} f(a_0 \cos\psi)\cos\psi\, d\psi$ and ϕ_0 is a constant of integration. In the x,x'-phase plane, the trajectories are ellipses with centers at the origin. Hence the first approximation of K & B is

$$x(t) = a_0 \cos\{[\omega + K(a_0)]t + \phi_0\} .$$

3. Suppose f is independent of x; that is, $x'' + \omega^2 x = \varepsilon f(x')$. Show that $\phi(t)$ is a constant ϕ_0. Thus the first approximation of K & B is given by

$$x(t) = a(t) \cos(\omega t + \phi_0) ,$$

where $a(t)$ may be calculated from Equations (6.30)(a). Therefore, the first approximation to a solution is nonperiodic with variable amplitude $a(t)$ and constant frequency $\omega/2\pi$.

4. Show that the first approximation of the K & B method to the solutions of the van der Pol equation

$$x'' + x = \varepsilon(1-x^2)x' ,$$

where $0 < \varepsilon < < 1$, given the initial amplitude $a(0) = a_0$, is

$$x(t) = \frac{a_0}{\sqrt{a_0^2/4 + e^{-\varepsilon t}(1-a_0^2/4)}} \sin (t + \phi_0) .$$

Using Definition 5.1, verify that for $0 < \varepsilon < < 1$, the limit cycle of the van der Pol equation is approximately a circle with radius 2. (See Figure 36, page 116.)

For additional reading see Aggarwal [1], Cesari [3], Lakin and Sanchez [10], and Ross [16].

CHAPTER VII

NUMERICAL METHODS

7.1. Introduction. In this chapter we consider the initial-value problem

$$y'(x) = f[x,y(x)] , \quad y(x_0) = y_0 \tag{7.1}$$

over some interval $[x_0,x_f]$ of x. Throughout this text it has been observed that an explicit closed-form solution of most nonlinear equations of this type is not obtainable. Various graphical and approximation methods of solution have already been considered. However, in many physical problems involving a differential equation of this form, it is desired to find a table of values of a solution y of Equation (7.1) for a finite range of values of x near the initial point x_0. For this reason, this chapter is devoted to numerical methods of solution of the initial-value problem (7.1).

The principal objective of this chapter is to present certain basic numerical methods which are easily programmed for a digital computer. The theoretical justification of these methods and a detailed discussion of their accuracy are beyond the scope of this text. It is assumed that the reader has an elementary knowledge of FORTRAN and the accessibility of a digital computer. All computations for this chapter were performed on an IBM System/360 computer and the graphical output was produced on a CalComp pen plotter.

7.2. Numerical Integration and Runge-Kutta Type Methods.

Three basic numerical methods of solution of the first-order equation (7.1) will now be considered. Each method will evaluate the function f at a finite number of grid points in the interval $[x_0,x_f]$ and combine these values in some manner so as to approximate a numerical solution of Equation (7.1) at these grid points. For simplicity, we will assume that the grid points are equally-spaced throughout the interval $[x_0,x_f]$. That is, we will approximate successively the values of the solution function y at the points $x_i = x_0 + ih$, $i = 1, 2, \ldots, N$ where the increment in x, $h = (x_f - x_0)/N$, is called the steplength or stepsize. These approximate values of y will be denoted y_i, $i = 1, 2, \ldots, N$.

The simplest method is that of Euler. Integration of both sides of Equation (7.1) over the interval $[x_0,x_1]$ yields

$$\int_{x_0}^{x_1} f(x,y)dx = \int_{x_0}^{x_1} \frac{dy}{dx}\,dx = y(x_1) - y(x_0) = y(x_1) - y_0$$

or

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x,y)dx \quad . \tag{7.2}$$

If we assume that f is a slowly varying function on the interval $[x_0,x_1]$, $f(x,y)$ can be approximated by its value $f(x_0,y_0)$ at the left endpoint x_0. Then

$$y(x_1) \approx y_0 + \int_{x_0}^{x_1} f(x_0,y_0)dx$$

$$= y_0 + f(x_0,y_0)(x_1 - x_0) \tag{7.3}$$

$$= y_0 + hf(x_0,y_0) \ .$$

Proceeding in this manner, one can find an approximate value y_i of y at the point x_i in terms of y_{i-1} by the formula

$$y_i = y_{i-1} + hf(x_{i-1},y_{i-1}) \ , \ i = 1,2, \ldots, N \ . \tag{7.4}$$

Euler's method is exact only if f is a polynomial of degree 0 or 1. Otherwise, the validity of the assumption that f varies slowly on an interval of length h clearly depends upon the size of h. Thus, in general, Euler's method is not very practical because of the small stepsize required to obtain reasonable accuracy.

The <u>second-order Runge-Kutta method</u> (also called Heun's method or modified Euler's method) replaces $f(x,y)$ in Equation (7.2) by its average value at the endpoints. Thus the general formula for this method is

$$y_i = y_{i-1} + \frac{h}{2}[f(x_{i-1},y_{i-1}) + f(x_{i-1}+h,y_{i-1}+h\cdot f(x_{i-1},y_{i-1}))] \ , \tag{7.5}$$

$$i = 1, 2, \ldots, N \ .$$

Although, in general, one can use a larger stepsize h for this method than with Euler's method to obtain the same degree of accuracy, a disadvantage is that the derivative f(x,y) must be evaluated twice for each step of the integration.

The method which is probably used more than any other is the fourth-order Runge-Kutta method. The general formula for this method is

$$y_i = y_{i-1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \ , \qquad (7.6)$$

$$i = 1, 2, \ldots, N$$

where

$$k_1 = hf(x_{i-1}, y_{i-1})$$

$$k_2 = hf(x_{i-1} + \tfrac{1}{2}h \ , \ y_{i-1} + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_{i-1} + \tfrac{1}{2}h \ , \ y_{i-1} + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_{i-1} + h, \ y_{i-1} + k_3) \ .$$

This method takes a weighted sum of derivatives in the interval in an attempt to obtain greater accuracy. The major disadvantage of this method, however, is that four derivative evaluations are

required for each step of integration. This might require considerable more computer time for those problems in which the function f is very complicated.

To compare the results of these three methods, a first-order initial-value problem with an exact closed-form solution has been chosen.

EXAMPLE 7.1. Obtain a numerical solution of the initial-value problem

$$y'(x) = \sin(2x) - y(x)\tan(x),\quad y(0) = 1 \tag{7.7}$$

on the interval $[0,1.5]$ with $h = 0.1$ for each of the three methods.

SOLUTION. The exact solution of (7.7) can be shown to be

$$y(x) = 3\cos x - 2\cos^2 x\ .$$

The exact values as well as the computed values for each of the three methods are shown in Table IV. These results are plotted in Figure 37 to provide a comparison. It is clear that the fourth-order Runge-Kutta method is the most accurate, and Euler's method is least accurate.

TABLE IV

SOLUTION OF EQUATION (7.7) ON [0,1.5] WITH $h = 1/10$

I	X	Exact Solution	Euler's Method	Runge-Kutta Order Two	Runge-Kutta Order Four
1	0.1	1.00495	1.00000	1.00492	1.00495
2	0.2	1.01914	1.00983	1.01900	1.01914
3	0.3	1.04067	1.02830	1.04033	1.04067
4	0.4	1.06648	1.05296	1.06583	1.06648
5	0.5	1.09245	1.08018	1.09140	1.09244
6	0.6	1.11365	1.10531	1.11209	1.11365
7	0.7	1.12456	1.12290	1.12237	1.12456
8	0.8	1.11932	1.12686	1.11636	1.11932
9	0.9	1.09203	1.11079	1.08817	1.09203
10	1.0	1.03705	1.06820	1.03212	1.03705
11	1.1	0.94929	0.99277	0.94310	0.94928
12	1.2	0.82447	0.87856	0.81680	0.82445
13	1.3	0.65939	0.72013	0.64986	0.65934
14	1.4	0.45212	0.51228	0.43987	0.45199
15	1.5	0.20221	0.24877	0.18222	0.20130

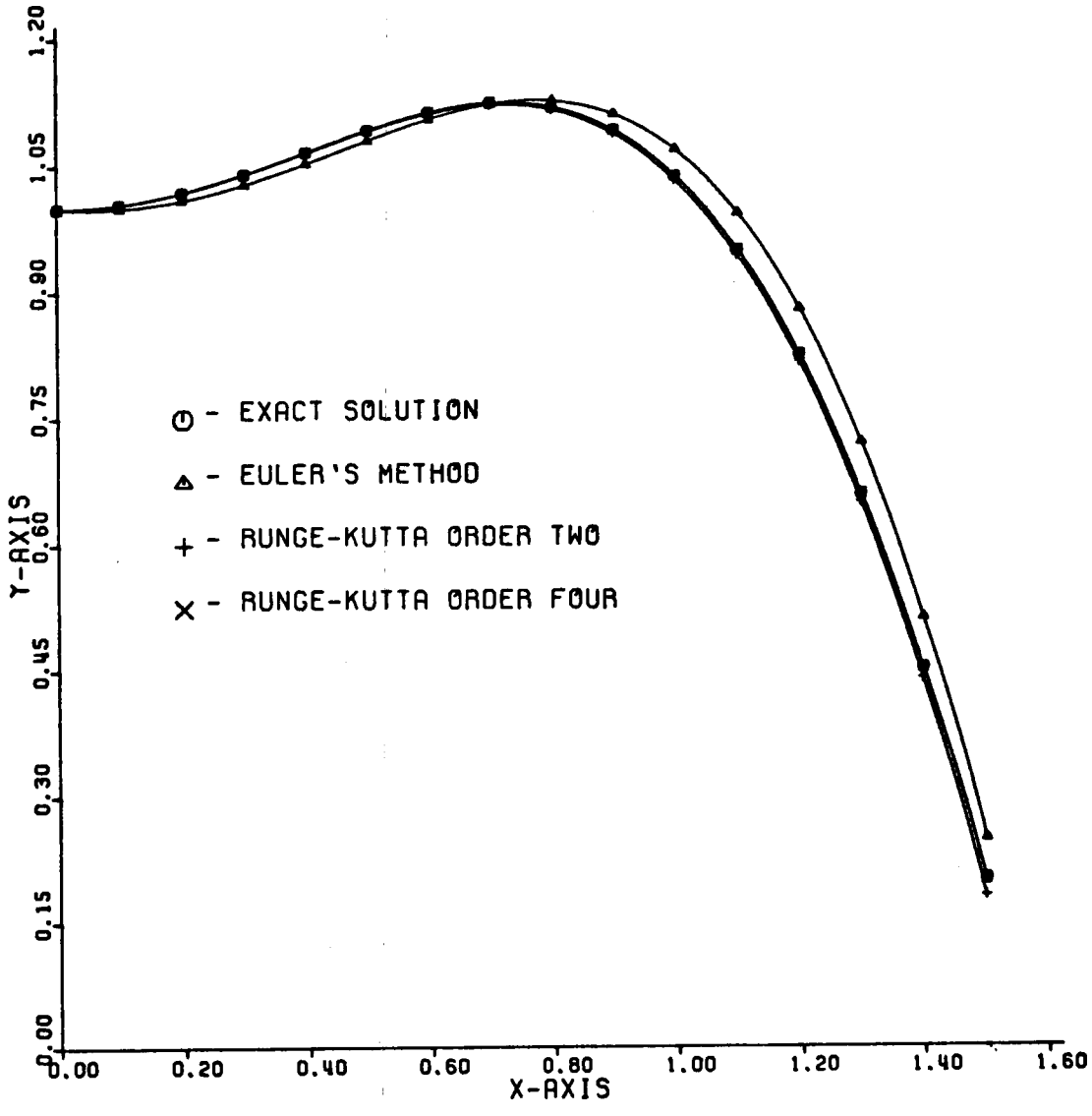


Figure 37. Smooth curves drawn through the data points of Table IV.

A. PROBLEM LIST 19

1. Write a FORTRAN program to obtain a numerical solution of the initial-value problem (7.1) over an interval $[x_0,x_f]$ using each of the three methods outlined in this section. For generality read in x_0 , x_f , y_0 and h in the main program, use separate subroutines for each method, and use a function subprogram to evaluate $f(x,y)$ as needed.

2. Using the FORTRAN program of Problem 1, obtain a numerical solution of each of the following initial-value problems with the indicated stepsizes using each of the three methods. Modify the program to calculate the absolute deviation between the numerical solution and the analytical solution for each step if the exact solution is given.

a. $y' = -2y$, $y(0) = 1$, $x_f = 1$,
   $h = 1/8, 1/10, 1/16$, $(y(x) = e^{-2x})$;

b. $y' = x - y$, $y(0) = 1$, $x_f = 1$,
   $h = 1/10, 1/16, 1/32$, $(y(x)) = 2e^{-x} + x - 1)$;

c. $xy' = x - y$, $y(2) = 2$, $x_f = 3$,
   $h = 1/8, 1/16, 1/32$;

d. $y^2y' = -xy^3 + 1$, $y(1) = 1$,
   $x_f = 1.5$, $h = 1/32, 1/64$;

e. $x^2y' = 1 - xy - x^2y^2$, $y(1) = -1$,
   $x_f = 2$, $h = 1/8, 1/16, 1/32$;

f. $y' = 1 + (x + x^2)e^{-y}$, $y(0) = 0$,
   $x_f = 1$, $h = 1/8, 1/16, 1/32$;

g. $y' = y^2e^{-x} + 1/2y$, $y(0) = 0.5$,

$x_f = 1$, $h = 1/8, 1/16$, $(y(x) = \frac{1}{2}e^x)$.

7.3. Sources of Error. It must be emphasized that this chapter is intended to provide only a brief introduction to the topic of numerical analysis. However, it is felt that the reader should at least be introduced to the types of errors which occur in numerical analysis.

These three numerical methods of solution of Equation (7.1) do not determine a function but rather the values of a function at a set of isolated points. Whenever a function defined on a continuum of points is approximated by one which is defined for only a finite number of points, a discretization, or truncation, error is incurred. The local discretization error is the error committed in the ith step from x_{i-1} to x_i in assuming that y and y' were known exactly for $x = x_{i-1}$. The local truncation error for Euler's method is of order h^2 , while that of the second- and fourth-order Runge-Kutta methods can be shown to be of order h^3 and h^5 , respectively. Thus we could expect to use a larger stepsize with the fourth-order Runge-Kutta method.

The quantity $\max_{0\le i\le N} |y(x_i) - y_i|$ is called the global discretization error of the numerical solution. It should be noted that the discretization error tends to accumulate as i increases since approximate values of y and y' are used in the successive approximations, resulting in considerably less accuracy at the latter steps of

integration than at the beginning.

Because computers deal with finite word lengths, in general a number $\hat{y}_i \neq y_i$ is obtained at the ith step, resulting in a <u>local round-off error</u>. The factors which affect the magnitude of such errors include:

1. the kind of arithmetic used in the computer (floating point or fixed point);
2. the method of rounding used by the machine; and
3. the number of bits used to specify each word.

While the tendency may be to choose a small stepsize to decrease the discretization error, this will result in more integration steps and may, therefore, increase the accumulated round-off error. Thus there is some danger in choosing too small a stepsize. It should be noted, however, that in modern, large-scale computers where a number is specified by a large number of bits, round-off errors tend to be small in comparison to discretization errors.

B. PROBLEM LIST 20

1. Estimate the value of h small enough to guarantee four-place accuracy in the solution of the following initial-value problems for each of the three numerical methods discussed in this chapter, and obtain a solution using the appropriate value of h for ten steps:

   a. $y' = 2y,\ y(0) = 1$;

   b. $y' = -y,\ y(0) = 1$;

   c. $y' = 3x + 2y,\ y(0) = 1$;

d. $y' = 2x - y$, $y(0) = 3$.

7.4. Numerical Solution of the Riccati Equation. Recall from Section 1.5 that the Riccati equation,

$$y'(x) + q(x)y(x) + r(x)[y(x)]^2 = p(x) \quad (7.8)$$

arises in the study of neutron transport. The critical length was defined to be that length x for which the expected number of right-moving particles increases without bound. Using the notation of this chapter, the critical length is that value x_c , $x_0 \le x_c \le x_f$, for which $y(x) \to \infty$ as $x \to x_c$. Given a sufficiently small stepsize h, the critical length may be approximated using a digital computer by computing successive values y_i until the program is interrupted by an overflow. The number of significant digits of accuracy will depend upon the stepsize used. Once a rough approximation of the critical length has been obtained, the stepsize can be decreased to improve the order of accuracy.

EXAMPLE 7.2. Use the fourth-order Runge-Kutta method to obtain the critical length to the nearest hundredth for the neutron transport problem

$$y'(x) = 4 + 4[y(x)]^2 , \quad y(0) = 0 . \quad (7.9)$$

SOLUTION. The computations for the stepsize $h = 1/64$ are shown in Table V. The program was interrupted by an overflow for $x = 0.40625$. Thus the critical length, to the nearest hundredth, was found to be 0.39. The exact solution of Equation (7.9) is

TABLE V

CALCULATION OF THE CRITICAL LENGTH FOR EQUATION (7.9) WITH $h = 1/64$

I	X	Runge-Kutta Order Four
1	0.01563	0.06258
2	0.03125	0.12566
3	0.04688	0.18973
4	0.06250	0.25534
5	0.07813	0.32309
6	0.09375	0.39363
7	0.10938	0.46773
8	0.12500	0.54630
9	0.14063	0.63044
10	0.15625	0.72148
11	0.17188	0.82114
12	0.18750	0.93160
13	0.20313	1.05573
14	0.21875	1.19742
15	0.23438	1.36207
16	0.25000	1.55741
17	0.26563	1.79493
18	0.28125	2.09257
19	0.29688	2.47990
20	0.31250	3.00954
21	0.32813	3.78493
22	0.34375	5.04140
23	0.35938	7.45398
24	0.37500	14.03599
25	0.39063	76.13565
26	0.40625	862277234.20332

$$y(x) = \tan (4x) .$$

Thus the actual critical length is $\pi/8$, which is 0.39 to the nearest hundredth.

EXAMPLE 7.3. Use the fourth-order Runge-Kutta method to obtain a numerical solution of the Riccati equation

$$(7.10) \qquad y'(x) = 2x^2y^2 + 4(x^2-x)y + 2x^2 \ , \ y(0) = 0$$

on the interval $[0,1.5]$ with stepsize $h = 1/16$. Then choose an appropriate stepsize to approximate the critical length to the nearest hundredth.

SOLUTION. An exact closed-form solution of this complicated nonlinear equation cannot be found; thus, as is often the case in physical problems, we have no analytical values with which to compare our results. The results of the fourth-order Runge-Kutta method with $h = 1/16$ are shown in Table VI. The method is then applied on the interval $[1.25, 1.50]$ with stepsize $h = 1/100$ to obtain the critical length. From Table VII, it is seen that the critical length is 1.33, to the nearest hundredth.

TABLE VI

SOLUTION OF EQUATION (7.10) ON [0,1.5] WITH h = 1/16

I	X	Runge-Kutta Order Four
1	0.0625	0.00016
2	0.1250	0.00129
3	0.1875	0.00429
4	0.2500	0.01002
5	0.3125	0.01922
6	0.3750	0.03260
7	0.4375	0.05082
8	0.5000	0.07456
9	0.5625	0.10463
10	0.6250	0.14204
11	0.6875	0.18816
12	0.7500	0.24497
13	0.8125	0.31547
14	0.8750	0.40436
15	0.9375	0.51936
16	1.0000	0.67414
17	1.0625	0.89512
18	1.1250	1.24085
19	1.1875	1.87216
20	1.2500	3.44694
21	1.3125	14.01245
22	1.3750	1409601.18017

TABLE VII

CALCULATION OF THE CRITICAL LENGTH FOR EQUATION (7.10) WITH h = 1/100

I	X	Runge-Kutta Order Four
0	1.25	3.90424
1	1.26	3.90424
2	1.27	4.56342
3	1.28	5.47962
4	1.29	6.84123
5	1.30	9.08026
6	1.31	13.45258
7	1.32	25.73135
8	1.33	158.04325
9	1.34	*********

C. PROBLEM LIST 21

1. Use the three methods of this chapter to solve each of the following Riccati equations with steplengths h = 1/16, 1/32, and 1/64. In each case, find the critical length to the nearest hundredth.

a. $y' - y^2 = 1 + x^2$, $y(0) = 0$;

b. $(1 + x^2)y' = 1 + y + xy^2$, $y(0) = 0$;

c. $y' + 3x^2y - 4xy^2 = 6x^3$, $y(0) = 0$;

d. $y' + 3e^{x^2}y - x^3y^2 = e^{2x^3}$, $y(0) = 0$.

See Aggarwal [1] and Conte [5] for a digital computer approach to obtaining numerical solutions. General discussions of numerical methods are also to be found in Birkhoff and Rota [2], Davis [7], Kaplan [9], and Ross [16]. For an advanced analysis of errors, see Ortega [12].

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