

To the Graduate Council:

I am submitting herewith a thesis written by Alice Ruth Darnell entitled "Weak Orders, Ordered Partitions, and Generalized Weak Orders." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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WEAK ORDERS, ORDERED PARTITIONS, AND GENERALIZED WEAK ORDERS

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## ABSTRACT

A binary relation  $R$  on  $X$  is a complete weak order if and only if it is reflexive, complete and transitive. A binary relation  $R$  on  $X$  is an asymmetric weak order if and only if it is asymmetric and negatively transitive. For any binary relation  $R$  on a nonempty set  $X$ , let  $aR$ ,  $cR$  and  $sR$  denote, respectively, the asymmetric part of  $R$ , the complement of  $R$ , and the symmetric part of  $R$ , that is,  $aR = \{xy | xy \in R \text{ and } yx \notin R\}$ ,  $cR = (X \times X) - R$ , and  $sR = \{xy | xy \in R \text{ and } yx \in R\}$ . When  $R$  is a complete weak order, equivalence classes of  $X$  are determined by the symmetric part of  $R$ . When  $R$  is an asymmetric weak order,  $X$  is partitioned into equivalence classes determined by the symmetric part of the complement of  $R$ . The partitions  $\hat{X} = X/sR$  and  $\hat{X} = X/scR$  induced, respectively, by a complete weak order  $R$ , and an asymmetric weak order  $R$ , have a natural linear ordering  $>$ . We show that there exists a one-to-one correspondence between complete weak orders, asymmetric weak orders and linearly ordered partitions of  $X$ . We enumerate the complete weak orders on an  $n$ -element set  $X$  by enumerating the ordered partitions of  $X$ .

A feature common to both types of weak orders already mentioned is used as the basis of a definition of a generalized weak order. When  $R$  is a generalized weak order,  $X$  is partitioned into equivalence classes determined by the symmetric part of the complement of the asymmetric part of  $R$ . The partition  $\hat{X} = X/scaR$  has a linear ordering  $>$  defined as follows: For all  $A, B \in \hat{X}$ ,  $A > B$  if

and only if  $A \times B \subseteq aR$ . A linearly ordered partition of  $X$  is a Fishburn partition if each of its equivalence classes is equipped with some symmetric relation. We show a one-to-one correspondence exists between generalized weak orders on  $X$  and Fishburn partitions on  $X$ .

Special types of generalized weak orders include the transitive generalized weak orders, the negatively transitive generalized weak orders, and the generalized weak orders that are both transitive and negatively transitive. We show that the transitive generalized weak orders are just those generalized weak orders with transitive symmetric part; the negatively transitive generalized weak orders are just those generalized weak orders, the symmetric part of whose complement is transitive; and the transitive, negatively transitive generalized weak orders are just the transitive, negatively transitive binary relations.

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## CHAPTER I

## WEAK ORDERS AND ORDERED PARTITIONS

Section 1.1. Review of Sets and Relations.

Let  $A$  be a given set and let  $a$  and  $b$  denote certain objects. When  $a$  is an element of  $A$ , we shall indicate this fact by writing  $a \in A$ ; when both  $a$  and  $b$  are elements of  $A$ , we shall write  $a, b \in A$  instead of  $a \in A$  and  $b \in A$ ; when  $b$  is not an element of  $A$ , we shall write  $b \notin A$ .

A set  $A$  is said to be a subset of a set  $X$ , denoted by  $A \subseteq X$ , if and only if  $x \in A$  implies  $x \in X$ . The set having no members at all is called the empty set and is denoted by  $\emptyset$ . If  $A$  and  $B$  are sets, the union of  $A$  and  $B$ , denoted by  $A \cup B$ , is defined by  $A \cup B = \{x | x \in A \text{ or } x \in B\}$ . The intersection of  $A$  and  $B$ , denoted by  $A \cap B$ , is defined by  $A \cap B = \{x | x \in A \text{ and } x \in B\}$ . Two sets are called disjoint if they have no elements in common, that is, if  $A \cap B = \emptyset$ . The complement of  $A$ ,  $A^C$ , is defined by  $A^C = \{x | x \in X \text{ and } x \notin A\}$ . The difference,  $A - B$ , of any two sets is defined by  $A - B = \{x | x \in A \text{ and } x \notin B\}$ , or equivalently,  $A - B = A \cap B^C$ .

We define  $A \times B = \{(x, y) | x \in A, y \in B\}$ . A binary relation  $R$  on a set  $X$  is a set of ordered pairs  $(x, y)$  with  $x \in X$  and  $y \in X$ , that is,  $R$  on  $X$  is a subset of  $X \times X$ .

If  $(x, y) \in R$ , we shall write  $xRy$  or  $xy \in R$ , where ordered pairs in  $X \times X$  are written by juxtaposition. Similarly,  $\sim xRy$  (it is false that  $x$  stands in the relation  $R$  to  $y$ ) means that  $xy \notin R$ . If  $R$  is a binary relation on  $X$  then for each  $(x, y)$  in the set

$X \times X$  either  $xRy$  or  $\sim xRy$  and not both. Because we are dealing with ordered pairs,  $(x, y)$  is not the same as  $(y, x)$  unless  $x = y$ .

The binary relations used will be assumed to have certain properties. A list of some of these follows:

### PROPERTIES OF BINARY RELATIONS

| A binary relation $R$ on $X$ is | provided that                                                                          |
|---------------------------------|----------------------------------------------------------------------------------------|
| reflexive                       | $xRx$ , for every $x \in X$                                                            |
| irreflexive                     | $\sim xRx$ , for every $x \in X$                                                       |
| symmetric                       | $xRy \rightarrow yRx$ , for every $x, y \in X$                                         |
| asymmetric                      | $xRy \rightarrow \sim yRx$ , for every $x, y \in X$                                    |
| antisymmetric                   | $(xRy \text{ and } yRx) \rightarrow x = y$ ,<br>for every $x, y \in X$                 |
| transitive                      | $(xRy \text{ and } yRz) \rightarrow xRz$ ,<br>for every $x, y, z \in X$                |
| negatively transitive           | $(\sim xRy \text{ and } \sim yRz) \rightarrow \sim xRz$ ,<br>for every $x, y, z \in X$ |
| complete                        | $x \neq y \rightarrow xRy \text{ or } yRx$<br>for every $x, y \in X$                   |

A relation  $R$  on a set  $X$  is called an equivalence relation on  $X$  when  $R$  is reflexive, symmetric and transitive. If  $a \in X$  , the elements  $y \in X$  satisfying  $yRa$  constitute a subset,  $[a]$  , of  $X$  , called an equivalence set or equivalence class. We recall that the distinct equivalence classes determined by  $R$  comprise a partition

of  $X$ , that is, a set  $\{A_1, A_2, \dots, A_n\}$  of nonempty subsets of  $X$  such that  $A_1 \cup A_2 \cup \dots \cup A_n = X$  and  $A_i \cap A_j = \emptyset$  whenever  $i \neq j$ .

## Section 1.2. Some Preliminary Theorems on Binary Relations.

For any binary relation  $R$  on a nonempty set  $X$ , let  $aR$ ,  $cR$ ,  $sR$  and  $dR$  denote, respectively, the asymmetric part of  $R$ , the complement of  $R$ , the symmetric part of  $R$ , and the dual of  $R$ , that is,  $aR = \{xy \mid xy \in R \text{ and } yx \notin R\}$ ,  $cR = (X \times X) - R$ ,  $sR = \{xy \mid xy \in R \text{ and } yx \in R\}$ , and  $dR = \{xy \mid yx \in R\}$ .

As our terminology suggests, it is easy to show that for any binary relation  $R$ ,  $aR$  is asymmetric and  $sR$  is symmetric. In addition, the following observations are relevant to our later work:

**Theorem 1.1.**  $R$  is asymmetric if and only if it is irreflexive and antisymmetric.

Proof: Assume that  $R$  is asymmetric. Then

|                                 |                                                          |
|---------------------------------|----------------------------------------------------------|
| $xRy \rightarrow \sim yRx$      | (definition of asymmetry)                                |
| $xRx \rightarrow \sim xRx$      | (let $x = y$ in the definition)                          |
| $\sim xRx \text{ or } \sim xRx$ | $(p \rightarrow q \leftrightarrow \sim p \text{ or } q)$ |
| $\sim xRx$                      | $((p \text{ or } \sim p) \leftrightarrow p)$             |

Thus,  $R$  is irreflexive. Moreover,

$xRy \text{ and } yRx \rightarrow x = y$  (this implication holds vacuously since the antecedent is never true)

Thus,  $R$  is antisymmetric.

Conversely, assume  $R$  is irreflexive and antisymmetric. Suppose  $R$  is not asymmetric, that is, suppose  $xRy$  and  $yRx$ , for some  $x, y \in X$ . Then  $x = y$  since  $R$  is antisymmetric. Thus  $xRx$  for some  $x$ , contradicting the irreflexivity of  $R$ . Hence  $R$  is asymmetric. ■

Theorem 1.2.  $R$  is reflexive if and only if  $cR$  is irreflexive.

Proof: Note that  $xy \in R$  if and only if  $xy \notin cR$ . If  $R$  is reflexive, then for all  $x \in X$ ,  $xx \in R$ , and hence  $xx \notin cR$ . Thus  $cR$  is irreflexive.

Conversely, if  $cR$  is irreflexive, then for all  $x \in X$ ,  $xx \notin cR$ , and hence  $xx \in R$ . Thus  $R$  is reflexive. ■

Theorem 1.3.  $R$  is antisymmetric if and only if  $cR$  is complete.

Proof: Suppose  $R$  is antisymmetric. Then,

$$\begin{array}{ll} (xRy \text{ and } yRx) \rightarrow x = y & \text{(definition of antisymmetry)} \\ \sim(\sim xRy \text{ or } \sim yRx) \rightarrow x = y & ((p \text{ and } q) \leftrightarrow \sim(\sim p \text{ or } \sim q)) \\ x \neq y \rightarrow \sim xRy \text{ or } \sim yRx & \text{(contrapositive of previous line)} \end{array}$$

Thus,  $cR$  is complete.

Conversely, assume  $cR$  is complete. Then,

$$\begin{array}{ll} x \neq y \rightarrow \sim xRy \text{ or } \sim yRx & \text{(definition of } cR \text{ complete)} \\ x \neq y \rightarrow \sim(xRy \text{ and } yRx) & (\sim p \text{ or } \sim q \leftrightarrow \sim(p \text{ and } q)) \\ (xRy \text{ and } yRx) \rightarrow x = y & \text{(contrapositive of previous line)} \end{array}$$

Hence,  $R$  is antisymmetric. ■

Theorem 1.4.  $R$  is complete if and only if  $cR$  is antisymmetric.

Proof: From Theorem 1.3,  $cR$  is complete if and only if  $R$  is antisymmetric. Since  $ccR = R$ ,  $R$  is complete if and only if  $cR$  is antisymmetric. ■

Theorem 1.5. The asymmetric part of the complement of  $R$  is identical to the asymmetric part of the dual of  $R$ , that is,  
 $acR = adR$ .

Proof:

$$\begin{aligned}
 xy \in acR &\leftrightarrow xy \in cR \text{ and } yx \notin cR && \text{(definition of } a) \\
 &\leftrightarrow \neg xRy \text{ and } yRx && \text{(definition of } c) \\
 &\leftrightarrow yx \notin dR \text{ and } xy \in dR && \text{(definition of } d) \\
 &\leftrightarrow xy \in adR && \text{(definition of } a)
 \end{aligned}$$

Since  $xy \in acR \leftrightarrow xy \in adR$ ,  $acR = adR$ . ■

Theorem 1.6. The operation  $d$  commutes with each of the operations  $a$ ,  $c$  and  $s$ , that is,  $d\gamma = \gamma d$  for all  $\gamma = \{a, c, s\}$ .

Proofs: (i) We must show  $da = ad$ .

$$\begin{aligned}
 xy \in daR &\leftrightarrow yx \in aR && \text{(definition of } d) \\
 &\leftrightarrow yx \in R \text{ and } xy \notin R && \text{(definition of } a) \\
 &\leftrightarrow xy \in dR \text{ and } yx \notin dR && \text{(definition of } d) \\
 &\leftrightarrow xy \in adR && \text{(definition of } a)
 \end{aligned}$$

Since  $xy \in daR \leftrightarrow xy \in adR$ ,  $daR = adR$  for all  $R$ . Hence  $da = ad$ .

(ii) We must show  $dc = cd$  .

$$\begin{aligned} xy \in dcR &\leftrightarrow yx \in cR \leftrightarrow yx \notin R && \text{(definitions of } d \text{ and } c) \\ &\leftrightarrow xy \notin dR \leftrightarrow xy \in cdR && \text{(definitions of } d \text{ and } c) \end{aligned}$$

Since  $xy \in dcR \leftrightarrow xy \in cdR$  ,  $dcR = cdR$  for all  $R$  . Hence  $dc = cd$  .

(iii) We must show  $ds = sd$  .

$$\begin{aligned} xy \in dsR &\leftrightarrow yx \in sR && \text{(definition of } d) \\ &\leftrightarrow yx \in R \text{ and } xy \in R && \text{(definition of } s) \\ &\leftrightarrow xy \in dR \text{ and } yx \in dR && \text{(definition of } d) \\ &\leftrightarrow xy \in sdR && \text{(definition of } s) \end{aligned}$$

Since  $xy \in dsR \leftrightarrow xy \in sdR$  ,  $dsR = sdR$  for all  $R$  . Hence  $ds = sd$  . ■

Theorem 1.7. A relation is transitive if and only if its dual is transitive.

Proof: Let  $T = dR$  , that is,  $xTy \leftrightarrow yRx$  . We must show that  $T$  is transitive when  $R$  is transitive. Assume  $xTy$  and  $yTz$  . We need to show  $xTz$  . Now,  $xTy$  and  $yTz \rightarrow yRx$  and  $zRy$  by definition of  $T$  . By transitivity of  $R$  ,  $zRx$  . But  $zRx \rightarrow xTz$  . Therefore, if a relation is transitive, its dual is transitive.

Conversely, we need to show that  $R$  is transitive when the dual of  $R$  is transitive. From the previous paragraph,  $R$  is transitive  $\rightarrow dR$  is transitive. By substituting  $dR$  for  $R$  ,  $dR$  is transitive  $\rightarrow ddR$  is transitive. But  $ddR = R$  for all  $R$  , since  $xy \in ddR \leftrightarrow yx \in dR \leftrightarrow xy \in R$  . Hence  $dR$  is transitive  $\rightarrow R$  is transitive. ■

Theorem 1.8.  $R$  is negatively transitive if and only if for all  
 $x, y, z \in X$  ,  $xRz \rightarrow (xRy \text{ or } yRz)$  .

Proof: By definition, negative transitivity of  $R$  is equivalent to  $\sim xRy$  and  $\sim yRz \rightarrow \sim xRz$  . The statement  $xRz \rightarrow (xRy \text{ or } yRz)$  is just the contrapositive of the former statement, and so the two are logically equivalent. ■

Theorem 1.9. If  $R$  is negatively transitive and asymmetric,  
then  $R$  is transitive.

Proof: Suppose  $xRy$  and  $yRz$  hold and suppose, by way of contradiction, that  $\sim xRz$  holds. Since  $R$  is asymmetric,  $yRz$  implies  $\sim zRy$  . Since  $R$  is negatively transitive,  $\sim xRz$  and  $\sim zRy$  implies  $\sim xRy$  , which is a contradiction. Thus  $R$  is transitive. ■

### Section 1.3. Complete Weak Orders and Asymmetric Weak Orders

A binary relation  $R$  on  $X$  is a complete weak order if and only if it is reflexive, complete and transitive. A binary relation  $R$  on  $X$  is an asymmetric weak order if and only if it is irreflexive, anti-symmetric and negatively transitive, or equivalently (by Theorem 1.1), asymmetric and negatively transitive. Traditionally, economists have used asymmetric weak orders as models of strict preference<sup>1</sup> and complete weak orders as models of preference-or-indifference. This suggests that there is a close connection between these two types of orders. The following three theorems make this connection explicit.

---

<sup>1</sup>The concept of an asymmetric weak order is closely related to the concept of an ordinal utility function. See the Appendix for a discussion of the connection.

Theorem 1.10. If  $R$  is a complete weak order then  $aR$  is an asymmetric weak order.

Proof: Let  $T = aR$ . We need to show that  $T$  is asymmetric and negatively transitive when  $R$  is reflexive, transitive and complete.

Asymmetry:

$$\begin{array}{ll}
 xTy \leftrightarrow xRy \text{ and } \sim yRx & \text{(by definition of } T) \\
 xRy \text{ and } \sim yRx \rightarrow xRy \text{ or } \sim yRx & ((p \text{ and } q) \rightarrow (p \text{ or } q)) \\
 \sim yTx \leftrightarrow xRy \text{ or } \sim yRx & \text{(by definition of } T) \\
 xTy \rightarrow \sim yTx & \text{(transitivity of } \rightarrow)
 \end{array}$$

Notice that no properties of  $R$  are used in showing that  $aR$  is asymmetric.

Negative Transitivity: Note that completeness and reflexivity of  $R$  imply that for all  $u$  and  $v$  (distinct or identical),  $\sim uRv \rightarrow vRu$ . Now, by definition of  $T$ ,  $\sim uTv \leftrightarrow \sim(uRv \text{ and } \sim vRu) \leftrightarrow \sim uRv \text{ or } vRu$ . By the above remark,  $(\sim uRv \text{ or } vRu) \rightarrow vRu$ , and hence  $\sim uTv \rightarrow vRu$ . Thus if  $\sim xTy$  and  $\sim yTz$ , then  $yRx$  and  $zRy$  and, hence,  $zRx$ , by transitivity of  $R$ . Thus, by definition of  $T$ ,  $\sim xTz$ . ■

Theorem 1.10\*. If  $R$  is a complete weak order then  $cR$  is an asymmetric weak order.

Proof: Let  $T = cR$ . We need to show that  $T$  is asymmetric and negatively transitive when  $R$  is reflexive, transitive and complete.

Asymmetry: By Theorems 1.2 and 1.4,  $T$  is irreflexive and antisymmetric respectively. Therefore,  $T$  is asymmetric by Theorem 1.1.

Negative Transitivity: If  $xy, yz \in R$ , then  $xz \in R$  by transitivity of  $R$ . Since  $ccR = R$ , we have  $xy, yz \in ccR$  implies  $xz \in ccR$ . Equivalently,  $xy, yz \in cT$  implies  $xz \in cT$ , thus  $T$  is negatively transitive. ■

Theorem 1.11. If  $R$  is an asymmetric weak order then  $R \cup scR$  is a complete weak order.

Proof: Let  $T = R \cup scR$ . Then

$$xy \in T \leftrightarrow xy \in R \text{ or } xy \in scR \quad (\text{definition of } T)$$

$$xy \in T \leftrightarrow xy \in R \text{ or } (xy \in cR \text{ and } yx \in cR) \quad (\text{definition of } s)$$

$$xTy \leftrightarrow xRy \text{ or } (\sim xRy \text{ and } \sim yRx) \quad (\text{definition of } c)$$

$$xTy \leftrightarrow (xRy \text{ or } \sim xRy) \text{ and } (xRy \text{ or } \sim yRx) \quad (\text{distributivity of "or" over "and"})$$

$$xTy \leftrightarrow xRy \text{ or } \sim yRx \quad (p \text{ and } q \leftrightarrow q, \text{ if } p \text{ is a tautology})$$

We need to show that if  $R$  is asymmetric and negatively transitive, then  $T = R \cup scR$  is reflexive, complete and transitive.

We may prove completeness and reflexivity simultaneously by showing that for all  $x$  and  $y$  (distinct or identical)  $xTy$  or  $yTx$ . Now, by the above observation,  $(xTy \text{ or } yTx) \leftrightarrow [(xRy \text{ or } \sim yRx) \text{ or } (yRx \text{ or } \sim xRy)]$ . Since  $[(xRy \text{ or } \sim yRx) \text{ or } (yRx \text{ or } \sim xRy)] \leftrightarrow [(xRy \text{ or } \sim xRy) \text{ or } (yRx \text{ or } \sim yRx)]$ , by

associativity of "or" and  $[(xRy \text{ or } \sim xRy) \text{ or } (yRx \text{ or } \sim yRx)]$  is a tautology, we have  $xTy \text{ or } yTx$  for all  $x, y \in X$ .

Transitivity: By the above remarks,  $(xTy \text{ and } yTz) \rightarrow [(xRy \text{ or } \sim yRx) \text{ and } (yRz \text{ or } \sim zRy)] \rightarrow [(xRy \text{ and } yRz) \text{ or } (xRy \text{ and } \sim zRy) \text{ or } (\sim yRx \text{ and } yRz) \text{ or } (\sim yRx \text{ and } \sim zRy)]$ . Since  $R$  is asymmetric,  $(xRy \text{ and } yRz)$  implies  $(\sim yRx \text{ and } \sim zRy)$ ;  $(xRy \text{ and } \sim zRy)$  implies  $(\sim yRx \text{ and } \sim zRy)$ ; and  $(\sim yRx \text{ and } yRz)$  implies  $(\sim yRx \text{ and } \sim zRy)$ . Hence,  $(xTy \text{ and } yTz)$  implies  $(\sim zRy \text{ and } \sim yRz)$  implies  $\sim zRx$ , by negative transitivity of  $R$ . Since  $xTz \leftrightarrow (xRz \text{ or } \sim zRx)$ , it follows that  $xTz$ . Therefore,  $T$  is transitive.

We have shown that given an asymmetric weak order  $R$ ,  $R \cup cR$  is a complete weak order. ■

Theorem 1.11\*. If  $R$  is an asymmetric weak order then  $cR$  is a complete weak order.

Proof: Let  $T = cR$ . We must show that  $T$  is reflexive, complete and transitive when  $R$  is asymmetric and negatively transitive.

Reflexivity of  $T$  follows from Theorems 1.1 and 1.2.

Completeness of  $T$  follows from Theorems 1.1 and 1.3.

Transitivity: If  $xy, yz \in cR$ , then  $xz \in cR$  by negative transitivity of  $R$ . Equivalently, since  $T = cR$ ,  $xy, yz \in T$  implies  $xz \in T$ . Thus  $T$  is transitive. ■

Theorem 1.12. There exists a one-to-one correspondence between complete weak orders on  $X$  and asymmetric weak orders on  $X$ .

Proofs: (i) Let  $C$  be the set consisting of all complete weak orders, and  $A$  be the set consisting of all asymmetric weak orders on  $X$ . By Theorem 1.10 there exists a function  $f: C \rightarrow A$ ,

namely,  $f(R) = aR$  where  $R \in C$ . Similarly, by Theorem 1.11, there exists a function  $g: A \rightarrow C$ , namely,  $g(R) = R \cup scR$  where  $R \in A$ .

To show that a one-to-one correspondence exists between  $C$  and  $A$ , we will show 1) that given  $R \in C$ ,  $g(f(R)) = aR \cup scaR = R$ , and 2) that given  $R \in A$ ,  $f(g(R)) = a(R \cup scR) = R$ . In other words, we will show that the function  $g$  is the inverse of function  $f$ , and vice versa.

1) let  $R \in C$ , so that  $R$  is complete, reflexive and transitive. We need to show that  $R = aR \cup scaR$ , that is,  $xy \in R \leftrightarrow xy \in aR$  or  $xy \in scaR$ . Now  $(xy \in aR \text{ or } xy \in scaR) \leftrightarrow$

$$\begin{array}{ll}
 (xRy \text{ and } \neg yRx) \text{ or} & \text{(definition of } a \text{ and } sca) \\
 [(\neg xRy \text{ or } yRx) \text{ and} & \\
 (\neg yRx \text{ or } xRy)] & \\
 \leftrightarrow [(xRy \text{ and } \neg yRx) \text{ or} & \text{(distributivity of "or" over "and")} \\
 (\neg xRy \text{ or } \neg yRx)] \text{ and} & \\
 [(xRy \text{ and } \neg yRx) \text{ or} & \\
 (\neg yRx \text{ or } xRy)] & \\
 \leftrightarrow [(xRy \text{ and } \neg yRx) \text{ or} & (\neg p \text{ or } q \leftrightarrow \neg(p \text{ and } \neg q)) \\
 \neg(xRy \text{ and } \neg yRx)] \text{ and} & \\
 [(xRy \text{ and } \neg yRx) \text{ or} & \\
 (\neg yRx \text{ or } xRy)] & \\
 \leftrightarrow (xRy \text{ and } \neg yRx) \text{ or} & (p \text{ and } q \leftrightarrow q, \text{ if } p \text{ is} \\
 (\neg yRx \text{ or } xRy) & \text{a tautology)} \\
 \leftrightarrow [(\neg yRx \text{ or } xRy) \text{ or } xRy] & \text{(distributivity of "or" over "and")} \\
 \text{and } [(\neg yRx \text{ or } xRy) \text{ or } \neg yRx] & \\
 \leftrightarrow (\neg yRx \text{ or } xRy) \text{ and} & ((\neg p \text{ or } q) \text{ or} \\
 (\neg yRx \text{ or } xRy) & q \leftrightarrow (\neg p \text{ or } q)) \\
 \leftrightarrow \neg yRx \text{ or } xRy & (p \text{ and } p \leftrightarrow p) \\
 \leftrightarrow xRy & \text{(by completeness and} \\
 & \text{reflexivity of } R)
 \end{array}$$

Thus  $R = aR \cup scaR$ .

(2) Let  $R \in A$ , so that  $R$  is asymmetric and negatively transitive. We must show that  $R = a(R \cup scR)$ , that is,  
 $xy \in R \leftrightarrow xy \in R \cup scR$  and  $yx \notin R \cup scR$ . Now,  $(xy \in R \cup scR$  and  $yx \notin R \cup scR) \leftrightarrow (xy \in R$  or  $xy \in scR)$  and  $\sim[yx \in R$  or  $(yx \notin R$  and  $xy \notin R)] \leftrightarrow [xy \in R$  or  $(xy \in cR$  and  $yx \in cR)]$  and  $[yx \notin R$  and  $(yx \in R$  or  $xy \in R)] \leftrightarrow [xRy$  or  $(\sim xRy$  and  $\sim yRx)]$  and  $[(yx \notin R$  and  $yx \in R)$  or  $(yx \notin R$  and  $xy \in R)] \leftrightarrow [(xRy$  or  $\sim xRy)$  and  $(xRy$  or  $\sim yRx)]$  and  $(yx \notin R$  and  $xy \in R) \leftrightarrow (xRy$  or  $\sim yRx)$  and  $(\sim yRx$  and  $xRy) \leftrightarrow [xRy$  and  $(\sim yRx$  and  $xRy)]$  or  $[\sim yRx$  and  $(\sim yRx$  and  $xRy)] \leftrightarrow (xRy$  and  $\sim yRx)$  or  $(\sim yRx$  and  $xRy) \leftrightarrow (xRy$  and  $\sim yRx) \leftrightarrow xy \in aR$ . But  $xy \in R \leftrightarrow xy \in aR$ , since  $R$  is asymmetric. That is,  $R = aR$  showing that  $R = a(R \cup scR)$ .

(ii) Let  $A$  and  $C$  be defined as in (i) above. By Theorem 1.10\*, there exists a function  $f: C \rightarrow A$ , namely  $f(R) = cR$  where  $R \in C$ . Similarly by Theorem 1.11\*, there exists a function  $g: A \rightarrow C$ , namely,  $g(R) = cR$  where  $R \in A$ .

Given  $R \in C$ ,  $g(f(R)) = c(cR) = R$ ; and given  $R \in A$ ,  $f(g(R)) = c(cR) = R$ .

In both (i) and (ii) above, we have shown that  $f$  and  $g$  are inverse functions of each other where  $f: C \rightarrow A$  and  $g: A \rightarrow C$ . Thus  $A$  and  $C$  are in one-to-one correspondence. Furthermore, the sets  $A$  and  $C$  are in one-to-one correspondence with the linearly ordered partitions of  $X$ , as will be shown in Section 1.4. ■

#### Section 1.4. A, C and Ordered Partitions

When  $R$  is a complete weak order, equivalence classes of  $X$  are determined by the symmetric part of  $R$ , that is,  $sR$  is an equivalence relation on  $X$ .

Theorem 1.13. Given a complete weak order  $R$  on  $X$ ,  $sR$  is an equivalence relation on  $X$ .

Proof: We need to show that  $T = sR$  is reflexive, symmetric and transitive when  $R$  is reflexive, transitive and complete. By definition,  $xRy \leftrightarrow xRy$  and  $yRx$ .

Reflexivity:

$$\begin{aligned} xTx &\leftrightarrow xRx \text{ and } xRx && \text{(definition of } T) \\ &\leftrightarrow xRx && (p \text{ and } p \leftrightarrow p) \end{aligned}$$

Since  $R$  is reflexive,  $sR$  is reflexive. Thus,  $xTx$  for all  $x \in X$ .

Symmetry:

$$\begin{aligned} xTy &\rightarrow (xRy \text{ and } yRx) && \text{(definition of } T) \\ &\rightarrow (yRx \text{ and } xRy) && \text{(commutativity of "and")} \\ &\rightarrow yTx && \text{(definition of } T) \end{aligned}$$

Thus  $xTy \rightarrow yTx$ , showing  $T$  is symmetric.

Transitivity:

$$\begin{aligned}
 xTy \text{ and } yTz &\rightarrow (xRy \text{ and } yRx) \quad (\text{definition of } T) \\
 &\quad \text{and } (yRz \text{ and } zRy) \\
 &\rightarrow (xRy \text{ and } yRz) \quad (\text{associativity of "and"}) \\
 &\quad \text{and } (zRy \text{ and } yRx) \\
 &\rightarrow xRz \text{ and } zRx \quad (\text{transitivity of } R) \\
 &\rightarrow xTz \quad (\text{definition of } T)
 \end{aligned}$$

Hence,  $T$  is transitive. ■

When  $R$  is an asymmetric weak order,  $X$  is partitioned into equivalence classes determined by the symmetric part of the complement of  $R$ , that is,  $scR$  is an equivalence relation on  $X$ .

Theorem 1.14. Given an asymmetric weak order  $R$  on  $X$ ,  $scR$  is an equivalence relation on  $X$ .

Proof: We need to show that  $T = scR$  is reflexive, symmetric and transitive when  $R$  is asymmetric and negatively transitive. By definition,  $xTy \leftrightarrow \sim xRy$  and  $\sim yRx$ .

Reflexivity:

$$\begin{aligned}
 xTx &\rightarrow \sim xRx \text{ and } \sim xRx \quad (\text{definition of } R) \\
 &\rightarrow \sim xRx \quad (\sim p \text{ and } \sim p \rightarrow \sim p)
 \end{aligned}$$

Since  $R$  is irreflexive by Theorem 1.1, for all  $x \in X$ ,  $\sim xRx$ .

Hence,  $xTx$  for all  $x \in X$ , and so  $T$  is reflexive.

Symmetry:

$$\begin{aligned}
 xTy &\rightarrow \sim xRy \quad \text{and} \quad \sim yRx && \text{(definition of } T) \\
 &\rightarrow \sim yRx \quad \text{and} \quad \sim xRy && \text{(commutativity of "and")} \\
 &\rightarrow yTx && \text{(definition of } T)
 \end{aligned}$$

Thus,  $T$  is symmetric.

Transitivity:

$$\begin{aligned}
 xTy \text{ and } yTz &\rightarrow [(\sim xRy \text{ and } \sim yRx) && \text{(definition of } T) \\
 &\quad \text{and } (\sim yRz \text{ and } \sim zRy)] \\
 &\rightarrow (\sim xRy \text{ and } \sim yRz) && \\
 &\quad \text{and } (\sim zRy \text{ and } \sim yRx) && \text{(associativity of "and")} \\
 &\rightarrow \sim xRz \text{ and } \sim zRx && \text{(negative transitivity of } R) \\
 &\rightarrow xTz && \text{(definition of } T)
 \end{aligned}$$

Therefore,  $T$  is transitive. ■

Now the partitions  $\hat{X} = X/sR$  and  $\hat{X} = X/scR$  induced respectively, by a complete weak order  $R$ , and an asymmetric weak order  $R$ , have a natural linear ordering  $>$ . In the former case we have  $A > B$ , where  $A, B \in X/sR$ , if and only if  $xy \in aR$ , for all  $x \in A$  and for all  $y \in B$ . In the latter case, we have  $A > B$ , where  $A, B \in X/scR$ , if and only if  $xy \in R$ , for all  $x \in A$  and for all  $y \in B$ . Conversely, it is clear that any linearly ordered partition of a set  $X$  gives rise in a natural way to a complete weak order and a companion asymmetric weak order. It follows easily that the set of linearly ordered partitions of  $X$ , the set of complete weak orders on  $X$ , and the set of asymmetric weak orders on  $X$  all have the

same cardinality. In particular, we may enumerate the complete (equivalently, asymmetric) weak orders on an  $n$ -element set  $X$ , by enumerating the ordered partitions of  $X$ .

### Section 1.5 Enumeration of Ordered Partitions

For  $n \geq 1$  let  $P(n)$  denote the number of ordered partitions of an  $n$ -element set. The number of such partitions, the initial block of which has cardinality  $k$ , is  $\binom{n}{k} P(n-k)$ , where we stipulate  $P(0) = 1$ . Summing over  $k$  yields the recurrence

$$(1) \quad P(n) = \sum_{k=1}^n \binom{n}{k} P(n-k); \quad P(0) = 1$$

We list the first 14 values<sup>2</sup> of  $P(n)$ , calculated by means of (1), below.

| $n$ | $P(n)$             |
|-----|--------------------|
| 1   | 1                  |
| 2   | 3                  |
| 3   | 13                 |
| 4   | 75                 |
| 5   | 541                |
| 6   | 4,683              |
| 7   | 47,293             |
| 8   | 545,835            |
| 9   | 7,087,261          |
| 10  | 102,247,563        |
| 11  | 1,622,632,573      |
| 12  | 28,091,567,595     |
| 13  | 526,858,348,381    |
| 14  | 10,641,342,970,443 |

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<sup>2</sup>Gross, O.A., "Preferential Arrangements", 69(1962), 4-8, American Mathematical Monthly.

Now we derive an exponential generating function of  $P$  by the method appearing in [3]. Let  $F$  and  $G$  be arithmetic functions mapping the non-negative integers into the real numbers, and let

$$F + G(n) = F(n) + G(n)$$

$$\text{and } F * G(n) = \sum_{k=0}^n \binom{n}{k} F(k) G(n - k)$$

Equipped with these operations, the set of arithmetic functions is an integral domain with multiplicative identity  $I$ , where  $I(0) = 1$  and  $I(n) = 0$  if  $n > 0$ .  $F$  has a multiplicative inverse if and only if  $F(0) \neq 0$ . As is familiar, this ring of arithmetic functions is isomorphic with the ring of formal exponential series

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

equipped with ordinary addition and multiplication, the isomorphism being given by the mapping  $\psi$ , where

$$\psi(F) = \sum_{n=0}^{\infty} F(n) \frac{x^n}{n!}$$

is the exponential generating function of the arithmetic function  $F$ . In order to minimize computation with formal series in the determination of the exponential generating function of  $P$ , we work "symbolically" in the ring of arithmetic functions until the last possible step, and then apply the isomorphism  $\psi$  to yield analytic results.

Define the arithmetic function  $Z$  by  $Z(n) = 1$  for all  $n \geq 0$ . Then we may express the recurrence relation (1) as an equation in the ring of arithmetic functions as follows:

$$P = ((Z - I) * P) + I.$$

Solving the above for  $P$  we have

$$P = \frac{I}{2I - Z}.$$

Now  $\psi(I) = 1$  and  $\psi(Z) = e^x$ , so

$$(2) \quad \psi(P) = \sum_{n=0}^{\infty} P(n) \frac{x^n}{n!} = \frac{1}{2 - e^x}.$$

We may use (2) to develop a series expansion for  $P(n)$ . By Taylor's Theorem, if  $n \geq 1$ ,

$$\begin{aligned} P(n) &= D^n (2 - e^x)^{-1} \big|_{x=0} \\ &= \frac{1}{2} D^n (1 - \frac{1}{2} e^x)^{-1} \big|_{x=0} \\ &= \frac{1}{2} D^n \sum_{k=0}^{\infty} (\frac{1}{2} e^x)^k \big|_{x=0} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} D^n (\frac{1}{2} e^x)^k \big|_{x=0} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} k^n 2^{-k}, \end{aligned}$$

which also holds for  $n = 0$ , if we stipulate  $0^0 = 1$ . Additional information about the numbers  $P(n)$ , including congruences satisfied by these numbers, may be found in the article by Gross [2].

## CHAPTER II

## GENERALIZED WEAK ORDERS

Section 2.1. Introduction to Generalized Weak Orders

In this chapter we study a class of binary relations, the generalized weak orders, which furnishes a common generalization of the class of complete weak orders and the class of asymmetric weak orders. Generalized weak orders were introduced in 1979 by P.C. Fishburn, a mathematical economist at Bell Laboratories [1]. In order to motivate the choice of the defining properties of these orders, we recall the following: If  $R$  is an asymmetric weak order then  $aR = R$  is negatively transitive, i.e.,  $caR = cR$  is transitive. Moreover, if  $R$  is a complete weak order, then  $aR$  is an asymmetric weak order (see Chapter I) and so  $caR$  is transitive. Thus the following definition is natural:

Definition: A binary relation  $R$  is a generalized weak order if and only if  $caR$  is transitive.

In other words, the generalized weak orders are just those binary relations  $R$  such that  $aR$  is an asymmetric weak order. Hence, if we denote the set of all binary relations on  $X$  by  $B$ , the set of asymmetric weak orders on  $X$  by  $A$  and the generalized weak orders on  $X$  by  $G$ , then  $G = \phi^{-1}(A)$ , the inverse image of  $A$  with respect to the mapping  $\phi: B \rightarrow B$  given by  $\phi(R) = aR$ .

Since an asymmetric weak order is transitive, as well as negatively transitive, if  $R$  is a generalized weak order, then  $aR$  is transitive and negatively transitive. However, a generalized weak

order  $R$  may itself fail to satisfy every one of the following conditions: transitivity, reflexivity, irreflexivity, antisymmetry, completeness, transitivity of  $cR$ , transitivity of  $sR$ , and transitivity of  $scR$ . To see this, consider the following generalized weak order  $R$  on  $\{1, 2, 3, \dots, 13\}$ , given by the incidence matrix  $M = (m_{ij})$  where  $m_{ij} = 1$  if  $iRj$  and  $m_{ij} = 0$  if  $\sim iRj$ :

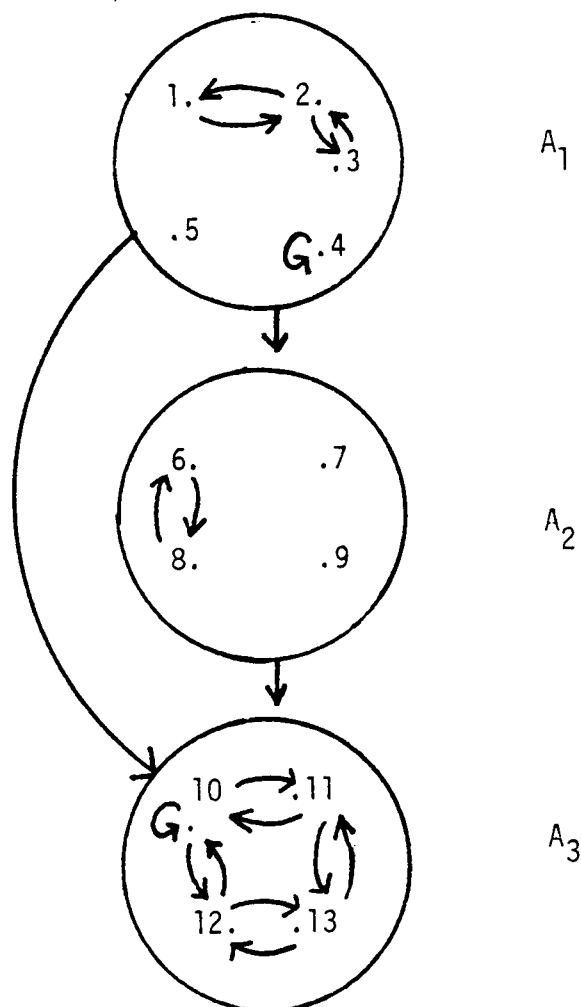
|           | <u>1</u> | <u>2</u> | <u>3</u> | <u>4</u> | <u>5</u> | <u>6</u> | <u>7</u> | <u>8</u> | <u>9</u> | <u>10</u> | <u>11</u> | <u>12</u> | <u>13</u> |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|-----------|-----------|-----------|-----------|
| <u>1</u>  | 0        | 1        | 0        | 0        | 0        | 1        | 1        | 1        | 1        | 1         | 1         | 1         | 1         |
| <u>2</u>  | 1        | 0        | 1        | 0        | 0        | 1        | 1        | 1        | 1        | 1         | 1         | 1         | 1         |
| <u>3</u>  | 0        | 1        | 0        | 0        | 0        | 1        | 1        | 1        | 1        | 1         | 1         | 1         | 1         |
| <u>4</u>  | 0        | 0        | 0        | 1        | 0        | 1        | 1        | 1        | 1        | 1         | 1         | 1         | 1         |
| <u>5</u>  | 0        | 0        | 0        | 0        | 0        | 1        | 1        | 1        | 1        | 1         | 1         | 1         | 1         |
| <u>6</u>  | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 1        | 0        | 1         | 1         | 1         | 1         |
| <u>7</u>  | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 1         | 1         | 1         | 1         |
| <u>8</u>  | 0        | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 1         | 1         | 1         | 1         |
| <u>9</u>  | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 1         | 1         | 1         | 1         |
| <u>10</u> | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 1         | 1         | 1         | 0         |
| <u>11</u> | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 1         | 0         | 0         | 1         |
| <u>12</u> | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 1         | 0         | 0         | 1         |
| <u>13</u> | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0         | 1         | 1         | 0         |

Referring to the above incidence matrix of  $R$  we can see that

1.  $R$  is not transitive because, for example,  $1R2$  and  $2R3$ , but  $\sim 1R3$ ;
2.  $R$  is not reflexive because, for example,  $\sim 1R1$ ;

3.  $R$  is not irreflexive because, for example,  $4R4$  ;
4.  $R$  is not antisymmetric because, for example,  $2R3$  and  $3R2$  , but  $2 \neq 3$  ;
5.  $R$  is not complete because, for example,  $\sim 1R3$  and  $\sim 3R1$  ;
6.  $T = cR$  is not transitive because, for example,  $1T4$  and  $4T2$  , but  $\sim 1T2$  ;
7.  $U = sR$  is not transitive because, for example,  $1U2$  and  $2U3$  , but  $\sim 1U3$  ;
8.  $V = scR$  is not transitive because, for example,  $6V7$  and  $7V8$  , but  $\sim 6V8$  .

There is an alternative way of specifying the foregoing weak order  $R$  , as follows:



The above specifies  $R$  on  $\{1, 2, 3, \dots, 13\}$  by the rule:  $iRj$  if and only if there is an arrow pointing from  $i$  to  $j$  or an arrow pointing from a circle containing  $i$  to a circle containing  $j$ . Note that there is an ordered partition of  $\{1, 2, 3, \dots, 13\}$ ,  $A_1 = \{1, 2, 3, 4, 5\}$ ,  $A_2 = \{6, 7, 8, 9\}$ ,  $A_3 = \{10, 11, 12, 13\}$  associated with the above weak order. This is always the case, as we shall see in the next section.

## Section 2.2 Fishburn Partitions and Generalized Weak Orders

Let  $R$  be a generalized weak order on  $X$ . Since  $aR$  is an asymmetric weak order, it follows from Theorem 1.14 that  $scaR$  is an equivalence relation on  $X$ . Denote by  $\hat{X}$  the partition  $X/scaR$  of  $X$  induced by  $scaR$ , and define a binary relation  $>$  on  $\hat{X}$  as follows: For all  $A, B \in \hat{X}$ ,  $A > B$  if and only if for all  $x \in A$  and for all  $y \in B$ ,  $xy \in aR$ , i.e., if and only if  $A \times B \subseteq aR$ .

Theorem 2.1. The binary relation  $>$  defined on  $\hat{X}$  above is a linear order, i.e.,  $>$  is irreflexive, transitive and complete.

Proof: Irreflexivity. Let  $A \in \hat{X}$ . For any  $x, y \in A$ ,  $xy \in scaR$  and so  $xy \notin aR$ . Hence  $\neg A > A$ .

Transitivity. Let  $A, B, C \in \hat{X}$  and suppose that  $A > B$  and  $B > C$ . Then for all  $x \in A$ , and for all  $y \in B$ , and for all  $z \in C$ ,  $xy \in aR$  and  $yz \in aR$ . It follows by transitivity of  $aR$  that  $xz \in aR$  for all  $x \in A$  and for all  $z \in C$ . Hence  $A > C$ .

Completeness. Let  $A, B \in \hat{X}$ . Suppose that  $\sim A > B$  and  $\sim B > A$ . We shall show that  $A = B$ . Since  $\sim A > B$ , there exists  $x_0 \in A$  and there exists  $y_0 \in B$  such that  $x_0 y_0 \notin aR$ . Since  $\sim B > A$ , there exists  $x_1 \in A$  and there exists  $y_1 \in B$  such that  $y_1 x_1 \notin aR$ . We show that  $y_0 x_0 \notin aR$ , so that  $x_0 y_0 \in scaR$ , from which it follows that  $x_0 \in A$  and  $y_0 \in B$  belong to the same equivalence class of  $\hat{X}$ , i.e., that  $A = B$ . Since  $y_0, y_1 \in B$ ,  $y_0 y_1 \notin aR$  and since  $y_1 x_1 \notin aR$ , it follows by negative transitivity of  $aR$  that  $y_0 x_1 \notin aR$ . Since  $x_0, x_1 \in A$ ,  $x_1 x_0 \notin aR$ , and it follows by negative transitivity of  $aR$  that  $y_0 x_0 \notin aR$ , as desired. ■

For  $x \in A$  and  $y \in B$ , where  $A$  and  $B$  are distinct equivalence classes of  $\hat{X}$ , we have, by the foregoing theorem, either  $xy \in aR$  or  $yx \in aR$ . On the other hand, if  $x$  and  $y$  are elements of the same equivalence class of  $\hat{X}$ , then  $xy \in R$  implies that  $yx \in R$ , i.e., the restriction of  $R$  to an equivalence class of  $\hat{X}$  is a symmetric relation, as proved below.

Theorem 2.2. Let  $R$  be a generalized weak order on  $X$  and let  $X = \hat{X}/scaR$ . For each  $A \in \hat{X}$ , the restriction of  $R$  to  $A$  is a symmetric relation.

Proof: Let  $x, y \in A$  and suppose that  $xy \in R$ . Since  $xy \in scaR$ ,  $xy \notin aR$ . If it were the case that  $yx \notin R$ , we would have  $xy \in aR$ , a contradiction. Hence  $yx \in R$ . ■

Let us call a linearly ordered partition of  $X$  a Fishburn partition if each of its equivalence classes is equipped with some symmetric relation. The above two theorems show that a generalized weak order on  $X$  gives rise to a Fishburn partition of  $X$ , which we call the Fishburn partition associated to  $R$ . Conversely, each

Fishburn partition of  $X$ , gives rise to a generalized weak order on  $X$ , as shown in the following theorem.

Theorem 2.3. Let  $(\tilde{X}, \succ, \{R_A: A \in \tilde{X}\})$  be a Fishburn partition of the set  $X$ , where each  $A \in \tilde{X}$  is equipped with a symmetric relation  $R_A$ . Then the relation  $R = (\bigcup_{\substack{A, B \in \tilde{X} \\ A \succ B}} (A \times B)) \cup (\bigcup_{A \in \tilde{X}} R_A)$  is

a generalized weak order on  $X$ .

Proof: Clearly,  $aR = \bigcup_{\substack{A, B \in \tilde{X} \\ A \succ B}} (A \times B)$ . Suppose that  $xy \notin aR$

and  $yz \notin aR$ . Let  $x \in A$ ,  $y \in B$ , and  $z \in C$ , where  $A, B, C \in \tilde{X}$ . Then  $(A = B \text{ or } B \succ A)$  and  $(B = C \text{ or } C \succ B)$ . Since  $\succ$  is a linear order on  $\tilde{X}$ , it follows that  $A = C$  or  $C \succ A$ . In either case  $xz \notin aR$ . Hence  $aR$  is negatively transitive, and  $R$  is a generalized weak order. ■

We call the relation  $R$  of Theorem 2.3 the generalized weak order associated to the Fishburn partition  $(\tilde{X}, \succ, \{R_A: A \in \tilde{X}\})$ . It is tedious but straightforward to prove that given a generalized weak order  $R$  on  $X$ , the generalized weak order associated to the Fishburn partition associated to  $R$  is  $R$ ; and given a Fishburn partition  $(\tilde{X}, \succ, \{R_A: A \in \tilde{X}\})$  of  $X$ , the Fishburn partition associated to the generalized weak order associated to  $(\tilde{X}, \succ, \{R_A: A \in \tilde{X}\})$  is  $(\tilde{X}, \succ, \{R_A: A \in \tilde{X}\})$ . This theorem, the proof of which we omit, implies that there is a one-to-one correspondence between generalized weak orders on  $X$  and Fishburn partitions of  $X$ . This correspondence can be exploited to enumerate generalized weak orders. Details may be found in [3].

We conclude this section with a theorem which exhibits for an arbitrary relation  $R$  the structure of the relation  $scaR$ .

Theorem 2.4. For every binary relation  $R$ ,  $scaR = sR \cup scR$ .

Proof:

$$\begin{aligned} xy \in scaR &\leftrightarrow (xy \notin aR) \text{ and } (yx \notin aR) \\ &\leftrightarrow (xy \notin R \text{ or } yx \in R) \text{ and } (yx \notin R \text{ or } xy \in R) \\ &\leftrightarrow (xy \in cR \text{ and } yx \in cR) \text{ or } (xy \in cR \text{ and } xy \in R) \\ &\quad \text{or } (yx \in R \text{ and } yx \in cR) \text{ or } (yx \in R \text{ and } xy \in R) \end{aligned}$$

Since  $(p \text{ or } q \leftrightarrow p, \text{ if } q \text{ is contradictory})$ , we have

$$\begin{aligned} &\leftrightarrow (xy \in cR \text{ and } yx \in cR) \text{ or } (yx \in R \text{ and } xy \in R) \\ &\leftrightarrow xy \in scR \text{ or } xy \in sR \\ &\leftrightarrow xy \in sR \cup scR. \quad \blacksquare \end{aligned}$$

In particular, when  $R$  is a generalized weak order the above theorem exhibits the equivalence relation  $scaR$  as the union of relations  $sR$  and  $scR$ . When  $R$  is an asymmetric weak order,  $sR = \emptyset$  and  $scaR = scR$ ; and when  $R$  is a complete weak order  $scR = \emptyset$  and  $scaR = sR$  (cf. Theorems 1.13 and 1.14).

### Section 2.3. Special Types of Generalized Weak Orders

We are already familiar with two special types of generalized weak orders, namely, the asymmetric weak orders and the complete weak orders. In this section we study some additional special types of generalized weak orders including

- a. the transitive generalized weak orders
- b. the negatively transitive generalized weak orders
- c. the transitive and negatively transitive generalized weak orders.

We require as preliminaries the following sequence of Lemmas:

Lemma 1.  $R$  is transitive  $\rightarrow sR$  and  $aR$  are transitive.

Proof: We first show  $sR$  is transitive given  $R$  is transitive.

Suppose  $xy, yz \in sR$ . Then by definition of  $s$ , we have  $(xRy$  and  $yRx)$  and  $(yRz$  and  $zRy)$  which is equivalent to  $(xRy$  and  $yRz)$  and  $(zRy$  and  $yRx)$  by commutativity of "and". By transitivity of  $R$ ,  $xRz$  and  $zRx$ , thus  $xz \in sR$ . Therefore,  $R$  is transitive  $\rightarrow sR$  is transitive.

Now we show that  $aR$  is transitive when  $R$  is transitive.

Suppose  $xy, yz \in aR$ . By definition of  $aR$ , we have  $xRy$  and  $yRz$  which implies  $xRz$  by transitivity of  $R$ . If  $zRx$ , then, since  $xRy$ , it follows that  $zRy$ , contradicting  $yz \in aR$ . Thus  $zx \in cR$ , and so  $xz \in aR$ . Thus,  $R$  is transitive  $\rightarrow aR$  is transitive. ■

The following three lemmas all postulate transitivity of  $sR$  among their hypotheses, and assert the transitivity of  $R$  as conclusion. In the proof of each lemma we shall therefore need to show that  $(xy \in R$  and  $yz \in R) \rightarrow xz \in R$ . Now, given any binary relation  $R$  on  $X$  and any  $u, v \in X$ , if  $uv \in R$ , then  $uv \in sR$  or  $uv \in aR$ . Hence, the condition  $*(xy \in R$  and  $yz \in R)$  is equivalent to the disjunction of the following four assertions:

- (i)  $xy \in sR$  and  $yz \in sR$
- (ii)  $xy \in aR$  and  $yz \in sR$
- (iii)  $xy \in sR$  and  $yz \in aR$
- (iv)  $xy \in aR$  and  $yz \in aR$ .

Now, if  $*$  holds in virtue of the fact that condition (i) is the case, then by the postulated transitivity of  $sR$ , it follows that  $xz \in sR$ , and hence that  $xz \in R$ , as desired. Thus the proofs below omit the above case, and treat cases (ii), (iii), and (iv) as two overlapping cases: (1)  $xy \in aR$  and (2)  $yz \in aR$ . The resulting transitivity assertions then reduce to

- I.  $(xy \in aR \text{ and } yz \in R) \rightarrow xz \in R$
- II.  $(xy \in R \text{ and } yz \in aR) \rightarrow xz \in R$ .

Lemma 2.  $cR$  and  $sR$  are transitive  $\rightarrow R$  is transitive.

Proof I: Suppose  $(xy \in aR \text{ and } yz \in R)$ . By definition of  $aR$ ,  $yx \in cR$ . We wish to show  $xz \in R$ . Suppose, to the contrary, that  $xz \in cR$ . Then by transitivity of  $cR$ ,  $yz \in cR$ , contradicting  $yz \in R$ . Hence  $xz \in R$ , as desired. II The proof that  $xy \in R$  when  $(xy \in R \text{ and } yz \in aR)$  is similar. ■

Lemma 3.  $caR$  and  $sR$  are transitive  $\rightarrow R$  is transitive.

Proof I: Suppose  $(xy \in aR \text{ and } yz \in R)$ . Since  $yz \in R$ , we have  $zy \in caR$ . We wish to show  $xz \in R$ . Suppose to the contrary, that  $xz \in cR$ . Then since  $xz \notin R$ ,  $xz \notin aR$ , i.e.,  $xz \in caR$ . Then by transitivity of  $caR$ ,  $xy \in caR$ , contradicting

$xy \in aR$  . Hence  $xz \in R$  , as desired. II The proof that  $xz \in R$  when  $(xy \in R$  and  $yz \in aR)$  is similar. ■

Lemma 4.  $aR$  ,  $sR$  , and  $scR$  are transitive  $\rightarrow R$  is transitive.

Proof I: Suppose  $(xy \in aR$  and  $yz \in R)$  . We need here to treat separately the two sub-cases Ia  $(xy \in aR$  and  $yz \in aR)$  and Ib  $(xy \in aR$  and  $yz \in sR)$  . Suppose that Ia holds. We wish to show that  $xz \in R$  . By transitivity of  $aR$  , we have  $xz \in aR$  , hence  $xz \in R$  . Suppose that Ib holds. We again wish to show that  $xz \in R$  . Suppose, to the contrary, that  $xz \in cR$  . Then either (i)  $zx \in aR$  or (ii)  $zx \in cR$  . If (i)  $zx \in aR$  , then  $zy \in aR$  by transitivity of  $aR$  . Thus  $yz \notin R$  and so  $yz \notin sR$  , contradicting  $yz \in sR$  . If (ii)  $zx \in cR$  , then  $xz \in scR$  , and  $zx \in scR$  . By transitivity of  $scR$  , it follows that  $zz \in scR$  , hence that  $zz \in cR$  . But since  $yz \in sR$  ,  $zy \in sR$  , and transitivity of  $sR$  implies  $zz \in sR$  . Hence  $zz \in R$  , contradicting  $zz \in cR$  . Hence  $xz \in R$  , as desired.

II The proof that  $xz \in aR$  when  $(xy \in R$  and  $yz \in aR)$  is similar. ■

Next we exhibit a complementary form of each of the previous three lemmas.

Lemma 2'.  $R$  and  $scR$  are transitive  $\rightarrow cR$  is transitive.

Proof: By replacing  $R$  by  $cR$  in Lemma 2, we have  $ccR$  and  $scR$  are transitive  $\rightarrow cR$  is transitive. Since  $ccR = R$  , we have the desired result. ■

Lemma 3'.  $caR$  and  $scR$  are transitive  $\rightarrow cR$  is transitive.

Proof: By substituting  $cR$  for  $R$  in Lemma 3, we obtain  $cacR$  and  $scR$  are transitive  $\rightarrow cR$  is transitive. By Theorem 1.5,  $acR = adR$ , and hence  $cacR = cadR$ . By Theorem 1.6,  $cadR = cdaR = dcaR$  and hence  $cacR = dcaR$ . Thus ( $dcaR$  and  $scR$  transitive  $\rightarrow cR$  transitive). But by Theorem 1.7,  $dcaR$  is transitive if  $caR$  is transitive. Hence  $caR$  and  $scR$  are transitive  $\rightarrow cR$  is transitive. ■

Lemma 4'.  $aR$ ,  $sR$  and  $scR$  are transitive  $\rightarrow cR$  is transitive.

Proof: By replacing  $R$  by  $cR$  in Lemma 4, we have ( $acR$ ,  $scR$  and  $sccR$  are transitive)  $\rightarrow cR$  is transitive. By Theorem 1.5,  $acR = adR$ , and since  $ccR = R$ , we have ( $daR$ ,  $scR$  and  $sR$  are transitive)  $\rightarrow cR$  is transitive. By Theorem 1.7,  $aR$  is transitive  $\rightarrow daR$  is transitive. Thus  $aR$ ,  $sR$  and  $scR$  are transitive  $\rightarrow cR$  is transitive. ■

Our next Lemma shows that a transitive, negatively transitive binary relation is necessarily a generalized weak order.

Lemma 5.  $R$  and  $cR$  are transitive  $\rightarrow caR$  is transitive.

Proof: Given that  $R$  and  $cR$  are transitive, we are to prove that  $caR$  is transitive, which by Theorem 1.8 is equivalent to proving that  $xz \in aR \rightarrow (xy \in aR \text{ or } yz \in aR)$ . Suppose  $xz \in aR$  and  $xy \in caR$ . Then either (i)  $xy \in scR$  or (ii)  $yx \in R$ .

(i)  $xy \in scR \leftrightarrow xy$ ,  $yx \in cR$ . Since  $xz \in aR$  we have  $zx \in cR$  which implies  $zy \in cR$  by transitivity of  $cR$ . If  $yz \in cR$ , then by transitivity of  $cR$ ,  $xz \in cR$ . This contradicts  $xz \in aR$ , so  $yz \in R$ , hence  $yz \in aR$ .

(ii)  $yx \in R$  hence  $yz \in R$  by transitivity of  $R$ . If  $zy \in R$ , then  $zx \in R$ , contrary to  $xz \in aR$ , hence  $zy \in cR$ . Thus  $(yz \in R \text{ and } zy \in cR) \rightarrow yz \in aR$ .

The proof that  $xz \in aR$  and  $yz \in caR \rightarrow xy \in aR$  is similar. ■

Lemma 6. (i)  $cR$  is transitive  $\rightarrow aR$  is transitive and  
(ii)  $caR$  is transitive  $\rightarrow aR$  is transitive.

Proof: Replacing  $R$  with  $cR$  in Lemma 1 yields  $cR$  transitive  $\rightarrow acR$  transitive. By Theorems 1.5 and 1.6,  $acR = adR = daR$ , and so  $cR$  transitive  $\rightarrow daR$  transitive. But (Theorem 1.7)  $daR$  transitive  $\rightarrow aR$  transitive. Hence  $cR$  is transitive  $\rightarrow aR$  is transitive.

(ii) If  $caR$  is transitive,  $aR$  is an asymmetric weak order. (i.e., negatively transitive and asymmetric). Hence by Theorem 1.9, with  $aR$  for  $R$ ,  $aR$  is transitive. ■

Lemma 7.  $R$  is complete and transitive  $\rightarrow cR$  is transitive.

Proof: Assume that  $xy, yz \in cR$ . We need to show  $xz \in cR$  when  $R$  is complete and transitive. If  $x = y$  or  $y = z$ , then there is nothing to prove, so assume  $x \neq y$  and  $y \neq z$ . By completeness of  $R$ ,  $yx, zy \in R$ . If  $xz \in R$ , then  $yz \in R$  by transitivity of  $R$ . But  $yz \in R$  contradicts  $yz \in cR$ . Hence  $xz \in cR$ . ■

Lemma 8.  $R$  is reflexive and  $scR$  is transitive  $\rightarrow R$  is complete.

Proof: If  $R$  is not complete, then  $xy \in scR$  for some  $x \neq y$ . Then  $xx \in scR$  by transitivity of  $scR$ . But  $xx \in cR$  since

$xx \in scR$  . This contradicts reflexivity. Thus  $R$  is complete when  $R$  is reflexive and  $scR$  is transitive. ■

Because reflexivity and irreflexivity are complementary, and completeness and antisymmetry are complementary by Theorems 1.2 and 1.3 respectively, the following pair of lemmas follow immediately from Lemmas 7 and 8 by substituting  $cR$  for  $R$  .

Lemma 7'.  $R$  is antisymmetric and  $cR$  is transitive  $\rightarrow R$  is transitive.

Lemma 8'.  $R$  is irreflexive and  $sR$  is transitive  $\rightarrow R$  is antisymmetric.

We are now ready to prove some interesting characterizations of the special types of generalized weak orders mentioned at the beginning of this section.

Theorem 2.5. The following statements are equivalent:

1.  $R$  is a transitive generalized weak order (i.e.,  $caR$  and  $R$  are transitive).
2.  $caR$  and  $sR$  are transitive.
3.  $aR$ ,  $scaR$  , and  $R$  are transitive.

Proof: By Lemma 3,  $2 \rightarrow 1$  . By Lemma 1, if  $R$  is transitive, then  $aR$  is transitive and (substituting  $scR$  for  $R$ ) if  $caR$  is transitive then  $scaR$  is transitive. Hence  $1 \rightarrow 3$  . Finally,  $3 \rightarrow 2$  by Lemma 2' with  $aR$  substituted for  $R$  and by Lemma 1. ■

In particular, the equivalence of 1 with 2 above means that the transitive generalized weak orders are just those generalized weak orders with transitive symmetric part.

Theorem 2.6. The following statements are equivalent:

1.  $R$  is a negatively transitive generalized weak order (i.e.,  $caR$  and  $cR$  are transitive).
2.  $caR$  and  $scR$  are transitive.
3.  $aR$  ,  $scaR$  and  $cR$  are transitive.

Proof: By Lemma 3' ,  $2 \rightarrow 1$  . Also  $1 \rightarrow 3$  , for if  $caR$  is transitive,  $scaR$  is transitive by Lemma 1 (with  $caR$  substituted for  $R$ ). And since 1 implies that  $aR$  is an asymmetric weak order,  $aR$  is transitive by Theorem 1.9. Finally,  $3 \rightarrow 2$  by Lemma 2' (with  $aR$  substituted for  $R$ ) and by Lemma 1 (with  $cR$  substituted for  $R$ ) . ■

In particular, the equivalence of 1 with 2 above means that the negatively transitive generalized weak orders are just those generalized weak orders, the symmetric part of whose complement is transitive.

Theorem 2.7. The following statements are equivalent:

1.  $R$  is a transitive, negatively transitive generalized weak order (i.e.,  $caR$  ,  $R$  and  $cR$  are transitive).
2.  $caR$  ,  $sR$  , and  $scR$  are transitive.
3.  $aR$  ,  $sR$  , and  $scR$  are transitive.
4.  $R$  and  $scR$  are transitive.
5.  $R$  and  $cR$  are transitive.
6.  $cR$  and  $sR$  are transitive.

Proof: By Theorems 2.5 and 2.6,  $1 \rightarrow 2$  . Since 2 implies that  $aR$  is an asymmetric weak order, it follows from Theorem 1.9 that  $aR$  is transitive. Hence  $2 \rightarrow 3$  . By Lemma 4,  $3 \rightarrow 4$  . By Lemma 2' ,

$4 \rightarrow 5$  . By Lemma 1,  $5 \rightarrow 6$  . Finally, given 6, it follows from Lemma 1 (with  $cR$  substituted for  $R$ ) that  $scR$  is transitive. By Lemma 2,  $R$  is transitive. Hence, by Lemma 5,  $caR$  is transitive. Thus  $6 \rightarrow 2$  . ■

In particular, the equivalence of 1 and 5 above means that the transitive, negatively transitive generalized weak orders are just the transitive, negatively transitive binary relations.

Theorem 2.8. The following statements are equivalent:

1.  $R$  is a complete weak order (i.e., a reflexive, complete transitive relation).
2.  $R$  and  $cR$  are transitive and  $R$  is reflexive.
3.  $caR$  and  $scR$  are transitive and  $R$  is reflexive.

Proof: By Lemma 7,  $1 \rightarrow 2$  . By Theorem 2.7, parts 2 and 5,  $2 \rightarrow 3$  . By Lemma 8, 3 implies that  $R$  is complete, hence that  $scR = \emptyset$  . By Lemma 1 (with  $caR$  substituted for  $R$ )  $scaR$  is transitive. Now for every binary relation  $R$  ,  $scaR = sR \cup scR$  (Theorem 2.4). Since  $scR = \emptyset$  ,  $scaR = sR$  , and so  $sR$  is transitive. It then follows from Lemma 3 that  $R$  is transitive. Hence  $3 \rightarrow 1$  . ■

In particular, the equivalence of 1 and 3 above means that the complete weak orders are just the reflexive generalized weak orders having transitive symmetric complements.

Theorem 2.9. The following statements are equivalent:

1.  $R$  is an asymmetric weak order (i.e., an asymmetric, negatively transitive relation).

2.  $R$  and  $cR$  are transitive and  $R$  is irreflexive.

3.  $caR$  and  $sR$  are transitive and  $R$  is irreflexive.

Proof: By Theorem 1.1, 1 implies that  $R$  is antisymmetric and irreflexive. By Lemma 7',  $R$  is thus transitive. Hence  $1 \rightarrow 2$ . By Theorem 2.7, parts 2 and 5,  $2 \rightarrow 3$ . Finally, it follows from 3 and Lemma 8', that  $R$  is antisymmetric. Along with irreflexivity of  $R$ , this implies, by Theorem 1.1, that  $R$  is asymmetric. Hence  $aR = R$  and  $caR = cR$ , and so  $cR$  is transitive. Thus  $3 \rightarrow 1$ . ■

In particular, the equivalence of 1 and 3 above means that the asymmetric weak orders are just the irreflexive generalized weak orders with transitive symmetric parts.

Furthermore, Theorems 2.8 and 2.9 furnish axiomatizations of complete weak orders and asymmetric weak orders which differ in a single axiom: The complete weak orders are just the reflexive, transitive, negatively transitive binary relations and the asymmetric weak orders the irreflexive, transitive, negatively transitive relations.

## LIST OF REFERENCES

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1. Fishburn, P.C., "Transitivity." The Review of Economic Studies 46(1), No. 142(1979), 163 - 173.
2. Gross, O.A., "Preferential Arrangements." American Mathematical Monthly 69(1962), 4 - 8.
3. Wagner, C.G., "Enumeration of Generalized Weak Orders." To appear.

## APPENDIX

If  $R$  is a binary relation on  $X$ , a function  $f: X \rightarrow \mathbb{R}$  satisfying  $aRb \leftrightarrow f(a) > f(b)$  is called an ordinal utility function.

The following theorem establishes necessary and sufficient conditions for  $R$  to possess an ordinal utility function, when  $X$  is a finite set.

Theorem: Given a binary relation  $R$  on the finite set  $X$ , there exists a real-valued function  $f$  on  $X$  such that for all  $a, b \in X$ ,  $aRb \leftrightarrow f(a) > f(b)$  if and only if  $R$  is asymmetric and negatively transitive.

Proof: Suppose that there exists a real-valued function  $f$  on  $X$  such that for all  $a, b \in X$ ,  $aRb \leftrightarrow f(a) > f(b)$ . We need to show that  $R$  is an asymmetric weak order.

Asymmetry: If  $aRb$ , then  $f(a) > f(b)$ , hence not  $f(b) > f(a)$ , and  $\sim bRa$ .

Negative Transitivity: If  $\sim aRb$  and  $\sim bRc$ , then  $\sim[f(a) > f(b)]$  and  $\sim[f(b) > f(c)]$ , so  $f(a) \leq f(b)$  and  $f(b) \leq f(c)$ . It follows that  $f(a) \leq f(c)$ , so  $\sim[f(a) > f(c)]$  or  $\sim aRc$ .

Conversely, suppose  $R$  is an asymmetric weak order. The proof that there exists a real-valued function  $f$  on  $X$  such that for all  $a, b \in X$ ,  $aRb \leftrightarrow f(a) > f(b)$  is constructive. We define  $f(x)$  as follows:  $f(x)$  = the number of  $y$  in  $X$  such that  $xRy$ . Let us begin by illustrating this construction. Suppose  $X = \{a, b, c, d, e\}$  and  $R = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e)\}$ . Notice that  $R$  is asymmetric and negatively transitive. The function  $f$  defined previously is

is given by:  $f(a) = 3$  (since  $aRc$ ,  $aRd$ , and  $aRe$ ),  $f(b) = 3$ ,  $f(c) = 2$ ,  $f(d) = 0$ ,  $f(e) = 0$ .

To prove formally that a function  $f$ , defined by  $f(x) =$  the number of  $y$  in  $X$  such that  $xRy$ , always satisfies  $aRb \leftrightarrow f(a) > f(b)$ , note that  $R$  is transitive by Theorem 1.9. We now verify  $aRb \leftrightarrow f(a) > f(b)$ .

We first show  $aRb \rightarrow f(a) > f(b)$ . If  $aRb$ , then by transitivity of  $R$ ,  $bRy$  implies  $aRy$  for every  $y$ . Thus the number of  $y$  such that  $aRy$  is at least as big as the number of  $y$  such that  $bRy$ . It follows that  $f(a) \geq f(b)$ . Moreover,  $aRb$  but  $\neg bRb$  since an asymmetric weak order is irreflexive (see Theorem 1.1). Thus  $f(a) > f(b)$ .

Now we must show  $f(a) > f(b) \rightarrow aRb$ . We do this by showing its contrapositive,  $\neg aRb \rightarrow \neg[f(a) > f(b)]$ . If  $\neg aRb$ , then  $\neg bRy$  implies  $\neg aRy$ , by negative transitivity. Hence  $aRy \rightarrow bRy$  so  $f(b) \geq f(a)$ , hence  $\neg[f(a) > f(b)]$ . Thus,  $f(a) > f(b) \rightarrow aRb$ . This proves  $aRb \leftrightarrow f(a) > f(b)$  and completes the proof of this theorem. ■

## VITA

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