POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCLIX

JAN ROSIŃSKI

Bilinear random integrals

WARSZAWA 1987
PAŃSTWOWE WYDAWNICTWO NAUKOWE

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PRINTED IN POLAND

ISBN 83-01-07186-9 ISSN 0012-3862

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

I. Introduction

Various kinds of stochastic integrals are known in the probability theory. Historically, the first one is due to Wiener [46]. Wiener's integral was generalized by Ito in two directions, first by omitting the restriction that the integrand was a deterministic function [10], second, by constructing a multiple Wiener integral [11]. The stochastic integral appearing in the theory of stationary processes was introduced by Kolmogorov [19], [20]. A stochastic Stieltjes integral with respect to a stochastic process with independent increments was first used in [12]. The integral representation of processes with independent increments is due to Lévy [25]. In [26] Lévy studied a stochastic integral of a deterministic function with respect to a stochastic process with independent increments (having no Gaussian component) as a generalization of a random series with independent summands, and characterized the set of all integrable functions.

The idea of random set functions has first appeared in Bochner's paper [5]. Random set functions taking independent values on disjoint sets were investigated by Prekopa [34]. Urbanik [42], [43] used completely additive random set functions of such type in representation theorems in the prediction theory for strictly stationary processes. A random integral viewed as an integral with respect to an L^0-valued random measure was defined by Urbanik and Woyczyński in [44]. They have characterized the spaces of integrable functions as certain Orlicz spaces. Our work arose from questions concerning generalizations of those results Banach spaces.

There are no "natural Banach space generalizations" of the object $\int f dM$, where f is a function and M is a random measure. For instance, the cases where f is a vector valued function and M takes values in a space of real random variables, or conversely, where f is a real function and M takes values in a space of random vectors and many others (see Examples A–J, Section 3.3) can be regarded as "natural Banach space generalizations" of the real random integral. In this paper we consider a more general situation that includes these two examples as particular cases; i.e. we deal with a random integral of the form

$$\int_T f \cdot dM$$

where f is a deterministic function, M takes values in a space of Y-valued random vectors and $\cdot: X \times Y \to Z$ is a bilinear form. Here X, Y and Z are Banach spaces.

According to the theory of vector measures we deal with a Bartle type integral. For the existence of that integral it is usually assumed that the set

$$\{\sum_{j=1}^{n} x_j \cdot M(A_j)\colon \|x_j\| \leqslant 1, \quad \bigcup_{j=1}^{n} A_j = T,\ n \geqslant 1\}$$

is bounded in probability (see [30]). Unfortunately, in the fundamental case where M is generated by increments of a real Wiener process $w(t)$, $t \in [0, 1]$ and "$\cdot$" stands for the usual multiplication of a vector by a scalar, the above condition does not hold in general (see Example 3.4.1). Therefore the Wiener integral of a Banach space valued function can not be obtained using the classical vector measure theory approach (see [48]).

In this paper we adapt Bartle's definition of a bilinear integral [4] to define a random integral and to find necessary and sufficient integrability conditions. Since the fundamental properties of the space of integrable functions depend on $(X, Y, Z, \cdot, M)$, we give our characterization in terms of infinitely divisible probability measures on Banach spaces. This approach to random integrals was used in [8], [38], [37] and [35] in some special cases.

Section 2 contains basic information on infinitely divisible probability measures on Banach spaces and random measures.

General theory of bilinear random integrals is given in Section 3. We prove that the set of all integrable functions is a metrizable complete function space containing simple functions densely (Corollary 3.2.3). We introduce the notion of an abstract random integral (Definition 3.2.7) and we show that our random integral is the largest extension of that object. Subsection 3 contains the main result of this section (Theorem 3.3.6). Weaker forms of random integrals are considered at the end of this subsection. In subsection 4 we develop a martingale method of approximation by simple functions in the space of all integrable functions and give some contraction principles. The first one is expressed in terms of a conditional integral and the second one in terms of multiplication by a bounded real function. In subsection 5 we study stable symmetric random integrals in a general form.

Random integrals of Banach space valued functions with respect to real valued random measures are studied in Section 4. Random integrals of this type constitute a natural generalization of random series with Banach space valued coefficients. Gaussian and stable random integrals are studied in subsection 2. Comparison theorem and its applications for semi-stable measures end the paper.

The author is indebted to prof. K. Urbanik for stimulating discussions and benevolent patronage and to prof. Nguyen Zui Tien for helpful suggestions and discussions during the preparation of this paper.

II. Preliminaries

1. Infinitely divisible probability measures on Banach spaces. In this paper by a measure on a Banach space we shall mean a countably additive Borel set function μ taking values in $[0, \infty]$ and satisfying the tightness condition: for every Borel set B

$$\mu(B) = \sup\{\mu(K)\colon\ K \subset B,\ K \text{ is compact}\}.$$

In this subsection we recall some definitions and facts concerning infinitely divisible laws on Banach spaces, which can be found in [3] Chapter III and in [2].

Let E be a Banach space, E^* its topological dual and $\mathscr{B}(E)$ the field of Borel subsets of E. A probability measure (p.m.) μ on E is said to be *infinitely divisible* if for every n there exists a p.m. μ_n such that

$$\mu = \underbrace{\mu_n * \ldots * \mu_n}_{n\text{-times}}.$$

Gaussian p.m. and a *Poissonian p.m.* are the most fundamental examples of infinitely divisible p.m.'s. The characteristic functional of a Gaussian p.m. μ has the form

$$\hat{\mu}(x^*) = \exp\{i\langle x^*, a\rangle - \tfrac{1}{2}Q(x^*)\}, \qquad x^* \in E^*,$$

where $a \in E$ and $Q\colon E^* \to \boldsymbol{R}_+$ is a quadratic form. The parameters a and Q are called the *mean* and the *variance of* μ, respectively. A quadratic form $Q\colon E^* \to \boldsymbol{R}_+$ is said to be a *Gaussian variance* if there exists a p.m. γ on E such that

$$\hat{\gamma}(x^*) = \exp\{-\tfrac{1}{2}Q(x^*)\}.$$

Now let ν be a finite measure on E. A p.m. $\operatorname{Pois}(\nu)$ is defined by

$$\operatorname{Pois}(\nu) = e^{-\nu(E)} \sum_{k=0}^{\infty} \frac{\nu^{*k}}{k!}$$

and is called a *Poissonian measure generated by* ν.

A measure m on E is said to be a *Lévy measure* if for each $x^* \in E^*$ $\int_E \min\{\langle x^*, x\rangle^2, 1\}\, m(dx) < \infty$, $m(\{0\}) = 0$ and

(2.1) $$\varphi_m(x^*) = \exp\Big\{\int_E [e^{i\langle x^*, x\rangle} - 1 - i\langle x^*, x\rangle \alpha(\|x\|)]\, m(dx)\Big\}$$

is the characteristic functional of some p.m. on E. The function α is defined by

$$\alpha(t) = \begin{cases} 1 & \text{if } t \leqslant 1, \\ t^{-1} & \text{if } t > 1. \end{cases}$$

If m is a Lévy measure then by $c\,\mathrm{Pois}(m)$ we denote the p.m. defined by the equality $c\,\mathrm{Pois}(m)\hat{}(x^*) = \varphi_m(x^*)$, $x^* \in E$. Every finite measure v such that $v(\{0\}) = 0$ is a Lévy measure and

$$c\,\mathrm{Pois}(v) = \delta_a * \mathrm{Pois}(v) \tag{2.2}$$

where $a = -\int_E x\alpha(\|x\|)\,v(dx)$. If μ is a Lévy measure and v is a measure such that $v(\{0\}) = 0$ and $\mu(B) \geqslant v(B)$ for every $B \in \mathscr{B}(E)$ then v is a Lévy measure. Hence it follows easily that a measure m is a Lévy measure if and only if $\tilde{m}(B) = \frac{1}{2}[m(B)+m(-B)]$ is a Lévy measure.

Every infinitely divisible p.m. μ on E has the form

$$\mu = \delta_a * \gamma * c\,\mathrm{Pois}(m) \tag{2.3}$$

where $a \in E$, γ is a symmetric Gaussian measure and m is a Lévy measure. Conversely, every p.m. μ given by (2.3) is infinitely divisible. We shall write

$$\mu = [a, Q, m]$$

if (2.3) holds; here Q is the variance of γ. This representation of μ is unique.

Usually, instead of function α in (2.1), the indicator of a ball with center in the origin of E is used. However, contering by a continuous function is more useful for our purposes since in this case the following lemma can be proven:

LEMMA 2.1.1. *Let $\mu_n = [a^{(n)}, Q^{(n)}, m^{(n)}]$ be infinitely divisible p.m.'s on E, $n \geqslant 0$. If $\mu_n \Rightarrow \mu_0$ then $a^{(n)} \to a^{(0)}$ as $n \to \infty$.*

Proof. Let $\mu = [a, Q, m]$ be an infinitely divisible p.m. and let $\tau \in C(m) = \{r > 0\colon m\{\|x\| = r\} = 0\}$ and $\tau \leqslant 1$. Then we can write

$$\mu = \delta_{a_\tau} * \gamma * c_\tau \mathrm{Pois}\, m,$$

where $a_\tau = a - \int_{\|x\| > \tau} x\alpha(\|x\|)\,m(dx)$ and γ is a Gaussian symmetric p.m. with variance Q (see [2], § 1). By the general converse central limit theorem ([2], Th.2.10) we have

$$a_\tau = \lim_{k\to\infty} k \int_{\|x\| \leqslant \tau} x\mu^{1/k}(dx). \tag{2.4}$$

We turn to the assumptions of the lemma. Let $\tau \in \bigcap_{n=0}^{\infty} C(m^{(n)})$ and $\tau \leqslant 1$. By (2.4) for every $n \geqslant 1$ there exists k_n such that

$$\Big\|a_\tau^{(n)} - k_n \int_{\|x\| \leqslant \tau} x\mu_n^{1/k_n}(dx)\Big\| < 1/n.$$

Since $\mu_n = \mu_n^{1/k_n} * \ldots * \mu_n^{1/k_n}$($k_n$-times), we have by the general converse central limit theorem $k_n \int_{\|x\| \leq \tau} x\mu_n^{1/k_n}(dx) \to a_\tau^{(0)}$ and $m^{(n)}|\{\|x\| > \tau\} \Rightarrow m^{(0)}|\{\|x\| > \tau\}$. Thus

$$a^{(n)} = a_\tau^{(n)} + \int_{\|x\| > \tau} x\alpha(\|x\|)\, m^{(n)}(dx) \to a^{(0)} \quad \text{as } n \to \infty.$$

The next fact concerns the moments of infinitely divisible p.m.'s; it was proved in a more general form in [1], [15] and [24], and it will be frequently used in this paper.

PROPOSITION 2.1.2. *Let μ be an infinitely divisible p.m. and m be its Lévy measure. Then for every $p > 0$ $\int_E \|x\|^p \mu(dx) < \infty$ if and only if $\int_{\|x\| > 1} \|x\|^p m(dx) < \infty$.*

2. Random measures. In this paper, $(\Omega, \mathscr{F}, P)$ will stand for a probability space. A random element in a metric space X is a Borel measurable function $\xi\colon \Omega \to X$ such that there exists a separable topologically complete subset $X_0 \subset X$ such that $P\{\xi \in X_0\} = 1$. Thus the distribution of ξ defined by

$$\mathscr{L}(\xi)(B) = P\{\xi \in B\}, \quad B \in \mathscr{B}(X)$$

is a Radon p.m. on X. By $L^0(\Omega, \mathscr{F}, P; X)$ we shall denote the set of all random elements defined on $(\Omega, \mathscr{F}, P)$ with values in X.

Let $\mathscr{R}$ be a δ-ring of subsets of a set T(i.e. a ring closed under countable intersections) and let G be an abelian metric semigroup. A function

$$M\colon \mathscr{R} \to \mathscr{L}^0(\Omega, \mathscr{F}, P; G)$$

is said to be a *random measure* if for every pairwise disjoint sequence $\{A_n\} \subset \mathscr{R}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$ the series $\sum_{n=1}^{\infty} M(A_n)$ converges in P and

$$\sum_{n=1}^{\infty} M(A_n) = M\Big(\bigcup_{n=1}^{\infty} A_n\Big) \text{ a.s.}$$

By definition, a random measure can be viwed as a stochastic process; hence question on the existence of regular versions are natural and important in this subject. We say a random measure M is regular if for every $\omega \in \Omega$, $M(\cdot)(\omega)$ is a G-valued measure on $(T, \mathscr{R})$. A point process is a good example of a regular random measure; here $G = \{0, 1, 2, \ldots\}$. It is known that if T is a locally compact Polish space, $\mathscr{R}$ is the set of all conditionally compact Borel subsets of T and $G = \boldsymbol{Z}_+$ or $\boldsymbol{R}_+$ then every random measure has a regular version (see [17], Th. 5.4). On the other hand, there are simple examples of random measures having no regular versions. For example, let $\{w(t)\colon 0 \leqslant t \leqslant 1\}$ be a Brownian motion. Define

$$M((a, b)) = w_b - w_a, \quad 0 \leqslant a < b \leqslant 1.$$

Then the function M extends to a random measure on $\mathscr{B}([0, 1])$ (see [34], Part I). M has no regular version, since almost all trajectories of a Brownian motion have infinite variation. A study of such type random measures requires a different approach from one taken e.g. in [17].

Let (T, Σ, λ) be an abstract measure space and $\Sigma_f = \{A \in \Sigma\colon \lambda(A) < \infty\}$. Let $\nu = [a, Q, m]$ be an infinitely divisible p.m. on a Banach space E. We say that

$$M\colon \Sigma_f \to \mathscr{L}^0(\Omega, \mathscr{F}, P; E)$$

is a *random measure on* (T, Σ, λ) *generated by* ν if

(i) M is a random measure on (T, Σ_f);

(ii) for every pairwise disjoint sets $\{A_k\}_{k=1}^n \subset \Sigma_f$ the random vectors $\{M(A_k)\}_{k=1}^n$ are independent;

(iii) $\mathscr{L}(M(A)) = [\lambda(A)a, \lambda(A)Q, \lambda(A)m]$, $A \in \Sigma_f$.

Note that, in view of (i) and (ii), condition (iii) determines finite dimensional distributions of the stochastic process $\{M(A)\colon A \in \Sigma_f\}$. The existence of a random measure on any measure space generated by an arbitrary infinitely divisible p.m. follows from Kolmogorov's consistency theorem.

For brievity we shall write $M \sim [a, Q, m]$ when (i), (ii) and (iii) hold and (T, Σ, λ) is fixed. If M is generated by $\mathrm{Pois}(m)$ then $M \sim [\int_E x\alpha(\|x\|)\, m(dx), 0, m_0]$, where $m_0(B) = m(B \setminus \{0\})$, $B \in \mathscr{B}(E)$. A random measure $M \sim [0, Q, 0]$ is called a *Gaussian* (*symmetric*) *random measure.*

Remark 2.2.1. Since the statements of this paper depend in fact on finite dimensional distributions of the stochastic process $\{M(A)\colon A \in \Sigma_f\}$, and do not depend on a concrete form of M, we can choose such versions of M which are convenient for proofs. Further, we shall use the following versions of the random measure generated by $\nu = [a, Q, m]$. Let $M_g \sim [0, Q, 0]$ and $M_p \sim [a, 0, m]$ be independent random measures on (T, Σ, λ). Then $M = M_g + M_p$ is a random measure on (T, Σ, λ) generated by ν. Now we define a second version of M useful in this paper. Let

$$M_0 \sim [a - \int_{\|x\| > 1} x\|x\|^{-1}\, m(dx), Q, m_0]$$

and

$$M_1 \sim [\int_{\|x\| > 1} x\|x\|^{-1}\, m(dx), 0, m_1]$$

be independent random measures on (T, Σ, λ), where $m_0(B)$

$= m(B \cap \{\|x\| \leqslant 1\})$, $m_1(B) = m(B \cap \{\|x\| > 1\})$, $B \in \mathscr{B}(E)$. Then for every $A \in \Sigma_f$ and $p > 0$, $E\|M_0(A)\|^p < \infty$ (see Proposition 2.1.2), M_1 is a Poissonian random measure generated by m_1 and $M = M_0 + M_1$ is a random measure on (T, Σ, λ) generated by ν.

III. Bilinear random integral

1. Definition and necessary conditions for the existence of a random integral. Let X, Y, Z be real Banach spaces. By x^*, y^*, z^*, possibly with a subspace of $L(Y, Z)$, $x(y) = b(x, y)$, $y \in Y$, the dual operator to x will be Y, Z. Let

$$b\colon X \times Y \to Z$$

be a fixed bilinear continuous form. Since X may be considered as a subspace of $L(Y, Z)$, $x(y) = b(x, y)$, $y \in Y$, the dual operator to x will be denoted by x', $x'\colon Z^* \to Y^*$. For simplicity of notation, the value $b(x, y)$ will be written as $x \cdot y$.

Let (T, Σ, λ) be σ-finite measure space and $\Sigma_f = \{A \in \Sigma\colon \lambda(A) < \infty\}$. Let $\nu = [a, Q, m]$ be an infinitely divisible p.m. on Y and M be a random measure on (T, Σ, λ) generated by ν.

We define the random integral of a simple function. Let $f\colon T \to X$ be a simple function, i.e. $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$, where $x_j \in X$ and $A_j \in \Sigma_f$ are pairwise disjoint. For every $A \in \Sigma$ we put

$$\int_A f \cdot dM = \sum_{j=1}^{n} x_j \cdot M(A_j \cap A).$$

Definition 3.1.1. A function $f\colon T \to X$ is said to be *M-integrable* if there exists a sequence of simple functions $\{f_n\}$ such that

(i) $f_n \to f$ λ-a.e.,

(ii) for every $A \in \Sigma$ the sequence $\{\int_A f_n \cdot dM\}$ converges in P.

If f is M-integrable then we put

$$\int_A f \cdot dM = P - \lim_{n \to \infty} \int_A f_n \cdot dM, \qquad A \in \Sigma,$$

where $\{f_n\}$ satisfies (i) and (ii).

Let

$$K(y^*) = \log \hat{\nu}(y^*) \tag{3.1}$$
$$= i\langle y^*, a\rangle - \tfrac{1}{2} Q(y^*) + \int_Y [e^{i\langle y^*, y\rangle} - 1 - i\langle y^*, y\rangle \alpha(\|y\|)]\, m(dy).$$

Then for a simple function f we have

$$E\exp\{i\langle z^*, \int_T f\cdot dM\rangle\} = \prod_{j=1}^{n} E\exp\{i\langle z^*, x_j\cdot M(A_j)\rangle\}$$
$$= \prod_{j=1}^{n} \exp\{K(x_j' z^*)\lambda(A_j)\}$$
$$= \exp\{\int_T K\circ(f' z^*)\,d\lambda\}.$$

LEMMA 3.1.2. *The random integral is well defined.*

Proof. It is sufficient to show that if a sequence of simple functions $\{f_n\}$ converges λ-a.e. to zero and for every $A\in\Sigma$ the sequence $\{\int_A f_n\cdot dM\}$ converges in P then for every $A\in\Sigma$, $P-\lim_{n\to\infty}\int_A f_n\cdot dM = 0$. Put $N_n(A) = \int_A f_n\cdot dM$, $A\in\Sigma$. N_n is a vector measure in $L^0(\Omega, \mathscr{F}, P; Z)$ absolutely continuous with respect to λ and for every $A\in\Sigma$, $N(A) = \lim_{n\to\infty} N_n(A)$ exists in $L^0(\Omega, \mathscr{F}, P; Z)$. Hence by the Hahn–Saks–Vitali theorem N is a vector measure in $L^0(\Omega, \mathscr{F}, P; Z)$ absolutely continuous with respect to λ. Let $A\in\Sigma$. We decompose A into pairwise disjoint sets, $A = \bigcup_{k=0}^{\infty} A_k$, $A_k\in\Sigma_f$, such that $\lambda(A_0) = 0$ and for every $k\geqslant 1$, $f_n\to 0$ uniformly on A_k as $n\to\infty$. Then for every $z^*\in Z^*$ and $k\geqslant 1$

$$E\exp\{i\langle z^*, N(A_k)\rangle\} = \lim_{n\to\infty}\exp\{\int_{A_k} K\circ(f_n' z^*)\,d\lambda\} = 1.$$

Hence

$$N(A) = N(A_0) + \sum_{k=1} N(A_k) = 0.$$

COROLLARY 3.1.3. *If f is M-integrable then $A\to\int_A f\cdot dM$ is a Z-valued independently scattered random measure on (T, Σ).*

LEMMA 3.1.4. *If f is M-integrable then for every $z^*\in Z^*$ we have $\int_T |K\circ(f' z^*)|\,d\lambda < \infty$ and*

$$\mathscr{L}(\int_T f\cdot dM)^\wedge(z^*) = \exp\{\int_T K\circ(f' z^*)\,d\lambda\}.$$

Proof. Let $\{f_n\}$ satisfy (i) and (ii) of Definition 3.1.1. Fix $z^*\in Z^*$. We have $K\circ(f_n' z^*)\to K\circ(f' z^*)$ λ-a.e. and for every $A\in\Sigma$

$$\mu(A) = \lim_{n\to\infty}\int_A K\circ(f_n' z^*)\,d\lambda = \lim_{n\to\infty}\log\big[\mathscr{L}(\int_A f_n\cdot dM)^\wedge(z^*)\big]$$

exists. By the Hahn–Saks–Vitali theorem μ is a complex valued measure absolutely continuous with respect to λ. Hence, by the Radon–Nikodym theorem, there exists a complex valued function h such that $\int_T |h|\, d\lambda < \infty$ and for every $A \in \Sigma$

$$\int_A h d\lambda = \mu(A).$$

Decompose T into pairwise disjoint sets, $T = \bigcup_{k=0}^{\infty} A_k$, $A_k \in \Sigma_f$, such that $\lambda(A_0) = 0$ and for every $k \geqslant 1$, $K \circ (f_n' z^*) \to K \circ (f' z^*)$ uniformly on A_k as $n \to \infty$. Then for every $A \in \Sigma$ and $k \geqslant 1$

$$\int_{A_k \cap A} h d\lambda = \int_{A_k \cap A} K \circ (f' z^*)\, d\lambda.$$

Hence $K \circ (f' z^*) = h$ λ-a.e., which ends the proof.

By definition, a random integral is a Z-valued random vector with an infinitely divisible distribution. We are going to find its components. First we define a function

$$V(x) = x \cdot a + \int_Y x \cdot y [\alpha(\|x \cdot y\|) - \alpha(\|y\|)]\, m(dy), \qquad x \in X. \tag{3.2}$$

The integral in the above formula exists in the Bochner sense; this follows from the first part of the following lemma.

LEMMA 3.1.5. *Let $v(x, y) = x \cdot y [\alpha(\|x \cdot y\|) - \alpha(\|y\|)]$. Then $v = 0$ provided that $\|y\| \leqslant \min\{(\|b\|\,\|x\|)^{-1}, 1\}$ or $x = 0$, moreover $\|v\| \leqslant 1 + \|b\|\,\|x\|$, where $\|b\| = \sup\{\|x \cdot y\| : \|x\| \leqslant 1, \|y\| \leqslant 1\}$. The function V given by (3.2) is uniformly continuous and bounded on every bounded ball in X.*

Proof. We prove the first assertion. We set

$$A_1 = \{(x, y) : \|x \cdot y\| \leqslant 1, \|y\| \leqslant 1\}, \quad A_2 = \{(x, y) : \|x \cdot y\| > 1, \|y\| \leqslant 1\},$$
$$A_3 = \{(x, y) : \|x \cdot y\| \leqslant 1, \|y\| > 1\}, \quad A_4 = \{(x, y) : \|x \cdot y\| > 1, \|y\| > 1\}.$$

Now we have the following implications:

If $(x, y) \in A_1$ then $v(x, y) = 0$.

If $(x, y) \in A_2$ then $\|v(x, y)\| = \|x \cdot y(\|x \cdot y\|^{-1} - 1)\| \leqslant 1 + \|x \cdot y\| \leqslant 1 + \|b\|\,\|x\|$.

If $(x, y) \in A_3$ then $\|v(x, y)\| = \|x \cdot y(1 - \|y\|^{-1})\| \leqslant \|x \cdot y\| + \|x \cdot y\|\,\|y\|^{-1} \leqslant 1 + \|b\|\,\|x\|$.

If $(x, y) \in A_4$ then $\|v(x, y)\| = \|x \cdot y(\|x \cdot y\|^{-1} - \|y\|^{-1})\| \leqslant 1 + \|b\|\,\|x\|$.

If $\|y\| \leqslant \min\{1, (\|b\|\,\|x\|)^{-1}\}$ then $\|x \cdot y\| \leqslant \|b\|\,\|x\|\,\|y\| \leqslant 1$ and $\|y\| \leqslant 1$, hence $(x, y) \in A_1$ and so $v(x, y) = 0$. The proof of the first assertion this lemma is complete. We prove the second one. Let $r > 0$, $x_{ni} \in X$, $\|x_{ni}\| \leqslant r$, $n \in N$, $i = 1, 2$ and $\|x_{n1} - x_{n2}\| \to 0$. We shall show that $\|V(x_{n1}) - V(x_{n2})\| \to 0$

as $n \to \infty$. Since, by the first assertion, for $\|x\| \leqslant r$ we have

$$\int_Y v(x, y)\, m(dy) = \int_{\|y\| > R} v(x, y)\, m(dy),$$

where $R = \min\{(\|b\|\, r)^{-1}, 1\}$, it is enough to prove that

$$x_{n1}\, \alpha(\|x_{n1} \cdot y\|) - x_{n2}\, \alpha(\|x_{n2} \cdot y\|) \to 0 \quad \text{as } n \to \infty.$$

Using the inequality $|\alpha(t) - \alpha(s)| \leqslant |t - s|$ for $s, t \geqslant 0$, we get

$$\begin{aligned}
&\|x_{n1}\, \alpha(\|x_{n1} \cdot y\|) - x_{n2}\, \alpha(\|x_{n2} \cdot y\|)\| \\
&\quad \leqslant \|(x_{n1} - x_{n2})\, \alpha(\|x_{n1} \cdot y\|)\| + \|x_{n2}\, [\alpha(\|x_{n1} \cdot y\|) - \alpha(\|x_{n2} \cdot y\|)]\| \\
&\quad \leqslant (1 + r\, \|b\|\, \|y\|)\, \|x_{n1} - x_{n2}\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

The uniform boundedness of V for $\|x\| \leqslant r$ follows from the inequality

$$\|V(x)\| \leqslant \|a\|\, \|b\|\, r + (1 + \|b\|\, r)\, m(\{\|y\| > R\}).$$

The proof of the lemma is complete.

We now compute the distribution of a random integral of a simple function $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$, where $x_j \in E$, $A_j \in \Sigma_f$ are pairwise disjoint.

$$\begin{aligned}
\mathscr{L}\Big(\int_T f \cdot dM\Big)^{\wedge}(z^*) &= \prod_{j=1}^{n} E \exp\{i \langle x_j' z^*, M(A_j)\rangle\} \\
&= \exp\Big\{i \sum_{j=1}^{n} \langle z^*, x_j \cdot a\rangle \lambda(A_j) - \tfrac{1}{2} \sum_{j=1}^{n} Q(x_j' z^*)\, \lambda(A_j) \\
&\qquad + \sum_{j=1}^{n} \int_Y [e^{i\langle z^*, x_j \cdot y\rangle} - 1 - i \langle z^*, x_j \cdot y\rangle\, \alpha(\|y\|)]\, m(dy)\, \lambda(A_j)\Big\} \\
&= \exp\Big\{i \Big\langle z^*, \int_T V \circ f\, d\lambda\Big\rangle - \tfrac{1}{2} \int_T Q \circ (f' z^*)\, d\lambda \\
&\qquad + \int_T \int_Y [e^{i\langle z^*, f \cdot y\rangle} - 1 - i \langle z^*, f \cdot y\rangle\, \alpha(\|f \cdot y\|)]\, m(dy)\, d\lambda\Big\}.
\end{aligned}$$

Hence

$$\mathscr{L}\Big(\int_T f \cdot dM\Big) = [a_f, Q_f, \mu_f] \tag{3.3}$$

where

$$\begin{aligned}
a_f &= \int_T V \circ f\, d\lambda, \\
Q_f(z^*) &= \int_T Q \circ (f' z^*)\, d\lambda, \quad z \in Z, \\
\mu_f(B) &= \lambda \times m(\{f \cdot y \in B \setminus \{0\}\}), \quad B \in \mathscr{B}(Z).
\end{aligned} \tag{3.4}$$

We can ask whether representation (3.3) holds for every M-integrable function, i.e. whether Q_f is a Gaussian quadratic form, μ_f is a Lévy measure and how to understand the integral in the definition of a_f. The answer to these questions is not immediate, since even on the real line the convergence of infinitely divisible p.m.'s does not imply the convergence of their Gaussian and centered Poissonian components.

In view of Remark 2.2.1 we may assume in what follows that

$$M = M_g + M_p$$

where M_g and M_p are specified in that remark.

LEMMA 3.1.6. *If f is M-integrable then it is both M_g- and M_p-integrable. Moreover, if a sequence $\{f_n\}$ of simple functions satisfies* (i) *and* (ii) *of Definition* 3.1.1 *then for every $A \in \Sigma$*

$$\int_A f_n \cdot dM_g \to \int_A f \cdot dM_g \quad \text{and} \quad \int_A f_n \cdot dM_p \to \int_A f \cdot dM_p$$

in P as $n \to \infty$.

Proof. Let $m_1 < n_1 < m_2 < n_2 < \ldots$ be natural numbers. Then for every $A \in \Sigma$ we have

$$\int_A (f_{n_k} - f_{m_k}) \cdot dM_g + \int_A (f_{n_k} - f_{m_k}) \cdot dM_p \to 0$$

in P as $k \to \infty$. By the symmetry argument $\int_A (f_{n_k} - f_{m_k}) \cdot dM_g \to 0$ and $\int_A (f_{n_k} - f_{m_k}) \cdot dM_p \to 0$. Hence for every $A \in \Sigma$ the sequences $\{\int_A f_n \cdot dM_g\}$ and $\{\int_A f_n \cdot dM_p\}$ converge in P.

Now let f be an M-integrable function and $\{f_n\}$ be a sequence of simple functions satisfying (i) and (ii) of Definition 3.1.1. By (3.3) we have for every $A \in \Sigma$

$$\mathscr{L}\Big(\int_A f_n \cdot dM\Big) = [a_{f_n 1_A}, Q_{f_n 1_A}, \mu_{f_n 1_A}],$$

where the components on the right are given by (3.4).

LEMMA 3.1.7. *Let f and $\{f_n\}$ as above. Then $V \circ f$ is Pettis integrable and for every $A \in \Sigma$*

$$\int_A V \circ f d\lambda = \lim_{n \to \infty} \int_A V \circ f_n d\lambda.$$

Proof. It was proved by Pettis ([33], Th. 5.1) that Dunford and Pettis integrals coincide. By the continuity of V (Lemma 3.1.5), $V \circ f_n \to V \circ f$ λ-a.e. Hence we finish the proof showing that for every $A \in \Sigma$, $\lim_{n \to \infty} \int_A V \circ f_n d\lambda$ exists.

The sequence $\{\mathscr{L}(\int_A f_n \cdot dM)\}$ converges weakly; thus by Lemma 2.1.1 $a_{f_n 1_A} = \int_A V \circ f_n \, d\lambda$ converge as $n \to \infty$.

LEMMA 3.1.8. *Let f and $\{f_n\}$ be as in Lemma 3.1.7 and let γ_{f_n} be a symmetric Gaussian p.m. on Z with variance Q_{f_n}. Then for every $z^* \in Z^*$, $Q_f(z^*) = \int_T Q \circ (f' z^*) \, d\lambda < \infty$; Q_f is the variance of a Gaussian symmetric p.m. γ_f and*

$$\gamma_{f_n} \Rightarrow \gamma_f \quad \text{as } n \to \infty.$$

Proof. By Lemma 3.1.6. for every $A \in \Sigma$ and $z^* \in Z^*$ $\lim_{n\to\infty} \int_A Q \circ (f_n' z^*) \, d\lambda$ exists. Since $Q \circ (f_n' z^*) \to Q \circ (f' z^*)$ λ-a.e., by a standard application of the Hahn–Saks–Vitali and Radon–Nikodym theorems (see the proof of Lemma 3.1.4) we get that $\int_T Q \circ (f' z^*) \, d\lambda = \lim_{n\to\infty} \int_T Q \circ (f_n' z^*) \, d\lambda$. In view of Lemma 3.1.6 $\gamma_{f_n} = \mathscr{L}(\int_T f_n \cdot dM_g)$ converge to $\gamma_f = \mathscr{L}(\int_T f \cdot dM_g)$; we hence conclude that Q_f is the variance of γ_f.

LEMMA 3.1.9. *Let f and $\{f_n\}$ be as in Lemma 3.1.7. Then μ_f given by* (3.4) *is a Lévy measure and*

$$c\,\mathrm{Pois}(\mu_{f_n}) \Rightarrow c\,\mathrm{Pois}(\mu_f).$$

Proof. For every $n \geqslant 1$ define random measures

$$N_n(A) = \int_A f_n \cdot dM_p - \int_A V \circ f_n \, d\lambda, \quad A \in \Sigma.$$

Then $\mathscr{L}(N_n(A)) = c\,\mathrm{Pois}(\mu_{f_n 1_A})$. By Lemmas 3.1.6 and 3.1.7 we have for every $A \in \Sigma$

$$N_n(A) \to N(A) = \int_A f \cdot dM_p - \int_A V \circ f d\lambda$$

in P. Now we choose an increasing sequence $\{A_k\} \subset \Sigma_f$ such that $A_k \nearrow A_\infty$, $\lambda(T \setminus A_\infty) = 0$ and $A_k \subset \{\|f\| \leqslant k\}$. By Lemmas 3.1.6 and 3.1.4 for every $k \geqslant 1$ we have

$$\begin{aligned} E \exp\{i \langle z^*, N(A_k)\rangle\} &= \exp\Big\{\int_{A_k} \big[\langle z^*, f \cdot y\rangle + \int_Y [e^{i\langle z^*, f\cdot y\rangle} \\ &\qquad - 1 - i\langle z^*, f \cdot y\rangle \alpha(\|y\|)]\, m(dy)\big]\, d\lambda - i\langle z^*, \int_{A_k} V \circ f d\lambda\rangle\Big\} \\ &= \exp\Big\{\int_{A_k} \int_Y [e^{i\langle z^*, f\cdot y\rangle} - 1 - i\langle z^*, f\cdot y\rangle \alpha(\|f \cdot y\|)]\, m(dy)\, d\lambda\Big\}. \end{aligned}$$

Hence $\mu_{f 1_{A_k}}$ is a Lévy measure and $c\,\mathrm{Pois}(\mu_{f 1_{A_k}}) = \mathscr{L}(N(A_k))$. Since $\mu_{f 1_{A_k}} \nearrow \mu_{f 1_{A_\infty}} = \mu_f$ and $N(A_k) \to N(A_\infty) = N(T)$ in P, we conclude that μ_f is a Lévy measure and $c\,\mathrm{Pois}(\mu_f) = \mathscr{L}(N(T))$. This ends the proof.

Summarizing the last three lemmas we get the necessary conditions for the existence of the random integral.

THEOREM 3.1.10. *If f is M-integrable then*

(i) *f is strongly measurable and $a_f = \int_T V \circ f d\lambda$ exists in the Pettis sense;*

(ii) *for every $z^* \in Z^*$, $Q_f(z^*) = \int_T Q \circ (f' z^*) d\lambda < \infty$ and Q_f is a Gaussian variance;*

(iii) *$\mu_f(B) = \lambda \times m(\{f \cdot y \in B \setminus \{0\}\})$, $\mathscr{B} \in \mathscr{B}(Z)$, is a Lévy measure.*

Moreover,

$$\mathscr{L}(\int_T f \cdot dM) = [a_f, Q_f, \mu_f].$$

COROLLARY 3.1.11. *Let $H_m(x) = m(\{y \in Y: \|x \cdot y\| > 1\})$, $x \in X$. If f is M-integrable then for every $\varepsilon > 0$ $\int_T H_m(\varepsilon f) d\lambda < \infty$. If for M-integrable functions $\{f_n\}$, $\int_T f_n \cdot dM \to 0$ in P then for every $\varepsilon > 0$, $\int_T H_m(\varepsilon f_n) d\lambda \to 0$ as $n \to \infty$.*

Proof. By (iii) of Theorem 3.1.10 we have

$$\int_T H_m(\varepsilon f) d\lambda = \mu_f(\|z\| > \varepsilon^{-1}\}) < \infty.$$

For the proof of the second statement observe that for every $\varepsilon > 0$, $\mu_{f_n}(\{\|z\| > \varepsilon^{-1}\}) \to 0$ by Th.1.10 in [2].

2. Topology in the space of M-integrable functions. Let $\mathscr{L}(M)$ denote the set of all M-integrable functions $f: T \to X$, where functions which are equal λ-a.e. are regarded as equal. $\mathscr{L}(M)$ is a linear space.

DEFINITION 3.2.1. Let $f_n, f \in \mathscr{L}(M)$, $n \geq 1$. We say that $\{f_n\}$ *converges to f in $\mathscr{L}(M)$ if $f_n \to f$* in λ on every set of finite measure λ and for every $A \in \Sigma$, $\int_A f_n \cdot dM \to \int_A f \cdot dM$ in P.

Let $\|\cdot\|_{L^0(\lambda;Z)}$ and $\|\cdot\|_{L^0(P;Z)}$ be F-norms in $L^0(T, \Sigma, \lambda; X)$ (equipped with the topology of convergence in λ on every set of finite measure) and $L^0(\Omega, \mathscr{F}, P; Z)$, respectively. Define an F-norm in $\mathscr{L}(M)$ by

$$\|f\|_M = \|f\|_{L^0(\lambda;X)} + \sup_{A \in \Sigma} \left\| \int_A f \cdot dM \right\|_{L^0(P;Z)}, \quad f \in \mathscr{L}(M).$$

THEOREM 3.2.2. *Let $f_n, f \in \mathscr{L}(M)$, $n \geq 1$. Then $f_n \to f$ in $\mathscr{L}(M)$ if and only if $\|f_n - f\|_M \to 0$ as $n \to \infty$.*

Proof. Put $h_n = f_n - f$. It is enough to prove that if $h_n \to 0$ in $\mathscr{L}(M)$ then $\|h_n\|_M \to 0$. First we show that $\sup_{A \in \Sigma} \left\| \int_A V \circ h_n d\lambda \right\| \to 0$ Since, for every $A \in \Sigma$, $h_n \mathbf{1}_A$ is M-integrable, we get by Theorem 3.1.10 (i) and Lemma 2.1.1 that $\int_A V \circ h_n d\lambda \to 0$. Let $\tilde{\lambda}$ be a finite measure on (T, Σ) equivalent to λ. Take

$\varepsilon > 0$. By the Hahn–Saks–Vitali theorem there exists $\delta > 0$ such that if $\tilde{\lambda}(A) < \delta$ then $\sup_n \|\int_A V \circ h_n \, d\lambda\| < \varepsilon$. By the Egorov theorem there exists $A_\varepsilon \in \Sigma_f$ such that $\tilde{\lambda}(T \setminus A_\varepsilon) < \delta$ and $h_n \to 0$ uniformly on A_ε. Hence for every $A \in \Sigma$

$$\|\int_A V \circ h_n \, d\lambda\| \leqslant \varepsilon + \|\int_{A \cap A_\varepsilon} V \circ h_n \, d\lambda\| \leqslant \varepsilon + \int_{A_\varepsilon} \|V \circ h_n\| \, d\lambda,$$

and by the Dominated Convergence Theorem we get $\limsup_{n \to \infty} \sup_{A \in \Sigma} \|\int_A V \circ h_n \, d\lambda\| \leqslant \varepsilon$. We have showed that $\sup_{A \in \Sigma} \|\int_A V \circ h_n \, d\lambda\| \to 0$ as $n \to \infty$. Now let

$$N_n(A) = \int_A h_n \cdot dM - \int_A V \circ h_n \, d\lambda, \quad A \in \Sigma.$$

To end the proof it suffices to show that

$$c_n = \sup_{A \in \Sigma} \|N_n(A)\|_{L^0(P;Z)} \to 0 \text{ as } n \to \infty.$$

For every n choose $A_n \in \Sigma$ such that

$$c_n < \|N_n(A_n)\|_{L^0(P;Z)} + 1/n.$$

Since $N_n(T) \to 0$ in P, there exist $z_n \in Z$ such that $N_n(A_n) + z_n \to 0$ in P. We have $\mathscr{L}[N_n(A_n) + z_n] = [z_n, Q_{h_n 1_{A_n}}, \mu_{h_n 1_{A_n}}]$; therefore by Lemma 2.1.1 $z_n \to 0$ and so $c_n \to 0$ as $n \to \infty$. The proof of the theorem is complete.

COROLLARY 3.2.3. *$\mathscr{L}(M)$ is a Fréchet space such that the natural embedding of $\mathscr{L}(M)$ into $L^0(\Omega, \mathscr{F}, P; Z)$ is continuous and the set of all simple functions lies in $\mathscr{L}(M)$ densely. Moreover, the mapping*

$$\mathscr{L}(M) \ni f \to \int_T f \cdot dM \in L^0(\Omega, \mathscr{F}, P; Z)$$

is linear and continuous.

Condition (ii) in Definition 3.1.1 is rather complicated for checking. The next theorem shows that it may be reduced to the stochastic convergence of the random integrals on all space T and certain convergence of Pettis integrals (conditions (iii) and (ii) below).

THEOREM 3.2.4. *Let $f_n \in \mathscr{L}(M)$ and f: $T \to X$. Then $f \in \mathscr{L}(M)$ and $f_n \to f$ in $\mathscr{L}(M)$ if and only if*

(i) *$f_n \to f$ in λ on every set of finite measure,*

(ii) *for every $A \in \Sigma$, $\lim_{n,m \to \infty} \int_A V \circ (f_n - f_m) \, d\lambda = 0$,*

(iii) *the sequence $\{\int_T f_n \cdot dM\}$ converges in P.*

Proof. Assume that $f_n \to f$ in $\mathscr{L}(M)$. Since for every $A \in \Sigma$ $\mathscr{L}(\int_A (f_n - f_m) \cdot dM) \Rightarrow \delta_0$, we get (ii) by Theorem 3.1.10 (i) and Lemma 2.1.1.

(i) and (iii) follow by Definition 3.2.1. We now prove the converse. By Corollary 3.2.3 it is enough to show that $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}(M)$. Let $A \in \Sigma$ and $m_1 < n_1 < m_2 < n_2 < \ldots$ be natural numbers. Put $g_k = f_{n_k} - f_{m_k}$, $k \geq 1$. Since

$$\int_A g_k \cdot dM + \int_{T \setminus A} g_k \cdot dM = \int_T g_k \cdot dM \to 0 \quad \text{in } P$$

by (iii), there exist $z_k \in Z$ such that $\int_A g_k \cdot dM + z_k \to 0$ in P and by Theorem 3.1.10 (i) and Lemma 2.1.1 we have $\int_A V \circ g_k \, d\lambda + z_k \to 0$. In view of (ii), $z_k \to 0$, and therefore $\int_A g_k \cdot dM \to 0$ in P, which ends the proof.

The next theorem is useful in the case of $\lambda(T) = \infty$.

THEOREM 3.2.5. *$f \in \mathscr{L}(M)$ if and only if there exists an increasing sequence $\{A_n\} \subset \Sigma$ such that $\lambda(T \setminus \bigcup A_n) = 0$ and*

(i) *$f\mathbf{1}_{A_n} \in \mathscr{L}(M)$ for every n,*

(ii) *$\{\mathscr{L}(\int_T f\mathbf{1}_{A_n} \cdot dM)\}$ is shift relatively compact,*

(iii) *$V \circ f$ is Pettis integrable.*

Proof. Put $f_n = f\mathbf{1}_{A_n}$, $S_n = \int_T f_n \cdot dM$. Since $\{S_n\}$ has independent increments, by the Ito–Nisio theorem ([13], Th. 3) there exists a sequence $\{z_n\} \subset Z$ such that $\{S_n + z_n\}$ converges a.s.. In view of Theorem 3.1.10, $\mathscr{L}(S_n + z_n) = [a_{f_n} + z_n, Q_{f_n}, \mu_{f_n}]$, where $a_{f_n} = \int_{A_n} V \circ f d\lambda$; hence by Lemma 2.1.1, $\{a_{f_n} + z_n\}$ converges. (iii) yields $a_{f_n} = \int_T V \circ f d\lambda$ as $n \to \infty$; therefore $\{z_n\}$ converges and finally, $\{S_n\}$ converges a.s.. To complete the proof it is sufficient to check condition (ii) of Theorem 3.2.4. Since $A \to \int_A V \circ f d\lambda$ is a vector measure, we have

$$\int_A V \circ (f_n - f_m) \, d\lambda = \int_{A \cap (A_n \triangle A_m)} V \circ f d\lambda \to 0$$

as $n, m \to \infty$, which ends this proof.

Using similar arguments as in Lemmas 3.1.6, 3.1.7, 3.1.8 and 3.1.9 we can get the following theorem whose proof is omitted.

THEOREM 3.2.6. *Let $f_n \in \mathscr{L}(M)$*

$$\mathscr{L}(\int_T f_n \cdot dM) = \delta_{a_{f_n}} * \gamma_{f_n} * c \operatorname{Pois}(\mu_{f_n}), \quad n = 0, 1, \ldots$$

If $f_n \to f_0$ in $\mathscr{L}(M)$ then

(i) $a_{f_n} \to a_{f_0}$,

(ii) $\gamma_{f_n} \to \gamma_{f_0}$,

(iii) $c \operatorname{Pois}(\mu_{f_n}) \Rightarrow c \operatorname{Pois}(\mu_{f_0})$.

Now we introduce the notion of an abstract random integral to compare it with our random integral.

DEFINITION 3.2.7. Let F be a linear subspace of $L^0(T, \Sigma, \lambda; X)$ such that

(a) F is a metrizable vector space such that the natural embedding of F into $L^0(T, \Sigma, \lambda; X)$ is continuous,

(b) the set of all simple functions is contained in F as a dense subset,

(c) for every sequence of simple functions $\{f_n\}$ and $A \in \Sigma$, if $f_n \to 0$ in F then $f_n \mathbf{1}_A \to 0$ in F.

Any linear continuous mapping

$$I\colon F \to L^0(\Omega, \mathscr{F}, P; Z)$$

such that for every simple function $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$ we have

$$I(f) = \sum_{j=1}^{n} x_j \cdot M(A_j)$$

is called an *abstract random integral.*

Having X, Y, Z, b and M fixed, one can study the set $\mathscr{I}$ of all abstract random integrals. Let us introduce an order in $\mathscr{I}$. For $I_1, I_2 \in \mathscr{I}$ we shall write $I_1 \prec I_2$ if the domain of I_1 is contained in the domain of I_2 and the natural embedding is continuous. It is easy to see that if $I_1 \prec I_2$ then I_2 is an extension of I_1. The next theorem says that our random integral is the largest extension of abstract random integrals; moreover it is the greatest element in $\mathscr{I}$ with respect to the above order.

THEOREM 3.2.8. *Let I be an abstract random integral with a domain F. Then $F \subset \mathscr{L}(M)$, the natural embedding is continuous and for every $f \in F$, $I(f) = \int_T f \cdot dM$.*

Proof. Let $f \in F$. By (a) and (b) there exists a sequence of simple functions $\{f_n\}$ such that $f_n \to f$ in F and in $L^0(T, \Sigma, \lambda; X)$. Let $A \in \Sigma$ and $m_1 < n_1 < m_2 < n_2 < \ldots$ be natural numbers. Since $f_{n_k} - f_{m_k} \to 0$ in F as $k \to \infty$, we have by (c) $(f_{n_k} - f_{m_k}) \mathbf{1}_A \to 0$ in F. By the continuity of I we get

$$\int_A f_{n_k} \cdot dM - \int_A f_{m_k} \cdot dM = I[(f_{n_k} - f_{m_k}) \mathbf{1}_A] \to 0$$

in P. Thus $f \in \mathscr{L}(M)$. Now we prove that the embedding of F into $\mathscr{L}(M)$ is continuous. Let $f_n \to 0$ in F. For every n there exists a sequence of simple functions $\{g_{n_k}\}_{k=1}^{\infty}$ such that $g_{n_k} \to f_n$ in F and in $L^0(T, \Sigma, \lambda; X)$ as $k \to \infty$. By (c) and the continuity of I we get for every $A \in \Sigma$ and $n \geqslant 1$

$$\int_A (g_{nk} - g_{nj}) \cdot dM = I[(g_{nk} - g_{nj}) \mathbf{1}_A] \to 0$$

in P as $k, j \to \infty$. Therefore using Theorem 3.2.2 we infer that $\{g_{nk}\}_{k=1}^{\infty}$ is a

Cauchy sequence in $\mathscr{L}(M)$ and by the completeness of $\mathscr{L}(M)$ there exists $h_n \in \mathscr{L}(M)$ such that $g_{nk} \to h_n$ in $\mathscr{L}(M)$ as $k \to \infty$. But $g_{nk} \to h_n$ also in $L^0(T, \Sigma, \lambda; X)$; hence $h_n = f_n$ λ-a.e.. Consequently, for every $n \geq 1$ we can find k_n such that $\|f_n - g_{nk_n}\|_F < 1/n$ and $\|f_n - g_{nk_n}\|_M < 1/n$. Because $g_{nk_n} \to 0$ in F, we shall finish the proof showing that $g_{nk_n} \to 0$ in $\mathscr{L}(M)$. But this follows easily by (a), (c) and the continuity of I, since our random integral and the abstract random integral coincide on simple functions. The proof is complete.

In view of the last theorem we can say that the essential difference between abstract random integrals lies in the topologies in which we complete the set of simple functions. The topology of $\mathscr{L}(M)$ is the weakest admissible topology. Taking stronger topologies we get stronger properties of the random integral, e.g. continuity in the pth mean etc. It is interesting to compare results from [43] and [44] where random integrals were considered as continuous operators taking values in $L^1(\Omega, \mathscr{F}, P)$ and $L^0(\Omega, \mathscr{F}^0, P)$, respectively.

3. Characterization of M-integrable functions. In the case of $X = Y = Z = \boldsymbol{R}$ and M being symmetric, Urbanik and Woyczyński [44] have characterized $\mathscr{L}(M)$ to be a certain Orlicz space. In the vector case the structure of $\mathscr{L}(M)$ is more complicated. For example, considering the random integral with respect to a white noise on $[0, 1]$ of $C[0, 1]$-valued functions we arrive at the conclusion that $\mathscr{L}(M)$ does not include all bounded measurable functions and moreover it is not locally convex (see Ch. IV, Ths 4.2.2 and 4.2.13). In the real case $\mathscr{L}(M) = L^2[0, 1]$. To get a general characterization of elements of $\mathscr{L}(M)$, we resort to the theory of infinitely divisible p.m.'s on Banach spaces. Our characterization is given in the terms of Gaussian covariances and Lévy measures; more precisely, we prove the converse to Theorem 3.1.10.

First we prove several auxiliary lemmas, which can be of interest for themselves.

LEMMA 3.3.1. *Let* $\lambda(T) < \infty$ *and* $E\|M(T)\| < \infty$. *Then for every* $A, B \in \Sigma$, $A \subset B$, *we have*

$$E[M(A)|M(B)] = \frac{\lambda(A)}{\lambda(B)} M(B) \quad a.s.$$

Proof. It suffices to show that for every $y_1^*, y_2^* \in Y^*$

$$E\langle y_1^*, M(A)\rangle e^{i\langle y_2^*, M(B)\rangle} = \frac{\lambda(A)}{\lambda(B)} E\langle y_1^*, M(B)\rangle e^{i\langle y_2^*, M(B)\rangle}.$$

Put

$$f(y^*, C) = E\exp\{i\langle y^*, M(C)\rangle\} = \exp\{\lambda(C)K(y^*)\},$$

where K is defined by (3.1). Then

$$E\langle y_1^*, M(C)\rangle e^{i\langle y_2^*, M(C)\rangle} = -i\nabla_{y_1^*} f(y_2^*, C)$$
$$= -i\lim_{t\to 0}\frac{f(y_2^*+ty_1^*, C)-f(y_2^*, C)}{t} = -if(y_2^*, C)\lambda(C)\nabla_{y_1^*}K(y_2^*).$$

Thus

$$\begin{aligned} E\langle y_1^*, M(A)\rangle e^{i\langle y_2^*, M(B)\rangle} &= E\langle y_1^*, M(A)\rangle e^{i\langle y_2^*, M(A)\rangle} f(y_2^*, B\setminus A)\\ &= -if(y_2^*, A)\lambda(A)\nabla_{y_1^*}K(y_2^*) f(y_2^*, B\setminus A)\\ &= -if(y_2^*, B)\lambda(A)\nabla_{y_1^*}K(y_2^*)\\ &= \frac{\lambda(A)}{\lambda(B)} E\langle y_1^*, M(B)\rangle e^{i\langle y_2^*, M(B)\rangle}. \end{aligned}$$

Let Σ_0 be a sub-σ-field of Σ. By $M(\Sigma_0)$ we shall denote the sub-σ-field of $\mathscr{F}$ generated by random vectors $M(A)$, $A\in\Sigma_0$. In the case of $\lambda(T)<\infty$, $\lambda(f|\Sigma_0)$ will denote the conditional integral of a Bochner integrable function f given Σ_0.

LEMMA 3.3.2. *Assume that* $\lambda(T)<\infty$ *and* $E\|M(T)\|^p<\infty$, $p\geqslant 1$. *Let* $f\colon T\to X$ *be a Bochner integrable function and* $\{\Sigma_n\}$ *be an increasing sequence of finite sub-*σ*-fields of* Σ. *Then* $\{\int_T \lambda(f|\Sigma_n)\cdot dM;\, M(\Sigma_n)\}$ *is a martingale in* $L^p(\Omega, \mathscr{F}, P; Z)$.

Proof. $\lambda(f|\Sigma_n)$ is M-integrable because it is a simple function. Let $\{A_{nj}\}_{j=1}^{k_n}$ denote that set of all atoms of Σ_n. Decompose $\{1, \ldots, k_{n+1}\}$ into non-empty subsets

$$J_{nk} = \{1\leqslant j\leqslant k_n\colon A_{n+1,j}\subset A_{nk}\},\quad k=1,\ldots,k_n.$$

Put $x_{nj} = \dfrac{1}{\lambda(A_{nj})}\displaystyle\int_{A_{nj}} fd\lambda$ if $\lambda(A_{nj})>0$ and $x_{nj}=0$ in the other case. Then by Lemma 3.3.1 we get

$$\begin{aligned} &E\big[\int_T \lambda(f|\Sigma_{n+1})\cdot dM\,|\,M(\Sigma_n)\big]\\ &\qquad= \sum_{j=1}^{k_{n+1}} E[x_{n+1,j}\cdot M(A_{n+1,j})|M(\Sigma_n)]\\ &\qquad= \sum_{k=1}^{k_n}\sum_{j\in J_{nk}} E[x_{n+1,j}\cdot M(A_{n+1,j})|M(A_{nk})]\\ &\qquad= \sum_{k=1}^{k_n}\sum_{j\in J_{nk}} x_{n+1,j}\frac{\lambda(A_{n+1,j})}{\lambda(A_{nk})}\cdot M(A_{nk}) = \int_T \lambda(f|\Sigma_n)\cdot dM \quad \text{a.s.} \end{aligned}$$

LEMMA 3.3.3. *Assume that* $\lambda(T) < \infty$ *and* $E\|M(T)\| < \infty$. *Then there exists a constant* C *such that for every simple function* $f\colon T \to X$ *and* $z^* \in Z^*$ *we have*

$$E\left|\langle z^*, \int_T f\cdot dM\rangle\right| \leqslant C\|z^*\|\left[\left(\int_T \|f\|^2 d\lambda\right)^{1/2} + \int_T \|f\|\, d\lambda\right].$$

Proof. We may assume that $M = M_0 + M_1$ where M_0 and M_1 are specified in Remark 2.2.1. We have

$$EM_0(A) = \lambda(A)\{a - \int_{\|y\|>1} y\|y\|^{-1} m(dy)\} = \lambda(A)\, a_1.$$

Put $\tilde{M}_0(A) = M_0(A) - \lambda(A)\, a_1$. Hence we get the following decomposition of M,

$$M(A) = \lambda(A)\, a_1 + \tilde{M}_0(A) + M_1(A),\ A \in \Sigma.$$

Let $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$. We have

$$\left\|\int_T f\cdot d(\lambda a_1)\right\| = \left\|\sum_{j=1}^{n} x_j\cdot \lambda(A_j)\, a_1\right\| \leqslant \|b\|\,\|a_1\| \int_T \|f\|\, d\lambda,$$

$$E\left\|\int_T f\cdot dM_1\right\| \leqslant \|b\| \sum_{j=1}^{n} \|x_j\|\, E\|M_1(A_j)\| \leqslant \|b\| \int_{\|y\|>1} \|y\|\, m(dy) \int_T \|f\|\, d\lambda$$

and

$$E\left\langle z^*, \int_T f\cdot d\tilde{M}_0\right\rangle^2 = \sum_{j=1}^{n} E\langle z^*, x_j\cdot \tilde{M}_0(A_j)\rangle^2 = \sum_{j=1}^{n} E\langle x_j' z^*, \tilde{M}_0(A_j)\rangle^2.$$

Now,

$$E\langle y^*, \tilde{M}_0(A)\rangle^2 = -\left[\frac{d^2}{dt^2} E e^{i\langle ty^*, \tilde{M}_0(A)\rangle}\right]_{t=0}$$
$$= \left[Q(y^*) + \int_{\|y\|\leqslant 1} \langle y^*, y\rangle^2 m(dy)\right]\lambda(A) \leqslant C_0^2 \|y^*\|^2 \lambda(A)$$

where C_0 depends only on Q and m (see [49], Th. 1A). Thus

$$E\left\langle z^*, \int_T f\cdot d\tilde{m}_0\right\rangle^2 \leqslant C_0^2 \|b\|^2 \|z^*\|^2 \int_T \|f\|^2 d\lambda$$

and finally we get the required inequality with $C = \|b\|\,\|a_1\| + C_0\|b\| + \|b\| \int_{\|y\|>1} \|y\|\, m(dy)$.

LEMMA 3.3.4. *Assume that* $\lambda(T) < \infty$ *and* $E\|M(T)\| < \infty$. *Let* $f\colon T \to X$ *be a bounded strongly measurable function such that* Q_f *and* μ_f *given by* (3.4) *are a Gaussian variance and a Lévy measure, respectively. Then* $f \in \mathscr{L}(M)$.

Moreover, if $\{\Sigma_n\}$ is an increasing sequence of finite sub-σ-fields of Σ such that the λ-completion of $\sigma(\bigcup_{n=1}^{\infty}\Sigma_n)$ contains $f^{-1}(\mathscr{B}(X))$ then for every $p>0$

$$\lambda(f\,|\,\Sigma_n)\to f \text{ a.e. and in } L^p(T,\Sigma,\lambda;\,X)$$

and for every $A\in\Sigma$

$$\int_A \lambda(f\,|\,\Sigma_n)\cdot dM \to \int_A f\cdot dM \text{ in } L^1(\Omega,\mathscr{F},P;\,Z).$$

Proof. Since $\lambda(f\,|\,\Sigma_n)$ is a simple function, it is M-integrable. $\lambda(f\,|\,\Sigma_n)\to f$ a.e. and in $L^p(T,\Sigma,\lambda;X)$ by the well-known martingale convergence theorem. Put $U_n=\int_T \lambda(f\,|\,\Sigma_n)\cdot dM$. In view of Lemma 3.3.3 for every $z^*\in Z^*$ the sequence $\{\langle z^*, U_n\rangle\}$ converges in $L^1(\Omega,\mathscr{F},P)$. Write $L(z^*)=\lim_{n\to\infty}\langle z^*, U_n\rangle$. By Dominated Convergence Theorem we get

$$E\exp\{iL(z^*)\}=\exp\{i\langle z^*, a_f\rangle-\tfrac{1}{2}Q_f(z^*)$$
$$+\int_Z [e^{i\langle z^*,z\rangle}-1-i\langle z^*, z\rangle\alpha(\|z\|)]\,\mu_f(dz)\}.$$

Hence by our assumption L is decomposable, i.e. there exists a random vector $U\colon\Omega\to Z$ such that $\langle z^*, U\rangle=L(z^*)$ a.s. for every $z^*\in Z^*$. We now prove that $E\|U\|<\infty$. We have

$$\int_{\|z\|>1}\|z\|\,\mu_f(dz)=\iint_{\{\|f\cdot y\|>1\}}\|f\cdot y\|\,m(dy)\,d\lambda\leqslant\iint_{\|y\|>k^{-1}}k\|y\|\,m(dy)\,d\lambda$$
$$=k\lambda(T)\int_{\|y\|>k^{-1}}\|y\|\,m(dy),$$

where $k=\|b\|\operatorname{ess\,sup}_T\|f\|$. By Lemma 2.1.2, $\int_{\|z\|>1}\|z\|\,\mu_f(dz)<\infty$ and so $E\|U\|<\infty$. Using Lemma 3.3.2 we get for every $z^*\in Z^*$ and $n\geqslant 1$

$$\langle z^*, E(U\,|\,M(\Sigma_n))\rangle=E(L(z^*)\,|\,M(\Sigma_n))$$
$$=\lim_{n\to\infty}E[\langle z^*, U_m\rangle\,|\,M(\Sigma_n)]$$
$$=\lim_{m\to\infty}\langle z^*, E[U_m\,|\,M(\Sigma_n)]\rangle$$
$$=\langle z^*, U_n\rangle \text{ a.s.}$$

Thus $E(U\,|\,M(\Sigma_n))=U_n$ a.s. and by the martingale convergence theorem $U_n\to U$ a.s. and in $L^1(\Omega,\mathscr{F},P;Z)$. Now, as in the proof of Lemma 3.3.3, we may assume that $M(A)=\lambda(A)a_1+\tilde{M}_0(A)+M_1(A)$, $A\in\Sigma$. Since for every simple function $g\colon T\to X$ and $A\in\Sigma$ we have

$$\left\|\int_A g\cdot d(\lambda a_1)\right\|\leqslant\|b\|\,\|a_1\|\int_A\|g\|\,d\lambda,$$

$$E\left\|\int_A g\cdot dM_1\right\|\leqslant\|b\|\int_{\|y\|>1}\|y\|\,m(dy)\int_A\|g\|\,d\lambda$$

and

$$E\left\|\int_A g\cdot d\tilde{M}_0\right\| \leqslant E\left\|\int_T g\cdot d\tilde{M}_0\right\| \leqslant E\left\|\int_T g\cdot dM\right\| + \left\|\int_T g\cdot d(\lambda a_1)\right\| + E\left\|\int_T g\cdot dM_1\right\|,$$

it follows that $\lim_{n,m\to\infty} E\left\|\int_A [\lambda(f|\Sigma_n)-\lambda(f|\Sigma_m)]\cdot dM\right\| = 0$, which completes the proof.

LEMMA 3.3.5. *Assume that $\lambda(T) < \infty$ and let M be generated by* Pois(m) $(m(Y) < \infty)$. *Then*

$$\mathscr{L}(M) = L^0(T, \Sigma, \lambda; X)$$

and $\mathscr{L}(\int_T f\cdot dM) = \operatorname{Pois}(\mu_f)$, *where μ_f is given by* (3.4).

Proof. Let

$$F(t) = \int_Y \min\{t\|y\|, 1\}\, m(dy), \qquad t \geqslant 0.$$

F is a subadditive bounded non-decreasing continuous function and for every $A\in\Sigma$ we have

$$\begin{aligned} E\min\{t\|M(A)\|, 1\} &= e^{-\lambda(A)m(Y)} \sum_{k=0}^{\infty} \frac{\lambda^k(A)}{k!} \int_Y \min\{t\|y\|, 1\}\, m^{*k}(dy) \\ &\leqslant e^{-\lambda(A)m(Y)} \sum_{k=0}^{\infty} \frac{\lambda^k(A)}{k!} k m^{k-1}(Y) \int_Y \min\{t\|y\|, 1\}\, m(dy) \\ &= F(t)\,\lambda(A). \end{aligned}$$

Hence, for every simple function $f = \sum_{j=1}^{n} x_1 \mathbf{1}_{A_j}$ and $A\in\Sigma$ we get

$$\begin{aligned} E\min\left\{\left\|\int_A f\cdot dM\right\|, 1\right\} &\leqslant \sum_{j=1}^{n} E\min\{\|x_j\cdot M(A\cap A_j)\|, 1\} \\ &\leqslant \sum_{j=1}^{n} F(\|b\|\,\|x_j\|)\,\lambda(A\cap A_j) \\ &= \int_A F(\|b\|\,\|f\|)\, d\lambda. \end{aligned}$$

Let $f\in L^0(T, \Sigma, \lambda; X)$. There exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ a.e. Therefore, for every $A\in\Sigma$ we have

$$\limsup_{n,m\to\infty} E\min\left\{\left\|\int_A (f_n - f_m)\cdot dM\right\|, 1\right\} \leqslant \limsup_{n,m\to\infty} \int_A F(\|b\|\,\|f_n - f_m\|)\, d\lambda = 0,$$

which proves that $f\in\mathscr{L}(M)$. Since the inequality

$$E\min\{\|\int_A f\cdot dM\|, 1\}\leqslant \int_A F(\|b\|\,\|f\|)\,d\lambda$$

holds true for every strongly measurable function f, we conclude that the topologies of $\mathscr{L}(M)$ and of $L^0(T,\Sigma,\lambda;X)$ coincide.

Now we can prove the main result of this chapter.

THEOREM 3.3.6. *A function $f\colon T\to X$ is M-integrable if and only if*

(i) *f is strongly measurable and $a_f=\int_T V\circ f d\lambda$ exists in the Pettis sense,*

(ii) *for every $z^*\in Z^*$, $Q_f(z^*)=\int_T Q\circ(f'z^*)\,d\lambda<\infty$ and Q_f is a Gaussian covariance,*

(iii) *$\mu_f(B)=\lambda\times m(\{f\cdot y\in B\setminus\{0\}\})$, $B\in\mathscr{B}(Z)$ is a Lévy measure.*

Moreover,

$$\mathscr{L}(\int_T f\cdot dM)=[a_f, Q_f, \mu_f].$$

Proof. Necessity follows directly from Theorem 3.1.10. We prove sufficiency. In view of Remark 2.2.1 we may assume that $M=M_0+M_1$, where M_0 and M_1 are specified in that remark. Let $\{A_n\}\subset\Sigma$ be an increasing sequence of subsets of T such that $\lambda(A_n)<\infty$, $\lambda(T\setminus\bigcup_{n=1}^{\infty}A_n)=0$ and $A_n\subset\{\|f\|\leqslant n\}$. Since $Q_{f1_{A_n}}\leqslant Q_f$ and $\mu_{f1_{A_n}}\leqslant\mu_f$, we see that $Q_{f1_{A_n}}$ is a Gaussian variance and $\mu_{f1_{A_n}}$ is a Lévy measure. Hence by Lemma 3.3.4 $f\mathbf{1}_{A_n}\in\mathscr{L}(M_0)$, and so there exists a sequence of simple functions $\{g_{nk}\}_{k=1}^{\infty}$ vanishing outside A_n and such that $g_{nk}\to f\mathbf{1}_{A_n}$ in $\mathscr{L}(M_0)$ as $k\to\infty$. In view of Lemma 3.3.5, $f\mathbf{1}_{A_n}\in\mathscr{L}(M_1)$ and for every $A\in\Sigma$ we have $\int_A g_{nk}\cdot dM_1\to\int_A f\mathbf{1}_{A_n}\cdot dM_1$ in P. Hence the sequence $\{\int_A g_{nk}\cdot dM\}_{k=1}^{\infty}$ converges in P, which proves that $f\mathbf{1}_{A_n}\in\mathscr{L}(M)$. Since

$$[a_{f1_{A_n}}, Q_{f1_{A_n}}, \mu_{f1_{A_n}}]*[a_{f1_{T\setminus A_n}}, Q_{f1_{T\setminus A_n}}, \mu_{f1_{T\setminus A_n}}]=[a_f, Q_f, \mu_f],$$

$\{\mathscr{L}(\int_T f\mathbf{1}_{A_n}\cdot dM)\}$ is shift relatively compact and by Theorem 3.2.5 $f\in\mathscr{L}(M)$. This ends the proof.

COROLLARY 3.3.7. *If f is M-integrable then $\int f\cdot dM$ is a $\sigma\{M(A)$: $A\in f^{-1}(\mathscr{B}(X))\}$-measurable random vector.*

Proof. Indeed, by Lemma 3.3.4 $f\mathbf{1}_{A_n}$ may be approximated by its conditional integrals given finite sub-σ-fields of $f^{-1}(\mathscr{B}(X))$ (see the proof of Theorem 3.3.6).

COROLLARY 3.3.8. *Assume that* $M \sim [a, 0, m]$. *If* $f\colon T \to X$ *is strongly measurable,* $V \circ f$ *is Pettis integrable and*

$$\int_T \int_Y \min\{\|f \cdot y\|, 1\}\, m(dy)\, d\lambda < \infty$$

then $f \in \mathscr{L}(M)$.

Proof. We have

$$\int_Z \min\{\|z\|, 1\}\, \mu_f(dz) = \int_T \int_Y \min\{\|f \cdot y\|, 1\}\, m(dy)\, d\lambda;$$

hence by Theorem 6.3 (i) [3, Chap.3] μ_f is a Lévy measure. Theorem 3.3.6 yields the assertion.

By Theorem 3.3.6 we get the following comparison theorem for random integrals.

THEOREM 3.3.9. *Let* $M^{(i)} \sim [a^{(i)}, Q^{(i)}, m^{(i)}]$, $i = 1, 2$, *be random measures on* (T, Σ, λ) *and* $\lambda(T) < \infty$. *Assume that there exist* $c, r > 0$ *such that*

$$Q^{(1)} \leqslant c Q^{(2)}$$

and for every $B \in \mathscr{B}(Y)$

$$\tilde{m}^{(1)}(B \cap \{0 < \|y\| \leqslant r\}) \leqslant c\tilde{m}^{(2)}(B \cap \{0 < \|y\| \leqslant r\}),$$

where $\tilde{m}^{(i)}(B) = \frac{1}{2}[m^{(i)}(B) + m^{(i)}(-B)]$, $i = 1, 2$.

Then every M_2*-integrable function* f *is* M_1*-integrable provided that* $\int_T V^{(1)} \circ f d\lambda$ *exists in the Pettis sense.*

Proof. Since

$$Q_f^{(1)}(z^*) = \int_T Q^{(1)} \circ (f' z^*)\, d\lambda \leqslant c \int_T Q^{(2)} \circ (f' z^*)\, d\lambda = c Q_f^{(2)}(z^*)$$

and $Q_f^{(2)}$ is a Gaussian variance, $Q_f^{(1)}$ is also a Gaussian variance. Define

$$m_r^{(i)}(B) = m^{(i)}(B \cap \{0 < \|y\| \leqslant r\}),$$
$$m^{(i)(r)}(B) = m^{(i)}(B \cap \{\|y\| > r\}), \qquad B \in \mathscr{B}(Y)$$

and

$$\mu_{f,r}^{(i)}(B) = \lambda \times m_r^{(i)}(\{f \cdot y \in B \setminus \{0\}\}),$$
$$\mu_f^{(i)(r)}(B) = \lambda \times m^{(i)(r)}(\{f \cdot y \in B \setminus \{0\}\}), \qquad B \in \mathscr{B}(Z),$$

$i = 1, 2$. Since $\mu_f^{(1)(r)}$ is a finite measure, it is a Lévy measure. It remains to prove that $\mu_{f,r}^{(1)}$ is a Lévy measure. For every $B \in \mathscr{B}(Z)$, $0 \notin B$, we have

$$\tilde{\mu}_{f,r}^{(1)}(B) = \tfrac{1}{2}\{\tilde{\mu}_{f,r}^{(1)}(B) + \mu_{f,r}^{(1)}(-B)\}$$
$$= \int_T \tilde{m}_r^{(1)}(\{y\colon f \cdot y \in B\})\, d\lambda$$
$$\leqslant c \int_T \tilde{m}_r^{(2)}(\{y\colon f \cdot y \in B)\, d\lambda = c\tilde{\mu}_{f,r}^{(2)}(B),$$

and because $\tilde{\mu}^{(2)}_{f,r}$ is a Lévy measure, $\tilde{\mu}^{(1)}_{f,r}$ is also a Lévy measure, which completes the proof.

The next theorem gives other conditions for the existence of the random integral in the terms of functions K and V given by (3.1) and (3.2), respectively.

THEOREM 3.3.10. *A function $f\colon T \to X$ is M-integrable if and only if*

(i) *f is strongly measurable and $V \circ f$ is Pettis integrable,*

(ii) *for every $z^* \in Z^*$, $\int_T |K \circ (f' z^*)|\, d\lambda < \infty$,*

(iii) *$\varphi_f(z^*) = \exp\{\int_T K \circ (f' z^*)\, d\lambda\}$ is the characteristic functional of a p.m. on Z.*

Moreover

$$\mathscr{L}\big(\int_T f \cdot dM\big)\hat{}\,(z^*) = \varphi_f(z^*), \quad z^* \in Z^*.$$

Proof. Necessity follows by Lemmas 3.1.4 and 3.1.7. We prove sufficiency. Let μ be a p.m. on Z such that $\hat{\mu} = |\varphi_f|^2$. We have

$$\hat{\mu}(z^*) = \exp\{-Q_f(z^*) + \int_Z [\cos\langle z^*, z\rangle - 1]\, \tilde{\mu}_f(dz)\},$$

where $\tilde{\mu}_f(B) = \mu_f(B) + \mu_f(-B)$. Thus, for every $z^* \in Z^*$, $Q_f(z^*)$ exists and there exists a cylindrical measure γ such that

$$\hat{\gamma}(z^*) = \exp\{-Q_f(z^*)\}.$$

Also,

$$\sup_{t \in [0,1]} \int_Z (1 - \cos\langle t z^*, z\rangle)\, \tilde{\mu}_f(dz) < \infty;$$

hence

$$\infty > \int_0^1 \int_Z (1 - \cos t\langle z^*, z\rangle)\, \tilde{\mu}_f(dz)\, dt$$

$$= \int_Z \left(1 - \frac{\sin\langle z^*, z\rangle}{\langle z^*, z\rangle}\right) \tilde{\mu}_f(dz) \geqslant \tfrac{1}{10} \int_Z \min\{1, \langle z^*, z\rangle^2\}\, \tilde{\mu}_f(dz)$$

and there exists a cylindrical measure ϱ such that

$$\hat{\varrho}(z^*) = \exp\{\int_Z (\cos\langle z^*, z\rangle - 1)\, \tilde{\mu}_f(dz)\}.$$

Since $\mu = \gamma * \varrho$, γ and ϱ being symmetric, γ and ϱ extend to p.m.'s on Z (see [36]). Therefore Q_f is a Gaussian variance and $\tilde{\mu}_f$ is a Lévy measure. Hence μ_f is a Lévy measure. An appeal to Theorem 3.3.6 completes the proof.

We now give the form of the characteristic functional φ_f of a random integral $\int_T f\cdot dM$ in certain important cases.

EXAMPLE A. $X = Y = Z = \boldsymbol{R}$, $x\cdot y = xy$.

$$\varphi_f(s) = \exp\{\int_T K(sf(t))\lambda(dt)\}, \quad s\in \boldsymbol{R}.$$

EXAMPLE B. $X = Y = Z = \boldsymbol{C}$ (here $\boldsymbol{C}$ is identified with $\boldsymbol{R}^2$), $x\cdot y = xy = \operatorname{Re}(x)\operatorname{Re}(y) - \operatorname{Im}(x)\operatorname{Im}(y) + i[\operatorname{Re}(x)\operatorname{Im}(y) + \operatorname{Im}(x)\operatorname{Re}(y)]$.

$$\varphi_f(z) = E\exp\{i\operatorname{Re}(\bar{z}\int_T f dM)\} = \exp\{\int_T K(\overline{f(t)}\,z)\lambda(dt), \quad z\in \boldsymbol{C}.$$

EXAMPLE C. $X = Z = E$, E is a Banach space, $Y = \boldsymbol{R}$, $x\cdot y = yx$, $x\in E$, $y\in \boldsymbol{R}$.

$$\varphi_f(x^*) = \exp\{\int_T K(\langle x^*, f(t)\rangle)\lambda(dt)\}, \quad x^*\in E^*.$$

EXAMPLE D. $X = \boldsymbol{R}$, $Y = Z = E$, E is a Banach space, $x\cdot y = xy$, $x\in \boldsymbol{R}$, $y\in E$.

$$\varphi_f(x^*) = \exp\{\int_T K(f(t)x^*)\lambda(dt)\}, \quad x^*\in E^*.$$

EXAMPLE E. X and Y in the duality $\langle\cdot,\cdot\rangle$, $Z = \boldsymbol{R}$, $x\cdot y = \langle x, y\rangle$.

$$\varphi_f(s) = \exp\{\int_T K(sf(t))\lambda(dt)\}, \quad s\in \boldsymbol{R}.$$

EXAMPLE F. $U\colon X\to Z$ is a fixed continuous linear operator, $Y = \boldsymbol{R}$, $x\cdot y = yUx$, $x\in X$, $y\in \boldsymbol{R}$.

$$\varphi_f(z^*) = \exp\{\int_T K(\langle z^*, Uf(t)\rangle)\lambda(dt), \quad z^*\in Z^*.$$

EXAMPLE G. $X = \boldsymbol{R}$, $U\colon Y\to Z$ is a fixed continuous linear operator, $x\cdot y = xUy$, $x\in \boldsymbol{R}$, $y\in Y$.

$$\varphi_f(z^*) = \exp\{\int_T K(f(t)U^*z^*)\lambda(dt)\}, \quad z^*\in Z^*.$$

EXAMPLE H. $X = L(E, F)$, $Y = E$, $Z = F$, where E and F are real Banach spaces, $x\cdot y = x(y)$, $x\in L(E, F)$, $y\in E$.

$$\varphi_f(z^*) = \exp\{\int_T K([f(t)]'(z^*))\lambda(dt)\}, \quad z^*\in F^*,$$

where x' is the dual operator of $x\in L(E, F)$.

EXAMPLE I. $X = E$, $Y = L(E, F)$, $Z = F$ where E and F are real Banach spaces, $x\cdot y = y(x)$, $x\in E$, $y\in L(E, F)$.

$$\varphi_f(z^*) = \exp\{\int_T K(f(t)\otimes z^*)\lambda(dt)\}, \quad z^*\in F^*,$$

where $(x\otimes z^*)(y) = \langle z^*, y(x)\rangle$, $x\in E$, $y\in L(E, F)$, $z^*\in F^*$.

EXAMPLE J. $X = Y = Z = C[0, 1]$, $[x\cdot y](s) = x(s)y(s)$, $s\in[0, 1]$.

$$\varphi_f(z^*) = \exp\{\int_T K([f(t)]'z^*)\lambda(dt)\}, \quad z^*\in[C[0, 1]]^* = \mathscr{M}[0, 1],$$

where $[x'z^*](C) = \int_C x(s)z^*(ds)$, $C\in\mathscr{B}([0, 1])$, $x\in C[0, 1]$, $z^*\in\mathscr{M}[0, 1]$.

Theorem 3.3.6 gives a full characterization of $\mathscr{L}(M)$ in all cases when conditions a Gaussian variance and a Lévy measure are known. This is the case e.g. if Z is a Hilbert space.

COROLLARY 3.3.11. *Let $Z = H$ be a Hilbert space. Then $f\in\mathscr{L}(M)$ if and only if*

(i) *f is strongly measurable and $V\circ f$ is Pettis integrable,*

(ii) *for every $h\in H$*

$$\int_T Q\circ(f'h)\,d\lambda < \infty$$

and

$$\sum_\alpha\int_T Q\circ(f'h_\alpha)\,d\lambda < \infty \quad \textit{for some orthonormal basis } \{h_\alpha\} \textit{ in } H,$$

(iii) $\int_T\int_Y \min\{1, \|f(t)\cdot y\|^2\}\,m(dy)\,\lambda(dt) < \infty$.

As regards Example A, the above corollary gives similar conditions for the existence of the random integral to those obtained by Lévy [26]. We may also use this corollary in the situations described in Examples B and E. In particular, if $X = E^*$, $Y = E$ and $Z = \boldsymbol{R}$, where E is a Banach space, we get

COROLLARY 3.3.12. *Let $f\colon T\to E^*$ and $M\sim[a, Q, m]$ be an E-valued random measure on (T, Σ, λ). Then $\int\langle f, dM\rangle$ exists if and only if*

(i) *f is strongly measurable and $\int_T|V\circ f|\,d\lambda < \infty$, where*

$$V(x^*) = \langle x^*, a\rangle + \int_E\langle x^*, x\rangle[\alpha(|\langle x^*, x\rangle|) - \alpha(\|x\|)]\,m(dx),$$

(ii) $\int_T Q\circ f d\lambda \to \infty$,

(iii) $\int_T\int_E \min\{1, \langle f(t), x\rangle^2\}\,m(dx)\,\lambda(dt) < \infty$.

In the case of operator-valued functions the strong measurability condition is rather unconvenient and too strong. We can weaken it as follows. Observe that if f is M-integrable then for every $z^*\in Z^*$, $\langle z^*, \int_A f\cdot dM\rangle = \int_A\langle f'z^*, dM\rangle$ a.s. for every $A\in\Sigma$. Moreover, the above corollary gives a full characterization of the right hand side integral. This suggests the following.

DEFINITION 3.3.13. A function $f\colon T \to X$ is *weak-M-integrable* if
(i) for every $z^* \in Z^*$, $\int \langle f' z^*, dM\rangle$ exists,
(ii) for every $A \in \Sigma$ there exists a Z-valued random vector U_A such that for every $z^* \in Z^*$

$$\langle z, U_A\rangle = \int_A \langle f' z^*, dM\rangle \text{ a.s.}$$

If f is weak-M-integrable then we write

$$w-\int_A f \cdot dM = U_A, \qquad A \in \Sigma.$$

The following theorem shows that in fact the weak random integral differs from the ordinary one only in the measurability of the integrand.

THEOREM 3.3.14. *Assume that Z is separable. Then $f\colon T \to X$ is weak-M-integrable if and only if*
(i) *for every $z^* \in Z^*$, $f' z^*\colon T \to Y$ is strongly measurable,*
(ii) *$V \circ f$ is Pettis integrable,*
(iii) *Q_f and μ_f are a Gaussian variance and a Lévy measure, respectively.*
Moreover,

$$\mathscr{L}\big(w-\int_T f \cdot dM\big) = [a_f, Q_f, \mu_f],$$

where a_f, Q_f and μ_f are given by (3.4).

Proof. First we prove that (i) implies that the function $(t, y) \to f(t) \cdot y$ is $(\Sigma \times \mathscr{B}(Y), \mathscr{B}(Z))$-measurable. Indeed, for every $z^* \in Z^*$, $(t, y) \to \langle z^*, f(t) \cdot y\rangle = \langle f'(t) z^*, y\rangle$ is measurable in $t \in T$ and continuous in $y \in Y$, hence it is $\Sigma \times \mathscr{B}(Y)$-measurable. By Pettis' theorem $(t, y) \to f(t) \cdot y$ is a Borel function.

Assume that f is weak-M-integrable. First we show that μ_f is well defined and $V \circ f$ is a Borel function. By definition, condition (i) is satisfied. Hence $(t, y) \to f(t) y$ is a Borel function and so μ_f is well defined. Using the Fubini and Pettis theorems we get that $t \to \int_T f(t) \cdot y \big[\alpha\big(\|f(t) \cdot y\|\big) - \alpha(\|y\|)\big] m(dy)$ is Borel measurable: thus $V \circ f$ is a Borel function.

Using Theorem 3.3.10 we obtain that for every $z^* \in Z^*$ $\int_T |K \circ (f' z^*)|\, d\lambda < \infty$ and

$$E \exp\{i \langle z^*, U_T\rangle\} = E \exp\{i \int_T \langle f' z^*, dM\rangle\}$$
$$= \exp\{\int_T K \circ (f' z^*)\, d\lambda\},$$

where K is given by (3.1). Hence, proceeding as in the proof of Theorem 3.3.10 we obtain that Q_f is a Gaussian variance and μ_f is a Lévy measure.

Since

$$K\circ(f'z^*) = i\langle z^*, V\circ f\rangle - \tfrac{1}{2}Q\circ(f'z^*) + \int_Y [e^{i\langle z^*, fy\rangle} - 1 - i\langle z^*, fy\rangle\alpha(\|fy\|)]\, m(dy)$$

and

$$\int_T \Big|\int_Y [e^{i\langle z^*, fy\rangle} - 1 - i\langle z^*, fy\rangle\alpha(\|fy\|)]\, m(dy)\Big|\, d\lambda$$

$$\leqslant [2+\|z^*\|]\,\mu_f(\|z\|>1) + \tfrac{1}{2}\int_{\|z\|\leqslant 1}\langle z^*, z\rangle^2\,\mu_f(dz) < \infty,$$

we have

$$\int_T |\langle z^*, V\circ f\rangle|\, d\lambda < \infty \quad \text{for every } z^*\in Z^*.$$

Therefore there exists $\xi_A \in Z^*$ such that

$$\xi_A(z^*) = \int_A \langle z^*, V\circ f\rangle\, d\lambda \quad \text{for every } z^*\in Z^*.$$

To prove that $\xi_A \in Z^*$, it is enough to show that if $z_n^* \to 0$ in the $\sigma(Z^*, Z)$-topology then $\xi_A(z_n^*) \to 0$. By the continuity of a characteristic functional on each bounded ball in the $\sigma(Z^*, Z)$-topology we get for every $s\in \boldsymbol{R}$

$$\lim_{n\to\infty} e^{is\xi_A(z_n^*)} = \lim_{n\to\infty} E\exp\{i\langle sz_n^*, U_A\rangle\}\,[\hat{\gamma}_{f1_A}(sz_n^*)\, c\operatorname{Pois}(\mu_{f1_A})\hat{}\,(sz_n^*)]^{-1} = 1;$$

hence $\xi_A(z_n^*) \to 0$.

Now we prove the converse. Let $z^*\in Z^*$. First we show that $f'z^*$ satisfies (i), (ii) and (iii) of Corollary 3.3.12. (ii) and (iii) of that corollary follow immediately by our (iii). We show (i). We have

$$\int_T \Big|\langle f'z^*, a\rangle + \int_Y \langle f'z^*, y\rangle[\alpha(|\langle f'z^*, y\rangle|) - \alpha(\|y\|)]\, m(dy)\Big|\, d\lambda$$

$$\leqslant \int_T |\langle z^*, V\circ f\rangle|\, d\lambda + \int_Z |\langle z^*, z\rangle[\alpha(|\langle z^*, z\rangle|) - \alpha(\|z\|)]|\,\mu_f(dz)$$

and in view of the first assertion of Lemma 3.1.5 in the case of $X = Z^*$, $Y = Z$ we get

$$\int_Z |\langle z^*, z\rangle[\alpha(|\langle z^*, z\rangle|) - \alpha(\|z\|)]|\,\mu_f(dz)$$

$$\leqslant (1+\|z^*\|)\,\mu_f\{\|z\| > \min[\|z^*\|^{-1}, 1]\},$$

which proves (i) of Corollary 3.3.11. Now let us consider a linear stochastic process $z^* \mapsto \int_A \langle f'z^*, dM\rangle$, where $A\in\Sigma$ is fixed. Since $Q_{f1_A} \leqslant Q_f$ and

$\mu_{f1_A} \leqslant \mu_f$, we conclude that Q_{f1_A} and μ_{f1_A} are a Gaussian variance and a Lévy measure, respectively. Now,

$$E\exp\{i\int_A \langle f'z^*, dM\rangle\} = \hat{\nu}_{f1_A}(z^*), \qquad z^* \in Z^*,$$

where $\nu_{f1_A} = [a_{f1_A}, Q_{f1_A}, \mu_{f1_A}]$; hence $z^* \mapsto \int_A \langle f'z^*, dM\rangle$ is decomposable, i.e. there exists a Z-valued random vector U_A such that $\langle z^*, U_A\rangle = \int_A \langle f'z^*, dM\rangle$ a.s. for every $z^* \in Z^*$. The proof of the theorem is complete.

Using similar methods to those applied in the proofs of Theorems 3.3.10 and 3.3.14 one can get the following theorem whose proof is omitted.

THEOREM 3.3.15. *Assume that Z is separable. Then $f\colon T \to X$ is weak-M-integrable if and only if*

(i) *for every $z^* \in Z^*$, $f'z^*\colon T \to Y^*$ is strongly measurable,*

(ii) *$V \circ f$ is Pettis integrable,*

(iii) *for every $z^* \in Z^*$, $\int_T |K \circ (f'z^*)|\, d\lambda < \infty$,*

(iv) *$\varphi_f(z^*) = \exp\{\int_T K \circ (f'z^*)\, d\lambda\}$ is the characteristic functional of a p.m. on Z.*

Moreover,

$$\mathscr{L}(w - \int_T f \cdot dM) = \varphi_f(z^*), \qquad z^* \in Z^*.$$

Note an interesting corollary, which we get in the case of $X = Z = E$ and $Y = \boldsymbol{R}$ (Example C).

COROLLARY 3.3.16. *Let E be a separable Banach space and $M \sim [a, Q, m]$ be a real-valued random measure on (T, Σ, λ). Then $f\colon T \to E$ is weak-M-integrable if and only if f is M-integrable.*

The above result can be treated as a generalization of the theorem of Pettis saying that Dunford and Pettis integrals coincide ([33], Th.5.1). The generalization consists in replacing deterministic measures by random ones; the deterministic case is included in $Q = 0$ and $m = 0$.

On the other hand, as far as a random integral is a generalization of a random series with Banach space valued coefficients, the above corollary gives a version of Ito–Nisio theorem for random integrals (see [13], Th.2).

Similar corollaries result from considering Examples D, F and G.

4. Approximation by simple functions and some contraction principles. We begin with a study of the approximation problem in $\mathscr{L}(M)$. By Definitions 3.1.1 and 3.2.1 simple functions lie densely in the space of all M-integrable functions. The problem is how to choose, for any concrete function $f \in \mathscr{L}(M)$, simple functions $\{f_n\}$ such that $f_n \to f$ in $\mathscr{L}(M)$. The following example shows that uniform approximation fails.

EXAMPLE 3.4.1. (see [38], Example 3.2). Consider the random integral of Banach space valued functions with respect to a real valued random measure (Example C). Let $T = [0, 1]$, $\Sigma = \mathscr{B}([0, 1])$, λ – the Lebesgue measure, $M \sim [0, 1, 0]$ a white noise, $E = l^p$, $1 \leqslant p < 2$. Let $\{e_n\}$ stand for the standard basis in l^p. Put

$$f_n = n^{-r/p} \sum_{k=1}^{n} e_k \mathbf{1}_{((k-1)/n, k/n]},$$

where $r \in (0, 1-p/2)$ is fixed. Then

$$\sup_{t\in[0,1]} \|f_n(t)\| = n^{-r/p} \to 0 \quad \text{as } n \to \infty$$

and

$$E\Big\|\int_0^1 f_n\, dM\Big\|^p = n^{-r} \sum_{k=1}^{n} E\big|M\{((k-1)/n, k/n]\}\big|^p$$
$$= c_p n^{1-p/2-r} \mapsto \infty \quad \text{as } n \to \infty.$$

Since $\int_0^1 f_n\, dM$, $n \geqslant 1$ have Gaussian distributions, the sequence $\{\int_0^1 f_n\, dM\}$ diverges in P. Finally, the function $f \equiv 0$ is M-integrable, $f_n \to f$ uniformly and $\{f_n\}$ diverges in $\mathscr{L}(M)$. Thus the uniform convergence of functions does not imply the convergence of random integrals, in general.

Note that using Egorov's theorem it is easy to show that for every M-integrable function there exists a uniformly convergent sequence of simple functions which diverges in $\mathscr{L}(M)$.

Now we observe that the approximation problem can be reduced to the case where $\lambda(T) < \infty$ and f is bounded. In reality let $f \in \mathscr{L}(M)$. Choose $A_n \in \Sigma$ so that $\lambda(A_n) < \infty$, $A_n \subset \{\|f\| \leqslant n\}$, $A_n \nearrow A_\infty$ and $\lambda(T \setminus A_\infty) = 0$. Put $f_n = f\mathbf{1}_{A_n}$. Then for every $A \in \Sigma$ the sequence $\{\int_A f_n \cdot dM\}$ has independent increments and converges a.s. to $\int_A f \cdot dM$. Moreover, if $E\|\int_A f \cdot dM\|^p < \infty$ then it also converges in $L^p(\Omega, \mathscr{F}, P; Z)$. Therefore it is sufficient to consider only the approximation of f_n by simple functions, i.e. the approximation of bounded M-integrable function in the case $\lambda(T) < \infty$.

The following theorem shows that a martingale type approximation is applicable for our purposes.

THEOREM 3.4.2. *Assume that* $\lambda(T) < \infty$ *and* $E\|M(T)\|^p < \infty$, $0 \leqslant p < \infty$. *Let f be a bounded M-integrable function and $\{\Sigma_n\}$ be a sequence of finite sub-σ-fields of Σ such that the λ-completion of $\sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$ contains $f^{-1}(\mathscr{B}(X))$. Then* *for every* $A \in \Sigma$

$$\int_A \lambda(f|\Sigma_n)\cdot dM \to \int_A f\cdot dM \quad in\ L^p(\Omega, \mathcal{F}, P; Z).$$

Proof. Put $f_n = \lambda(f|\Sigma_n)$. Without loss of generality we may assume that $M = M_0 + M_1$, where M_0 and M_1 were specified in Remark 2.2.1. Since f is both M_0 and M_1 integrable, we have, according to Lemma 3.3.4,

$$E\Big(\int_T f\cdot dM_0 | M_0(\Sigma_n)\Big) = \int_T f_n\cdot dM_0 \text{ a.s.}$$

On the other hand, using Proposition 2.1.2 it is easy to show that $E\|\int_T f\cdot dM_0\|^r < \infty$ for every $r \geq 1$. Hence $\int_T f_n\cdot dM_0 \to \int_T f\cdot dM_0$ a.s. and in $L^r(\Omega, \mathcal{F}, P; Z)$ for every $r \geq 1$. Set $\tilde{M}_0(A) = M_0(A) - EM_0(A) = M_0(A) - \lambda(A)\,a_1$ where $a_1 = a - \int_{\|y\|>1} y\|y\|^{-1}\, m(dy)$. Then for every $A \in \Sigma$

$$\int_A f_n\cdot d(\lambda a_1) \to \int_A f\cdot d(\lambda a_1)$$

and

$$\begin{aligned}
\Big[E\|\int_A (f_n - f)\cdot d\tilde{M}_0\|^r\Big]^{1/r} &\leq \Big[E\|\int_T (f_n - f)\cdot d\tilde{M}_0\|^r\Big]^{1/r} \\
&\leq \Big[E\|\int_T (f_n - f)\cdot dM_0\|^r\Big]^{1/r} + \|\int_T (f_n - f)\cdot d(\lambda a_1)\| \\
&\leq \Big[E\|\int_T (f_n - f)\cdot dM_0\|^r\Big]^{1/r} + \|a_1\|\,\|b\| \int_T \|f_n - f\|\, d\lambda \to 0
\end{aligned}$$

as $n \to \infty$. Therefore we have for every $A \in \Sigma$

$$\int_A f_n\cdot dM_0 = \int_A f_n\cdot d(\lambda a_1) + \int_A f_n\cdot d\tilde{M}_0 \to \int_A f\cdot dM_0$$

in $L^r(\Omega, \mathcal{F}, P; Z)$, for any $r \geq 1$. To end this proof it is enough to show that $\int_A f_n\cdot dM_1 \to \int_A f\cdot dM_1$ in $L^p(\Omega, \mathcal{F}, P; Z)$. In view of Lemma 3.3.5, for every strongly measurable function $g\colon T \to X$ we have

$$\begin{aligned}
E\|\int_T g\cdot dM_1\|^p &= e^{-\mu_{1,g}(Z)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_Z \|z\|^p\, \mu_{1,g}^{*k}(dz) \\
&= e^{-\mu_{1,g}(Z)} \sum_{k=1}^{\infty} \frac{1}{k!} \int_Z \cdots \int_Z \|z_1 + \ldots + z_k\|^p\, \mu_{1,g}(dz_1)\ldots\mu_{1,g}(dz_k) \\
&= e^{-\mu_{1,g}(Z)} \sum_{k=1}^{\infty} \frac{1}{k!} k^{p+1}\, [\mu_{1,g}(Z)]^{k-1} \int_Z \|z\|^p\, \mu_{1,g}(dz) \\
&\leq C \int_T \|g\|^p\, d\lambda,
\end{aligned}$$

where $C = e^{-r} \sum_{k=0}^{\infty} \frac{(k+1)^p}{k!} r^k \|b\|^p \int_{\|y\|>1} \|y\|^p m(dy)$, whereas $r = \mu_{1,g}(Z) = \lambda(T)\, m\{\|y\| > 1\}$. Since $\int_{\|y\|>1} \|y\|^p m(dy) < \infty$ by Proposition 2.1.2, we have $C < \infty$ and the theorem follows by the martingale convergence theorem.

Assume that $\lambda(T) = 1$. Then the conditional integral $\lambda(\cdot|\Sigma_0)$ is a natural contraction operator on $L^p(T, \Sigma, \lambda; X)$, for every $p \in [1, \infty]$, and it is an interesting question whether M-integrability is preserved under this operation? The following theorem shows that in some sense operations of a random integral and of a conditional integral commute.

THEOREM 3.4.3. *Assume that $\lambda(T) < \infty$. Let f be an M-integrable function such that $\int_T \|f\|\, d\lambda < \infty$ and $E\left\|\int_T f \cdot dM\right\| < \infty$. Then for every sub-$\sigma$-field Σ_0 of Σ the conditional integral $\lambda(f|\Sigma_0)$ is M-integrable and*

$$\int_T \lambda(f|\Sigma_0) \cdot dM = E\Big(\int_T f \cdot dM \,|\, M(\Sigma_0)\Big) \text{ a.s.} \tag{3.5}$$

Proof. The proof is divided into several steps, in which the theorem is proved under some additional assumptions. These additional assumptions are specified in the headline of each step.

Step I. f is bounded, $E\|M(T)\| < \infty$ and Σ_0 is finite.

Choose an increasing sequence $\{\mathscr{A}_n\} \subset \Sigma$ of finite σ-fields such that $\mathscr{A}_1 = \Sigma_0$ and the λ-completion of $\sigma\big(\bigcup_{n=1}^{\infty} \mathscr{A}_n\big)$ contains $f^{-1}(\mathscr{B}(X))$. Then, by Lemma 3.3.4 and Theorem 3.4.2, $\{\int_T \lambda(f|\mathscr{A}_n) \cdot dM \colon M(\mathscr{A}_n)\}$ is a martingale which converges in $L^1(\Omega, \mathscr{F}, P; Z)$ to $\int_T f \cdot dM$. Hence

$$E\Big\{\int_T f \cdot dM \,|\, M(\Sigma_0)\Big\} = \lim_{n\to\infty} E\Big\{\int_T \lambda(f|\mathscr{A}_n) \cdot dM \,|\, M(\mathscr{A}_1)\Big\}$$
$$= \int_T \lambda(f|\Sigma_0) \cdot dM.$$

Step II. f is bounded and $E\|M(T)\| < \infty$.

Let $\mathfrak{A} = \{\mathscr{A}\colon \mathscr{A}$ is a finite sub-σ-field of $\Sigma_0\}$. $\mathfrak{A}$ is a partially ordered set by the ordering $\mathscr{A}_1 \prec \mathscr{A}_2$ if $\mathscr{A}_1 \subset \mathscr{A}_2$. Define generalized sequences

$$\mathfrak{A} \ni \mathscr{A} \to \lambda(f|\mathscr{A}) \in L^1(T, \Sigma, \lambda; X)$$

and

$$\mathfrak{A} \ni \mathscr{A} \to \int_T \lambda(f|\mathscr{A}) \cdot dM \in L^1(\Omega, \mathscr{F}, P; Z).$$

By Step I, for every $\mathscr{A} \in \mathfrak{A}$ we have $\int_T \lambda(f|\mathscr{A}) \cdot dM = E\big(\int_T f \cdot dM \,|\, M(\mathscr{A})\big)$.

Using the martingale convergence theorem we see that for every increasing sequence $\mathscr{A}_1 \prec \mathscr{A}_2 \prec \ldots$, $\mathscr{A}_n \in \mathfrak{A}$, there exists the limit $\lim_{n\to\infty} \lambda(f|\mathscr{A}_n)$ and

$$\lim_{n\to\infty} \int_T \lambda(f|\mathscr{A}_n)\cdot dM = \lim_{n\to\infty} E\Big(\int_T f\cdot dM | M(\mathscr{A}_n)\Big).$$

Hence, by a standard argument, there exist the limits of the generalized sequences

$$g = \lim_{\mathscr{A}\in\mathfrak{A}} \lambda(f|\mathscr{A})$$

and

$$G = \lim_{\mathscr{A}\in\mathfrak{A}} \int_T \lambda(f|\mathscr{A})\cdot dM.$$

Moreover, there exists an increasing sequence $\{\mathscr{A}_n\} \subset \mathfrak{A}$ such that

$$g = \lim_{n\to\infty} \lambda(f|\mathscr{A}_n)$$

and

$$G = \lim_{n\to\infty} \int_T \lambda(f|\mathscr{A}_n)\cdot dM.$$

On the other hand, it is easy to check that

$$\lim_{\mathscr{A}\in\mathfrak{A}} \lambda(f|\mathscr{A}) = \lambda(f|\Sigma_0)$$

and

$$\lim_{\mathscr{A}\in\mathfrak{A}} E\Big(\int_T f\cdot dM|M(\mathscr{A})\Big) = E\Big(\int_T f\cdot dM|M(\Sigma_0)\Big).$$

Hence

$$\lambda(f|\mathscr{A}_n) \to \lambda(f|\Sigma_0) \quad \text{in} \quad L^1(T, \Sigma, \lambda; X)$$

and

$$\int_T \lambda(f|\mathscr{A}_n)\cdot dM \to E\Big(\int_T f\cdot dM|M(\Sigma_0)\Big)$$

in $L^1(\Omega, \mathscr{F}, P; Z)$ as $n \to \infty$. Thus, by Theorem 3.2.4, $\lambda(f|\Sigma_0) \in \mathscr{L}(M)$ and (3.5) holds.

Step III. $E\|M(T)\| < \infty$.

Put $\tilde{M}(A) = M(A) - EM(A) = M(A) - \lambda(A)a_1$, $A \in \Sigma$, where $a_1 = a + \int_{\|y\|>1} y(1-\|y\|^{-1})\,m(dy)$. Choose an increasing sequence $\{A_n\}$ such that $\bigcup_n A_n = T$ and $A_n \subset \{\|f\| \leqslant n\}$ and put $f_n = f\cdot \mathbf{1}_{A_n}$. By Step II, for every $n \geqslant 1$ we have $\lambda(f_n|\Sigma_0) \in \mathscr{L}(\tilde{M})$ and

$$\int_T \lambda(f_n|\Sigma_0)\cdot d\tilde{M} = E\Big[\int_T f_n\cdot d\tilde{M}|\tilde{M}(\Sigma_0)\Big] \to E\Big[\int_T f\cdot d\tilde{M}|\tilde{M}(\Sigma_0)\Big]$$

in $L^1(\Omega, \mathscr{F}, P; Z)$, since $\{\int_T f_n \cdot d\tilde{M}\}$ has independent increments. But $E\int_A \lambda(f_n|\Sigma_0)\cdot d\tilde{M} = 0$ for every $A \in \Sigma$; hence, for every $A\in\Sigma$ there exists the limit $\lim_{n\to\infty} \int_A \lambda(f_n|\Sigma_0)\cdot d\tilde{M}$ in $L^1(\Omega, \mathscr{F}, P; Z)$. Therefore $\lambda(f|\Sigma_0)\in\mathscr{L}(\tilde{M})$, whence $\lambda(f|\Sigma_0)\in\mathscr{L}(M)$ and

$$\begin{aligned} E\{\int_T f\cdot dM|M(\Sigma_0)\} &= E\{\int_T f\cdot d\tilde{M}|\tilde{M}(\Sigma_0)\} + \int_T f\cdot d(\lambda a_1)\\ &= \int_A \lambda(f|\Sigma_0)\cdot d\tilde{M} + \int_T \lambda(f|\Sigma_0)\cdot d(\lambda a_1)\\ &= \int_T \lambda(f|\Sigma_0)\cdot dM. \end{aligned}$$

Step IV. Without any additional assumptions.

For any fixed $r>0$ put $m_r = m|\{\|y\|\leqslant r\}$ and $m^{(r)} = m|\{\|y\|>r\}$. Let $M_r \sim [a - \int_{\|y\|>r} y\alpha(\|y\|)\,m(dy), Q, m_r]$ and $M^{(r)} \sim [\int_{\|y\|>r} y\alpha(\|y\|)\,m(dy), 0, m^{(r)}]$ be independent random measures on (T, Σ, λ). By the same arguments as in Remark 2.2.1 we may assume that $M = M_r + M^{(r)}$. Since $E\|M_r(T)\| < \infty$ and $M^{(r)}$ is generated by $\mathrm{Pois}(m^{(r)})$, we conclude in view od Step III and Lemma 3.3.5 that $\lambda(f|\Sigma_0)\in\mathscr{L}(M_r)$,

$$E[\int_T f\cdot dM_r|M_r(\Sigma_0)] = \int_T \lambda(f|\Sigma_0)\cdot dM_r$$

and $\lambda(f|\Sigma_0)\in\mathscr{L}(M^{(r)})$. Therefore $\lambda(f|\Sigma_0)\in\mathscr{L}(M)$ and

$$\begin{aligned} &E[\int_T f\cdot dM|M_r(\Sigma_0)\vee M^{(r)}(\Sigma_0)]\\ &\quad = E[\int_T f\cdot dM_r|M_r(\Sigma_0)] + E[\int_T f\cdot dM^{(r)}|M^{(r)}(\Sigma_0)]\\ &\quad = \int_T \lambda(f|\Sigma_0)\cdot dM - \int_T \lambda(f|\Sigma_0)\cdot dM^{(r)} + E[\int_T f\cdot dM^{(r)}|M^{(r)}(\Sigma_0)]. \end{aligned}$$

$\mathscr{L}(\int_T \lambda(f|\Sigma_0)\cdot dM^{(r)})$ is a Poissonian p.m. on Z (see Lemma 3.3.5), Thus we get

$$\begin{aligned} E\|\int_T \lambda(f|\Sigma_0)\cdot dM^{(r)}\| &\leqslant \int_T\int_Y \|\lambda(f|\Sigma_0)\cdot y\|\,m^{(r)}(dy)\,d\lambda\\ &= \int_{\|y\|>r} \{\int_T \|\lambda(f|\Sigma_0)\cdot y\|\,d\lambda\}\,m(dy)\\ &\leqslant \int_{\|y\|>r} \{\int_T \|f\cdot y\|\,d\lambda\}\,m(dy) = \varepsilon_r, \end{aligned}$$

where the last inequality follows from the Jensen inequality, since for every fixed $y \in Y$ the function $X \ni x \mapsto \|x \cdot y\| \in \boldsymbol{R}_+$ is convex. Similarly,

$$E\|E[\int_T f \cdot dM^{(r)} | M^{(r)}(\Sigma_0)]\| \leqslant E\|\int_T f \cdot dM^{(r)}\| \leqslant \varepsilon_r.$$

Since $E\|\int_T f \cdot dM^{(1)}\| < \infty$, using Lemma 3.3.5 we get

$$E\|\int_T f \cdot dM^{(1)}\| \geqslant e^{-\lambda(T) m^{(1)}(Y)} \int_T \int_Y \|f \cdot y\| \, m^{(1)}(dy) \, d\lambda = c\varepsilon_1,$$

and hence $\varepsilon_1 < \infty$. Finally,

$$E\|E[\int_T f \cdot dM | M(\Sigma_0)] - \int_T \lambda(f|\Sigma_0) \cdot dM\|$$

$$= E\|E[\{E[\int_T f \cdot dM | M_r(\Sigma_0) \vee M^{(r)}(\Sigma_0)] - \int_T \lambda(f|\Sigma_0) \cdot dM\} | M(\Sigma_0)]\| \leqslant 2\varepsilon_r,$$

which proves (3.5), since r is arbitrary and $\varepsilon_r \to 0$ as $r \to \infty$ by the Dominated Convergence Theorem. The proof is complete.

We return to the first contraction problem. Let

$$\mathscr{L}^p(M) = \mathscr{L}(M) \cap L^p(T, \Sigma, \lambda; X), \quad 0 \leqslant p \leqslant \infty.$$

$\mathscr{L}^p(M)$ is a linear subspace of $L^p(T, \Sigma, \lambda; X)$ and the question is whether $\mathscr{L}^p(M)$ is invariant with respect to a conditional integral operator $(1 \leqslant p \leqslant \infty)$. We can answer this problem in the affirmative, in general, only for $p = \infty$, although in some particular cases it is possible to get better results (see Theorems 3.5 and 3.5.5)

THEOREM 3.4.4. *Assume that* $\lambda(T) < \infty$. *Then* $\mathscr{L}^\infty(M)$ *is an invariant subspace of* $L^\infty(T, \Sigma, \lambda; X)$ *with respect to a conditional integral operator.*

Proof. By Lemma 3.3.5 it is enough to prove that $\lambda(f|\Sigma_0) \in \mathscr{L}(M_0)$, where M_0 is specified in Remark 2.2.1. Thus in view of Theorem 3.4.3 it suffices to check that $E\|\int_T f \cdot dM_0\| < \infty$. Putting $k = \operatorname{ess\,sup} \|f\|$ we get

$$\iint_{\{\|f \cdot y\| > 1\}} \|f \cdot y\| \, m_1(dy) \, d\lambda \leqslant k\|b\| \lambda(T) m\{(\|b\| k)^{-1} < \|y\| \leqslant 1\} < \infty,$$

which by Proposition 2.1.2 ends the proof.

The next type of contraction is that given by the multiplication

$$f \mapsto \varphi f,$$

where φ is a measurable real-valued function such that $\sup|\varphi| \leqslant 1$, and we ask whether $\varphi f \in \mathscr{L}(M)$, provided $f \in \mathscr{L}(M)$.

THEOREM 3.4.5. *If* f *is* M-*integrable then for every bounded real-valued* *measurable function φ, φf is M-integrable. Moreover, if M is symmetric then for every seminorm $q\colon Z \to [0, \infty]$ such that $\{q(z) \leqslant 1\}$ is closed we have*

$$P\{q(\int_T \varphi f \cdot dM) > \varepsilon\} \leqslant 2P\{\|\varphi\|_\infty q(\int_T f \cdot dM) > \varepsilon\}, \tag{3.6}$$

where $\varepsilon > 0$ and $\|\varphi\|_\infty = \operatorname{ess\,sup}|\varphi|$.

Proof. Assume that M is symmetric and let $\varphi = \sum_{j=1}^{n} c_1 \mathbf{1}_{A_j}$, $c_j \in \boldsymbol{R}$, $A_j \in \Sigma$ are pairwise disjoint and $0 \leqslant \lambda(A_j) \leqslant \infty$. Put $\xi_j = \int_{A_j} f \cdot dM$. Then by Kwapień's inequality (see [39], Th.1.2) we get

$$\begin{aligned} P\{q(\int_T \varphi f \cdot dM) > \varepsilon\} &= P\{q(\sum_{j=1}^{n} c_j \xi_j) > \varepsilon\} \\ &\leqslant 2P\{\max_j |c_j|\, q(\sum_{j=1}^{n} \xi_j) > \varepsilon\} = 2P\{\|\varphi\|_\infty q(\int_T f \cdot dM) > \varepsilon\}. \end{aligned}$$

Thus (3.6) holds for φ with a finite rank. If φ is arbitrary then we can choose a sequence of finite rank functions $\{\varphi_n\}$ such that $\|\varphi - \varphi_n\|_\infty \mapsto 0$. Then putting in (3.6) $q = \|\cdot\|$ we get for every $A \in \Sigma$

$$P\{\|\int_A (\varphi_n - \varphi_m) f \cdot dM\| > \varepsilon\} \leqslant 2P\{\|\varphi_n - \varphi_m\|_\infty \|\int_A f \cdot dM\| > \varepsilon\} \mapsto 0$$

as $n, m \mapsto \infty$. Therefore $\varphi f \in \mathcal{L}(M)$ and

$$\begin{aligned} P\{q(\int_T \varphi f \cdot dM) > \varepsilon\} &= P\{\int_T \varphi f \cdot dM \in q^{-1}(\varepsilon, \infty)\} \\ &\leqslant \liminf_{n\to\infty} P\{\int_T \varphi_n f \cdot dM \in q^{-1}(\varepsilon, \infty)\} \\ &\leqslant 2 \liminf_{n\to\infty} P\{\|\varphi_n\|\, q(\int_T f \cdot dM) > \varepsilon\} \\ &\leqslant 2P\{\|\varphi\|\, q(\int_T f \cdot dM) \geqslant \varepsilon\}. \end{aligned}$$

Since the function

$$2P\{\|\varphi\|\, q(\int_T f \cdot dM) > \varepsilon\} - P\{q(\int_T \varphi f \cdot dM) > \varepsilon\}$$

is right continuous in ε and non-negative except for countably many ε's, it is non-negative for every $\varepsilon > 0$. This completes the proof of the second assertion of the theorem.

Observe that for the proof of the first assertion it is sufficient to show that for every sequence $\{\varphi_n\}$ of measurable real-valued finite rank functions such that $\|\varphi_n\|_\infty \to 0$ and for every $f \in \mathcal{L}(M)$ the sequence $\{\int_T \varphi_n f \cdot dM\}$

converges to 0 in probability. On the product probability space $(\Omega\times\Omega, \mathscr{F}\times\mathscr{F}, P\times P)$ we define a symmetrization of M by $M(A)[(\omega_1,\omega_2)] = M(A)(\omega_1)-M(A)(\omega_2)$, $A\in\Sigma$. Then $f\in\mathscr{L}(M^s)$ provided $f\in\mathscr{L}(M)$ and by (3.6) just proved $\int_T \varphi_n f\cdot dM^s\to 0$ in $P\times P$. Therefore there exists a sequence $\{z_n\}\subset Z$ such that $z_n+\int_T \varphi_n f\cdot dM\to 0$ in P; hence, by Lemma 2.1.1, $z_n+\int_T V\circ(\varphi_n f)\,d\lambda\to 0$ as $n\to\infty$. Thus it is enough to prove that

$$\lim_{n\to\infty}\int_T V\circ(\varphi_n f)\,d\lambda = 0.$$

Since $V\circ f$ is Pettis integrable, $\lim_{n\to\infty}\int_T \varphi_n V\circ f d\lambda = 0$ and so it is sufficient to show that

$$\lim_{n\to\infty}\int_T [V\circ(\varphi_n f)-\varphi_n V\circ f]\,d\lambda = 0.$$

Let $\varepsilon\in(0,1)$ and let φ be a measurable finite rank function such that $\|\varphi\|_\infty<\varepsilon$. We have

$$\begin{aligned}\int_T [V\circ(\varphi f)-\varphi V\circ f]\,d\lambda &= \int_T\int_Y \varphi f\cdot y\,\{\alpha(\|\varphi f\cdot y\|)-\alpha(\|f\cdot y\|)\}\,m(dy)\,d\lambda\\ &= \iint_{\{1<\|f\cdot y\|<|\varphi|^{-1}\}} \varphi f\cdot y\,[1-\|f\cdot y\|^{-1}]\,m(dy)\,d\lambda\\ &+\iint_{\{\|f\cdot y\|>|\varphi|^{-1}\}} \varphi f\cdot y\,[\|\varphi f\cdot y\|^{-1}-\|f\cdot y\|^{-1}]\,m(dy)\,d\lambda = I_1(\varphi)+I_2(\varphi).\end{aligned}$$

Further,

$$\begin{aligned}\|I_1(\varphi)\| &\leqslant \iint_{\{1<\|f\cdot y\|<|\varphi|^{-1}\}} \|\varphi f\cdot y\|\,m(dy)\,d\lambda\\ &= \iint_{\{1<\|f\cdot y\|<\varepsilon^{-1}\}} \|\varphi f\cdot y\|\,m(dy)\,d\lambda+\iint_{\{\varepsilon^{-1}\leqslant\|f\cdot y\|<|\varphi|^{-1}\}} \|\varphi f\cdot y\|\,m(dy)\,d\lambda\\ &\leqslant \|\varphi\|_\infty\varepsilon^{-1}\mu_f\{1<\|z\|<\varepsilon^{-1}\}+\mu_f\{\|z\|\geqslant\varepsilon^{-1}\}\end{aligned}$$

and

$$\|I_2(\varphi)\|\leqslant 2\mu_f\{\|z\|\geqslant\|\varphi\|_\infty^{-1}\}.$$

Thus for every $\varepsilon\in(0,1)$

$$\begin{aligned}&\limsup_{n\to\infty}\Big\|\int_T [V\circ(\varphi_n f)-\varphi_n V\circ f]\,d\lambda\Big\|\\ &\leqslant \limsup_{n\to\infty}[\|\varphi_n\|_\infty\varepsilon^{-1}\mu_f\{1<\|z\|<\varepsilon^{-1}\}+\mu_f\{\|z\|\geqslant\varepsilon^{-1}\}+2\mu_f\{\|z\|\geqslant\|\varphi_n\|_\infty^{-1}\}]\\ &=\mu_f\{\|z\|\geqslant\varepsilon^{-1}\}.\end{aligned}$$

Letting $\varepsilon\to 0$ we finish the proof.

Using the Closed Graph Theorem, we get immediately the following result.

LEMMA 3.4.6. *Let $f \in \mathscr{L}(M)$. Then the operator T: $L^\infty(T, \Sigma, \lambda) \to \mathscr{L}(M)$, $T\varphi = \varphi f$, is continuous.*

From Theorem 3.4.5 we obtain also the following corollary.

COROLLARY 3.4.7. *Consider a random integral of real-valued functions with respect to a Banach space valued random measure (Example D). If $\lambda(T) < \infty$ then every bounded measurable function is M-integrable.*

Proof. $f = \mathbf{1}_T$ is M-integrable.

5. Stable symmetric random integrals. This section is devoted to a study of bilinear random integrals generated by symmetric stable random measures. We begin with some basic facts concerning stable p.m.'s on Banach spaces. The characteristic functional of a symmetric stable p.m. μ on E can be written in the form

$$\hat{\mu}(x^*) = \exp\{-\int_S |\langle x', x\rangle|^p \sigma(dx)\}, \qquad x^* \in E^*, \tag{3.7}$$

where σ is a finite symmetric measure on the unit sphere S of E and $p \in (0, 2]$ (see [3], Ch.3 and [23]). σ is called the *spectral measure* of μ and p is called the *order* of μ. We say that μ is *p-stable* p.m. on E with the spectral measure σ (and write $\mu = \mu_{p,\sigma}$) if (3.7) holds. 2-stable p.m.'s coincide with Gaussian p.m.'s and in this case we shall utilize the following form of $\hat{\mu}$,

$$\hat{\mu}(x^*) = \exp\{-\tfrac{1}{2}\langle x^*, Rx^*\rangle\}, \qquad x^* \in E^*,$$

where R: $E^* \to E$ is the covariance operator of μ.

Banach space valued Gaussian random integrals have been studied by N. V. Thu [40] (a real integrand and a vector integrator as in Example D), Hoffmann–Jørgensen [7], Rosiński and Suchanecki [38] (a vector integrand and a real integrator as in Example C) and by Mamporia [28] (an operator integrand and a vector integrator as in Example H).

Theorems 3.3.6 and 3.4.3 yield the following statements concerning Gaussian random integrals (see also Theorem 3.4.4).

THEOREM 3.5.1. *Let M be a random measure on (T, Σ, λ) generated by a Gaussian p.m. on Y with the covariance operator R (or equivalently $M \sim [0, Q, 0]$ where $Q(y^*) = \langle y^*, Ry^*\rangle$). Then $f \in \mathscr{L}(M)$ if and only if f is strongly measurable and*

$$\varphi_f(z^*) = \exp\{-\tfrac{1}{2}\int_T \langle z^*, f(t)Rf'(t)z^*\rangle \lambda(dt)\}$$

is the characteristic functional of a Radon p.m. on Z.

THEOREM 3.5.2. *Let M be as the above. If $\lambda(T) < \infty$ and Σ_0 is a sub-σ-field of Σ then the conditional integral $\lambda(f|\Sigma)$ is an M-integrable function provided that $f \in \mathscr{L}^1(M)$.*

When investigating stable non-Gaussian random integrals (i.e. such that $p < 2$) we find out that this case is quite different from the Gaussian one. This is most apparent when integrands take values in a Banach space and M is a real stable random measure; but this case will be studied in detail in the next section. Here we consider general properties of bilinear stable random integrals.

If $\mu = \mu_{p,\sigma}$ and $p < 2$ then $\mu = [0, 0, m]$, where

$$m(B) = c_p \int_0^\infty \int_S \mathbf{1}_B(uy)\,\sigma(dy)\frac{du}{u^{1+p}}, \qquad B \in \mathscr{B}(Y),$$

c_p being a numerical constant.

We compute

$$\begin{aligned} H_m(x) &= m(\{y \in Y\colon \|x\cdot y\| > 1\}) \\ &= c_p \int_0^\infty \int_S \mathbf{1}_{\{y:\,\|x\cdot y\|>1\}}(uy)\,\sigma(dy)\frac{du}{u^{1+p}} \\ &= \frac{c_p}{p}\int_S \|x\cdot y\|^p\,\sigma(dy) \end{aligned}$$

and define a seminormed vector space $L_{p,\sigma}$ consisting of all strongly measurable functions $f\colon T \to X$ such that

$$\|f\|_{p,\sigma} = \Big\{\int_T\int_S \|f(t)\cdot y\|^p\,\sigma(dy)\,\lambda(dt)\Big\}^{1/\max\{p,1\}} < \infty;$$

(as usual, functions which are equal λ-a.e. are not distinguishable).

THEOREM 3.5.3. *Let M be a random measure on (T, Σ, λ) generated by a stable p.m. $\mu_{p,\sigma}$ on Y and $p < 2$. Then*

(i) $f \in \mathscr{L}(M)$ if and only if f is strongly measurable and

$$\varphi_f(z^*) = \exp\Big\{-\int_T\int_S |\langle z^*, f(t)\cdot y\rangle|^p\,\sigma(dy)\,\lambda(dt)\Big\}, \qquad z^* \in Z^*,$$

is the characteristic functional of a Radon p.m. on Z,

(ii) $\mathscr{L}(M) \subset L_{p,\sigma}$,

(iii) if $p < 1$ then $\mathscr{L}(M) = L_{p,\sigma}$.

Proof. Applying Theorem 3.3.10 we get (i). (ii) follows by Corollary 3.1.11. To obtain (iii) we compute

$$\int_T\int_Y \min\{\|f\cdot y\|, 1\}\, m(dy)\, d\lambda = c_p \int_T\int_S\int_0^\infty \min\{u\|f\cdot y\|, 1\}\frac{du}{u^{1+p}}\sigma(dy)\, d\lambda$$

$$= c_p\left(\frac{1}{1-p}+\frac{1}{p}\right)\int_T\int_S \|f\cdot y\|^p\,\sigma(dy)\,d\lambda.$$

Thus by Corollary 3.3.8 $L_{p,\sigma} \subset \mathscr{L}(M)$, which completes the proof.

EXAMPLE 3.5.4. Let X and Y be in the duality $\langle\cdot,\cdot\rangle$, Example E. Then $\mathscr{L}(M) = L_{p,\sigma}$. Indeed,

$$\varphi_f(s) = \exp\{-|s|^p \int_T\int_S |\langle f(t), y\rangle|^p\,\sigma(dy)\,\lambda(dt)\}, \qquad s\in \boldsymbol{R},$$

is a characteristic function if and only if $\int_T\int_S |\langle f, y\rangle|^p\,\sigma(dy)\,d\lambda < \infty$,

THEOREM 3.5.5. *Let M be as in Theorem* 3.5.3, $\lambda(T) < \infty$ *and let Σ_0 be a sub-σ-field of Σ. If $f\in \mathscr{L}^1(M)$ and $p\neq 1$ then $\lambda(f|\Sigma_0)\in\mathscr{L}^1(M)$.*

Proof. Since a random integral by a p-stable random measure is a p-stable random vector and it possesses all moments less than p, by Theorem 3.4.3 we establish the theorem for $1<p<2$. To complete the proof, we observe that for $0<p<1$

$$L^1(T, \Sigma, \lambda; X) \subset L_{p,\sigma} = \mathscr{L}(M).$$

For $p=1$, the open question is whether $\lambda(f|\Sigma_0)\in\mathscr{L}^1(M)$ provided that $f\in\mathscr{L}^1(M)$. In this case we can only prove the following result. Let $\mathrm{LLOG}(X)$ denote the space of all strongly measurable functions $f\colon T\to X$ such that $\int_T \|f\|\log(1+\|f\|)\,d\lambda<\infty$.

THEOREM 3.5.6. *Let M be a 1-stable random measure and* $\lambda(T)<\infty$. *Let Σ_0 be a sub-σ-field of Σ. If $f\in\mathscr{L}(M)\cap \mathrm{LLOG}(X)$ then $\lambda(f|\Sigma_0)\in\mathscr{L}(M)\cap\mathrm{LLOG}(X)$.*

Proof. By Lemma 3.3.5 it is enough to prove that $\lambda(f|\Sigma_0)\in\mathscr{L}(M_0)$, where M_0 was specified in Remark 2.2.1. Thus in view of Theorem 3.4.3 it suffices to check that $E\|\int_T f\cdot dM_0\|<\infty$. We have

$$\iint_{\{\|f\cdot y\|>1\}} \|f\cdot y\| m_0(dy)\,d\lambda \leqslant \|b\|\int_T \int_{\{(\|b\|\,\|f\|)^{-1}<\|y\|\leqslant 1\}} \|f\|\,\|y\|\,m(dy)$$

$$\leqslant \|b\|\,c_p\,\sigma(S)\int_{\{\|f\|>\|b\|^{-1}\}} \|f\|\log(\|b\|\,\|f\|)\,d\lambda<\infty,$$

which in view of Proposition 2.1.2 completes the proof.

We end this section with the following comparison theorem.

THEOREM 3.5.7. $\lambda(T) < \infty$. *Assume that* $0 < p_1 < p_2 < 2$ *and* $\sigma_1 \leqslant c\sigma_2$ *for some* $c > 0$. *If* M_i *is a random measure generated by* $\mu_i = \mu_{p_i,\sigma_i}$, $i = 1, 2$, *then* $\mathscr{L}(M_2) \subset \mathscr{L}(M_1)$.

Proof. We shall prove this statement using Theorem 3.3.9. We have for $\mu_i = [0, 0, m_i]$, $i = 1, 2$, and $B \in \mathscr{B}(\{y \in Y: 0 < \|y\| \leqslant 1\}$

$$m_1(B) = c_{p_1} \int_0^1 \int_S \mathbf{1}_B(uy)\,\sigma_1(du)\frac{du}{u^{1+p_1}}$$

$$\leqslant cc_{p_1} \int_0^1 \int_S \mathbf{1}_B(uy)\,\sigma_2(du)\frac{du}{u^{1+p_1}} \leqslant cc_{p_1}\, m_2(B).$$

IV. Random integrals of Banach space valued functions with respect to real valued random measures

1. Immediate corollaries from a general theory of random integrals and examples. Let E be a separable Banach space. In this chapter we will study random integrals of the form $\int_T f dM$, where $f\colon T \to E$ and M is a random measure generated by an infinitely divisible p.m. on the real line. Since a covariance operator on $\boldsymbol{R}$ is characterized by one real number $\sigma^2 \geqslant 0$, we shall write $M \sim [a, \sigma^2, m]$ to represent the distribution of M. $\mathscr{L}_E(M)$ will stand for the space of all M-integrable functions. It is a Fréchet space with the F-norm

$$\|f\|_M = \|f\|_{L_E^0(\lambda)} + \sup_{A \in \Sigma} \Big\|\int_A f dM\Big\|_{L_E^0(P)}$$

(see Theorem 3.2.2). Formula (3.2) in this case reads

$$V(x) = ax + \int_{-\infty}^{\infty} v\,[\alpha(\|vx\|) - \alpha(|v|)\,xm(dv)], \qquad x \in E.$$

We adapt the characterization theorem (Theorem 3.3.6) to this case as follows.

THEOREM 4.1.1. *Let* $M \sim [a, \sigma^2, m]$ *be a random measure on* (T, Σ, λ). *A measurable function* $f\colon T \to E$ *is* M-*integrable if and only if*

(i) $a_f = \int_T V \circ f d\lambda$ *exists in the Pettis sense,*

ii) $Q_f(x^*) = \int_T \sigma^2 \langle x^*, f\rangle^2 d\lambda$ *is a Gaussian variance,*

(iii) $\mu_f(B) = \lambda \times m\left(\{(t, u) \in T \times \boldsymbol{R}\colon f(t)u \in B \setminus \{0\}\}\right)$ *is a Lévy measure on E.*
If f is M-integrable then $\mathscr{L}\left(\int\limits_T f dM\right) = [a_f, Q_f, \mu_f]$.

Using the above theorem one can get a full description of $\mathscr{L}_E(M)$ spaces provided that conditions characterizing pre-Gaussian functions and Lévy measures are known, for example if $E = l_p$ (see [45] and [49]).

Urbanik and Woyczyński [44] have characterized $\mathscr{L}_{\boldsymbol{R}}(M)$ as certain Orlicz spaces (assuming additionally that M is symmetric and $\lambda(T) < \infty$) (see also [43]). Their characterization can be easily extended only to the case where E is a Hilbert space. Utilizing notions of the geometry of Banach spaces one can get more information about $\mathscr{L}_E(M)$.

Let $M \sim [a, \sigma^2, m]$ and put

$$\Phi_M(u) = \sigma^2 u^2 + \int\limits_{-\infty}^{\infty} \min\{1, |uv|^2\}\, m(dv).$$

Let $L_E(\Phi_M)$ denote the Orlicz space of all measurable functions $f\colon T \to E$ such that $\int\limits_T \Phi_M(c\|f\|)\, d\lambda < \infty$ for some $c > 0$.

THEOREM 4.1.2 (a) *Assume that E is of type 2. If* $f \in L_E(\Phi_M)$ *and* $V \circ f$ *is Pettis integrable then* $f \in \mathscr{L}_E(M)$.

(b) *Assume that E is of cotype 2. If* $f \in \mathscr{L}_E(M)$ *then* $f \in L_E(\Phi_M)$ *and* $V \circ f$ *is Pettis integrable.*

Proof. We observe that $f \in L_E(\Phi_M)$ if and only if $\int\limits_T \sigma^2 \|f\|^2\, d\lambda < \infty$ and

$$\int\limits_E \min\{1, \|x\|^2\}\, \mu_f(dx) = \int\limits_T \int\limits_{-\infty}^{\infty} \min\{1, \|f(t)u\|^2\}\, m(du)\, \lambda(dt) < \infty.$$

These conditions are sufficient in spaces of type 2 [necessary in spaces of cotype 2] for σf and μ_f to be, respectively, a pre-Gaussian function be a Lévy measure (see [3], Chapter III, Theorems 7.5, 7.6 and 8.16). Using Theorem 4.1.1 we end the proof.

As regards Gaussian random measures, we obtain the converse to Theorem 4.1.2 (see Theorems 4.2.2 and 4.2.3). Also using other notions from the geometry of Banach spaces, such as p and φ types and cotypes etc., one can get results similar to the above characterizing $\mathscr{L}_E(M)$ for some special M's. However, in general, it is imposible to obtain integral type conditions discribing functions from $\mathscr{L}_E(M)$. This follows simply from the fact that, in general, pre-Gaussian functions, as well as Lévy measures, cannot be characterized in that manner, i.e. by integral conditions.

On the other hand, Corollary 3.1.11 gives an interesting condition on f ensured by M-integrability. For a Lévy measure m on $\boldsymbol{R}$ put

$$H_m(u) = m\left((-u^{-1}, u^{-1})\right), \qquad u > 0,$$

and $H_m(0) = 0$. Let $L_E(H_m)$ denote the Orlicz space of all measurable functions $f\colon T \to E$ such that $\int_T H_m(c\|f\|)\,d\lambda < \infty$ for some $c > 0$.

PROPOSITION 4.1.3. *Let $M \sim [a, \sigma^2, m]$. Then $\mathscr{L}_E(M) \subset L_E(H_m)$. Moreover, if $\int_T f_n\,dM \xrightarrow{P} 0$ then for every $c > 0$ we have $\int_T H_m(cf_n)\,d\lambda \to 0$ as $n \to \infty$.*

Using Corollary 3.3.8 we can also give a sufficient condition for M-integrability; however, it is applicable only for certain classes of random measures.

Let m be a Lévy measure on $\boldsymbol{R}$ such that $\int_{-1}^{1} |u|\,m(du) < \infty$. Put

$$F_m(v) = \int_{-\infty}^{\infty} \min\{1, v|u|\}\,m(du), \qquad v \geqslant 0.$$

Let $L_E(F_m)$ denote the Orlicz space of all measurable functions $f\colon T \to E$ such that $\int_T F_m(c\|f\|)\,d\lambda < \infty$ for some $c > 0$.

PROPOSITION 4.1.4. *Assume that $M \sim [a, 0, m]$ and $\int_{-1}^{1} |u|\,m(du) < \infty$. If $f \in L_E(F_m)$ and $V \circ f$ is Pettis integrable then $f \in \mathscr{L}_E(M)$.*

Let us now recall another form of the characterization theorem useful when the function $K = \log \hat{v}$ is given (see Theorem 3.3.10). We have in this case

$$K(u) = iau - \tfrac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} [e^{iuv} - 1 - iuv\alpha(|v|)]\,m(dv).$$

THEOREM 4.1.5. *A measurable function $f\colon T \to E$ is M-integrable if and only if*

(i) *$V \circ f$ is Pettis integrable,*

(ii) *for every $x^* \in E^*$, $\int_T |K(\langle x^*, f\rangle)|\,d\lambda < \infty$,*

(iii) *$\varphi_f(x^*) = \exp\{\int_T K(\langle x^*, f\rangle)\,d\lambda\}$ is the characteristic functional of a p.m. on E.*

Moreover, if f is M-integrable then $\mathscr{L}(\int_T f\,dM)\hat{}\,(x^*) = \varphi_f(x^*)$.

In this chapter we shall examine the following examples of random measures.

(a) A white noise $M \sim [0, 1, 0]$.

(b) A standard p-stable random measure

$$K(u) = -|u|^p, \qquad 0 < p < 2.$$

(c) A stable random measure

$$K(u) = ibu - c|u|^p \left\{1 + i\beta \frac{u}{|u|}\,\omega(|u|, p)\right\},$$

where

$$\omega(|u|, p) = \begin{cases} \operatorname{tg} \dfrac{\pi p}{2} & \text{if } p \neq 1, \\ \dfrac{2}{\pi} \log|u| & \text{if } p = 1, \end{cases}$$

$c \geqslant 0$, $|\beta| \leqslant 1$ and $0 < p < 2$.

(d) A semi-stable symmetric random measure

$$K(u) = \sum_{n=-\infty}^{\infty} b^{-pn} \int_{[1, b^{-1}]} [\cos(b^n uv) - 1]\, q(dv),$$

where $0 < b < 1$, $0 < p < 2$ and q is a finite Borel measure (see [21]).

(e) A gamma random measure

$$K(u) = -\beta \log(1 - iu/\theta),$$

where $\beta, \theta > 0$ are parameters. In this case the distribution of $M(A)$ has the density

$$f(x) = \frac{\theta^{\beta\lambda(A)}}{\Gamma(\beta\lambda(A))} x^{\beta\lambda(A)-1} e^{-\theta x}, \quad x > 0$$

and $f(x) = 0$ if $x \leqslant 0$.

(f) A symmetric gamma random measure

$$K(u) = -\beta \log(1 + u^2/\theta^2),$$

where $\beta, \theta > 0$ are parameters. The distribution of $M(A)$ arises by symmetrization of a gamma random measure given above.

We will give a full characterization of the spaces of all functions integrable with respect to a gamma or a symmetric gamma distribution. If M is a gamma random measure with the parameters β and θ then $M \sim [a_{\beta,\theta}, 0, m_{\beta,\theta}]$, where $a_{\beta,\theta} = \beta \int_0^\infty e^{-\theta u}(1+u^2)^{-1}\, du$ and $m_{\beta,\theta}$ is concentrated on $(0, \infty)$ with the density $m_{\beta,\theta}\, du = \beta e^{-\theta u} u^{-1}\, du$ (see [27], Ch. V, Table 5). Hence

$$V_{\beta,\theta}(x) = \{\beta \int_0^\infty e^{-\theta u} [(1+u^2)^{-1} + \alpha(u\|x\|) - \alpha(u)]\, du\} x, \quad x \in E,$$

and

$$H_{m_{\beta,\theta}}(u) = \beta \int_{u^{-1}}^{\infty} e^{-\theta v} v^{-1}\, dv, \quad u > 0.$$

The symmetrization of M, i.e. the symmetric gamma random measure with the parameters β and θ, will be denoted by M^s. Thus $M^s \sim [0, 0, \tilde{m}_{\beta,\theta}]$, where $\tilde{m}_{\beta,\theta}(B) = m_{\beta,\theta}(B) + m_{\beta,\theta}(-B)$, $B \in \mathscr{B}(\boldsymbol{R})$ and

$$H_{\tilde{m}_{\beta,\theta}}(u) = 2\beta \int_{u^{-1}}^{\infty} e^{-\theta v} v^{-1}\,dv, \quad u > 0.$$

PROPOSITION 4.1.6. *Assume that $\lambda(T) < \infty$. Let M^s be a symmetric gamma random measure with the parameters β and θ. Then $f \in \mathscr{L}_E(M^s)$ if and only if $\int_T \log(1 + \|f\|)\,d\lambda < \infty$.*

Proof. It is easy to check that

$$H_{\tilde{m}_{\beta,\theta}}(u) \sim 2\beta \log(1+u) \quad \text{as } u \to \infty.$$

Necessity follows hence by the inclusion $\mathscr{L}_E(M^s) \subset L_E(H_{\tilde{m}_{\beta,\theta}})$ proved in Proposition 4.1.3. We prove the converse by Proposition 4.1.4. We have

$$F_{m_{\beta,\theta}}(v) = 2\beta \int_0^{\infty} \min\{1, vu\} e^{-\theta u} u^{-1}\,du \sim 2\beta \log(1+u) \quad \text{as } u \to \infty.$$

This completes the proof.

PROPOSITION 4.1.7. *Assume that $\lambda(T) < \infty$. Let M be a gamma random measure with the parameters β and θ. Then $f \in \mathscr{L}_E(M)$ if and only if f is Pettis integrable and $\int_T \log(1 + \|f\|)\,d\lambda < \infty$.*

Proof. Using the fact that μ is a Lévy measure if and only if $\tilde{\mu}(B) + \mu(-B)$ is so and applying the characterization theorem we conclude that $f \in \mathscr{L}_E(M)$ if and only if $f \in \mathscr{L}_E(M^s)$ and $V_{\beta,\theta} \circ f$ is Pettis integrable. Hence it remains to prove that $V_{\beta,\theta} \circ f$ is Pettis integrable if and only if f is. Observe that

$$V_{\beta,\theta}(x) = \big(c_{\beta,\theta} + \delta(\|x\|)\big)x, \quad x \in E,$$

where

$$c_{\beta,\theta} = \beta \int_0^{\infty} e^{-\theta u}[(1+u^2)^{-1} - \alpha(u)]\,du < 0$$

and

$$\delta(v) = \beta \int_0^{\infty} e^{-\theta u} \alpha(uv)\,du, \quad v \geqslant 0.$$

$\delta\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is bounded, continuous and $\delta(v) \to 0$ as $v \to \infty$. Hence there exists $v_0 < 0$ such that for every $v > v_0$, $\delta(v) < \frac{1}{2}|c_{\beta,\theta}|$. Assume that f is Pettis integrable. Since $\delta(\|f\|)$ is bounded, $\delta(\|f\|)f$ is Pettis integrable, hence so is $V \circ f$. For the converse, write $\varphi(t) = [c_{\beta,\theta} + \delta(\|f(t)\|]^{-1}$ if $\|f(t)\| > v_0$ and $\varphi(t)$

$= 0$ in the other case. Since φ is a bounded measurable function and $\varphi V \circ f = f\mathbf{1}_{\{\|f\|>v_0\}}$, the integrability of $V \circ f$ implies the integrability of f. The proof if complete.

Every random measure M generates the linear space of infinitely divisible random vectors of the form

$$\{\int_T f dM \colon f \in \mathscr{L}_E(M)\}.$$

We can also consider the linear space

$$[M]_E = \{\sum_{j=1}^{n} x_j M(A_j) \colon x_j \in E,\ A_j \in \Sigma_f,\ n \in N\}.$$

Let $\mathrm{cl}_0 [M]_E$ denote the closure of $[M]_E$ in $L_E^0(\Omega, \mathscr{F}, P)$. We prove the following

THEOREM 4.1.8. *Let $M \sim [a, Q, m]$ be a random measure such that $m \neq 0$ and $V \equiv 0$. Then*

$$\mathscr{L}_E(M) \ni f \mapsto \int_T f dM \in \mathrm{cl}_0 [M]_E$$

is an isomorphism.

Proof. Put $I(f) = \int_T f dM$. In view of the definition of the random integral we have $I(f) \in \mathrm{cl}_0 [M]_E$. By assumption there exists $t_0 > 0$ such that $H_m(t) > 0$ for $t \geqslant t_0$. If $I(f) = 0$ then by Proposition 4.1.3 $\int_T H_m(c\|f\|)\, d\lambda = 0$ for every $c > 0$. Hence $f = 0$ λ-a.e., which proves that I is one-to-one. Since I is continuous, it is enough to prove that I is "onto". Let $X \in \mathrm{cl}_0 [M]_E$. There exists a sequence $\{X_n\} \subset [M]_E$ such that $X_n \to X$ in P. Since $X_n = \sum_{j=1}^{k_n} x_{nj} M(A_{nj})$, where $x_{nj} \in E$ and $A_{nj} \in \Sigma_f$ are pairwise disjoint for $1 \leqslant j \leqslant k_n$, we can write $X_n = \int_T f_n dM$, where $f_n = \sum_{j=1}^{k_n} x_{nj} \mathbf{1}_{A_{nj}}$ and by Proposition 4.1.3 we get

$$\lim_{n,m\to\infty} \int_T H_m(c\|f_n - f_m\|)\, d\lambda = 0 \quad \text{for every } c > 0.$$

Thus for every $\varepsilon > 0$

$$\lambda\{\|f_n - f_m\| \geqslant \varepsilon\} \leqslant \lambda\{H_m(\varepsilon^{-1} t_0 \|f_n - f_m\|) \geqslant H_m(t_0)\}$$

$$\leqslant \frac{1}{H_m(t_0)} \int_T H_m(\varepsilon^{-1} t_0 \|f_n - f_m\|)\, d\lambda \to 0$$

as $n, m \to \infty$. Theorem 3.2.4 completes the proof.

It is interesting to note that the assumptions $m \neq 0$ and $V \equiv 0$ are both necessary in the above theorem. Indeed, if M is a white noise, i.e. $M \sim [0, 1, 0]$, then by Proposition 4.2.7 I is an isomorphism only under some additional assumptions on E. Now, if $m \neq 0$ but $V \not\equiv 0$, as e.g. is the case when M is a stable random measure with the parameters $(p, 1, 1, 0)$, $0 < p < 1$, then it can be shown that I is not an isomorphism even in the case $E = \boldsymbol{R}$ (see Proposition 4.2.14). Indeed, it is enough to choose a sequence $\{f_n\}$ such that $\int_T |f_n|\, d\lambda < \infty$ for every $n \leqslant 1$ and $\int_T f_n\, d\lambda \to 0$, $\int_T |f_n|\, d\lambda \to \infty$, $\int_T |f_n|^p\, d\lambda \to 0$. Then

$$\mathscr{L}\big(\int_T f_n dM\big) = [0, 0, \mu_{f_n}] \Rightarrow [0, 0, 0];$$

hence $I(f_n) \to 0$ but $f_n \not\to 0$ in $\mathscr{L}_{\boldsymbol{R}}(M)$, because $\mathscr{L}_{\boldsymbol{R}}(M) = L^1$ in this case.

Random integrals, which can be treated as a generalization of random series, give clear representations of some clases of infinitely divisible random vectors.

PROPOSITION 4.1.9. *Let $T = \boldsymbol{R}_+$ and $M \sim [0, 0, m]$, where $m = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, i.e. M is generated by the symmetrization of a Poisson process. Then for every symmetric Lévy measure μ on E there exists $f\colon \boldsymbol{R}_+ \to E$ such that*

$$\mathscr{L}\big(\int_{\boldsymbol{R}_+} f dM\big) = c\,\mathrm{Pois}(\mu).$$

Proof. Put $a_n = \mu\{1/n \leqslant \|x\| < 1/(n-1)\} < \infty$ and $b_n = a_1 + \ldots + a_n$, $n \geqslant 1$. Let l denote the Lebesgue measure. By a standard argument there exist Borel functions $g_n\colon [b_{n-1}, b_n) \to E$ such that $l(\{t \in [b_{n-1}, b_n)\colon g_n(t) \in B\}) = \mu(B \cap \{1/n \leqslant \|x\| < 1/(n-1)\})$ for every $B \in \mathscr{B}(E)$, $n \geqslant 1$. We set

$$f(t) = \sum_{n=1}^{\infty} g_n(t)\,\mathbf{1}_{[b_{n-1}, b_n)}(t), \qquad t \in \boldsymbol{R}.$$

Then for every $B \in \mathscr{B}(E)$ we have

$$\mu_f(B) = l \times m\big(\{(t, u)\colon f(t)u \in B \setminus \{0\}\}\big) = \tfrac{1}{2} l\big(\{t\colon f(t) \in B \setminus \{0\}\}\big)$$

$$+ \tfrac{1}{2} l\big(\{t\colon f(t) \in -B \setminus \{0\}\}\big) = \tfrac{1}{2} \sum_{n=1}^{\infty} l\big(\{t \in [b_{n-1}, b_n)\colon f(t) \in B \setminus \{0\}\}\big)$$

$$+ \tfrac{1}{2} \sum_{n=1}^{\infty} l\big(\{t \in [b_{n-1}, b_n)\colon f(t) \in -B \setminus \{0\}\}\big) = \mu(B).$$

An appeal to Theorem 4.1.1 completes the proof.

2. Gaussian and stable random integrals. In this section we study properties of the spaces $\mathscr{L}_E(M_p)$, where M_p denotes a standard p-stable random measure if $0 < p < 2$ and M_2 is white noise. Some properties, as e.g. the contraction principles, were proved in a more general form in Chapter 3, § 5 and we do not rewrite them. To establish relations between $\mathscr{L}_E(M_p)$ and L_E^p we use the notions of types and cotypes. Applications to the theory of stable measures on Banach spaces end this section.

N. Z. Tien [41] has considered the space S_E^p of all measurable functions $f\colon T \to E$ such that

$$x^* \to \exp\{-\int_T |\langle x^*, f(t)\rangle|^p \lambda(dt)\}$$

is the characteristic functional of a p.m. on E, $0 < p \leqslant 2$, however without any reference to a random integral. From Theorem 4.1.5 we get immediately

PROPOSITION 4.2.1. *$\mathscr{L}_E(M_p) = S_E^p$ for $0 < p \leqslant 2$.*

This above fact was established in [35] for $0 < p < 2$ and under some additional assumptions in [37]. The case $p = 2$ was stated in [8] and [38].

We will need the notations of stable types and cotypes of Banach spaces. Let $\{\theta_n^{(p)}\}$ be a sequence of independent real valued standard p-stable random variables, $0 < p \leqslant 2$, i.e. such that $E\exp\{iu\theta_n^{(p)}\} = \exp\{-b_p|u|^p\}$, where $b_p = 1$ if $0 < p < 2$ and $b_2 = 1/2$. We say that E is of *stable type* p if and only if for every sequence $\{x_n\} \subset E$ such that $\sum \|x_n\|^p < \infty$ the random series $\sum \theta_n^{(p)} x_n$ converges a.s. We say that E is of *stable cotype* p if $\sum \|x_n\|^p < \infty$ for every sequence $\{x_n\} \subset E$ such that $\sum \theta_n^{(p)} x_n$ converges a.s.. E is of stable type p [stable cotype p] if and only if for every $0 < r < p^*$ there exists a constant C such that for every finite sequence $\{x_n\} \subset E$

$$\left(E\left\|\sum \theta_n^{(p)} x_n\right\|^r\right)^{1/r} \leqslant C\left(\sum \|x_n\|^p\right)^{1/p} \qquad \left[\left(\sum \|x_n\|^p\right)^{1/p} \leqslant C\left(E\left\|\sum \theta_n^{(p)} x_n\right\|^r\right)^{1/r}\right]$$

where $p^* = p$ for $0 < p < 2$ and $p^* = \infty$ if $p = 2$. Every Banach space is of stable type p for $0 < p < 1$ and of stable cotype p for $0 < p < 2$. A Banach space is of stable type 2 [stable cotype 2] if and only if it is of Rademacher type 2 [Rademacher cotype 2]. In that case we say simply that E is of type 2 [cotype 2]. All these facts and other detailes concerning of types and cotypes of Banach spaces can be found e.g. in [47].

The next theorem was proved in [38] (see also [8]).

THEOREM 4.2.2. *Assume that λ is not atomic. Then the following are equivalent*:

(i) $L_E^\infty \cap L_E^2 \subset \mathscr{L}_E(M_2)$,

(ii) $L_E^2 \subset \mathscr{L}_E(M_2)$,

(iii) *E is of type 2.*

Proof. (iii) $\Rightarrow$ (ii). If f is a simple function, $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$, then

$$E\left\|\int_T f\,dM_2\right\|^2 = E\left\|\sum_{j=1}^{n} x_j M_2(A_j)\right\|^2 \leqslant C\sum \|x_j\|^2 \lambda(A_j) = c\int_T \|f\|^2\,d\lambda,$$

where C is a universal constant. This establishes (ii). Obviously (ii) $\Rightarrow$ (i). We now prove that (i) $\Rightarrow$ (iii). Let $\{\theta_n^{(2)}\}$ be a sequence of independent $N(0,1)$ random variables. Let $\{x_n\} \subset E$ be such that $\sum \|x_n\|^2 = 1$ and $x_n \neq 0$ for every $n \geqslant 1$. Let $T_0 \in \Sigma_f$ be a non-zero set such that $\lambda | T_0$ is continuous. We choose a pairwise disjoint sequence $\{A_n\} \subset \Sigma$ such that $\bigcup A_n = T_0$ and $\lambda(A_n) = \lambda(T_0)\|x_n\|^2$. Define $f = \sum \|x_n\|^{-1} x_n \mathbf{1}_{A_n}$. Then $f \in L_E^\infty \cap L_E^2$ and by (i) $f \in \mathscr{L}_E(M_2)$. Since

$$\exp\{-\lambda(T_0) \sum_{n=1}^{\infty} \langle x^*, x_n\rangle^2\} = E\exp\{i\langle x^*, \int_T f dM_2\rangle\},$$

the series $\sum \theta_n^{(2)} x_n$ converges a.s. by Ito–Nisio's theorem.

The following result is a generalization of Proposition 6.2 in [38] when we put in (i) e.g. $\Phi(u) = \log(1+u)$, and of Theorem 3.9 in [8] if we take $\Phi(u) = u^p$.

THEOREM 4.2.3. *Assume that λ is not atomic. Then the following are equivalent*:

(i) $\mathscr{L}_E(M_2) \subset L_E(\Phi)$ *for some non-decreasing function* Φ: $\mathbf{R}_+ \to \mathbf{R}_+$ *such that* $\Phi(+\infty) = \infty$,

(ii) $\mathscr{L}_E(M_2) \subset L_E^2$,

(iii) E *is of cotype* 2.

Proof. (iii) $\Rightarrow$ (ii). If f is a simple function, $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$, then

$$\int_T \|f\|^2 d\lambda = \sum_{j=1}^{n} E\|x_j M_2(A_j)\|^2 \leqslant CE\Big\|\sum_{j=1}^{n} x_j M_2(A_j)\Big\|^2 = CE\Big\|\int_T f dM_2\Big\|^2.$$

This shows (ii) because simple functions are dense in $\mathscr{L}_E(M)$. Obviously (ii) $\Rightarrow$ (i). We now prove that (i) $\Rightarrow$ (iii). In the proof we use a technical lemma which says that there exists a sequence of positive numbers $\{c_k\}$ such that $\sum c_k < \infty$ and, for every $\varepsilon > 0$, $\sum \Phi(\varepsilon c_k^{-1/2}) c_k = \infty$. We first finish the proof of the theorem and then we prove the lemma. Assume, to the contrary that E is not of cotype 2, i.e. there exists a sequence $\{x_n\} \subset E$, $x_n \neq 0$, such that $\sum \|x_n\|^2 = \infty$ and $\sum \theta_n^{(2)} x_n$ converges a.s.. Put $M = \sup_n \|x_n\|^2 < \infty$.

Define $n_0 = 0$,

$$n_k = \max\{n\colon M \leqslant \sum_{j=n_{k-1}+1}^{n} \|x_j\|^2 < 2M\}$$

and $b_j = c_k$ for $n_{k-1} < j \leqslant n_k$, $k \geqslant 1$. Let $T_0 \in \Sigma_f$ be a non-zero set such that $\lambda | T_0$ is continuous. Since

$$0 < \sum_{j=1}^{\infty} b_j \|x_j\|^2 = \sum_{k=1}^{\infty} \sum_{j=n_{k-1}+1}^{n_k} c_k \|x_j\|^2 < 2M \sum_{k=1}^{\infty} c_k < \infty,$$

we can choose a pairwise disjoint sequence $\{A_j\} \subset \Sigma$ such that $\bigcup A_j = T_0$ and $\lambda(A_j) = cb_j \|x_j\|^2$ where $c > 0$ is a constant. Define $f = \sum_{j=1}^{\infty} \lambda^{-1/2}(A_j) x_j \mathbf{1}_{A_j}$. We have $f \in \mathscr{L}_E(M_2)$ but for every $\varepsilon > 0$.

$$\int_T \Phi(\varepsilon \|f\|) \, d\lambda = \sum_{j=1}^{\infty} \Phi(\varepsilon \lambda^{-1/2}(A_j) \|x_j\|) \lambda(A_j)$$

$$= c \sum_{j=1}^{\infty} \Phi(\varepsilon c^{-1/2} b_j^{-1/2}) b_j \|x_j\|^2$$

$$\geqslant cM \sum_{k=1}^{\infty} \Phi(\varepsilon c^{-1/2} c_j^{-1/2}) c_k = \infty .$$

Thus $f \notin L_E(\Phi)$, which completes the proof.

LEMMA 4.2.4. *Let Φ: $\mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\Phi(+\infty) = +\infty$. Then there exists a sequence $\{c_k\}$ of positive numbers such that $\sum c_k < \infty$ and, for every $\varepsilon > 0$, $\sum \Phi(\varepsilon c_k^{-1/2}) c_k = \infty$.*

Proof. Put $m_0 = n_0 = 0$ and $n_k = n_{k-1} + m_k$, where

$$m_k = \min\{m: \Phi(m^{1/2}) \geqslant k^2\}, \quad k = 1, 2, \ldots$$

Define $c_j = m_k^{-1} k^{-2}$ for $n_{k-1} < j \leqslant n_k$, $k \geqslant 1$. Then $\sum_{j=1}^{\infty} c_j = \sum_{k=1}^{\infty} k^{-2} < \infty$ and for every $\varepsilon > 0$

$$\sum_{j=1}^{\infty} \Phi(\varepsilon c_j^{-1/2}) c_j = \sum_{k=1}^{\infty} \Phi(\varepsilon k m_k^{1/2}) k^{-2} \geqslant \sum_{k \geqslant \varepsilon^{-1}} \Phi(m_k^{1/2}) k^{-2} = \infty .$$

The proof is complete.

The following result shows that if E is of cotype 2 and $\lambda(T) < \infty$ then $\mathscr{L}_E(M_2)$ is the smallest space among all $\mathscr{L}_E(M)$ spaces. Corollary 4.2.9 says that this is true only for spaces of cotype 2.

THEOREM 4.2.5. *Assume that E is of cotype 2 and $\lambda(T) < \infty$. Then for every symmetric random measure M we have $\mathscr{L}_E(M_2) \subset \mathscr{L}_E(M)$.*

Proof. Let $f \in \mathscr{L}_E(M_2)$. In view of Remark 2.2.1 and Lemma 3.3.5, it is enough to prove that $f \in \mathscr{L}_E(M_0)$, where M_0 is the measure occuring in that remark. Let $\mu_{0,f}(B) = \lambda \times m\{(t, u): |u| \leqslant 1, f(t) u \in B \setminus \{0\}\}$. We have

$$\int_E \langle x^*, x \rangle^2 \mu_{0,f}(dx) = \int_T \int_{-1}^{1} \langle x^*, f(t) u \rangle^2 du \lambda(dt) = \tfrac{2}{3} \int_T \langle x^*, f \rangle^2 d\lambda .$$

Hence $\mu_{0,f}$ is a Lévy measure by Corollary 7.14 in [3]. The proof is complete.

In the sequel we shall need the following lemma.

LEMMA 4.2.6. *Assume that E is not of cotype 2 and λ is not atomic. Then there exists $T_0 \in \Sigma$ such that $\lambda | T_0$ is continuous, $0 < \lambda(T_0) < \infty$, and a sequence f_n: $T_0 \to E$ such that $\int_{T_0} f_n dM_2 \to 0$ in P, while $\inf_{t \in T_0} \|f_n(t)\| \to \infty$ as $n \to \infty$.*

Proof. Let $T_0 \in \Sigma_f$ be a non-zero set such that $\lambda | T_0$ is continuous. By assumption, there exists a sequence $\{x_n\} \subset E$, $x_n \neq 0$, such that $a_n = \sum_{j=1}^{n} \|x_j\|^2 \to \infty$ and $\sum \theta_n^{(2)} x_n$ converges a.s.. For every $n \geqslant 1$ let $\{A_{nj}\}_{j=1}^n$ be a partition of T_0 such that $\lambda(A_{nj}) = \lambda(T_0) a_n^{-1} \|x_j\|^2$. Put

$$f_n = a_n^{1/4} \sum_{j=1}^{n} \|x_j\|^{-1} x_j \mathbf{1}_{A_{nj}}.$$

Then we have

$$\mathscr{L}\big(\int_{T_0} f_n dM_2\big) = \mathscr{L}\big(\lambda^{1/2}(T_0) a_n^{-1/4} \sum_{j=1}^{n} \theta_j^{(2)} x_j\big) \Rightarrow \delta_0$$

and $\|f_n(t)\| = a_n^{1/4} \to \infty$ for every $t \in T_0$. The proof is complete.

Now we pass to stable random integrals. If $0 < p < 2$ then $M_p \sim [0, 0, m_p]$ and we have

$$H_{m_p}(u) = m_p\big((-u^{-1}, u^{-1})^c\big) = 2c_p u^p.$$

For $0 < p \leqslant 2$ we set

$$[M_p]_E = \{\sum_{j=1}^{\infty} x_j M_p(A_j) \colon x_j \in E,\ A_j \in \Sigma_f,\ n \in N\}.$$

$[M_p]_E$ is a linear space consisting of p-stable random vectors. $[M_p]_E$ can be treated as a subspace of $L_E^r(\Omega, \mathscr{F}, P)$ for $0 \leqslant r < p^*$, where $p^* = p$ if $p < 2$ and $p^* = \infty$ if $p = 2$. All relative topologies of $L_E^r(\Omega, \mathscr{F}, P)$ $0 \leqslant r < p^*$, coincide on $[M_p]_E$ (see [7], Th.6.1). Hence the closures $\mathrm{cl}_r[M_p]_E$ of $[M_p]_E$ in $L_E^r(\Omega, \mathscr{F}, P)$ are identical for every $0 \leqslant r < p^*$ and the Fréchet spaces coinsisting of p-stable random vectors are isomorphic.

The first part of the next theorem was proved in [35] (Th.6.1).

THEOREM 4.2.7. *Let $0 < p \leqslant 2$ and $0 \leqslant r < p^*$. Consider the linear injection I_p: $\mathscr{L}_E(M_p) \to \mathrm{cl}_r[M_p]_E$, $I_p(f) = \int_T f dM_p$.*

Then for every $p \in (0, 2)$, I_p is an isomorphism. If E is of cotype 2 then I_2 is an isomorphism. If λ is not atomic then I_2 is an isomorphism if and only if E is of cotype 2.

Proof. The case $0 < p < 2$ follows immediately from Theorem 4.1.8 and the above discussion.

We now consider $p = 2$. If $I_2(f) = 0$ then for every $x^* \in E^*$, $\int_T \langle x^*, f \rangle^2 d\lambda$

$= 0$. Since E is separable, $f = 0$ λ-a.e. Thus I_2 is one-to-one and of course continuous.

Assume that E is of cotype 2. We prove that E is „onto". Let $X \in \mathrm{cl}_r[M_2]_E$. There exists a sequence $\{X_n\} \subset [M]_E$ such that $X_n \to X$ in $L_E^r(\Omega, \mathcal{F}, P)$. Since $X_n = \sum_{j=1}^{k_n} x_{nj} M_2(A_{nj})$, where $x_{nj} \in E$ and $A_{nj} \in \Sigma_f$ are pairwise disjoint for $1 \leqslant j \leqslant k_n$, we can write $X_n = \int_T f_n dM_2$, where $f_n = \sum_{j=1}^{k_n} x_{nj} \mathbf{1}_{A_{nj}}$ and by Theorem 4.2.3 (ii) $\lim_{n,m\to\infty} \int_T \|f_n - f_m\|^2 d\lambda = 0$. Thus Theorem 3.2.4 gives $f \in \mathcal{L}_E(\mathcal{M}_2)$ and $X = \int_T f dM_2$.

Assume contrary to the assertion, that I_2 is an isomorphism and E is not of cotype 2. Let $\{f_n\}$ be a sequence with properties as in Lemma 4.2.6. Then $I_2(f_n) = \int_T f_n dM_2 \to 0$ in $\mathrm{cl}_r[M_2]_E$, while $f_n \not\to 0$ in $\mathcal{L}_E(M_2)$. A contradiction.

Corollary 4.2.8. *For* $0 < r < p < 2$

$$f \to \Big(E\Big\|\int_T f dM_p\Big\|^r\Big)^{1/r}$$

is a quasi-norm (a norm if $r \geqslant 1$*) on* $\mathcal{L}_E(M_p)$*. Hence* $\mathcal{L}_E(M_p)$ *is a Banach space for* $1 < p < 2$*. If E is of cotype 2 then* $\mathcal{L}_E(M_2)$ *is a Banach space with the norm*

$$f \to \Big(E\Big\|\int_T f dM_2\Big\|^r\Big)^{1/r}, \quad 1 \leqslant r < \infty.$$

If for some $1 \leqslant r < \infty$ *the norm* $f \to \big(E\|\int_T f dM_2\|^r\big)^{1/r}$ *on* $\mathcal{L}_E(M_2)$ *is complete and* λ *is not atomic then E is of cotype* 2.

The next result was proved in [35] (Th.6.4).

Theorem 4.2.9. *Let* $0 < p < 2$. *Then*

(i) $\mathcal{L}_E(M_p) \subset L_E^p$,

(ii) $\mathcal{L}_E(M_p) = L_E^p$ *for* $0 < p < 1$,

(iii) $\mathcal{L}_E(M_{p_2}) \subset \mathcal{L}_E(M_{p_1})$ *provided that* $\lambda(T) < \infty$ *and* $0 < p_1 \leqslant p_2 \leqslant 2$.

Proof. By Proposition 4.1.3 we get (i). To establish (ii) we use Proposition 4.1.4. Since

$$F_{m_p}(v) = 2c_p\Big(\frac{1}{1-p} + \frac{1}{p}\Big)v^p,$$

we have $L_E^p \subset \mathcal{L}_E(M_p)$ and (i) completes the proof. (iii) follows by Theorem 3.5.7.

Corollary 4.2.10. *Assume that* λ *is not atomic. Then* $\mathcal{L}_E(M_2)$

$\subset \mathscr{L}_E(M_p)$ *for some (for every)* $p \in (0, 2)$ *if and only if* E *is of cotype* 2.

Proof. If $\mathscr{L}_E(M_2) \subset \mathscr{L}_E(M_p)$ then $\mathscr{L}_E(M_2) \subset L_E^p$ and we apply Theorem 4.2.3. The converse follows by Theorem 4.2.5.

The following result has been obtained in [29] and [37].

THEOREM 4.2.11. *If* E *is of stable type* p, $p \in (0, 2)$ *then* $\mathscr{L}_E(M_p) = L_E^p$. *If* λ *is not atomic and* $\mathscr{L}_E(M_p) = L_E^p$ *then* E *is of stable type* p.

Proof. For a simple function $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$ and $0 < r < p$ we have

$$(E\|\int_T f dM_p\|^r)^{1/r} = (E\|\sum_{j=1}^{n} x_j M_p(A_j)\|^r)^{1/r}$$

$$\leqslant C\Big(\sum_{j=1}^{n} \|x_j\|^p \lambda(A_j)\Big)^{1/p} = C\Big(\int_T \|f\|^p d\lambda\Big)^{1/p}.$$

This proves the first claim of the theorem. Now, let $\mathscr{L}_E(M_p) = L_E^p$ and $T_0 \in \Sigma_f$ be a non-zero set such that $\lambda | T_0$ is continuous. Let $\{x_n\} \subset E$ be such that $\sum \|x_n\|^p < \infty$, $x_n \neq 0$, and let $\{A_n\}_{n=1}^{\infty}$ be a partition of T_0 such that $\lambda(A_n) = \lambda(T_0)\big[\sum_{j=1}^{\infty} \|x_j\|^p\big]^{-1} \|x_n\|^p$. We define $f = \sum_{n=1}^{\infty} \lambda^{-1/p}(A_n) x_n \mathbf{1}_{A_n}$. Since $\int_T \|f\|^p d\lambda = \sum_{n=1}^{\infty} \|x_n\|^p < \infty$, we have $f \in \mathscr{L}_E(M_p)$ and

$$E \exp\{i\langle x^*, \sum_{j=1}^{n} \theta_j^{(p)} x_j\rangle\} \to \exp\{-\int_T \langle x^*, f\rangle^p d\lambda\}.$$

Proposition 4.2.1 and Ito–Nisio's theorem complete the proof.

Now we consider questions concerning the existence of homogeneous complete norms on $\mathscr{L}_E(M_1)$ and on $\mathscr{L}_E(M_2)$, respectively. By Theorem 4.2.11, if E is of stable type 1 then $\mathscr{L}_E(M_1) = L_E^1$ and it is of course a Banach space in that case.

THEOREM 4.2.12. *Assume that there exists a sequence of pairwise disjoint sets* $\{A_n\}_{n=1}^{\infty} \subset \Sigma$ *such that* $0 < \lambda(A_n) < \infty$ *and* $\lambda(\bigcup_{n=1}^{\infty} A_n) < \infty$. *Then the following statements are equivalent*:

(i) $\mathscr{L}_E(M_1)$ *is isomorphic to a Banach space,*

(ii) E *is of stable type* 1.

Proof. Obviously, in view of Theorem 4.2.11, it is enough to prove that (i) $\Rightarrow$ (ii). First we show that, if for every sequence $\{x_n\} \subset E$, $\sum \theta_n^{(1)} x_n$ is a.s. bounded provided that $\sum \|x_n\| < \infty$, then E is of stable type 1. Put

$$F = \{\{x_n\} \subset E\colon \sum \theta_n^{(1)} x_n \text{ is a.s. bounded}\}.$$

Then F is a Fréchet space with the norm

$$\|\{x_n\}\| = \sup_n \Big(E\Big\|\sum_{j=1}^{n} \theta_j^{(1)} x_j\Big\|^r\Big)^{1/r}, \qquad \{x_n\}\in F,$$

where $0<r<1$. Hence, by the Closed Graph Theorem, E is of stable type 1 if and only if $l_E^1 \subset F$.

Let $\{x_n\}\in l_E^1$, $x_n \neq 0$ for $n\geqslant 1$. Put

$$f_n = [\|x_n\|\,\lambda(A_n)]^{-1}\, x_n \mathbf{1}_{A_n}, \qquad n\geqslant 1.$$

Then $\{f_n\}_{n=1}^{\infty} \subset \mathscr{L}_E(M_1)$ is a bounded sequence in $\mathscr{L}_E(M_1)$ and by (i) the sequence $\{\sum_{j=1}^{n} \|x_j\|\, f_j\}_{n=1}^{\infty}$ is bounded in $\mathscr{L}_E(M_1)$. Hence, by Corollary 4.2.8,

$$\sup_n \Big(E\Big\|\sum_{j=1}^{n} \theta_j^{(1)} x_j\Big\|^r\Big)^{1/r} = \sup_n \Big(E\Big\|\int_T \Big(\sum_{j=1}^{n} \|x_j\|\, f_j\Big)\, dM_1\Big\|^r\Big)^{1/r} \to \infty$$

which completes the proof.

If E is of cotype 2 then $\mathscr{L}_E(M_2)$ is a Banach space by Corollary 4.2.8. The converse follows from the next theorem, which generalizes Proposition 6.3 in [38].

THEOREM 4.2.13. *Assume that λ is not atomic. Then the following statements are equivalent*:

(i) *$\mathscr{L}_E(M_2)$ is locally convex,*

(ii) *E is of cotype 2.*

Proof. By Corollary 4.2.8 (ii) $\Rightarrow$ (i). We prove that (i) $\Rightarrow$ (ii). Assume, to the contrary, that E is not of cotype 2. Let T_0 and $\{f_n\}$ be as in Lemma 4.2.6. Put $b_n = E\|\int_{T_0} f_n\, dM_2\| \to 0$, $k_n = [b_n^{-1/2}]$. Let $\{A_{nj}\}_{j=1}^{k_n}$ be a partition of T_0 such that $\lambda(A_{nj}) = \lambda(T_0)\, k_n^{-1}$. We set $g_{nj} = k_n f_n \mathbf{1}_{A_{nj}}$. Then

$$\lambda\{g_{nj}\neq 0\} \leqslant k_n^{-1}\lambda(T_0) \to 0$$

and

$$E\Big\|\int_T g_{nj}\, dM_2\Big\| = k_n E\Big\|\int_{A_{nj}} f_n\, dM_2\Big\| \leqslant k_n b_n \leqslant b_n^{1/2} \to 0.$$

Let $\mathscr{U}$ be a convex neighbourhood of 0 in $\mathscr{L}_E(M_2)$. There exists $n_0\geqslant 1$ such that, for every $n\geqslant n_0$, $g_{nj}\in\mathscr{U}$ for $j=1,\ldots,k_n$. Hence $f_n = \frac{1}{k_n}\sum_{j=1}^{k_n} g_{nj}\in\mathscr{U}$. We proved that $f_n\to 0$ in $\mathscr{L}_E(M_2)$, hence in λ, which gives a contradiction.

Now we consider general stable random measures. Recall that M is a stable random measure with the parameters (p, b, c, β) if

$$K(u) = -ibu - c|u|^p\Big\{1 + i\beta\frac{u}{|u|}\,\omega(|u|, p)\Big\},$$

where

$$\omega(|u|, p) = \begin{cases} \operatorname{tg} \dfrac{\pi p}{2} & \text{if } p \neq 1, \\ \dfrac{2}{\pi} \log |u| & \text{if } p = 1, \end{cases}$$

$b \in R$, $c > 0$, $|\beta| \leqslant 1$ and $0 < p < 2$. On the other hand, we can present K in the canonical form

$$K(u) = iau + c_1 \int_0^{\infty} (e^{iux} - 1 - iux\alpha(x)) \frac{dx}{x^{1+p}}$$

$$+ c_2 \int_{-\infty}^{0} (e^{iux} - 1 - iux\alpha(x)) \frac{dx}{|x|^{1+p}}$$

(see [3], Ch.II, Cor.6.6). Obviously, there exists a one-to-one correspondence between (b, c, β) and (a, c_1, c_2). By routine computations (see e.g. the proof of Th.5.7.3 in [27]) we infer that $a = b + \dfrac{c_1 - c_2}{p(1-p)}$ for $p \neq 1$ and that $\beta = 0$ if and only if $c_1 = c_2$. Now it is easy to show that

$$V(x) = \begin{cases} ax + \dfrac{c_1 - c_2}{p(1-p)} (\|x\|^{p-1} - 1)\, x & \text{for } p \neq 1 \\ ax + [(c_2 - c_1) \log \|x\|]\, x & \text{for } p = 1. \end{cases}$$

The following theorem gives a full characterization of the space of all functions integrable with respect to a general stable measure. It also generalizes some results of [32], where only certain classes of Banach spaces where considered.

THEOREM 4.2.14. *Let M be a stable random measure with the parameters (p, b, c, β). Then $f \in \mathscr{L}_E(M)$ if and only if*

(i) *$f \in L_E^p$, provided $0 < p < 1$ and $b = 0$,*
(ii) *$f \in L_E^p$ and f is Pettis integrable, provided $0 < p < 1$ and $b \neq 0$,*
(iii) *$f \in S_E^1$, provided $p = 1$ and $\beta = 0$,*
(iv) *$f \in S_E^1$ and $f \log \|f\|$ is Pettis integrable, provided $p = 1$ and $\beta \neq 0$,*
(v) *$f \in S_E^p$, provided $1 < p < 2$ and $b = 0$,*
(iv) *$f \in S_E^p$ and f is Pettis integrable, provided $1 < p < 2$ and $b \neq 0$.*

Proof. Using the fact that μ is a Lévy measure if and only if $\tilde{\mu}(B) = \mu(B) + \mu(-B)$ is so and the characterization theorem we conclude that $f \in \mathscr{L}_E(M)$ if and only if $f \in \mathscr{L}_E(M_p) = S_E^p$ and $V \circ f$ is Pettis integrable.

Let $0 < p < 1$. Since

$$V(x) = \left(a - \frac{c_1 - c_2}{p(1-p)}\right)x + \frac{c_1 - c_2}{p(1-p)}\|x\|^{p-1}x = bx + \frac{c_1 - c_2}{p(1-p)}\|x\|^{p-1}x$$

and by Theorem 4.2.9 (ii) $\mathscr{L}_E(M_p) = L_E^p$, hence $f \in \mathscr{L}_E(M)$ if and only if $f \in L_E^p$ and bf is Pettis integrable. This proves (i) and (ii).

Now let $p = 1$. We have that $f \in \mathscr{L}_E(M)$ if and only if $f \in S_E^1$ and $af + (c_2 - c_1)\log\|f\|$ is Pettis integrable. Since $S_E^1 \subset L_E^1$ and $\beta = 0$ if and only if $c_1 = c_1$, we obtain (iii) and (iv).

Let $1 < p < 2$. Since $V(x) = bx + \dfrac{c_1 - c_2}{p(1-p)}\|x\|^{p-1}x$, we have $f \in \mathscr{L}_E(M)$ if and only if $f \in S_E^p \subset L_E^p$ and bf is Pettis integrable. That proves (v) and (vi) and completes the proof of the theorem.

We now give some applications of stable random integrals to the theory of stable measures on Banach spaces. Put

$$\Omega_{p,\beta}(u) = |u|^p\left\{1 + i\beta\frac{u}{|u|}\omega(|u|, p)\right\},$$

$0 < p < 2$ and $|\beta| \leqslant 1$. It is well known that the characteristic functional of a stable p.m. on a Banach space E has the form

$$\hat{\mu}(x^*) = \exp\{i\langle x^*, a\rangle - \int_S \Omega_{p,\beta}(\langle x^*, x\rangle)\sigma(dx)\},$$

where σ is a finite Borel measure on the unit sphere S of E. σ is called the *spectral measure* of μ, p the *order* of μ and β the *parameter of asymmetry* of μ.

THEOREM 4.2.15. *Let M be a stable random measure with the parameters $(p, 0, 1, \beta)$ on an atomless probability space (T, Σ, λ). Then for every stable p.m. μ of order p and with the parameter of asymmetry β there exist a function $f \in \mathscr{L}_E(M)$ and an $a \in E$ such that*

$$\mathscr{L}\left(\int_T f\,dM\right) = \mu * \delta_a.$$

Proof. Let σ be the spectral measure of μ. We may assume that $\sigma(S) > 0$; in the other case we put $f = 0$. By the standard arguments there exists a Borel function $g\colon T \to E$ such that $\lambda\{t \in T\colon g(t) \in B\} = \dfrac{\sigma(B \cap S)}{\sigma(S)}$, $B \in \mathscr{B}(E)$. Put $f(t) = \sigma^{1/p}(S)g(t)$, $t \in T$. Then

$$\exp\left\{\int_T K(\langle x^*, f\rangle)\,d\lambda\right\} = \exp\left\{-\int_T \Omega_{p,\beta}(\langle x^*, f\rangle)\,d\lambda\right\}$$
$$= \exp\{i\langle x^*, d_p\rangle - \int_S \Omega_{p,\beta}(\langle x^*, x\rangle)\sigma(dx)\},$$

where $d_p = 0$ for $p \neq 1$ and $d_1 = \frac{2\beta}{\pi} \log \sigma(S) \int_S x\sigma(dx)$. Theorem 4.1.5 concludes the proof.

THEOREM 4.2.16. *Let μ be a symmetric p-stable p.m. on E with the spectral measure σ. Then there exist symmetric p-stable measures μ_n with spectral measures σ_n, $n \geqslant 1$, such that*

(i) *σ_n has a finite support,*

(ii) *$\sigma_n \Rightarrow \sigma$,*

(iii) *$\mu_n \Rightarrow \mu$ as $n \to \infty$.*

Proof. Put $T = S$, $\Sigma = \mathscr{B}(S)$, $\lambda = \sigma$ and let M_p be a standard p-stable random measure on (T, Σ, λ). Consider $f\colon T \to E, f(x) = x$. By Theorem 4.1.5 $f \in \mathscr{L}_E(M_p)$ and $\mathscr{L}(\int_T f dM_p) = \mu$. Hence, there exists a sequence of simple functions $\{g_n\}$ such that $g_n \to f$ a.e. and $\int_T g_n dM_p \to \int_T f dM_p$ in P. Set $A_n = \{\sup_{k \geqslant n} \|g_k\| \leqslant 3/2\}$ and $f_n = g_n \mathbf{1}_{A_n}$. Then $A_n \nearrow A_\infty$, where $\lambda(T \setminus A_\infty) = 0$ and

$$\int_T (f_n - f)\, dM_p = \int_T \mathbf{1}_{A_n}(g_n - f)\, dM_p - \int_{A_n^c} f dM_p.$$

By inequality (3.6) in Theorem 3.4.5, $\int_T \mathbf{1}_{A_n}(g_n - f)\, dM_p \to 0$ in P and $\int_{A_n^c} f dM_p \to 0$ in P by Corollary 3.1.3. Thus we have found a uniformly bounded sequence of simple functions $\{f_n\}$ such that $f_n \to f$ a.s. and $\int_T f_n dM_p \to \int_T f dM_p$ in P. Putting $\mu_n = \mathscr{L}(\int_T f_n dM_p)$ we get (iii). We compute σ_n. Since $f_n = \sum_{j=1}^{k_n} x_{nj} \mathbf{1}_{A_{nj}}$, where $x_{nj} \in E$, $x_{nj} \neq 0$ and $A_{nj} \in \Sigma$ are pairwise disjoint, $1 \leqslant j \leqslant k_n$, hence $\sigma_n = \sum_{j=1}^{k_n} p_{nj} \delta_{y_{nj}}$ satisfy (i); here $p_{nj} = \|x_{nj}\|^p \sigma(A_{nj})$ and $y_{nj} = \|x_{nj}\|^{-1} x_{nj}$. We prove (ii). For every continuous bounded function $\varphi\colon S \to \boldsymbol{R}$ we have

$$\lim_{n\to\infty} \int_S \varphi(x)\, \sigma_n(dx) = \lim_{n\to\infty} \sum_{j=1}^{k_n} \varphi\left(\frac{x_{nj}}{\|x_{nj}\|}\right) \|x_{nj}\|^p \sigma(A_{nj})$$

$$= \lim_{n\to\infty} \int_{\{\|f_n\| > 0\}} \varphi\left(\frac{f_n}{\|f_n\|}\right) \|f_n\|^p d\lambda = \int_S \varphi(x)\, \sigma(dx)$$

by the Dominated Convergence Theorem. The proof is complete.

3. Comparison theorem and some applications. In the previous sections we often used the fact that $\mathscr{L}_E(M) \subset L_E(H_m)$. Here we show an other interesting property of H_m. Namely, observing the behaviour of the functions H_m for various random measures M we can compare the appropriate spaces of M-integrable functions $\mathscr{L}_E(M)$. The function H_m will be called the control function of the random measure M.

As it was showed in Proposition 4.1.7 and Theorem 4.2.14, the study of a random integral in the case of non-symmetric random measure M reduces to the study of a random integral with respect to the symmetrization M^s of M and certain non-random Pettis integral. Therefore in this section we shall consider only symmetric random measures and for simplicity we shall write $M \sim [\sigma^2, m]$ instead of $M \sim [0, \sigma^2, m]$, when m is symmetric.

The next lemma has a rather technical character. It is based on the Kingmen's construction of a Poisson point process on an abstract measure space (see [18], Th.7).

LEMMA 4.3.1. *Assume that* $0 < \lambda(T) < \infty$. *Let* $\{\tau_n\}_{n=0}^{\infty}$ *be a sequence of independent random elements in* (T, Σ) *with the common distribution* $\frac{\lambda}{\lambda(T)}$; *let* $\{\xi_n\}_{n=1}^{\infty}$ *be a sequence of independent identicaly distributed random variables with the distribution* ν *and* N *be a random variable with the Poisson distribution with the parameter* $L > 0$. *Assume that* $\{\tau_n\}$, $\{\xi_n\}$ *and* N *are independent and put* $\xi_0 \equiv 0$. *Then*

$$M(A) = \sum_{n=0}^{N} \xi_n \delta_{\tau_n}(A), \qquad A \in \Sigma,$$

is a random measure on (T, Σ, λ) *such that* $M \sim \left[0, 0, \frac{L}{\lambda(T)} \nu_0\right]$, *where* $\nu_0(B) = \nu(B \setminus \{0\})$, $B \in \mathscr{B}(\boldsymbol{R})$.

Proof. By a routine computation of the characteristic functional of $(M(A_1), \ldots, M(A_n))$ for pairwise disjoint $A_1, \ldots, A_n \in \Sigma$ we get the proof.

LEMMA 4.3.2. *Let* $M_i \sim [0, m_i]$ *be such that* $m_i(\boldsymbol{R}) < \infty$, $i = 1, 2$. *Assume that for every* $u \geqslant 0$

$$H_{m_1}(u) \leqslant k H_{m_2}(u),$$

where $k \in \boldsymbol{N}$ *is a constant. Then for every measurable semi-norm* q: $E \to [0, \infty]$ *and* $\varepsilon > 0$

$$P\{q(\int_T f dM_1) > \varepsilon\} \leqslant 2kP\{kq(\int_T f dM_2) > \varepsilon\}.$$

Proof. In view of lemma 4.3.1 and without loss of generality we may assume that

$$M_i = \sum_{n=0}^{N} \xi_n^{(i)} \delta_{\tau_n},$$

where N has the Poisson distribution with the parameter $L = k\lambda(T)\, m_2(\boldsymbol{R})$,

$$\mathscr{L}(\xi_n^{(1)}) = p\frac{m_1}{m_1(\boldsymbol{R})} + (1-p)\,\delta_0, \quad \text{while} \quad p = \frac{m_1(\boldsymbol{R})}{km_2(\boldsymbol{R})} = \frac{H_{m_1}(+\infty)}{kH_{m_2}(+\infty)} \leqslant 1$$

and

$$\mathscr{L}(\xi_n^{(2)}) = \frac{1}{k}\frac{m_2}{m_2(\boldsymbol{R})} + \left(1 - \frac{1}{k}\right)\delta_0.$$

We have for every $\varepsilon > 0$

$$P\{|\xi_n^{(1)}| \geqslant \varepsilon\} = p\frac{m_1\big((-\varepsilon, \varepsilon)^c\big)}{m_1(\boldsymbol{R})} \leqslant \frac{m_2\big((-\varepsilon, \varepsilon)^c\big)}{m_2(\boldsymbol{R})} = kP\{|\xi_n^{(2)}| \geqslant \varepsilon\}.$$

Thus $\{\xi_n^{(1)}\}$ is dominated by $\{\xi_n^{(2)}\}$ and by Theorem 1.3 ([39]) we get

$$\begin{aligned} P\{q(\int_T f dM_1) > \varepsilon\} &= E\big[P\{q(\sum_{n=0}^{N} \xi_n^{(1)} f(\tau_n)) > \varepsilon | N, \{\tau_n\}\}\big] \\ &\leqslant 2kE\big[P\{kq(\sum_{n=0}^{N} \xi_n^{(2)} f(\tau_n)) > \varepsilon | N, \{\tau_n\}\}\big] \\ &= 2kP\{kq(\int_T f dM_2) > \varepsilon\}. \end{aligned}$$

We now formulate the main result of this section.

THEOREM 4.3.3. *Let $M_i \sim [\sigma_i^2, m_i]$ be symmetric random measures on (T, Σ, λ), $i = 1, 2$. Assume that there exists constants $c, k, l > 0$ such that*

$$\sigma_1^2 \leqslant c\sigma_2^2$$

and

$$H_{m_1}(u) \leqslant kH_{m_2}(lu) \quad \textit{for every } u > 0.$$

Then for every Banach space E

$$\mathscr{L}_E(M_2) \subset \mathscr{L}_E(M_1).$$

Proof. Without loss of generality we may assume that $k \in N$ and $l = 1$. Indeed, let $M_3 \sim [\sigma_2^2, m_3]$ be a random measure on (T, Σ, λ), where $m_3(B) = m_2(l^{-1}B)$, $B \in \mathscr{B}(\boldsymbol{R})$. Then

$$H_{m_1}(u) \leqslant ([k]+1)\, H_{m_3}(u) \quad \text{for every } u > 0$$

and $\mathscr{L}_E(M_2) = \mathscr{L}_E(M_3)$ by Theorem 4.1.1. Let $f \in \mathscr{L}_E(M_2)$. Put

$$\mu_i(B) = \lambda \times m_i\big(\{f(t)u \in B \setminus \{0\}\}\big), \quad B \in \mathscr{B}(E), \ i = 1, 2.$$

In view of Theorem 4.1.1, μ_2 is a Lévy measure and it is sufficient to prove that μ_1 is a Lévy measure. Choose a sequence $\{T_j\} \in \Sigma_f$ such that $T_j \nearrow T$ and

$\lambda(T_j) > 0$. We set

$$m_i^j(F) = m_i\big(F \cap (-j^{-1}, j^{-1})^c\big), \quad F \in \mathscr{B}(\boldsymbol{R}),$$

$$\mu_i^j(B) = \lambda \times m_i^j\big(\{t \in T_j,\ f(t)u \in B \setminus \{0\}\}\big), \quad B \in \mathscr{B}(E),$$

where $j \in N$, $i = 1, 2$. Let $M_i^j \sim [0, m_i^j]$ be a random measure on $(T_j, \Sigma_{|T_j}, \lambda_{|T_j})$. We observe that

$$\mathscr{L}\big(\int_{T_j} f dM_i^j\big) = \mathrm{Pois}(\mu_i^j)$$

and

$$H_{m_1^j}(u) \leqslant k H_{m_2^j}(u) \quad \text{for every } u > 0.$$

Let $\varepsilon > 0$ be fixed. Since $\mu_2 \geqslant \mu_2^j$ for every $j \in N$, the set $\{\mathrm{Pois}(\mu_2^j)\colon j \in N\}$ is tight and consequently there exists an absolutely convex compact set $C \subset E$ such that

$$[\mathrm{Pois}(\mu_2^j)](C^c) < \frac{\varepsilon}{2k} \quad \text{for every } j \in N.$$

Let q be the gauge of C. Then by Lemma 4.3.2 we get

$$\begin{aligned}[\mathrm{Pois}(\mu_1^j)][(kC)^c] &= P\{q\big(\int_{T_j} f dM_1^j\big) > k\} \\ &\leqslant 2kP\{q\big(\int_{T_j} f dM_2^j\big) > 1\} = 2k[\mathrm{Pois}(\mu_2^j)](C^c) < \varepsilon.\end{aligned}$$

Hence $\{\mathrm{Pois}(\mu_1^j)\colon j \in N\}$ is tight. Since $\mu_1^j \nearrow \mu$ as $j \to \infty$, we obtain that μ_1 is a Lévy measure (see [3], Chap. III, Th.4.7). The proof is complete.

Applying Lemma 4.3.2 to the random measures M_i^j defined above and letting $j \to \infty$ we get the following

PROPOSITION 4.3.4. *Let $M_i \sim [0, m_i]$ be symmetric random measures on (T, Σ, λ) such that for every $u > 0$*

$$H_{m_1}(u) \leqslant k H_{m_2}(u)$$

for some constant $k \in N$. Then for every seminorm $q\colon E \to [0, \infty]$ such that $\{x\colon q(x) \leqslant 1\}$ is closed and for every $f \in \mathscr{L}_E(M_2)$ we have

$$P\{q\big(\int_T f dM_1\big) > \varepsilon\} \leqslant 2kP\{kq\big(\int_T f dM_2\big) > \varepsilon\}, \quad \varepsilon > 0.$$

In the case of $\lambda(T) < \infty$ it is enough to study only the behaviour of H_m at infinity; we now prove this fact.

THEOREM 4.3.5. *Assume that $\lambda(T) < \infty$. Let $M_i \sim [\sigma_i^2, m_i]$ be symmetric random measures on (T, Σ, λ), $i = 1, 2$. Assume that there exist constants $c, k, l, u_0 > 0$ such that*

$$\sigma_1^2 \leqslant c\sigma_2^2$$

and

$$H_{m_1}(u) \leqslant kH_{m_2}(lu) \quad \textit{for every } u \geqslant u_0.$$

Then for every Banach space E

$$\mathscr{L}_E(M_2) \subset \mathscr{L}_E(M_1).$$

Proof. Put $r_1 = u_0^{-1}$, $r_2 = lu_0^{-1}$ and define

$$m_i^0(F) = m_i(F \cap (-r_i, r_i)) + p_i \delta_{-r_i}(F) + p_i \delta_{r_i}(F),$$

where $F \in \mathscr{B}(\boldsymbol{R})$, $p_i = m_i([r_i, \infty])$ and $i = 1, 2$. Then we have

$$H_{m_1^0}(u) \leqslant kH_{m_2^0}(lu)$$

for every $u \geqslant 0$. Let $M_i^0 \sim [\sigma_i^2, m_i^0]$ be a symmetric random measure on (T, Σ, λ), $i = 1, 2$. Since a finite measure is Lévy, by Theorem 4.1.1 we get $\mathscr{L}_E(M_i^0) = \mathscr{L}_E(M_i)$ for $i = 1, 2$. Theorem 4.3.3 ends the proof.

We now describe the class of all control functions.

PROPOSITION 4.3.6. *Let H: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a non-decreasing right-continuous function such that $H(0) = 0$. Then $H = H_m$ for some symmetric Lévy measure m on $\boldsymbol{R}$ if and only if*

$$\int_1^\infty \frac{H(u)}{u^3}\, du < \infty.$$

Proof. Let m be a symmetric measure on $\boldsymbol{R}$ such that $m((-u, u)^c) = H(u^{-1})$, $u > 0$. We have

$$\int_0^1 u^2\, m(du) = 2\int_0^1 \left(\int_0^u v\,dv\right) m(du) = 2\int_0^1 \left(\int_v^1 v\,m(du)\right) dv$$

$$-2\int_0^1 v\,m([v, 1)\, dv = \int_1^\infty \frac{H(u)}{u^3}\, du - \tfrac{1}{2} H(1).$$

Hence

$$\int_{-\infty}^{\infty} \min\{1, u^2\}\, m(du) = 2\int_1^\infty \frac{H(u)}{u^3}\, du,$$

which yields the assertion.

We conclude this section with some applications to the theory of semi-stable p.m.'s on Banach spaces. Let M be a semi-stable symmetric random measure, i.e.

$$K(u) = \sum_{n=-\infty}^{\infty} b^{-pn} \int_{[1,b^{-1}]} [\cos(b^n uv) - 1]\, q(dv) \tag{4.1}$$

where $0 < b < 1$, $0 < p < 2$ and q is a finite non-zero Borel measure. Then $M \sim [0, m]$, where

$$m(F) = \sum_{n=-\infty}^{\infty} b^{-pn} \int_{[1,b^{-1}]} \mathbf{1}_F(b^n v)\, q(dv), \qquad F \in \mathscr{B}(\boldsymbol{R})$$

(see [21]). The author is grateful to W. Krakowiak for the following estimation of H_m (oral communication).

LEMMA 4.3.7. *Let M be a semi-stable random measure with K given by (4.1). Then for every $u > 0$*

$$C_1 u^p \leqslant H_m(u) \leqslant C_2 u^p$$

where $C_1 = 2b^p(1-b^p)^{-1} q([1, b^{-1}])$ and $C_2 = 2b^{-p}(1-b^p)^{-1} q([1, b^{-1}])$.

Proof. We have

$$\tfrac{1}{2} H_m(u) = m([u^{-1}, \infty]) = \sum_{n=-\infty}^{\infty} b^{-pn} \int_{[1,b^{-1}]} \mathbf{1}_{[u^{-1},\infty]}(b^n v)\, q(dv).$$

Choose $N_0 \in \boldsymbol{Z}$ such that

$$b^{N_0+1} < u^{-1} \leqslant b^{N_0}.$$

Then we have for $n \geqslant N_0 + 2$

$$\int_{[1,b^{-1}]} \mathbf{1}_{[u^{-1},\infty]}(b^n v)\, q(dv) = 0$$

and for $n \leqslant N_0$

$$\int_{[1,b^{-1}]} \mathbf{1}_{[u^{-1},\infty]}(b^n v)\, q(dv) = q([1, b^{-1}]).$$

Thus

$$\begin{aligned}
\tfrac{1}{2} H_m(u) &= \sum_{n=-\infty} b^{-pn} q([1, b^{-1}]) + b^{-p(N_0+1)} \int_{[1,b^{-1}]} \mathbf{1}_{[u^{-1},\infty]}(b^{N_0+1} v)\, q(dv) \\
&= b^{-pN_0}(1-b^p)^{-1} q([1, b^{-1}]) + b^{-p(N_0+1)} q([u^{-1} b^{-(N_0+1)}, b^{-1}])
\end{aligned}$$

and hence

$$2(1-b^p)^{-1} q([1, b^{-1}])\, b^{-pN_0} \leqslant H_m(u) \leqslant 2(1-b^p)^{-1} q([1, b^{-1}])\, b^{-p(N_0+1)}.$$

Since $b^{-pN} \leqslant u^p < b^{-p(N_0+1)}$, the proof is complete.

Since $u \to 2c_p u^p$ is the control function of a standard p-stable random measure, by Theorem 4.3.3 we conclude that every function which is integrable with respect to a semi-stable random measure is also integrable with respect to a standard p-stable measure with the same order p, and conversely. Thus Proposition 4.2.1 yields

THEOREM 4.3.8. *Let M be a semi-stable random measure with K given by (4.1). Then*

$$\mathscr{L}_E(M) = S_E^p \quad \text{for every } p \in (0, 2).$$

Hence, by Proposition 4.3.8 we obtain immediately

THEOREM 4.3.9. *Let M be a semi-stable random measure with K given by (4.1). Then for every semi-norm $q\colon E \to [0, \infty]$ such that the set $\{x\colon q(x) \leqslant 1\}$ is closed, every $f \in S_E^p$ and $\varepsilon > 0$ we have*

$$AP\{Aq(\int_T f dM_p) > \varepsilon\} \leqslant P\{q(\int_T f dM) > \varepsilon\} \leqslant BP\{Bq(\int_T f dM_p) > \varepsilon\},$$

where M_p is a standard p-stable random measure, $A, B > 0$ depending only on M and p.

COROLLARY 4.3.10. *Let M and M_p be as above. Then for every $r \in (0, p)$ there exist $A_r, B_r > 0$ such that*

$$A_r[E\|\int_T f dM_p\|^r]^{1/r} \leqslant [E\|\int_T f dM\|^r]^{1/r} \leqslant B_r[E\|\int_T f dM_p\|^r]^{1/r}$$

for every $f \in S_E^p$.

COROLLARY 4.3.11. *Let M and M_p be as above. Then for every $r \in [0, p)$, $[M]_E \subset L_E^r(\Omega, \mathscr{F}, P)$ and the topologies of $L_E^r(\Omega, \mathscr{F}, P)$ coincide on $[M]_E$.*

COROLLARY 4.3.12. *Let M and M_p be as above. Then for every $r \in [0, p)$ the spaces $\mathrm{cl}_r[M]_E$ and $\mathrm{cl}_r[M_p]_E$ are isomorphic.*

Theorem 4.2.15 shows that in the case where (T, Σ, λ) is an atomless probability space the set

$$\{\int_T f dM_p\colon f \in S_E^p\}$$

forms a linear space of "all" symmetric p-stable random vectors. Now we consider a similar problem for semi-stable p.m.'s. Recall that a symmetric p.m. μ is called a *semi-stable* p.m. if there exist $0 < b < 1$ and $0 < c < 1$ such that

$$[\hat{\mu}(x^*)]^c = \hat{\mu}(bx^*) \text{ for all } x^* \in E^*. \tag{4.2}$$

The unique real solution p of $b^p = c$ is called the *order* of μ (we assume that μ is non-degenerate). Let M be a semi-stable random measure with K given by (4.1). Then for every $f \in S_E^p$

$$\mathscr{L}(\int_T f dM)\hat{}(bx^*) = \exp\{\int_T K(b\langle x^*, f\rangle) d\lambda\}$$
$$= \exp\{\int_T [\sum_{n=-\infty}^{\infty} b^{-np} \int_{[1,b^{-1}]} [\cos(b^{n+1}\langle x^*, f\rangle u) - 1] q(du) d\lambda\}$$
$$= \exp\{b^p \int_T K(\langle x^*, f\rangle) d\lambda\} = [\mathscr{L}(\int_T f dM)\hat{}(x^*)]^c$$

where $c = b^p$. Thus $\int_T f dM$ is a semi-stable random vector in E.

Let $M_{p,b}$ denote a symmetric semi-stable random measure with $q = \delta_1$, i.e.

$$K(u) = \sum_{n=-\infty}^{\infty} b^{-np}[\cos(b^n u) - 1]. \tag{4.3}$$

THEOREM 4.3.13. *Let $T = \boldsymbol{R}_+$. For every p.m. μ on E statisfying* (4.2) *there exists a function f: $\boldsymbol{R}_+ \to E$ such that $f \in S_E^p$ and*

$$\mathscr{L}(\int_{\boldsymbol{R}_+} f dM_{p,b}) = \mu.$$

Proof. Let μ satisfy (4.2). By [21], Corollary 2, there exists a finite measure σ on $D = \{x \in E: 1 \leqslant \|x\| \leqslant b^{-1}\}$ such that

$$\hat{\mu}(x^*) = \sum_{n=-\infty}^{\infty} b^{-pn} \int_D [\cos(b^n\langle x^*, x\rangle) - 1] \sigma(dx), \qquad x^* \in E^*.$$

Let f: $\boldsymbol{R}_+ \to E$ be a Borel function such that for every $B \in \mathscr{B}(E)$

$$|\{t \in [0, \sigma(D)]: f(t) \in B\}| = \sigma(B \cap D)$$

and $f(t) \equiv 0$ for $t > \sigma(D)$. Theorem 4.1.5 concludes the proof.

Let $\Gamma_{p,b}$ (Γ_p, resp.) denote the set of all σ-finite Borel measures μ on E such that the function

$$x^* \mapsto \exp\{\sum_{n=-\infty}^{\infty} b^{-pn} \int_E [\cos(b^n\langle x^*, x\rangle) - 1] \mu(dx)\}$$

($x^* \mapsto \exp\{-\int_E |\langle x^*, x\rangle|^p \mu(dx)\}$, resp.) is the characteristic functional of a p.m. on E ($0 < p < 2$, $0 < b < 1$). The elements of $\Gamma_{p,b}$ (Γ_p, resp.) will be called *representing measures* of (p, b)-semi-stable p.m.'s (p-stable p.m.'s, resp.). To describe the classes of all p-stable p.m.'s or (p, b)-semi-stable p.m.'s on a Banach space it is sufficient to characterize the sets of all representing measures Γ_p and $\Gamma_{p,b}$, respectively. The following corollary says that for this purpose it is enough to study only p-stable p.m.'s.

COROLLARY 4.3.14. *$\Gamma_{p,b} = \Gamma_p$ for every $0 < p < 2$ and $0 < b < 1$.*

Proof. Let μ be a σ-finite Borel measure on E and let $M_{p,b}$ be (as above) a random measure on $(E, \mathscr{B}(E), \mu)$. Consider $f\colon E \to E$, $f(x) = x$, $x \in E$. Then $f \in S_E^p = S_E^p(E, \mathscr{B}(E), \mu)$ if and only if $\mu \in \Gamma_p$ and by Theorem 4.1.5 $f \in \mathscr{L}_E(M_{p,b})$ if and only if $\mu \in \Gamma_{p,b}$. A recall of Theorem 4.3.8 completes the proof.

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ISSN 0012-3862

ISBN 83-01-07186-9