Lesson Plan Collection

Let's Get Physical
Teaching Mathematics Through the Lens of Physics

Mathematical Emphasis

All lesson plans in this collection have a real-world connection to physics and physical phenomena and are designed with the intent to spark student interest and inquiry. It is hoped that teachers use them to supplement more traditional strategies for teaching the mathematical content standards addressed in the lessons.

While each lesson addresses content in the *Tennessee State Mathematics Standards*, the lessons also strongly emphasize the *Standards of Mathematical Practice* detailed below:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Each lesson plan will report the course, content standards, and practices emphasized in the lesson. It will also report where in the *Tennessee State Science Standards* the content is likely to appear.

The purpose of the instructional activities in these lesson plans is to focus on *mathematical connections to science*, not the science principles themselves. For this reason, units may at times be ignored in intermediate steps, and number precision may be rounded for mathematical clarity. Units will be restored on final answers if they are necessary.

Organization of Lesson Plans

The first couple of pages of each lesson plan introduce the topic, standards, practices, and objectives. Grade and course descriptors are included, although the teacher is encouraged to use the lessons in other courses in which they may be applicable. A list of materials will be given. A basic instructional outline is presented, followed by ancillary material that includes much more detail and includes suggestions for guiding student exploration. Sample teacher questions and prompts, as well as possible student answers, are included.

Lesson Plan Collection

Table of Contents

1. **Journey of a Bess Bug**
   - *8th grade – Measurement, Geometry and Functions*
     - 8.G.B.6: Pythagorean theorem
     - 8.F.B.4: Functions that model linear relationships
2. **Speedy Athletes**
   - *7th grade – Rates and Estimation, Measurement and Geometry*
     - 7.RP.A.1: Compute rates
     - 7.EE.B.3: Solve multi-step real-world problems
     - 7.G.A.1: Solve problems with scale drawings
   - *8th grade – Geometry*
     - 8.G.B.6: Pythagorean theorem
3. **Rolling Things**
   - *Algebra I – Systems of Equations*
     - A1.A.REI.C.4: Write and solve system of linear equations
     - A1.A.REI.D.6: Explain intersection of graphs as solution
   - *Algebra II – Systems of Equations*
     - A2.A.REI.C.5: Solve system of linear and quadratic equation
4. **Falling Things**
   - *Algebra I – Parabolas, Quadratics*
     - A1.F.IF.B.3: Interpret key features of graphs
     - A1.F.IF.C.8: Compare properties of functions represented different ways
5. **Basketball Shot**
   - *Algebra I – Parabolic and Linear Functions and Graphs*
     - A1.F.IF.B.3: Interpret key features of graphs
     - A1.F.IF.B.3: Calculate and interpret rate of change
     - A1.F.IF.B.3: Use scatterplot and linear regression to solve problems
   - *Calculus – Integrals and Derivatives of Polynomials*
     - C.D.CD.A.1: Represent derivative graphically, numerically, and analytically
     - C.D.CD.A.2: Interpret derivative as rate of change
     - C.I.AI.B.4: Use definite integral to find area
6. **Springy Things**
   - *Pre-Calculus – Parabolic and Linear Functions and Graphs*
     - P.F.TF.A.3: Use unit circle to describe symmetry
     - P.F.TF.A.4: Choose trig functions to model periodic behavior
   - *Calculus –Trigonometric Derivatives*
     - C.D.AD.A.2: Calculate derivative of basic functions

7. Pendulum Timers

Integrated Math I – Descriptive Modeling and Statistically Relevant Differences

M1.N.Q.A.2: Identify appropriate variables for descriptive modeling

M1.N.Q.A.3: Choose appropriate level of accuracy based on measurement

Statistics – Modeling with Bivariate Data

S.ID.B.12: Display and describe scatterplot, use regression tools

S.ID.B.32: Identify trends in data, transform and model data

8. Bending Light

8th Grade Math – Measurement, Angles, Trig Ratios

G.SRT.C.8: Use trigonometric ratios

Pre-Calculus – Parabolic and Linear Functions and Graphs

P.G.AT.A.1: Use trigonometric ratios

9. Simple Circuits

7th grade – Ratios and Proportions

7.RP.A.2: Proportional Relationships

Pre-Calculus – Rational Functions

P.F.IF.A.2: Real-world rational problems

P.F.IF.A.7: Graph rational functions

10. Charging and Discharging

Pre-Calculus – Exponential Functions

P.F.IF.A.2: Analyze real-world exponential functions

Calculus – Exponential Functions and Natural Logarithms

C.D.LF.A.2: Estimate limits of functions

C.D.LF.A.3: Draw a sketch that illustrates limit

C.I.AI.A.2: Use substitution of variables to calculate antiderivatives

C.I.AI.B.6: Use integrals to solve problems (exponential decay)

11. Coffee Filter Drop

Integrated Math III – Logarithms and Linear Models

M3.A.CED.A.3: Rearrange formulas to highlight quantity of interest

M3.F.IF.A.1: Interpret key features of graphs

M3.F.LE.A.2: Using logarithms, evaluate using technology

Pre-Calculus – Logarithms and Exponents

P.N.NE.A.1: Laws of exponents and logarithms

P.N.NE.A.2: Inverse relationship between exponents and logarithms

P.F.BF.A.2: Use functions to get new functions

Journey of a Bess Bug

Lesson Plan Overview

Lesson Description

When Bess Bugs go on a journey, the paths they trace can be delightfully unpredictable. Equipped with graph paper, rulers, string, tape, patience, and creativity, students can collect data and apply mathematical reasoning to determine the distance between the starting and ending points of a Bess Bug's journey as well as the length of the actual path the Bess Bug took to get there! Add a stopwatch or timer, and students can calculate the average speed and velocity.

The method is simple. Students will gently place a Bess Bug on a large sheet of graph paper and encourage it to move by nudging its rear end. Students will mark the path the bug takes as it crawls across the paper while using an appropriate device to measure the time. Using the Pythagorean Theorem and/or the distance formula, students will calculate their bug's displacement (distance between the starting and ending points) and verify their calculation with a direct measurement. Determining the path length is a bit harder if the Bess Bug has not walked in a straight line, but that is where the string comes in handy. Average speed (path length divided by time) and average velocity (displacement divided by time) can be determined once path length and displacement are known.

This lesson is appropriate for 8th grade mathematics, but may also be used in 7th grade.

Essential Question(s)

How can we use simple tools to determine how far and how fast a Bess Bug walks?

Math Standards

8th Grade Mathematics

8.G.B.6: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

8.F.B.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Math Practices

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.

Let's Get Physical: Teaching Math Through the Lens of Physics

Peggy Bertrand (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark (Principal Investigator, Math Department, Project Manager)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Science Standards

PWC.PS2.1: Investigate, measure, calculate, and analyze the relationship among position, displacement, velocity, acceleration, and time.

PWC.PS2.6: Understand that the two-dimensional movement of an object can be explained as a combination of its horizontal and vertical components of motion.

Lesson Objectives

TSW devise a method to collect and tabulate data for the position of a Bess Bug as a function of time

TSW use the Pythagorean theorem to determine the displacement of a Bess Bug by calculating the distance between two points

TSW calculate the average speed and average velocity of the Bess Bug by calculating rate of change of position

Materials

- Bess Bugs (found outdoors in decomposing logs, or purchased from Carolina Biological Supply for $50 for 12 bugs)
- Large sheets of graph paper, markers, yarn or string, rulers or meter sticks.
- Stopwatches

Basic Instructional Plan

1. Review or introduce the concepts of distance, displacement, speed, and velocity.
2. Demonstrate how to handle a Bess Bug by removing one from the container and placing it on a sheet of graph paper so students can observe it wandering about. Return the bug to the container.
3. Have students brainstorm how to collect data to determine distance, displacement, average speed, and average velocity of a Bess Bug as it journeys across the graph paper.
4. Have students collect data using the methods they have discussed.
5. Have students analyze their Bess Bug data to determine distance, displacement, average speed, and average velocity.
6. Have students compare results for various Bess Bugs and explain their calculations.

Ancillary Materials

1. Distance, Displacement, Speed, and Velocity of Bess Bugs
2. Experimental Procedure
3. Post-Lab Analysis

Distance, Displacement, Speed, and Velocity of Bess Bugs

Physics Terminology

In physics, certain common words have very precise meanings that may differ from our ordinary definitions. For example, the term *distance* refers to the total length of a path traveled by an object. This is different from *displacement*, which is defined as the distance between the starting and ending points of an object's motion. *(Note that displacement in physics includes direction, but that will not be considered here.)* In mathematics, the distinction between distance and displacement is usually not made. For example, the "distance formula" is used to calculate what physicists refer to as displacement.

Speed and velocity likewise have precise meanings. Average speed is distance traveled by an object divided by the time it took; hence, average speed describes how fast an object is moving. Average velocity is displacement divided by time, and describes how fast an object has moved from the starting point of its motion to the ending point of its motion. Average velocity does not account for non-linear variations in an object's path. This can lead to interesting paradoxes; for example, if an Olympic sprinter runs once around an oval track, the sprinter has a high average speed but zero average velocity! This is because the starting and ending points of the sprinter's path are the same.

About Bess Bugs

Bess Bugs (more correctly called Bess Beetles or *Odontotaenius disjunctis*) are large, docile beetles that are sometimes kept as pets. They live outdoors in decomposing wood. They can be collected from fallen trees, or they can be purchased from a biology lab supply company[1]. Keep your Bess Bugs alive and comfortable by placing them in an enclosure filled with wood bark. Use a spray bottle to mist their environment daily to keep it moist but not sopping wet. Bess Beetles look rather frightening, but they do not bite. They may squeak when afraid. It is not necessary to feed them as they eat the decomposing wood, and interestingly, their own droppings.



Figure 1

Bess Bug (more correctly called Bess Beetle)

Handling Bess Bugs is easy. Do not be afraid. They fit neatly in the palm of a hand. Simply lift one out of the container, and transport it gently to the large sheet of graph paper where you will run your experiment.

[1] Carolina Biological is a good source: http://www.carolina.com

Experimental Procedure

Pre-Lab Planning

The teacher should provide students with the definition of the terms *distance*, *displacement*, *speed*, and *velocity* given in the preceding section. These definitions should remain on the board for reference during the lab.

Students can then brainstorm how they will collect data to determine distance, displacement, speed, and velocity of their Bess Bug. Some guiding teacher questions appear below, with possible student responses.

a) What measurements will you make to determine distance and displacement?

For distance, we will need to measure the entire length of the path taken by the bug.

For displacement, we need the distance between the starting end ending points.

b) How will you perform distance measurements?

We will record the position of the Bess Bug as it moves. We can take video and use it to draw the path. Another method is to put dots on the graph paper just behind the bug as it moves. We can sketch the path later by smoothly connecting the dots (points).

Measuring the path length may be tricky if the path is curvy. We may need something flexible, like a string, to measure the length of the entire path. We can measure little short distances from one point to the other and add them up to get total path length.

c) How will you perform displacement measurements?

Displacement is easy to determine. If we mark just the starting and ending points of the Bess Bug, we can measure the distance between these points to get the displacement

d) What additional data do you need to determine speed and velocity?

Because speed is distance/time and velocity is displacement/time, we need to measure time.

e) How will you collect time data?

We will use a video camera to record the dots being placed, because we can always get the time each dot was placed from the video.

If we don't have a video camera, we can take times with a stopwatch. Student 1 will mark the path with dots, Student 2 will call out the times for each dot, and Student 3 will record the times in a table. This may be error prone if the bug moves fast.

Data Collection

This is the high point of the lesson!

Students gently place a Bess Bug on a large sheet of graph paper, mark its position, and start timing. The bug should begin to crawl. If it does not, a gentle nudge on the rear end should get it started.

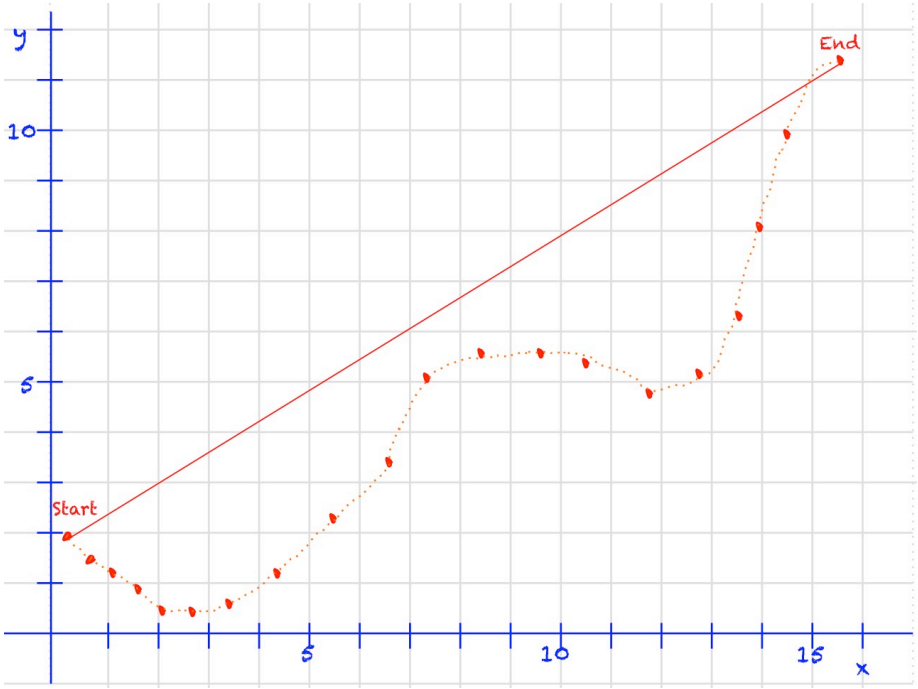
As the bug crawls, students collect data. The teacher is strongly encouraged to have students video the action, because at a minimum the video will provide accurate time data. Even if video is being recorded, it is a good idea for students to physically mark the Bess Bug's path on graph paper and analyze the path on the paper. Because video analysis software makes things very automatic, the manual method will be better at reinforcing graph-reading skills.

The teacher might encourage students to collect data for several Bess Bug journeys, and record them using different colored markers.

Post-Lab Analysis

Preparing the Graph for Analysis

Students have recorded the journey of a Bess Bug on a large sheet of graph paper as a series of points as shown in Figure 2 below. Points are numbered sequentially from start (Point 1) to end (Point 20). Students extracted time data from a video clip and tabulated the time values. They have also connected the dots by sketching a smooth curve, and connected start and end points with a straight line. To facilitate subsequent analysis, students have drawn a Cartesian coordinate axis on the graph paper.



Point	Time (s)
start, 1	0. 6
2	1.2
3	2.0
4	2.6
5	3.2
6	4.0
7	4.2
8	5.1
9	6.0
10	6.4
11	7.3
12	8.1
13	9.2
14	9.8
15	10.3
16	11.0
17	11.3
18	12.0
19	12.4
end, 20	12.8

Figure 2
Graph and Table Showing Journey of Bess Bug

Teacher Tips for Student Guidance in Measuring Distance

The students should measure length with a scale consistent with the graph paper grid. For example, quadrille graph paper has a ¼ inch grid, so the measuring device should be marked in quarters of an inch. If students don't have an appropriately scaled ruler, they can make a paper ruler out of the graph paper and measure distance in grid units.

Distances between adjacent points should be measured and tabulated. In another column, these can be added to calculate total distance. Curvy paths can be tricky, so students will need to be creative. If the points are close enough, straight-line measurement may give good approximate distances. If the line segments are too curvy, string can be laid along the path, marked, and then straightened out to measure the distance.

An illustration of student data appears below. Two columns have been added to the table in Figure 2, one to hold incremental distance measurements, and one to hold cumulative distance.

Point #	Time (s)	Distance between points (grid units)	Total Distance (grid units)
1	0. 6	0.0	0.0
2	1.2	0.6	0.6
3	2.0	0.5	1.1
4	2.6	0.6	1.7
5	3.2	0.6	2.3
6	4.0	0.6	2.9
7	4.2	0.7	3.6
8	5.1	1.0	4.6
9	6.0	1.6	6.2
10	6.4	1.7	7.9
11	7.3	1.8	9.7
12	8.1	1.2	10.9
13	9.2	1.2	12.1
14	9.8	0.9	13.0
15	10.3	1.3	14.3
16	11.0	1.0	15.3
17	11.3	1.4	16.7
18	12.0	1.8	18.5
19	12.4	2.0	20.5
20	12.8	2.2	22.7

Figure 3
Table Showing Time and Distance Data

Teacher Tips for Student Guidance in Determining Displacement

Direct measurement is the easiest and most obvious way to determine displacement, but the students can also calculate it. Once students have determined displacement by one method, the teacher should encourage them to verify their results using at least one other method.

1. Direct measurement
   This involves simply measuring the length of the straight line connecting the starting point to the ending point using a ruler with a scale appropriate for the graph paper.

2. Calculation with Pythagorean Theorem
   Students draw a right triangle for which the displacement is the hypotenuse. They measure the length of the legs of the triangle, and use the Pythagorean theorem to calculate the length of the hypotenuse.

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

3. Calculation using Distance Formula
   While this method is very similar to using the Pythagorean Theorem, using the so-called distance formula requires students to read points from the graph. The distance formula is often presented as:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The figure below represents a graph that has been set up to facilitate displacement calculations. The student has constructed a right triangle for the Pythagorean Theorem and determined the coordinates of the starting and ending points.

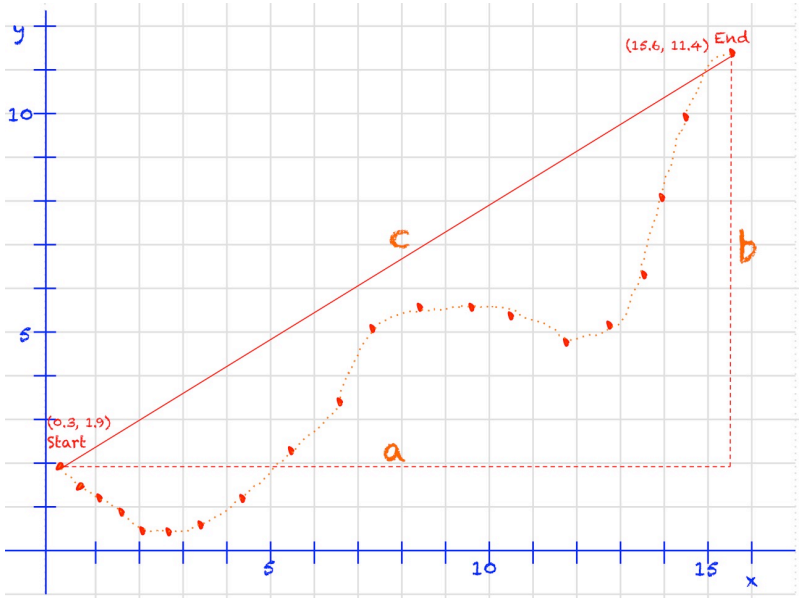


Figure 4
Graph For Displacement Calculations

Comparing Distance and Displacement

After students have completed distance and displacement measurement, the teacher can ask the following questions to prompt student discourse.

a) How well do your measured and calculated values for displacement agree? What could be the cause of any differences?

Our measurements and calculations are close, but not exactly the same. This is because there may be some error in measurement.

Note: This would be an ideal opportunity to have each student show their graphs and calculations to the class. A gallery walk would work well here. Students can show their Bess Bug graphs and include their measured and calculated values on the graph.

b) When you compare the total distance your bug traveled to its displacement, what do you notice? Back up your comparisons with specific evidence.

The distance is bigger than the displacement! (Students should cite their measurements)

Note: Whole class tabulation on the whiteboard of distance and displacement data will reveal a general trend.

c) Based on our class data, can we write a general mathematical statement for distance and displacement?

distance > displacement

d) Can distance ever be the same as displacement?

Yes, if a bug walks in an absolutely straight line.

e) Can we adjust our mathematical statement?

distance ≥ displacement

Comparing Speed and Velocity

Once students have measured distance and displacement they are ready to calculate, speed and velocity. Here are some guiding questions for the teacher and possible student answers.

a) Is your bug walking at a constant speed? Cite evidence from your data.

No, my bug isn't walking at a constant speed. Constant speed is a rate of change, and the ratio below would need to be constant:

distance between points / time between points

Also, if I graph total distance versus time, it is not a straight line.

Note: Students should cite data for their particular bug. It is possible that within experimental error, some bugs may be walking at approximately constant speed. Be sure that students use elapsed time, and not total time, in the ratio calculation above.

b) How would you calculate average speed of your bug?

We can calculate average speed by averaging all the individual speed calculations. A much easier way is to take the total distance of the path and divide it by the total time.

c) How would you calculate average velocity of your bug?

We can calculate average velocity by dividing displacement of the bug by the total time.

f) When you compare the average speed and average velocity of your bug, what do you notice? Back up your comparisons with specific evidence.

The average speed is bigger than the average velocity. (Students should cite their measurements)

Note: Whole class tabulation on the whiteboard of distance and displacement data will reveal a general trend.

g) Based on our class data, can we write a general mathematical statement for average speed and average velocity?

average speed > average velocity

h) Can average speed ever be the same as average velocity?

Yes, if a bug walks in an absolutely straight line.

i) Can we adjust our mathematical statement?

average speed ≥ average velocity

Speedy Athletes

Lesson Plan Overview

Lesson Description

We know that an Olympic sprinter running at full speed is faster than an NFL football player, but exactly how much faster? Keep in mind that the sprinter has trained for years to run in lightweight clothing down a track, carrying nothing in his hands. He is free to run just as fast as he can in a straight line! The football player is wearing full pads and a helmet, carrying a football, and dodging large opponents who fling themselves at him with impressive force. How much do these challenges decrease the football player's average velocity as he dashes toward the end zone? How much slower is the football player than the sprinter?

In this lesson, we will analyze two video clips. The first is of Usain Bolt as he runs the 100-meter sprint in the Beijing Olympics. The second is of Ted Ginn Jr. as he returns a kickoff for a touchdown for the Miami Dolphins. Based on visual clues in the videos, the students will determine the average velocity of both athletes and compare them.

This lesson is appropriate for 7th grade mathematics. The activities reinforce conceptual understanding of ratios and similar triangles. Students will need to use skills of estimation. It is also appropriate for 8th graders studying the Pythagorean Theorem.

Essential Question(s)

How can we use a scale drawing to determine an actual distance?
How can we use ratios to convert from one unit system to another?
How can we calculate the average velocity as a rate?
How can we use the Pythagorean Theorem to determine a distance? (8th grade)

Math Standards

7th Grade Mathematics

7.RP.A.1: Compute unit rates associated with ratios of fractions, including ratios of lengths, areas, and other quantities measured in like or different units.

7.EE.B.3: Solve multi-step real-world and mathematical problems posed with positive and negative rational numbers presented in any form (whole numbers, fractions, and decimals).

7.G.A.1: Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

8th Grade Mathematics

8.G.B.6: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Math Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
5. Use appropriate tools strategically.

Science Standards

PWC.PS2.1: Investigate, measure, calculate, and analyze the relationship among position, displacement, velocity, acceleration, and time.

PWC.PS2.6: Understand that the two-dimensional movement of an object can be explained as a combination of its horizontal and vertical components of motion.

Lesson Objectives

TSW analyze video clips of sporting events, collecting evidence for distance and time calculations.

TSW estimate distance when it is not given exactly, using visual evidence and similar triangles.

TSW compute average velocity from distance and time data.

TSW compare and evaluate average velocity and speed, converting units when necessary.

TSW determine the length of a hypotenuse using similar triangles and/or Pythagorean theorem.

Materials

- Videos of Usain Bolt and Ted Ginn that can be played on a classroom projector.
- Scale drawing of a football field.
- Stopwatches or timers, rulers, calculators.

Basic Instructional Plan

1. Provide students with scale drawings of a 100-meter track and a 100-yard long football field.
2. Show a video of Usain Bolt's Olympic run, and guide students as they calculate average velocity based on evidence in video clip.
3. Show a video of Ted Ginn's kick-off return for a touchdown, and guide students as they calculate average velocity based on evidence in video clip.
4. Have students compare average velocity for the two runs, after appropriate unit conversion.

Ancillary Materials

1. Collecting Data for Velocity Calculations
2. Speedy Athletes Handout and Key
3. Video Clips (Usain Bolt[1] and Ted Ginn[2])

[1] Usain Bolt's run, live video, at https://www.youtube.com/watch?v=qslbf8L9nl0.

[2] Ted Ginn's kickoff returns, live video and replay, https://www.youtube.com/watch?v=89dApPANy7Y.

Collecting Data for Velocity Calculations

Physics Terminology

In physics, average velocity is a rate defined as displacement per unit time. Velocity describes how fast an object moves from the starting point of its motion to the ending point of its motion. Velocity is not exactly the same thing as speed, but it is a related concept.

When an athlete runs down a straight track, the athlete's velocity and speed are identical because the athlete runs in a straight line along the shortest possible path. On the other hand, when an athlete dodges around to avoid being tackled, velocity will be less than speed. The path the athlete takes is longer because the athlete deviates from straight-line motion. Average velocity does not account for the longer path taken by athletes that are dodging this way and that, and is less than the speed in the case of non-linear motion.

(Note: Another difference between speed in velocity is that velocity also includes direction. However, we will ignore direction and simply focus on the numeric value of velocity in this lesson.)

Video of Usain Bolt Winning the World Record

The video clip of Usain Bolt setting a world record at the Beijing Olympics has all the information within the video that students need to calculate his record-setting average speed. You can find several different clips of Bolt's Beijing performance on YouTube; a link to a good one is footnoted in the materials list.

Because Bolt runs in a straight line, his average speed and average velocity are identical. Bolt pulls away from the field, finishing 100 meters in 9.68 seconds for a new world record. His average speed and average velocity (the same since Bolt was running in a straight line) were 10.33 m/s.

Video of Ted Ginn Jr. Scoring Touchdowns on Kick-Off Returns

In this particular game, Ginn scores not one, but two, touchdowns on 100+ yard kick-off returns. A good video is footnoted in the materials list.

Analysis of Ginn's video is more complicated than analysis of Bolt's. First, Ginn is not running in a straight line when he scores either touchdown. His position on each goal line must be estimated from evidence in the video. The time it took him to run the length of the field is not directly reported in the video. The students must figure out a way to time the runs. They can use a stopwatch, or use the time showing on the video-player application.

Note that Ginn's first kick-off return is probably better for student analysis. A replay of the run is shown from a different angle, so students should have enough visual information to get a reasonable estimate of the points at which he begins and ends his run. Be prepared to show the video clip several times, so students can estimate positions and time the run from one goal line to the other.

Handout and Key

The student handout and key provide a framework to guide student work. This lesson is strongly tied to real-world mathematics, and underscores the importance of similar triangles and ratios.

Speedy Athletes Handout

1) In the space below, write an equation that you can use to calculate velocity for Usain Bolt and Ted Ginn. Define what the variables in the equation mean.

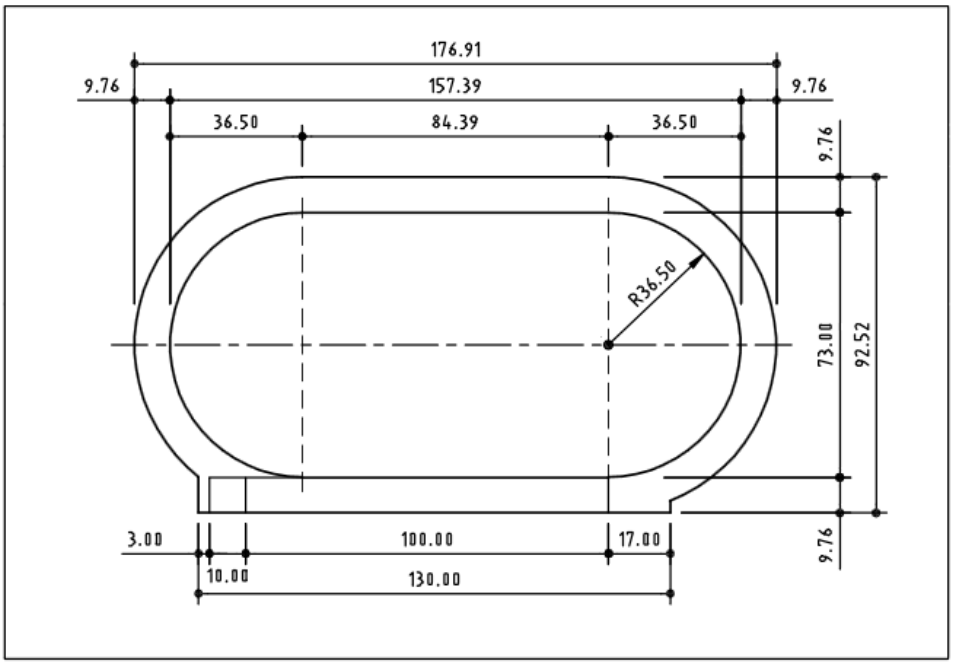


Figure 1.2.3a - Shape and dimensions of the 400m Standard Track (Radius 36.50m)
(Dimensions in m) **Source -** The IAAF Track and Field Facilities Manual 2008 (the Manual)

2) In the diagram above of an Olympic track, can you find a straight 100-meter section where Usain Bolt ran his race? Mark it on the diagram above.
3) What numeric data do you need to calculate Usain's velocity? Record it below, with units.

4) Calculate Usain's velocity below. Show all your work and include units.

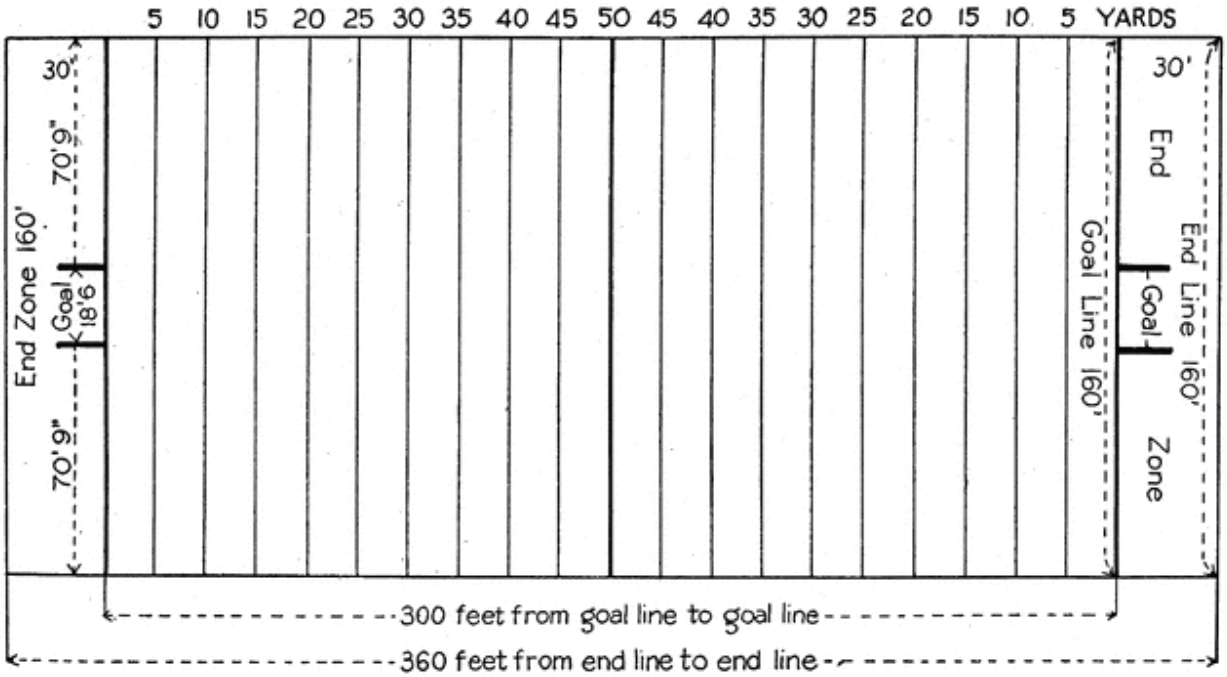


DIAGRAM OF A FOOTBALL FIELD

Image from SportsCourtDimensions.com

5) On the diagram above, mark and label the approximate starting and ending points for Ted's kick-off return. Draw a straight line between the starting and ending points. Construct a right triangle with your straight line as the hypotenuse.
6) Using a ruler, your triangle, and clues in the figure above, calculate the <u>actual distance</u> in feet between Ted's starting and ending points. This is Ted's displacement.

7) Track athletes know that 1600 meters very close to 1 mile, and 1 mile is 5,280 feet. Using this information, convert Ted's displacement to meters.

8) Now that you have calculated displacement in meters, what additional data do you need to calculate Ted's average velocity? Record it below.

9) Calculate Ted's velocity below. Show all your work and include units.

10) How does Ted Ginn's velocity compare to Usain Bolt's velocity? What are some reasons for the difference?

Speedy Athletes Key

1) In the space below, write an equation that you can use to calculate velocity for Usain Bolt and Ted Ginn. Define what the variables in the equation mean.

$v = \frac{d}{t}$ where v is velocity, d is displacement, and t is time

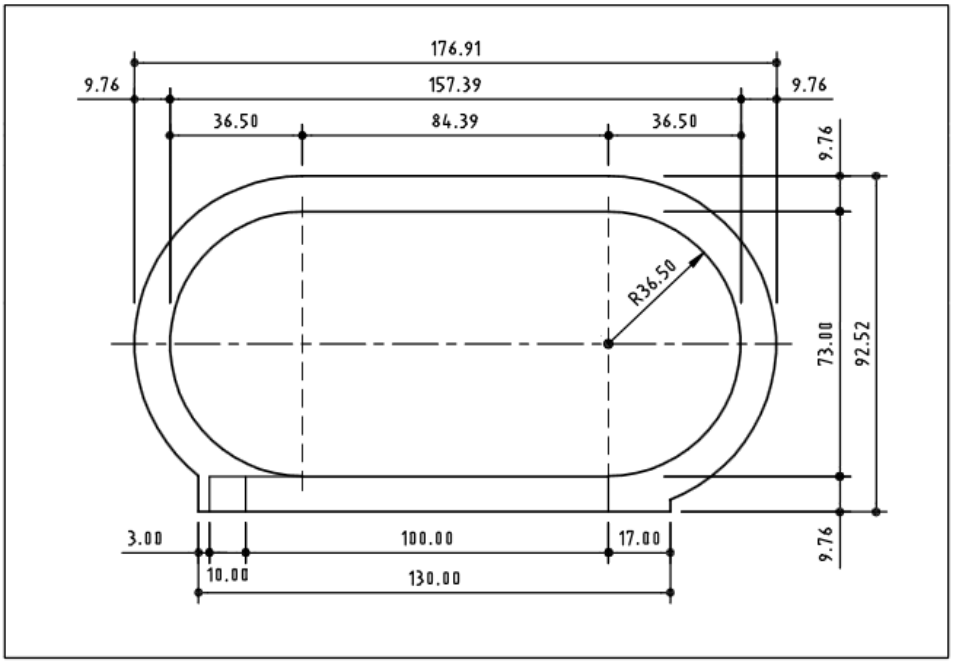


Figure 1.2.3a - Shape and dimensions of the 400m Standard Track (Radius 36.50m)
(Dimensions in m) **Source -** The IAAF Track and Field Facilities Manual 2008 (the Manual)

2) In the diagram above of an Olympic track, can you find a straight 100-meter section where Usain Bolt ran his race? Mark it on the diagram above. (100.00 meter length at bottom of figure)
3) What numeric data do you need to calculate Usain's velocity? Record it below, with units.

Need time, t, and displacement, d. From video, t = 9.69 s. From track, d = 100 m.

4) Calculate Usain's velocity below. Show all your work and include units.

$$v = \frac{d}{t} = \frac{100.00\ m}{9.69\ s} = 10.32\ m/s$$

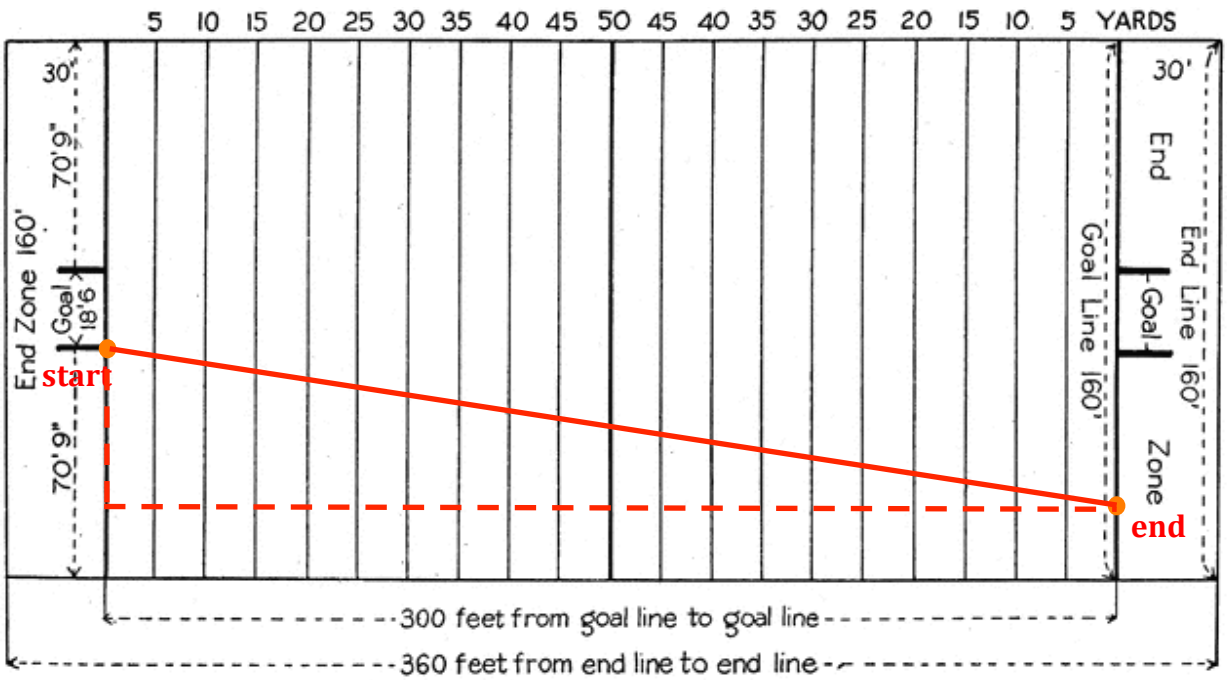


DIAGRAM OF A FOOTBALL FIELD

Image from SportsCourtDimensions.com

5) On the diagram above, mark and label the approximate starting and ending points for Ted's kick-off return. Draw a straight line between the starting and ending points. Construct a right triangle with your straight line as the hypotenuse.
6) Using a ruler, your triangle, and clues in the figure above, calculate the <u>actual distance</u> in feet between Ted's starting and ending points. This is Ted's displacement.

Measuring with a metric ruler yields 13.3 cm horizontal distance from between end zones. We see from the figure, this represents 300 feet. The length of the hypotenuse is 13.5 cm. Setting up the ratio for similar triangles:

$\frac{d}{300\,feet} = \frac{13.5cm}{13.3cm}$ and solving this for the unknown yields d = 304.5 feet.

(NOTE: 8th grade teachers can use this as an application of the Pythagorean Theorem.)

7) Track athletes know that 1600 meters very close to 1 mile, and 1 mile is 5,280 feet. Using this information, convert Ted's displacement to meters.

$\frac{d}{304.5\,feet} = \frac{1600m}{5280'}$ and solving this for the unknown yields d = 92.27 m.

8) Now that you have calculated displacement in meters, what additional data do you need to calculate Ted's average velocity? Record it below.

We need time, t. If we time the run, we find t = 13.5 seconds.

9) Calculate Ted's velocity below. Show all your work and include units.

$$v = \frac{d}{t} = \frac{92.27\ m}{13.5\ s} = 6.83\ m/s$$

10) How does Ted Ginn's velocity compare to Usain Bolt's velocity? What are some reasons for the difference?

Ted Ginn has a lower velocity than Usain Bolt. However, he has many more challenges than Usain Bolt. He is not running in a straight line because he is dodging other players. He is wearing much heavier clothing (pads and a helmet), and he is carrying a football. He slows down near the end of his run. Even so, he is running about 68% of Usain Bolt's world record velocity. Note that Ted Ginn's speed is faster than his velocity, but we don't know the exact path he is running so we cannot calculate speed. If we could calculate Ted's speed, we would find him closer than 68% to Usain's world record speed.

Rolling Things

Lesson Plan Overview

Lesson Description

Predicting the point of a collision between two moving cars, or determining where and when a fast car will catch up with a slow one, are examples of very practical real-world physics problems. In the Algebra I classroom, the motion of two objects moving with constant velocity can be described with a system of equations. The point that solves both equations simultaneously is the place where the objects meet. If one of the two objects is accelerating and its motion can be described with a quadratic equation, this becomes an Algebra II problem.

Students will use video analysis and probeware to investigate the motion of two rolling objects with different velocities. They will tabulate and graph data, construct equations representing each object's motion, then solve the system of equations to predict the time and horizontal position at which the paths of the objects intersect.

The bulk of the lesson is appropriate for Algebra I. An extension is provided for Algebra II.

Essential Question(s)

How can we model the position of the object as a function of time graphically and in equation form?

How can we use equations and graphs to predict where the paths of the objects will intersect?

Math Standards

Algebra I

A1.A.REI.C.4: Write and solve a system of linear equations in context.

A1.A.REI.D.6: Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the approximate solutions using technology.

Algebra II (extension)

A2.A.REI.C.5: Solve a system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

Math Practices

2. Reason abstractly and quantitatively.
4. Model with mathematics.
7. Look for and make use of structure.

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Science Standards

PHYS.PS2.1: Investigate and evaluate the graphical and mathematical relationship (using either manual graphing or computers) of one-dimensional kinematic parameters (distance, displacement, speed, velocity, acceleration) with respect to an object's position, direction of motion, and time.

Lesson Objectives

TSW analyze motion to develop graphs and equations describing the motion.

TSW solve systems of equations to predict a point of intersection.

Materials

- Computers or tablets with video analysis software installed.[1]
- Video clips containing two moving objects with different velocities and starting positions; e.g., toy vehicles moving toward and away from each other
- Digital video cameras and tripods
- Balls, carts, ramps, etc., to create examples of motion in which one object is moving at constant acceleration and the other is moving at constant velocity
- Meter sticks, and other measuring tools
- Motion sensors and data acquisition software (for Algebra II extension)

Basic Instructional Plan (Algebra I and II)

1. Students prepare or obtain a video clip of two objects moving at constant velocity and import it into a video analysis program.
2. Students perform regression analysis of experimental *x* versus *t* data for each moving object to develop motion equations and associated graphs.
3. Students analytically solve the system of equations describing the two moving objects to predict the position and time at which their paths intersect.
4. Students verify their predictions by comparing the analytically determined point of intersection with the point of intersection on the graphs of motion.

Instructional Plan Extension (Algebra II)

Objects that accelerate as they roll down ramps offer an additional challenge because they require students to consider quadratic equations. Students investigate two moving objects, one that is accelerating and one that is moving at constant velocity. They then perform the same type of analysis on the data as described in the Basic Instructional Plan.

Ancillary Materials

1. Constant Velocity Motion
2. Cars Rolling In The Same Direction
3. Cars Rolling In Opposite Directions
4. Accelerated and Constant Velocity Motion

[1] Vernier Logger Pro, PASCO Capstone, etc. Software must be able to analyze two objects in the same video

Constant Velocity Motion

Concepts

The general form of a linear equation in slope-intercept form is represented by

(1) $$y = mx + b$$

where *m* is the slope and *b* the vertical intercept. Constant velocity motion in physics is represented by linear equations analogous to the one above, but with variables more appropriate to the real-world context. When an object moves at constant velocity, the following represents its general equation of motion:

(2) $$x = vt + x_o$$

The constant *v* in Equation (2) represents velocity, or the rate of change of position with respect to time. The constant x_o represents initial position of the object at time $t = 0$. The domain is time, *t*, and the range is position, *x*. Position depends on time and is therefore a function of time; hence, $x = f(t)$. When Equation (2) is graphed, the slope represents the velocity *v* of the object and the vertical intercept x_o represents the initial position.

Even though Equation (2) is analogous to Equation (1), Algebra I students usually find it more difficult to conceptualize. It can be challenging to visualize the domain because time *t* cannot actually be seen. The range *x* is the position along whatever axis the object is moving. Even if the object is moving horizontally, the range is graphed on the vertical axis, which students can find confusing. Using video analysis techniques to collect position versus time data for moving objects can help students conceptualize time-domain equations and graphs.

Videos of Constant Velocity Motion

Students can create their own video clips for this laboratory with a cell phone or inexpensive video camera; however, true constant velocity motion can sometimes be a bit difficult to obtain. Students can roll balls across the floor or in tracks, but if friction or other resistive forces are significant the objects will decelerate and the motion equations may not be very linear. A friendly nearby physics teacher may possess some constant velocity carts with a couple of different speed settings. If that is the case, ask to borrow these carts to make videos. Be sure to include a meter stick in the video for calibration purposes. Alternatively, you can use the video clips on the USB memory drive provided with this course.

Cars Rolling In The Same Direction

Collecting Data From The Video

For this investigation, students will analyze the motion of two toy cars, each having a different constant velocity but moving in the same direction. The video used in the example below is available on the USB memory drive distributed at the *Teaching Mathematics Through the Lens of Physics* workshop and is entitled *Toy Cars Same Direction.*[2] The video has been imported into Vernier Logger Pro.[3] The coordinate system has been defined by positioning the yellow coordinate axes at a desired location, and the scale has been defined by dragging the green line over the meter stick and typing in the measurement it represents. For our purposes, the green line is calibrated as 100 cm. Figure 1 below illustrates the set-up. The large red car is entirely visible and is moving to the right. A second smaller car is just barely visible at the far left and is also moving to the right.

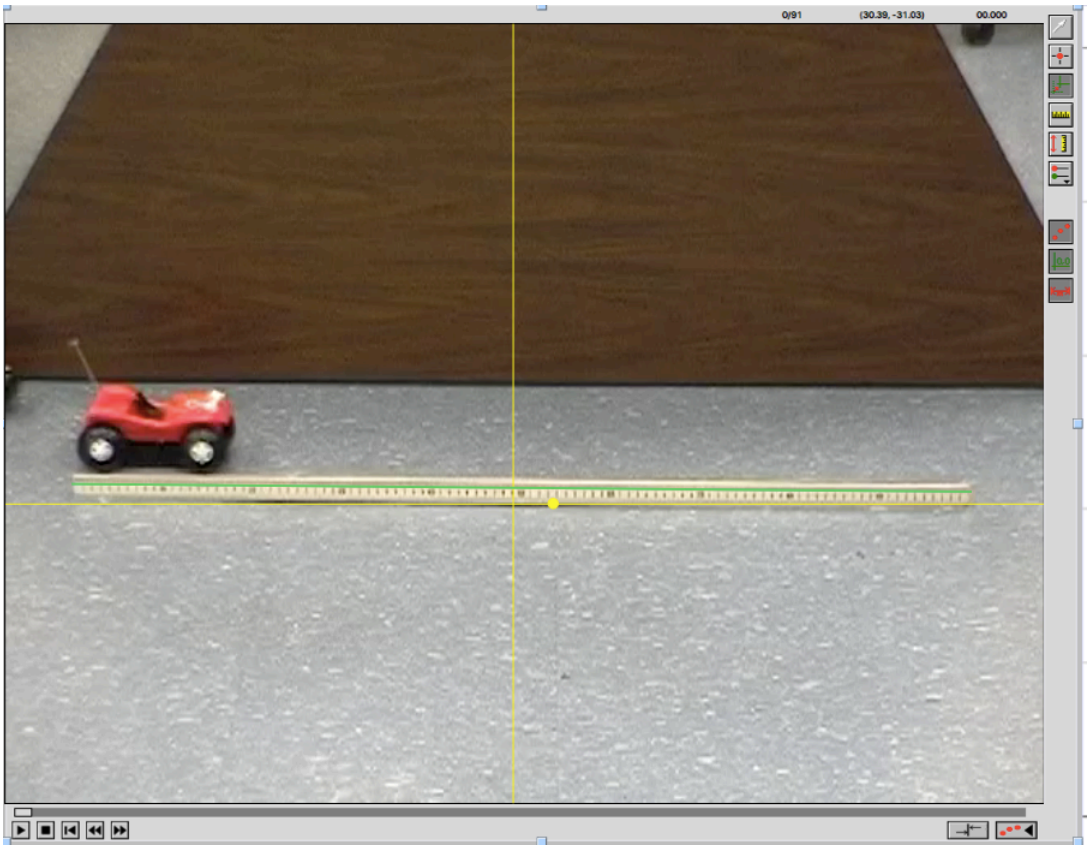


Figure 1
Set-Up for Constant Velocity Cars (Toy Cars Same Direction)

Logger Pro allows for the separate collection of position versus time data for each of the two cars. For our purposes, the first data set will be for the big car, and the second for the smaller car. To collect data, identify a reference point on the front of the car being analyzed and click on it with the cross-hair cursor. A dot is placed at the center of the cross-hair cursor and the video automatically advances by one frame. As dots are placed on the video, the points representing their locations are stored in a data table and plotted on a graph as *(t, x)* coordinate pairs. Note that controls on the

[2] *Obtained from Joel Bryan, Ph.D., at Ball State University, http://jabryan.iweb.bsu.edu/VideoAnalysis/*
[3] More complete Logger Pro instructions are available from Vernier.

video allow you to skip one or more frames as you plot the data; this is convenient for slow-moving objects such as the cars in the video.

Figure 2 shows a screenshot after data on both cars has been collected.

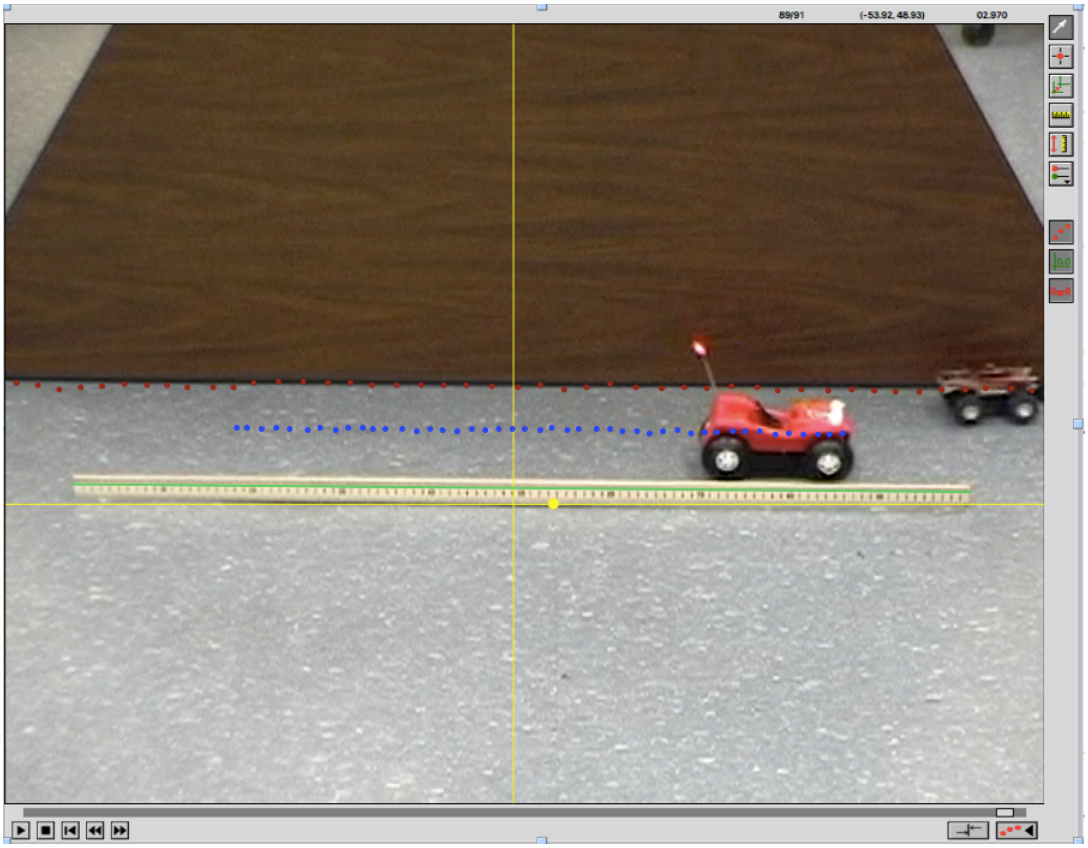


Figure 2
Video Analysis of Constant-Velocity Cars Showing Position Data

The blue dots represent a position on the front of the big car at various times, and the red dots similarly represent a position on the front of the small car. For both cars, a dot was placed and a data point recorded every other frame.

Figure 3 below shows the plotted data points, best-fit linear graphs, and regression equations corresponding to the data from the video analysis. Two graphs are shown, one representing each car. The green data points are for the smaller car and the red data points are for the larger car. The text boxes show the constants (slope m and vertical intercept b) associated with each linear regression equation. Remember that for a motion equation, the slope m is also the velocity v and the vertical intercept b is also the initial position x_o. The units on the constants support this association.

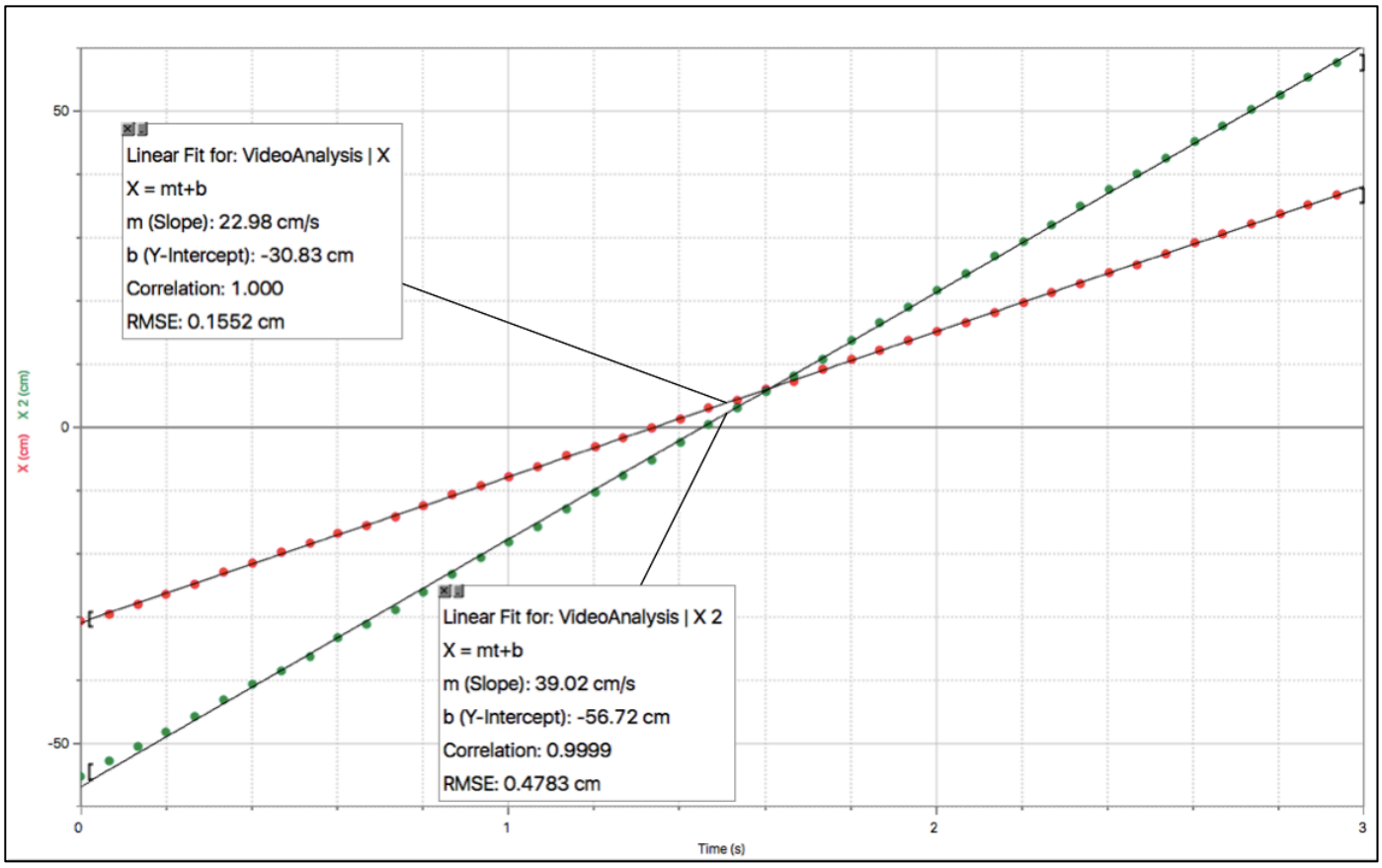


Figure 3
Graphs and Regression Equations for Constant Velocity Cars

Algebraic Analysis

The teacher can guide students through an algebraic analysis using the equations and graphs shown in Figure 3. Questions and prompts should be both conceptual and analytical. Sample prompts with possible correct student responses are shown below.

a) What is the graphical evidence that the carts are both moving at constant velocity?

Each graph is linear. The slope of the line is constant in each graph. Because the slope represents the velocity, both cars are moving at constant velocity.

b) What is the graphical evidence that the carts are moving in the same direction?

The slope of each line is positive, so the position of each car is increasingly positive on the x-axis as time elapses. This means that each car is moving in the positive direction and therefore both cars are moving in the same direction.

c) Which car is traveling with a larger velocity? Use evidence from the graphs.

The smaller car is traveling faster than the bigger car because slope of its position versus time graph is larger. A larger slope means a faster rate of change of position with respect to time and a larger velocity.

d) Which car is traveling with a larger velocity? Use evidence from the equations.[4]

The equation of motion for the bigger car obtained from Figure 3 is:

(3) $$x = 22.98\,{}^{cm}\!/_{s}\,t - 30.83cm$$

The equation of motion for the smaller car obtained from Figure 3 is:

(4) $$x = 39.02\,{}^{cm}\!/_{s}\,t - 56.72cm$$

The velocity is the constant that multiplies t term; therefore, the velocity of the small car in Equation (4) is larger than the velocity of the bigger car in Equation (3).

e) Using the equations of motion, determine the point at which the paths of the two cars will intersect horizontally.

The paths will intersect when they have the same horizontal position at the same time. This (t, x) point is the solution to the system of equations represented by (3) and (4) above. At this point, x is the same for both cars. So eliminating x as a variable can be done by setting (3) and (4) equal to each other:

$$22.98\,{}^{cm}\!/_{s}\,t - 30.83cm = 39.02\,{}^{cm}\!/_{s}\,t - 56.72cm$$

Solving this equation for time yields t = 1.614 seconds. Substituting this time value into Equation (3) yields x = 6.260 cm and into Equation (4) yields x = 6.258 cm. This is very close agreement given the experimental nature of the equations; the average value is x = 6.259 cm. The point of intersection is therefore very close to (1.614 s, 6.259 cm), given to three significant figures.

f) How does the graph support your solution to the system of equations?

The point of intersection of the two graphs represents the solution to the system of equations. Figure 3 shows that the graphs intersect at a point that could indeed be represented by the solution determined analytically in (e) above of (1.614 s, 6.259 cm).

[4] Note that equations of motion in the algebraic analysis above are shown with units on the constants. If the inclusion of units causes too much challenge, allow your students to work without units and restore the units on the final answers.

Cars Rolling In Opposite Directions

Predicting the Differences

A second video clip of the same two cars moving in opposite directions, *Toy Cars Opposite Directions*,[5] is provided on the USB memory drive for comparative analysis. It is recommended that the teacher begin by playing the video clip for the students and asking them to describe the similarities and differences they expect in the graphs and equations of motion when video analysis is performed on this video clip.

Students who conceptually grasp motion graphs will realize that the slope of the graph for the smaller car will be very similar if not identical to that shown in Figure 3. Likewise, the equation of motion for the smaller car will have a very similar numeric value for slope, since slope represents the velocity *v*. It is less clear that the intercept will be the same, since the intercept is highly sensitive to the placement of the origin of the coordinate system. The slope, on the other hand, is not dependent on the origin.

Because the bigger car is now moving in the opposite direction, the students should recognize that the slope of its motion graph will be negative, but the absolute value of the slope will be the same as that in Figure 3. The equation of motion for the bigger car will also have a negative numeric value for the slope. Because the slope represents the velocity, we would expect that *v* for bigger car in the second clip is very similar to –*v* for the same car the first clip. It is clear that the vertical intercept will be much more positive than it was in the first clip, since the car starts far to the right and moves to the left. It is hard to make an exact prediction about the vertical intercept, as it is highly dependent on the location of the origin.

Verifying Predictions

Once students have predicted how the graphs and equations of motion in the second video clip will compare to those in the first video clip, they should perform video analysis on the second clip to test their predictions.

Algebraic Analysis

Students should repeat the Algebraic Analysis detailed on pages 6-7 on the second video clip to reinforce and practice the skills emphasized in the analysis.

[5] *Obtained from Joel Bryan, Ph.D., at Ball State University, http://jabryan.iweb.bsu.edu/VideoAnalysis/*

Accelerated and Constant Velocity Motion

Concepts

We have seen that when an object moves at constant velocity, its position as a function of time is:

(2) $$x = vt + x_o$$

If an object moves with uniform acceleration, its position as a function of time is a bit more complicated:

(5) $$x = \tfrac{1}{2}at^2 + v_o t + x_o$$

In Equation (5), a is acceleration, v_o is initial velocity, and x_o is initial position. This equation is analogous to the general form of a quadratic equation:

(6) $$x = Ax^2 + Bx + C$$

Because Algebra II students are expected to solve a system of equations consisting of one linear equation and one quadratic equation, the following experiment would be a valuable real-world connection in an Algebra II classroom.

Data Collection

Students set up two ramps, one that is level and another at a slight (<5°) angle. Each ramp is equipped with a motion sensor as shown in Figure 4 below. One student gives the cart on the level track a gentle push, setting it into constant velocity motion. A second student releases the cart on the angled track from rest, allowing it to roll with constant acceleration down the track. A third student starts data acquisition. Both motion sensors collect position versus time data for their respective carts simultaneously.

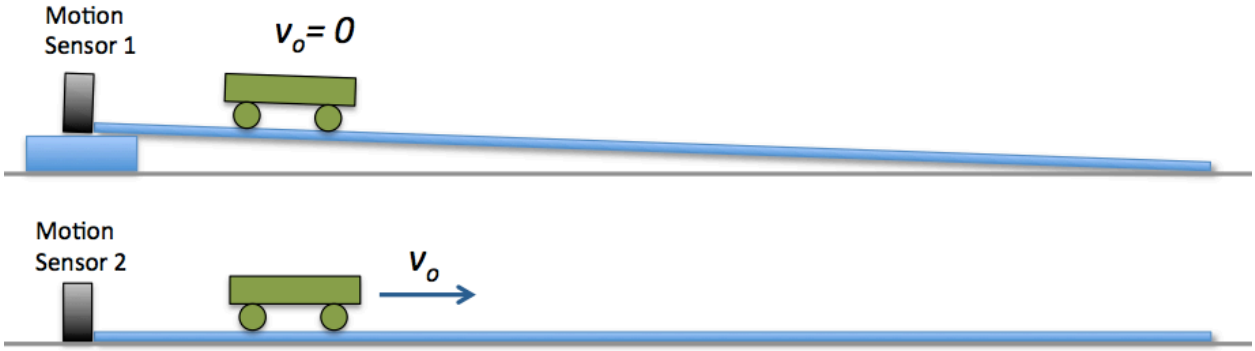


Figure 4
Experimental Set-Up for Algebra II Experiment

Figure 5 shows the resulting position versus time graphs for the two moving carts. The top graph is from Motion Sensor 1 on the angled track, and the bottom graph is from Motion Sensor 2 on the level track. The motion sensors measure the distance between themselves and the back ends of the carts. The software was set up to trigger data collection automatically when Motion Sensor 1 detected that the cart on the angled track was 0.20 m from the sensor and moving away from it.

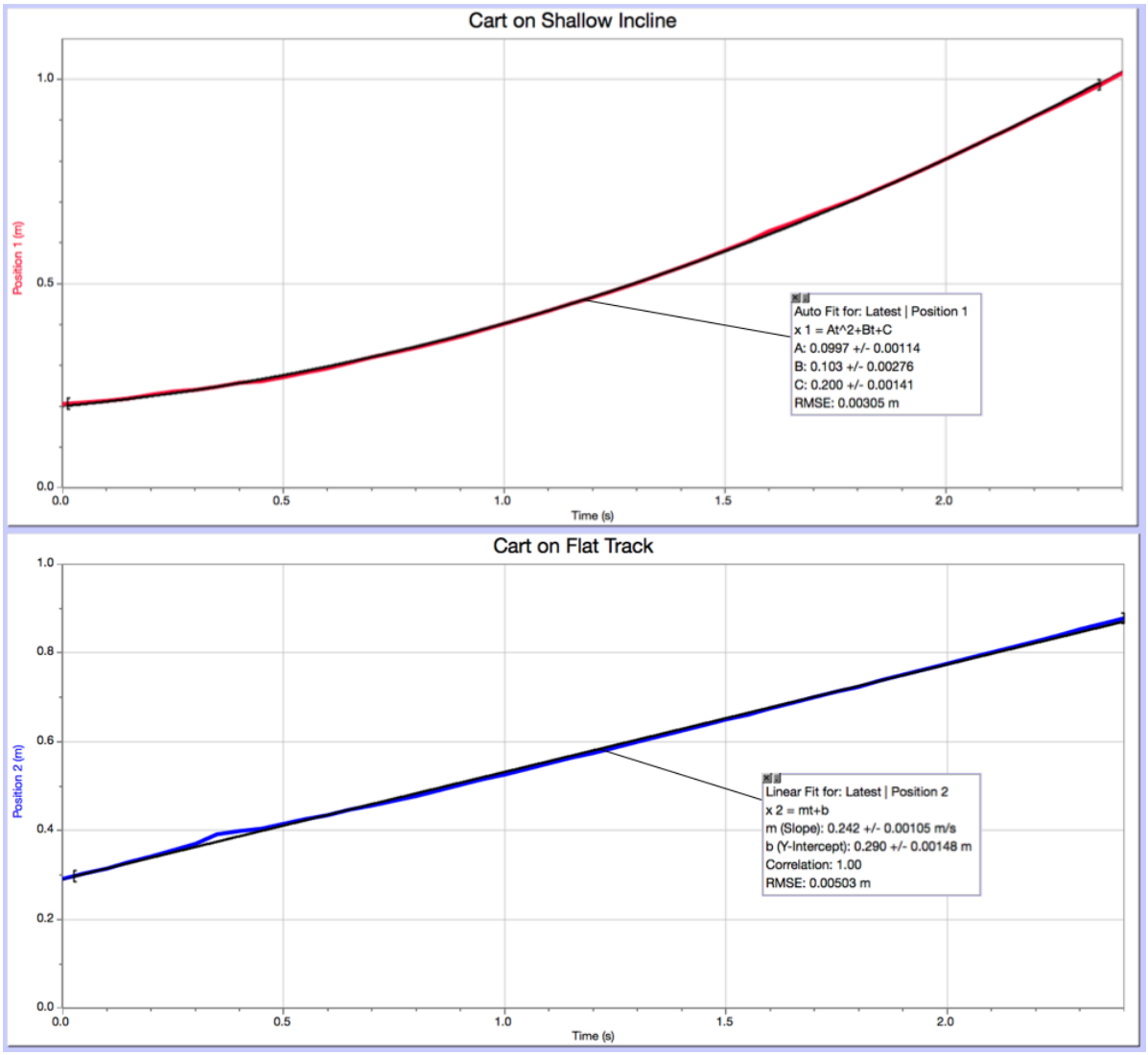


Figure 5
Position versus Time for Accelerating Cart and Constant Velocity Cart

Algebra II Analysis

Based on the regression analysis, the motion of the cart on the angled track is described by Equation (7):

(7) $$x = 0.0997\,\tfrac{m}{s^2}t^2 + 0.103\,\tfrac{m}{s}t + 0.200\,m$$

The motion of the cart on the level track is described by Equation (8):

(8) $$x = 0.242\,\tfrac{m}{s}t + 0.290\,m$$

These equations can be solved simultaneously to determine (a) the *x*-coordinate at which the back ends of the carts are at the same position and (b) the time at which that occurs.

Note: Hopefully, at least a few students will realize that distance down the ramp cannot exactly be compared with horizontal distance. The coordinate system of one cart is rotated with respect to the other. Because the ramp angle is so small, a "small angle approximation" has been made, and the results are only approximately correct.

Falling Things

Lesson Plan Overview

Lesson Description

In this lesson, students will observe several examples of accelerated and non-accelerated motion. They will first observe the motion and write verbal descriptions of it. They will then predict time-domain graphs of the motion by sketching qualitative graphs. Only after making predictions will students use motion sensors to collect data that verifies or refutes them.

All experiments will involve one-dimensional motion. Equations representing the motion will be either linear or quadratic.

This lesson is accessible to students studying Algebra I.

Essential Question(s)

How can we use graphs to describe the motion of an object that is moving along a straight line under various conditions?

Math Standards

Algebra I

A1.F.IF.B.3: For a function that models a relationship between two quantities, interprets key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

A1.F.IF.C.8 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

i) Tasks have a real-world context.

*ii) Tasks are limited to **linear functions**, **quadratic functions**, square root functions, cube root functions, piecewise-defined functions (including step functions and absolute value functions), and exponential functions with domains in the integers.*

Math Practices

2. Reason abstractly and quantitatively.
4. Model with mathematics.
8. Look for and express regularity in repeated reasoning

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Science Standards

PHYS.PS2.1: Investigate and evaluate the graphical and mathematical relationship (using either manual graphing or computers) of one-dimensional kinematic parameters (distance, displacement, speed, velocity, acceleration) with respect to an object's position, direction of motion, and time.

Lesson Objectives

TSW predict the shape of graphs that describe motion with constant velocity and/or accelerated motion and test their predictions against graphs collected in an experiment.

TSW connect graphs of motion with observation, verbal descriptions, and equations.

Materials

- Motion sensor and data acquisition device, such as those from Vernier or PASCO
- Tennis balls and/or other things that can be both dropped and rolled on a level floor
- Cart track and dynamics carts
- Vinyl threshold (available at a home supply store) to make a tennis ball track (optional)
- Handouts for recording predictions and verifications

Note: An alternate method for obtaining the experimental graphs if motion sensors are not available is to perform videoanalysis as described in the lesson ***Basketball Shot****.*

Basic Instructional Plan

The teacher provides a handout with instructions for student investigation. The student will:

1. observe an object set in motion according to a set of instructions
2. making a qualitative sketch of the graph of motion and explain the sketch in words
3. select an equation that would be associated with the graph of motion
4. collect experimental data to support or refute their qualitative graph and equation selection

Ancillary Materials

1. Motion Graphs for Falling Objects
2. Qualitative Motion Graphing Lab
3. Student Skill Development
4. Student Handout for Lab Exploration
5. Answer Key

Motion Graphs For Falling Objects

Terminology

Physics tells us that when an object does not experience a net force, it moves with constant velocity. An object that is moving with constant velocity is an ideal real-world connection for the linear equation.

By comparison, when an object moves under the influence of a constant net force, it experiences constant acceleration. A common example of constant acceleration is an object that falls under the influence of gravity. Physicists call this free-fall. Another example is an object that rolls on a ramp. Gravity causes this object to accelerate as well, but not with full gravitational acceleration. We can still say that the object is in a sense "falling", but not free falling! Falling objects, whether in free fall or "falling" by rolling on a ramp, are ideal for investigating parabolas and quadratic equations.

General Principles of Time Domain Graphing

Motion graphs are time-domain, and this can present a new conceptual challenge for students who are accustomed to graphing with exclusively (*x, y*) coordinates. When students are first grappling with time-domain graphing to describe motion they are actually observing, it is best if the teacher is consistent where possible to avoid unnecessary confusion. For our 1-dimensional motion experiments, we will always define an *x*-axis for the object to move along, even when the motion is vertical. A second spatial axis (*y*-axis) is unnecessary.[1] We will define the origin of our *x*-axis by the position of a motion sensor, and define the positive *x*-direction as pointed away from the sensor. Because the lab is qualitative in nature, exact measurements are not required.

Students will need to acknowledge that time *t* is the independent variable, and position *x* is the dependent variable. One way to explain this is that when an object is moving, position depends on time and not the other way around. If we change the motion of an object, we do not change the rate at which time passes.[2] Therefore, we can with confidence say that position *x* is a function of time *t* and write $x = f(t)$.[3] Correct understanding of functional dependencies is necessary for students to correctly assign horizontal and vertical axes to the appropriate variables.

For each physical experiment or demonstration we run, we must know some details about the *x*-axis our object will move along, specifically the location of the origin and the positive *x*-direction. In the interest of consistency, we will define the origin as the position of a motion sensor, and the positive *x*-direction as pointed away from the motion sensor. This should be an easy convention for your students to remember, and they should be able to visually identify motion as toward, or away from, the origin.

---

[1] For 2-dimensional motion (such as that described in the Basketball Shot lesson), two spatial coordinates (*x* and *y*) as well as time (*t*) are required; however, that is a more advanced lesson.

[2] Apologies to Einstein, but we will ignore you in Algebra I.

[3] In my first physics class, I had a huge *ah-ha* moment I still vividly remember when my teacher said "position is a function of time" and wrote $x = f(t)$ on the board. I suddenly had a conceptual breakthrough in my understanding of functions that had eluded me through several mathematics classes.

Motion With Constant Velocity

Although our goal is to experiment with and predict qualitative graphs for falling objects that accelerate, for comparison purposes let's begin with a review of constant velocity motion and the linear graphs associated with it, such as the ones shown in Figure 1 below. Note that the qualitative graphs sketched by the students will not have a scale on their axes.

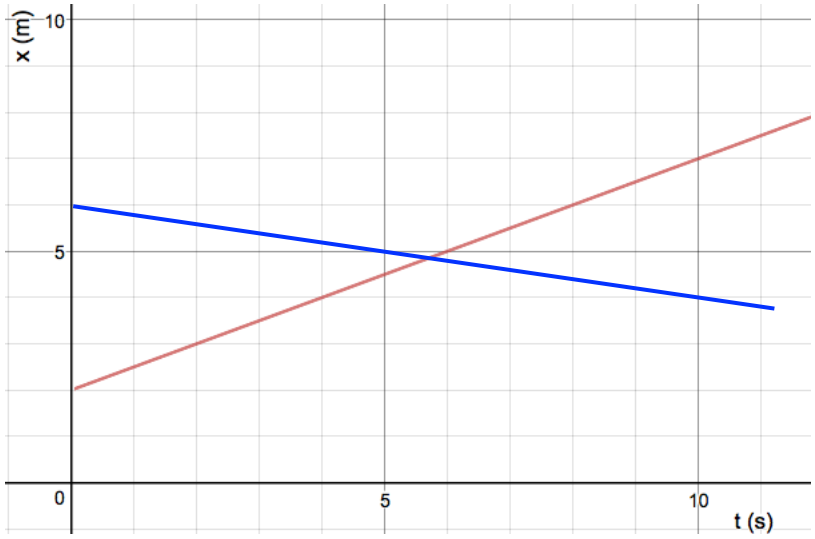


Figure 1

Objects Moving With Constant Velocity

The general equation for constant-velocity motion is $x = vt + x_o$, where the slope of the line is the velocity v of the object and x_o is the starting position at time $t = 0$. The faster the object moves, the steeper the slope. If an object moves in the negative x-direction, the slope is negative. This motion equation can be compared to the more familiar general form of a linear equation, $y = mx + b$. For the graph above using the time domain and motion along an x-axis, we can convert the general form to $x = mt + b$.

By examining the two curves in Figure 1, the student can draw some conclusions about the motion of the objects. Qualitatively, the object described by the red curve moves in the positive direction, heading away from the origin with a constant velocity. The object described by the blue curve starts farther from the origin and moves with a constant negative velocity toward it. The slope of the blue line is shallower, which means the object is not moving as fast. If the objects move at the same time, they may collide at the position and time indicated by the intersection of their graphs.

Motion With Constant Non-Zero Acceleration

By comparison, when an object moves along the x-axis axis with constant non-zero acceleration, the graph of its position versus time is parabolic, as shown in Figure 2 below.

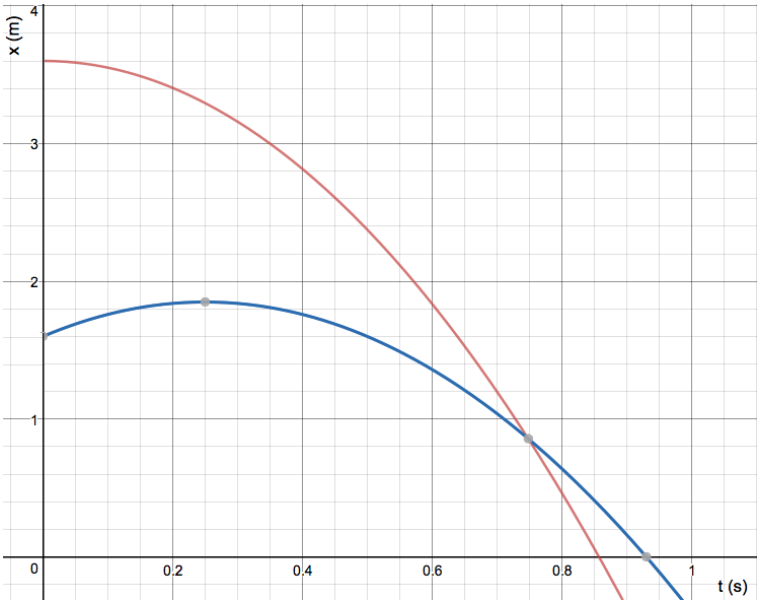


Figure 2

An Object Moving With Constant Non-Zero Acceleration

The general equation for 1-dimensional motion with constant acceleration is $x = \frac{1}{2}at^2 + v_o t + x_o$, where a is acceleration, v_0 is initial velocity, and x_0 is initial position. This equation is analogous to the general form of a quadratic equation, $y = Ax^2 + Bx + C$, which in the time domain is $x = At^2 + Bt + C$

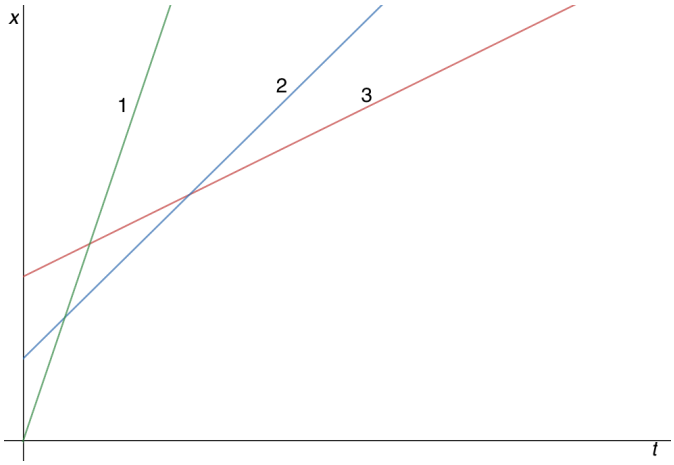
These curves are significantly more challenging to interpret conceptually, but let's try. We'll begin with the red curve. It describes an object that begins its motion at a positive *x*-position, meaning that *C* is positive. The object is moving in the negative direction because as time elapses its position is getting less positive (hence more negative). Not only that, the position is changing more and more rapidly as time elapses. This indicates higher and higher velocities in the negative direction. How do we know this? If we look at the change in position for small time intervals, this change is bigger and bigger as time elapses. The acceleration, *a*, is therefore negative because the velocity is getting more and more negative. We would predict that if the red parabola is fit with a quadratic equation, the *A* term would be negative. The *B* term is the hardest to predict, but it seems that we may be at the peak of the parabola. Parabolas symmetric about the vertical axis would have *B* = 0. (Note that the actual equation for the quadratic representing the red curve is $x = -4.9\frac{m}{s^2}t^2 + 3.6m$).

In comparison with the red curve, the blue curve depicts an object that starts at closer to the origin and initially moves in the positive direction, indicating an initial positive velocity. The object gradually slows to a momentary stop and then moves in the negative direction with higher and higher negative velocities. The acceleration is therefore negative, but less negative than for the red curve because the change in negative velocity from one interval in time to the next is less extreme. Therefore, the *A* term is negative, but not as negative as for the red curve. The *B* term is positive because the initial velocity is positive. And the *C* term is positive. (Note that the actual equation for this quadratic representing the blue curve is $x = -4.0\frac{m}{s^2}t^2 + 2.0\frac{m}{s}t + 1.6m$).

Student Skill Development

Reviewing some examples of qualitative graphs prior to the laboratory will help prepare students for the challenging task of sketching the graphs on their own. The teacher can show the sample graphs below to the students and ask them to describe and compare the motion in words, select an appropriate motion equation, and rank the constants in order. Each sample graph below includes a possible correct verbal description as well as the appropriate general equation of motion. Constants in the equation are ranked in numeric order as a way of comparing the motion of the objects graphed on the same axis.

Sample 1: Constant Positive Velocity



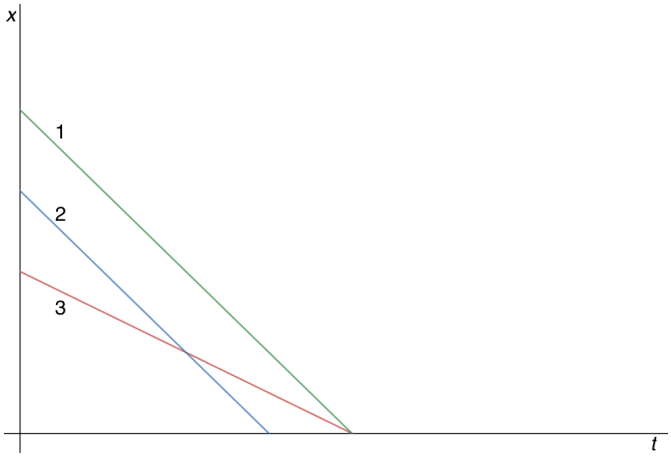
All objects are moving away from the origin. Graph 1 illustrates motion that is the fastest and has the highest positive velocity. Graph 3 illustrates the slowest object, and Graph 2 is in the middle. The object associated with Graph 3 begins at the origin and the other two objects start at other positive values of *x*.

All three graphs can be described by the equation: $x = vt + x_o$

Ranking of *v*: 3 < 2 < 1

Ranking of x_o: 1 < 2 < 3

Sample 2: Constant Negative Velocity



All objects are moving toward the origin. Graph 1 and Graph 2 depict objects with the same negative velocity; they are moving equally fast in the negative direction. These two objects are moving faster, with a greater negative velocity, than the object associated with Graph 3. All three objects have positive initial starting points.

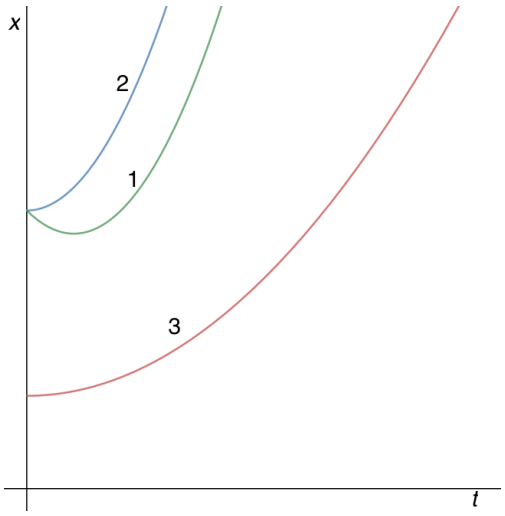
All three graphs can be described by the equation: $x = vt + x_o$

Ranking *v*: 2 = 1 < 3 *

Ranking x_o: 3 < 2 < 1

**Note that the velocities in 2 and 3 are more negative and are ranked as less than the slope in 3 even though the speed is faster for these objects.*

Sample 3: Positive acceleration

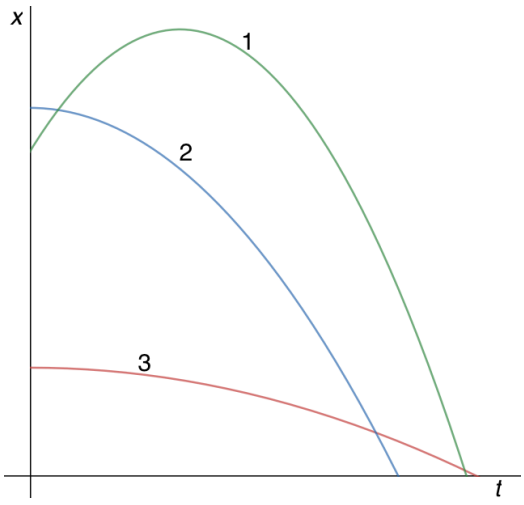


Objects 2 and 3 move away from the origin the entire time with continually increasing positive velocity, so the multiplier of t^2 and the objects' acceleration must be positive. A positive coefficient on the squared variable is also a characteristic of concave up parabolas. Object 3's parabola is wider, so the coefficient of t^2 and hence its acceleration is smaller than that of Object 2 indicating it is changing its velocity less rapidly. The parabolas are symmetric about the vertical axis, so v_o (the coefficient of t) must be zero. Object 1 moves toward the origin with negative initial velocity v_o but slows as it reaches a minimum positive value and reverses direction. It then moves increasingly faster in the positive direction indicating increasingly positive velocity and positive acceleration. Based on the curvature of the parabolas, Objects 1 and 2 appear to have the same acceleration, as well as the same initial x-value.

All three graphs can be described by: $x = \frac{1}{2}at^2 + v_o t + x_o$

Ranking a: 3 < 2 = 1
Ranking v_o: 1 < 2 = 3*
Ranking x_o: 3 < 1 = 2
**Note that the initial velocity of Object 1 is negative and the other two are zero, hence the ranking.*

Sample 3: Negative acceleration



Objects 2 and 3 move toward the origin the entire time with continually increasing negative velocity, so the multiplier of t^2 and the objects' acceleration must be negative. A negative coefficient of the squared variable is also a characteristic of concave down parabolas. Object 3's parabola is wider, so the coefficient of t^2 and hence its acceleration has a smaller absolute value (is less negative) than that of Object 2. The parabolas are symmetric about the vertical axis, so v_o as the coefficient for t must be zero. Object 1 moves away from the origin with positive initial velocity v_o but slows as it reaches a maximum positive value and reverses direction. It then moves faster and faster in the negative direction indicating increasingly negative velocity and negative acceleration. None of the parabolas seem to indicate the same value of acceleration.

All three graphs can be described by: $x = \frac{1}{2}at^2 + v_o t + x_o$

Ranking a: 1 < 2 < 3*
Ranking v_o: 2 = 3 < 1
Ranking x_o: 3 < 1 < 2
**Note that the acceleration terms are all negative and they are ranking in number line order, which is the inverse of the acceleration magnitude (absolute value).*

Qualitative Motion Graphing Lab

General Procedure

In this lab, students observe motion of a ball by running very simple experiments. On a provided handout, they are given an illustration and instructions for each experiment that includes the proposed position of a motion sensor that defines the origin of the *x*-axis. As a ball moves away from the motion sensor, it moves away from zero to more positive *x*-coordinates. The students run each experiment, sketch a qualitative graph describing the motion they have observed, and explain why they drew the graphs as they did. They also select an equation of motion by choosing between linear and quadratic forms and identifying constants as positive, negative, or zero. The final step is to repeat the experiment using the motion sensor to collect actual experimental graphs.

Instructions for setting up and using motion sensors are available from the vendor. If time is constrained, the teacher could collect experimental motion graphs prior to the laboratory to show to the students afterward. However, if time permits, students love to use this technology, adapt readily to it, and would find collecting their own experimental evidence quite engaging.

This lab can be done as a sequence of teacher-led demonstrations or as individual student exploration. It lends itself most naturally to a station activity, where each station requires students to perform and analyze a different experiment. Stations should be set up with the necessary equipment prior to class, and students should be given plenty of time to collaborate as they experiment. The teacher may find it particularly effective to implement a collaborative strategy such as Jigsaw or Gallery Walk to facilitate meaningful collaboration.

Experiment Setup Instructions

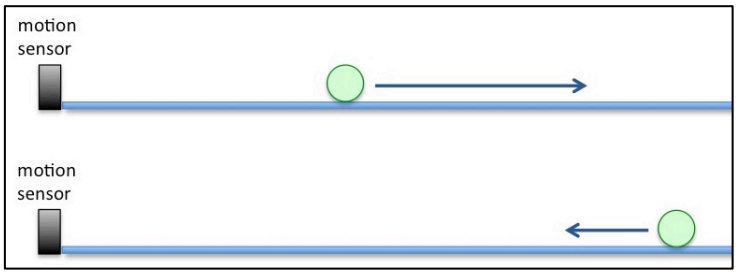
Each of the illustrations on the following handout shows the setup for each suggested experiment. The position of the motion sensor is shown, although it is not necessary to have the motion sensor in place during student experimentation. Students simply need to know where the sensor would be, since it defines the origin and the direction of the positive *x*-axis.

Note that especially for the Experiment 3 the sample rate of the motion sensor must be adjusted so it rapid enough to capture the motion. Most motion sensors have a default sample rate of 10 Hz, which is generally not be sufficient when gravitational acceleration is in play. A sample rate of between 60 to 100 Hz is much better for free fall.

Student Handout for Lab Exploration

Experiment 1

Roll a tennis ball or a cart on a level track in two ways: (1) roll it away from the motion sensor at a fairly high speed and (2) roll it toward the motion sensor at a lower speed, but not low enough you observe it slowing down.



Draw two qualitative graphs describing 1 and 2 on the axes below. Clearly label the graphs.

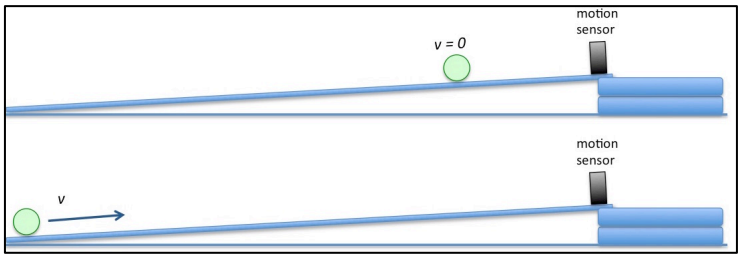


Explain why you drew the graphs as you did.

Select a motion equation, either $x = vt + x_o$ or $x = \frac{1}{2}at^2 + v_o t + x_o$, that satisfies both graphs. Rank the constants in the equations in numeric order, from least to greatest. Explain your ranking.

Experiment 2

Roll a tennis ball or a cart on a ramp in two ways: (1) hold the ball or cart at rest and release it so it rolls down the ramp away from the sensor and (2) roll the ball or cart up the ramp so it stops before reaching the sensor and then rolls back down.

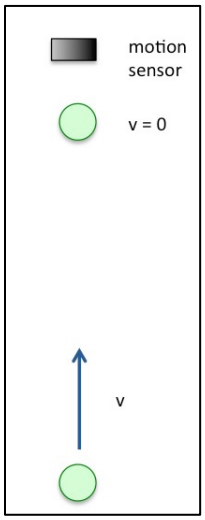


Draw two qualitative graphs describing 1 and 2 on the axes below. Clearly label the graphs.



Explain why you drew the graphs as you did.

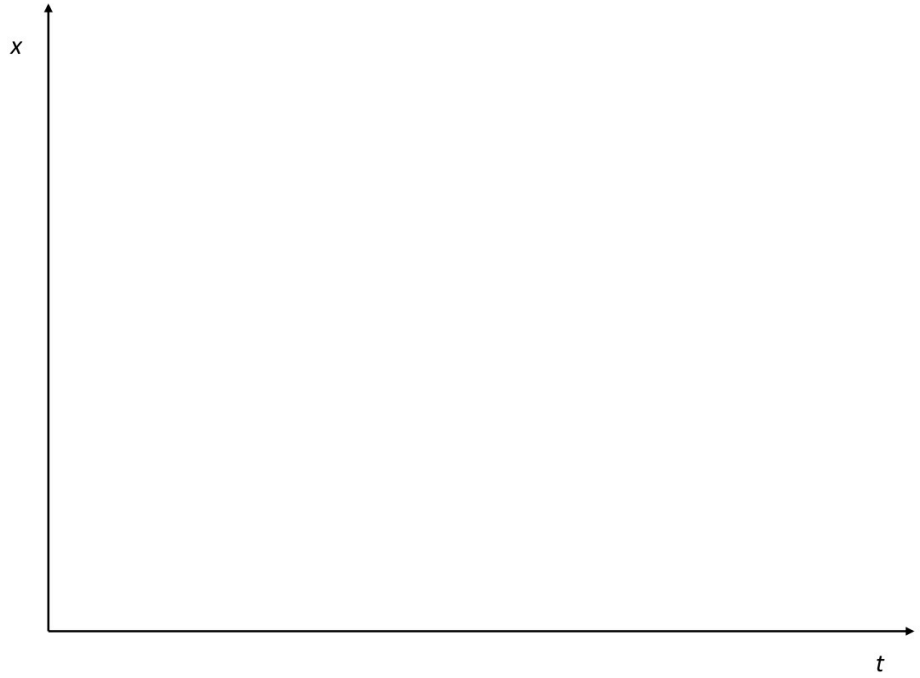
Select a motion equation, either $x = vt + x_o$ or $x = \frac{1}{2}at^2 + v_o t + x_o$, that satisfies both graphs. Rank the constants in the equations in numeric order, from least to greatest. Explain your ranking.



Experiment 3

Drop a tennis ball in two ways: (1) hold the ball at rest beneath the motion sensor and release it so it falls freely and (2) toss the ball up toward the sensor so it stops before reaching the sensor and then falls back down.

Draw two qualitative graphs describing 1 and 2 on the axes below. Clearly label the graphs.



Explain why you drew the graphs as you did.

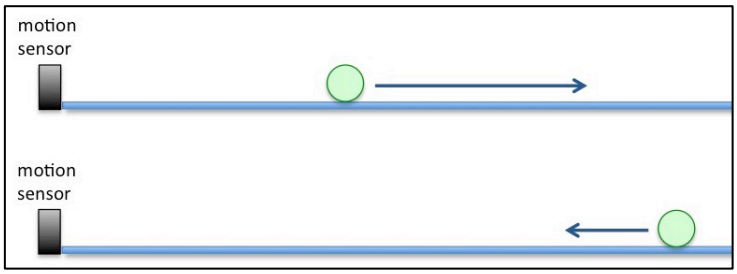
Select a motion equation, either $x = vt + x_o$ or $x = \frac{1}{2}at^2 + v_o t + x_o$, that satisfies both graphs. Rank the constants in the equations in numeric order, from least to greatest. Explain your ranking.

Explain how the graphs and equations compare to those in Experiment 2.

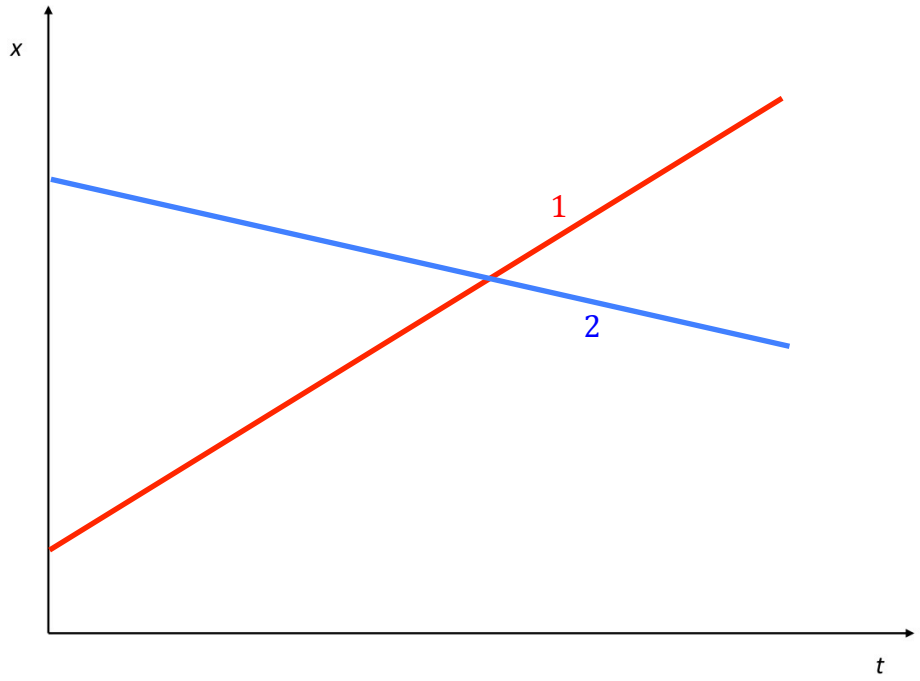
Answer Key

Experiment 1

Roll a tennis ball or a cart on a level track in two ways: (1) roll it away from the motion sensor at a fairly high speed and (2) roll it toward the motion sensor at a lower speed, but not low enough you observe it slowing down.



Draw two qualitative graphs describing 1 and 2 on the axes below. Clearly label the graphs.



Explain why you drew the graphs as you did.

Object 1 starts closer to the origin and moves away from the origin so the velocity and slope is positive. Object 2 starts farther from the origin and moves toward it, so the velocity and slope must be negative. Object 2 is moving slower than Object 1, so the slope cannot have as large and absolute value and graph is less steep.

Select a motion equation, either $x = vt + x_o$ or $x = \frac{1}{2}at^2 + v_o t + x_o$, that satisfies both graphs. Rank the constants in the equations in numeric order, from least to greatest. Explain your ranking.

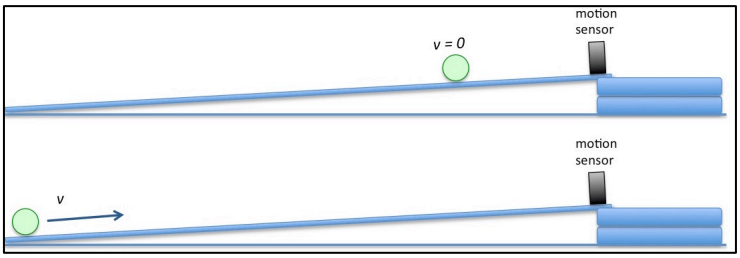
$x = vt + x_o$

v: 2 < 1 (2 is negative and 1 is positive)

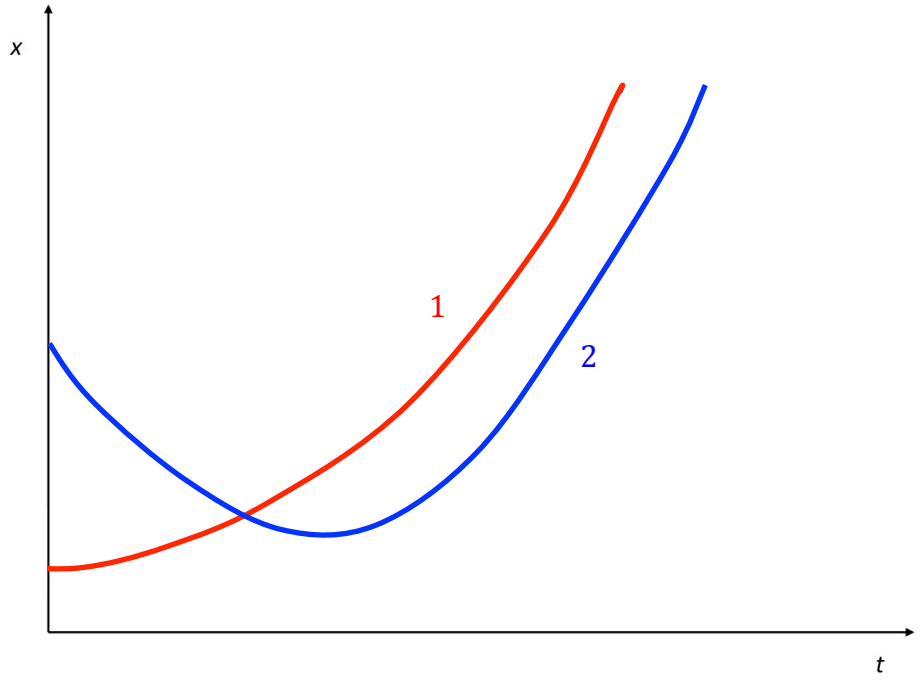
x_o: 1 < 2 (1 starts closer to the origin than 2)

Experiment 2

Roll a tennis ball or a cart on a ramp in two ways: (1) hold the ball or cart at rest and release it so it rolls down the ramp away from the sensor and (2) roll the ball or cart up the ramp so it stops before reaching the sensor and then rolls back down.



Draw two qualitative graphs describing 1 and 2 on the axes below. Clearly label the graphs.



Explain why you drew the graphs as you did.

Object 1 starts closer to the origin from rest and moves away from the origin (positive direction) faster and faster with increasingly positive velocity. Therefore it has positive acceleration. Object 2 starts farther from the origin and initially moves toward it, so the initial velocity must be negative. However, it gradually slows to a stop and slowly reverses direction. After that happens, its motion looks a lot like that of Object 1. Beginning with its momentary rest, it accelerates away from the origin (positive direction) faster and faster with increasingly positive velocity.

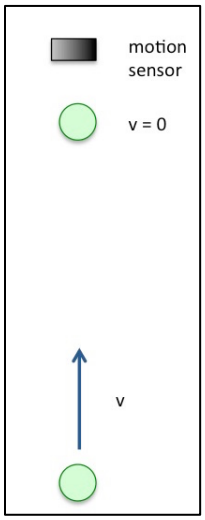
Select a motion equation, either $x = vt + x_o$ or $x = \frac{1}{2}at^2 + v_o t + x_o$, that satisfies both graphs. Rank the constants in the equations in numeric order, from least to greatest.

$x = \frac{1}{2}at^2 + v_o t + x_o$

a: 1 = 2 (this is the hardest to get, but the clue that Object 2 momentarily stops and then it rolls down is very helpful)

v_o: 2 < 1 (2 is negative and 1 is zero)

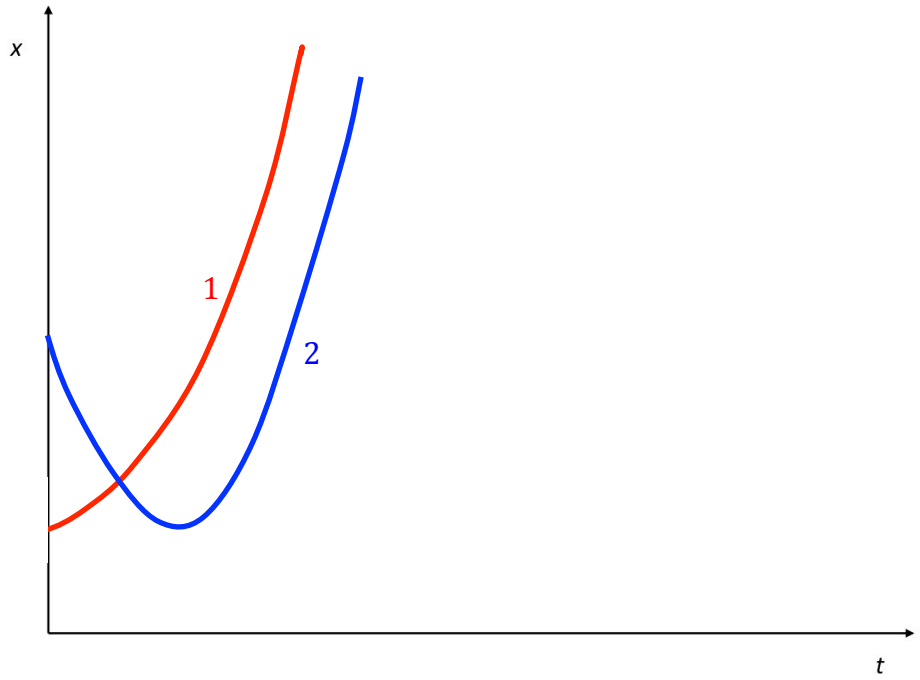
x_o: 1 < 2 (1 starts closer to the origin than 2)



Experiment 3

Drop a tennis ball in two ways: (1) hold the ball at rest beneath the motion sensor and release it so it falls freely and (2) toss the ball up toward the sensor so it stops before reaching the sensor and then falls back down.

Draw two qualitative graphs describing 1 and 2 on the axes below. Clearly label the graphs.



Explain why you drew the graphs as you did.

Object 1 starts closer to the origin than Object 2. Object 1 is dropped from rest it accelerates rapidly away from the origin (positive direction) with increasingly positive velocity and positive acceleration. Object 2 starts farther from the origin and initially moves toward it, so the initial velocity must be negative. However, it slows to a momentary stop and reverses direction. After that happens, its motion resembles that of Object 1 as it accelerates away from the origin (positive direction) with increasingly positive velocity.

Select a motion equation, either $x = vt + x_o$ or $x = \frac{1}{2}at^2 + v_o t + x_o$, that satisfies both graphs. Rank the constants in the equations in numeric order, from least to greatest.

$x = \frac{1}{2}at^2 + v_o t + x_o$

a: 1 = 2 (both are gravitational acceleration)
v_o: 2 < 1 (2 is negative and 1 is zero)
x_o: 1 < 2 (1 starts closer to the origin than 2)

Explain how the graphs and equations compare to those in Experiment 2.

These graphs are very similar to those in Experiment 2. The acceleration is much larger, but still in the positive direction (away from the motion sensor). Because the acceleration is larger, the parabolas in Experiment 3 must be narrower than those in Experiment 2.

Basketball Shot

Lesson Plan Overview

Lesson Description

Students will analyze video of moving objects, specifically those in projectile motion. Using *video analysis* to perform frame by frame measurement of an object's position over time, they will collect, tabulate, and graph data describing the coordinates of the object's position at various times. They will analyze the data to define a function for the parabolic trajectory of the projectile. They will extend the analysis to time-domain functions and their graphs for position and velocity.

Much of this lesson is accessible to students studying Algebra I. The calculus extension is appropriate for advanced students.

Essential Question(s)

How can we define the object's *trajectory* in graphical and equation form?

How can we describe how position and velocity vary with time in two-dimensional projectile motion?

Math Standards

Algebra I

A1.F.IF.B.3: For a function that models a relationship between two quantities, interprets key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

A1.F.IF.B.5: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph

A1.S.ID.B.4: Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. (a) Fit a function to the data; use functions fitted to the data to solve problems in the context of the data. Emphasize linear, quadratic, and exponential models.

i) Tasks have a real-world context.

*ii) Tasks are limited to **linear functions**, **quadratic functions**, square root functions, cube root functions, piecewise-defined functions (including step functions and absolute value functions), and exponential functions with domains in the integers.*

Calculus Extension

C.D.CD.A.1: Represent and interpret the derivative of a function graphically, numerically, and analytically.

C.D.CD.A.2: Interpret the derivative as an instantaneous rate of change.

C.I.AI.B.4: Use a definite integral to find the area of a region.

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Math Practices

2. Reason abstractly and quantitatively.
4. Model with mathematics.
8. Look for and make use of structure.

Science Standards

PHYS.PS2.1: Investigate and evaluate the graphical and mathematical relationship (using either manual graphing or computers) of one-dimensional kinematic parameters (distance, displacement, speed, velocity, acceleration) with respect to an object's position, direction of motion, and time.

PHYS.PS2.13: Develop a model to predict the range of a two-dimensional projectile based upon its starting height, initial velocity, and angle at which it was launched.

Lesson Objectives

TSW interpret key features of linear and parabolic motion graphs obtained by regression analysis.

TSW connect graphs of position versus time and velocity versus time to draw conclusions.

(Extension)

TSW determine the relationship between derivative and integral graphs and functions.

Materials

- Digital video cameras and tripods (cell phones can be used instead of cameras)
- Balls, meter sticks, and other measuring tools
- Computers or tablets with video analysis software installed.

Basic Instructional Plan

1. Students produce video clips of projectile motion[1].
2. Students import the video clips into a program that allows them to perform video analysis.
3. Students perform regression analysis of experimental *x*, *y*, and *t* data collected from the videos to develop various functions, such as *y(x)*, *y(t)*, and *x(t)* and their associated graphs.
4. Students analyze the graphs and functions they have developed to meet the measurable objectives and address concepts described in the standards.

Instructional Plan Extension (Calculus)

Advanced students will use calculus skills to relate experimental position-versus-time and velocity-versus-time graphs and functions to each other.

Ancillary Materials

1. The Algebra of Trajectory
2. Time Domain Functions
3. Graphical and Analytical Calculus Explorations

[1] Sample videos of projectile motion are available from vendor if student-collected data is impractical.

The Algebra of Trajectory

Terminology

When someone throws or otherwise launches an object under the influence of gravity, it becomes what is known as a *projectile*. If the object experiences uniform gravity and doesn't have any sort of parachute or any other characteristic that makes it experience significant air resistance, the type of motion it experiences is called *free fall*. Projectiles in free fall experience uniform (or constant) acceleration in the downward direction, and no acceleration at all in the horizontal direction.

The path of a projectile in free fall near the surface of the earth is parabolic, unless the object is thrown straight upward or straight downward. Students can become more familiar with the mathematics of parabolas by exploring the trajectory of a projectile.

Data Collection

Figure 1 shows the trajectory of a basketball collected using Vernier Logger Pro[2]. The specific video shown in the figure comes from a collection of Logger Pro sample files, but any video of a thrown projectile can be used. In fact, students will find it more enjoyable to record their own video clips using a camera or smart phone on a tripod. Students will need to place an object of known length in the plane of the anticipated motion to serve as a measurement calibration device. In the figure below, that object is a 2-meter track on the floor of the gym.

After the video is imported into the Logger Pro program, a coordinate system is defined by positioning the yellow coordinate axes at a desired location. The scale is defined by overlaying the green measurement line over an object of known length and entering the length. To collect the actual data, blue dots are added to the display by touching the center of a moving object with a cursor as the video automatically advances one frame at a time. The program automatically calculates and tabulates the *(x, y)* coordinate of each blue dot and pairs the resulting point with the time *t* corresponding to the frame on the screen when the dot is added.

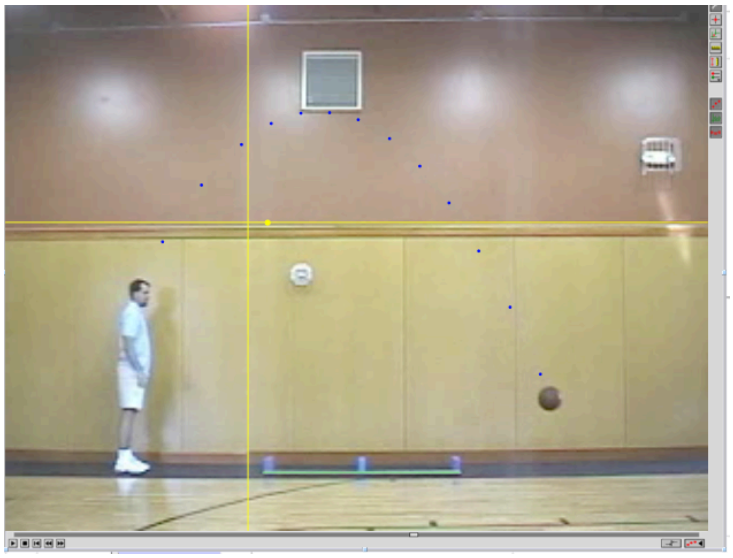


Figure 1
Collecting Trajectory Data using Logger Pro by Vernier

[2] Vendors provide specific instructions for using videoanalysis software. All examples shown in this lesson were collected using Logger Pro by Vernier.

As data is captured in this manner, Logger Pro produces a table such as that shown below in Figure 2. Note that in addition to *t*, *x*, and *y* data, Logger Pro has also calculated velocity components in the orthogonal *x* and *y* directions; these appear in the rightmost two columns.

Trajectory Data				
t (s)	*x (m)*	*y (m)*	*v_x (m/s)*	*v_y (m/s)*
0.15	-0.87	-0.20	3.00	3.97
0.28	-0.47	0.38	3.02	3.43
0.41	-0.07	0.80	3.02	2.67
0.51	0.23	1.01	3.01	1.67
0.61	0.53	1.11	2.98	0.61
0.71	0.83	1.12	2.96	-0.36
0.82	1.12	1.05	3.02	-1.33
0.92	1.44	0.85	3.07	-2.33
1.02	1.75	0.58	3.05	-3.29
1.12	2.05	0.20	3.03	-4.31
1.22	2.34	-0.29	3.08	-5.22
1.32	2.66	-0.86	3.12	-5.96
1.42	2.97	-1.55	3.12	-6.48

Figure 2

Tabulated Trajectory Data from Basketball Shot

Data Analysis

While Logger Pro provides many internal tools for analysis, the data can also be exported. For the analysis below, the second and third columns from Figure 2 have been copied into Excel. The data is then graphed on an *x*, *y* scatterplot and fit with a second-order polynomial using the Excel trendline feature, as shown below:

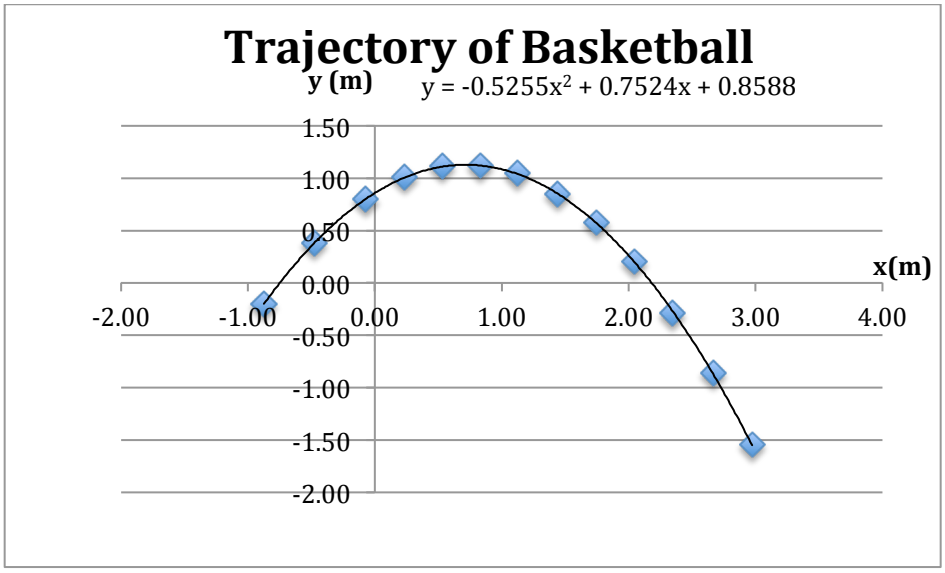


Figure 3

Scatterplot and Trendline Fit of Trajectory Data

Figures 2 and 3 together present three different representations of trajectory data - a table of numeric data, a graph, and an equation. Students can be asked to find key features of the parabolic trajectory using each of these three representations. Key features could include the roots of the parabola, the *y*-intercept, and the vertex. Students could compare estimates obtained, and discuss the advantages and disadvantages of each of the representations.

Below are examples of how to use the table, the graph, and the equation to obtain the roots, *y*-intercept, and vertex of the parabola defining the trajectory of the basketball.

a) Using Tabulated Data (Figure 2)

Students should notice it is not so convenient to get an exact numeric value for a key feature from the tabulated data. There are at least three common approaches students might use to extract key features of the parabolic function using data from the table.

1. Pick the closest number they can given what they know of the key feature.
2. Pick two numbers and claim the actual number is in between them.
3. Take approach (b) further and average the two numbers to get a more accurate estimate through the process of data interpolation.

The examples below illustrate how students might use these three approaches to extract numeric descriptions of the key features of the parabola using data from the table in Figure 2:

Roots: The roots occur where the y-coordinate changes from positive to negative or vice-versa. The roots therefore must exist between x-coordinates of -0.87 and -0.47 m, as well as between 2.05 and 2.34 m. [This is an example of approach 2]

y-Intercept: The function has a y-intercept where x-coordinate changes from negative to positive. The y-intercept is between 0.80 and 1.01; the mean of these two numbers is 0.91. Therefore, 0.91 m is closest to the y-intercept. [This is an example of approach 3]

Vertex: The vertex is the point at which the y-value is largest, which is (0.83 m, 1.12 m) for the tabulated data. [This is an example of approach 1]

b) Using the Graphical Representation (Figure 3)

Students should notice that they cannot get highly precise numbers from the graph. However, they can very quickly spot key features of the parabola from its graphical representation, and they can conveniently extract approximate numeric data. Of course, the ease of reading a graph depends somewhat on how large it is, and how well the grid lines are marked.

Students should not imply too much precision when reading values from graphs. The numbers below are restricted to one decimal place for that reason.

Roots: Roots appear to be at approximately -0.7 m and 2.2 m on the x-axis.

y-Intercept: The y-intercept appears to be at about 0.8 m.

Vertex: The vertex appears to be at approximately (0.7 m, 1.1 m).

c) Using the Equation (Figure 3)

The parabolic trajectory of the basketball can be modeled using a second-order polynomial function of the form $y = ax^2 + bx + c$. When the Excel trendline feature is used to fit the experimental data with such a polynomial, it yields $y = -0.5255x^2 + 0.7524x + 0.8588$. Students can use this equation to calculate the roots, y-intercept, and vertex of the parabola. They should expect these quantities to be consistent with those estimated from the table and graph.

Students may think that equations facilitate calculation of "exact" numeric quantities, and in many cases they would be correct. However, it is important that students recognize that in this

specific case involving a real-world parabolic trajectory, calculated values obtained from the equation are still approximate. This is because regression analysis of experimental data that inevitably has some measurement error associated with it results in an equation which is necessarily a close approximation of the actual trajectory. Calculation of key features using the equation will therefore yield approximate results, although these results may be closer to the actual values than those estimated from the table or graph.

The examples below illustrate how students might use the equation to extract numeric quantities for the key features of the parabola:

Roots: The student can use the quadratic formula to calculate that for this equation, x = -0.75 m and x = 2.18 m are the roots. Calculators equipped with CAS packages can be used to solve the equation more conveniently and, of course, yield the same roots.

y-Intercept: The y-intercept is the c term of the quadratic function, or 0.86 m.

Vertex: Students can calculate the x-coordinate of the vertex using the formula, $-b/2a$*. The constants b and a are extracted from the trendline equation, where a = -0.5255 and b = 0.7524. Substituting these quantities into the vertex formula will yield an x-coordinate of the vertex of 0.72 m. Substitution of this x-coordinate into the quadratic function will yield a y-coordinate of 1.13 m. Hence the vertex is at the point (0.72 m, 1.13 m).*

Time-Domain Functions

Logger Pro allows for graphical analysis within the program, and provides tools for exploring real-world algebra, pre-calculus, and calculus concepts involving rate of change. A student can analyze video of the basketball shot, or any free-fall video for that matter, to explore position as a function of time, as well as first and second time derivatives of position, velocity and acceleration. To do this, the student analyzes the vertical and horizontal data separately.

Position and Velocity as Functions of Time

Horizontal motion of a projectile in the time domain

To analyze horizontal motion, the student instructs Logger Pro to graph horizontal data in the time domain and subsequently uses curve-fitting tools to fit the data using regression analysis. Figure 4 shows the resulting graphs for the basketball shot video.

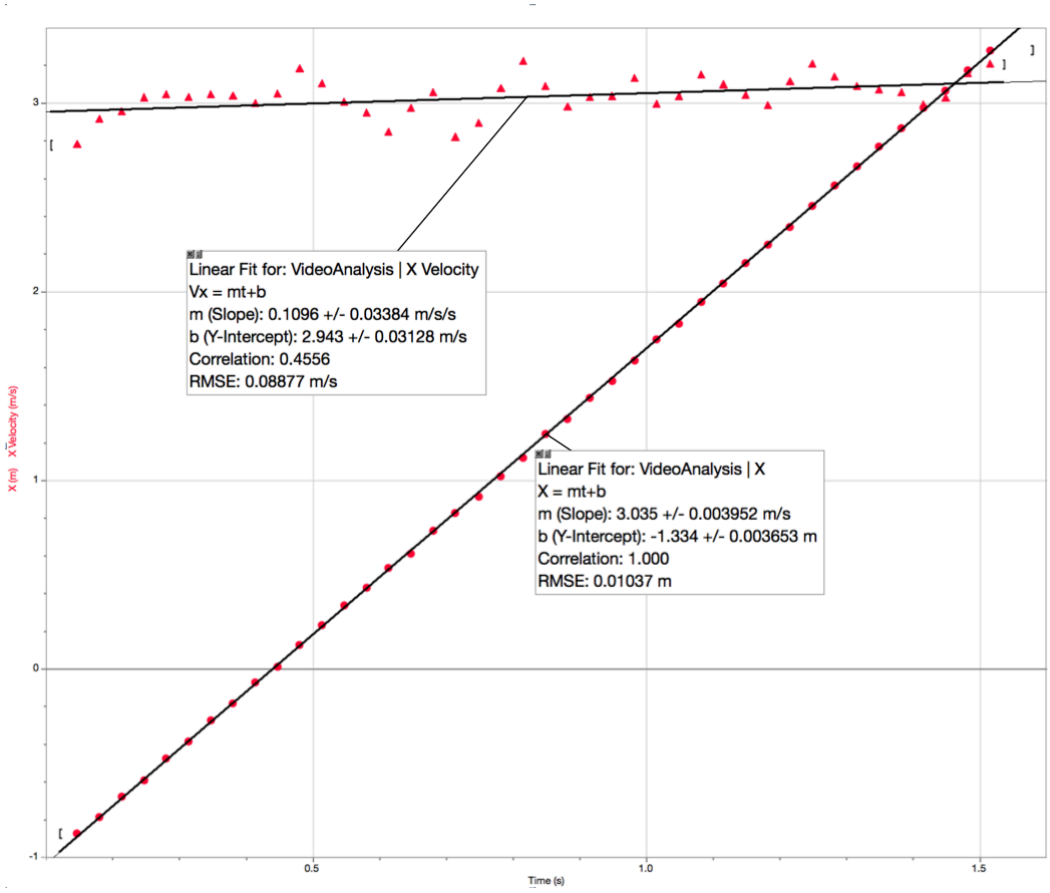


Figure 4

Position and Velocity as Functions of Time in the Horizontal Direction

The graph labeled *VideoAnalysis | X* is a graph of the *x(t)* function, where *x* is the horizontal position of the center of the basketball and *t* is time. The slope of *x(t)*, $\Delta x/\Delta t$, is the horizontal velocity component, v_x. This is consistent with the definition of velocity as the rate of change of position with respect to time. The rate of change of a linear function such as *x(t)* is of course represented by the constant slope of its graph.

It gets more interesting if the student decides to produce the graph labeled *VideoAnalysis | X Velocity*, a graph of horizontal velocity versus time. Some confusion and even frustration may result. The student may reasonably expect that because the slope of $x(t)$ is constant, the slope of $v_x(t)$ will be zero. Notice that this graph does not, as expected, have a slope that is exactly zero! At this point, comparative analysis of horizontal and vertical motion could help the student feel more comfortable evaluating whether deviations in the horizontal velocity graph from its expected appearance are significant and meaningful, or whether they are due to some sort of error or uncertainty. This sort of issue must be resolved routinely in applied or "real-world" mathematics.

Vertical motion of a projectile in the time domain

Now let's turn our attention to the analysis of vertical motion. The student instructs Logger Pro to graph vertical data as a function of time, and again uses curve-fitting tools to perform regression analysis. Figure 5 shows the resulting graphs.

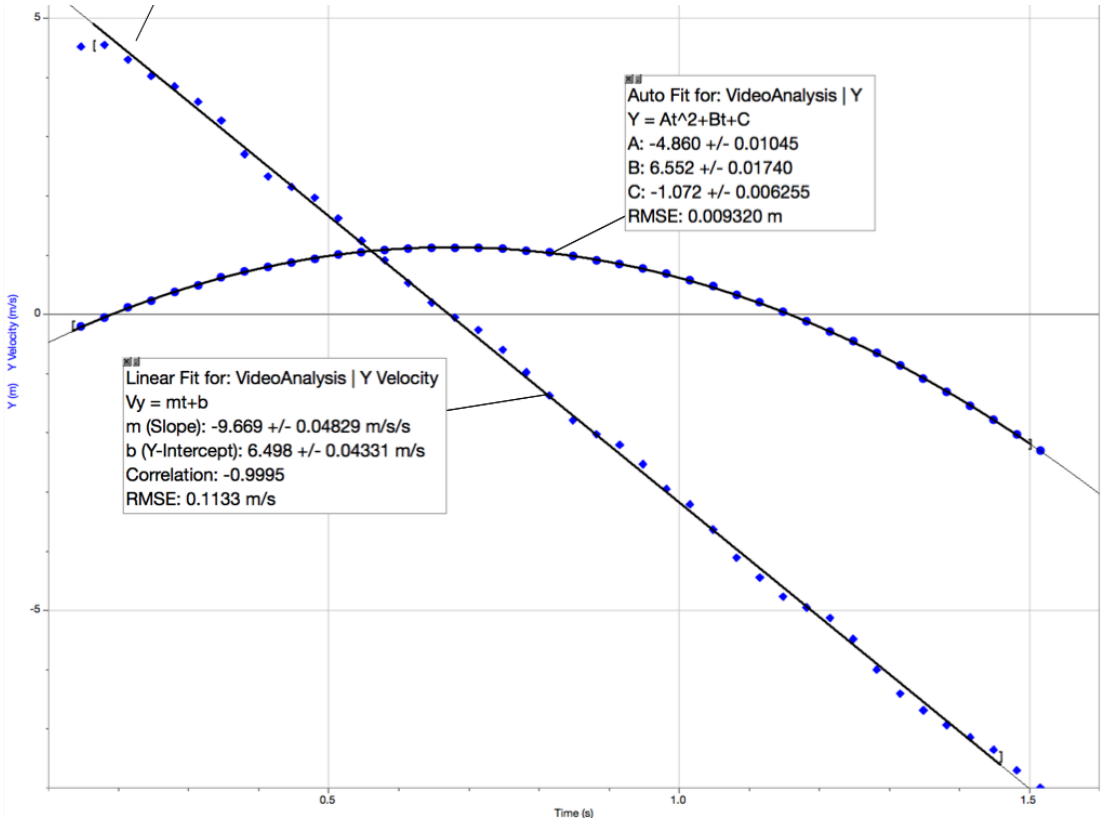


Figure 5

Position and Velocity as Functions of Time in the Vertical Direction

The graph labeled *VideoAnalysis | Y* represents $y(t)$, or vertical position as a function of time. It is apparent that this graph, unlike that of $x(t)$, does not have a constant slope. It has been fit with a second-order polynomial; i.e., a parabola. The graph labeled *VideoAnalysis | Y Velocity*, or $v_y(t)$, appears to be linear with an obvious negative slope. If the student compares $v_y(t)$ in Figure 5 with $v_x(t)$ in Figure 4, it should be clear that $v_x(t)$ comes close enough to having zero slope to assume that velocity is constant in the horizontal direction. More advanced students may be aware that the slope of the graph of $v_y(t)$ represents acceleration, or change in velocity with respect to time. This is calculated by the expression $\Delta v_x / \Delta t$.

Graphical and Analytical Calculus Explorations

The student can explore calculus concepts using the graphs shown in Figure 5. The teacher can guide student investigation using prompts such as the sample ones that follow.

a) Using the graphs, show how tangent lines and first derivatives are related.

The student draws a tangent line to the $y(t)$ curve at a specific time t and determines the slope of the tangent line. The student then compares this slope to the value of $v_y(t)$ evaluated at the same time. Using the graphs in Figure 5, the student might find for example that a tangent to $y(t)$ at 1.42 s has a slope of -7.14 m/s and that v_y(1.42 s) = -7.14 m/s, as expected.

b) Show how to use the $v_y(t)$ graph to help find the coordinates of the vertex of the parabola.

The student finds the time value at which $v_y(t)$ crosses the horizontal axis. In Figure 5, this is very close to 0.68 s. This is where the velocity is zero as its direction shifts from positive (upward) to negative (downward). To get the point representing the vertex, the student evaluates $y(t)$ at 0.68 s as shown below:

$$y(t) = -4.860t^2 + 6.552t - 1.072$$

$$y(0.68) = -4.860(0.68)^2 + 6.552(0.68) - 1.072 = 1.14\ m$$

The point (0.68, 1.14) therefore represents the vertex of the parabola.

c) Using the graphs, show how area under the curve can be used to estimate an integral.

The student can use the built-in Logger Pro area tool to find the area "under the curve" of the $v_y(t)$ function between t_1 and t_2 . This should approximate the definite integral of $v_y(t)dt$ evaluated from t_1 to t_2. The integration should equal the change in position, $y(t_2) - y(t_1)$.

Figure 6 illustrates this area analysis for the basketball shot video, where t_1 = 0.61 s, t_2 = 1.11 s, and the shaded area under the $v_y(t)$ curve is evaluated to be -0.63 m.

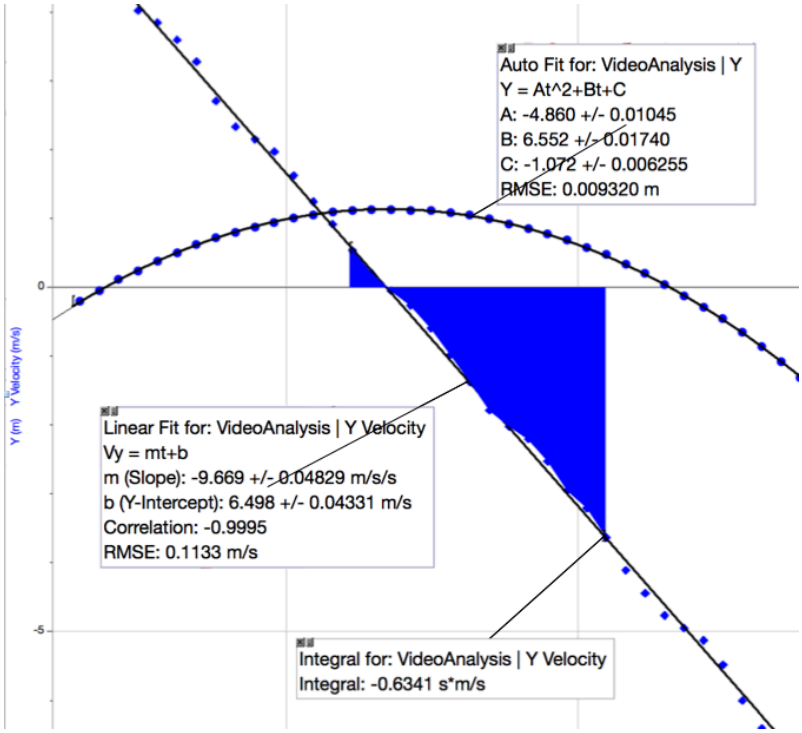


Figure 6

The student will hopefully notice that the area tool looks like it may yield an inaccurate result, as parts of the area "triangle" are not shaded. The student might then do a more careful area estimation by constructing two triangles, one with positive area (above the axis) and a larger one with negative area (below the axis), calculating their areas using the triangle area formula, and summing the areas. This method yields an integrated area almost 0.70 m, which is a bit larger than the shaded area on the graph, as expected.

But is the area equal to the change in position? If $y_1 = y(0.61\text{ s}) = 1.12$ m and $y_2 = y(1.11\text{ s}) = 0.21$ m, then the change in position is $y_2 - y_1 = -0.91$ m when calculated using the equation for the vertical position as a function of time. This does not appear to be very close to the expected change in position based on the area analysis, and clearly there is some discrepancy. This provides an opportunity to discuss the fact that graphical methods can be used to estimate integrals and derivatives, but that analytical techniques will be more accurate.

d) Using the equations from the regression analysis, illustrate the derivative relationship between velocity and position using analytical calculus.

Analytical analysis using equations may be easier for calculus students than the corresponding graphical analysis, and it will certainly lead to more accurate results. Taking the derivative of *y(t)* yields:

$$v_y(t) = \frac{dy(t)}{dt} = \frac{d}{dt}(-4.860t^2 + 6.552t - 1.072) = -9.720t + 6.552$$

This is close to $v_y(t) = -9.669t + 6.498$ produced by fitting the velocity data with a linear function. The acceleration appears to be about -9.7 m/s^2 in both cases, which students studying physics will recognize as near the acceleration due to gravity, -9.8 m/s^2.

e) Using the equations associated with the graphs, illustrate the integral relationship between velocity and position using analytical calculus.

Evaluating the definite integral of $v_y(t)dt$ between two points in time should yield the change in *y(t)* over that elapsed time. Using as the lower and upper limits of integration the same values $t_1 = 0.6133$ s and $t_2 = 1.114$ s as in prompt 3) above:

$$\int_{t_1}^{t_2} v_y(t)dt = \int_{0.6133}^{1.114} (-9.669t + 6.498)dt = -0.91\ m$$

Calculating the change in vertical position between these two points in time:

$$y(t) = -4.860t^2 + 6.552t - 1.072$$

$$y_1 = y(0.6122) = -4.860(0.6133)^2 + 6.552(0.6133) - 1.072 = 1.1183$$

$$y_2 = y(1.114) = -4.860(1.114)^2 + 6.552(1.114) - 1.072 = 0.1957$$

$$y_2 - y_1 = -0.92\ m$$

The change in vertical position calculated using the two methods is similar.

Springy Things

Lesson Plan Overview

Lesson Description

Students will use motion sensors to investigate the periodic motion produced by oscillating spring-mass systems. They will tabulate and graph position and velocity as functions of time, and identify key features on the graphs such as period, frequency, angular frequency, amplitude, and phase. Students will fit the motion data with sine and cosine functions, and examine how the phase differs between these two functions. They will also predict the graphical relationship between position and velocity in an oscillating spring-mass system, and test their prediction by manipulating the graphical axes.

The bulk of the lesson is accessible to students studying Precalculus. The extension exploring trigonometric derivatives is accessible to students in Calculus.

Essential Question(s)

What trigonometric functions best model the motion of an oscillating spring-mass system?

How can we interpret time-domain graphs that look like sine and cosine functions?

What connections can we make when we graph position versus velocity for an oscillator?

Math Standards

Pre-Calculus

P.F.TF.A.3: Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

P.F.TF.A.4: Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Calculus

C.D.AD.A.2: Calculate the derivative of basic functions (power, exponential, logarithmic, and trigonometric).

Math Practices

1. Make sense of problems and persevere in solving them.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precisison.
8. Look for and express regularity in repeated reasoning

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Science Standards

PHYS.PS2.1: Investigate and evaluate the graphical and mathematical relationship (using either manual graphing or computers) of one-dimensional kinematic parameters (distance, displacement, speed, velocity, acceleration) with respect to an object's position, direction of motion, and time.

PHYS.PS4.1: Know wave parameters (i.e., velocity, period, amplitude, frequency, angular frequency) as well as how these quantities are defined in the cases of longitudinal and transverse waves.

Lesson Objectives

TSW model oscillators with time-domain sine and cosine functions.

TSW extract key features from graphs and functions describing spring-mass systems.

TSW connect velocity versus position for an oscillating spring with the unit circle.

(Extension)

TSW explore derivatives using experimental data from oscillating spring-mass systems.

Materials

- Computers, tablets or other data collections devices equipped with motion sensors
- Lab stands equipped with horizontal pendulum/spring clamps
- Lab springs and weights that hang vertically from the pendulum/spring clamps

Basic Instructional Plan

Students will do the following:

1. Set up a lab stand and horizontal clamp. Attach a spring to the clamp so it hangs vertically and attach a mass to the bottom of the spring.
2. Place a motion sensor directly beneath the mass so it is not too close to the mass.
3. Pull the mass downward gently and release it so the mass-spring system oscillates vertically with low amplitude. Turn on the motion sensor to collect position versus time and velocity versus time data as the spring oscillates.
4. Fit each set of time-domain data with a sine function.
5. Extract information from the graph (e.g., amplitude, period, frequency, angular frequency, phase).
6. Convert between sine and cosine functions by adjusting the phase angle.
7. Predict the shape of a graph of velocity versus position, and test the prediction by re-graphing the data. Explain the shape of the graph

Instructional Plan Extension (Calculus)

Derivatives of sine and cosine functions are explored using data collected in this experiment.

Ancillary Materials

1. Simple Harmonic Motion
2. Laboratory Investigation
3. Sine and Cosine Functions
4. Connecting Equations and Graphs
5. Trigonometric Derivatives

Simple Harmonic Motion

Physics Concepts and Terminology

Periodic motion is repetitive with a predictable cycle. The period of such motion is defined as the fixed time interval *T* over which the motion repeats. Examples of periodic motion include the rotation of the earth on its axis with a period of 24 hours, the revolution of the earth about the sun with a period of 1 year, and the rotation of the second had of a clock with a period of one minute.

There are many different forms of periodic motion. The specific type we will investigate in this lesson is referred to as *simple harmonic motion* and can be mathematically modeled with sinusoidal (sine or cosine) functions. An object that moves with simple harmonic motion is referred to as a *simple harmonic oscillator*. Sine and cosine functions are used to model the behavior of simple harmonic oscillators such as pendulums, spring-mass systems, and other timing devices, as well as wave phenomena such as sound and light.

A force capable of producing simple harmonic motion in a mechanical system functions as a restoring force that always attempts to pull an object back to some favored, or *equilibrium,* position. The magnitude of a restoring force increases in proportion to how far the object is displaced from equilibrium. To explain how such a force produces motion resembling a sine or cosine function requires a thorough understanding of Newton's Second Law, as well as a familiarity with second-order differential equations, and is therefore well beyond the scope of this lesson.

Our investigation will be limited to basic trigonometry and perhaps a little differential calculus for those students ready for it.

Springs and Simple Harmonic Motion

Springs can be used to produce simple harmonic motion in objects attached to them. Think about how a spring works. A given spring has a preferred equilibrium length. If a spring at equilibrium is compressed, it pushes outward in an attempt to return to equilibrium. If it is stretched, it pulls inward. The force exerted by a spring is therefore a restoring force.

If one end of a vertical spring is attached to a fixed support and the other end is attached to a mass, the spring pulls upward on the mass while gravity pulls downward on it. Gravity exerts a constant force on the mass, but the spring force is dependent on how much the spring is extended or compressed. When the mass hangs motionless and undisturbed, the spring is stretched just the right amount so that the upward spring force and downward gravitational force are balanced and cancel each other out. This is the equilibrium position of the spring-mass-earth system.

If someone displaces the mass downward or upward from its equilibrium position, the spring force no longer exactly cancels the gravitational force. If the mass is then released while still displaced from equilibrium, it will oscillate up and down in simple harmonic motion due to the variable restoring spring force. Therefore, a spring hung vertically with a mass attached and then set in motion exhibits the behavior of a typical simple harmonic oscillator.

Laboratory Investigation

Setting Up the Equipment

Students should set up an apparatus similar to that shown in Figure 1 below. The spring hangs from a horizontal rod or clamp mounted on a rigid lab stand. A mass is hung from the bottom of the spring. The black rectangle directly below the spring represents a motion sensor. The sensor sends out sound pulses, which reflect back to the sensor when they bounce off the bottom of the mass. The time interval between the generation of a pulse and the detection of its reflection by the sensor is used to calculate the distance between sensor and object.[1]

Collecting the Data

The students initiate simple harmonic motion in the mass by gently displacing the mass a small distance from equilibrium and releasing it. After the mass is oscillating smoothly up and down, they should turn on the motion sensor for several seconds to collect data.

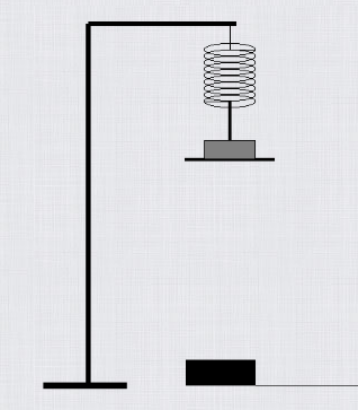


Figure 1
Vertical Spring, Mass, and Motion Sensor[2]

Experimental Graphs and Equations

Figure 2 below shows position versus time and velocity versus time graphs generated by the motion sensor, as well as the best-fit equations corresponding to the graphs. Position data is detected directly by the motion sensor, as described above. Velocity data is computed automatically by the sensor software from changes in position over very short time intervals. Velocity is therefore not as directly or accurately determined as position.

Most motion sensors can also report acceleration versus time, which is calculated from the change in the velocity data over short intervals of time. Because more calculation required, acceleration data can show significant rounding error and will not be used in this lesson.

---

[1] Complete instructions on operation of the motion sensor and display of the results are available from the vendor.

[2] Image from http://www1.lasalle.edu/~blum/p106wks/pl106_MassOnSpring.htm

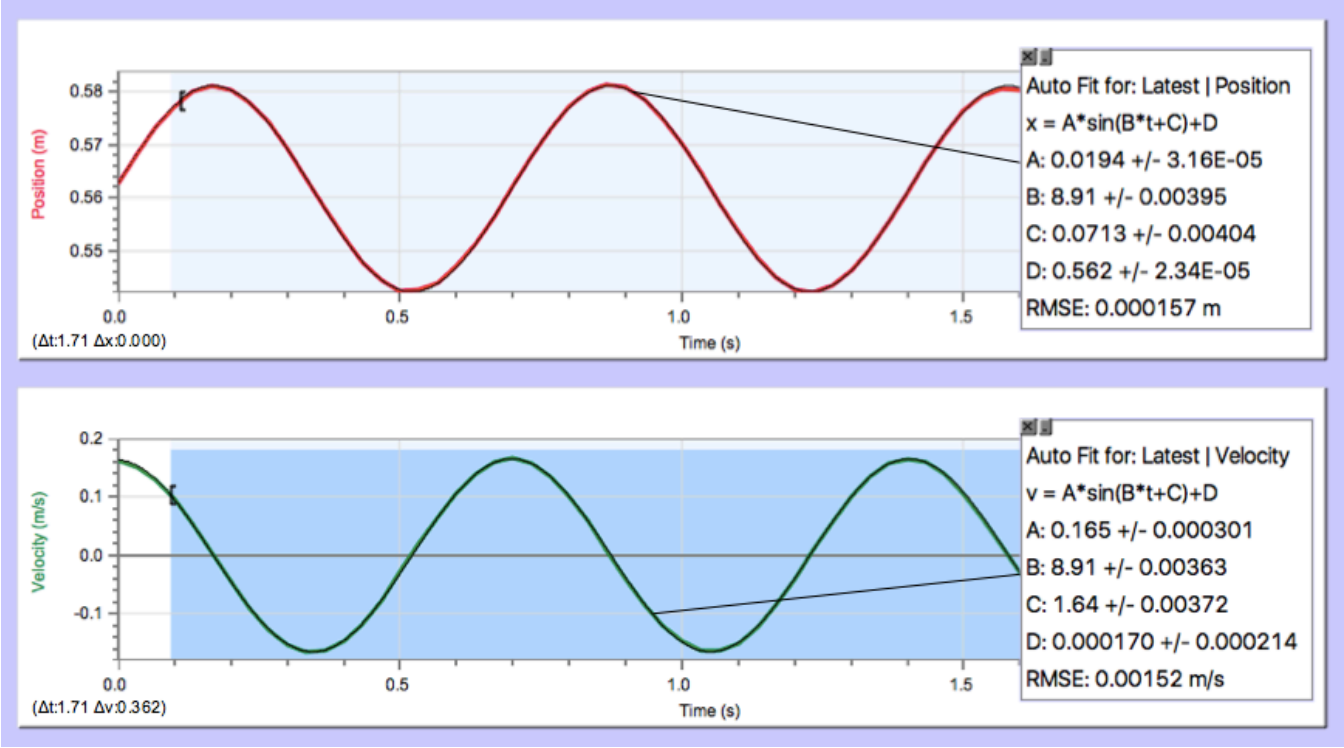


Figure 2

Position and Velocity Motion Graphs and Equations for an Oscillating Spring[3]

To correctly interpret the graphs, the coordinate system must be defined. Because the motion sensor measures how far away the mass is from the sensor, the *x*-axis points away from the sensor. Consequently, when the mass is moving away from the sensor, its position is getting more positive and its velocity is therefore positive. When the mass is moving toward the sensor, its position is getting less positive and its velocity is therefore negative.

The text boxes on the right-hand side of the graphs above show the equations that are a best fit for the data collected. The equations are reproduced below. Equation (1) is the position of the bottom of the mass as a function of time, and Equation (2) is the velocity of the mass as a function of time. To add a bit of clarity, only three significant figures are used in the representations below, and units are not included on the equation constants.

(1) $$x = 0.0194\sin(8.91t + 0.0713) + 0.562$$

(2) $$v = 0.165\sin(8.91t + 1.65)$$

[3] Motion data collected with a Vernier motion sensor and analyzed using Vernier Logger Pro.

Sine and Cosine Functions

Cartesian Coordinates

Once the experimental data have been obtained, students will compare equation and graphical representations of the motion and explain how they are connected. In order to facilitate this process, the teacher might recommend a review of sine and cosine functions. In Cartesian coordinates, the general form of sine and cosine functions can be written as:

(3) $$y = A\sin(Bt + C) + D \text{ and } y = A\cos(Bt + C) + D$$

Students can gain a better intuitive feel for the constants if they explore Equations (3) in a graphing program such as Desmos.[4] They would enter an equation, assign all the constants to sliders, and then explore how changing one constant at a time impacts the appearance of the graph.

It would also be especially valuable at this point for students to review how a constant representing the phase angle can be used to make sine and cosine functions equivalent. Sine and cosine functions are 90°, or π/2, apart in phase. This means therefore that:

(4) $$A\sin(Bx) = A\cos\left(Bx - \frac{\pi}{2}\right)$$

and

(5) $$A\sin\left(Bx + \frac{\pi}{2}\right) = A\cos(Bx)$$

Simple Harmonic Motion in the Time Domain

The examples above use familiar Cartesian coordinates. Because students are exploring motion, or changes in position and velocity with time, they must become comfortable with time domain sine and cosine functions, such as:

(6) $$x = A\sin(Bt + C) + D \text{ and } x = A\cos(Bt + C) + D$$

where *x* is a function of *t*.

In Equations (6), *A* is the amplitude, or maximum distance the graph varies from the midline. This corresponds to the maximum distance the mass- spring system moves away from the equilibrium point as it oscillates. *B* is angular frequency ω, which equals $2\pi/T$ where *T* is the period of the function. The angular frequency determines how rapidly the mass-spring system oscillates. *C* is the phase angle, which shifts the function horizontally along the *t*-axis, and *D* is the position of the midline and therefore shifts the function vertically along the *x*-axis.

[4] Available at no cost from Desmos at https://www.desmos.com/calculator

Connecting Equations and Graphs

Guiding Prompts for Student Inquiry and Analysis

The sample questions and prompts below are intended to help guide student inquiry. Possible correct student responses are also shown. The teacher is encouraged to add additional questions to help students make conceptual connections between the physical observation of the oscillating mass and the verbal, equation, and graphical representations that describe the motion.

a) Compare the graphs in Figure 2. Describe what the relative positions of the maxima, minima, and equilibrium points on the two graphs tell you about the relationship between position and velocity of the mass as it oscillates.

When position is at a maximum or minimum, velocity is zero. The means that when the mass has reached a high or low point, it slows momentarily to a stop as it reversed direction. When velocity is at a maximum or minimum, position is at the midline position. This means that the mass is going fastest when it is moving through an equilibrium point.

b) How do the amplitudes in Equations (1) and (2) correlate with specific features of the graphs in Figure 1? Include appropriate numbers from the equations and graphs in your argument.

From the velocity equation we get amplitude A of 0.165 m/s, which corresponds to a maximum of 0.165 m/s and minimum of -0.165 m/s on the graph. The midline on this graph is zero.

We can observe that the midline of the position graph is not zero. The position equation tells us the midline D is 0.562 m and the amplitude A is 0.0194 m. The maxima and minima therefore will not have the same absolute value as A. The maximum value is obtained by adding A to D to get 0.581 m. The minimum value is obtained by subtracting A from D to get 0.543 m. These calculated values are consistent with the maxima and minima observed on the graph.

c) Are the periods of the position and velocity functions the same? Use evidence from the equations to justify your claim.

Comparing Equations (1) and (2), it is clear that the periods are the same since the B terms (which are inversely proportional to the period) are identical. In both equations, the B term has a numeric value of 8.91.

d) How does the period from the position equation compare to the period obtained from the position graph?

The period is the time that elapses as the function repeats one cycle. One way to determine the period using a graph is to subtract the time at which one minimum occurs from the time at which the next minimum occurs. On either graph, the period appears to be just over 0.7 seconds.

Using the B term from the equations to calculate the period T:

$$B = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{B} = \frac{2\pi}{8.91} = 0.705 s$$

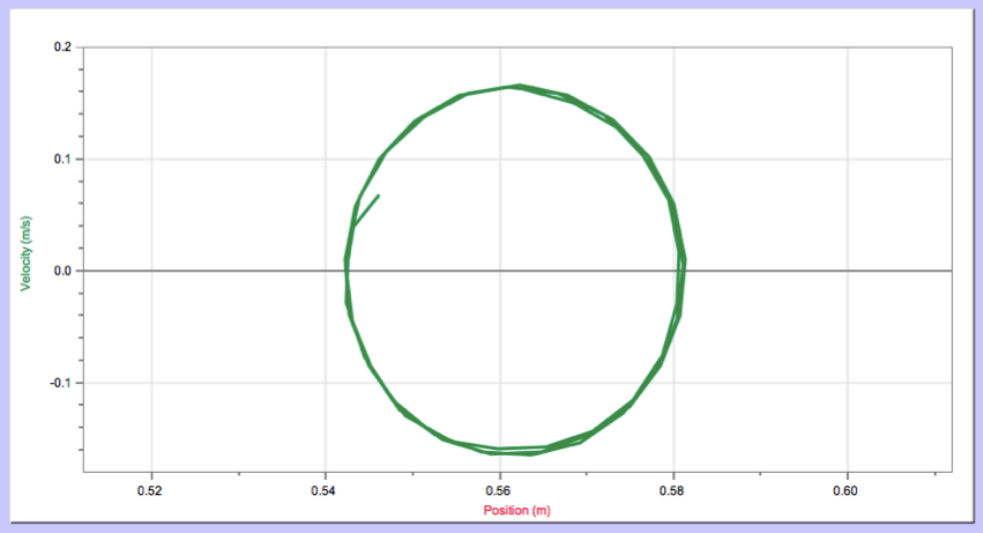
This period is consistent with the one estimated from the graphs.

e) Predict the appearance of a graph of velocity versus position. (Hint: Start by using numeric data from part (b) to establish a scale and plot four specific points. Sketch your prediction before you plot the actual graph.)

When velocity is at its maximum of 0.165 m/s or minimum of -0.165 m/s, position is at its midline of 0.562 m. When position is at its maximum of 0.581 m or minimum of 0.543 m, velocity is at its midline of 0. Thus the graph must have horizontal and vertical scales that can accommodate these four points: (0.562 m, 0.165 m/s), (0.562 m, -0.165 m/s), (0.581 m, 0) and (0.543 m, 0).

Because position and velocity vary smoothly between midline and extrema in an ongoing cycle, the graph should not show abrupt transitions between those four points. The smoothest connection of the four points would be a circle.

f) Graph velocity versus position[5] and compare the graph with your prediction. How might you adapt the position and velocity equations so when you graph velocity versus position you create a unit circle?



A unit circle has a radius of 1 and is centered on the origin. The functions for position and velocity could be adapted so they create a unit circle. Both functions would need to have amplitudes of 1.0 and a midline of zero.

g) Do the phase angle constants *C* in Equations (1) and (2) support the argument that if position is represented as a sine function, velocity could be represented by a cosine function (and vice versa)?

Sine and cosine functions are equivalent if the phase is different by $\pi/2$, as illustrated by Equations (4) and (5). The constant C in Equation (1) is 0.0713 and in Equation (2) it is 1.65. The difference between these two values is 1.58. This is very close to $\pi/2$, which has an approximately value of 1.57. This close agreement supports the idea that if one function is represented by a sine, the other can be represented by a cosine.

[5] This graph was created easily in Vernier Logger Pro by simply reassigning the time axis of the velocity graph in Figure 1 to position.

Calculus Extension

Ask your advanced students to use calculus to show the relationship between Equation (1) and Equation (2). They will hopefully remember that velocity is the time derivative of position, or:

(7) $$v = x' = \frac{dx}{dt}$$

The students' mathematical reasoning will hopefully begin with taking the derivative of position as a function of time, Equation (1), which is reproduced for convenience below:

(1) $$x = 0.0194\sin(8.91t + 0.0713) + 0.562$$

The time derivative of Equation (1) yields a velocity equation:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left[0.0194\sin(8.91t + 0.0713) + 0.562)\right]$$

$$v = (8.91)\left[0.0194\cos(8.91t + 0.0713)\right]$$

(8) $$v = 0.173\cos(8.91t + 0.0713)$$

Is Equation (8) is equivalent to Equation (2), the previously established velocity equation reproduced for convenience below?

(2) $$v = 0.165\sin(8.91t + 1.65)$$

We must first convert Equation (2) to its cosine form. Using Equation (4) as a model:

$$v = 0.165\cos\left(8.91t + 1.65 - \frac{\pi}{2}\right)$$

Combining constants, we get:

(9) $$v = 0.165\cos(8.91t + 0.0792)$$

Comparison of Equation (8) with Equation (9) shows that they agree closely but not exactly. The amplitude values of 0.173 and 0.165 differ by about 5%, and the phase angles of 0.0792 and 0.0713 differ by about 11%. Most scientists would be quite happy with the agreement and declare Equations (8) and (9) equivalent within the constraints of experimental and computational uncertainty.

The student of pure mathematics may not be so quick to pronounce equivalency, as textbook mathematical derivations generally avoid approximations. The standards of what constitutes a mathematical proof are much more rigorous than showing close agreement in constants. When students apply mathematics to a real-world context using measured quantities and decimal approximations, they should not expect the type of precision they have come to expect from textbook examples. Experimental uncertainties in measurement are always a factor in science and engineering applications. Rounding and limitations in computational precision can amplify measurement errors. Students working with real-world mathematics must learn to account for such uncertainties when evaluating the correctness of the mathematics they explore.

Pendulum Timers

Lesson Plan Overview

Lesson Description

Upon what factors does the period of a pendulum depend? What descriptive mathematical model best describes a pendulum's behavior? Can multiple correct mathematical models be developed? How do the descriptive models compare with the established theoretical model? In this lesson, students will plan and carry out investigations to answer these questions.

In a preliminary investigation, students will observe how the period of simple pendulums varies with mass and length. The preliminary investigation should help them identify pendulum length as the most appropriate quantity to use as the independent variable when developing their descriptive model. Students will collect bi-variant data to create a mathematical model that describes period as a function of length, and they will present their model in graphical and equation form. To assess the quality of their completed model, students will be randomly given a specific period, and they will use their mathematical model to design and construct a pendulum with that period.

The first investigation, entitled *Preliminary Investigation for Variable Selection*, is well aligned with the listed standards from Integrated Mathematics I. Statistics teachers may choose to skip this part of the lesson if pressed for time. Students in both Statistics and Integrated Mathematics I courses will find the second investigation, entitled *Creating and Using a Mathematics Model*, equally valuable.

Essential Question(s)

How do we choose the most important variables when constructing a mathematical model?

How do we create a model that can then be used by others?

How well does our descriptive mathematical model agree with a derived theoretical model?

Math Standards

Integrated Math I

M1.N.Q.A.2: Identify, interpret, and justify appropriate quantities for the purpose of descriptive modeling.

M1.N.Q.A.3: Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Statistics

S.ID.B.12: For bivariate measurement data, be able to display a scatterplot and describe its shape; use technological tools to determine regression equations and correlation coefficients.

S.ID.B.13: Identify trends in bivariate data; find functions that model the data and that transform the data so that they can be modeled.

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Math Practices

3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
6. Attend to precision.
7. Look for and make use of structure.

Science Standards

PHYS.PS4.1: Know wave parameters (i.e., velocity, period, amplitude, frequency, angular frequency) as well as how these quantities are defined in the cases of longitudinal and transverse waves.

Lesson Objectives

TSW determine an independent variable that significantly changes a pendulum's period in a predictable way.

TSW create a mathematical model consisting of a graph and an equation that describes the period of the pendulum as a function of the independent variable.

TSW use a mathematical model to construct a pendulum with a specific period.

TSW evaluate their descriptive mathematical model against an established theoretical model.

Materials

- Lab stands equipped with pendulum/spring clamps
- String and washers to build pendulums
- Meter sticks, stopwatches or other timers, mass balance (optional)
- Calculators
- Graph paper, or software that will create *x*-*y* scatterplots and perform regression analysis

Basic Instructional Plan

The goal is to create a model for the period of a pendulum so it can be used to construct a pendulum with a specific period. To accomplish this, students will do the following:

1. Build a pendulum using lab stand, clamp, string, and from 1 to 4 washers.
2. Measure the period of pendulums, varying the mass and length to determine which variable impacts the period in a predictable way.
3. Collect bi-variant data to model the period as a function of the independent variable selected.
4. Develop a mathematical model consisting of a graph and a regression equation that fits the bi-variant data.
5. Test the model by building a pendulum with a specific period.

Ancillary Materials

1. Periodic Motion and Timers
2. Preliminary Investigation for Variable Selection
3. Creating and Using a Descriptive Mathematical Model
4. Comparison of Descriptive and Derived Models

Periodic Motion and Timers

Concepts and Terminology

Timers with well-defined and reproducible periods, such as watches and clocks, have helped people tell time for centuries. The simple pendulum is a fairly accurate mechanical timer that is easy to build and operate. The motion of a swinging pendulum approximates the simple harmonic motion exhibited by an oscillating spring, provided the amplitude of the pendulum's swing is small. This is why a swinging pendulum is the mechanical heart of a grandfather clock.

Not all pendulums qualify as "simple". A simple pendulum has a small "bob" with significantly more mass than the very lightweight, non-stretchable string to which it is attached. The other end of the string is attached to a fixed pivot point from which the pendulum can swing. When the pendulum hangs motionless from its pivot, it is at its equilibrium position. When the pendulum bob is displaced slightly from equilibrium and released, it swings back and forth with a reproducible period. Simple pendulums gradually lose energy and stop swinging, but while they swing the period is fairly constant.

Pendulum Mathematics

The period of a simple pendulum is the time T it takes for the pendulum to swing through one complete cycle. At the end of a period, the pendulum bob has arrived back at the same place it was at the beginning of a period. It is important to emphasize this because for some mysterious reason, students frequently misidentify the period as the time it takes for the pendulum to swing from one side to the opposite side; thus, they think the period is half as long as it really is.

The period T of a simple pendulum of length L can be predicted by Equation (1), which is derived from Newton's Second Law of Motion using second-order differential equations and complex trigonometry:

(1) $$T = 2\pi\sqrt{\frac{L}{g}}$$

The derivation of Equation (1) is clearly beyond the scope of most high school physics or math classes, but the equation is not difficult to understand or use. It shows that the period T of a simple pendulum is proportional to the square root of its length L measured from the point of attachment to the center of mass of the bob. The constant g is the acceleration due to gravity of 9.8 m/s^2. The equation gives no indication that the period depends on pendulum mass or the amplitude of oscillation. Experiments show that, indeed, for small amplitude pendulums only length affects the period.

To make this lesson truly inquiry-based, students should not be given Equation (1) prior to their investigation, although students who have previously studied physics may be aware of it. After students have successfully developed and used their descriptive mathematical model, the teacher should provide them with the opportunity to compare the descriptive model to the derived equation above.

Preliminary Investigation for Variable Selection

Setting Up the Equipment

Students should set up an apparatus similar to that shown in Figure 1 below. The string used to make the pendulum should be lightweight, strong, and not too stretchy. The pendulum bob should be a small relatively massive object with an easily identifiable center of mass. Many teachers find small metal washers to be ideal for constructing the pendulum bob. The students can tie from between one and four washers to one end of the string, thus varying the mass of the bob in a convenient way.

The end of the string opposite the bob is tied to a horizontal rod or clamp mounted on a rigid lab stand. Use of a pendulum clamp for the horizontal rod allows for easy adjustment of the string length. However, if you don't have a pendulum clamp or even a lab stand, do not worry. It is possible to do this lab with students holding the pendulums and acting as lab stands, provided the points of suspension of the pendulums are held very still as the pendulums oscillate.

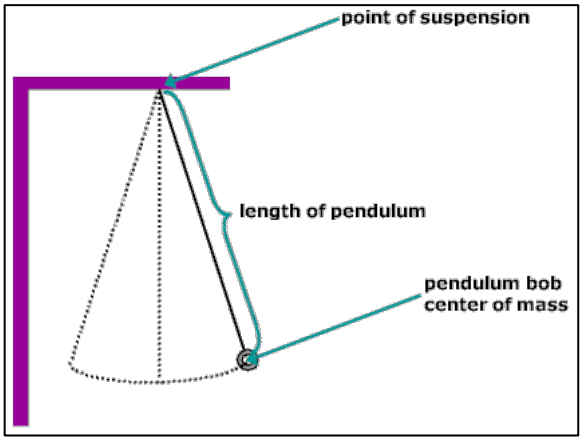


Figure 1[1]
Simple Pendulum

Collecting Data

A given pendulum has a highly reproducible period; however, if the pendulum length is changed the period will be altered according to Equation (1). The period is therefore the dependent variable and length is the independent variable. The mass of the pendulum bob should not have an appreciable effect on the period.

Students who have not studied physics will not know this. Most students will claim that any change to the pendulum (length or mass) will alter the period. In the preliminary investigation described in the next section, students should test this claim. They should measure and tabulate the period, length, and mass of various pendulum configurations in order to determine if mass or length should be selected as the independent variable upon which the period most clearly depends. They are effectively justifying the selection of variables for the descriptive modeling process that will follow the preliminary investigation.

There is no need for the preliminary investigation to take a lot of time and semi-quantitative evaluation will likely be sufficient. One way to speed up the preliminary investigation is to have one

[1] From Physics Lab Online:
http://dev.physicslab.org/Document.aspx?doctype=2&filename=OscillatoryMotion_LabProPendulums.xml

group of students adjust the pendulum length while holding mass constant, and a second group adjust the mass while holding length constant. Both groups measure and record the impact of their adjustments on the period. When the two groups compare results, they will see that length, and not mass, has a significant impact on period.

Below are some tips for the teacher as students collect and tabulate data:

Length Measurements: Pendulum length should be measured from the pivot (or point of suspension) to the center of mass of the bob. If washers are used as the bob, length should be measured from the pivot to the center of the washer. Because the students will eventually compare their models to Equation (1), it is best if the meter is used as the unit for length.

Mass Measurements: The mass of the pendulum is the mass of the washers used to create the bob. The mass of one washer can be measured on a mass balance, and the mass of the bob will be a multiple of that. Alternatively, the unit "washer" can be used for mass. Because mass will not be identified as the independent variable, the unit for mass in this experiment is irrelevant.

Period Measurements: The students should initiate pendulum oscillation by gently displacing the bob a small distance horizontally from equilibrium and releasing it. Using a stopwatch, they time a given number of oscillations (ten is a good number) and use this information to calculate the period. Note that not all students will realize that timing error plays a large role when a single period only is timed. These students will attempt to time one oscillation, which for short pendulums is less than one second long. The teacher might suggest timing multiple oscillations if students do not realize this on their own.

Selecting the Independent Variable

The preliminary results should clearly indicate that pendulum length significantly impacts the period while the mass does not. Therefore, length should be used as the independent variable. Period is of course the dependent variable. After running the preliminary investigation, students will have practiced and refined their data collection methods and should be ready to create their mathematical model.

Creating and Using a Descriptive Mathematical Model

Collecting Data

Students should now collect and tabulate period versus length data for simple pendulums using the methods practiced in the previous section. It is not necessary to vary mass this time, as it has already been eliminated as an appropriate independent variable.

One way to make data collection far less time consuming is for the teacher to assign each team of students a single pendulum of a specific length to analyze. Data from the entire class can then be recorded in a single table to provide students with a common data set to use when creating and analyzing their mathematical models.

Power-Fit Model

Figure 2 contains a table of sample whole-class data for a simple pendulum. Each team of two or three students has run multiple trials and has provided the average value they have measured for the period, as well as a carefully measured length for the pendulum they are investigating. The data was graphed in Excel on a scatterplot and regression analysis was performed. It was found that the power function displayed on the chart was the best fit for the data.

Length, L (m)	*Period, T (s)*
0.10	0.63
0.20	0.90
0.40	1.26
0.70	1.69
0.90	1.90
1.20	2.20
1.51	2.46
1.75	2.66
2.00	2.83
2.50	3.17

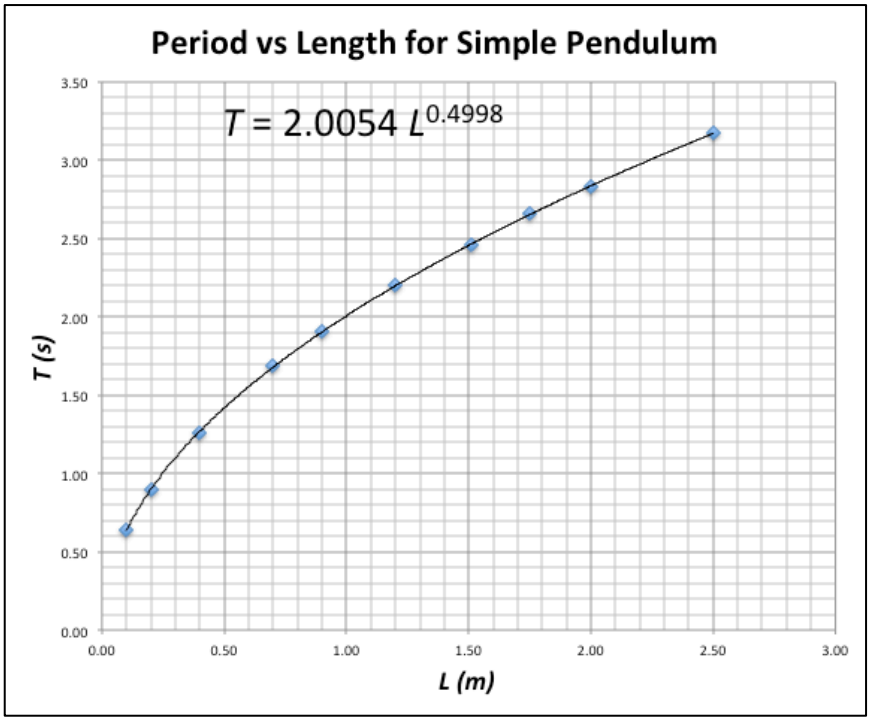


Figure 2
Simple Pendulum Data with Power Trendline Fit

The descriptive model in Figure 2 can now presumably be used to accomplish the task of building a pendulum with a desired period if the period is in within the range of the data. The model includes an equation and a graph, either of which can be used to determine the required length of the pendulum. Obtaining *L* from the graph should be the easier task. The student finds the desired value of *T* on the vertical axis, draws a horizontal line until it intersects the curve, and then draws a vertical line from the curve to the horizontal axis to find the associated value of *L*.

However, students may be stymied if they try to solve the regression equation in Figure 2 for L in order to calculate its value from a given T. Students may not know what to do with a term raised to a power of 0.4998. Part of the confusion in working with the equation in Figure 2 arises from the inappropriate number of significant figures provided by the power regression. One especially valuable standard in Integrated Mathematics I requires that students recognize appropriate accuracy in measured quantities. This laboratory clearly addresses that standard.

The teacher should ask the students to evaluate whether the equation in Figure 2 implies excessive accuracy given the meter sticks and stopwatches used to collect the data. Hopefully, students will conclude that meter sticks can be read with confidence to at most a millimeter. Because it is not easy to judge to the exact position of the center of mass of a pendulum, length measurements include additional error. Stopwatch timing includes error due to human reflexes, so timing to three significant figures is difficult. Students are justifiable in rounding the constants in the equation to more appropriate accuracy. If rounded to two significant digits, the line of best fit becomes:

(2) $$T = 2.0L^{0.50}$$

which students may recognize as equivalent to:

(3) $$T = 2.0\sqrt{L}$$

which can be solved for L by squaring both sides and rearranging as follows:

(4) $$T^2 = 4.0L$$

(5) $$L = \frac{T^2}{4.0}$$

Equation (5) can be easily used to determine the length of a pendulum with a given period simply by substituting the known value of the period T and calculating the required length L.

Linearized Model

In physics and engineering modeling in particular, students are often tasked with converting non-linear mathematical models into linear ones through a process called linearization. There are many advantages to linearization, not the least of which is that linear models are easy to use and interpret. The slope and intercept of a linear equation are very easy to determine, and various constants can be easily extracted from them. Furthermore, the coefficient of determination (R^2) can be calculated for a linear regression model to evaluate how closely the equation fits the data, but R^2 is invalid for non-linear regression.

Modeling at its best is exploratory with a heavy emphasis on graphing. When presented with a plot consisting of data points that are clearly non-linear, the student tries different approaches to linearize the plot. Modeling encourages trial and error. Graphing calculators and programs such as Excel make modeling faster and easier.

Let's describe the possible thought process of a hypothetical student using modeling to linearize the data in Figure 2. The student observes that as L increases, so does T. However, the plotted data displays a concave-down shape. This means that as L gets larger, each additional incremental increase ΔL results in a smaller incremental increase ΔT. To straighten the curve, the student decides to perform an operation on T to amplify its increase as L increases. If the student first tries to multiply T values by a constant C where $C > 1$, this will indeed increase T values but the line will not

become straight. If the student then tries squaring the T values and graphs T^2 versus L, a straight line will result. *Voila!*

Figure 3 illustrates the linearized model for a simple pendulum that results when T^2 is graphed as a function of L. It is clear that T^2 increases linearly with L. Because the graph appears to go through the origin, T^2 is proportional to L.

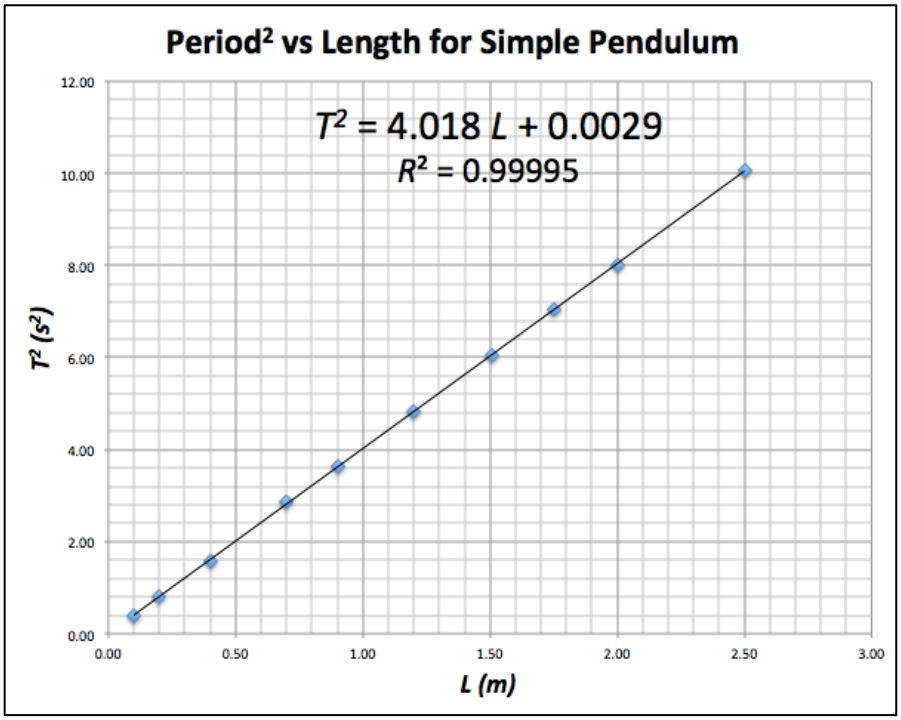


Figure 3
Linearized Pendulum Data with Linear Fit

If the regression equation shown on the graph is reported to two significant figures, which is more appropriate given measurement precision, we get a relationship that is identical to Equation (4):

(4) $$T^2 = 4.0L$$

Note that linearization could also be accomplished by graphing T as a function of $\sqrt{L}$.

Comparison of Descriptive and Derived Models

Recall the derived equation for the period of a simple pendulum as a function of its length:

(1) $$T = 2\pi\sqrt{\frac{L}{g}}$$

Squaring both sides of the equation yields:

(6) $$T^2 = \frac{4\pi^2}{g}L$$

Recalling Equation (4):

(4) $$T^2 = 4.0L$$

If the linearized descriptive model represented by Equation (4) and the theoretical model represented by Equation (6) are both correct, the following relationship between constants in the two equations must be true:

(7) $$\frac{4\pi^2}{g} = 4.0$$

Numeric substitution and simplification shows Equation (7) to be true within the bounds of experimental error.

Bending Light

Lesson Plan Overview

Lesson Description

A light ray in a transparent medium such as air travels in a straight line until it strikes an object in its path. If the object has a smooth shiny surface, the ray will reflect off of it in a direction dictated by the Law of Reflection. If the object is a transparent material such as glass or water, most of the ray will refract into the glass in accordance with Snell's Law of Refraction, although some of the ray will also reflect.

The Laws of Reflection and Refraction give students an opportunity practice geometry and trigonometry skills.

Essential Question(s)

How can we determine the measurement of angles without using a protractor?

What can we conclude about angles when light reflects off a surface or refracts through a material?

Math Standards

Geometry

G.SRT.C.8: Solve triangles. A) Know and use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems

Precalculus (Applied Trigonometry Standards)

P.G.AT.A.1: Use the definitions of the six trigonometric ratios as ratios of sides in a right triangle to solve problems about lengths of sides and measures of angles.

Math Practices

3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
8. Look for and express regularity in repeated reasoning.

Science Standards

PHYS.PS4: 3: Understand that the reflection, refraction, and transmission of waves at an interface between two media can be modeled on the basis of characteristics of specific wave parameters and parameters of the medium.

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Lesson Objectives

TSW use trigonometric ratios to determine angle measurements

TSW draw conclusions about angles when light reflects off of mirrors

TSW draw conclusions about angles when light refracts into water and glass

Materials

- Semicircular glass Petri dishes and water
- Low power laser[1] or flashlight with card-stock collimator (alternate method)
- Graph paper or white copy paper
- Calculator
- Straight edge ruler
- Protractor
- Laser with diffraction grating (for extension)

As an alternative, cardboard and straight pins can be used instead of a laser.

Basic Instructional Plan

Students will do the following:

1. Place a test object (mirror, Petri dish with water, glass block) on graph paper so one face of the object is aligned with the grid on the paper
2. Trace the object
3. Use a laser to direct light onto the object
4. Trace the path of the laser before and after it strikes the object
5. Determine the angles of incidence, reflection, and/or refraction using trigonometric ratios

Draw conclusions based on angle measurements

Safety Note

***If lasers are used by students, they should be low power Class I or II devices from a reputable vendor. For Class IIIa lasers, students should wear eye protection. Teachers must monitor carefully that students use lasers safely and do not direct the laser light into their eyes*.**

Ancillary Materials

1. Introduction to Ray Optics

[1] For student use, do not exceed Class II lasers of of < 1 mW unless eye protection is provided. Alternatively, a collimated beam from a cell phone flashlight can be used; directions will be given.

Introduction to Ray Optics

Terminology

Reflection and refraction of light when it interacts with objects is highly predictable and can be analyzed by mathematics students using simple geometry. Protractors can be used to measure the various angles involved. If students have been introduced to trigonometric ratios (sine, cosine, tangent, and their inverse functions), they will find that protractors are not necessary to determine the angles as they can calculate them based on right triangle construction and measurement of the side lengths of the triangles.

Reflection

Let's take a closer look at the angles involved in ray optics. Reflection of light from a plane mirror is illustrated in Figure 1.

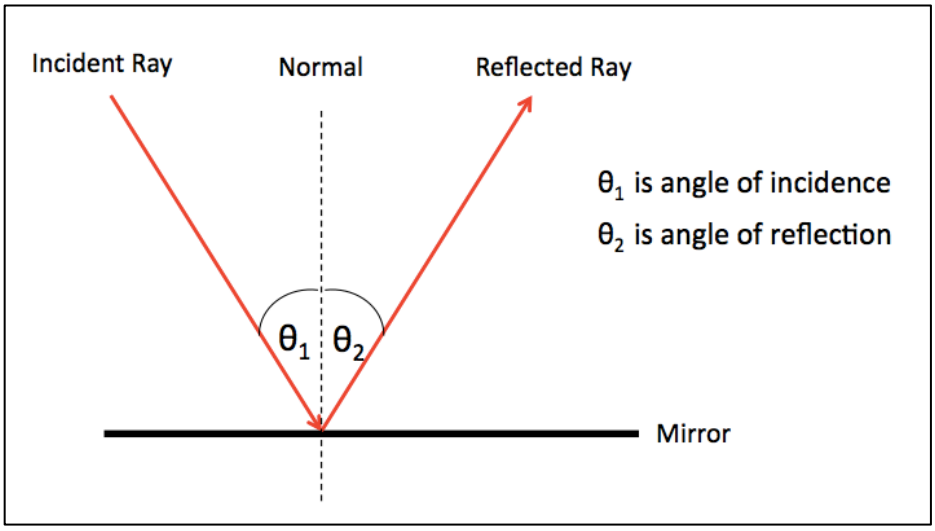


Figure 1
Reflection of Light from a Plane Mirror

The incident ray in Figure 1 describes the path of the light before striking the mirror, and the reflected ray describes the path after reflection. The angle of incidence, θ_1, and the angle of reflection, θ_2, are defined relative to a normal (or perpendicular) to the mirror's surface.

Measurement of these angles relative to a normal and not the mirror itself may seem an unnecessary complication. For plane mirrors, it would indeed be more straightforward to measure an angle defined by the light ray and the mirror surface. However, complications arise when mirrors are NOT planar. It is difficult to define a tangent line to a curved mirror's surface from which to measure angles. The normal is much easier to define. For a common spherical mirror, the normal passes from the surface through the center of curvature and is thus very straightforward to determine.

The *Law of Reflection* is intuitive and mathematically simple. It states that the angle of incidence is equal to the angle of reflection. This is easy to demonstrate experimentally provided measurement error is taken into account. In equation form:

(1) $$\theta_1 = \theta_2$$

Refraction

Refraction of light is more complicated mathematically than reflection and gives students a chance to exercise additional trigonometry skills. The reflection through a semicircular Petri dish filled with water is illustrated in Figure 2.

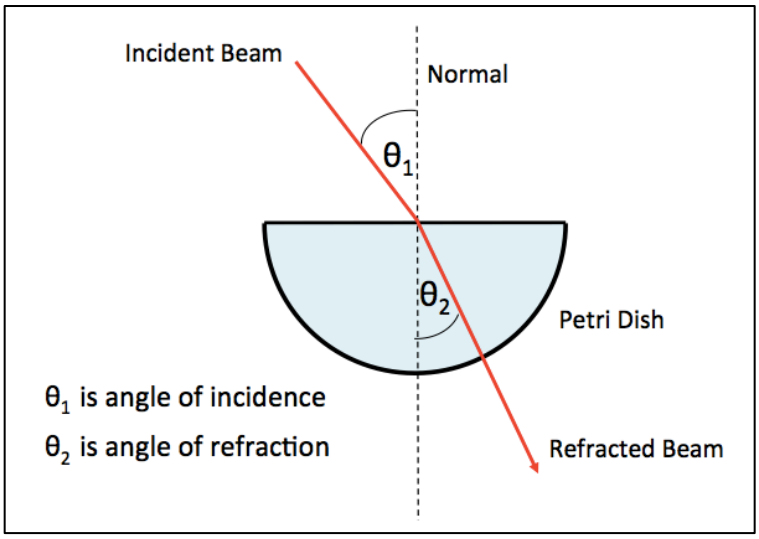


Figure 2
Refraction of Light through Semicircular Dish of Water

The incident ray describes the path of the light before striking the dish, and the reflected ray describes the path after refraction. The angles of incidence, θ_1, and refraction, θ_2, are measured relative to a normal (or perpendicular) to the flat side of the dish.

In the figure above, line bends "toward the normal" when it enters the dish. Mathematically, *Snell's Law of Refraction* describes the relationship between the angles:

(2) $$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

In this equation, the constants n_1 and n_2 together determine the degree of bending of the light as it passes from one medium into the next. The constant n is referred to as the *index of refraction* and depends upon the characteristics of a transparent medium. Equation (2) predicts that when light passes from one medium into another, the light ray will make a smaller angle relative to the normal in the medium with the higher index of refraction. Air has an index of refraction of about 1.0, and most other transparent materials have indices of refraction that are appreciably greater than 1.0. The index of refraction of various materials is readily available[2].

Without refraction, eyesight would be impossible. The cornea and lens system of the eye has a higher refractive index than air. Together, these two structures bend light entering the eye so it makes images on the retina that can be sent to the brain. For those of us that do not have perfect vision, corrective lenses made of glass can correct the degree of bending so that we can see more clearly.

Refraction is clearly more complicated than reflection. At certain angles and for certain combinations of materials, light will refuse to refract and undergo "total internal reflection" instead. We will restrain our enthusiasm for such phenomena and tailor our experiments appropriately to reinforce mathematical principles of simple trigonometry and geometry.

---

[2] From Hyperphysics at Georgia State Univ.: http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/indrf.html

Reflection and Refraction Laboratory

Experimental Set Up

Figure 3 illustrates how the laser and water dish can be used investigate both reflection and refraction. The laser is aimed at a central point on the flat surface of the dish. A portion of the light bounces off the flat surface and forms the reflected ray, but the majority of the light travels into the water dish and out the curved surface to form the refracted ray.

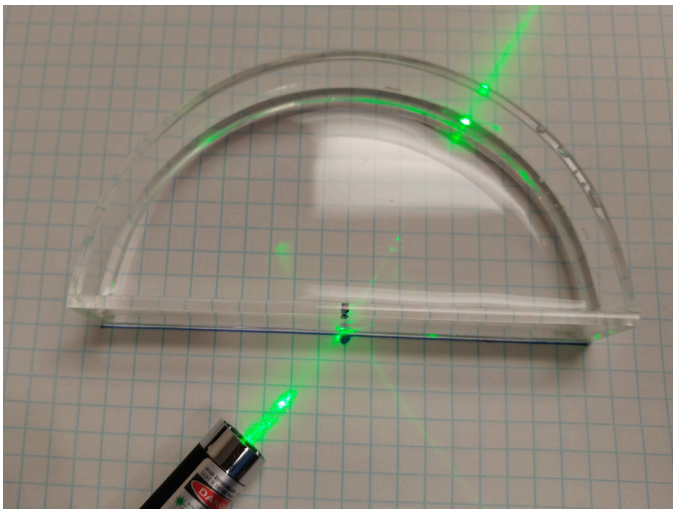


Figure 3
Reflection and Refraction Set-up

Light Sources

Lasers make ideal light sources for these experiments because the beams are collimated. Laser beams are not usually visible in air, but the laser in Figure 3 is diffuse enough that it is easily visible, making it even more ideal. The laser in Figure 3 is a Class IIIa green laser of <5mW. It is recommended that students use eye protection with such a laser. As an alternative, students can use low-power Class II red lasers of <1mW without eye protection, or they can use a flashlight beam if they shine it through a vertical slit in cardstock prior to directing it to the water dish. Simply cut a slit on the edge of card stock and place it on the graph paper to allow only narrow beam of light to strike the dish.

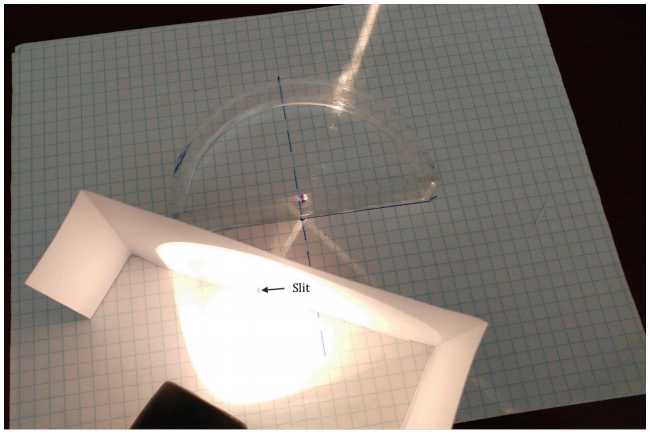


Figure 4
Slit in Card Stock for Flashlight as Light Source

The lab works best with students in teams. The general procedure follows:

1. Measure the flat surface of the dish and use a permanent marker to mark the midpoint.
2. Fill the dish with water and place it on graph paper so the midpoint is on a graph paper line as shown in Figure 3. This line is normal (perpendicular) to the flat surface.
3. Direct the light source toward the midpoint.
4. Mark the incident, reflected, and refracted rays with at least two points each so they can be reconstructed with a straight edge. Use a pencil or sharp marker to make the marks.
5. If desired, repeat the experiment at different angles.

As an alternative to using light beams, students can collect reflection and refraction data by sticking pins in cardboard so the pins and reflected appear aligned; this method is demonstrated for reflection[3] and refraction[4] in the linked sources.

After collecting data, the students should have sketches similar to those shown in Figure 4 below. You can see where the student has marked the laser beam by placing dots on the paper. Line segments representing the rays and the normal were drawn with a straightedge after the dish was removed from the paper.

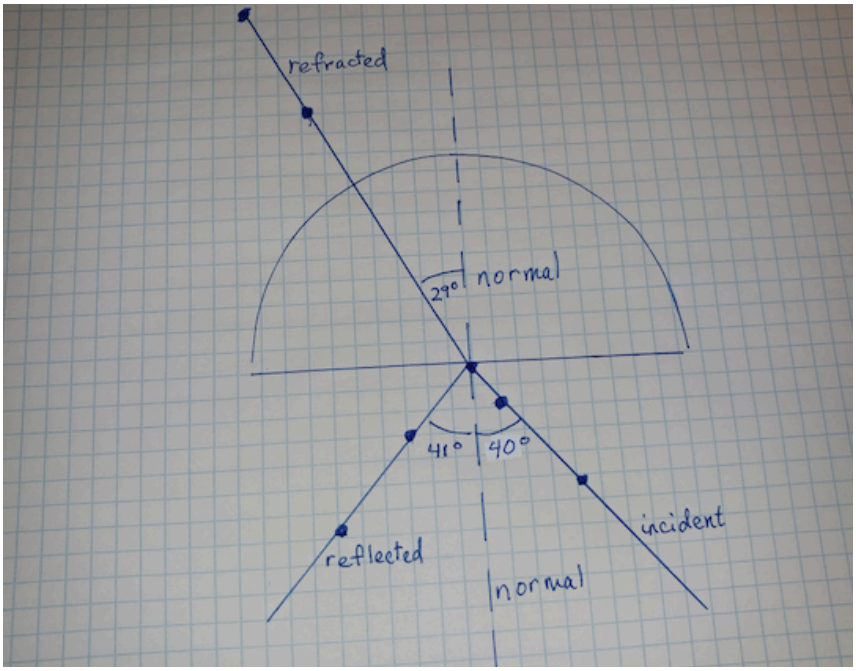


Figure 5
Sample Student Data

[3] Nur Huda Ismail, https://www.youtube.com/watch?v=fMDGuwolFm0

[4] http://www.batesville.k12.in.us/Physics/PhyNet/Optics/Refraction/Labs/Refraction_Lab.html

Reflection and Refraction Data Analysis

Angle Determination

Students can measure the angles of incidence and reflection relative to the normal with a protractor. They can also calculate the angles by constructing right triangles and carefully measuring the length of the triangle sides. Trigonometry (inverse sine, inverse cosine, or inverse tangent functions) can then be used to calculate each pair of incident and reflection angles. The teacher might want to have students determine angles using both methods, and then compare measured and calculated angle quantities.

Reflection

If the whole class tabulates their determinations for the angles of incidence and their associated angles of reflection in a common table, they can discuss what the results imply regarding the pattern exhibited by reflection. It should be evident that within measurement error, each reflection produces an angle of incident that is roughly equal to the angle of reflection. The Law of Reflection is easy to demonstrate in this manner.

Refraction

Figures 3, 4, and 5 all show how light bends toward the normal as it enters the semi-circular dish. The angle of incidence is clearly greater than the angle of refraction. The light does not bend as it exits the dish because it exits along a normal. The angles of incidence and refraction on exiting the dish are both zero.

Students can again compare their data with that of other students. If the whole class tabulates their determinations for the angles of incidence and their associated angles of refraction in a common table, this will facilitate further investigation. Students should see the pattern that the angle of incidences are all greater than the angles of refraction.

As we have seen, Snell's Law of Refraction can be used for refraction calculations.

(2) $$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

From the measured angles of incidence and refraction, we can estimate the index of refraction of the prism. For the data in Figure 5, with $\theta_1 = 40^o$ and $\theta_2 = 29^o$, the index of refraction is calculated as follows:

$$n_2 = \frac{n_1 \sin\theta_1}{\sin\theta_2} = \frac{(1.0)\sin 40^o}{\sin 39^o} = 1.33$$

This compares favorably with the reported index of refraction of water of 1.33.

Alternatively, if given the index of refraction of 1.33, students could calculate the expected angle of refraction from the angle of incidence and then test experimentally to see how well calculation and experimental measurement agree. This approach gives the student additional practice using inverse trigonometric functions.

Diffraction Patterns

Laser Diffraction

Lasers can form distinct patterns when they are shone through slits or a fine mesh. The physics name for this is *diffraction* followed by *interference*. The material through which the laser shines is called a *diffraction grating*. A sample diffraction pattern formed by a green Class IIIa laser is shown below in Figure 6. The pattern is directed onto a sheet of quadrille graph paper.

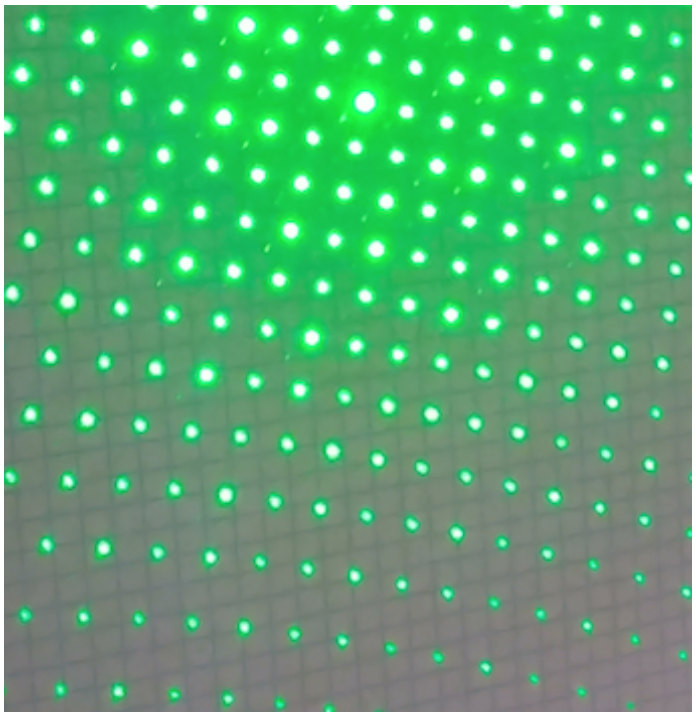


Figure 6
Diffraction Data

Math students could certainly analyze the pattern and investigate what changes it. Can they devise a descriptive model for the spacing?

For example, if the laser is situated at measured distances from graph paper and photographs are taken of the resulting patterns (which will change as distance changes), students could propose a descriptive mathematical model for the spacing of the pattern as a function of the distance of the source from the paper. They could collect information from points on the graph paper to use as evidence to support their model.

For reference, the equation for diffraction along one axis is:

(3) $$n\lambda = d\sin\theta$$

Without going too deeply into the physics, n is an integer reflecting how far a given spot is from the central bright spot, λ is the laser wavelength, and d is the spacing between openings in the diffraction grating. Math students would be most interested in exploring $\sin\theta$, which is the ratio of the distance between dots to the distance between the screen and the diffraction grating. This equation is used for diffraction through vertical slits to produce a single row of dots, but does it work for the pattern shown above? Math students could answer this question.

Simple Circuits

Lesson Plan Overview

Lesson Description

The voltage, current, and resistance in a simple series circuit are easy quantities to measure. Students can begin by using visual evidence, such as the brightness of bulbs, to make preliminary claims regarding which quantities in a circuit are proportional. They can then measure and tabulate data and use it to test their claims and develop an equation that models the behavior of a simple circuit.

At a more advanced level, Precalculus students can model resistance and current in a simple circuit using a rational function of the form $y = C/x$, and convert the rational form into a linear model.

Essential Question(s)

What proportional relationships can we find between measured quantities in a simple circuit?

What circuit data does the rational equation $y = C/x$ model, and what advantage do we gain when we convert the rational model into a linear model? *(Precalculus)*

Math Standards

7th Grade Mathematics

7.RP.A.2: Recognize and represent proportional relationships between quantities.
- (a) Decide whether two quantities are in a proportional relationship (e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin).
- (b) Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
- (c) Represent proportional relationships by equations.

Precalculus

P.F.IF.A.2: Analyze qualities of exponential, polynomial, logarithmic, trigonometric, and **rational** functions and solve real-world problems that can be modeled with these functions (by hand and with appropriate technology).

P.F.IF.A.7: Graph rational functions, identifying zeros, asymptotes (including slant), and holes (when suitable factorizations are available) and showing end-behavior.

Math Practices

1. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
6. Attend to precision.

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Science Standards

PHYS.PS3.11: Investigate Ohm's law ($I=V/R$) by conducting an experiment to determine the relationships between current and voltage, current and resistance, and voltage and resistance.

Lesson Objectives

TSW draw identify proportional relationships in experimental data for a simple circuit.

TSW model simple circuit data using rational and linear models (Precalculus only)

Materials

- Vernier student circuit board[1]
- Alternative - discrete circuit components
  - 1.5 V batteries in battery holders
  - Small incandescent light bulbs
  - Resistors
  - Wires with connecting clips
  - Digital Multi-Meter or voltage and current sensors from a probeware vendor.[2]
- Alternative - mobile technology:
  - PhET Ohm's Law simulation on USB drive for workshop; also available online.[3]
- Calculators and paper, and graph paper

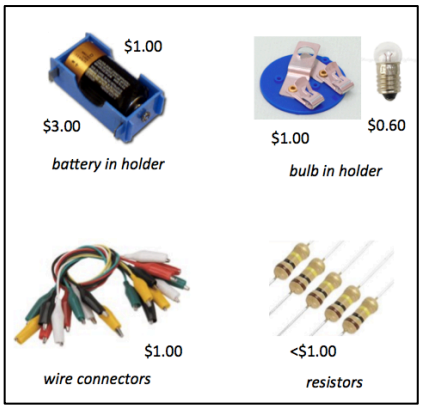


Figure 1
Discrete Component Circuit Kit[4]

Basic Instructional Plan:

1. As a demonstration, the teacher will build series circuits consisting of 1-2 light bulbs and 1-2 batteries. Based on brightness comparison, students will make claims about proportional (and inversely proportional) quantities in simple circuits.
2. Students will build simple circuits and collect and tabulate potential, current, and resistance data. As an alternative, they can use the simulation from PhET to collect the data.
3. Students will test and justify claims of proportionality with evidence from calculations and graphs using their data.
4. Precalculus students will use a rational function of the form $y = c/x$ to model resistance and current data, and linearize the data to compare the usefulness of rational and linearized forms

Ancillary Materials

1. Introduction to Single-Loop Circuits
2. Investigation of Proportional Relationships in Circuits
3. Inverse Relationships and Rational Functions in Circuit Data (Precalculus only)

---

[1] Vernier (Part VCB2)

[2] Available from Vernier, PASCO, and other vendors of probeware

[3] *Ohm's Law* from Univ. of Colorado, Boulder: https://phet.colorado.edu/en/simulation/resistance-in-a-wire

[4] Approximate costs per unit from vendors that supply general science lab materials

Introduction to Single-Loop Circuits

Terminology and Characteristics

The physics of circuits is fraught with terminology and convention. Because this is a mathematics lesson, we'll just introduce the basics.

The battery supplies electrical energy to the circuit by providing what physicists call potential difference but almost everyone informally calls *voltage*. Batteries in common usage provide very constant voltage until near the end of their lives. Commercial AAA, AA, C, and D cells are all 1.5 Volt batteries; the physical size of the battery is correlated with how much energy it can provide and for how long. That's why big flashlights use D cells and calculators use AA or AAA cells. When batteries are put together by connecting the positive end of one to the negative end of the next, the voltages add together. Thus four D cells put together head-to-tail provide 6 Volts, and six AA cells arranged in this manner provide 9 Volts.

When wires and other components provide an uninterrupted conduction path from one terminal of a battery to the other through a circuit, *current* flows in the circuit. Circuit components such as light bulbs provide *resistance* to current. The more resistance, the lower the current. Current I is a dependent variable that depends on both voltage V and resistance R as defined by *Ohm's Law*:

(1) $$I = V/R$$

In our investigation, we will use only simple single-loop series circuits. We want to model how the quantities of voltage, resistance, and current are mathematically related to each other in these circuits by doing some very simple measurements followed by mathematical analysis.

Series Circuit Construction

Let's take a look at how a single-loop series circuit can be constructed using a battery and light bulbs:

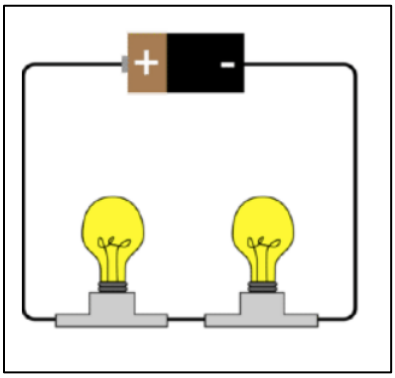


Figure 2
Circuit with One Battery and Two Bulbs

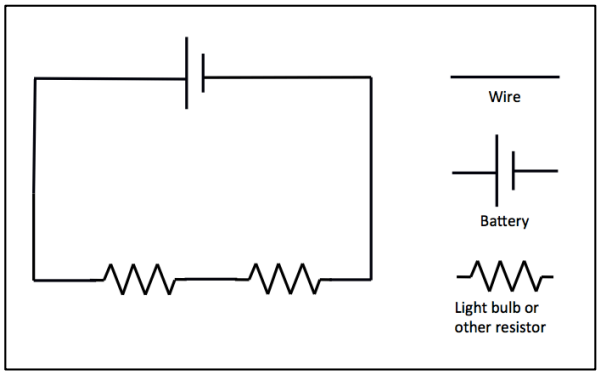


Figure 3
Circuit Diagram for Figure 2

An illustration of a single-loop circuit with one battery and two bulbs is shown in Figure 2.[5] In Figure 3, the circuit is diagrammed using commonly acceptable symbols. Addition of more batteries is accomplished by connecting the positive terminal of one battery to the negative terminal of another. Addition of bulbs is accomplished by connecting one bulb directly to another (there is no positive and negative terminals to worry about with bulbs). In this lab, we want series circuits with no junctions and only one loop.

[5] From Institute for Artificial Intelligence http://data.allenai.org/tqa/electric_circuits_L_0759/

To measure the voltage provided by the battery (or batteries) and the current in our circuit we would use sensors arranged as shown below:

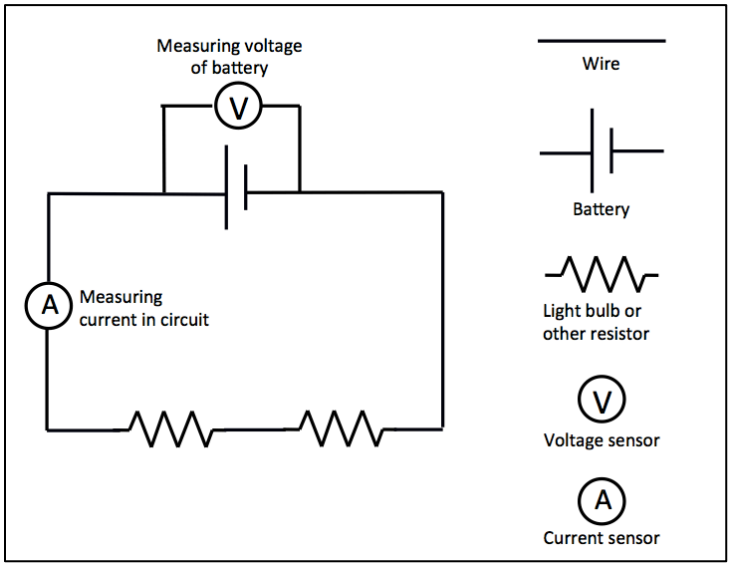


Figure 5
Sensors Measure Current and Voltage in Single-Loop Circuit

Current is the same everywhere in a single-loop circuit and is a property of the circuit as a whole. This means that the current sensor can be put anywhere in the circuit, provided the current flows through the sensor, and it will read the same value.

Voltage, on the other hand, is not the same everywhere in the circuit. Voltage is related to energy change from one point in the circuit to another. We will measure voltage over the battery or batteries, which add energy to the circuit. The voltage sensor must be placed so its positive probe is on the positive end of the battery connected to the rest of the circuit, and its negative probe is on the negative end of the battery connected to the rest of the circuit.

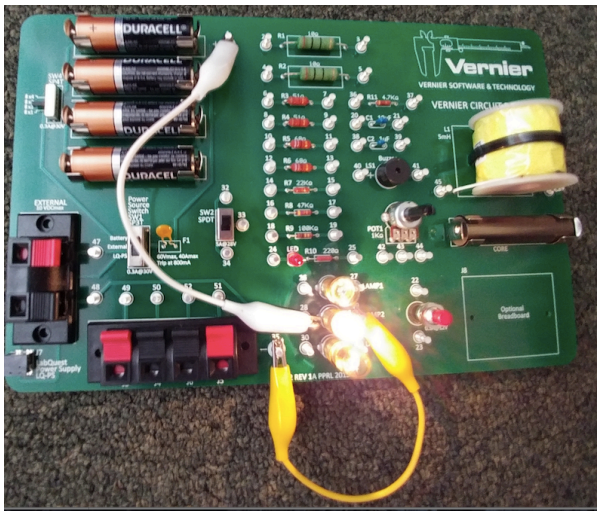
Finally, students will need to know something about the units used for circuit measurements. Voltage is measured in volts (V), current in amperes (A), and resistance in ohms (Ω).

Investigation of Proportional Relationships in Circuits

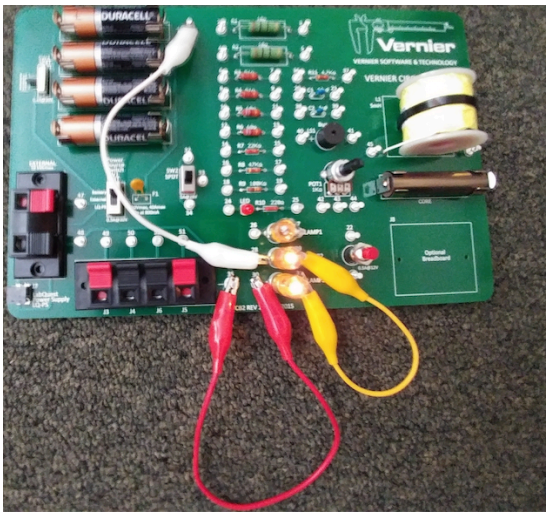
Preliminary Semi-Quantitative Demonstration

These four quick demonstrations are intended to get students thinking about relationships that *might be proportional*, as opposed to ones that *could not be proportional*. The teacher builds four series circuits and with different numbers of batteries and bulbs as shown in Figure 6 below. Circuit 1 has one battery and one bulb. Students describe the "brightness" of bulbs in circuits 2, 3, and 4 as compared to circuit 1. This is a semi-quantitative exercise, as there is no method provided for the students to actually measure brightness.

Figure 6 shows how the circuits are wired using the Vernier circuit board for one and two bulb configurations, respectively. The switch on the side allows for easy selection of the battery number. The table contains possible student data.



Vernier Board with One Bulb



Vernier Board with Two Bulb

Circuit	***# Batteries NB***	***# Bulbs NL***	***Brightness, B***		
			Verbal	***Variable***	***Numeric***
1	1	1	Normal	B	100%
2	1	2	Dim	B/2	50%
3	2	2	Normal	B	100%
4	2	1	Very Bright	2B	200%

Figure 6
Sample Preliminary Investigation Data

Below are some sample teacher questions with possible student responses. The teacher may need to remind students that for a comparison between two variables to be valid, all other variables must be held constant. For example, when considering how brightness varies with bulb number, the number of batteries must be held constant. When considering how brightness varies with battery number, the number of bulbs must be held constant.

a) Could bulb brightness be proportional to the number of bulbs? Justify your answer.

Brightness cannot be proportional to number of bulbs. Circuits 1 and 2 each have one battery. Circuit 2 has more bulbs than Circuit 1 but the bulbs are dimmer. If brightness were proportional to bulb number, the bulbs in Circuit 2 would be brighter than in Circuit 1.

b) Could bulb brightness be proportional to the number to the number batteries? Justify your answer.

Brightness could be proportional to number of batteries. Circuits 1 and 4 each have one bulb. Circuit 4 has more batteries, and its bulb is brighter. Because brightness increases with number of batteries, it could be proportional to number of batteries. To know for sure, we would need to calculate a common ratio between number of batteries and brightness of a single bulb.

c) How could the NB/NL ratio predict brightness?

A ranking of NB/NL appears below:

$$\frac{NB}{NL} = \frac{1}{2}(\textit{for dim circuit 2}) = \frac{1}{1}(\textit{for normal circuits 1 and 3}) = \frac{2}{1}(\textit{for very bright circuit 4})$$

It seems that the greater NB/NL, the brighter the bulb.

Quantitative Investigation of Voltage, Current, and Resistance

Students are now tasked with investigating quantitative measurements of voltage, resistance, and current to identify proportional relationships. To accomplish this, they can build circuits and perform measurements to collect the data they will analyze. Another possibility is to have the teacher collect the data and have the students perform only the mathematical analysis.

For this portion of the investigation, accuracy is important; therefore, commercially available resistors of known value should be used instead of light bulbs. Resistors are inexpensive and have a tolerance of within 5% of the listed value, whereas the resistance of light bulbs can vary with their age and other factors.

Figure 7 shows how the Vernier circuit board can be set up for experiments with voltage, resistance, and current. The battery bank is connected to one of the resistors in the middle of the board; resistance values are printed next to the resistors. The voltage sensor is connected so it measures voltage across the batteries, and the current sensor is measuring current in the circuit.

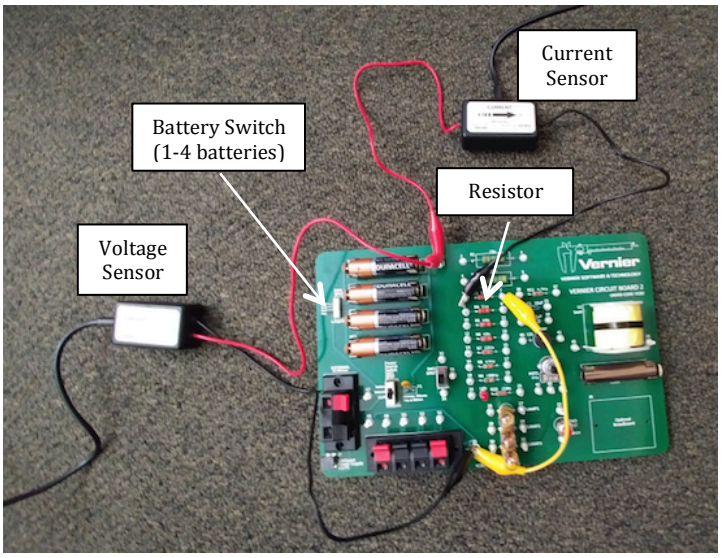


Figure 7
Setting Up Vernier Circuit Board With Batteries and Resistors

Below are two tables of data representative of actual circuits. The resistors that were used are common in high school circuit labs using discrete components. Battery voltage was generated using one or more D or AA cells as the voltage source.

Battery Voltage, V (V)	Resistance, R (Ω)	Current, I (A)
1.5	100	0.01500
1.5	220	0.00692
1.5	330	0.00449
1.5	560	0.00262
1.5	1000	0.00153

Figure 8
Actual Circuit Data with Constant V

Battery Voltage, V (V)	Resistance, R (Ω)	Current, I (A)
1.5	330	0.00452
3.0	330	0.00911
4.5	330	0.01370
6.0	330	0.01810
7.5	330	0.02270

Figure 9
Actual Circuit Data with Constant R

Note that the PhET HML5 simulation *Ohm's Law* mimics an actual circuit and can be used in lieu of collecting physical data. The simulation will generate values similar to those above, but also allows students to use fractions of a battery voltage. This simulation will run on an iPad.

Some middle schoolers may not be ready to analyze decimal numbers such as those in Figures 8 and 9. Simplified data is tabulated in Figures 10 and 11 below. The currents of 1 ampere and above are quite dangerous, so students would never work with actual circuits in this range. The teacher would calculate data in easy-to-analyze numeric ranges using Equation (1).

Battery Voltage, V (V)	Resistance, R (Ω)	Current, I (A)
20	2	10
20	4	5
20	5	4
20	10	2
20	20	1

Figure 10
Simple Circuit Data with Constant V

Battery Voltage, V (V)	Resistance, R (Ω)	Current, I (A)
10	5	2
20	5	4
30	5	6
40	5	8
50	5	10

Figure 11
Simple Circuit Data with Constant R

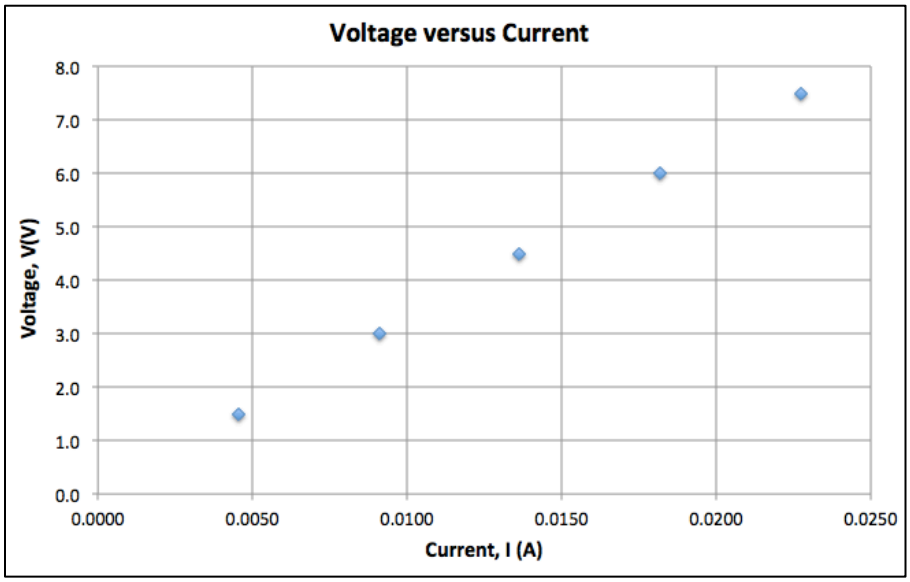
Once an appropriate data set is identified, students analyze it to search for proportional relationships. Here are some sample teacher questions to guide analysis, along with possible student responses. The responses shown below assume analysis of the data in Figures 6 and 7.

a) Is the relationship between voltage and current proportional? Use numerical and graphical evidence to justify your answer.

Voltage is proportional to current. In Figure R is constant and V/I calculations yield:

1) $\frac{1.5V}{0.00452A} = 332\frac{V}{A}$ *2)* $\frac{3.0V}{0.00911A} = 329\frac{V}{A}$ *3)* $\frac{4.5V}{0.0137A} = 328\frac{V}{A}$ *4)* $\frac{6.0V}{0.0181A} = 331\frac{V}{A}$ *5)* $\frac{7.5V}{0.00227A} = 330\frac{V}{A}$

While there is some variation in the calculated ratio, it is constant within experimental error. A graph of V versus I appears linear and should extend through the origin, as shown below:

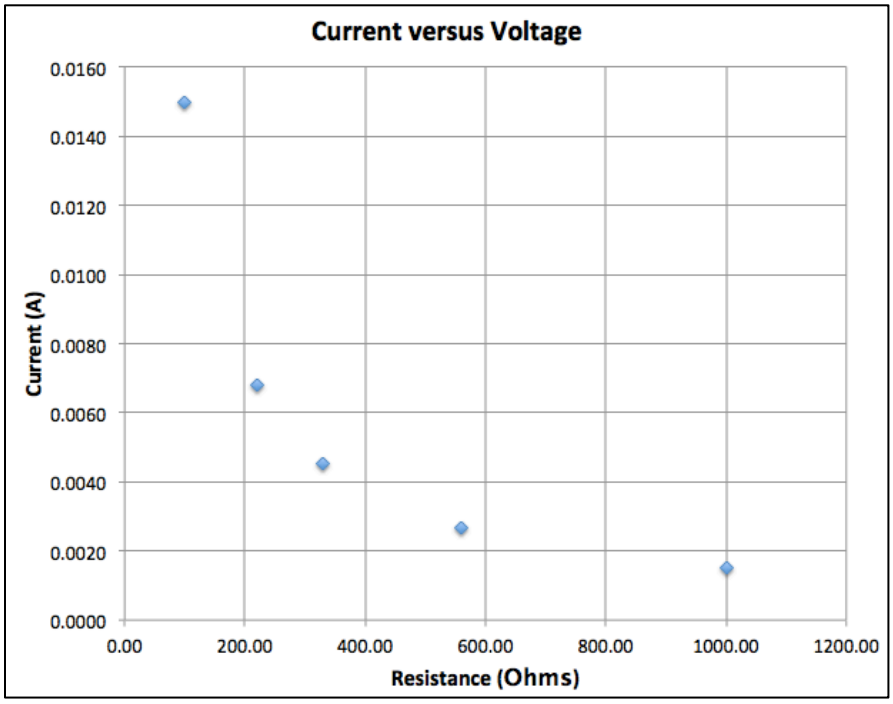


b) Is the relationship between current and resistance proportional? Use numerical and graphical evidence to justify your answer.

Current is NOT proportional to resistance. The data in Figure 6 can be used to show this because V is constant but R and I are varied. The ratio R/I does not yield a constant ratio. For the first three circuits:

$$1)\ \frac{100\Omega}{0.00151A} = 66{,}225\,{}^{\Omega}\!/_{A} \quad 2)\ \frac{220\Omega}{0.00692A} = 317{,}919\,{}^{\Omega}\!/_{A} \quad 3)\ \frac{330\Omega}{0.00449A} = 734{,}967\,{}^{\Omega}\!/_{A}$$

A graph of I versus R does not show a proportional arrangement. The data is not linear and does not pass through the origin.



c) From your answers above, write an equation relating *V*, *I*, and *R*.

The ratios from part (a) show that

$$\frac{V}{I} = R$$

This is algebraically equivalent to Ohm's Law in Equation (1).

Comparing the linear graph in part (a) to the general form $y = mx + b$, we can write:

$$V = IR$$

which is another variant of Ohm's Law in Equation (1).

Inverse Relationships and Rational Functions in Circuits

Precalculus students can extend this lesson beyond proportional versus non-proportional relationships. In precalculus, inverse functions and rational functions are included in the standards.

An Excel curve-fit of current versus resistance data from Figure 6 appears below:

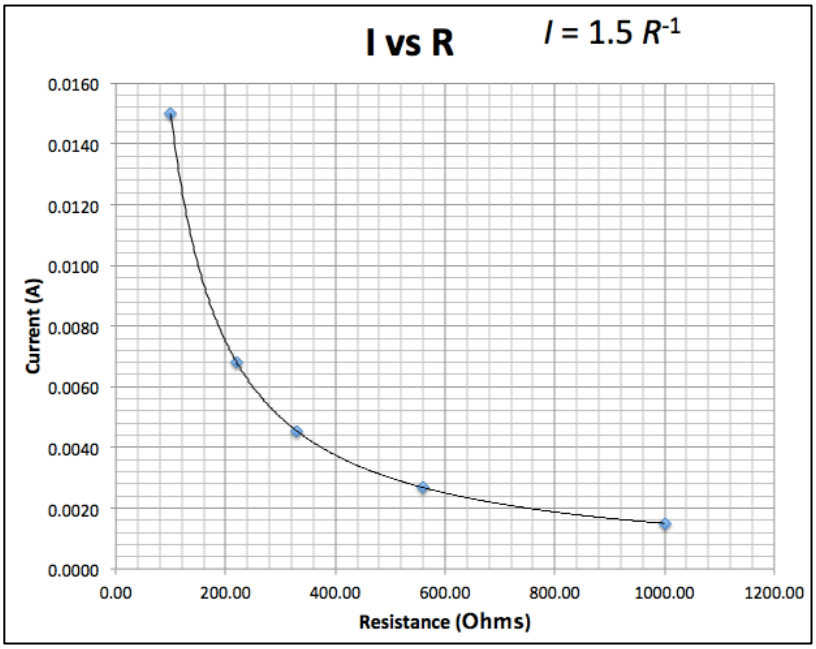


Figure 12
Current versus Resistance

A power fit trendline was found to best fit the data; it yields the function shown on the graph. Students may recognize the shape of the graph as representative of domain and range elements that are inversely related to each other. Because all data points are positive, only the first quadrant is graphed.

From the function shown on the graph, we can see that:

$$I = \frac{1.5}{R}$$

and therefore:

$$IR = 1.5$$

This shows that current and resistance are inversely related. A closer look at Figure 6 will reveal that the constant 1.5 is the voltage of the battery. Therefore, the equation above supports Ohm's Law, Equation (1).

We can linearize the data from Figure 6 by graphing current I by the reciprocal of resistance $1/R$. The linearized graph is shown in Figure 11. Note that a linear trendline fits the data. When the linear trendline is interpreted in terms of the range I and domain $1/R$, the resulting equation is identical to that in Figure 10.

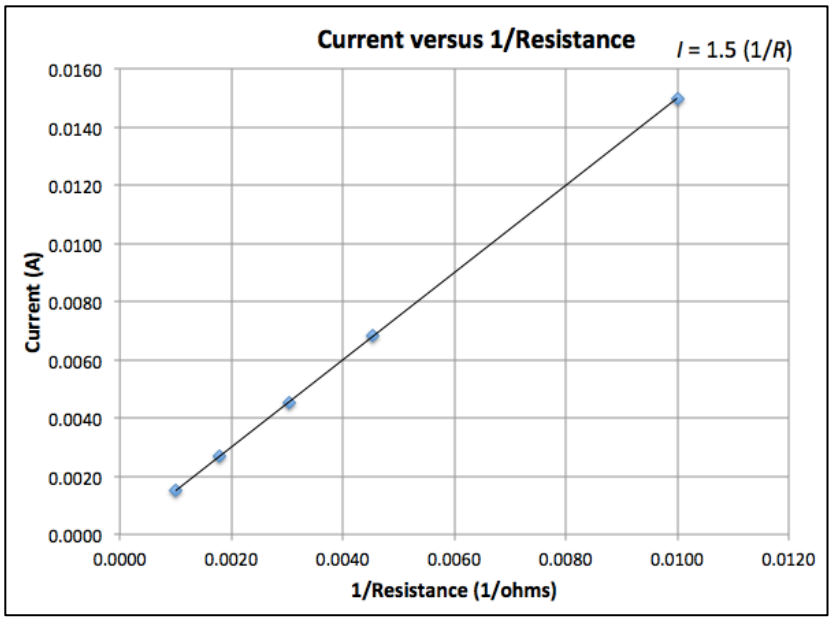


Figure 13
Current versus 1/Resistance

Because the graph in Figure 11 is linear, it is easier to extract the slope and intercept.

Charging and Discharging

Lesson Plan Overview

Lesson Description

Capacitors are electrical devices that, when in circuits with batteries and resistors, result in voltage and current graphs that are beautiful examples of exponential functions. Circuits with resistors and capacitors are called RC circuits for short. Students in Precalculus can use RC circuits as real-world applications of exponential and logarithmic functions.

Calculus or AP® Calculus students can use limits to explain RC circuit graphs. Very advanced students can derive RC circuit equations using differential equations and integration.

Essential Question(s)

How can exponential functions be used to describe voltage and current in an RC circuit?

How can I develop an exponential model for an RC circuit using integration? *(Calculus only)*

Math Standards

Precalculus

P.F.IF.A.2: Analyze qualities of **exponential**, polynomial, logarithmic, trigonometric, and rational functions and solve real-world problems that can be modeled with these functions (by hand and with appropriate technology).

Calculus

C.F.LF.A.2: Estimate limits of functions (including one-sided limits) from graphs or tables of data. Apply the definition of a limit to a variety of functions, including piecewise functions.

C.F.LF.A.3: Draw a sketch that illustrates the definition of the limit; develop multiple real-world scenarios that illustrate the definition of the limit.

C.I.AI.A.2: Use substitution of variables to calculate antiderivatives (including changing limits for definite integrals).

C.I.AI.B.6: Use integrals to solve a variety of problems (e.g., distance traveled by a particle along a line, **exponential** growth/decay).

Math Practices

1. Reason abstractly and quantitatively.
4. Model with mathematics.
6. Attend to precision.
8. Look for and express regularity in repeated reasoning

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)* =

Science Standards

AP® Physics C Electricity and Magnetism Course Objectives C.3.b.1-3: Students should understand discharging and charging of a capacitor through a resistor so they can calculate and interpret the time constant of the circuit, sketch and interpret graphs, and write expressions for current as a function of time.

Lesson Objectives

TSW identify RC circuit charge and discharge graphs as examples of exponential functions.

TSW calculate the capacitive time constant RC and correlate it with a constant in the equations and a specific point in charge/discharge graphs.

TSW derive functions that describe current in an RC circuit and/or voltage across the resistor and capacitor as functions of time (*Calculus only*).

Materials

- Vernier student circuit board[1]
- Alternative - discrete circuit components
  - A 1.5 V battery in battery holder
  - 1 or 2 resistors (100,000 – 200,000 Ω)
  - Non-polar capacitor (100 – 300 μF)
  - Wires with connecting clips
  - Digital Multi-Meter for measuring current, or voltage and current sensors from a probeware vendor.[2]
- Calculators and paper, and graph paper

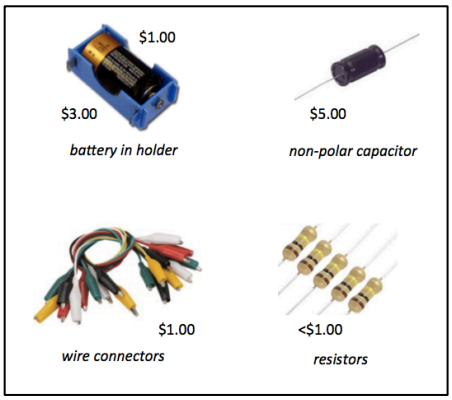


Figure 1
Sample RC Circuit Kit[3]

Basic Instructional Plan:

1. Students will build RC circuits and collect graphs of current and voltage as a functions of time for charge and discharge.
2. Students will fit the graphs with exponential functions.
3. Students will correlate the capacitive time constant $\tau = RC$ with a constant in the equations and a point on the graph.
4. Students will derive equations for voltage across the resistor and/or capacitor as functions of time (Calculus only).

Ancillary Materials

1. Introduction to RC Circuits
2. Exploring Functions for Charge and Discharge in RC Circuits
3. Derivation of RC Circuit Equations (Calculus only)

---

[1] Vernier (Part VCB2)

[2] Available from Vernier, PASCO, and other vendors of probeware

[3] Approximate costs per unit from vendors that supply general science lab materials

Introduction to RC Circuits

Terminology and Characteristics

Capacitors are simple structures. They consist of two closely spaced conducting *plates* separated by an insulating material. Capacitors "store" electrical energy by accumulating excess positive charge on one plate and excess negative charge on the other. The *capacitance, C*, describes how much charge and energy a capacitor can hold. The unit of capacitance is the *farad*.

Circuits with *resistors* and *capacitors* are called RC circuits. Batteries and switches are used in RC circuits as well. When a switch in an RC circuit is placed in one position, the battery *charges* the capacitor. When the switch is moved to another position, the capacitor *discharges*. Depending on the value of capacitance *C* and resistance *R*, the charge and discharge cycles occur with a predictable *time constant* τ that is numerically equal to the product of *R* and *C*. Thus, RC circuits are useful as electronic timing devices.

Exponential Equation for Current

For the following equations, we assume charge cycles begin with a completely uncharged capacitor and discharge cycles begin with a completely charged capacitor. This will simplify our constants. *I* represents current, V_o battery voltage, *R* resistance in ohms, and *C* capacitance in farads.

The current follows this equation:

(1) $$I = \pm\frac{V_o}{R}\left(e^{-t/RC}\right)$$

The $\pm$ sign is used to indicate current direction, which depends on whether the capacitor is charging or discharging.

Voltage across the resistor is proportional to current according to Ohm's Law, $V = IR$, hence:

(2) $$V = \pm V_o\left(e^{-t/RC}\right)$$

The capacitor voltage equations are a bit more complicated. When being charged, the voltage across the capacitor increases until it reaches the battery voltage, hence:

(3) $$V = V_o\left(1 - e^{-t/RC}\right)$$

When being discharged, the voltage decreases until it reaches zero.

(4) $$V = V_o\left(e^{-t/RC}\right)$$

Because the voltage across the capacitor does not switch sign during charge and discharge, we will assume a positive value.

Calculus students can derive these equations, as will be explored in the final section of this lesson plan.

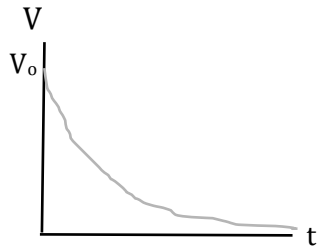
Exploring Functions for Charge and Discharge in RC Circuits

Predictions of Charge and Discharge Graphs

Precalculus students should be able to predict the shape of charge and discharge graphs for RC circuits by examining Equations (1) through (4) and considering intercepts, asymptotes, concavity, and end behavior.

Some guiding questions for the teacher and possible responses from students in Calculus appear below. Precalculus students would not be expected to use limits in their justifications, but would provide more informal and qualitative reasoning that address end behavior. Student sketches of graphs are intended to be qualitative, and not derived from numeric substitutions into the equations.

a) Sketch the shape of a graph of Equation (4). Justify your sketch based on end behavior.



Equation (4) is $V = V_o\left(e^{-t/RC}\right)$. When $t = 0$, term $e^{-t/RC}$ is 1.

Substitution into Equation (4) shows the vertical intercept is V_o.

Also, $\lim_{t\to\infty}\left(e^{-t/RC}\right) = \lim_{t\to\infty}\left(\frac{1}{e^{t/RC}}\right) = 0$.

Therefore, the horizontal axis $\left(V_o = 0\right)$ is an asymptote.

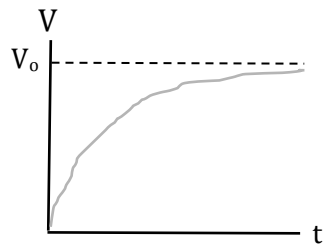
b) How will the graph of Equation (2) compare to the graph of Equation (4)?

The graph of Equation (2) will look identical to the graph of Equation (4) if it is a positive quantity, or it will be the reflection of the graph of Equation (4) across the *t* axis if it is a negative quantity.

c) How will the graph of Equation (1) compare to the graph of Equation (2)?

The graph of Equation (1) will look similar in shape to the graph of Equation (2), however the vertical intercept will have an absolute value of V_o/R instead of V_o.

d) Sketch the shape of a graph of Equation (3). Justify your sketch.



Equation (3) is $V = V_o\left(1 - e^{-t/RC}\right)$. When $t = 0$, term $e^{-t/RC}$ is 1.

Substitution into Equation (3) shows the vertical intercept is zero.

Also, $\lim_{t\to\infty}\left(e^{-t/RC}\right) = \lim_{t\to\infty}\left(\frac{1}{e^{t/RC}}\right) = 0$.

Therefore, the function approaches a horizontal asymptote of V_o.

RC Circuit Construction

After predicting the appearance of graphs, students can build an RC circuit and collect data to test their predictions. The circuit diagram below shows an RC circuit with a battery, capacitor, resistor, and switch. The diagram also shows where current and voltage sensors can be placed.

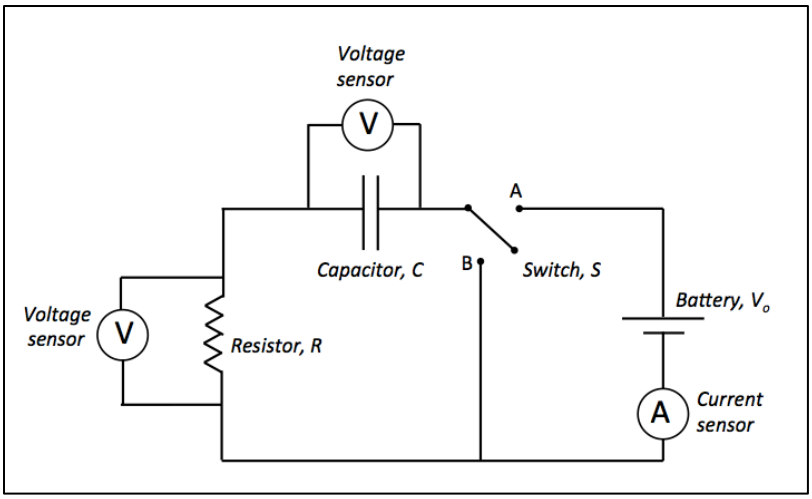


Figure 2
RC Circuit Schematic

A photograph of an actual RC Circuit set up using a Vernier Circuit Board. Voltage sensors can be added across the resistor and capacitor, which are in the locations shown. Either yellow connector can be replaced with a current sensor to measure current in the circuit. The switch is used to change from charge to discharge cycle.

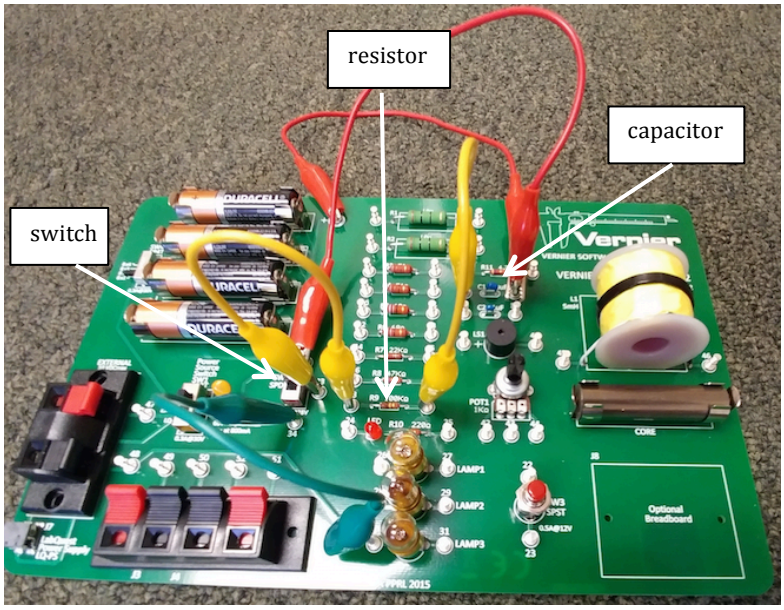


Figure 3
RC Circuit

Data Collection

Figure 4 shows graphs of voltage data collected from an actual RC circuit. Voltage across the resistor is shown by the red curve and voltage across the capacitor is shown by the green curve. Charging starts at 5 seconds and continues until it switches to discharging just at about 109 seconds. The change in the graph shape is obvious when charging and discharging begin.

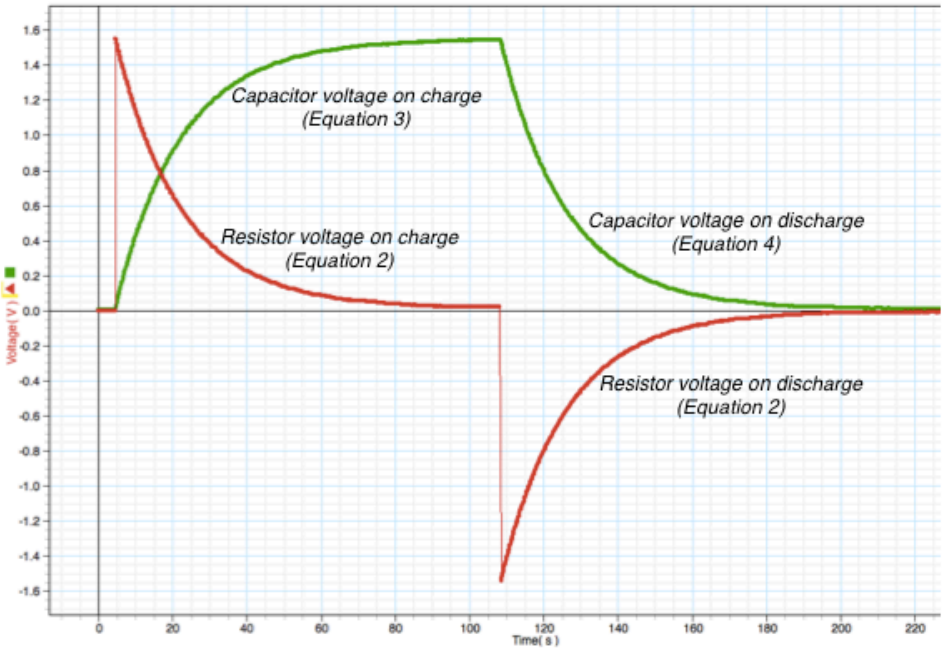


Figure 4

RC Circuit Data[4] *(R = 15,000 Ω , C = 0.001 F, and V_o = 1.50 V)*

Comparison of Predictions and Experiment

Students can readily see that the charge and discharge curves are similar to their predicted graphs, but the position of the graphs on the time axes is shifted. If domain data were adjusted so that each event started at time zero, the experimental curves would be consistent with those predicted by Equations (2) through (4). Because we have not measured current data, we cannot test Equation (1).

To see how well theoretical and experimental graphs agree, students can graph the equations on a graphing calculator or in a program such as Desmos[5]. Figure 5 shows a graph of Equation (3) in Desmos, where the experimental values *R*, *C*, and V_o shown under Figure 4 were assigned to the equation constants. The graph has the expected shape, but how well does it fit the experimental data?

To find out, two experimental points were selected from Figure 4 and, after adjusting for a time delay in the experimental by subtracting 5 seconds from each domain element, were plotted on the same axes with the theoretical curve. These are the red and black points that agree nicely with the theoretical line.

[4] Data Collected with PASCO voltage sensors and graphed in Data Studio

[5] See Desmos at https://www.desmos.com/calculator

The purple point is an illustration of the *capacitive time constant, τ,* of which physics teachers are very fond. When $t = \tau = RC$, the voltage should be about 63.2% of its asymptotic value. This is illustrated by:

(5) $$V = V_o\left(1 - e^{-RC/RC}\right) = V_o\left(1 - e^{-1}\right) = 0.632V_o$$

For the circuit in question

(6) $$\tau = RC = (15{,}000\Omega)(0.001F) = 15s$$

Therefore, at a time of 15 after charging starts, the voltage should be

(7) $$(0.632)(V_o) = (0.632)(1.53V) = 0.967V$$

The point representing voltage at time *RC* is indicated on the graph by the purple point.

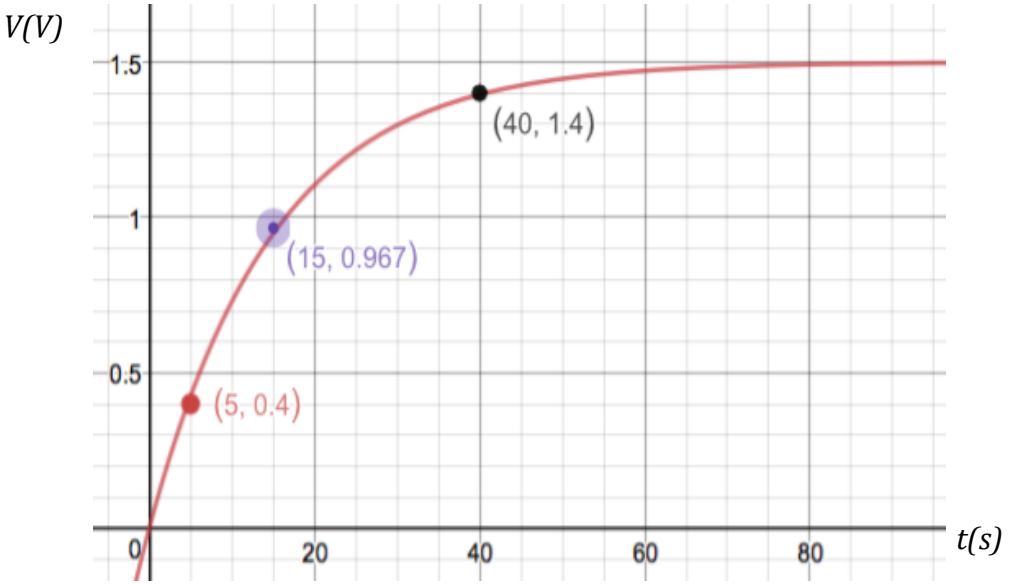


Figure 5
Desmos Graph of Equation (3)

The analysis above is just for Equation (3). Students should be encouraged to compare the other equations with the experimental data in this manner as well.

Derivation of RC Circuit Equations

Capacitor Dischargingе: Physics

Sufficiently advanced students in Calculus or AP® Calculus can derive Equations (1) through (4). In this section, we will derive Equation (1) as it applies to the discharge of a fully charged capacitor through a resistor.

First let's very briefly introduce the physics that precedes the mathematics. Physicists use a conservation of energy strategy called *Kirchoff's Loop Rule* as their starting point for RC circuit derivations. For a circuit in which a capacitor is discharging through a resistor, there is no battery in the loop and therefore:

(8) $$0 = V_R + V_C = IR + \frac{Q}{C}$$

Equation (8) basically says that the voltage across the resistor V_R and capacitor V_C, when added together, yield zero. The voltage across the resistor is $V_R=IR$ according to Ohm's law, where I is current and R is resistance. The voltage across the capacitor is $V_C=Q/C$, where Q is the charge on the capacitor and C is the capacitance. Because charge Q and capacitance C are both positive quantities, we expect I to be negative for Equation (8) to be true.

Current I is the derivative of charge with respect to time, so:

(9) $$0 = \left(\frac{dQ}{dt}\right)R + \frac{Q}{C}$$

Because the capacitor is discharging, Q on the capacitor is decreasing, so we know that dQ/dt is indeed negative, as it must be if Equation (9) is to be true.

Capacitor Discharging: Mathematics

Equation (9) is where the physics ends and the mathematics begins. Rearrangement of Equation (9) followed by separation of variables is next:

$$\left(\frac{dQ}{dt}\right)R = -\frac{Q}{C}$$

(10) $$\frac{dQ}{Q} = -\frac{dt}{RC}$$

Setting up a definite integral:

(11) $$\int_{Q_o}^{Q} \frac{dQ}{Q} = -\frac{1}{RC}\int_{0}^{t} dt$$

The lower limits are the initial charge Q_o on the capacitor and the initial time of zero. The upper limits are variables, as we are deriving a function. (If this is troubling in terms of mathematics convention, use appropriate alternative notation.) Integration yields:

(12) $$\ln\left(\frac{Q}{Q_o}\right) = -\frac{t}{RC}$$

By taking an exponential of both sides, Equation (12) can be converted to:

(13) $$\frac{Q}{Q_o} = e^{-t/RC}$$

or

(14) $$Q = Q_o e^{-t/RC}$$

Look familiar? This is the exponential decay pattern observed in Equations (1), (2) and (4).

As current flows during a discharge cycle, the charge Q on the capacitor decreases with time. Current is moving charge, which in this case is coming from the capacitor. So we define current I as dQ/dt. This leads us to:

(15) $$I = \frac{dQ}{dt} = \frac{d}{dt}\left(Q_o e^{-t/RC}\right) = -\frac{Q_o}{RC} e^{-t/RC}$$

Remembering that $V=Q/C$, we can substitute to get:

(16) $$I = -\frac{V_o}{R} e^{-t/RC}$$

Voila! This is the negative version of Equation (1)! Remember that the sign indicates current direction. The negative sign indicates that the current is moving in the negative direction when the capacitor discharges.

Capacitor Charging: Physics

This derivation is a quite a bit more complicated than the one for capacitor discharge. This is because the Kirchhoff Loop Rule starting point has an additional constant, the voltage across the battery V_o.

(17) $$0 = V_o + V_R + V_C$$

If the voltage across the battery V_o is considered to be a positive quantity, the other two quantities, V_R and V_C, will be negative. Substituting the appropriate constants:

(18) $$0 = V_o - IR - \frac{Q}{C}$$

The differential equation is:

(19) $$0 = V_o - \left(\frac{dQ}{dt}\right)R - \frac{Q}{C}$$

Capacitor Charging: Mathematics

Here comes lots of fun if you like integration by substitution (and who doesn't?). We need to begin by separating variables, which is not as easy this time. Taking it a step at a time can help avoid errors in algebra. Multiplying both sides of Equation (19) by C:

(20) $$0 = V_o C - \left(\frac{dQ}{dt}\right)RC - Q$$

Some more algebra to put Q and dQ terms on the left and t on the right:

$$\left(\frac{dQ}{dt}\right)RC = V_oC - Q$$

$$\frac{dQ}{dt} = \frac{V_oC - Q}{RC}$$

(21)
$$\frac{dQ}{V_oC - Q} = \frac{dt}{RC}$$

The integral is:

(22)
$$\int_0^Q \frac{dQ}{V_oC - Q} = \frac{1}{RC}\int_0^t dt$$

In Equation (22), the lower limit of integration on the left indicates that there is no charge on the capacitor at the beginning of the charge cycle. The upper limit is charge Q on the capacitor at time t.

Solving this integral can by done by substitution. Define a variable u such that:

(23)
$$u = V_oC - Q$$

The differential relationship is therefore:

(24)
$$du = -dQ \text{ or } dQ = -du$$

The upper and lower limits of integration, according to Equation (23) are now:

(25)
$$V_oC - Q \text{ and } V_oC$$

Using Equations (23) – (25), we can rewrite Equation (22) as:

(26)
$$-\int_{V_oC}^{V_oC-Q} \frac{du}{u} = \frac{1}{RC}\int_0^t dt$$

Integration yields:

(27)
$$-\ln\frac{V_oC - Q}{V_oC} = \frac{t}{RC} \quad \text{or} \quad \ln\frac{V_oC - Q}{V_oC} = -\frac{t}{RC}$$

Taking an exponential of both sides:

(28)
$$\frac{V_oC - Q}{V_oC} = e^{-t/RC}$$

Algebraic rearrangement yields:

$$V_oC - Q = V_oCe^{-t/RC}$$

$$-Q = V_oCe^{-t/RC} - V_oC$$

(29) $$Q = V_o C\left(1 - e^{-t/RC}\right)$$

Which is looking oddly familiar. Dividing both sides by *C*, we get:

(30) $$\frac{Q}{C} = V_o\left(1 - e^{-t/RC}\right)$$

It was stated earlier that $Q/C = V$, so we can make our final substitution:

$$V = V_o\left(1 - e^{-t/RC}\right)$$

And we have derived Equation (3), the voltage across the capacitor as a function of time during a charge cycle.

We also derive current during the charge cycle. Because $I = dQ/dt$, taking a derivative of Equation (29) should yield current.

(31) $$I = \frac{dQ}{dt} = \frac{d}{dt} V_o C\left(1 - e^{-t/RC}\right)$$

This yields the following:

(32) $$I = V_o C\left(\frac{-e^{-t/RC}}{-RC}\right) = \frac{V_o C}{RC} e^{-t/RC}$$

which when simplified yields:

$$I = \frac{V_o}{R}\left(e^{-t/RC}\right)$$

This is the positive version of Equation (1).

Coffee Filter Drop

Lesson Plan Overview

Lesson Description

When a single coffee filter is dropped, it appears to fall at constant terminal velocity. When additional filters are nested inside the first, the terminal velocity increases with each filter added to the stack.

Students will collect experimental data to construct a function that describes the relationship between terminal velocity and mass. This will require the use of logarithms and the application of logarithm rules. This lesson is best used in Pre-calculus or Integrated Math III. It demonstrates the usefulness of logarithms, and reinforces concepts of the linear equation, *x-y* scatter plots, and regression analysis.

Essential Question(s)

How can logarithms be used to determine an unknown power and constant in an equation?

Math Standards

Pre-Calculus

P.N.NE.A.1: Use the laws of exponents and logarithms to expand or collect terms in expressions; simplify expressions or modify them in order to analyze them or compare them.

P.N.NE.A.2: Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

P.F.BF.A.2: Develop an understanding of functions as elements that can be operated upon to get new functions: addition, subtraction, multiplication, division, and composition of functions.

Integrated Mathematics III

M3.A.CED.A.3: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *i) Tasks have a real-world context. ii) Tasks are limited to polynomial, rational, absolute value, exponential, or logarithmic functions.*

M3.F.IF.A.1: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *i) Tasks have a real-world context. ii) Tasks may involve polynomial, exponential, and logarithmic functions.*

M3.F.LE.A.2: For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Let's Get Physical! Teaching Mathematics through the Lens of Physics

Peggy Bertrand, Ph.D. (Investigator, VolsTeach, Lesson Plan Author)
L. Jeneva Clark, Ph.D. (Principal Investigator, Math Department, Project Director)
University of Tennessee, Knoxville
Funded by an ITQ (*Improving Teacher Quality)* grant from THEC (*Tennessee Higher Education Commission)*

Math Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Model with mathematics.
6. Attend to precision.

Science Standards

PHYS.PS2.12: Use experimental evidence to demonstrate that air resistance is a velocity dependent drag force that leads to terminal velocity.

Lesson Objectives

TSW use logarithms and their rules to linearize a data set and determine unknown constants in an equation.

TSW apply exponents and logarithms as inverse functions.

Materials

- Box of standard coffee filters (the kind for 12-cup drip coffee makers is best)
- Mass balance, meter sticks, timers (or cell phones)
- Graph paper and straight edge

A place from which to drop coffee filter stacks. Must be relatively free from air currents.

Basic Instructional Plan

1. Students prepare for the laboratory by doing the pre-lab activities, which include:
   a) Learning the basic terminology and the purpose of the laboratory
   b) Using logarithms to linearize $v^n = \frac{g}{b}m$ and rearranging the resulting equation such that n and b appear in the slope and intercept, respectively, of the new equation.
   c) Designing a procedure that will allow appropriate data collection
2. Students perform laboratory data collection, which includes:
   a) Determining the mass of a single coffee filter.
   b) Dropping stacks of a coffee filters from a measured height and recording the time it takes the stacks to fall to the ground.
   c) Tabulating data and calculating required additional quantities
3. Students perform post-lab analysis activities, including:
   a) Linearizing velocity and mass data using logarithm operator.
   b) Graphing the linearized data.
   c) Extracting the slope and intercept from the graph and using them to determine constants.
   d) Answering post-lab analysis questions.

Ancillary Materials

1. Terminal Velocity and Logarithms
2. Experimental Procedure
3. Data Analysis
4. Simplifying the Math with Non-Standard Units

Terminal Velocity and Logarithms

Terminology

In introductory physics, more often than not it is assumed that a falling object experiences a constant downward gravitational force. As a result, the object experiences constant downward acceleration and a constantly increasing downward velocity. However, under certain circumstances *air resistance* (also known as *drag*) creates a significant *upward* force on a falling object that simply cannot be overlooked or ignored.

The drag force is in the opposite direction of an object's velocity. Its magnitude increases as the object's velocity increases. Drag also depends upon the shape and cross-sectional area of the object, the material from which it is made, and the characteristics of the medium through which it falls. As drag increases, acceleration decreases. If an object falls long enough, drag will cause its acceleration to diminish until it is zero. At that point, the object has reached its *terminal* velocity.

Mathematics Connection

In a physics class, acceleration of a falling object that experiences drag is derived using Newton's Second Law of motion. Our *mathematical* exploration will begin with the result of the physics derivation, and appears below:

$$a = g - \frac{b}{m} v^n \tag{1}$$

The object's acceleration is a, gravitational acceleration is g, the mass of the object is m, a constant representing the object's tendency to cause drag is b, and the velocity of the object is v. Note that velocity is raised to a power n. For fast-moving objects falling through air, n is typically 2; for slow moving objects falling through liquids, n is typically 1.

Equation (1) shows that, whatever the integer value of n, acceleration will decrease as velocity increases until it disappears entirely. At that point, with zero acceleration, the object falls at terminal velocity v_T. Substitution into Equation (1) yields:

$$0 = g - \frac{b}{m} v_T^n \tag{2}$$

which can be rearranged to show velocity as a function of mass:

$$v_T^n = \left(\frac{g}{b} \right) m \tag{3}$$

Pre-Lab Mathematical Analysis

The purpose of the laboratory is to determine n and b for coffee filters falling at terminal velocity using a linearization technique. Since terminal velocity depends on mass according to Equation (3), we can adjust mass m and measure the effect on terminal velocity v_T. From the resulting data, we will determine n and b. (Note that g is a well-established constant of about 9.8 m/s^2.)

Coffee filters work well for this laboratory because they can be nested inside of each other, thus presenting a convenient way to adjust the mass of a falling body without changing its shape or surface characteristics and thus impacting b. Most students are familiar with these filters. A drawing appears below:



The challenge for mathematics students is to develop a linear model using Equation (3) as the starting point. There are two variables in the equation, v_T and m. The constant g has a well-established value. The two unknown constants that must be determined are n and b.

Using Logarithms to Develop the Linear Model

We will begin by addressing the algebra requirement, namely the development of a linear equation from Equation (3) that contains n within the slope and b within the vertical intercept. The teacher will likely need to provide significant guidance to the students with questions and prompts such as those below. Possible correct student answers and work are provided.

a) The terminal velocity of the coffee filter stack increases with the number of filters in the stack. Identify the independent and dependent variables, and translate that into the language of functions.

The independent variable is the number of coffee filters, because that is what is adjusted. That isn't in the equation, but the mass of the coffee filters is. So mass, m, is the independent variable. The dependent variable is terminal velocity, v_T, because that is what is changing as the mass changes. We can say terminal velocity is a function of mass.

b) Suppose we want the power n to appear in the equation as a multiplier rather than an exponent. What type of function can we use?

The logarithm function can be used make an exponent into a multiplier. The function must be applied to both sides of the equation. The example below uses natural logarithms, but logarithms of any base work just as well.

$$\ln(v_T^n) = \ln\left[\left(\frac{g}{b}\right)m\right] \quad (4)$$

Applying the power rule of logarithms results in the following:

$$n\ln(v_T) = \ln\left[\left(\frac{g}{b}\right)m\right] \quad (5)$$

c) Apply other logarithm rules and algebraic rearrangement to the equation above so that n is contained in the slope and b is contained in the intercept of a linear equation.

Applying the product rule of logarithms to the right side of the equation results in the following:

$$n\ln(v_T) = \ln(m) + \ln(g) - \ln(b) \quad (6)$$

The resulting equation is then rearranged to the $y = mx + b$ (slope-intercept) form of a linear equation as follows:

$$\ln(v_T) = \frac{1}{n}\ln(m) + \frac{1}{n}\ln(g) - \frac{1}{n}\ln(b) \quad (7)$$

$$\ln(v_T) = \frac{1}{n}\ln(m) + \frac{1}{n}\ln\left(\frac{g}{b}\right)$$

(8)

d) Identify the dependent and independent variables, the slope, and the vertical intercept in Equation (8). Explain how you will graph the equation so it is linear.

Encourage students to write the general form of a linear equation in slope-intercept form directly above Equation (8). Have them identify and clearly highlight analogous constants and variables. An example appears below:

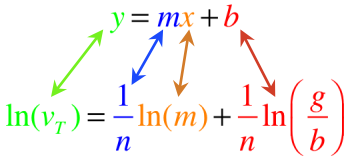


Figure 1

Comparison of Logarithmic Equation with Generalized Linear Equation

For the generalized linear equation $y = mx + b$, data points are represented by (x, y), where x is the independent variable and y is the dependent variable. The horizontal axis is the x-axis, and the vertical axis is the y-axis. By analogy, each data point for the linearized logarithmic equation above is represented by $(\ln(m), \ln(v_T))$. The horizontal axis is for the independent variable $\ln(m)$, and the vertical axis is for the dependent variable $\ln(v_T)$.

A Word About Units

Note that in the subsequent experimental section, all measurements are reported in SI units. Distance is reported in meters and mass in kilograms. All mass values for coffee filter stacks are therefore much less than 1.00 kg. A consequence of this is the negative natural log values recorded in the table. This presents an interesting added complexity in graphing!

If you as a mathematics teacher wish to simplify the laboratory somewhat, mass units of "grams" or even of "coffee filters" are perfectly acceptable and will not result in exclusively negative outputs to the logarithm function. While the use of non-standard units are guaranteed to cause great unhappiness and distress for a physics teacher, for the purposes of this mathematical investigation, any units at all will work. Alternative units will not affect the determination of n whatsoever. They will impact the numeric value of b, but if units within a class are consistent, b will be consistent between various groups.

Experimental Procedure

Data Collection

Students collaboratively design a simple experiment to collect data for eventual graphing. This should not be difficult if they have properly defined the mathematics and understand what variables they will need to plot. Guidance for the teacher is provided below.

a) The independent variable, ln(m)

Students determine the mass of stacks of between 1 to 5 coffee filters. The mass of a typical coffee filter is less than a gram. Unless you have access to a mass balance that reads accurately to a 10th of a gram, students should measure the mass of many filters and calculate the average.

b) The dependent variable, ln(v_T)

Students decide what data they need to collect so they can determine v_T for each stack of coffee filters. Dropping the filters from a measured height d and measuring the time t it takes for the filters to hit the floor is the easiest method, since $v_T = d/t$.

c) The data table

Column headings should include mass m of the coffee filter stack, the distance the filters fall d, and the time it takes to fall t. Units should be included in headers and tables should have space for multiple trials. Columns for calculated values, v_T, ln(m), and ln(v_T) should also be included.

Students drop coffee filter stacks from a pre-determined height and measure the time they take to fall. They record data in the table, and calculate necessary values. Note that for stacks of more than five coffee filters, timing errors and the breakdown of the assumption that coffee filters reach terminal velocity quickly add error. It is also acceptable to use 4 filters as the maximum stack size if your students cannot drop the filters from a height of 2 meters or more.

Sample Data

Figure 2 below shows an example of student data collected during a coffee filter laboratory experiment at Oak Ridge High School in Oak Ridge, TN. For each trial, coffee filters were dropped from a height of 9.45 m in an atrium at the school. The filters experienced no significant air currents as they fell. The values shown in the table are averages taken over four trials.

Number of filters	Mass, m (kg)	Fall distance, d (m)	Average fall time, t (s)	Terminal velocity, v_T (m/s)	ln(m)	ln(v_T)
1	0.00072	9.45	7.91	1.20	-7.24	0.178
2	0.00144	9.45	5.75	1.64	-6.55	0.497
3	0.00215	9.45	4.58	2.07	-6.14	0.725
4	0.00287	9.45	3.96	2.39	-5.86	0.870
5	0.00359	9.45	3.68	2.57	-5.63	0.943

Figure 2
Student Data and Calculated Values for Coffee Filter Drop using SI Units

Data Analysis

Graphing the Linearized Function

Students should now graph plot the dependent variable as a function of the independent variable to produce a scatterplot, and then fit the data with a line of best fit. Students can do this manually or with a program such as Excel. The results of tabulating and graphing the data from the table in Figure 2 are shown below in Figure 3. After the data were plotted, a linear regression was performed to generate the equation shown on the chart.

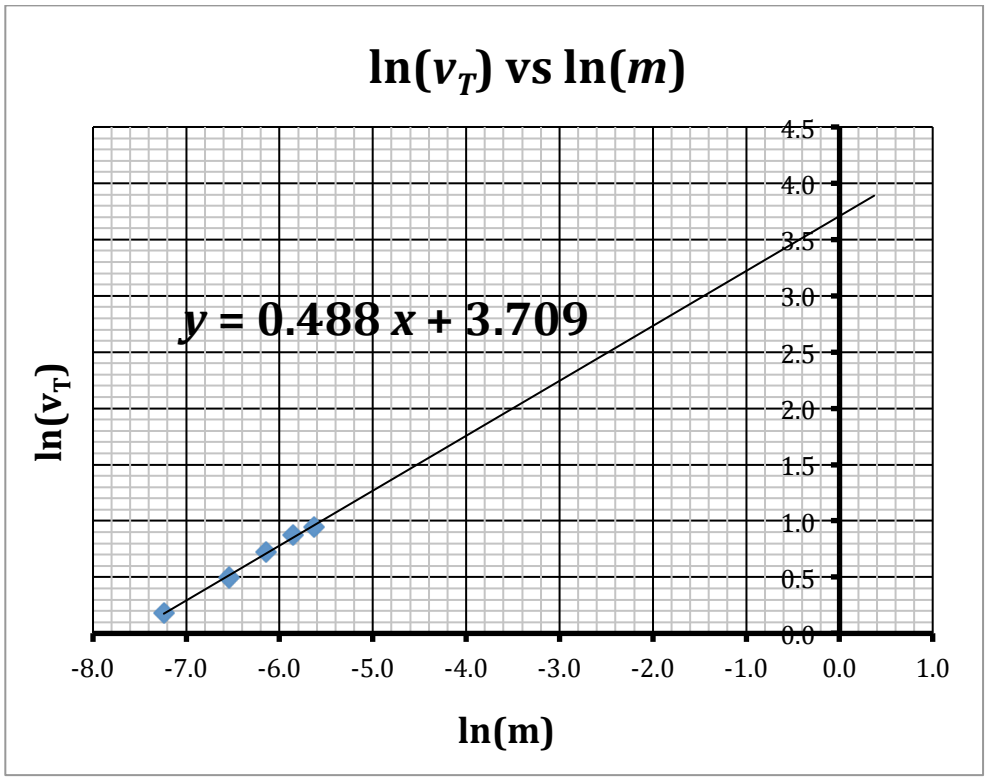


Figure 3
Linearized Graph of Experimental Data for Falling Coffee Filters Using SI Units

Extracting Information From a Linear Graph

The teacher can now guide students to extract the constants *n* and *b* from the linear equation defined by the line of best fit. In their pre-lab mathematical analysis, students should have already set the stage for this; however, they will likely need some gentle reminders regarding using logarithms and exponential functions as well as some guidance in doing the associated algebraic manipulations. However, the process must begin by comparing Equation (8) and the regression equation given in Figure 3.

a) Facilitating comparison

In Figure 1, we compared Equation (8) with the generalized form of a linear equation in slope-intercept format. If instead of $y = mx + b$, we use the more specific regression equation from Figure 3, $y = 0.488\,x + 3.709$, the comparison looks like this:

$$y = 0.488x + 3.709$$
$$\ln(v_T) = \frac{1}{n}\ln(m) + \frac{1}{n}\ln\left(\frac{g}{b}\right)$$

Figure 4

Comparison of Logarithmic Equation with Linear Regression Equation

Once students recognize the similarity between the two equations in Figure 4, they are conceptually ready to determine the unknown constants.

b) Using the trendline fit to determine *n*

The teacher can now ask the students how they would determine n given the illustration in Figure 4. Hopefully, students will readily set up the following equality:

$$0.488 = \frac{1}{n}$$

Solving for n is now fairly trivial:

$$n = \frac{1}{0.488} = 2.05$$

Students will likely need to be reminded at this point that n must be an integer equal to either 1 or 2. Therefore:

$$n = 2$$

c) Using the trendline fit to determine *b*

The determination of the constant b from the illustration in Figure 4 will be significantly more challenging. Based on the figure, students will likely be able to write the following equation as the starting point:

$$3.709 = \frac{1}{n}\ln\left(\frac{g}{b}\right)$$

After that, students may get stuck. Isolating b on one side of the equation will require the use of not only algebra with literals, but also application of rules of logarithms and exponentials. One step-by-step solution path appears below, beginning with multiplication of both sides of the equation by n:

$$[3.709]n = \left[\frac{1}{n}\ln\left(\frac{g}{b}\right)\right]n$$

This is followed by simplification:

$$3.709n = \ln\left(\frac{g}{b}\right)$$

Applying the exponential function to both sides of the equation yields:

$$e^{3.709n} = \frac{g}{b}$$

Taking the reciprocal of both sides:

$$e^{-3.709n} = \frac{b}{g}$$

Solving for b yields:

$$b = g(e^{-3.709n})$$

Given that $g = 9.8$ m/s^2 and $n = 2$, numeric substitution yields:

$$b = 0.00588\ kg/m$$

Note that b has units! The teacher can direct the curious and scientifically inclined student to Equation (3) to figure out how the units of b are determined. *(The units on the left- and right-hand side of the equation must be identical. Because velocity is squared on the left-hand side, the units must be m²/s². To have units of m²/s² on the right-hand side, b must have units of kg/m.)*

Manual Graphing

Note that while the graph and linear regression equation in Figure 3 were done using Excel, graphing by hand is quite adequate for the analysis performed in this laboratory. Students need only graph paper and a ruler to plot the data and draw a line of best fit. This gives students more practice in standard graphing skills such as plotting data points, drawing a line of best fit, calculating the slope of the line, and reading the graph to determine the vertical intercept. Students will need a calculator equipped with logarithm and exponential functions to perform the analysis. When students analyze falling coffee filter data using manual rather than computer-based graphing techniques, it underscores the value of linearization in a much more powerful way.

Simplifying the Math With Non-Standard Units

Figures 5 and 6 below illustrate the analysis using "coffee filters" rather than kilograms as the mass unit. While this might be frowned upon in a science lab, mathematics teachers may prefer this approach as it limits graphing to first quadrant and doesn't require messy negative logarithms.

Number of filters	Mass, m (in coffee filters)	Fall distance, d (m)	Average fall time, t (s)	Terminal velocity, v_T (m/s)	ln(m)	ln(v_T)
1	1.00	9.45	7.91	1.20	0.000	0.178
2	2.00	9.45	5.75	1.64	0.693	0.497
3	3.00	9.45	4.58	2.07	1.099	0.725
4	4.00	9.45	3.96	2.39	1.386	0.870
5	5.00	9.45	3.68	2.57	1.609	0.943

Figure 5

Student Data and Calculated Values for Coffee Filter Drop using "Coffee Filter" as Mass

The data in the last two columns in Figure 5 is graphed in Figure 6. The slope in Figure 6 is the same as in Figure 3; therefore, n is still 2. The intercept is clearly different, which indicates that b depends on the unit system used for measurement. The observant student may notice that n is a quantity that has no units and conclude that it is therefore not sensitive to a unit system. On the other hand, b has units and will therefore have a different numeric value for different combinations of mass and/or distance units.

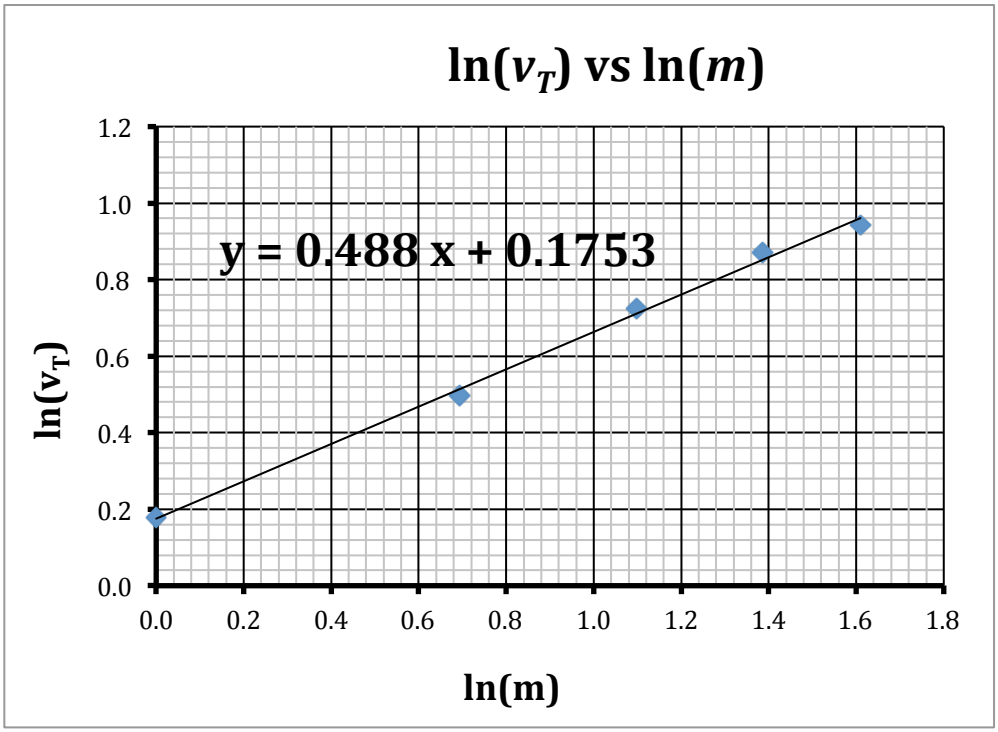


Figure 6

Linearized Graph of Experimental Data for Falling Coffee Filters Using Alternate Units