To the graduate council:

I am submitting herewith a dissertation written by Reid Marshall Davis entitled "Covers, q-Binomial Series, and q-Lattices." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Carl G. Wagner, Major Professor

We have read this dissertation and recommend its acceptance:

Accepted for the Council:

Associate Vice Chancellor
and Dean of The Graduate School

COVERS, q-BINOMIAL SERIES, AND q-LATTICES

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Reid Marshall Davis

August 1991

Soli Deo Gloria

To God alone the glory

But whatever was to my profit I now consider loss for the sake of Christ.
(Phillipians 3 : 7, NIV)

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ABSTRACT

An *ordered cover* of a finite set S is a finite sequence of nonempty subsets of S with union S. If the subsets are pairwise disjoint, then the cover is an *ordered partition*. An ordered cover of a finite vector space V is a finite sequence of nontrivial subspaces of V with sum V. If the sum is, in fact, a direct sum, then the cover is an ordered partition of V.

Certain power series are generating functions for ordered partitions (in both senses—i.e., for finite sets and finite vector spaces). This dissertation develops a new class of infinite series that are generating functions for ordered *covers* (as distinct from partitions) in both senses. These series are elements of the *algebra of q-binomial series* that results from equipping the set of arithmetic functions (sequences of complex numbers) with the usual addition and scalar multiplication and with a new multiplication, the *q-Newton product*.

In a particular sort of binomial poset, the *q-lattice*, the union of sets and the sum of vector spaces are both special cases of the join of lattice elements. Thus q-lattices unify and generalize several pairs of identities involving, on the one hand, binomial coefficients, and on the other hand, q-binomial coefficients.

Formal q-binomial series are defined in terms of the join of q-lattice elements. As a consequence, particular q-binomial series turn out to be generating functions both for ordered covers of sets and ordered covers of vector spaces.

When equipped with suitable operations, the collection of complex-valued functions on the intervals of a q-lattice becomes an algebra—the *covering algebra* of the lattice. A subalgebra—the *reduced covering algebra*—is isomorphic to the algebra of q-binomial series. These results provide an algebraic underpinning for q-binomial series much like that provided by incidence algebras and reduced incidence algebras for classical generating functions.

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LIST OF SYMBOLS

The following is a list of special symbols (and a very few terms) used in the dissertation. Each is listed with the number of the section in which it first appears (the letters A and B denote appendices) and a definition or description or perhaps the name of the symbol. A few symbols receive two definitions in the text and thus appear twice in the list.

Symbols that denote homomorphisms appear with the domain and codomain indicated.

The order of presentation is as follows:

(1) Expressions involving neither letters nor numbers
(2) Expressions involving Greek letters
(3) Expressions in which a number is the "primary" symbol, arranged in numerical order
(4) Expressions in which Roman, script, or boldface letters are the "primary" symbols, arranged "alphabetically."

When listings in category (4) involve variables, the variables used are the "typical" ones. For instance, the covering coefficient symbol appears as $\left\{ {n \atop r,s} \right\}_q$ and is alphabetized under n.

Symbol/term	Section	Explanation
$\subseteq$	1.1	containment (perhaps improper) of sets
$\subset$	A	proper containment of sets
$\wedge$	2.3	meet of lattice elements
$\vee$	2.3	join of lattice elements
$*$	4.1	the Cauchy product of arithmetic functions (power series)
$\cdot$	4.1	scalar multiplication (in any algebra)
$\diamond$	4.2	the q-Newton product of arithmetic functions
$\diamond$	5.1	the multiplication of the covering algebra $(\mathcal{C}(L), +, \diamond, .)$
$\cdot$	6.1	the Schur (pointwise) product of arithmetic functions
$\cdot$	8	the Schur (pointwise) product on $\mathcal{C}(L)$
$\hat{}: \mathcal{R}(L) \to \mathbf{C}^{\mathbf{N}}$	5.3	The isomorphism from $(\mathcal{R}(L), +, \diamond, .)$ to $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$

δ_{nk}	4.3	the Kronecker delta (= 1 if $n = k$; = 0 otherwise)
Δ	8	the finite difference operator on $\mathcal{P}$
Δ_q	8	the q-finite difference operator on $\mathcal{P}$
$\phi: \mathbf{C} \to \mathbf{C}^{\mathbf{N}}$	4.2	the monomorphism from $\mathbf{C}$ to the algebras $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ and $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$
$\phi_c: \mathcal{P} \to \mathbf{C}$	4.4	the evaluation homomorphism on $(\mathcal{P}, +, *, .)$
$\phi'_c: \mathcal{P} \to \mathbf{C}$	4.4	the evaluation homomorphism on $(\mathcal{P}, +, \diamond, .)$
ν	5.1	the Newton function of the covering algebra (inverse of the zeta function)
$\psi: \mathcal{P} \to \mathcal{P}$	4.4	the isomorphism between $(\mathcal{P}, +, *, .)$ and $(\mathcal{P}, +, \diamond, .)$
$\rho(z)$	2.1	the rank of z in a graded poset
$\rho_x(z)$	2.5	the rank of z in the interval $[x, z]$
ζ	5.1	the zeta-function of the covering algebra ($\zeta = 1$ on all intervals)
$\hat{0}$	2.1	the least element in a poset
0^0	3.4	by convention $0^0 = 1$ throughout the dissertation
$\hat{1}$	2.1	the greatest element in a poset
2^S	2.3	the power set (set of all subsets) of a set S
$A(n)$	2.2	the number of elements of rank one (atoms) in an n-interval of a binomial poset
$B(n)$	2.2	the factorial function of a binomial poset—the number of maximal chains in an n-interval
$B(n)$	A	the n-th Bell number—the number of subspaces of an n-space
$\mathbf{C}$	1.1	the complex numbers
$\mathcal{C}(L)$	5.1	the set of complex-valued functions on Int(L)—the set on which the covering algebra of L is defined
$C(n)$	A	the number of covers of an n-set
$\mathbf{C}^{\mathbf{N}}$	4.1	the set of arithmetic functions $\{F: \mathbf{N} \to \mathbf{C}\}$
$C(n; A_1, \ldots, A_k)$	7.2	the (relaxed) k-covers of an n-interval in which the rank of the ith term must be in A_i
$C(n, k)$	7.2	the k-covers of an n-interval in a q-lattice
$C(n, k)$	A	the number of k-covers of an n-set
$C_0(n, k)$	7.2	the relaxed (rank zero elements allowed) k-covers of an n-interval in a q-lattice

$\mathbf{C}[X]$	1.1	the set $\mathcal{P}$; informally the algebra of complex polynomials
$\mathbf{C}[[X]]$	1.1	the set $\mathbf{C}^{\mathbf{N}}$; informally the algebra of complex power series
$\deg P$	1.1	the degree of the polynomial P
$\dim V$	1.2	the dimension of the vector space V
$D_q F$	8	the Eulerian derivative of F
$f^{\diamond n}$	5.1	the nth power of $f \in \mathcal{C}(L)$ under the multiplication $\diamond$
$F^*(P)$	8	the polynomial $\sum_{n=0}^{\deg F} F(n)P^{*n}$
$F^\diamond(P)$	8	the polynomial $\sum_{n=0}^{\deg F} F(n)P^{\diamond n}$
F_q	1.2	the finite field with q elements (q implicitly a power of a prime)
$GF(q)$	A	the finite field with q elements (same as F_q)
$\mathrm{Int}(L)$	5.1	the set of intervals in the lattice L
k-cover	7.2	a k-tuple of elements of positive rank from an interval $[x, y]$ with join y
k-cover	A	a k-block cover of a finite set
$k^{\underline{n}}$	1.1	the number $k(k-1)\ldots(k-n+1)$
$l(P)$	2.1	the length of the poset P
$L_n(q)$	A	the lattice of subspaces of $V_n(q)$
$M(n,k)$	A	the number of minimal k-covers of an n-set
$M(n,k,j)$	A	the number of minimal k-covers of an n-set that cover j points of the set uniquely
N	6.2	the Newton function of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ (the inverse of the zeta function)
$\mathbf{N}$	1.1	the non-negative integers
$[n]$	1.1	the set $\{1, 2, ..., n\}$
$\binom{n}{k}_q$	1.2	the number of k-subspaces of an n-space
$\begin{bmatrix} n \\ k \end{bmatrix}$	2.2	the number of elements of rank k in an n-interval of a binomial poset
$\binom{n}{n_1, n_2, \ldots, n_k}$	B	a (generalized) multinomial coefficient of order n
$\left\{ {n \atop r_1, \ldots, r_k} \right\}_q$	4.2	a multi-covering coefficient
$\left\{ {n \atop r, s} \right\}$	1.1	the number of ways to write an n-set as an ordered union of an r-subset and an s-subset
$\left\{ {n \atop r, s} \right\}_q$	1.2	the number of ways to write an n-space as an ordered sum of an r-subspace and an s-subspace

$\left\{ {n \atop r,s} \right\}_q$	4.2	a covering coefficient
n-interval	2.2	an interval of length n in a binomial poset
$n_q^{\underline{k}}$	3.2	the quantity $(n_q)(n_q - 1_q)\dots(n_q - (k-1)_q)$
n_q	3.2	the number $(q^{n-1} + q^{n-2} + \cdots + 1)$
$n!_q$	3.2	the number $n_q(n-1)_q \dots 1_q$
n-set/subset	1.1	a finite set/subset with n elements
n-space/subspace	1.2	an n-dimensional vector space/subspace over a finite field
$\mathbf{P}$	2.2	the set of positive integers
$\mathcal{P}$	4.4	the set of polynomials—the finitely nonzero arithmetic functions
$\left\langle {P \atop n} \right\rangle_q$	8	the arithmetic function that results from replacing X with the polynomial P in the definition of $\left\langle {X \atop n} \right\rangle_q$
q	3.1	the number $A(2) - 1$ in a locally modular binomial poset
$\mathcal{R}(L)$	5.2	the functions $f \in \mathcal{C}(L)$ that are constant on intervals of equal length—the set on which the reduced covering algebra is defined
$\|S\|$	1.1	the cardinality of the set S
$S_2(n,k)$	A	a Stirling number of the second kind—the number of partitions of an n-set into k-blocks
span{S}	A	the vector subspace spanned by the set S
$U \oplus V$	A	the direct sum of the vector spaces U and V
V_1/V_2	B	the quotient group V_1/V_2 viewed as a vector space, where V_1 is a finite vector space and V_2 is a subspace of V_1
val$F$	1.1	the position of the first nonzero term in the arithmetic function F
$V_n(q)$	A	the unique (up to isomorphism) n-space over F_q
$\lfloor x \rfloor$	A	the greatest integer $\leq x$
$\lceil x \rceil$	A	the smallest integer $\geq x$
X	4.3	the element $(0,1,0,0,\dots) \in \mathbf{C}^{\mathbf{N}}$—the indeterminate in the algebra of formal power series and the algebra of formal binomial series
$X^{\underline{n}}$	1.1	the polynomial $X(X-1)\dots(X-n+1)$
X^{*n}	4.3	the nth Cauchy power of X

$\binom{X}{n}$	1.1	the polynomial $X^{\underline{n}}/n!$
$\left\langle {X \atop n} \right\rangle_q$	4.3	the q-binomial polynomial of order n—the arithmetic function defined by $\frac{1}{n_q^{\underline{n}}}\big(X \diamond (X - 1_q) \diamond \cdots \diamond (X - (n-1)_q)\big)$
$\begin{bmatrix} X \\ n \end{bmatrix}$	8	the polynomial $\frac{(X-1)(X-q)\ldots(X-q^{n-1})}{(q^n-1)(q^n-q)\ldots(q^n-q^{n-1})}$
$[x, y]$	2.1	the interval $\{z \in P \mid x \leq z \leq y\}$ in a poset P
Z	6.2	the zeta function of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ ($Z(n) = 1$ for all n)

Introduction

Three themes, tightly interwoven, form the body of this dissertation: ordered covers, q-binomial series, and q-lattices.

An ordered cover of an n-set S is a finite sequence of subsets of S with union S. If the subsets are pairwise disjoint, then the ordered cover receives the name *ordered partition.* Both these concepts have analogues involving subspaces of a finite vector space. Appendix A gives an extensive description of combinatorial problems that arise under the general label of *covers*—ordered or unordered.

Generating functions provide a useful tool for studying ordered covers. Classical generating functions, in particular exponential and Eulerian generating functions, address the enumeration of ordered partitions, as Appendix B details. The enumeration of ordered covers, as distinct from ordered partitions, requires a new sort of generating function, namely q-binomial series.

As Doubilet, Rota, and Stanley [4, 16] show, classical generating functions share a unifying background of binomial posets and incidence algebras. A special sort of binomial poset, namely the q-lattice, also unifies q-binomial series. Further, q-lattices provide the context for a more general definition of *ordered cover* that encompasses the other definitions of that term as special cases.

Thus each of the three themes provokes consideration of the other two.

The eight sections of the dissertation have the following goals: Section 1 shows several binomial coefficient and q-binomial coefficient identities that motivate and direct the search for q-binomial series. Section 2 gives a brief introduction to partially ordered sets and lattices, introducing some new concepts along the way. Section 3 defines q-lattices, develops some of the basic notation associated with them, and shows their power to unify results about sets and results about finite vector spaces. Section 4 defines the algebra of q-binomial series and explores its algebraic and topological properites. Section 5 introduces an incidence algebra-like structure—the covering algebra—and shows how it generates the algebra of q-binomial series. Section 6 makes a slight detour to present Newton inversion—an analogue of Möbius inversion—in the algebra of q-binomial series. Section 7 returns to the main discussion and indeed completes it by showing how q-binomial series serve as generating functions for ordered covers in a q-lattice. Finally, Section 8 briefly examines questions which the dissertation touches on but fails to answer, defining in the process a finite difference operator which seems well-suited to the study of q-binomial series.

Appendices A and B contain definitions and concepts associated with generating functions, finite vector spaces, covers, and the analogues of covers in finite vector spaces. Otherwise the material in the main body of the dissertation is self-contained and should be accessible to anyone moderately familiar with vector spaces and algebras.

1. Motivation

1.1 Set identities

Classical generating functions, as appendix B illustrates, count ordered partitions or objects analogous to ordered partitions. They provide no information about the weaker concept of a cover and its analogues. Certain identities suggest, however, the existence of a new sort of generating function that does give information about covers.

For $n \in \mathbf{N} = \{0, 1, 2, \dots\}$ let T be a set with n elements (henceforth an *n-set*), and for $r, s \in \mathbf{N}$ define $\left\{ {n \atop r,s} \right\}$ to be the number of ordered pairs of subsets of T for which $|R| = r$ and $|S| = s$ and $R \cup S = T$. Note that $\left\{ {n \atop r,s} \right\} = 0$ whenever $r > n$ or $s > n$ or $r + s < n$. For other values of n, r, and s the coefficient $\left\{ {n \atop r,s} \right\}$ has a simple formula which appears in Section 4 in a more general setting.

The following identity has an easy combinatorial proof.

Theorem 1.1. *For* $k, r, s \in \mathbf{N}$,

$$\binom{k}{r}\binom{k}{s} = \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\} \binom{k}{n}. \tag{1.1}$$

Proof. Let T be a k-set. Then both sides of the equation count the number of ordered pairs (R, S) of subsets of T such that $|R| = r$ and $|S| = s$.

The left-hand side counts these in the obvious fashion. There are $\binom{k}{r}$ ways to choose R and $\binom{k}{s}$ ways to choose S.

The right-hand side counts the ordered pairs according to the cardinality of $R \cup S$. Let $n \in \{0, 1, \dots, r + s\}$. Then there are $\binom{k}{n}$ n-subsets of T. For a fixed $A \subseteq T$ with $|A| = n$ there are $\left\{ {n \atop r,s} \right\}$ ordered pairs (R, S) such that $R \cup S = A$ with $|R| = r$ and $|S| = s$. Thus, there are $\left\{ {n \atop r,s} \right\}\binom{k}{n}$ ordered pairs (R, S) for which $|R \cup S| = n$ with $|R| = r$ and $|S| = s$. Summing over all relevant values of n yields the desired result.

□

One of the usual formulas for $\binom{k}{n}$ is

$$\binom{k}{n} = \frac{k^{\underline{n}}}{n!}$$

where

$$k^{\underline{n}} = k(k-1)\dots(k-n+1).$$

Let $\mathbf{C}[X]$ be the algebra of polynomials over the field of complex numbers, and define $X^{\underline{n}}$ for $n \in \mathbf{N}$ by

$$X^{\underline{n}} = \prod_{i=0}^{n-1} (X - i) \tag{1.2}$$

Then, $X^{\underline{n}}$ and

$$\binom{X}{n} = \frac{X^{\underline{n}}}{n!} \tag{1.3}$$

are polynomials in $\mathbf{C}[X]$ of degree n.

Replacing the variable k by the indeterminate X in equation (1.1) produces a statement—as yet unproved—about polynomials. A common way of establishing such polynomial identities relies on the following algebraic facts. For $n \in \mathbf{N}$, a nonzero polynomial in $\mathbf{C}[X]$ of degree n has at most n distinct zeroes. Let $f, g \in \mathbf{C}[X]$ have degree n or less, and suppose that there exist distinct $c_1, c_2, \ldots, c_{n+1} \in \mathbf{C}$ such that $f(c_i) = g(c_i)$ for all $i \in [n+1] = \{1, 2, \ldots, n+1\}$. Then the polynomial $f - g$ has degree n or less and has $n+1$ distinct zeroes. Hence $f - g \equiv 0$. In other words $f = g$ as polynomials.

Applying this to the polynomials $\binom{X}{r}\binom{X}{s}$ and $\sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\} \binom{X}{n}$ where $r, s, n \in \mathbf{N}$, we see that these polynomials of degree at most $r+s$ agree, by Theorem (1.1), on an infinite set of values, namely on $\mathbf{N}$. Consequently

$$\binom{X}{r}\binom{X}{s} = \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\} \binom{X}{n} \tag{1.4}$$

as polynomials.

The identities

$$X^r X^s = X^{r+s}$$

and

$$\begin{aligned} \frac{X^r}{r!}\frac{X^s}{s!} &= \frac{1}{r!\,s!} X^{r+s} \\ &= \frac{(r+s)!}{r!\,s!} \frac{X^{r+s}}{(r+s)!} \\ &= \binom{r+s}{r} \frac{X^{r+s}}{(r+s)!} \end{aligned}$$

for $r, s \in \mathbf{N}$ lead to the usual rules for the multiplication of polynomials, ordinary generating functions, and exponential generating functions, as well as to the combinatorial significance of such multiplication. The polynomial identity (1.4) does the same for expressions of the form $\left(\sum_{i=0}^{n} a_i \binom{X}{i}\right)$, where $a_i \in \mathbf{C}$ for all $i \in \{0, 1, \ldots, n\}$.

Theorem 1.2. *If $m, n \in \mathbf{N}$ and $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in \mathbf{C}$, then*

$$\left(\sum_{i=0}^{m} a_i \binom{X}{i}\right)\left(\sum_{i=0}^{n} b_i \binom{X}{i}\right) = \sum_{i=0}^{m+n}\left(\sum_{r=0}^{i}\sum_{s=i-r}^{i} \left\{ {i \atop r,s} \right\} a_r b_s\right)\binom{X}{i}. \tag{1.5}$$

Proof.

$$\begin{aligned}
\left(\sum_{i=0}^{m} a_i \binom{X}{i}\right)\left(\sum_{i=0}^{m} b_i \binom{X}{i}\right) &= \sum_{\substack{0\le r\le m\\ 0\le s\le n}} a_r b_s \binom{X}{r}\binom{X}{s} \\
&= \sum_{\substack{0\le r\le m\\ 0\le s\le n}} a_r b_s \left(\sum_{i=0}^{r+s} \left\{ {i \atop r,s} \right\}\binom{X}{i}\right) \\
&= \sum_{i=0}^{m+n}\left(\sum_{\substack{0\le r\le m\\ 0\le s\le n\\ r+s\ge i}} \left\{ {i \atop r,s} \right\} a_r b_s\right)\binom{X}{i} \\
&= \sum_{i=0}^{m+n}\left(\sum_{r=0}^{i}\sum_{s=i-r}^{i} \left\{ {i \atop r,s} \right\} a_r b_s\right)\binom{X}{i},
\end{aligned}$$

where we have used the fact that $\left\{ {i \atop r,s} \right\} = 0$ whenever $r > i$ or $s > i$ or $r+s < i$ to get the last equality. All indices take only nonnegative values in the above summations. □

The right-hand side of (1.5) can also be written as

$$\sum_{i=0}^{m+n}\left(\sum_{r=0}^{i}\sum_{s=0}^{i} \left\{ {i \atop r,s} \right\} a_r b_s\right)\binom{X}{i}, \tag{1.6}$$

since $\left\{ {i \atop r,s} \right\} = 0$ in all the additional terms.

Since $\deg \binom{X}{n} = n$ for all $n \in \mathbf{N}$, the polynomials $\binom{X}{n}$ form a vector space basis for $\mathbf{C}[X]$ over $\mathbf{C}$. In other words, every polynomial in $\mathbf{C}[X]$ can be written in the form $\sum_{i=0}^{n} a_i \binom{X}{i}$ for some $n \in \mathbf{N}$ and $a_0, a_1, \ldots, a_n \in \mathbf{C}$. Consequently equation (1.5) provides a new way of drawing combinatorial information out of the multiplication of polynomials.

If equation (1.2) involved infinite sums, then sums of the form $\sum_{n=0}^{\infty} a_n \binom{X}{n}$, where $a_0, a_1, a_2, \ldots \in \mathbf{C}$, would be generating functions for covers. Expressions of

the form $\sum_{n=0}^{\infty} a_n \binom{X}{n}$, however, are as yet undefined. In particular they are not formal power series in general, as the following shows:

If $f = \sum_{n=0}^{\infty} a_n X^n \in \mathbf{C}[[X]]$, then the power series valuation *val* (called *degree* by some authors) of f is

$$\mathrm{val} f = \begin{cases} \infty, & \text{if } f \equiv 0 \\ \min\{n \in \mathbf{N} \mid a_n \neq 0\}, & \text{otherwise.} \end{cases}$$

Let $f_n \in \mathbf{C}[[X]]$ for all $n \in \mathbf{N}$. By I. Niven [13], the formal series $\sum_{n=0}^{\infty} f_n$ is defined if and only if $\lim_{n\to\infty} \mathrm{val} f_n = \infty$. Note, however, that

$$\mathrm{val}\binom{X}{n} = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n \in \mathbf{P}. \end{cases}$$

Thus $\sum_{n=0}^{\infty} a_n \binom{X}{n} \in \mathbf{C}[[X]]$ if and only if the a_n are finitely nonzero—i.e. if and only if

$$\left|\{n \in \mathbf{N} \mid a_n \neq 0\}\right| < \infty.$$

So the use of expressions like $\sum_{n=0}^{\infty} a_n \binom{X}{n}$ requires a formal theory of such expressions. The development of that theory occupies much of the remainder of this dissertation. Indeed, the theory that results deals with a much more general class of expressions of this type.

1.2 Finite vector space identities

Recall that if $n, r \in \mathbf{N}$ and q is a power of a prime, then the q-binomial coefficient $\binom{n}{r}_q$ is the number of r-dimensional subspaces (henceforth *r-subspaces*) of an n-dimenstional vector space (henceforth *n-space*) over F_q. Appendix A gives a brief introduction to q-binomial coefficients.

Using q-binomial coefficients it is easy to develop a q-analog of Theorem 1.1. Let $n, r, s \in \mathbf{N}$, and let V be an n-space over the finite field F_q. Define $\left\{ {n \atop r,s} \right\}_q$ to be the number of ordered pairs (R, S) of subspaces of V for which $\dim R = r$ and $\dim S = s$ and $R + S = V$. Note that $\left\{ {n \atop r,s} \right\}_q = 0$ if $r > n$ or $s > n$ or $r + s < n$.

Theorem 1.3. *For all $k, r, s \in \mathbf{N}$*

$$\binom{k}{r}_q \binom{k}{s}_q = \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\}_q \binom{k}{n}_q. \tag{1.7}$$

Proof. Let V be a k-space over F_q. Then both sides of (1.7) count the number of ordered pairs (R, S) with R an r-subspace of V and S an s-subspace of V.

The left-hand side counts in the obvious fashion. There are $\binom{k}{r}_q$ ways to choose R and $\binom{k}{s}_q$ ways to choose S.

The right-hand side counts according to the dimension of $R+S$. Let $n \in \{0, 1, \ldots, r+s\}$. Then V has $\binom{k}{n}_q$ subspaces of dimension n. If A is one of these subspaces, then there are $\left\{ {n \atop r,s} \right\}_q$ ordered pairs (R, S) with $\dim R = r$ and $\dim S = s$ and $R+S = A$. As the sum of subspaces is well-defined, there are $\left\{ {n \atop r,s} \right\}_q \binom{k}{n}_q$ ordered pairs (R, S) with $\dim R = r$ and $\dim S = s$ and $\dim(R+S) = n$. Summing over all relevant values of n yields the desired result.

□

Is there a way to generalize equation (1.7) into a polynomial identity analogous to the way Theorem 1.2 generalizes Theorem 1.1? The procedure of "replacing k by X" which leads to Theorem 1.2 does not produce a polynomial for q-binomial coefficients. Nevertheless, equation (1.7) does indeed have a polynomial generalization. Its derivation, however, occurs in a much more general context and, once again, occupies much of the remainder of the dissertation.

2. Poset and Lattice Background

2.1 Partially ordered sets

This is a brief introduction to partially ordered sets and lattices. The definitions related to binomial posets are Stanley's [16]. His book also serves as the reference for the standard definitions given here.

A *partially ordered set* (henceforth *poset*) is a set P with a binary relation, usually denoted $\leq$ (or $\leq_P$ when necessary for clarity), which is reflexive, antisymmetric, and transitive.

If $x, y \in P$, then x and y are *comparable* if $x \leq y$ or $y \leq x$. Otherwise x and y are *incomparable*.

As a shorthand notation write $x < y$ to mean $x \neq y$ and $x \leq y$.

If $x \in P$ such that for all $y \in P$ the relation $x \leq y$ holds, then denote x by $\hat{0}$. If $x' \in P$ has the same property, then $x \leq x'$ and $x' \leq x$, so $x = x'$ by antisymmetry. Thus $\hat{0}$ is unique. Similarly, if $y \in P$ and for all $x \in P$ the relation $x \leq y$ holds, then denote y by $\hat{1}$.

Call a poset Q a *subposet* of P if $Q \subseteq P$ and if for all $x, y \in Q$ the condition $x \leq_Q y$ if and only if $x \leq_P y$ holds.

If $x, y \in P$ with $x \leq y$, define the (closed) *interval*

$$[x, y] = \{z \in P \mid x \leq z \leq y\}.$$

This interval is a subposet of P—i.e., it inherits the partial ordering of P. The poset P is *locally finite* if every interval in P has finite cardinality.

If $x, y \in P$, then y *covers* x if $x < y$ and $[x, y] = \{x, y\}$—in other words, if "there is nothing between x and y." Note that this poset-theoretical *cover* is unrelated to the sort of cover (i.e. a cover of a set or vector space) that forms the primary theme of this dissertation!

A *chain* or *totally ordered set* is a poset in which no pair of elements is incomparable. If P is a poset and $C \subseteq P$ is a chain, then C is *maximal* if there is no chain $C' \subseteq P$ such that $C \subset C'$.

If C is a finite chain, the *length* of C is $l(C) = |C| - 1$. In general, if P is a finite poset, then the *length* of P is

$$l(P) = \max\{l(C) \mid C \text{ is a chain in } P\}.$$

For the length of a finite interval $[x, y]$ write $l(x, y)$ instead of $l([x, y])$.

If all maximal chains in a finite poset P have the same length n, then P is *graded of rank* n. For such a poset define a *rank function* $\rho: P \to \{0, 1, \ldots, n\}$ as follows: Let $y \in P$ and let $C = \{x_0 < x_1 < \cdots < x_n\}$ be a maximal chain of P which includes y. Then $\rho(y) = l(x_0, y)$. In other words, if $i \in \{0, 1, \ldots, n\}$ and $y = x_i$, then $\rho(y) = i$. That is, y has *rank* i.

For the same $y \in P$ suppose $C' = \{x'_0 < x'_1 < \cdots < x'_n\}$ is another maximal chain of P and that $y = x_j$ for some $j \in \{0, 1, \ldots, n\}$. Without loss of generality suppose $j \leq i$. Then the chain $x_0 < \cdots < (x_i = x'_j) < \cdots < x'_n$ has length $i + (n - j) = n + (i - j) \geq n$. But no chain in P has length greater than n—by finiteness of P every chain is contained in a maximal chain—so $n + (i - j) = n$. That is, $i = j$. So the rank function is well-defined.

Suppose x and y are members of a graded poset P. Then y covers x if and only if $x < y$ and $\rho(y) = \rho(x) + 1$. This observation follows almost trivially from the definition of rank, but it proves extremely useful in later proofs.

2.2 Binomial posets

A poset P is called a *binomial poset* if it satisfies the following three conditions:

(i) The poset P is locally finite with $\hat{0}$ and contains an infinite chain.
(ii) Every interval $[x, y]$ of P is graded. If $l(x, y) = n$, call $[x, y]$ an n-interval.
(iii) For all $n \in \mathbf{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains. Call $B(n)$ the *factorial function* of P.

Stanley [16, pp. 142–143] lists several examples of binomial posets. For our purposes the most interesting are $\mathbf{N}$ with the standard ordering; the collection of finite subsets of an infinite set, partially ordered by set inclusion; and the collection of finite-dimensional subspaces of an infinite-dimensional vector space over a finite field, again partially ordered by set inclusion.

Let P be a binomial poset and let $[x, y]$ be an n-interval in P for some $n \in \mathbf{N}$. Since P is a binomial poset, the interval $[x, y]$ is graded and thus has a rank function. For $k \in \mathbf{N}$ define $\begin{bmatrix} n \\ k \end{bmatrix}$ to be the number of elements of rank k in $[x, y]$. Let z be an element of rank i in $[x, y]$ for some $i \in \{0, 1, \ldots, n\}$. Every maximal chain of P which includes z consists of one of the $B(i)$ maximal chains in the i-interval $[x, z]$ and one of the $B(n - i)$ maximal chains in the $(n - i)$-interval $[z, y]$. Thus $B(i)B(n - i)$ maximal chains of $[x, y]$ include z. Every maximal chain in $[x, y]$ includes exactly one of the $\begin{bmatrix} n \\ i \end{bmatrix}$ elements of rank i. Thus $[x, y]$, and indeed any n-interval of P, has

$\begin{bmatrix} n \\ i \end{bmatrix} B(i)B(n-i)$ maximal chains. Consequently,

$$\begin{bmatrix} n \\ i \end{bmatrix} B(i)B(n-i) = B(n),$$

or, more usefully,

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{B(n)}{B(i)B(n-i)}.$$

The second identity gives a nice formula for $\begin{bmatrix} n \\ i \end{bmatrix}$ and shows that that the value of $\begin{bmatrix} n \\ i \end{bmatrix}$ depends only on the length of the interval, not on the particular interval under consideration.

Stanley gives the above formula for $\begin{bmatrix} n \\ i \end{bmatrix}$ and shows further that

$$B(n) = A(n)A(n-1)\dots A(1),$$

where $A(i) = \begin{bmatrix} i \\ 1 \end{bmatrix}$ for $i \in \mathbf{P} = \{1, 2, 3, \dots\}$. In the binomial poset of finite subsets of $\mathbf{N}$ listed above, $B(n) = n!$ and

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{n!}{i!\,(n-i)!} = \binom{n}{i},$$

thus explaining the terminology *factorial function* and *binomial poset.*

2.3 Lattices

Let P be a poset and $x, y \in P$. If there exists $z \in P$ (necessarily unique) such that $x \leq z$ and $y \leq z$ and for which $z \leq w$ whenever w is a member of P with $x \leq w$ and $y \leq w$, call z the *join* or *supremum* or *least upper bound* of x and y and write $x \vee y = z$.

Similarly, if there exists $z \in P$ such that $z \leq x$ and $z \leq y$ and for which $w \leq z$ whenever w is a member of P with $w \leq x$ and $w \leq y$, call z the *meet* or *infimum* or *greatest lower bound* of x and y and write $x \wedge y = z$.

A poset L in which every pair of elements has a join and a meet is called a *lattice.* Thus in a lattice, meet and join are binary operations which are easily seen to be commutative and associative.

Examples.

1. For $n \in \mathbf{N}$ let $L = 2^{[n]}$, the power set of $[n]$, be partially ordered by set inclusion. Then L is a lattice in which

$$R \vee S = R \cup S \qquad \text{and} \qquad R \wedge S = R \cap S$$

for all $R, S \in L$.

2. Let $L = \mathbf{P}$ ordered by divisibility (i.e., $m \leq n$ if and only if $m \mid n$). Then L is a lattice in which

$$m \vee n = \text{lcm}(m, n) \qquad \text{and} \qquad m \wedge n = \gcd(m, n)$$

for all $m, n \in L$.

3. Let L be any totally ordered set. Then L is a lattice in which

$$x \vee y = \max\{x, y\} \qquad \text{and} \qquad x \wedge y = \min\{x, y\}$$

for all $x, y \in L$.

2.4 Modular lattices

Let L be a finite lattice. If L is graded (as a poset) and

$$\rho(x) + \rho(y) = \rho(x \wedge y) + \rho(x \vee y) \qquad \text{for all } x, y \in L,$$

then L is a *modular lattice*. As Stanley [16, p. 103] shows, these two conditions on L are equivalent to the following condition: For all $x, y \in L$ the elements x and y both cover $x \wedge y$ if and only if $x \vee y$ covers both x and y.

Examples.

1. For $n \in \mathbf{N}$ let $L = 2^{[n]}$ ordered by set inclusion. Then L is a modular lattice with $\rho(R) = |R|$ for all $R \in L$. Modularity follows from the fact that if $R, S \in L$, then

$$|R \cup S| = |R| + |S| - |R \cap S|.$$

2. For $n \in \mathbf{N}$ and V an n-space over F_q, let L be the collections of subspaces of V ordered by inclusion. Then L is a modular lattice with $\rho(U) = \dim U$ for all $U \in L$. Modularity follows from the fact that if $U, W \in L$, then

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

3. Let L be a finite chain. Then L is a modular lattice with $\rho(x) = l(\hat{0}, x)$. Modularity follows from the fact that if $x, y \in L$ and, without loss of generality, $x \leq y$, then

$$\rho(x) + \rho(y) = \rho(x \wedge y) = \rho(x \vee y)$$

since $x \wedge y = x$ and $x \vee y = y$.

2.5 New concepts

A lattice is *locally modular* if every interval in L is a modular lattice.

If L is a binomial poset which is also a lattice, then L is a *binomial lattice.*

Obviously every interval in a lattice is itself a lattice. If L is a binomial lattice, then by local finiteness every interval in L is a finite lattice. If in addition L is locally modular, we shall call L a *locally modular binomial lattice.* The promised results about covers and generating functions grow out of the properties of locally modular binomial lattices.

Let L be a locally modular binomial lattice, and let $z \in L$. Then z has a rank in the interval $[\hat{0}, z]$; denote this rank by $\rho_{\hat{0}}(z)$. If $[x, y]$ is a fixed interval in L and $z \in [x, y]$, then z also has a rank within $[x, y]$. Denote this rank by $\rho_x(z)$ or simply $\rho(z)$ if the context is clear. Note that if $y' \in L$ with $y \leq y'$, then the rank of z in $[x, y']$ is the same as its rank in $[x, y]$, so $\rho_x(z)$ is well-defined. In particular $\rho_{\hat{0}}(z)$ is well-defined regardless of whether we view z as being in $[\hat{0}, z]$ or in $[\hat{0}, y]$ for some $y \in L$ with $z \leq y$.

Examples.

1. The set $\mathbf{N}$ with the natural ordering is a locally modular binomial lattice with factorial function $B(n) = 1$ for all $n \in \mathbf{N}$. If $m, n \in \mathbf{N}$, then $m \vee n = \max(m, n)$ and $m \wedge n = \min(m, n)$. If $r \in [m, n]$, then $\rho_{\hat{0}}(r) = r$ and $\rho_m(r) = r - m$.
2. Let L be the collection of finite subsets of an infinite set, ordered by inclusion. Then L is a locally modular binomial lattice with factorial function $B(n) = n!$ for all $n \in \mathbf{N}$. If $R, S \in L$, then $R \vee S = R \cup S$ and $R \wedge S = R \cap S$. If $T \in [R, S]$, then $\rho_{\hat{0}}(T) = |T|$ and $\rho_R(T) = |T| - |R|$.
3. Let L be the collection of finite-dimensional subspaces of an infinite-dimensional vector space over F_q, ordered by set inclusion. Then L is a locally modular binomial lattice with factorial function

$$B(n) = (q^{n-1} + q^{n-2} + \cdots + 1) \ldots (q + 1)(1) \qquad \text{for all } n \in \mathbf{N}.$$

If $U, V \in L$, then $U \vee V = U + V$ (vector space sum) and $U \wedge V = U \cap V$. If $W \in [U, V]$, then $\rho_{\hat{0}}(W) = \dim W$ and $\rho_U(W) = \dim W - \dim U = \dim(W/U)$.

3. The q-Lattice

3.1 The factorial function

Let L be a locally modular binomial lattice and suppose that $[x,y]$ is an n-interval in L with $n \geq 2$. Choose $z_0 \in [x,y]$ such that $\rho(z_0) = 1$ (i.e. $\rho_x(z_0) = 1$). If $z \in [x,y]$ with $z \neq z_0$ and $\rho(z) = 1$, then by modularity of $[x,y]$

$$\rho(z_0 \vee z) = \rho(z_0) + \rho(z) - \rho(z_0 \wedge z) = 1 + 1 - \rho(x) = 2.$$

In short, $z_0 \vee z$ covers both z and z_0. On the other hand, suppose $w \in [x,y]$ covers both z and z_0. Since $z \leq w$ and $z_0 \leq w$, the definition of *join* requires that $z \wedge z_0 \leq w$. Hence w cannot cover z and z_0 unless $w = z \wedge z_0$. Thus, except for z_0 itself, every element of rank one in $[x,y]$ is covered by exactly one element which also covers z_0.

Let $A = \{\text{rank-one elements of } [x,y]\} - \{z_0\}$ and let $B = \{\text{rank-two elements of } [x,y] \text{ which cover } z_0\}$. Each element in B covers $\begin{bmatrix}2\\1\end{bmatrix}$ rank-one elements of $[x,y]$, one of which is z_0. Thus the map $z \mapsto z \vee z_0$ from A to B is a $\left(\begin{bmatrix}2\\1\end{bmatrix} - 1\right)$-to-one surjection. Further $B = \{\text{rank-one elements of } [z_0,y]\}$, so $|B| = \begin{bmatrix}n-1\\1\end{bmatrix}$. Therefore the number of rank-one elements of $[x,y]$ is $|A| + 1 = |B|\left(\begin{bmatrix}2\\1\end{bmatrix} - 1\right) + 1 = \begin{bmatrix}n-1\\1\end{bmatrix}\left(\begin{bmatrix}2\\1\end{bmatrix} - 1\right) + 1$. This essentially proves the following lemma.

Lemma 3.1. *If L is a locally modular binomial lattice, then for all $n \in \mathbf{P}$*

$$\begin{bmatrix}n\\1\end{bmatrix} = \begin{bmatrix}n-1\\1\end{bmatrix}\left(\begin{bmatrix}2\\1\end{bmatrix} - 1\right) + 1.$$

Proof. Recall that

$$\begin{bmatrix}n\\k\end{bmatrix} = \text{the number of elements of rank } k \text{ in an } n\text{-interval.}$$

So $\begin{bmatrix}n\\k\end{bmatrix} = 0$ if $n < k$. Thus the result holds easily for $n = 1$.

For $n \geq 2$ the argument preceding the lemma shows that the right-hand side of the equation counts rank-one elements of an n-interval, which is precisely the value of $\begin{bmatrix}n\\1\end{bmatrix}$.

□

For locally modular binomial lattices the quantity $\begin{bmatrix}2\\1\end{bmatrix} - 1$ is extremely important. Henceforth for a given locally modular binomial lattice define

$$q = \begin{bmatrix}2\\1\end{bmatrix} - 1. \tag{3.1}$$

Doubilet, Rota, and Stanley [4, p.310] give the following result:

Theorem 3.1. *Let L be a locally modular binomial lattice. Then for all $n \in \mathbf{N}$*

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (q^{n-1} + q^{n-2} + \cdots + 1) = \sum_{i=0}^{n-1} q^i,$$

and thus

$$B(n) = \prod_{j=1}^{n} \sum_{i=0}^{j-1} q^i.$$

Proof. If $n = 0$, the result holds by the convention that an empty sum equals zero.

If $n = 1$, the result is trivial.

If $n = 2$, then $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = q + 1$ by definition of q.

Proceed by induction and apply Lemma 3.1 to see that

$$\begin{aligned} \begin{bmatrix} n+1 \\ 1 \end{bmatrix} &= \begin{bmatrix} n \\ 1 \end{bmatrix} q + 1 \\ &= \left(q \sum_{i=0}^{n-1} q^i \right) + 1 \\ &= \left(\sum_{i=1}^{n} q^i \right) + 1 \\ &= \sum_{i=0}^{(n+1)-1} q^i. \end{aligned}$$

□

In light of Theorem 3.1, we shall give locally modular binomial lattices the name *q-lattices*.

Thus far we have encountered three sorts of q-lattices:

1. An infinite chain is a q-lattice with $q = 0$.
2. The collection of finite subsets of an infinite set is a q-lattice with $q = 1$.
3. The collection of finite-dimensional subspaces of an infinite-diminsional vector space over the finite field F_q is a q-lattice with q a power of a prime number.

An apparently open question is that of whether other q-lattices exist. The work of Doubilet, Rota, and Stanley [4, p.311] on lattices of "full binomial type" seems to address this question in some fashion.

3.2 Notation

If L is a q-lattice and $n \in \mathbf{N}$, define

$$n_q = (q^{n-1} + q^{n-2} + \cdots + 1) = \sum_{i=0}^{n-1} q^i. \tag{3.2}$$

To avoid confusion with other subscripts, we use a literal "q" as the subscript never replacing q by a particular value—e.g. writing "n_q with $q = 2$" rather than "n_2". In practice no confusion arises from this convention.

In a further abuse of notation, for $n, k \in \mathbf{N}$ define

$$n!_q = \prod_{j=1}^{n} j_q \tag{3.3}$$

and

$$n^{\underline{k}}_q = \prod_{j=0}^{k-1} (n_q - j_q). \tag{3.4}$$

Note that

$$n^{\underline{n}}_q = q^{\binom{n}{2}} n!_q. \tag{3.5}$$

In the context of q-lattices, the numbers n_q, $n!_q$, and $n^{\underline{k}}_q$ behave in a fashion much like we have come to expect from n, $n!$, and $n^{\underline{k}}$ in a more conventional setting. The exploitation of this behavior produces a number of the results that follow.

For instance, let L be a q-lattice with factorial function $B(n)$. By Theorem 3.1

$$A(n) = \begin{bmatrix} n \\ 1 \end{bmatrix} = n_q \qquad \text{for all } n \in \mathbf{N}.$$

Thus

$$\begin{aligned} B(n) &= A(n)A(n-1)\ldots A(1) \\ &= n_q(n-1)_q \ldots 1_q \\ &= n!_q. \end{aligned}$$

Further, for all $n, k \in \mathbf{N}$ with $k \leq n$

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \frac{B(n)}{B(k)B(n-k)} \\ &= \frac{n!_q}{k!_q(n-k)!_q} \\ &= \frac{n^{\underline{k}}_q}{k^{\underline{k}}_q}. \end{aligned} \tag{3.6}$$

Note that the last equality, namely

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n^{\underline{k}}_q}{k^{\underline{k}}_q},$$

holds even without the condition $k \leq n$.

3.3 Combinatorial results

The well-known tendency of identities involving binomial coefficients and identities involving q-binomial coefficients to occur in analogous pairs (see Appendix A) finds its explanation in the fact that the identities depend on q-lattice properties rather than on properties of sets or finite vector spaces. This fact provides a powerful tool for showing that these pairs of identities are special cases of more general results.

The following theorem follows easily from equation (3.6) by algebraic manipulation. The combinatorial proof given below illustrates, however, the power of q-lattices to generalize familiar set and vector space arguments.

Theorem 3.2 (q-binomial recurrence). *Let L be a q-lattice. For all $n \in \mathbf{P}$ and $k \in \mathbf{N}$*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{3.7}$$

Proof. The result holds trivially if $n \leq k$ or $k = 0$, so assume $1 \leq k < n$.

Let $[x, y]$ be an n-interval in L. Then both sides of equation (3.7) count the number of elements of rank k in $[x, y]$.

The left-hand side does this by definition.

For the right-hand side proceed as follows. Choose $z_0 \in [x, y]$ with $\rho_x(z_0) = 1$. Define

$$A = \{z \in [z_0, y] \mid \rho_x(z) = k\}$$

and

$$B = \{z \in [x, y] - [z_0, y] \mid \rho_x(z) = k\}.$$

The number of rank k elements of $[x, y]$ is $|A| + |B|$.

Clearly,

$$|A| = \left|\{z \in [z_0, y] \mid \rho_{z_0}(z) = k-1\}\right| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Now let $C = \{w \in [z_0, y] \mid \rho_{z_0}(w) = k\}$, and define a mapping $\alpha: B \to L$ by $\alpha(z) = z_0 \vee z$. We wish to show that α is a (q^k)-to-one surjection onto C. Since $|C| = \begin{bmatrix} n-1 \\ k \end{bmatrix}$ this result will show that $|B| = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$.

Let $z \in B$. Since $z_0 \not\leq z$, the inequality $z_0 \wedge z < z_0$ holds. But the only element of $[x, y]$ less than z_0 is x, so $z_0 \wedge z = x$ and

$$\begin{aligned} \rho_x(z_0 \vee z) &= \rho_x(z_0) + \rho_x(z) - \rho_x(z_0 \wedge z) \\ &= 1 + k - 0 \\ &= k + 1. \end{aligned}$$

Also $z_0 \vee z \in [z_0, y]$ and $\rho_{z_0}(z_0 \vee z) = k$, so $\alpha(z) \in C$. Note that $\alpha(z)$ covers z (in $[x, y]$). Indeed, $\alpha(z)$ is, as we show below, the only element of C that covers z (in $[x, y]$), and so we may define $\alpha(z)$ equivalently as *the unique element of C that covers z* (in $[x, y]$).

Suppose $w' \in C$, distinct from w, also covers z. Since $\rho_x(w) = \rho_x(w') = k + 1$, the elements w and w' are incomparable and $\rho_x(w \wedge w') \leq k$. But $z \leq w$ and $z \leq w'$ and $\rho_x(z) = k$, so it must be the case that $w \wedge w' = z$. Now $z_0 \leq w$ and $z_0 \leq w'$, so $z_0 \leq w \wedge w' = z$. Thus $z_0 \leq z \leq y$ and $z \in [z_0, y]$. This is a contradiction since $z \in B$ implies $z \notin [z_0, y]$. Therefore w must be the only element in C that covers z.

If $w \in C$ and thus $\rho_x(w) = k + 1$, then

$$\begin{aligned} \Big|\{z \in [x, y] \mid w \text{ covers } z\}\Big| &= \Big|\{z \in [x, w] \mid \rho_x(z) = k\}\Big| \\ &= \begin{bmatrix} k+1 \\ k \end{bmatrix} \\ &= \frac{(k+1)!_q}{k!_q 1!_q} \\ &= (k+1)_q. \end{aligned}$$

Of these $\begin{bmatrix} k \\ k-1 \end{bmatrix} = k_q$ are in $[z_0, w]$ and hence in $[z_0, y]$. These are not in B and must be omitted. Thus w covers $(k+1)_q - k_q = q^k$ elements of B. So α is surjective and (q^k)-to-one.

It follows that

$$|B| = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Thus

$$\begin{bmatrix} n \\ k \end{bmatrix} = |A| + |B| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

□

In the special cases of (3.7), namely the binomial recurrence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

and the finite vector space q-binomial recurrence

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q,$$

for q a prime power, the above proof specializes as follows. In the first case count k-subsets of an n-set according to whether they contain a distinguished element of the n-set. In the second case count k-subspaces of an n-space V over F_q according to whether they contain a distinguished one-subspace of V.

3.4 Complementation

If L is a q-lattice, $[x, y]$ is an interval in L, and $z \in [x, y]$, call $z' \in [x, y]$ a *complement* of z (in $[x, y]$) if $z \wedge z' = x$ and $z \vee z' = y$. The following theorem about complements proves useful for a later result.

Theorem 3.3. *Let L be a q-lattice and $[x, y]$ an n-interval in L for some $n \in \mathbf{N}$. If $k \in \{0, 1, \ldots, n\}$ and $z \in [x, y]$ with $\rho(z) = k$, then*

$$\left| \{ z' \in [x, y] \mid z' \text{ is a complement of } z \} \right| = q^{k(n-k)}.$$

Proof. For $k \in \{0, 1, \ldots, n\}$, let $z \in [x, y]$ with $\rho(z) = k$. Further let

$$C = \{ z' \in [x, y] \mid z' \text{ is a complement of } z \}.$$

We wish to show that $|C| = q^{k(n-k)}$.

If $z' \in C$, then by modularity

$$\rho(z') + \rho(z) = \rho(z \wedge z') + \rho(z \vee z') = 0 + n,$$

so

$$\rho(z') = n - \rho(z) = n - k.$$

In particular, distinct elements of C are incomparable since they have the same rank.

Define

$$D = \{\text{chains } x = a_0 < a_1 < \cdots < a_{n-k} \text{ in } [x,y] \mid a_{n-k} \in C \text{ and } \rho(a_i) = i \text{ for all } i\},$$

and evaluate $|D|$ in two ways as a means to finding $|C|$.

First note that every chain in D is a maximal chain in the $(n-k)$-interval $[x,z']$ for some $z' \in C$. Conversely, if $z' \in C$, then every maximal chain of $[x,z']$ is in D. The factorial function $B(n-k) = (n-k)!_q$ gives the number of maximal chains in $[x,z']$ for each $z' \in C$, and no chain in D includes two distinct elements of C. Thus

$$|D| = |C|B(n-k) = |C|(n-k)!_q. \tag{3.8}$$

On the other hand note that $a_0 < a_1 < \cdots < a_{n-k}$ with $a_{n-k} \in C$ is in D if and only if $a_i \wedge z = x$ for all $i \in \{0,1,\ldots,n-k\}$. This is obvious except perhaps to see that if $i \in \{0,1,\ldots,n-k\}$, then $a_i \wedge z \leq z$ and $a_i \wedge z \leq a_i \leq a_{n-k}$. So $a_i \wedge z \leq a_{n-k} \wedge z = x$, which is to say $a_i \wedge z = x$, thus showing the "only if" statement.

Proceed by counting the ways to construct $a_0 < a_1 < \cdots < a_{n-k}$ in D directly: Choose a_0 in the one way possible. Then choose a_1, a_2, etc. If the first part of the chain $a_0 < a_1 < \cdots < a_i$ for $i \in \{0,1,\ldots,n-k-1\}$ has already been chosen, how many choices are there for a_{i+1}?

The interval $[a_i,y]$ is an $(n-i)$-interval and thus has $\begin{bmatrix} n-i \\ 1 \end{bmatrix} = (n-i)_q$ elements that cover a_i. These are all the elements of rank (i.e. ρ_x) $i+1$ which cover a_i in $[x,y]$. Not all of these elements, however, have the property that their meet with z is x. Exclusion of the elements that lack this property will leave the valid choices for a_{i+1}.

Let α cover a_i. Then $a_i \vee z \leq \alpha \vee z$, so

$$\rho(\alpha \vee z) \geq \rho(a_i \vee z) = \rho(a_i) + \rho(z) - \rho(a_i \wedge z) = i + k - \rho(x) = i + k.$$

Thus

$$\begin{aligned} \rho(\alpha \wedge z) &= \rho(\alpha) + \rho(z) - \rho(\alpha \vee z) \\ &\leq i + 1 + k - (i + k) \\ &= 1, \end{aligned}$$

which is to say

$$\rho(\alpha \wedge z) = 0 \quad \text{or} \quad \rho(\alpha \wedge z) = 1.$$

Let $A = \{\alpha \in [x,y] \mid \alpha \text{ covers } a_i \text{ and } \rho(\alpha \wedge z) = 1\}$. The valid choices for a_{i+1} are all the elements that cover a_i except for those in A. So the problem is to find $|A|$.

Let $B = \{\text{rank-one elements of } [x,z]\}$. If $\alpha, \alpha' \in A$ are distinct, then $(\alpha \wedge z) \wedge (\alpha' \wedge z) = (\alpha \wedge \alpha') \wedge z = a_i \wedge z = x$, so $\alpha \wedge z \neq \alpha' \wedge z$. Thus the map from A to B by $\alpha \mapsto \alpha \wedge z$ is injective. In particular, $|A| \leq |B|$.

On the other hand, let $\beta \in B$. Then $a_i \wedge \beta \leq a_i \wedge z = x$. Thus $\rho(a_i \vee \beta) = \rho(a_i) + \rho(\beta) - \rho(a_i \wedge \beta) = i+1$ and $\rho\big((a_i \vee \beta) \wedge z\big) = \rho(a_i \vee \beta) + \rho(z) - \rho\big(a_i \vee (\beta \vee z)\big) = i+1+k-\rho(a_i \vee z) = i+1+k-(i+k) = 1$, so $a_i \vee \beta \in A$. If $\beta' \in B$ is distinct from β, then $\rho\big((a_i \vee \beta) \vee (a_i \vee \beta')\big) = \rho\big(a_i \vee (\beta \vee \beta')\big) = \rho(a_i) + \rho(\beta \vee \beta') - \rho\big(a_i \wedge (\beta \wedge \beta')\big) = i+2-\rho(x) = i+2$ (note $a_i \wedge (\beta + \beta') \leq a_i \wedge z = x$), so $a_i \vee \beta \neq a_i \vee \beta'$. Thus the map from B to A by $\beta \mapsto a_i \vee \beta$ is injective. In particular $|B| \leq |A|$.

Therefore $|A| = |B| = \begin{bmatrix} k \\ 1 \end{bmatrix} = k_q$. So k_q of the $(n-i)_q$ elements that cover a_i cannot be element a_{i+1} in the chain.

Altogether then there are $\prod_{i=0}^{n-k-1} \big((n-i)_q - k_q\big)$ ways to construct a chain in D—that is

$$|D| = \prod_{i=0}^{n-k-1} \big((n-i)_q - k_q\big).$$

Combining this formula with (3.8) yields

$$|C|(n-k)!_q = \prod_{i=0}^{n-k-1} \big((n-i)_q - k_q\big). \tag{3.9}$$

Now use the following identity to manipulate the right-hand side of (3.9). If $r, s \in \mathbf{P}$ with $s < r$, then

$$\begin{aligned} r_q - s_q &= (q^{r-1} + q^{r-2} + \cdots + 1) - (q^{s-1} + q^{s-2} + \cdots + 1) \\ &= q^{r-1} + q^{r-2} + \cdots + q^s \\ &= q^s(q^{r-s-1} + q^{r-s-2} + \cdots + 1) \\ &= q^s(r-s)_q. \end{aligned} \tag{3.10}$$

From (3.9) then,

$$\begin{aligned} |C|(n-k)!_q &= \prod_{i=0}^{n-k-1} \big((n-i)_q - k_q\big) \\ &= \prod_{i=0}^{n-k-1} q^k (n-k-i)_q \end{aligned}$$

$$= q^{k(n-k)} \prod_{i=0}^{n-k-1} (n-k-i)_q$$
$$= q^{k(n-k)} \prod_{i=1}^{n-k} i_q$$
$$= q^{k(n-k)} (n-k)!_q.$$

That is,

$$|C|(n-k)!_q = q^{k(n-k)}(n-k)!_q. \tag{3.11}$$

Divide both sides of (3.11) by $(n-k)!_q$ (recall $0!_q = 1$) to get the desired result, adopting the convention $0^0 = 1$ in the case $q = 0$.

□

4. The Algebra of q-Binomial Series

4.1 The set of arithmetic functions

The set of (complex-valued) arithmetic functions $\mathbf{C}^{\mathbf{N}} = \{F : \mathbf{N} \to \mathbf{C}\}$ is simply the set of sequences of complex numbers. If $a_0, a_1, a_2, \ldots \in \mathbf{C}$, then the statements

$$F(n) = a_n \qquad \text{for all } n \in \mathbf{N}$$

and

$$F = (a_0, a_1, a_2, \ldots)$$

both describe the same arithmetic function F.

There are several ways to give $\mathbf{C}^{\mathbf{N}}$ the structure of an algebra over $\mathbf{C}$. Two of these possibilities appear shortly. The third appears in Section 6. All three, however, have the same addition and scalar multiplication. Namely, if $F, G \in \mathbf{C}^{\mathbf{N}}$ and $c \in \mathbf{C}$ define

$$(F + G)(n) = F(n) + G(n) \qquad \text{for all } n \in \mathbf{N}$$

and

$$(cF)(n) = (c.F)(n) = c(F(n)) \qquad \text{for all } n \in \mathbf{N}.$$

Either juxtaposition or a period denotes scalar multiplication, the latter appearing, however, only when clarity demands it.

Let $F, G \in \mathbf{C}^{\mathbf{N}}$ and define the *Cauchy product* $F * G$ of F and G by

$$(F * G)(n) = \sum_{i=0}^{n} F(i)G(n-i) \qquad \text{for all } n \in \mathbf{N}.$$

The quadruple $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ is well-known to be a $\mathbf{C}$-algebra and is, in fact, the algebra of formal power series. For a thorough development of formal power series, see I. Niven [13]. This is the first way to give $\mathbf{C}^{\mathbf{N}}$ the structure of a $\mathbf{C}$-algebra.

4.2 Newton multiplication

The second way to give $\mathbf{C}^{\mathbf{N}}$ the structure of a C-algebra involves a new multiplication and requires the following definition. Let L be a q-lattice. If $n, r, s \in \mathbf{N}$ and $[x, y]$ is an n-interval in L, define the *covering coefficient*

$$\left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q = \Big| \{ (z, w) \mid z, w \in [x, y],\ \rho(z) = r,\ \rho(w) = s, \text{ and } z \vee w = y \} \Big|. \tag{4.1}$$

The following theorem shows that covering coefficients are well-defined—i.e., they do not depend on the particular interval $[x, y]$ under consideration.

Theorem 4.1. *If L is a q-lattice and $n, r, s \in \mathbf{N}$, then*

$$\left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q = \begin{cases} \frac{n!_q q^{(n-r)(n-s)}}{(n-r)!_q (n-s)!_q (r+s-n)!_q}, & \text{if } r \le n,\ s \le n, \text{ and } r + s \ge n \\ 0, & \text{otherwise} \end{cases}. \tag{4.2}$$

Proof. The "otherwise" case of (4.2) is obvious, so assume $r \le n$, $s \le n$, and $r + s \ge n$.

The result follows immediately from showing that

$$\left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q (n-r)!_q (n-s)!_q (r+s-n)!_q = n!_q q^{(n-r)(n-s)}. \tag{4.3}$$

Let $[x, y]$ be an n-interval in L. The right-hand side of (4.3) is $q^{(n-r)(n-s)}$ times the number of maximal chains in $[x, y]$.

The left-hand side of (4.3) counts the same quantity, but in indirect fashion. Choose $z, w \in [x, y]$ with $\rho(z) = r$, $\rho(w) = s$, and $z \vee w = y$. There are $\left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q$ ways to do this, where $\left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q$ may depend on $[x, y]$.

Let $x = a_0 \le a_1 \le \cdots \le a_n = y$ be a maximal chain in $[x, y]$ that includes $z \wedge w$ and w. Clearly $a_s = w$ and $a_{r+s-n} = z \wedge w$ since $\rho(w) = s$ and $\rho(z \wedge w) = \rho(z) + \rho(w) - \rho(z \vee w) = r + s - n$ by modularity.

How many such chains are there? This chain being maximal is equivalent to the chain $a_0 \le a_1 \le \cdots \le a_{r+s-n}$ being maximal in $[x, z \wedge w]$, the chain $a_{r+s-n} \le a_{r+s-n+1} \le \cdots \le a_s$ being maximal in $[z \wedge w, w]$, and the chain $a_s \le a_{s+1} \le \cdots \le a_n$ being maximal in $[w, y]$. The interval $[x, z \wedge w]$ has length $r + s - n$ and thus has $B(r + s - n) = (r + s - n)!_q$ maximal chains. Similarly the interval $[z \wedge w, w]$ has $B\big(s - (r + s - n)\big) = (n - r)!_q$ maximal chains, and the interval $[w, y]$ has $B(n - s) = (n - s)!_q$ maximal chains. Altogether, then, $(r + s - n)!_q (n - r)!_q (n - s)!_q$ maximal chains in $[x, y]$ include $z \wedge w$ and w.

This quantity appears in (4.3) and gives rise to the following interpretation. The left-hand side of (4.3) counts the number of ways of choosing $z, w \in [x, y]$ with $\rho(z) = r$, $\rho(w) = s$, and $z \vee w = y$ and then forming a maximal chain of $[x, y]$ which includes $z \wedge w$ and w.

How many times does this procedure count a given maximal chain? A chain that includes $z \wedge w$ and w for the z and w chosen above also includes $z' \wedge w$ and w, where $\rho(z') = r$, $z' \vee w = y$, and $z' \wedge w = z \wedge w$. In short, this procedure counts a maximal chain that includes $z \wedge w$ and w once for each complement of w in the interval $[z \wedge w, y]$.

The interval $[z \wedge w, y]$ has length $n - (r + s - n) = 2n - (r + s)$, and $\rho_{z \wedge w}(w) = s - (r + s - n) = n - r$. So by Theorem 3.3, the element w has

$$q^{(n-r)\left((2n-(r+s))-(n-r)\right)} = q^{(n-r)(n-s)}$$

complements in $[z \wedge w, y]$.

Therefore this procedure counts every maximal chain in the interval $[x, y]$ exactly $q^{(n-r)(n-s)}$ times. The resulting quantity is the left-hand side of equation (4.3).

This establishes (4.3) as an identity, and the desired formula (4.2) follows immediately by division. In particular, (4.3) shows that $\left\{ {n \atop r,s} \right\}_q$ depends on n, r, s, and q, but not on the particular interval $[x, y]$ under consideration.

□

If L is a q-lattice with $q = 0$ and if $n, r, s \in \mathbf{N}$ with $r \leq n$, $s \leq n$, and $r + s \geq n$, then equation (4.2) becomes

$$\left\{ {n \atop r,s} \right\}_q = \begin{cases} 1, & \text{if } r = n \text{ or } s = n \\ 0, & \text{otherwise} \end{cases}.$$

Under the same conditions with $q = 1$,

$$\left\{ {n \atop r,s} \right\}_q = \frac{n!}{(n-r)!\,(n-s)!\,(r+s-n)!},$$

which is the trinomial coefficient $\binom{n}{n-r,n-s,r+s-n}$.

Let $F, G \in \mathbf{C}^{\mathbf{N}}$. Define the *q-Newton product* $F \diamond G$ of F and G by

$$(F \diamond G)(n) = \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\}_q F(r)G(s) \qquad \text{for all } n \in \mathbf{N}, \tag{4.4}$$

where the $\left\{ {n \atop r,s} \right\}_q$ are covering coefficients of a q-lattice as defined by equation (4.1).

The frequently-useful alternative definition

$$(F \diamond G)(n) = \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q F(r)G(s) \qquad \text{for all } n \in \mathbf{N} \tag{4.5}$$

is equivalent to (4.4) since $\left\{ \begin{matrix} n \\ r,s \end{matrix} \right\}_q = 0$ in all the additional terms.

The proof that Newton multiplication is associative requires the following definition and lemma:

Let L be a q-lattice and $[x, y]$ an n-interval of L for some $n \in \mathbf{N}$. If $k \geq 2$ and $n, r_1, r_2, \ldots, r_k \in \mathbf{N}$, define the *multi-covering coefficient*

$$\left\{ \begin{matrix} n \\ r_1, r_2, \ldots, r_k \end{matrix} \right\}_q = |A|, \tag{4.6}$$

where

$$A = \left\{ (z_1, \ldots, z_k) \;\middle|\; z_i \in [x, y] \text{ and } \rho(z_i) = r_i \text{ for } i \in [k], \text{ and } z_1 \vee \cdots \vee z_k = y \right\}.$$

Lemma 4.1. *Let L be a q-lattice and $[x, y]$ an n-interval of L for some $n \in \mathbf{N}$. If k is an integer with $k \geq 2$ and if $r_1, r_2, \ldots, r_{k+1}$ are all nonnegative integers, then*

$$\sum_{r=0}^{n} \left\{ \begin{matrix} r \\ r_1, r_2, \ldots, r_k \end{matrix} \right\}_q \left\{ \begin{matrix} n \\ r, r_{k+1} \end{matrix} \right\}_q = \left\{ \begin{matrix} n \\ r_1, r_2, \ldots, r_{k+1} \end{matrix} \right\}_q. \tag{4.7}$$

Proof. Let A be the set of $(k+1)$-tuples that the covering coefficient on the right-hand side of (4.7) counts by definition, and let $B = \left\{ (z, w) \;\middle|\; z, w \in [x, y] \text{ and } \rho(w) = r_{k+1} \text{ and } z \vee w = y \right\}$.

Define a mapping $\alpha : A \to B$ by $\alpha(z_1, z_2, \ldots, z_{k+1}) = (z_1 \vee z_2 \vee \cdots \vee z_k, z_{k+1})$. As the following argument shows, the left-hand side of (4.7) counts the elements of A according to their image under α.

Let $(z, w) \in B$ be fixed. Then $(z_1, z_2, \ldots, z_{k+1}) \in A$ maps to (z, w) under α if and only if $z_1 \vee z_2 \vee \cdots \vee z_k = z$ and $z_{k+1} = w$. If $r = \rho(z)$, then there are $\left\{ \begin{matrix} r \\ r_1, r_2, \ldots, r_k \end{matrix} \right\}_q$ such $(z_1, z_2, \ldots, z_k)$ since for $i \in [k]$ the z_i must be in the r-interval $[x, z]$. Consequently,

$$\left| \alpha^{-1}(z, w) \right| = \left\{ \begin{matrix} r \\ r_1, r_2, \ldots, r_k \end{matrix} \right\}_q.$$

The set B contains $\left\{ \begin{matrix} n \\ r, r_{k+1} \end{matrix} \right\}_q$ elements that have a first coordinate with rank r. Summing over all possible values of r yields

$$\left\{ \begin{matrix} n \\ r_1, r_2, \ldots, r_{k+1} \end{matrix} \right\}_q = |A| = \sum_{r=0}^{n} \left\{ \begin{matrix} r \\ r_1, r_2, \ldots, r_k \end{matrix} \right\}_q \left\{ \begin{matrix} n \\ r, r_{k+1} \end{matrix} \right\}_q.$$

□

Note that equation (4.7) establishes a recurrence which effectively defines the multi-covering coefficients in terms of covering coefficients. As the value of covering coefficients does not depend on the interval under consideration, neither does the value of multi-covering coefficients. That is, the multi-covering coefficients are well-defined.

Note also that the value of $\left\{ {n \atop r_1, r_2, \ldots, r_k} \right\}_q$ remains unchanged if the order of the r_i changes. This observation follows from the commutativity of the join operation.

Theorem 4.2. *If $\diamond$ is a q-Newton product and $F, G, H \in \mathbf{C}^{\mathbf{N}}$, then $(F \diamond G) \diamond H = F \diamond (G \diamond H)$.*

Proof. Let $F, G, H \in \mathbf{C}^{\mathbf{N}}$ and $n \in \mathbf{N}$. Then

$$\begin{aligned}
\big((F \diamond G) \diamond H\big)(n) &= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ {n \atop r, s} \right\}_q (F \diamond G)(r) H(s) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ {n \atop r, s} \right\}_q \left(\sum_{i=0}^{r} \sum_{j=0}^{r} \left\{ {r \atop i, j} \right\}_q F(i) G(j) \right) H(s) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ {n \atop r, s} \right\}_q \left(\sum_{i=0}^{n} \sum_{j=0}^{n} \left\{ {r \atop i, j} \right\}_q F(i) G(j) \right) H(s) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{s=0}^{n} F(i) G(j) H(s) \sum_{r=0}^{n} \left\{ {r \atop i, j} \right\}_q \left\{ {n \atop r, s} \right\}_q \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{s=0}^{n} F(i) G(j) H(s) \left\{ {n \atop i, j, s} \right\}_q, \qquad (4.8)
\end{aligned}$$

where the last step follows from Lemma 4.1.

Beginning with the same lemma and essentially reversing the previous steps,

$$\begin{aligned}
\sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{s=0}^{n} F(i) G(j) H(s) \left\{ {n \atop i, j, s} \right\}_q &= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{s=0}^{n} F(i) G(j) H(s) \sum_{r=0}^{n} \left\{ {n \atop i, r} \right\}_q \left\{ {r \atop j, s} \right\}_q \\
&= \sum_{i=0}^{n} \sum_{r=0}^{n} \left\{ {n \atop i, r} \right\}_q F(i) \left(\sum_{j=0}^{n} \sum_{s=0}^{n} \left\{ {r \atop j, s} \right\}_q G(j) H(s) \right) \\
&= \sum_{i=0}^{n} \sum_{r=0}^{n} \left\{ {n \atop i, r} \right\}_q F(i) \big((G \diamond H)(r)\big) \\
&= \big(F \diamond (G \diamond H)\big)(n).
\end{aligned}$$

□

The following three lemmas complete the demonstration that Newton multiplication together with the usual addition and scalar multiplication give $\mathbf{C}^{\mathbf{N}}$ the structure of a commutative $\mathbf{C}$-algebra.

Lemma 4.2. *If $\diamond$ is a q-Newton product and if $F, G \in \mathbf{C}^{\mathbf{N}}$, then $F \diamond G = G \diamond F$. That is, Newton multiplication is commutative.*

Proof. This follows easily from equation (4.5) and the fact that

$$\left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q = \left\{ \begin{matrix} n \\ s, r \end{matrix} \right\}_q \qquad \text{for all } n, r.s. \in \mathbf{N}.$$

□

Lemma 4.3. *If $\diamond$ is a q-Newton product and $F, G, H \in \mathbf{C}^{\mathbf{N}}$, then $F \diamond (G + H) = F \diamond G + F \diamond H$. That is, Newton multiplication distributes over addition.*

Proof. Let $n \in \mathbf{N}$. Then

$$\begin{aligned}
\big(F \diamond (G+H)\big)(n) &= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q F(r)(G+H)(s) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q \big(F(r)G(s) + F(r)H(s)\big) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q F(r)G(s) + \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q F(r)H(s) \\
&= (F \diamond G)(n) + (F \diamond H)(n).
\end{aligned}$$

□

Lemma 4.4. *If $\diamond$ is a q-Newton product, $F, G \in \mathbf{C}^{\mathbf{N}}$ and $c \in \mathbf{C}$, then $c(F \diamond G) = (cF) \diamond G$.*

Proof. If $n \in \mathbf{N}$, then

$$\begin{aligned}
\big(c(F \diamond G)\big)(n) = c\big((F \diamond G)(n)\big) &= c \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q F(r)G(s) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q c\big(F(r)\big)G(s) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q (cF)(r)G(s) \\
&= \big((cF) \diamond G\big)(n).
\end{aligned}$$

□

This lemma completes the proof of the following theorem.

Theorem 4.3. *If $\diamond$ is a q-Newton product then the quadruple $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ is a commutative algebra over $\mathbf{C}$.*

Let $U = (1, 0, 0, \dots) \in \mathbf{C}^{\mathbf{N}}$. If $F \in \mathbf{C}^{\mathbf{N}}$ and $n \in \mathbf{N}$, then

$$\begin{aligned}(U \diamond F)(n) &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q U(r)F(s) \\ &= \begin{Bmatrix} n \\ 0, n \end{Bmatrix}_q U(0)F(n) \\ &= (1)(1)F(n) \\ &= F(n).\end{aligned}$$

Therefore $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ has a multiplicative identity, namely U.

Under the same conditions define $\phi: \mathbf{C} \to \mathbf{C}^{\mathbf{N}}$ by $\phi(c) = cU$ for all $c \in \mathbf{C}$. The mapping ϕ is well-known to be a $\mathbf{C}$-algebra monomorphism between $\mathbf{C}$ and $(\mathbf{C}^{\mathbf{N}}, +, *, .)$. As it turns out, ϕ is also a $\mathbf{C}$-algebra monomorphism between $\mathbf{C}$ and $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$.

Theorem 4.4. *If $\diamond$ is a q-Newton product, then the mapping $\phi: \mathbf{C} \to \mathbf{C}^{\mathbf{N}}$ by $\phi(c) = cU$ for all $c \in \mathbf{C}$, is a $\mathbf{C}$-algebra monomorphism between $\mathbf{C}$ and $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$.*

Proof. By the remarks preceding this theorem, ϕ is injective and preserves addition and scalar multiplication. It suffices to show that ϕ preserves multiplication.

Let $a, b \in \mathbf{C}$. Then $\phi(ab) = (ab)U = a(bU) = a(U \diamond bU) = (aU) \diamond bU = \phi(a) \diamond \phi(b)$.

□

In light of Theorem 4.4 and the remarks preceding it, the natural practice arises of viewing the image of $\mathbf{C}$ under ϕ, that is the set

$$\phi(\mathbf{C}) = \{cU \mid c \in \mathbf{C}\} = \{(c, 0, 0, \dots) \mid c \in \mathbf{C}\}$$

as a copy of $\mathbf{C}$ in $\mathbf{C}^{\mathbf{N}}$. This leads to the convention of denoting the sequence $cU = (c, 0, 0, \dots)$, where $c \in \mathbf{C}$, simply by c, allowing the context to resolve ambiguities. In particular,

$$U = 1.U = 1,$$

where $1 \in \mathbf{C}$ the first time it appears and $1 = (1, 0, 0, \dots) \in \mathbf{C}^{\mathbf{N}}$ the second time it appears.

4.3 The indeterminate and q-binomial polynomials

Define $X \in \mathbf{C}^{\mathbf{N}}$ by

$$X = (0, 1, 0, 0, \dots).$$

Let

$$X^{*0} = 1 = (1, 0, 0, \dots),$$

and for $n \in \mathbf{P}$ let

$$X^{*n} = \overbrace{X * X * \cdots * X}^{n \text{ times}}.$$

With a suitable metric on $\mathbf{C}^{\mathbf{N}}$ and a few natural definitions (see section 4.5), the series $\sum_{n \geq 0} a_n X^{*n}$ converges to $(a_0, a_1, a_2, \dots)$. For this reason the algebra $(\mathbf{C}^{\mathbf{N}}, +, *, .)$, much more commonly denoted $(\mathbf{C}[[X]], +, *, .)$, is called the *algebra of formal power series.* See I. Niven [13] for a thorough development of formal power series.

A bit unexpectedly, if $\diamond$ is a q-Newton product with $q > 0$, then the elements of $\mathbf{C}^{\mathbf{N}}$ also have a "nice" representation as infinite series in the algebra $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$. The development of this idea and an investigation of the algebraic properties of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ occupy the remainder of this section.

Let $\diamond$ be a q-Newton product with $q > 0$. Adopting the convention that an empty Newton product is $1 \in \mathbf{C}^{\mathbf{N}}$, define the *q-binomial polynomial of order n*

$$\begin{aligned}\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q &= \frac{(X \diamond (X - 1_q) \diamond \cdots \diamond (X - (n-1)_q))}{(n_q (n_q - 1_q) \dots (n_q - (n-1)_q))} \\ &= \frac{1}{(n_q (n_q - 1_q) \dots (n_q - (n-1)_q))} (X \diamond (X - 1_q) \diamond \cdots \diamond (X - (n-1)_q)) \\ &= \frac{1}{n_q^{\underline{n}}} (X \diamond (X - 1_q) \diamond \cdots \diamond (X - (n-1)_q)) \qquad (4.9)\end{aligned}$$

for all $n \in \mathbf{N}$.

Clearly $\left\langle \begin{matrix} X \\ 0 \end{matrix} \right\rangle_q = 1 = (1, 0, 0, \dots)$ by the convention on empty Newton products, and $\left\langle \begin{matrix} X \\ 1 \end{matrix} \right\rangle_q = X = (0, 1, 0, 0, \dots)$. The following theorem characterizes $\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q$ in general.

Theorem 4.5. *For $n \in \mathbf{N}$ let $\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q$ be the q-binomial polynomial of order n where $q > 0$. For all $k \in \mathbf{N}$, then, $\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q (k) = \delta_{nk}$, where δ is the Kronecker delta. In other words*

$$\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q = (0, 0, \dots, 0, 1, 0, \dots) \in \mathbf{C}^{\mathbf{N}},$$

where the 1 *falls in the* n*th position, the initial position being considered position* 0.

Proof. The result holds for $n = 0, 1$.

Proceeding by induction, assume the result holds for some $n \in \mathbf{N}$. Then recalling equation (3.5)

$$\begin{aligned}
\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q &= \frac{1}{(n+1)_q^{\frac{n+1}{2}}}(X \diamond (X - 1_q) \diamond \cdots \diamond (X - n_q)) \\
&= \frac{1}{q^{n(n+1)/2}(n+1)!_q}(X \diamond (X - 1_q) \diamond \cdots \diamond (X - n_q)) \\
&= \left(\frac{1}{q^{n(n-1)/2}n!_q}(X \diamond (X - 1_q) \diamond \cdots \diamond (X - (n-1)_q))\right) \\
&\qquad \diamond \left(\frac{1}{q^n(n+1)_q}(X - n_q)\right) \\
&= \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q \diamond \left(\frac{1}{q^n(n+1)_q}(X - n_q)\right) \qquad (4.10)
\end{aligned}$$

Let

$$F = \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q \qquad \text{and} \qquad G = \left(\frac{1}{q^n(n+1)_q}(X - n_q)\right).$$

If $k \in \mathbf{N}$, then by (4.10)

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (k) = (F \diamond G)(k) = \sum_{r=0}^{k} \sum_{s=k-r}^{k} \left\{ \begin{matrix} k \\ r, s \end{matrix} \right\}_q F(r)G(s). \qquad (4.11)$$

By the induction hypothesis $F(r) = \delta_{nr}$. In particular, $F(r) = 0$ for $r < n$. Therefore, by (4.11), for all $k < n$

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (k) = 0.$$

Now assume $k \geq n$. Then equation (4.11) yields

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (k) = F(n) \sum_{s=k-n}^{k} \left\{ \begin{matrix} k \\ n, s \end{matrix} \right\}_q G(s). = \sum_{s=k-n}^{k} \left\{ \begin{matrix} k \\ n, s \end{matrix} \right\}_q G(s). \qquad (4.12)$$

Note that

$$G = \frac{1}{q^n(n+1)_q}(-n_q, 1, 0, 0, \dots).$$

That is, $G(s) = 0$ for all $s \geq 2$. So the summation in (4.12) is nonzero only if $k - n = 0$ or $k - n = 1$—in other words if $k = n$ or $k = n + 1$. Consider these two cases separately.

Case 1. Suppose $k = n$. From equation (4.12)

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (n) = \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ n,s \end{matrix} \right\}_q G(s)$$
$$= \left\{ \begin{matrix} n \\ n,0 \end{matrix} \right\}_q G(0) + \left\{ \begin{matrix} n \\ n,1 \end{matrix} \right\}_q G(1).$$

By the definition of covering coefficients

$$\left\{ \begin{matrix} n \\ n,0 \end{matrix} \right\}_q = 1,$$

and by equation (4.2)

$$\left\{ \begin{matrix} n \\ n,1 \end{matrix} \right\}_q = n_q.$$

So

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (n) = \left\{ \begin{matrix} n \\ n,0 \end{matrix} \right\}_q G(0) + \left\{ \begin{matrix} n \\ n,1 \end{matrix} \right\}_q G(1).$$
$$= (1)\left(\frac{-n_q}{q^n(n+1)_q}\right) + (n_q)\left(\frac{1}{q^n(n+1)_q}\right)$$
$$= 0.$$

Case 2. Suppose $k = n+1$. From equation (4.12)

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (n+1) = \sum_{s=1}^{n+1} \left\{ \begin{matrix} n+1 \\ n,s \end{matrix} \right\}_q G(s)$$
$$= \left\{ \begin{matrix} n+1 \\ n,1 \end{matrix} \right\}_q G(1)$$
$$= \left(\frac{(n+1)!_q\, q^{(1)(n)}}{1!_q\, n!_q\, 0!_q}\right)\left(\frac{1}{q^n(n+1)_q}\right)$$
$$= 1,$$

where the value of $\left\{ \begin{matrix} n+1 \\ n,1 \end{matrix} \right\}$ comes from equation (4.2).

Therefore

$$\left\langle \begin{matrix} X \\ n+1 \end{matrix} \right\rangle_q (k) = \begin{cases} 1, & \text{if } k = n+1 \\ 0, & \text{otherwise} \end{cases}.$$

□

4.4 Polynomials

Let $\mathcal{P} = \{F \in \mathbf{C}^{\mathbf{N}} \mid$ there exists an $N \in \mathbf{N}$ such that $F(n) = 0$ whenever $n > N\}$. In other words $\mathcal{P}$ is the set of finitely nonzero sequences in $\mathbf{C}^{\mathbf{N}}$. Then $(\mathcal{P}, +, *, .)$ is the algebra of polynomials over $\mathbf{C}$, more usually denoted $(\mathbf{C}[X], +, *, .)$.

If $F \in \mathcal{P}$ with $F \neq 0$, define the *degree* of F by $\deg F = \max\{n \in \mathbf{N} \mid F(n) \neq 0\}$. Further define $\deg 0 = -\infty$.

As it turns out, if $\diamond$ is a q-Newton product with $q > 0$, then the quadruple $(\mathcal{P}, +, \diamond, .)$ is also a $\mathbf{C}$-algebra and is isomorphic to $(\mathcal{P}, +, *, .)$. The proof of this isomorphism requires the following lemma.

Lemma 4.5. *Let L be a q-lattice with $q > 0$. For $n \in \mathbf{N}$ define $F_n \in \mathbf{C}^{\mathbf{N}}$ by*

$$F_0 = 1,$$

$$F_1 = X,$$

and

$$F_n = \frac{1}{n^{\underline{n}}_q}\big(X * (X - 1_q) * \cdots * (X - (n-1)_q)\big) \qquad \text{for } n \geq 2. \tag{4.13}$$

If $r, s \in \mathbf{N}$, then

$$F_r * F_s = \sum_{n=0}^{r+s} \left\{ {n \atop r, s} \right\}_q F_n. \tag{4.14}$$

Proof. For $c \in \mathbf{C}$, let $\phi_c : \mathcal{P} \to \mathbf{C}$ be the usual evaluation homomorphism from $(\mathcal{P}, +, *, .)$ to $\mathbf{C}$. That is to say, if $F = \sum_{i=0}^{N} a_i X^{*i} \in \mathcal{P}$ for some $N \in \mathbf{N}$ and complex constants $a_0, a_1, \ldots, a_N$, then

$$\phi_c(F) = \sum_{i=0}^{N} a_i c^i.$$

If $k, n \in \mathbf{N}$, then

$$\begin{aligned} \phi_{k_q}(F_n) &= \left(\frac{1}{n^{\underline{n}}_q}\right) k^{\underline{n}}_q \\ &= \begin{bmatrix} n \\ k \end{bmatrix} \end{aligned} \tag{4.15}$$

by equation (3.6). So if $r, s \in \mathbf{N}$, then Theorems 1.1 and 1.3 establish

$$\begin{aligned} \phi_{k_q}(F_r * F_s) &= \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} k \\ s \end{bmatrix} \\ &= \sum_{n=0}^{r+s} \left\{ {n \atop r, s} \right\}_q \begin{bmatrix} k \\ n \end{bmatrix} \\ &= \phi_{k_q}\left(\sum_{n=0}^{r+s} \left\{ {n \atop r, s} \right\}_q F_n \right) \end{aligned} \tag{4.16}$$

for all $k \in \mathbf{N}$. Those theorems, however, make use of set arguments and vector space arguments. A more general argument using properties of q-lattices suffices to establish equation (4.16) for all $k \in \mathbf{N}$. This argument, of course, then yields Theorems 1.1 and 1.3 as special cases. Namely, let $[x, y]$ be a k-interval in L. The left-hand side of (4.16) counts the number of ways to choose $z, w \in [x, y]$ such that $\rho(z) = r$ and $\rho(w) = s$. The right-hand side of (4.16) does the same according to the rank of $z \vee w$.

Both sides of equation (4.14) are polynomials of degree $r + s$ or less, but equation (4.16) holds for all $k \in \mathbf{N}$. This says that

$$\phi_c(F_r * F_s) = \phi_c\left(\sum_{n=0}^{r+s} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q F_n\right)$$

for an infinite collection of $c \in \mathbf{C}$, namely for any $c = k_q$, where $k \in \mathbf{N}$. Thus (4.14) is a polynomial identity.

□

Let $\diamond$ be a q-Newton product with $q = 0$ and let

$$F = (1, -1, 0, 0, \dots) \qquad \text{and} \qquad G = (0, 0, 1, 0, 0, \dots)$$

be in $\mathcal{P}$. If $n \in \mathbf{N}$, then

$$\begin{aligned}(F \diamond G)(n) &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q F(r)G(s) \\ &= \sum_{r=0}^{n} F(r)G(n) + \sum_{s=0}^{n-1} F(n)G(s).\end{aligned}$$

So

$$\begin{aligned}(F \diamond G)(0) &= F(0)G(0) = 0, \\ (F \diamond G)(1) &= F(0)G(1) + F(1)G(1) + F(1)G(0) \\ &= 0 + 0 + 0 = 0, \\ (F \diamond G)(2) &= F(0)G(2) + F(1)G(2) + F(2)G(2) + F(2)G(1) + F(2)G(0) \\ &= 1 - 1 + 0 + 0 + 0 = 0,\end{aligned}$$

and clearly $(F \diamond G)(n) = 0$ if $n \geq 3$. In other words $F \diamond G = 0$. Thus $(\mathcal{P}, +, \diamond, .)$ has zero divisors and is not isomorphic to $(\mathcal{P}, +, *, .)$ since $(\mathcal{P}, +, *)$ is an integral domain.

If $q > 0$, however, the situation is rather different.

Theorem 4.6. *Let $\diamond$ be a q-Newton product with $q > 0$, and for $n \in \mathbf{N}$ define F_n as in the previous lemma. Define $\psi: \mathcal{P} \to \mathcal{P}$ by*

$$\psi(F) = \sum_{n=0}^{N} a_n F_n, \tag{4.17}$$

where

$$F = (a_0, a_1, \ldots, a_N, 0, 0, \ldots) \in \mathcal{P}.$$

Then ψ is an isomorphism between $(\mathcal{P}, +, \diamond, .)$ and $(\mathcal{P}, +, *, .)$.

Proof. Since $\deg F_n = n$ for all $n \in \mathbf{N}$, the set $\{F_n \mid n \in \mathbf{N}\}$ is clearly a vector-space basis for $(\mathcal{P}, +, .)$. From this it follows easily that ψ is bijective.

It is also clear that ψ preserves addition and scalar multiplication.

Let $F, G \in \mathcal{P}$ with $N = \max(\deg F, \deg G)$. From (4.4) it is not hard to see that $\deg(F \diamond G) = \deg(F) + \deg(G)$, so $\deg(F \diamond G) \leq 2N$. Thus

$$\begin{aligned}
\psi(F \diamond G) &= \sum_{n=0}^{2N} \big((F \diamond G)(n)\big) F_n \\
&= \sum_{n=0}^{2N} \left(\sum_{r=0}^{N} \sum_{s=0}^{N} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q F(r) G(s) \right) F_n \\
&= \sum_{r=0}^{N} \sum_{s=0}^{N} F(r) G(s) \sum_{n=0}^{2N} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q F_n \\
&= \sum_{r=0}^{N} \sum_{s=0}^{N} \big(F(r) G(s)\big)(F_r * F_s) \qquad \text{(by Lemma 4.5)} \\
&= \left(\sum_{r=0}^{N} F(r) F_r \right) * \left(\sum_{s=0}^{N} G(s) F_s \right) \\
&= \psi(F) * \psi(G).
\end{aligned}$$

□

If $F = (a_0, a_1, \ldots, a_N, 0.0\ldots) \in \mathcal{P}$, then by Theorem 4.5

$$F = \sum_{n=0}^{N} a_n \left\langle {X \atop n} \right\rangle_q, \tag{4.18}$$

so the isomorphism of Theorem 4.6 can also be written as

$$\psi\left(\sum_{n=0}^{N} a_n \left\langle {X \atop n} \right\rangle_q \right) = \sum_{n=0}^{N} a_n F_n. \tag{4.19}$$

That is

$$\psi\left(\sum_{n=0}^{N} a_n \frac{1}{n!_q}\big(X \diamond (X-1_q) \diamond \cdots \diamond (X-(n-1)_q)\big)\right)$$
$$= \sum_{n=0}^{N} a_n \frac{1}{n!_q}\big(X * (X-1_q) * \cdots * (X-(n-1)_q)\big) \quad (4.20)$$

Note that $\psi(X) = X$, so ψ can also be written as

$$\psi\left(\sum_{n=0}^{N} a_n X^{\diamond n}\right) = \sum_{n=0}^{N} a_n X^{*n}, \quad (4.21)$$

where

$$X^{\diamond n} = \overbrace{X \diamond X \diamond \cdots \diamond X}^{n \text{ times}}.$$

If $c \in \mathbf{C}$ and ϕ_c is the evaluation homomorphism from $(\mathbf{C^N}, +, *, .)$ to $\mathbf{C}$, then the mapping $\phi_c' : \mathcal{P} \to \mathbf{C}$ by $\phi_c'(F) = \phi_c\big(\psi(F)\big)$ is a homomorphism from $(\mathbf{C^N}, +, \diamond, .)$ to $\mathbf{C}$. Call ϕ_c' the *evaluation homomorphism* from $(\mathbf{C^N}, +, \diamond, .)$ to $\mathbf{C}$.

4.5 Topology and convergence

I. Niven [13] defines a metric d on $\mathbf{C^N}$ as follows: Fix a real number $b \in (0,1)$. Define the valuation *val* on $\mathbf{C^N}$ by

$$\mathrm{val}(F) = \begin{cases} \min\{n \in \mathbf{N} \mid F(n) \neq 0\}, & \text{if } F \neq 0 \\ \infty, & \text{if F=0} \end{cases}$$

for $F \in \mathbf{C^N}$ (Niven calls this the *degree* of F). Use this to define an absolute value on $F \in \mathbf{C^N}$ by

$$|F| = b^{\mathrm{val}(F)}.$$

Finally define the metric d on $\mathbf{C^N}$ by

$$d(F,G) = |F - G| \quad (4.22)$$

for all $F, G \in \mathbf{C^N}$.

The sequence $\{F_n\}$ in $\mathbf{C^N}$ is *summable* if and only if there exists $F \in \mathbf{C^N}$ such that $\lim_{n\to\infty} \sum_{k=0}^{n} F_k = F$—that is, if and only if the sequence of partial sums $F_0 + F_1 + \cdots + F_n$ converges to F in the metric d as $n \to \infty$. In this case write

$\sum_{n=0}^{\infty} F_n = F$. It is easy to see that $\{F_n\}$ being summable is equivalent to requiring that the sequence $\{F_n\}$ converge to 0 in the metric d. This in turn is equivalent to requiring $\lim_{n\to\infty} \mathrm{val}(F_n) = \infty$.

Let $F = (a_0, a_1, a_n, \dots) \in \mathbf{C}^{\mathbf{N}}$. Since $\mathrm{val}(X^{*n}) = n$ for all $n \in \mathbf{N}$, the sequence $\{a_n X^{*n}\}$ is summable, and clearly

$$F = \sum_{n=0}^{\infty} a_n X^{*n}. \tag{4.23}$$

This is the reason for calling $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ the algebra of formal power series.

Now let $\diamond$ be a q-Newton product with $q > 0$. Theorem 4.5 shows that

$$\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q = X^{*n} \qquad \text{for all } n \in \mathbf{N},$$

so by (4.23) we can also write

$$F = \sum_{n=0}^{\infty} a_n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q. \tag{4.24}$$

In the light of this representation of the members of $\mathbf{C}^{\mathbf{N}}$, call the algebra $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ the *algebra of formal q-binomial series.*

Equation (4.24) suggests the following restatement of the definition of Newton multiplication. Let $\diamond$ be a q-Newton product with $q > 0$, and let

$$F = (a_0, a_1, a_2, \dots) \in \mathbf{C}^{\mathbf{N}} \qquad \text{and} \qquad G = (b_0, b_1, b_2, \dots) \in \mathbf{C}^{\mathbf{N}}.$$

Then

$$\begin{aligned} F \diamond G &= \left(\sum_{n=0}^{\infty} a_n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q \right) \diamond \left(\sum_{n=0}^{\infty} b_n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q a_r b_s \right) \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q. \end{aligned} \tag{4.25}$$

Formula (4.5) produces the equivalent statement

$$F \diamond G = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q a_r b_s \right) \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q. \tag{4.26}$$

4.6 Algebraic properties

Despite the existence of isomorphic subalgebras $(\mathcal{P}, +, *, .)$ and $(\mathcal{P}, +, \diamond, .)$, the $\mathbf{C}$-algebras $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ and $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ are not isomorphic. This follows immediately from the existence of zero divisors in $(\mathbf{C}^{\mathbf{N}}, +, \diamond)$ since $(\mathbf{C}^{\mathbf{N}}, +, *)$ is well-known to be an integral domain.

Theorem 4.7. *Let $\diamond$ be a q-Newton product. The ring $(\mathbf{C}^{\mathbf{N}}, +, \diamond)$ is not an integral domain.*

Proof. The remarks preceding Theorem 4.6 establish the result for $q = 0$, so assume $q > 0$.

Let $F = (-1, 1, 0, 0, \dots) \in \mathbf{C}^{\mathbf{N}}$. If F is a zero divisor of $(\mathbf{C}^{\mathbf{N}}, +, \diamond)$, then there is a nonzero $G \in \mathbf{C}^{\mathbf{N}}$ such that $F \diamond G = 0$. By formula (4.4) this equation says that

$$\sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r, s} \right\} F(r)G(s) = 0 \qquad \text{for all } n \in \mathbf{N}. \tag{4.27}$$

Solving (4.27) for G yields the recursive definition

$$G(0) = 0,$$
$$G(1) = 1,$$

and

$$G(n) = \frac{1}{\left(\left\{ {n \atop 1,n} \right\}_q - \left\{ {n \atop 0,n} \right\}_q \right)} \left(-\left\{ {n \atop 1, n-1} \right\}_q G(n-1) \right) \qquad \text{for } n \geq 2.$$

By equation (4.2), for $n \geq 2$

$$\left\{ {n \atop 1,n} \right\}_q - \left\{ {n \atop 0,n} \right\}_q = n_q - 1 = q^{n-1} + q^{n-2} + \cdots + q \neq 0,$$

so this definition involves no division by 0 since $q > 0$.

To verify that indeed $F \diamond G = 0$, note that

$$(F \diamond G)(0) = \left\{ {0 \atop 0,0} \right\}_q F(0)G(0) = (1)(-1)(0) = 0$$

and

$$\begin{aligned} F \diamond G(1) &= \left\{ {1 \atop 0,1} \right\}_q F(0)G(1) + \left\{ {1 \atop 1,0} \right\}_q F(1)G(0) + \left\{ {1 \atop 1,1} \right\}_q F(1)G(1) \\ &= (1)(-1)(1) + (1)(1)(0) + (1)(1)(1) \\ &= 0 \end{aligned}$$

and for $n \geq 2$

$$\begin{aligned}
(F \diamond G)(n) &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\} F(r)G(s) \\
&= \left\{ {n \atop 0,n} \right\}_q F(0)G(n) + \left\{ {n \atop 1,n-1} \right\}_q F(1)G(n-1) + \left\{ {n \atop 1,n} \right\}_q F(1)G(n) \\
&= \left\{ {n \atop 1,n-1} \right\}_q G(n-1) + \left(\left\{ {n \atop 1,n} \right\} - \left\{ {n \atop 0,n} \right\}_q \right) G(n) \\
&= \left\{ {n \atop 1,n-1} \right\}_q G(n-1) + \frac{\left(\left\{ {n \atop 1,n} \right\} - \left\{ {n \atop 0,n} \right\}_q \right)}{\left(\left\{ {n \atop 1,n} \right\} - \left\{ {n \atop 0,n} \right\}_q \right)} \left(-\left\{ {n \atop 1,n-1} \right\}_q G(n-1) \right) \\
&= \left\{ {n \atop 1,n-1} \right\}_q G(n-1) - \left\{ {n \atop 1,n-1} \right\}_q G(n-1) \\
&= 0.
\end{aligned}$$

As neither F nor G is 0, this shows that F is a zero divisor of $(\mathbf{C}^{\mathbf{N}}, +, \diamond)$.

□

Corollary. *Let $\diamond$ be a q-Newton product. The* **C**-*algebra $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ is not isomorphic to the* **C**-*algebra $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$.*

The units of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ have a simple characterization whose proof uses, in part, the same technique as that used to define G in the proof of Theorem 4.7.

Theorem 4.8. *Let $\diamond$ be a q-Newton product. An element F of $\mathbf{C}^{\mathbf{N}}$ is a unit of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ if and only if*

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) \neq 0 \qquad \text{for all } n \in \mathbf{N}.$$

Proof. Suppose $F \in \mathbf{C}^{\mathbf{N}}$ such that $\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) \neq 0$ for all $n \in \mathbf{N}$. Define $G \in \mathbf{C}^{\mathbf{N}}$ by

$$G(0) = \frac{1}{F(0)}$$

and

$$G(n) = -\left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) \right)^{-1} \left(\sum_{r=0}^{n-1} \sum_{s=0}^{n} \left\{ {n \atop r,s} \right\}_q G(r)F(s) \right) \qquad \text{for } n \in \mathbf{P}.$$

By Theorem 4.1 and equation (3.6) note that

$$\begin{aligned}\left\{ {n \atop r,s} \right\}_q &= \frac{n!_q\, q^{(n-r)(n-s)}}{(n-r)!_q\,(n-s)!_q\,(r+s-n)!_q} \\ &= \left(\frac{n!_q}{(n-r)!_q\, r!_q}\right)\left(\frac{r!_q}{(n-s)!_q\,(r-(n-s))!_q}\right) q^{(n-r)(n-s)} \\ &= \begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} r \\ n-s \end{bmatrix} q^{(n-r)(n-s)} \end{aligned} \tag{4.28}$$

whenever $n, r, s \in \mathbf{N}$ with $r \leq n$, $s \leq n$, and $r + s \geq n$. Of course a combinatorial proof of this identity also exists.

Equation (4.28) and the definition of G yield

$$(G \diamond F)(0) = \left\{ {0 \atop 0,0} \right\}_q G(0)F(0) = (1)(1)(1) = 1$$

and for $n \in \mathbf{P}$

$$\begin{aligned}(G \diamond F)(n) &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\}_q G(r)F(s) \\ &= G(n) \sum_{s=0}^{n} \left\{ {n \atop n,s} \right\}_q F(s) + \sum_{r=0}^{n-1} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\}_q G(r)F(s) \\ &= G(n) \sum_{s=0}^{n} \begin{bmatrix} n \\ n \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{(n-n)(n-s)} F(s) + \sum_{r=0}^{n-1} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\}_q G(r)F(s) \\ &= G(n) \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} F(s) + \sum_{r=0}^{n-1} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\}_q G(r)F(s) \\ &= \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i)\right)\left(G(n) + \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i)\right)^{-1} \sum_{r=0}^{n-1} \sum_{s=n-r}^{n} \left\{ {n \atop r,s} \right\}_q G(r)F(s)\right) \\ &= \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i)\right)(G(n) - G(n)) \\ &= 0. \end{aligned}$$

Therefore $G \diamond F = (1, 0, 0, \dots) = 1$, the multiplicative identity of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$. So F is a unit of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$.

Conversely, suppose that F is a unit of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ and there exists an $n \in \mathbf{N}$ such that $\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) = 0$. Define $H \in \mathbf{C}^{\mathbf{N}}$ by $H(i) = \delta_{ni}$ for all $i \in \mathbf{N}$, where δ is the Kronecker delta.

If $k \in \{0, 1, \ldots, n-1\}$, then

$$\begin{aligned}(F \diamond H)(k) &= \sum_{r=0}^{k} \sum_{s=k-r}^{k} \left\{ {k \atop r, s} \right\}_q F(r)\delta_{ns} \\ &= \sum_{r=0}^{k} \sum_{s=k-r}^{k} \left\{ {k \atop r, s} \right\}_q F(r)(0) \\ &= 0.\end{aligned}$$

Further, again by equation (4.28),

$$\begin{aligned}(F \diamond H)(n) &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r, s} \right\}_q F(r)\delta_{ns} \\ &= \sum_{r=0}^{n} \left\{ {n \atop r, n} \right\}_q F(r) \\ &= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} r \\ n-n \end{bmatrix} q^{(n-r)(n-n)} F(r) \\ &= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} F(r) \\ &= 0.\end{aligned}$$

Thus $(F \diamond H)(k) = 0$ for $k \in \{0, 1, \ldots, n\}$.

Since F is a unit, there exists $G \in \mathbf{C}^{\mathbf{N}}$ such that $G \diamond F = 1$. In particular

$$((G \diamond F) \diamond H)(n) = (1 \diamond H)(n) = 1.$$

On the other hand

$$\begin{aligned}(G \diamond (F \diamond H))(n) &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r, s} \right\}_q G(r)(F \diamond H)(s) \\ &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ {n \atop r, s} \right\}_q G(r)(0) \\ &= 0.\end{aligned}$$

This is a contradiction. Therefore if F is a unit, it cannot be the case that $\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) = 0$ for some $n \in \mathbf{N}$.

□

The second half of the preceding proof is essentially an argument about valuations and could easily be recast in that form as in the following sketch. The valuation of the Newton product of two q-binomial series cannot be smaller than the maximum of the valuations of the two factors (Equation (4.26) makes this clear.). In particular if $F, H \in \mathbf{C}^{\mathbf{N}}$, then $\mathrm{val}(F \diamond H) \geq \mathrm{val}(H)$. But if F is a unit of $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ with Newton multiplicative inverse G, then $\mathrm{val}(F \diamond H) = \mathrm{val}(H)$ as otherwise $\mathrm{val}(H) = \mathrm{val}(1 \diamond H) = \mathrm{val}\big(G \diamond (F \diamond H)\big) \geq \mathrm{val}(F \diamond H) > \mathrm{val}(H)$. On the other hand, if $F, H \in \mathbf{C}^{\mathbf{N}}$ with $\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) = 0$ and $\mathrm{val}(H) = n$, then $\mathrm{val}(F \diamond H) > n$, so F cannot be a unit.

It is interesting that this same sort of valuation argument produces the standard result that units in $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ must have nonzero constant terms.

5. The Underlying Structures

5.1 Covering algebras

Doubilet, Rota, and Stanley [4, 16] develop the theory of incidence algebras as a means of finding the common structures behind many of the generating functions which arise in combinatorics. Similar structures underlie q-binomial series. Indeed, these structures bear much the same relation to incidence algebras that $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ bears to $(\mathbf{C}^{\mathbf{N}}, +, *, .)$.

Let L be a locally finite lattice (always nonempty by assumption), and define

$$\mathrm{Int}(L) = \{\text{intervals of } L\} = \big\{[x,y] \;\big|\; x,y \in L \text{ and } x \leq y\big\}.$$

Further, define $\mathcal{C}(L) = \big\{\text{complex-valued functions on } \mathrm{Int}(L)\big\} = \big\{f\colon \mathrm{Int}(L) \to \mathbf{C}\big\}$.

Equip $\mathcal{C}(L)$ with the usual addition and scalar multiplication; namely, for all $f, g \in \mathcal{C}(L)$ and $c \in \mathbf{C}$ let

$$(f+g)(x,y) = f(x,y) + g(x,y)$$

and

$$(cf)(x,y) = (c.f)(x,y) = c\big(f(x,y)\big)$$

for all intervals $[x,y] \in \mathrm{Int}(L)$. The triple $(\mathcal{C}(L), +, .)$ is well-known to be a vector space over $\mathbf{C}$. Write $f(x,y)$ in place of the more awkward $f([x,y])$.

Now equip $\mathcal{C}(L)$ with a multiplication. If $f, g \in \mathcal{C}(L)$, we shall define $f \diamond g$ by

$$(f \diamond g)(x,y) = \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} f(x,z)g(x,w) \tag{5.1}$$

for all $[x,y] \in \mathrm{Int}(L)$. The next few lemmas explore the behavior of this multiplication.

Lemma 5.1. *Let L be a locally finite lattice. If $f, g \in \mathcal{C}(L)$, then $f \diamond g = g \diamond f$. In other words, the operation $\diamond$ is commutative.*

Proof. Let $[x,y] \in \mathrm{Int}(L)$ and $f, g \in \mathcal{C}(L)$. Then

$$\begin{aligned}(f \diamond g)(x,y) &= \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} f(x,z)g(x,w) \\ &= \sum_{\substack{w,z \in [x,y] \\ w \vee z = y}} g(x,w)f(x,z) \\ &= (g \diamond f)(x,y).\end{aligned}$$

□

Lemma 5.2. *Let L be a locally finite lattice. If $f, g, h \in \mathcal{C}(L)$, then $(f \diamond g) \diamond h = f \diamond (g \diamond h)$. In other words, the operation $\diamond$ is associative.*

Proof. Let $[x, y] \in \mathrm{Int}(L)$ and $f, g, h \in \mathcal{C}(L)$. Then

$$\begin{aligned}
\big((f \diamond g) \diamond h\big)(x, y) &= \sum_{\substack{z, w \in [x, y] \\ z \vee w = y}} \big((f \diamond g)(x, z)\big) h(x, w) \\
&= \sum_{\substack{z, w \in [x, y] \\ z \vee w = y}} \left(\sum_{\substack{a, b \in [x, z] \\ a \vee b = z}} f(x, a) g(x, b) \right) h(x, w) \\
&= \sum_{\substack{a, b, w \in [x, y] \\ a \vee b \vee w = y}} f(x, a) g(x, b) h(x, w) \\
&= \sum_{\substack{a, c \in [x, y] \\ a \vee c = y}} f(x, a) \sum_{\substack{b, w \in [x, c] \\ b \vee w = c}} g(x, b) h(x, w) \\
&= \sum_{\substack{a, c \in [x, y] \\ a \vee c = y}} f(x, a) \big((g \diamond h)(x, c)\big) \\
&= \big(f \diamond (g \diamond h)\big)(x, y).
\end{aligned}$$

□

Lemma 5.3. *Let L be a locally finite lattice. If $f, g, h \in \mathcal{C}(L)$, then $f \diamond (g + h) = (f \diamond g) + (f \diamond h)$. In other words, the operation $\diamond$ distributes over addition.*

Proof. Let $[x, y] \in \mathrm{Int}(L)$ and $f, g, h \in \mathcal{C}(L)$. Then

$$\begin{aligned}
\big(f \diamond (g + h)\big)(x, y) &= \sum_{\substack{z, w \in [x, y] \\ z \vee w = y}} f(x, z) \big((g + h)(x, w)\big) \\
&= \sum_{\substack{z, w \in [x, y] \\ z \vee w = y}} f(x, z) \big(g(x, w) + h(x, w)\big) \\
&= \sum_{\substack{z, w \in [x, y] \\ z \vee w = y}} f(x, z) g(x, w) + \sum_{\substack{z, w \in [x, y] \\ z \vee w = y}} f(x, z) h(x, w) \\
&= (f \diamond g)(x, y) + (f \diamond h)(x, y).
\end{aligned}$$

□

Lemma 5.4. *Let L be a locally finite lattice. If $f, g \in \mathcal{C}(L)$ and $c \in \mathbf{C}$ then $c(f \diamond g) = (cf) \diamond g$.*

Proof. Let $[x, y] \in \mathrm{Int}(L)$ and $f, g \in \mathcal{C}(L)$. If $c \in \mathbf{C}$, then

$$\begin{aligned}
\big(c(f \diamond g)\big)(x, y) &= c\big((f \diamond g)(x, y)\big) \\
&= c \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} f(x, z) g(x, w) \\
&= \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} cf(x, z) g(x, w) \\
&= \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} \big((cf)(x, z)\big) g(x, w) \\
&= \big((cf) \diamond g\big)(x, y).
\end{aligned}$$

□

Since $(\mathcal{C}(L), +, .)$ is a vector space, the previous lemmas complete the proof of the following:

Theorem 5.1. *If L is a locally finite lattice, then $(\mathcal{C}(L), +, \diamond, .)$ is a commutative algebra over $\mathbf{C}$.*

We shall call $(\mathcal{C}(L), +, \diamond, .)$ the *covering algebra* of L over $\mathbf{C}$.

The next two results give the multiplicative identity and characterize the units of the covering algebra of a locally finite lattice.

Proposition 5.1. *Let L be a locally finite lattice. The function $\delta \in \mathcal{C}(L)$ defined by*

$$\delta(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x < y \end{cases}$$

is the multiplicative identity of $(\mathcal{C}(L), +, \diamond, .)$.

Proof. Let $f \in \mathcal{C}(L)$ and $[x, y] \in \mathrm{Int}(L)$. Then

$$\begin{aligned}
(\delta \diamond f)(x, y) &= \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} \delta(x, z) f(x, w) \\
&= \delta(x, x) f(x, y) + \sum_{\substack{w \in [x,y] \\ x < z \leq y \\ z \vee w = y}} \delta(x, z) f(x, w)
\end{aligned}$$

$$= (1)f(x,y) + \sum_{\substack{w\in[x,y]\\ x<z\leq y\\ z\vee w=y}} (0)f(x,w)$$
$$= f(x,y).$$

□

Theorem 5.2. *Let L be a locally finite lattice and $f \in \mathcal{C}(L)$. The function f is a unit of $(\mathcal{C}(L),+,\diamond,.)$ if and only if for all $[x,y] \in \mathrm{Int}(L)$ the condition $\sum_{a\in[x,y]} f(x,a) \neq 0$ holds.*

Proof. Suppose that for all $[x,y] \in \mathrm{Int}(L)$ the condition $\sum_{a\in[x,y]} f(x,a) \neq 0$ holds.

Define a function $g \in \mathcal{C}(L)$ as follows: First define f on all intervals of length zero by setting

$$g(x,x) = \frac{1}{f(x,x)}$$

for all $x \in L$. Complete the definition of g recursively by specifying

$$g(x,y) = \left(\sum_{a\in[x,y]} f(x,a)\right)^{-1} \left(-\sum_{\substack{z,w\in[x,y]\\ x\leq w<y\\ z\vee w=y}} f(x,z)g(x,w)\right),$$

where $[x,y]$ is an interval in L of positive length n. Note that the hypothesis rules out the possibility of division by zero in this definition. (This is, of course, precisely the definition of g that results from solving the equation $(f\diamond g)(x,y) = \delta(x,y)$ for g.)

If $x \in L$, then

$$(f\diamond g)(x,x) = \sum_{\substack{z,w\in[x,x]\\ z\vee w=x}} f(x,z)g(x,w)$$
$$= f(x,x)g(x,x)$$
$$= f(x,x)\frac{1}{f(x,x)}$$
$$= 1.$$

If $[x,y] \in L$ with $x < y$, then

$$(f\diamond g)(x,y) = \sum_{\substack{z,w\in[x,y]\\ z\vee w=y}} f(x,z)g(x,w)$$

$$= \left(\sum_{z \in [x,y]} f(x,z) \right) g(x,y) + \sum_{\substack{z,w \in [x,y] \\ x \le w < y \\ z \vee w = y}} f(x,z) g(x,w)$$

$$= \frac{\left(\sum_{z \in [x,y]} f(x,z) \right)}{\left(\sum_{a \in [x,y]} f(x,a) \right)} \left(- \sum_{\substack{z,w \in [x,y] \\ x \le w < y \\ z \vee w = y}} f(x,z) g(x,w) \right)$$

$$+ \sum_{\substack{z,w \in [x,y] \\ x \le w < y \\ z \vee w = y}} f(x,z) g(x,w)$$

$$= - \sum_{\substack{z,w \in [x,y] \\ x \le w < y \\ z \vee w = y}} f(x,z) g(x,w) + \sum_{\substack{z,w \in [x,y] \\ x \le w < y \\ z \vee w = y}} f(x,z) g(x,w)$$

$$= 0.$$

In short, $f \diamond g = \delta$.

Conversely, suppose $f, g \in \mathcal{C}(L)$ with $f \diamond g = \delta$ and $\sum_{a \in [x,y]} f(x,a) = 0$ for some $[x,y] \in \mathrm{Int}(L)$. Then

$$\sum_{a \in [x,y]} (f \diamond g)(x,a) = \sum_{a \in [x,y]} \delta(x,a) = \delta(x,x) = 1.$$

On the other hand

$$\sum_{a \in [x,y]} (f \diamond g)(x,a) = \sum_{a \in [x,y]} \sum_{\substack{z,w \in [x,a] \\ z \vee w = a}} f(x,z) g(x,w)$$

$$= \sum_{z,w \in [x,y]} f(x,z) g(x,w)$$

$$= \left(\sum_{z \in [x,y]} f(x,z) \right) \left(\sum_{w \in [x,y]} g(x,w) \right)$$

$$= (0) \left(\sum_{w \in [x,y]} g(x,w) \right)$$

$$= 0.$$

This is a contradiction, so there must be no interval $[x,y] \in \mathrm{Int}(L)$ such that $\sum_{a\in[x,y]} f(x,a) = 0$.

□

The sequel shows the previous theorem to be a generalization of Theorem 4.8. In particular, the contradiction that proves the converse above can be adapted, albeit with much difficulty, to prove the corresponding implication in Theorem 4.8, and the "valuation" argument given there also has a counterpart here.

Let $f \in \mathcal{C}(L)$. We define powers of f under $\diamond$ by

$$f^{\diamond 0} = \delta$$
$$f^{\diamond 1} = f$$

and

$$f^{\diamond n} = \overbrace{f \diamond f \diamond \cdots \diamond f}^{n \text{ times}}.$$

for $n \geq 2$.

The function in $\mathcal{C}(L)$ which takes the value one on every interval of L has particular combinatorial significance. Call this the *zeta function* of $\mathcal{C}(L)$ and denote it ζ. As in the case of incidence algebras, powers of ζ count interesting quantities of L.

Proposition 5.2. *Let L be a locally finite lattice. For $n \in \mathbf{P}$ and $[x,y] \in \mathrm{Int}(L)$, define*

$$J(n;x,y) = \left|\left\{(z_1, z_2, \ldots, z_n) \;\middle|\; z_i \in [x,y] \text{ for } i \in [n], \text{ and } z_1 \vee z_2 \vee \cdots \vee z_n = y\right\}\right|.$$

Then $J(n;x,y) = \zeta^{\diamond n}(x,y)$.

Proof. Proceed by induction.

The result is trivial for $n = 1$. If $n = 2$, then

$$\begin{aligned}
\zeta^{\diamond 2}(x,y) &= (\zeta \diamond \zeta)(x,y) \\
&= \sum_{\substack{z,w\in[x,y] \\ z\vee w = y}} \zeta(x,z)\zeta(x,w) \\
&= \sum_{\substack{z,w\in[x,y] \\ z\vee w = y}} 1 \\
&= J(2;x,y).
\end{aligned}$$

Finally, if $\zeta^{\diamond n}(x,y) = J(n;x,y)$, then

$$\begin{aligned}
\zeta^{\diamond n+1}(x,y) &= (\zeta^{\diamond n} \diamond \zeta)(x,y) \\
&= \sum_{\substack{z,w\in[x,y] \\ z\vee w=y}} \zeta^{\diamond n}(x,z)\zeta(x,w) \\
&= \sum_{\substack{z,w\in[x,y] \\ z\vee w=y}} J(n;x,z)(1) \\
&= \sum_{\substack{z,w\in[x,y] \\ z\vee w=y}} \left(\sum_{\substack{z_1,\ldots,z_n\in[x,z] \\ z_1\vee\cdots\vee z_n=z}} 1 \right) \\
&= \sum_{\substack{z_1,\ldots,z_n,w\in[x,y] \\ z_1\vee\cdots\vee z_n\vee w=y}} 1 \\
&= J(n+1;x,y).
\end{aligned}$$

□

If $c \in \mathbf{C}$, then denote by c the element $c\delta$ of $\mathcal{C}(L)$, allowing context to clarify whether $c \in \mathbf{C}$ or $c \in \mathcal{C}(L)$. In particular, $\delta = 1 \in \mathcal{C}(L)$.

Proposition 5.3. *Let L be a locally finite lattice. If $n \in \mathbf{P}$ and $[x,y] \in \mathrm{Int}(L)$, then*

$$(\zeta - 1)^{\diamond n}(x,y) = \Big|\{(z_1,\ldots,z_n) \mid x < z_i \leq y \text{ for } i \in [n], \text{ and } z_1 \vee \cdots \vee z_n = y\}\Big|.$$

Proof. The proof differs from that of Proposition 5.2. in only minor details.

□

By Theorem 5.2 the zeta function has an inverse. We shall call this inverse the *Newton function* and denote it by ν. The Newton function plays a role in $(\mathcal{C}(L), +, \diamond, .)$ analogous to that played by the Möbius function in an incidence algebra.

If $f, g \in \mathcal{C}(L)$ with $g = \zeta \diamond f$, then $\nu \diamond g = \nu \diamond \zeta \diamond f = 1 \diamond f = f$. Conversely, if $f = \nu \diamond g$, then $\zeta \diamond f = \zeta \diamond \nu \diamond g = 1 \diamond g = g$. These observations prove the following inversion formula.

Theorem 5.3 (Newton Inversion). *Let $f, g \in \mathcal{C}(L)$. The following two propositions are equivalent:*

(i) For all $[x, y] \in \mathrm{Int}(L)$

$$g(x,y) = (\zeta \diamond f)(x,y) = \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} f(x,w).$$

(ii) For all $[x, y] \in \mathrm{Int}(L)$

$$f(x,y) = (\nu \diamond g)(x,y) = \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} \nu(x,z) g(x,w).$$

This result appears later in a special case that gives it a clearer combinatorial significance.

Solving $(\zeta \diamond \nu)(x, y) = \delta(x, y)$ for ν produces a recursive definition of ν. The next theorem gives the result.

Theorem 5.4. *If $\nu \in \mathcal{C}(L)$ has the recursive definition*

$$\nu(x,x) = 1 \qquad \text{for all } x \in L$$

and

$$\nu(x,y) = -\big|[x,y]\big|^{-1} \sum_{\substack{z,w \in [x,y] \\ x \le w < y \\ z \vee w = y}} \nu(x,w) \qquad \text{for all } x, y \in L \text{ with } x < y,$$

then $\zeta \diamond \nu = \delta$.

Proof. If $x \in L$, then

$$\begin{aligned}
(\zeta \diamond \nu)(x,x) &= \sum_{\substack{z,w \in [x,x] \\ z \vee w = x}} \zeta(x,z)\nu(x,w) \\
&= \zeta(x,x)\nu(x,x) \\
&= (1)(1) \\
&= \delta(x,x).
\end{aligned}$$

If $[x,y] \in \mathrm{Int}(L)$ with $x < y$, then

$$\begin{aligned}
(\zeta \diamond \nu)(x,y) &= \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} \zeta(x,z)\nu(x,w) \\
&= \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} \nu(x,w) \\
&= \sum_{z \in [x,y]} \nu(x,y) + \sum_{\substack{z,w \in [x,y] \\ x \leq w < y \\ z \vee w = y}} \nu(x,w) \\
&= \big|[x,y]\big|\nu(x,y) - \big|[x,y]\big|\nu(x,y) \\
&= 0 \\
&= \delta(x,y).
\end{aligned}$$

□

5.2 Reduced covering algebras

Let L be a q-lattice, and define $\mathcal{R}(L) = \big\{ f \in \mathcal{C}(L) \;\big|\; f(x,y) = f(x',y') \text{ whenever } [x,y],[x',y'] \in \mathrm{Int}(L) \text{ with } l(x,y) = l(x',y') \big\}$. Recall that $l(x,y)$ is the length of the interval $[x,y]$—that is, the length of a chain of maximum length in $[x,y]$. So $\mathcal{R}(L)$ is the set of all functions in $\mathcal{C}(L)$, the value of which, on a given interval, depends only on the length of the interval.

The set $\mathcal{R}(L)$ is precisely the subset of $\mathcal{C}(L)$ on which Doubilet, Rota, and Stanley [4, 16] define the reduced incidence algebra of the binomial poset L. They show that $(\mathcal{R}(L), +, .)$ is a subspace of the vector space $(\mathcal{C}(L), +, .)$. As the next lemma and theorem show, $\mathcal{R}(L)$ is also closed under the multiplication $\diamond$. So $(\mathcal{R}(L), +, \diamond, .)$ is a $\mathbf{C}$-subalgebra of $(\mathcal{C}(L), +, \diamond, .)$.

Lemma 5.5. *Let L be a q-lattice. If $f, g \in \mathcal{R}(L)$, then $f \diamond g \in \mathcal{R}(L)$.*

Proof. Let f, g be functions in $\mathcal{R}(L)$ and suppose that for all $n \in \mathbf{N}$ the value of f on an n-interval is a_n and the value of g on an n-interval is b_n, where $a_n, b_n \in \mathbf{C}$.

For some fixed $n \in \mathbf{N}$ suppose that $[x,y]$ and $[x',y']$ are both n-intervals in L. Then

$$(f \diamond g)(x,y) = \sum_{\substack{z,w \in [x,y] \\ z \vee w = y}} f(x,z)g(x,w)$$

$$
\begin{aligned}
&= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \sum_{\substack{z,w\in[x,y] \\ \rho(z)=r,\rho(w)=s \\ z\vee w=y}} f(x,z)g(x,w) \\
&= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \sum_{\substack{z,w\in[x,y] \\ \rho(z)=r,\rho(w)=s \\ z\vee w=y}} a_r b_s \\
&= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \begin{Bmatrix} n \\ r,s \end{Bmatrix}_q a_r b_s \\
&= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \sum_{\substack{z,w\in[x',y'] \\ \rho(z)=r,\rho(w)=s \\ z\vee w=y'}} f(x',z)g(x',w) \\
&= \sum_{\substack{z,w\in[x',y'] \\ z\vee w=y'}} f(x',z)g(x',w) \\
&= (f \diamond g)(x',y'),
\end{aligned}
$$

where $\begin{Bmatrix} n \\ r,s \end{Bmatrix}_q$ is a covering coefficient.

□

Theorem 5.5. *Let L be a q-lattice. Then the quadruple $(\mathcal{R}(L), +, \diamond, .)$ is a* **C***-subalgebra of the* **C***-algebra $(\mathcal{C}(L), +, \diamond, .)$.*

We shall call $(\mathcal{R}(L), +, \diamond, .)$ the *reduced covering algebra of L.*

5.3 Isomorphism to the algebra of q-binomial series

The appearance of covering coefficients in Lemma 5.5 and the dependence of q-binomial series and reduced covering algebras on q-lattices suggest a connection between these two algebras. Then the two algebras are, in fact, isomorphic.

Theorem 5.6. *Let L be a q-lattice. Define a mapping $\hat{}: \mathcal{R}(L) \to \mathbf{C}^{\mathbf{N}}$ as follows: If $f \in \mathcal{R}(L)$, then define $\hat{f} \in \mathbf{C}^{\mathbf{N}}$ by $\hat{f}(n) = f(x,y)$, where $n \in \mathbf{N}$ and $[x,y]$ is an n-interval in L. Then $\hat{}$ is an isomorphism between $(\mathcal{R}(L), +, \diamond, .)$ and $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$.*

Proof. The mapping $\hat{}$ is clearly a bijection.

Let $n \in \mathbf{N}$ and let $[x,y]$ be an n-interval in L.

If $f, g \in \mathcal{R}(L)$, then

$$\begin{aligned}\widehat{(f+g)}(n) &= (f+g)(x,y) \\ &= f(x,y) + g(x,y) \\ &= \hat{f}(n) + \hat{g}(n) \\ &= (\hat{f} + \hat{g})(n),\end{aligned}$$

so $\widehat{\ }$ preserves addition.

If $f, g \in \mathcal{R}(L)$, then by the proof of Lemma 5.5

$$\begin{aligned}\widehat{(f \diamond g)}(n) &= (f \diamond g)(x,y) \\ &= \sum_{r=0}^{n} \sum_{s=n-r}^{n} \left\{ \begin{matrix} n \\ r, s \end{matrix} \right\}_q \hat{f}(r)\hat{g}(s) \\ &= (\hat{f} \diamond \hat{g})(n),\end{aligned}$$

so $\widehat{\ }$ preserves multiplication.

If $f \in \mathcal{R}(L)$ and $c \in \mathbf{C}$, then

$$\widehat{(cf)}(n) = (cf)(x,y) = c\big(f(x,y)\big) = c\big(\hat{f}(n)\big) = (c\hat{f})(n),$$

so $\widehat{\ }$ preserves scalar multiplication.

□

This result parallels that of Doubilet, Rota, and Stanley [4, pp. 280-281], [16, p. 144] showing the isomorphism between an algebra of formal power series and the reduced incidence algebra of its corresponding binomial poset.

The reduced covering algebra also behaves nicely with respect to multiplicative inverses.

Theorem 5.7. *Let L be a q-lattice, and let $f, g \in \mathcal{C}(L)$ such that $f \diamond g = \delta$. If $f \in \mathcal{R}(L)$, then $g \in \mathcal{R}(L)$. In other words, if $f \in \mathcal{R}(L)$ is invertible in $(\mathcal{C}(L), +, \diamond, .)$, then its inverse is in $(\mathcal{R}(L), +, \diamond, .)$.*

Proof. Let $f \in \mathcal{R}(L)$ and $g \in \mathcal{C}(L)$ with $f \diamond g = \delta$. If $[x,y] \in \mathrm{Int}(L)$ is an n-interval for some $n \in \mathbf{N}$, then by Theorem 5.2

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \hat{f}(r) = \sum_{z \in [x,y]} f(x,z) \neq 0.$$

By Theorem 4.8, then, $\hat{f}$ is invertible in $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$. But $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ is isomorphic to $(\mathcal{R}(L), +, \diamond, .)$, so f is invertible in $(\mathcal{R}(L), +, \diamond, .)$. Since $\delta \in \mathcal{R}(L)$ and $f \diamond g = \delta$, the function g must be the inverse of f in $(\mathcal{R}(L), +, \diamond, .)$. In particular, $g \in \mathcal{R}(L)$.

□

Corollary. *Let L be a q-lattice, and let ζ and ν be the zeta-function and Newton function of $\mathcal{C}(L)$ respectively. Then $\zeta, \nu \in \mathcal{R}(L)$.*

Proof. Since ζ is in $\mathcal{R}(L)$ by definition, Theorem 5.7 also guarantees that $\nu \in \mathcal{R}(L)$. □

This corollary shows that under the given conditions the functions ζ and ν have images in $\mathbf{C}^{\mathbf{N}}$ under the isomorphism of Theorem 5.6. As a result, the algebra $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ also has a Newton inversion formula as the next section shows.

6. Newton Inversion

6.1 Arithmetic functions with the Schur product

The long-awaited third way to give $\mathbf{C}^{\mathbf{N}}$ the structure of a $\mathbf{C}$-algebra involves *Schur* (i.e. "pointwise") multiplication. Namely, if $F, G \in \mathbf{C}^{\mathbf{N}}$, define

$$(F \cdot G)(n) = F(n)G(n) \qquad \text{for all } n \in \mathbf{N}.$$

The quadruple $(\mathbf{C}^{\mathbf{N}}, +, \cdot, .)$ is well-known to be a $\mathbf{C}$-algebra. This algebra has the particularly nice property that zero divisors, units, and multiplicative inverses are recognizable by inspection.

Oddly enough, this third way of giving $\mathbf{C}^{\mathbf{N}}$ the structure of a $\mathbf{C}$-algebra is not algebraically distinct from the second way. That is, the algebras $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ and $(\mathbf{C}^{\mathbf{N}}, +, \cdot, .)$ are isomorphic! The following lemmas lead to a proof of the isomorphism.

Lemma 6.1. *Let L be a q-lattice. If $n \in \mathbf{N}$, then*

$$\sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{6.1}$$

Proof. If $n = 0$, then the left-hand side of (6.1) is

$$(-1)^0 q^0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$$

with the convention that $0^0 = 1$ in the case $q = 0$.

If $n > 0$, then by equation (3.7)

$$\begin{aligned}
\sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} &= \sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \left(\begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} \right) \\
&= \sum_{r=1}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \sum_{r=0}^{n-1} (-1)^r q^{r(r-1)/2} q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} \\
&= \sum_{r=0}^{n-1} (-1)^{r+1} q^{(r+1)(r)/2} \begin{bmatrix} n-1 \\ r \end{bmatrix} \\
&\qquad + \sum_{r=0}^{n-1} (-1)^r q^{(r+1)(r)/2} \begin{bmatrix} n-1 \\ r \end{bmatrix} \\
&= 0.
\end{aligned}$$

□

Lemma 6.2. *Let L be a q-lattice and let $n, r, s \in \mathbf{N}$ with $r \leq s \leq n$. Then*

$$\begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n-r \\ s-r \end{bmatrix}. \tag{6.2}$$

Proof. Let $[x, y]$ be an n-interval in L.

Both sides of (6.2) count the number of ways to form a chain $x \leq z \leq w \leq y$ such that $\rho(z) = r$ and $\rho(w) = s$. There are $\begin{bmatrix} n \\ s \end{bmatrix}$ ways to choose w and then $\begin{bmatrix} s \\ r \end{bmatrix}$ ways to choose $z \in [x, w]$. On the other hand there are $\begin{bmatrix} n \\ r \end{bmatrix}$ ways to choose z and then $\begin{bmatrix} n-r \\ s-r \end{bmatrix}$ ways to choose $w \in [x, y]$.

□

The next lemma gives a q-binomial inversion formula which encompasses both the usual binomial inversion and its q-analog as given by Goldman and Rota [6, p.244]. It is, thus, not a new result, but its proof requires no reference to sets or finite vector spaces, growing instead out of the properties of q-lattices.

Lemma 6.3 (q-binomial inversion). *Let L be a q-lattice. If the constants $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are all complex numbers, then the following two statements are equivalent:*

(i)

$$b_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} a_i \qquad \text{for all } n \in \mathbf{N},$$

and

(ii)

$$a_n = \sum_{i=0}^{n} (-1)^{n-i} q^{(n-i)(n-i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} b_i \qquad \text{for all } n \in \mathbf{N}.$$

Proof. Suppose the second statement holds. Then

$$\begin{aligned} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} a_i &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{i} (-1)^{i-j} q^{(i-j)(i-j-1)/2} \begin{bmatrix} i \\ j \end{bmatrix} b_j. \\ &= \sum_{j=0}^{n} b_j \sum_{i=j}^{n} (-1)^{i-j} q^{(i-j)(i-j-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} \\ &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} b_j \sum_{i=j}^{n} (-1)^{i-j} q^{(i-j)(i-j-1)/2} \begin{bmatrix} n-j \\ i-j \end{bmatrix} \\ &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} b_j \sum_{i=0}^{n-j} (-1)^{i} q^{i(i-1)/2} \begin{bmatrix} n-j \\ i \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} n \\ n \end{bmatrix} b_n$$
$$= b_n.$$

The next-to-last equality follows from Lemma 6.1. The identity $\begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n-j \\ i-j \end{bmatrix}$ is an application of Lemma 6.2.

The converse follows in a similar fashion, the identity $\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n \\ n-i \end{bmatrix}$ being useful in performing the necessary algebraic manipulations.

□

Theorem 6.1. *Let $\diamond$ be a q-Newton product and define a mapping $\theta\colon \mathbf{C}^{\mathbf{N}} \to \mathbf{C}^{\mathbf{N}}$ by*

$$\big(\theta(F)\big)(n) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) \tag{6.3}$$

for all $F \in \mathbf{C}^{\mathbf{N}}$ and $n \in \mathbf{N}$. Then θ is a $\mathbf{C}$-algebra isomorphism from $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ to $(\mathbf{C}^{\mathbf{N}}, +, \cdot, .)$. Further the inverse of θ is given by

$$\big(\theta^{-1}(G)\big)(n) = \sum_{i=0}^{n} (-1)^{n-i} q^{(n-i)(n-i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} G(i) \tag{6.4}$$

for all $G \in \mathbf{C}^{\mathbf{N}}$ and $n \in \mathbf{N}$.

Proof. Let $F \in \mathbf{C}^{\mathbf{N}}$. By Lemma 6.3 the equations $\theta^{-1}\big(\theta(F)\big) = F$ and $\theta\big(\theta^{-1}(F)\big) = F$ hold. Thus θ is bijective, and θ^{-1} is the inverse of θ.

Let $F, G \in \mathbf{C}^{\mathbf{N}}$ and $c \in \mathbf{C}$. If $n \in \mathbf{N}$, then

$$\begin{aligned} \big(\theta(F+G)\big)(n) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (F+G)(i) \\ &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) + \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} G(i) \\ &= \big(\theta(F)\big)(n) + \big(\theta(G)\big)(n) \\ &= \big(\theta(F) + \theta(G)\big)(n) \end{aligned}$$

and

$$\begin{aligned} \big(\theta(cf)\big)(n) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (cF)(i) \\ &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} c\big(F(i)\big) \\ &= c \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} F(i) \\ &= \big(c\theta(F)\big)(n) \end{aligned}$$

and

$$\begin{aligned}
\big(\theta(F \diamond G)\big)(n) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (F \diamond G)(i) \\
&= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \left(\sum_{r=0}^{i} \sum_{s=i-r}^{i} \begin{Bmatrix} i \\ r, s \end{Bmatrix}_q F(r)G(s) \right) \\
&= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \left(\sum_{r=0}^{n} \sum_{s=0}^{n} \begin{Bmatrix} i \\ r, s \end{Bmatrix}_q F(r)G(s) \right) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{Bmatrix} i \\ r, s \end{Bmatrix}_q \right) F(r)G(s) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} F(r)G(s) \qquad \text{(by equation (4.16))} \\
&= \left(\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} F(r) \right) \left(\sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} G(s) \right) \\
&= \big(\theta(F) \cdot \theta(G)\big)(n).
\end{aligned}$$

In other words, θ preserves addition, scalar multiplication, and multiplication.

Therefore θ is an isomorphism.

□

Since the units of $(\mathbf{C}^{\mathbf{N}}, +, \cdot, .)$ are the sequences with no nonzero terms, Theorem 6.1 yields Theorem 4.8 as a corollary.

6.2 A Newton inversion formula

Let L be a q-lattice, and let $Z, N \in \mathbf{C}^{\mathbf{N}}$ be the respective images under the isomorphism of Theorem 5.6 of the zeta-function and the Newton function of $\mathcal{R}(L)$. In other words, let $Z = \hat{\zeta}$ and $N = \hat{\nu}$. Call Z the zeta-function of $\mathbf{C}^{\mathbf{N}}$ and N the Newton function of $\mathbf{C}^{\mathbf{N}}$.

The zeta-function of $\mathbf{C}^{\mathbf{N}}$ has a simple description, namely

$$Z(n) = 1 \qquad \text{for all } n \in \mathbf{N}. \tag{6.5}$$

On the other hand the recursive formula for ν in Theorem 5.4 does not seem to produce a very enlightening formula for N.

As a means to finding a more helpful formula, for all $n \in \mathbf{N}$ define the quantity

$$G_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \tag{6.6}$$

If q is a power of a prime, then G_n is known as the nth *Galois number*. That is, G_n is the number of subspaces of an n-dimensional vector space over the finite field F_q. (See Goldman and Rota [7] for an investigation of the Galois numbers.). If $q = 1$, then G_n is simply the number of subsets of an n-set. That is, $G_n = 2^n$.

Let θ be the isomorphism of Theorem 6.1. Then $N = \theta^{-1}\big(\theta(Z)^{-1}\big)$, where $\theta(Z)^{-1}$ is the inverse of $\theta(Z)$ in $(\mathbf{C}^{\mathbf{N}}, +, \cdot, .)$.

If $n \in \mathbf{N}$, then

$$\begin{aligned} \theta(Z)^{-1}(n) &= \frac{1}{\displaystyle\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} Z(i)} \\ &= \frac{1}{\displaystyle\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}} \\ &= \frac{1}{G_n}. \end{aligned}$$

By Theorem 6.1, then, if $n \in \mathbf{N}$,

$$\begin{aligned} N(n) &= \theta^{-1}\big(\theta(Z)^{-1}\big)(n) \\ &= \sum_{i=0}^{n} (-1)^{n-i} q^{(n-i)(n-i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} \frac{1}{G_i}. \end{aligned}$$

This observation constitutes the proof of the following theorem.

Theorem 6.2. *If L is a q-lattice and $n \in \mathbf{N}$, then*

$$N(n) = \sum_{i=0}^{n} (-1)^{n-i} q^{(n-i)(n-i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} \frac{1}{G_i}, \tag{6.7}$$

where $G_i = \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix}$.

Corollary. *If $q = 1$, then*

$$N(n) = \left(-\frac{1}{2}\right)^n.$$

Proof. Let $n, i \in \mathbf{N}$. If $q = 1$, then $\begin{bmatrix} n \\ i \end{bmatrix} = \binom{n}{i}$ by equation (3.6). Thus

$$N(n) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{1}{2^i} = \left(-1 + \frac{1}{2}\right)^n = \left(-\frac{1}{2}\right)^n$$

by the binomial theorem. In this case, of course, $L =$ the lattice of finite subsets of an infinite set.

□

The Newton inversion formula of Theorem 5.3 also has, in effect, a special case in terms of Z and N.

Theorem 6.3. *Let $\diamond$ be a q-Newton product and $F, G \in \mathbf{C}^{\mathbf{N}}$. The following statements are equivalent:*

(i) For all $n \in \mathbf{N}$,

$$G(n) = (Z \diamond F)(n) = \sum_{r=0}^{n} \sum_{s=n-r}^{n} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q F(s).$$

(ii) For all $n \in \mathbf{N}$,

$$F(n) = (N \diamond G)(n) = \sum_{r=0}^{n} \sum_{s=n-r}^{n} \begin{Bmatrix} n \\ r, s \end{Bmatrix}_q N(r) G(s).$$

Proof. Let $f, g \in \mathcal{R}(L)$ such that $F = \hat{f}$ and $G = \hat{g}$. Since ζ and ν are also in $\mathcal{R}(L)$, apply the isomorphism $\hat{}$ to both statements in Theorem 5.3.

□

7. Combinatorial Results

7.1 Old results

Several examples have already illustrated how q-lattice identities generalize identities involving binomial coefficients or q-binomial coefficients. It remains to show how q-binomial series yield the results of Theorems 1.1–1.3 as special cases.

Let $\diamond$ be a q-Newton product with $q > 0$, and for $c \in \mathbf{C}$, let $\phi'_c: \mathbf{C}^{\mathbf{N}} \to \mathbf{C}$ be the evaluation homomorphism from $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ to $\mathbf{C}$. Since ϕ'_c is defined by way of the usual evaluation homomorphism for polynomials and the isomorphism ψ (see Theorem 4.6) between $(\mathcal{P}, +, \diamond, .)$ and $(\mathcal{P}, +, *, .)$, the mapping ϕ'_c has most of the properties of the usual evaluation homomorphism.

If $k, n \in \mathbf{N}$, then

$$\phi'_{k_q}\left(\left\langle {X \atop n} \right\rangle_q\right) = \frac{k(k-1_q)\dots(k-(n-1)_q)}{n(n-1_q)\dots(n-(n-1)_q)}.$$

So if q is a power of a prime,

$$\phi'_{k_q}\left(\left\langle {X \atop n} \right\rangle_q\right) = \binom{k}{n}_q, \tag{7.1}$$

and if $q = 1$,

$$\phi'_{k_q}\left(\left\langle {X \atop n} \right\rangle_q\right) = \binom{k}{n}. \tag{7.2}$$

The latter statement follows from the observation that $n_q = (q^{n-1} + q^{n-2} + \cdots + 1) = n$ if $q = 1$.

If q is a prime power, then the two definitions of the symbol $\left\{ {n \atop r,s} \right\}_q$ given in Section 1.2 and Section 4.1 are clearly equivalent. Further if $q = 1$, then the definition of the covering coefficient $\left\{ {n \atop r,s} \right\}_q$ is clearly equivalent to the definition of $\left\{ {n \atop r,s} \right\}$ given in Section 1.1.

If $r, s \in \mathbf{N}$, then equation (4.25) yields the special case

$$\left\langle {X \atop r} \right\rangle_q \diamond \left\langle {X \atop s} \right\rangle_q = \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\}_q \left\langle {X \atop n} \right\rangle_q. \tag{7.3}$$

If q is a power of a prime and $k \in \mathbf{N}$, equation (7.3) yields

$$\begin{aligned}\binom{k}{r}_q \binom{k}{s}_q &= \phi'_{k_q}\left(\left\langle {X \atop r}\right\rangle_q \diamond \left\langle {X \atop s}\right\rangle_q\right) \\ &= \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\}_q \phi'_{k_q}\left(\left\langle {X \atop n}\right\rangle_q\right) \\ &= \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\}_q \binom{k}{n}_q,\end{aligned}$$

which is Theorem 1.3. Similarly, if $q = 1$ and $k \in \mathbf{N}$, then (7.3) yields

$$\begin{aligned}\binom{k}{r}\binom{k}{s} &= \phi'_{k_q}\left(\left\langle {X \atop r}\right\rangle_q \diamond \left\langle {X \atop s}\right\rangle_q\right) \\ &= \sum_{n=0}^{r+s} \left\{ {n \atop r,s} \right\} \binom{k}{n},\end{aligned}$$

which is Theorem 1.1.

Another special case of equation (4.25) is the statement

$$\left(\sum_{i=0}^{m} a_i \left\langle {X \atop i}\right\rangle_q\right) \diamond \left(\sum_{i=0}^{n} b_i \left\langle {X \atop i}\right\rangle_q\right) = \sum_{i=0}^{m+n} \left(\sum_{r=0}^{i} \sum_{s=i-r}^{i} \left\{ {i \atop r,s} \right\}_q a_r b_s\right) \left\langle {X \atop i}\right\rangle_q, \quad (7.4)$$

where $m, n \in \mathbf{N}$ and the constants $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n$ are all complex constants. The degree argument to establish $m + n$ as the upper limit of the outer summation on the right-hand side follows easily from equation (4.25).

If $q = 1$, then Theorem 1.2 follows immediately by application of the isomorphism ψ from Theorem 4.6 to both sides of (7.4).

In short, q-binomial series are indeed the infinite series which Theorems 1.1–1.3 initially provoked us to seek.

7.2 Generating functions

Finally, then, it remains to show the usefulness of q-binomial series as generating functions. The following technical lemma leads immediately to the desired results.

Lemma 7.1. *Let $\diamond$ be a q-Newton product with $q > 0$, and let $k \in \mathbf{N}$ with $k \geq 2$. If $\sum_{n=0}^{\infty} a_{n,i} \left\langle {X \atop n} \right\rangle_q \in \mathbf{C}^{\mathbf{N}}$ for all $i \in [k]$, then*

$$\prod_{i=1}^{k} \left(\sum_{n=0}^{\infty} a_{n,i} \left\langle {X \atop n} \right\rangle_q \right) = \sum_{n=0}^{\infty} \left(\sum_{r_1=0}^{n} \cdots \sum_{r_k=0}^{n} \left\{ {n \atop r_1, \ldots, r_k} \right\}_q a_{r_1,1} \ldots a_{r_k,k} \right) \left\langle {X \atop n} \right\rangle_q, \quad (7.5)$$

where the product symbol here indicates a q-Newton product and $\left\{ {n \atop r_1,\ldots,r_k} \right\}_q$ is the multi-covering coefficient of equation (4.6).

Proof. The lemma holds for $k = 2$ by equation (4.26).

Proceed by induction. If the lemma holds for some $k \in \mathbf{N}$ with $k \geq 2$, then

$$\begin{aligned}
&\prod_{i=1}^{k+1} \left(\sum_{n=0}^{\infty} a_{n,i} \left\langle {X \atop n} \right\rangle_q \right) \\
&= \left(\prod_{i=1}^{k} \left(\sum_{n=0}^{\infty} a_{n,i} \left\langle {X \atop n} \right\rangle_q \right) \right) \diamond \left(\sum_{n=0}^{\infty} a_{n,k+1} \left\langle {X \atop n} \right\rangle_q \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ {n \atop r, s} \right\}_q \left(\sum_{r_1=0}^{r} \cdots \sum_{r_k=0}^{r} \left\{ {r \atop r_1, \ldots, r_k} \right\}_q a_{r_1,1} \ldots a_{r_k,k} \right) a_{s,k+1} \right) \left\langle {X \atop n} \right\rangle_q \\
&= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \sum_{s=0}^{n} \left\{ {n \atop r, s} \right\}_q \left(\sum_{r_1=0}^{n} \cdots \sum_{r_k=0}^{n} \left\{ {r \atop r_1, \ldots, r_k} \right\}_q a_{r_1,1} \ldots a_{r_k,k} a_{s,k+1} \right) \right) \left\langle {X \atop n} \right\rangle_q \\
&= \sum_{n=0}^{\infty} \left(\sum_{r_1=0}^{n} \cdots \sum_{r_k=0}^{n} \sum_{s=0}^{n} a_{r_1,1} \ldots a_{r_k,k} a_{s,k+1} \sum_{r=0}^{n} \left\{ {r \atop r_1, \ldots, r_k} \right\}_q \left\{ {n \atop r, s} \right\}_q \right) \left\langle {X \atop n} \right\rangle_q \\
&= \sum_{n=0}^{\infty} \left(\sum_{r_1=0}^{n} \cdots \sum_{r_k=0}^{n} \sum_{s=0}^{n} \left\{ {n \atop r_1, \ldots, r_k, s} \right\}_q a_{r_1,1} \ldots a_{r_k,k} a_{s,k_1} \right) \left\langle {X \atop n} \right\rangle_q .
\end{aligned}$$

The last equality follows from formula (4.7).

□

Let L be a q-lattice with $q > 0$, and let $[x, y]$ be an n-interval in L for some $n \in \mathbf{N}$. For $k \in \mathbf{N}$ with $q \geq 2$ define

$$C_0(n, k) = \left| \{ (z_1, \ldots, z_k) \mid x \leq z_i \leq y \text{ for } i \in [k] \text{ and } z_1 \vee \cdots \vee z_k = y \} \right|$$

and

$$C(n,k) = \Big|\{(z_1,\ldots,z_k) \mid x < z_i \le y \text{ for } i \in [k] \text{ and } z_1 \vee \cdots \vee z_k = y\}\Big|.$$

From the definition of multi-covering coefficients in equation (4.6) it is clear that

$$C_0(n,k) = \sum_{r_1=0}^{n} \cdots \sum_{r_k=0}^{n} \begin{Bmatrix} n \\ r_1,\ldots,r_k \end{Bmatrix}_q$$

and

$$C(n,k) = \sum_{r_1=1}^{n} \cdots \sum_{r_k=1}^{n} \begin{Bmatrix} n \\ r_1,\ldots,r_k \end{Bmatrix}_q,$$

and thus these quantities are independent of the particular interval $[x,y]$. Call the k-tuples that $C(n,k)$ counts the *ordered k-covers of an n-interval in L* and those that $C_0(n,k)$ counts the ***relaxed ordered k-covers of an n-interval in L.***

Let Z be the zeta-function of $\mathbf{C}^{\mathbf{N}}$. The following corollaries of Lemma 7.1 demonstrate the combinatorial significance of that lemma. They also provide our first example of q-binomial series as generating functions.

Corollary. *Let $\diamond$ be a q-Newton product with $q > 0$, and let $k \in \mathbf{N}$ with $k \ge 2$. Then*

$$Z^{\diamond k} = \left(\sum_{n=0}^{\infty} \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q\right)^{\diamond k} = \sum_{n=0}^{\infty} C_0(n,k) \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q. \tag{7.6}$$

Proof. In Lemma 7.1 let $a_{n,i} = 1$ for all $n \in \mathbf{N}$ and $i \in [k]$.

□

Corollary. *Let $\diamond$ be a q-Newton product with $q > 0$, and let $k \in \mathbf{N}$ with $k \ge 2$. Then*

$$(Z-1)^{\diamond k} = \left(\sum_{n=1}^{\infty} \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q\right)^{\diamond k} = \sum_{n=1}^{\infty} C(n,k) \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q. \tag{7.7}$$

Proof. In Lemma 7.1 let $a_{0,i} = 0$ and $a_{n,i} = 1$ for all $n \in \mathbf{P}$ and $i \in [k]$.

□

The above corollaries also follow from propositions 5.2 and 5.3 by way of the isomorphism $\hat{}$ in Theorem 5.6.

If $L =$ (the lattice of finite subsets of an infinite set), then $C(n,k)$ is simply the number of ordered k-covers (in the usual sense—i.e., the sense of Appendix A)

of an n-set, and $C_0(n,k)$ is the number of ordered k-covers of an n-set in which the empty set is allowed as a block. That is, from equation (A1.2),

$$C(n,k) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{2^j - 1}{k}$$

and

$$C_0(n,k) = (2^k - 1)^n.$$

These interpretations of $C(n,k)$ and $C_0(n,k)$ follow from the fact that the join operation in L is simply the union of sets.

Similarly if L = (the lattice of finite-dimensional subspaces of an infinite-dimensional vector space over the finite field F_q), where q is a power of a prime, then $C_0(n,k)$ is the number of ways of writing an n-space as an ordered sum of subspaces involving k summands. The quantity $C(n,k)$ counts the same ordered sums with the additional restriction that all summands must have nonzero dimension.

Thus $(Z-1)^{\diamond k}$ and $Z^{\diamond k}$ are generating functions for k-covers and relaxed k-covers of an n-interval. These results also explain the apparent similarity between ordered covers of a set and ordered decompositions of a finite vector space as a sum of subspaces (see Appendix B). Both are special cases of a more general construction.

Again let L be a q-lattice with $q > 0$, and let $[x,y]$ be an n-interval in L for some $n \in \mathbf{N}$. For $k \in \mathbf{N}$ with $k \geq 2$, let $A_1, \ldots, A_k \subseteq \mathbf{N}$, and define

$$C(n; A_1, \ldots, A_k) = \left| \{ (z_1, \ldots, z_k) \mid x \leq z_i \leq y \text{ and } \rho(z_i) \in A_i \text{ for } i \in [k] \text{ and } z_1 \vee \cdots \vee z_k = y \} \right|.$$

So $C(n; A_1, \ldots, A_k)$ counts ordered k-covers of an n-interval in which the rank of each term (i.e. the "size" of each block) is restricted to a particular subset of $\mathbf{N}$. As with $C_0(n,k)$ and $C(n,k)$, the quantity $C(n; A_1, \ldots, A_k)$ depends only on the length of $[x,y]$ and not on the particular interval itself since clearly

$$C(n; A_1, \ldots, A_k) = \sum_{\substack{r_1 \in A_1 \\ r_1 \leq n}} \cdots \sum_{\substack{r_k \in A_k \\ r_k \leq n}} \begin{Bmatrix} n \\ r_1, \ldots, r_k \end{Bmatrix}_q.$$

Lemma 7.1 produces the following very general result.

Theorem 7.1. *Let $\diamond$ be a q-Newton product with $q > 0$. For $k \in \mathbf{N}$ with $k \geq 2$ let $A_1, \ldots, A_k \subseteq \mathbf{N}$, and for $i \in [k]$ define $\chi_{A_i} : \mathbf{N} \to \{0,1\}$ by*

$$\chi_{A_i}(n) = \begin{cases} 1, & \text{if } n \in A_i \\ 0, & \text{if } n \notin A_i \end{cases}. \tag{7.8}$$

Then

$$\prod_{i=1}^{k}\left(\sum_{n=0}^{\infty}\chi_{A_i}(n)\left\langle {X \atop n}\right\rangle_q\right)=\sum_{n=0}^{\infty}C(n;A_1,\ldots,A_k)\left\langle {X \atop n}\right\rangle_q. \tag{7.9}$$

Proof. By Lemma 7.1

$$\begin{aligned}
&\prod_{n=1}^{k}\left(\sum_{n=0}^{\infty}\chi_{A_i}(n)\left\langle {X \atop n}\right\rangle_q\right)\\
&\qquad=\sum_{n=0}^{\infty}\left(\sum_{r_1=0}^{n}\cdots\sum_{r_k=0}^{n}\left\{ {n \atop r_1,\ldots,r_k}\right\}_q\chi_{A_1}(r_1)\cdots\chi_{A_k}(r_k)\right)\left\langle {X \atop n}\right\rangle_q\\
&\qquad=\sum_{n=0}^{\infty}\left(\sum_{\substack{r_1\in A_1\\ r_1\le n}}\cdots\sum_{\substack{r_k\in A_k\\ r_k\le n}}\left\{ {n \atop r_1,\ldots,r_k}\right\}_q\right)\left\langle {X \atop n}\right\rangle_q\\
&\qquad=\sum_{n=0}^{\infty}C(n;A_1,\ldots,A_k)\left\langle {X \atop n}\right\rangle_q.
\end{aligned}$$

□

8. Problems and Unanswered Questions

The results in this dissertation open the door for several lines of further research. They also raise a number of questions that they fail to answer. The following discussion briefly describes some of the possibilities and problems.

Thus far q-binomial series are purely formal (i.e. algebraic) objects. It would be useful to know whether they converge in any meaningful sense when the variable X is replaced by a suitable complex value z. The obvious approach is to limit z to an open subset of $\mathbf{C}$ and seek convergence in the usual sense in the complex plane. Another possibility is to restrict z to a q-adic number field and seek convergence in the q-adic norm. The goal, of course, is to apply the techniques of analysis to q-binomial series—a useful approach for obtaining asymptotic estimates of the growth of coefficients of generating functions (in the usual sense of power series).

The appearance of interesting combinatorial quantities (i.e. the covering coefficients $\left\{ {n \atop r,s} \right\}_q$) in products of q-binomial series—or equivalently, in products of functions in the reduced covering algebra $\mathcal{R}(L)$—occurs by design. That is, the definition of multiplication makes these quantities appear. Would other such "contrived" multiplications produce other interesting coefficients?

A related question arises from the following description of the development of q-binomial series for $q = 1$. For $n \in \mathbf{N}$ define $P_n \in \mathcal{P}$ by

$$P_0 = 1$$

and

$$P_n = X * (X-1) * \cdots * \big(X-(n-1)\big) \qquad \text{if } n \geq 1.$$

If $a_0, a_1, \ldots, a_N \in \mathbf{C}$ for some $N \in \mathbf{N}$ and if ϕ_c is the evaluation homomorphism for some $c \in \mathbf{C}$, then Theorem 1.2 provides a simple rule for multiplying objects of the form

$$\sum_{n=0}^{N} a_n \frac{P_n}{\phi_n(P_n)}.$$

This simple rule suggests a new multiplication on $\mathcal{P}$, namely the q-Newton product $\diamond$. The mapping $\psi\colon \mathcal{P} \to \mathcal{P}$ with the nice definition

$$\psi\big((a_0, a_1, \ldots, a_n, 0, 0, \ldots)\big) = \sum_{n=0}^{N} a_n \frac{P_n}{\phi_n(P_n)}^{\bullet}$$

turns out to be an isomorphism between $(\mathcal{P}, +, \diamond, .)$ and $(\mathcal{P}, +, *, .)$. Finally, the q-Newton product $\diamond$ has a natural extension to $\mathbf{C}^{\mathbf{N}}$ that produces an algebra $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$ with nice combinatorial properties.

The question, then, is whether another collection of polynomials, say $Q_n \in \mathcal{P}$ for $n \in \mathbf{N}$, induces the same process. That is, are there nice rules for multiplying objects of the form

$$\sum_{n=0}^{N} a_n \frac{Q_n}{\phi_n(Q_n)}$$

that lead to other combinatorially-interesting algebras on $\mathcal{P}$ and $\mathbf{C}^{\mathbf{N}}$?

One obvious attempt begins with the definition

$$\begin{bmatrix} X \\ n \end{bmatrix}_q = \frac{(X-1)*(X-q)*\cdots*(X-q^{n-1})}{(q^n-1)(q^n-q)\ldots(q^n-q^{n-1})} \tag{8.1}$$

where $n \in \mathbf{N}$ and q is a power of a prime. For $k \in \mathbf{N}$,

$$\phi_{q^k}\left(\begin{bmatrix} X \\ n \end{bmatrix}_q\right) = \binom{k}{n}_q,$$

a q-binomial coefficient. By Theorem 1.3 this yields the polynomial identity

$$\begin{bmatrix} X \\ r \end{bmatrix}_q \begin{bmatrix} X \\ s \end{bmatrix}_q = \sum_{n=0}^{r+s} \left\{ \begin{matrix} n \\ r,s \end{matrix} \right\}_q \begin{bmatrix} X \\ n \end{bmatrix}_q, \tag{8.2}$$

where $\left\{ \begin{matrix} n \\ r,s \end{matrix} \right\}_q$ is the usual covering coefficient. Equation (7.3) produces the same equation when combined with the observation that

$$\phi_{q^n}\left(\begin{bmatrix} X \\ n \end{bmatrix}_q\right) = \phi'_{n_q}\left(\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q\right)_q.$$

Indeed it appears that most of the development of q-binomial series could also be done using $\begin{bmatrix} X \\ n \end{bmatrix}_q$ instead of $\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q$, but with the devastating defect that $\begin{bmatrix} X \\ n \end{bmatrix}_q$ is undefined for $q = 1$.

The usual finite difference operator Δ has a q-analog, namely the Eulerian derivative. A natural question is whether the Eulerian derivative behaves "nicely" on $(\mathcal{P}, +, \diamond, .)$. The study of that question requires the following definitions.

Let $\diamond$ be a q-Newton product with $q > 0$, and let $F = (a_0, a_1, \ldots, a_N, 0, 0, \ldots) \in \mathcal{P}$ for some $N \in \mathbf{N}$. So

$$F = \sum_{n=0}^{N} a_n X^{*n} = \sum_{n=0}^{N} a_n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q.$$

Further, if $P \in \mathcal{P}$, define

$$\left\langle \begin{matrix} P \\ n \end{matrix} \right\rangle_q = \frac{1}{n_q^{\underline{n}}}\left(P \diamond (P - 1_q) \diamond \cdots \diamond (P - (n-1)_q)\right) \qquad \text{for all } n \in \mathbf{N}. \tag{8.3}$$

Finally, define

$$F^*(P) = \sum_{n=0}^{N} a_n P^{*n} \tag{8.4}$$

and

$$F^\diamond(P) = \sum_{n=0}^{N} a_n \left\langle \begin{matrix} P \\ n \end{matrix} \right\rangle_q. \tag{8.5}$$

In particular, then, $F = F^*(X) = F^\diamond(X)$.

The notation $F^*(X)$ is, of course, just the standard way of denoting polynomials and $F^\diamond(X)$ is the natural analog of this notation using Newton multiplication.

Identify $\mathbf{C}$ with its monomorphic image in both $(\mathbf{C}^{\mathbf{N}}, +, *, .)$ and $(\mathbf{C}^{\mathbf{N}}, +, \diamond, .)$—see Theorem 4.4—and let ϕ_c and ϕ'_c be the usual evaluation homomorphisms for some $c \in \mathbf{C}$. Then $F^*(c) = \phi_c(F)$ and $F^\diamond(c) = \phi'_c(F)$. Note that if $n \in \mathbf{N}$, then the use of the superscripts disinguishes $F^*(n)$ and $F^\diamond(n)$ from $F(n)$, the nth term of the sequence F.

If $F \in \mathcal{P}$, then the formula

$$\Delta F = F^*(X + 1) - F^*(X) \tag{8.6}$$

defines the usual finite difference operator $\Delta: \mathcal{P} \to \mathcal{P}$. Define "powers" of Δ by $\Delta^0 F = F$ and $\Delta^n F = \Delta\left(\Delta^{n-1} F\right)$ for $n \geq 2$. If

$$\binom{X}{n} = \frac{X * (X-1) * \cdots * \left(X - (n-1)\right)}{n!} \qquad \text{for } n \in \mathbf{P}, \tag{8.7}$$

then simple algebraic manipulation produces the well-known result that $\Delta\binom{X}{n} = \binom{X}{n-1}$. Similarly, if $F = \sum_{n=0}^{N} a_n \binom{X}{n}$, then $a_n = \phi_0\left(\Delta^n F\right)$. The last quantity is often written $\Delta^n F^*(0)$.

Now let q be a power of a prime. If $F \in \mathcal{P}$, then the Eulerian derivative of F, denoted $D_q F$, is defined by

$$X * D_q F = F^*(qX) - F^*(X). \tag{8.8}$$

Note that $D_q: \mathcal{P} \to \mathcal{P}$ is well-defined since $(\mathcal{P}, +, *, .)$ is an integral domain. In other words the solution to (8.8) is unique (it is easily seen to exist).

The Eulerian derivative operates nicely on the polynomials defined in (8.1). If $n \in \mathbf{P}$, then

$$\begin{aligned}\begin{bmatrix} qX \\ n \end{bmatrix}_q - \begin{bmatrix} X \\ n \end{bmatrix}_q &= \frac{\big(q^{n-1}(qX-1) - (X - q^{n-1})\big)(X-1)(X-q)\dots(X-q^{n-2})}{(q^n-1)(q^n-q)\dots(q^n-q^{n-1})} \\ &= X * \frac{1}{q^{n-1}} \left(\frac{(X-1)(X-q)\dots(X-q^{n-2})}{(q^{n-1}-1)(q^{n-1}-q)\dots(q^{n-1}-q^{n-2})} \right) \\ &= X * \frac{1}{q^{n-1}} \begin{bmatrix} X \\ n-1 \end{bmatrix}_q .\end{aligned}$$

So

$$D_q \begin{bmatrix} X \\ n \end{bmatrix}_q = \frac{1}{q^{n-1}} \begin{bmatrix} X \\ n-1 \end{bmatrix}_q . \tag{8.10}$$

The attempt to carry this idea over into $(\mathcal{P}, +, \diamond, .)$ by something like $X \diamond D_q^\diamond F = F^*(qX) - F^*(X)$ for $F \in \mathcal{P}$, does not seem to produce any useful results—i.e., $D_q^\diamond \left\langle {X \atop n} \right\rangle_q$ does not seem to have a "nice" form. And this definition is certainly uninspiring if $q = 1$. The difficulty seems to lie in the fact that $\left\langle {X \atop n} \right\rangle_q$ is written in "projective" form while the Eulerian derivative is designed for "affine" quantities. See Goldman and Rota [6, p.246] for a more precise description of "projective" and "affine" representations.

Another operator, however, produces nice results. Let $\diamond$ be a q-Newton product with $q > 0$. We define the *q-finite difference operator*, $\Delta_q \colon \mathcal{P} \to \mathcal{P}$ by

$$\big((q-1)X + 1\big) \diamond \Delta_q F = F^\diamond(qX+1) - F^\diamond(X) \qquad \text{for } F \in \mathcal{P}. \tag{8.10}$$

If $n \in \mathbf{P}$, then

$$\begin{aligned}\left\langle {qX+1 \atop n} \right\rangle_q &= \frac{1}{n_q^{\underline{n}}} \big((qX+1) \diamond ((qX+1) - 1_q) \diamond \dots \diamond ((qX+1) - (n-1)_q)\big) \\ &= \frac{1}{n_q^{\underline{n}}} (qX+1) \diamond (qX) \diamond \big((qX - (q)1_q) \diamond \dots \diamond \big(qX - (q)(n-2)_q\big) \\ &= \frac{q^{n-1}}{n_q^{\underline{n}}} (qX+1) \diamond X \diamond (X - 1_q) \diamond (X - 2_q) \diamond \dots \diamond \big(X - (n-2)_q\big).\end{aligned}$$

Define $G \in \mathcal{P}$ by $G = X \diamond (X - 1_q) \diamond \dots \diamond \big(X - (n-2)_q\big)$. Then

$$\begin{aligned}\left\langle {qX+1 \atop n} \right\rangle_q - \left\langle {X \atop n} \right\rangle_q &= \frac{1}{n_q^{\underline{n}}} \big((qX+1)(q^{n-1}) - (X - (n-1)_q)\big) \diamond G \\ &= \frac{1}{n_q^{\underline{n}}} (n_q) \big((q-1)X + 1\big) \diamond G\end{aligned}$$

$$= ((q-1)X+1) \diamond \left(\frac{G}{(n_q - 1_q)(n_q - 2_q)\dots(n_q - (n-1)_q)} \right)$$

$$= ((q-1)X+1) \diamond \left(\frac{1}{q^{n-1}} \right) \left\langle \begin{matrix} X \\ n-1 \end{matrix} \right\rangle_q . \tag{8.11}$$

Thus,

$$\Delta_q \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q = \frac{1}{q^{n-1}} \left\langle \begin{matrix} X \\ n-1 \end{matrix} \right\rangle_q \qquad \text{for } n \in \mathbf{P}. \tag{8.12}$$

Of course $\Delta_q \left\langle \begin{matrix} X \\ 0 \end{matrix} \right\rangle_q = \Delta_q 1 = 0$.

Now for $F \in \mathcal{P}$ define $\Delta_q^0 F = F$, and for $n \geq 2$ define $\Delta_q^n F = \Delta_q(\Delta_q^{n-1} F)$. From (8.12) if $n \in \mathbf{N}$, then $\Delta_q^n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q = 1/q^{n(n-1)/2}$, and if $k > n$, then $\Delta_q^k \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q = 0$. These observations together with the fact that

$$\left\langle \begin{matrix} 0 \\ n \end{matrix} \right\rangle_q = \phi_0' \left(\left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q \right) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \in \mathbf{P} \end{cases}$$

suffice to prove the following theorem.

Theorem 8.1 (q-Newton theorem). *Let $\diamond$ be a q-Newton product with $q > 0$. If $F = \sum_{n=0}^{N} a_n \left\langle \begin{matrix} X \\ n \end{matrix} \right\rangle_q$, then*

$$a_n = \phi_0'(\Delta^n F) = \Delta^n F^{\diamond}(0).$$

The definition of Δ_q is particularly nice in that if $q = 1$, then Δ_q is the usual finite difference operator and Theorem 8.1 is the same well-known result about finite differences of polynomials mentioned after equation (8.7). Actually, the preceding statement is not quite correct in that Δ_q is defined in terms of Newton multiplication, but Theorem 4.6 and equation (4.20) show that in this regard Newton multiplication and Cauchy multiplication display the same behavior.

Let L be a q-lattice. If $f, g \in C(L)$ and $[x, y] \in \mathrm{Int}(L)$, define *Schur multiplication* on $C(L)$ by

$$(f \cdot g)(x, y) = f(x, y) g(x, y).$$

Wagner [19] has shown an isomorphism between the **C**-algebras $(\mathcal{C}(l), +, \cdot, \cdot)$ and $(\mathcal{C}(L), +, \diamond, .)$. This isomorphism should provide a simpler development of the results in sections 5 and 6. In particular it should give Theorem 6.1 a much more natural proof.

One problem in the definition of q-binomial series is the exclusion of the case $q = 0$. This exclusion finds some justification in the fact that $(\mathcal{P}, +, \diamond, .)$ has zero divisors when $q = 0$, but a more complete justification would be helpful.

A much larger problem is the determination of the values of q for which q-lattices exist and the finding of a combinatorial or set-theoretic explanation of that result. This seems to demand a deep understanding of the containment relations among the subspaces of finite vector spaces—the sort of insight that would produce a purely set-theoretic proof of proposition A6.2, for instance.

Covers and partitions of finite sets correspond to sums and direct sums of finite vector spaces, both being special cases of covers of intervals in a q-lattice. The concept of a *minimal* cover of a set (see Appendix A) also has a q-lattice generalization. Any results on minimal covers of an interval in a q-lattice would be interesting.

Finally, is the definition of *cover* used herein—i.e., the join of elements in a particular class of lattices—the most general desirable? Could we drop the requirement of local modularity? Could we even drop the requirement that the poset be a lattice, thus allowing some collections of elements to cover more than one other element or no other element at all. One promising possibility, suggested by Wagner [19], is the study of binomial posets that are not lattices but in which every interval is a modular lattice. We can easily generate such structures by allowing two otherwise incomparable q-lattices to share a common least element.

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APPENDICES

APPENDIX A

A1. Classical Results

A1.1 Covers

Let S be a set of finite cardinality n. That is, $|S| = n$ for some $n \in \mathbf{N}$, where $\mathbf{N} = \{\text{non-negative integers}\} = \{0, 1, 2, \dots\}$. Call such an S an *n-set*. Denote by 2^S the power set of S. So $2^S = \{T \mid T \subseteq S\}$. If $\mathcal{B} = \{B_1, B_2, \dots, B_k\} \subseteq 2^S - \emptyset$ for some $k \in \mathbf{N}$ and $\cup_{i=1}^{k} B_i = S$, call $\mathcal{B}$ a *cover* of S or say that $\mathcal{B}$ *covers* S. The elements of $\mathcal{B}$ are *blocks* of the cover.

Let $C(n) = \left|\{\mathcal{B} \mid \mathcal{B} \text{ is a cover of } S\}\right|$ and $C(n,k) = \left|\{\mathcal{B} | \mathcal{B} \text{ is a cover of } S \text{ and } |\mathcal{B}| = k\}\right|$ for $n, k \in \mathbf{N}$. In other words $C(n)$ is the number of covers of an n-set, and $C(n,k)$ is the number of such covers that consist of k blocks (*k-covers* henceforth). Wagner [18, pp. 515–516] gives the following two results.

Proposition A1.1. *If* $n \in \mathbf{N}$, *then*

$$C(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} 2^{(2^j - 1)}. \tag{A1.1}$$

Proof. Count the number of subsets of $2^S - \emptyset$ in two ways. First observe that $\left|2^{(2^S - \emptyset)}\right| = 2^{(2^n - 1)}$. Second count subsets of $2^S - \emptyset$ according to their union—i.e. according to what they cover. For each $j \in [n]$, where $[n] = \{1, 2, \dots, n\}$ (so $[0] = \emptyset$) choose one of the $\binom{n}{j}$ subsets of S that has j elements (*j-subsets* henceforth) and then count the $C(j)$ covers of this subset. A cover covers only one set, and every subset of $2^S - \emptyset$ covers some subset of S. Summing over the relevant values of j yields $\left|2^{(2^S - \emptyset)}\right| = \sum_{j=0}^{n} \binom{n}{j} C(j)$.

These two enumerations produce the identity $2^{(2^n - 1)} = \sum_{j=0}^{n} \binom{n}{j} C(j)$.

By binomial inversion, then, $C(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} 2^{2^j - 1}$

□

Proposition A1.2. *If* $n, k \in \mathbf{N}$, *then*

$$C(n,k) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{2^j - 1}{k}. \tag{A1.2}$$

Proof. Proceed as in the previous theorem, this time counting only k-subsets of $2^S - \emptyset$. This yields the identity $\binom{2^n - 1}{k} = \sum_{j=0}^{n} \binom{n}{j} C(j,k)$, and the proposition again follows by binomial inversion.

□

Note that the empty set covers the empty set, so $C(0) = 1$, $C(0,0) = 1$, and $C(0,k) = 0$ for $k \in \mathbf{P}$, where $\mathbf{P} = \{\text{positive integers}\} = \{1, 2, 3, \dots\}$.

A1.2 Partitions

If $\mathcal{B}$ is a cover of an n-set S and the blocks of $\mathcal{B}$ are pairwise disjoint, call $\mathcal{B}$ a *partition* of S.

Partitions arise in many mathematical settings and have received much attention. The basic combinatorial facts about them are well-known, namely that the number of partitions of an n-set is the Bell number $B(n)$ and that the number of these partitions having exactly k blocks, for $k \in \mathbf{N}$, is the Stirling number of the second kind $S_2(n,k)$. (Rota [14] gives an exhaustive (through 1964) bibliography of literature about Bell numbers.) So $B(n) = \sum_{k=0}^{n} S_2(n,k)$.

The following is a "typical" derivation of the usual formula for $S_2(n,k)$.

Proposition A1.3. *If $n, k \in \mathbf{N}$, then*

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \qquad \text{for } n,k \in \mathbf{N}. \tag{A1.3}$$

Proof. Let S be an n-set, $A = \{\text{ordered partitions of } S \text{ consisting of } k \text{ blocks}\} = \{(B_1, B_2, \dots, B_k) \mid \{B_1, B_2, \dots, B_k\} \text{ is a partition of } S\}$, and $[k]^S_{surj} = \{f: S \to [k] \mid f \text{ is surjective}\}$. Define $\phi: [k]^S_{surj} \to A$ by $\phi(f) = (B_1, B_2, \dots, B_k)$, where $f \in [k]^S_{surj}$ and $B_i = f^{-1}(i) = \{s \in S \mid f(s) = i\}$. It is easy to see that ϕ is well-defined and bijective.

Now define $\sigma(n,k) = |A| = \left|[k]^S_{surj}\right|$ and $[k]^S = \{f: S \to [k]\}$. Calculate $\left|[k]^S\right|$ as follows: For $j = 0, 1, \dots, k$ choose a j-subset T of $[k]$. There are $\binom{k}{j}$ ways to choose T and $\sigma(n,j)$ functions in $[k]^s$ with range T. Summing over j yields $\left|[k]^S\right| = \sum_{j=0}^{k}\binom{k}{j}\sigma(n,j)$.

On the other hand it is well known that $\left|[k]^S\right| = \left|[k]\right|^{|S|} = k^n$, so $k^n = \sum_{j=0}^{k}\binom{k}{j}\sigma(n,j)$. By binomial inversion $\sigma(n,k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$.

The $k!$ ways (obviously distinct) to order the blocks of a k-block partition of S produce $k!$ ordered partitions. Distinct partitions produce distinct ordered partitions, so $\sigma(n,k) = k!\, S_2(n,k)$. Therefore

$$S_2(n,k) = \frac{1}{k!}\sigma(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

□

The generating functions for $B(n)$ and $\sigma(n,k)$ are known to be

$$e^{e^x-1} = \sum_{n\geq 0} B(n)\frac{x^n}{n!}$$

and

$$(e^x-1)^k = \sum_{n\geq 0} \sigma(n,k)\frac{x^n}{n!}$$

respectively.

A2. Wagner's Survey

A2.1 Minimal covers

This section gives several results from Wagner's survey [18] of enumeration problems involving restricted covers.

A cover $\mathcal{B}$ of an n-set S is *minimal* if no proper subset of $\mathcal{B}$ covers S.

If $B \subseteq S$, then B *covers* the points of S which are in B. If $\mathcal{B}$ covers S, with $B \in \mathcal{B}$, and every point of S covered by B is covered by at least one other block of $\mathcal{B}$, then $\mathcal{B} - \{B\}$ still covers S—that is, $\mathcal{B}$ is not minimal. Hence if $\mathcal{B}$ is a minimal cover of S and $B \in \mathcal{B}$, then B must cover some point of S that no other block covers. We say that B and $\mathcal{B}$ cover this point *uniquely*.

For $n,k,j \in \mathbf{N}$ let $M(n,k,j)$ be the number of minimal k-covers of an n-set S that cover j points of S uniquely.

Proposition A2.1. *For* $n,k,j \in \mathbf{N}$,

$$M(n,k,j) = \binom{n}{j} S_2(j,k)(2^k-k-1)^{n-j}, \qquad (A2.1)$$

with the convention $0^0 = 1$.

Proof. Count the minimal k-covers of an n-set S that cover j points uniquely, as follows: Choose a j-subset of S. These are the uniquely-covered points. Partition these j points into k blocks. Then distribute the remaining $n-j$ elements of S among the k blocks in such a manner that each point is in at least two blocks.

There are $\binom{n}{j}$ ways to perform the first step, $S_2(j,k)$ ways to perform the second, and $(2^k-k-1)^{n-j}$ ways to perform the third (A collection of k blocks has 2^k subcollections. Omitting the empty subcollection and subcollections with exactly one block leaves 2^k-k-1 subcollections with at least two blocks.)

Therefore $M(n,k,j) = \binom{n}{j} S_2(j,k)(2^k-k-1)^{n-j}$.

□

Define $M(n,k)$ to be the number of minimal k-covers of an n-set.

Corollary. *If $n, k \in \mathbf{N}$, then*

$$M(n,k) = \sum_{j=0}^{n} \binom{n}{j} S_2(j,k)(2^k - k - 1)^{n-j} \qquad (A2.2)$$

with the convention $0^0 = 1$.

A2.2 r-Minimal covers

A cover $\mathcal{B}$ of an n-set S is *r-minimal*, for $r \in \mathbf{P}$, if for all $\mathcal{C} \subset \mathcal{B}$ such that $|\mathcal{C}| \geq r$, the set $\mathcal{B} - \mathcal{C}$ does cover not S—that is, when the removal of r blocks from $\mathcal{B}$ always uncovers at least one point of S. Clearly *one-minimal* is synonymous with *minimal*.

Wagner asks the question, "What is the maximum number of blocks in an r-minimal cover of an n-set S?" For minimal covers the obvious answer, as he notes, is n since each block covers a point of S uniquely. The only minimal cover of S that attains this maximum is $\big\{\{s\} \mid s \in S\big\}$, the *singleton cover* of S.

For two-minimal covers Wagner gives the following unexpected answer with a different proof.

Proposition A2.2. *A two-minimal cover of an n-set S has at most n blocks.*

Proof. Let $\mathcal{B}$ be a two-minimal cover of an n-set S. If $\mathcal{B}$ is minimal, then $|\mathcal{B}| \leq n$, so assume $\mathcal{B}$ is not minimal. Then there exists $B \in \mathcal{B}$ such that $\mathcal{B} - \{B\}$ covers S.

Since $\mathcal{B}$ is two-minimal, removal of any block of $\mathcal{B} - \{B\}$ uncovers an element of S. In other words, $\mathcal{B} - \{B\}$ is minimal. So $\big|\mathcal{B} - \{B\}\big| \leq n$.

If $\big|\mathcal{B} - \{B\}\big| = n$, then $\mathcal{B} - \{B\}$ is the singleton cover of S. As B can be neither the empty set nor a singleton subset of S, there are distinct $s_1, s_2 \in B$. But in that case $\{s_1\}, \{s_2\} \in \mathcal{B}$ and $\mathcal{B} - \big\{\{s_1\}, \{s_2\}\big\}$ covers S. This contradicts $\mathcal{B}$ being two-minimal.

Therefore $\big|\mathcal{B} - \{B\}\big| \leq n - 1$, and $|\mathcal{B}| \leq n$.

□

Wagner states that this question about r-minimal covers is still open for general r. Proposition A6.1, however, answers the question in a large number of cases.

A2.3 Unnested covers

A cover $\mathcal{B}$ of an n-set S is *unnested* if no block of $\mathcal{B}$ properly contains another block.

Several obvious combinatorial problems arise—enumeration of unnested covers with k-blocks, for instance. Sperner's Lemma [15] solves one of these, namely the determination of the maximum number of blocks in an unnested cover:

Consider the collection of subsets of an n-set S. Partially order this collection by set inclusion—that is $A \leq B$ when $A \subseteq B$ for members A and B of the collection. An *antichain* is a collection of incomparable elements of the resulting partially ordered set (henceforth *poset*—see the main text of the dissertation for an introduction to posets), that is, a collection of subsets of S in which no subset contains another properly.

An antichain of maximum size is obviously an unnested cover of S. Sperner's Lemma states that the number of subsets in such an antichain is no greater than $\binom{n}{\lfloor n/2 \rfloor}$. A slightly elaborated version of D. Lubell's elegant proof [12] of Sperner's Lemma follows:

Theorem A2.1. *Let Γ be an antichain in the collection of subsets of an n-set S for $n \in \mathbf{N}$. Then*

$$\sum_{A \in \Gamma} \binom{n}{|A|}^{-1} \leq 1. \tag{A2.3}$$

(Note: Since $\binom{n}{\lfloor n/2 \rfloor}$ is the largest binomial coefficient of order n, Theorem A2.1 shows that

$$\begin{aligned} |\Gamma| = \sum_{A \in \Gamma} 1 &= \binom{n}{\lfloor \frac{1}{2}n \rfloor} \sum_{A \in \Gamma} \binom{n}{\lfloor \frac{1}{2}n \rfloor}^{-1} \\ &\leq \binom{n}{\lfloor \frac{1}{2}n \rfloor} \sum_{A \in \Gamma} \binom{n}{|A|}^{-1} \\ &\leq \binom{n}{\lfloor \frac{1}{2}n \rfloor}, \end{aligned}$$

which is Sperner's lemma.)

Proof. There are $n!$ maximal chains in the poset of subsets of S. If $A \subseteq S$, then $|A|!\,(n - |A|)!$ of these chains include A. If $A, B \in \Gamma$ with $A \neq B$, then no maximal chain contains both A and B (as otherwise $A \subseteq B$ or $B \subseteq A$). The number of maximal chains that include an element of Γ is no greater than the total number of

maximal chians. That is, $\sum_{A\in\Gamma} |A|!\,(n-|A|)! \le n!$. Dividing by $n!$ yields

$$\sum_{A\in\Gamma} \binom{n}{|A|}^{-1} = \sum_{A\in\Gamma} \frac{|A|!\,(n-|A|)!}{n!} \le 1. \qquad (A2.4)$$

□

The collection of all $\lfloor n/2 \rfloor$-subsets of S has $\binom{n}{\lfloor n/2 \rfloor}$ members and is obviously an antichain (and in particular an unnested cover of S). Thus the maximum number of blocks in an unnested cover of an n-set is $\binom{n}{\lfloor n/2 \rfloor}$. That is, the largest unnested cover attains the bound set by Sperner's Lemma.

A2.4 r-Unnested covers

Call a cover $\mathcal{B}$ of an n-set S an *r-unnested cover*, for $r \in \mathbf{P}$, if $\mathcal{B}$ has no subset $\{B_0, B_1, \ldots, B_r\}$ such that $B_0 \subset B_1 \subset \cdots \subset B_r$.

This concept generalizes unnested covers and provokes the same combinatorial questions. In particular Erdös [5] shows that the maximum number of blocks in an r-unnested cover of an n-set equals the sum of the r largest binomial coefficients of order n.

A2.5 Partitions of type r

Finally Wagner mentions *partitions of type r*. These are covers of an n-set S in which every block has at least r elements, for some $r \in \mathbf{P}$, and every r-subset of S is contained in exactly one block. J. Hartmanis [8] first defined this concept. Evidently the enumeration questions related to partitions of type r have received little attention.

A3. Clarke's Approach

R.J. Clarke [3] looks at the enumeration of k-block covers of an n-set S in the four cases that arise from taking the blocks to be labelled or unlabelled (equivalently, ordered or unordered) and from considering the elements of S to be distinguishable or indistinguishable. His definition of *cover* allows the empty set as a block and allows repetition of blocks within a cover. His unordered, non-minimal covers are thus multisets rather than sets.

The enumeration of unlabelled covers of a set of indistinguishable objects appears to be enormously hard—intuitively it bears similarities to the enumeration of

partitions of an integer. (If $n \in \mathbf{N}$, a *partition* of n is a multiset of positive integers with sum n, repetitions included.) Clarke's results about such covers would thus seem to be the most interesting part of his work.

A4. Covers of Doubles by Triples

In private communication, S. Mulay suggests the following cover-like idea. Let S be an n-set, and let $\mathcal{A} \subseteq 2^S$ have the property that every element of $\mathcal{A}$ has cardinality three and every two-subset of S belongs to at least one element of $\mathcal{A}$. Call $\mathcal{A}$ a *cover of the doubles of S by triples*—a *cover of doubles by triples* for short. The idea generalizes to covers of r-tuples by a-tuples for $r, a \in \mathbf{N}$

If $\mathcal{A}$ is a cover of doubles by triples, call $\mathcal{A}$ *minimal* if no proper subset of $\mathcal{A}$ is a cover of doubles by triples. Two results on minimal covers of doubles by triples are among the new results in section A6.

A5. Finite Vector Space Background

A5.1 Introduction

It is well-known that combinatorial results about subspaces of a finite vector space resemble those about subsets of a finite set in a striking fashion. This suggests an analogue (a generalization, actually, as the text of this dissertation shows) of covers in the setting of a finite vector space. The attempt to find general results that unify results on sets and results on vector spaces lies behind much of this dissertation.

This section is a brief introduction to finite vector spaces and the combinatorial results about them. Several sources [6, 7, 10, 16] serve as general references for this introduction, but all material presented here is common knowledge unless otherwise noted.

If $q \in \mathbf{P}$, then a field with q elements exists if and only if $q = p^n$ for some $p, n \in \mathbf{P}$ with p prime. For fixed q this field is unique up to isomorphism. Call it the *finite field with q elements* and denote it by F_q or $GF(q)$.

For $n \in \mathbf{N}$ there is an n-dimensional vector space (henceforth *n-space*) over F_q. The simplest example of this is F_q^n, the set of n-tuples of elements from F_q, with coordinatewise addition and the usual scalar multiplication (i.e., $c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$). Denote this vector space by $V_n(q)$. Any n-space over F_q is isomorphic to $V_n(q)$. Infinite-dimensional vector spaces over F_q also exist—for in-

stance, sequences of elements from F_q with the addition and scalar multiplication above.

Proposition A5.1. *If V is an n-dimensional vector space over F_q, for $n \in \mathbf{N}$, then $|V| = q^n$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then $V = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n \mid a_1, a_2, \ldots, a_n \in F_q\}$. There are q ways to choose each of the n coefficients in these linear combinations, and distinct choices yield distinct elements of V. Therefore $|V| = q^n$.

□

A finite dimensional vector space over a finite field has, thus, finitely many points. Call such vector spaces *finite vector spaces.*

A5.2 q-Binomial coefficients

For $n, k \in \mathbf{N}$ let V be an n-space over F_q, and define $\binom{n}{k}_q$ to be the number of k-dimensional subspaces (henceforth *k-subspaces*) of V. The numbers $\binom{n}{k}_q$ are called *q-binomial coefficients* or *Gaussian coefficients*. The similarity in notation and terminology between these coefficients and the usual binomial coefficients reflects the similarity of their behavior.

Proposition A5.2. *If $n, k \in \mathbf{N}$,*

$$(q^k - 1)(q^k - q)\ldots(q^k - q^{k-1})\binom{n}{k}_q = (q^n - 1)(q^n - q)\ldots(q^n - q^{k-1}). \quad (A5.1)$$

Proof. Let V be an n-space over F_q. Count in two ways the k-tuples of linearly independent vectors in V.

First method: Choose $v_1 \in V - \{0\}$. For $i = 2, 3, \ldots, k$ choose $v_i \in V - \text{span}\{v_1, v_2, \ldots, v_{i-1}\}$, where $\text{span}\{v_1, v_2, \ldots, v_{i-1}\}$ is the subspace of V spanned by $\{v_1, v_2, \ldots, v_{i-1}\}$. Since $|V| = q^n$ and $|\text{span}\{v_1, v_2, \ldots, v_{i-1}\}| = q^{i-1}$, for $i \in [k]$ there are $q^n - q^{i-1}$ ways to choose v_i.

Thus there are $(q^n - 1)(q^n - q)\ldots(q^n - q^{k-1})$ ways to choose the k-tuple $(v_1, v_2, \ldots, v_k)$.

Second method: Count the sequences $(v_1, v_2, \ldots, v_k)$ according to the subspace $\text{span}\{v_1, v_2, \ldots, v_k\}$. Since $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set, the subspace $\text{span}\{v_1, v_2, \ldots, v_k\}$ is a k-subspace of V. There are $\binom{n}{k}_q$ such subspaces. Now enumerate the k-tuples of linearly independent vectors (i.e., ordered bases)

from a fixed k-subspace. The argument of the first method shows that there are $(q^k-1)(q^k-q)\dots(q^k-q^{k-1})$ of these.

Altogether, then, there are $(q^k-1)(q^k-q)\dots(q^k-q^{k-1})\binom{n}{k}_q$ ways to choose $(v_1, v_2, \dots, v_k)$.

Both methods count the same set, thus establishing equation (A5.1).

□

Corollary. *If* $n, k \in \mathbf{N}$,

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})} \tag{A5.2}$$

The identity $(q^n-1) = (q-1)(q^{n-1}+q^{n-2}+\cdots+1)$, proves useful in manipulating (A5.2) into other forms:

$$\begin{aligned}
\binom{n}{k}_q &= \frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})} \\
&= \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)} \cdot \frac{q^{\frac{(k-1)k}{2}}}{q^{\frac{(k-1)k}{2}}} \\
&= \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)} \qquad (A5.3) \\
&= \frac{(q^{n-1}+q^{n-2}+\cdots+1)\dots(q^{n-k}+q^{n-k-1}+\cdots+1)}{(q^{k-1}+q^{k-2}+\cdots+1)\dots(q+1)(1)} \qquad (A5.4)
\end{aligned}$$

A5.3 Results about the lattice of subspaces

In general, if V is a vector space and $U_1, U_2, \dots, U_k$ are subspaces of V for $k \in \mathbf{N}$, define the *sum* $U_1 + U_2 + \cdots + U_k$ by

$$U_1 + U_2 + \cdots + U_k = \text{span}\{U_1 \cup U_2 \cup \cdots \cup U_k\}.$$

Call this a *direct sum* and write $U_1 \oplus U_2 \oplus \cdots \oplus U_k$ or $\sum_{i=1}^{k} U_i$ if

$$\text{for all } i \in [k] \qquad U_i \cap (U_1 + U_2 + \cdots + \widehat{U}_i + \cdots + U_k) = \{0\},$$

where the notation $\widehat{U}_i$ indicates the omission of U_i from the sum.

At last, with the introduction to finite vector spaces behind us and the definitions of *sum* and *direct sum* in hand, we are ready to define a cover of a finite vector

space. Namely, a *cover* (by subspaces) of the finite vector space V is a collection of subspaces of V whose sum is V.

As we will see, results about covers of vector spaces bear a strong resemblance to those about covers of sets. In addition, if we restrict our consideration to covers in which the covered vector space is the direct sum of the subspaces in the cover, then the results are much like those for partitions of a set.

Theorem A2.1 (Sperner's Lemma) has a nice analogue for covers of finite vector spaces. We give it here, with necessary background, to illustrate the similarities that exist between the two sorts of covers.

If U and V are subspaces of $V_n(q)$, define $U \leq V$ whenever $U \subseteq V$. This establishes a partial ordering on the set of subspaces of $V_n(q)$. Under this partial ordering the subspaces of $V_n(q)$ form a lattice. Call this lattice $L_n(q)$ (for an introduction to lattices see the main text of the dissertation).

The number of maximal chains in $L_n(q)$ proves to be an important quantity in the study of $V_n(q)$. The following proposition provides an easy way to find this quantity.

Proposition A5.3. *For $n, k \in \mathbf{N}$, let $U \in L_n(q)$ with $\dim U = k$. For $r \in \mathbf{N}$*

$$\left|\{V \in L_n(q) \mid U \leq V \text{and} \dim V = k + r\}\right| = \binom{n-k}{r}_q .$$

Proof. The quotient group $V_n(q)/U$ is well-known to be an $(n-k)$-space over F_q, the r-subspaces of which are in one-to-one correspondence with the $(k+r)$-subspaces of $V_n(q)$ by $V \mapsto V/U$, where V is a $(k+r)$-subspace of $V_n(q)$. The number of r-subspaces of $V_n(q)/U$ is $\binom{n-k}{r}_q$.

□

Corollary. *Under the above hypotheses, by equation A5.2*

$$\begin{aligned}\left|\{V \in L_n(q) \mid U \leq V \text{and} \dim V = k+1\}\right| &= \binom{n-k}{1}_q \\ &= \frac{q^{n-k}-1}{q-1} \\ &= (q^{n-k-1} + q^{n-k-2} + \cdots + 1) \\ &= \sum_{i=0}^{n-k-1} q^i .\end{aligned}$$

Proposition A5.4. *The number of maximal chains in $L_n(q)$ is*

$$\prod_{k=0}^{n-1} \binom{n-k}{1}_q = \prod_{k=0}^{n-1} \left(\sum_{i=0}^{n-k-1} q^i \right)$$
$$= (q^{n-1} + q^{n-2} + \cdots + 1)(q^{n-2} + q^{n-3} + \cdots + 1) \ldots (q+1)(1).$$

Proof. A maximal chain in $L_n(q)$ has the form

$$\{0\} = V_0 \leq V_1 \leq \cdots \leq V_n = V_n(q),$$

where V_k is a k-subspace of $V_n(q)$ for $k = 0, 1, \ldots, n$. By the corollary there are $\binom{n}{1}_q$ ways to choose V_1, then $\binom{n-1}{1}_q$ ways to choose V_2, and in general $\binom{n-k+1}{1}_q$ ways to choose V_k for $k = 1, 2, \ldots, n$. Therefore there are $\prod_{k=1}^{n} \binom{n-k+1}{1}_q = \prod_{k=0}^{n-1} \binom{n-k}{1}_q$ such chains.

□

Let $\mathcal{V}$ be a cover of $V_n(q)$. We say that $\mathcal{V}$ is *unnested* if no block of $\mathcal{V}$ properly contains another block of $\mathcal{V}$.

As in the case of covers by subsets, we would like to know the maximum number of blocks $\mathcal{V}$ can contain. Again such a cover is an antichain of maximum size in $L_n(q)$. A result like Sperner's lemma would provide the desired information, so proceeding in exact analogy with the statement and proof of Theorem A2.1, we have the following result.

Theorem A5.1. *Let Γ be an antichain in $L_n(q)$. Then*

$$\sum_{V \in \Gamma} \binom{n}{\dim V}_q^{-1} \leq 1.$$

Proof. There are $(q^{n-1} + q^{n-2} + \cdots + 1)(q^{n-2} + q^{n-3} + \cdots + 1) \ldots (q+1)(1)$ maximal chains in $L_n(q)$. For $V \in \Gamma$ with $\dim V = k$, the number of these chains that include V is

$$\left[(q^{k-1} + q^{k-2} + \cdots + 1) \ldots (q+1)(1)\right] \left[(q^{n-k-1} + q^{n-k-2} + \cdots + 1) \ldots (q+1)(1)\right]$$

since each such chain consists of a maximal chain in the interval $[\{0\}, V]$ and a maximal chain in the interval $[V, V_n(q)]$. As no maximal chain in $L_n(q)$ includes more than one member of Γ,

$$\sum_{V \in \Gamma} \left[\prod_{j=1}^{\dim V} \left(\sum_{i=0}^{j-1} q^i \right) \right] \left[\prod_{j=1}^{n-\dim V} \left(\sum_{i=0}^{j-1} q^i \right) \right] \leq \prod_{j=1}^{n} \left(\sum_{i=0}^{j-1} q^i \right). \qquad (A5.5)$$

Dividing by the right-hand side of (A5.5) yields

$$\sum_{V\in\Gamma} \frac{\left[\prod_{j=1}^{\dim V}\left(\sum_{i=0}^{j-1} q^i\right)\right]\left[\prod_{j=1}^{n-\dim V}\left(\sum_{i=0}^{j-1} q^i\right)\right]}{\prod_{j=1}^{n}\left(\sum_{i=0}^{j-1} q^i\right)} \leq 1. \qquad (A5.6)$$

Finally, simplifying the left-hand side of (A5.6) and recalling (A5.4) we see that

$$\sum_{V\in\Gamma} \binom{n}{\dim V}_q^{-1} = \sum_{V\in\Gamma} \left(\frac{\prod_{j=n-\dim V+1}^{n}\left(\sum_{i=0}^{j-1} q^i\right)}{\prod_{j=1}^{\dim V}\left(\sum_{i=0}^{j-1} q^i\right)}\right)^{-1} \leq 1.$$

□

From equation (A5.4) it is easy to see that $\binom{n}{\lfloor n/2\rfloor}_q$ is the largest q-binomial coefficient of order n. Thus we have the following corollary:

Corollary (q-Sperner's Lemma). *Let Γ be an antichain in $L_n(q)$. Then*

$$|\Gamma| \leq \binom{n}{\lfloor n/2\rfloor}_q.$$

Proof.

$$\begin{aligned}|\Gamma| = \sum_{V\in\Gamma} 1 &= \binom{n}{\lfloor n/2\rfloor}_q \sum_{V\in\Gamma} \binom{n}{\lfloor n/2\rfloor}_q^{-1} \\ &\leq \binom{n}{\lfloor n/2\rfloor}_q \sum_{V\in\Gamma} \binom{n}{\dim V}_q^{-1} \\ &\leq \binom{n}{\lfloor n/2\rfloor}_q.\end{aligned}$$

□

Goldman and Rota [6, p.257] mention the existence of this result and its similarity to Lubell's proof of Sperner's lemma.

If $\mathcal{V}$ is an unnested cover of $V_n(q)$, then $\mathcal{V}$ is an antichain in $L_n(q)$ and hence has at most $\binom{n}{\lfloor n/2\rfloor}_q$ blocks. On the other hand, the collection of all $\lfloor n/2\rfloor$-subspaces of $V_n(q)$ has $\binom{n}{\lfloor n/2\rfloor}_q$ elements and is obviously an unnested cover of $V_n(q)$. Hence, the maximum number of blocks in an unnested cover of $V_n(q)$ is $\binom{n}{\lfloor n/2\rfloor}_q$. That is, the largest unnested cover of $V_n(q)$ attains the bound set by the q-sperner's Lemma.

A5.4 q-Binomial identities in the limit $q \to 1$

In the limit as $q \to 1$, with q regarded as a real variable, identities involving q-binomial coefficients tend to become identities involving binomial coefficients. Goldman and Rota [6] investigate this well-known phenomenon and give numerous examples of it, the simplest of which comes from the following identity.

Proposition A5.5. *For $n, k \in \mathbf{N}$,*

$$\binom{n}{k}_q = \binom{n}{n-k}_q \tag{A5.7}$$

Proof. This follows from (A5.4) by algebraic manipulation. □

Note that

$$\begin{aligned}\lim_{q\to 1}\binom{n}{k}_q &= \lim_{q\to 1}\frac{(q^{n-1}+q^{n-2}+\cdots+1)\ldots(q^{n-k}+q^{n-k-1}+\cdots+1)}{(q^{k-1}+q^{k-2}+\cdots+1)\ldots(q+1)(1)}\\ &= \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots(1)}\\ &= \binom{n}{k}.\end{aligned}$$

Taking the limit of both sides of equation (A5.7) as $q \to 1$ produces the well-known identity

$$\binom{n}{k} = \binom{n}{n-k}. \tag{A5.8}$$

Note, however, that this does not prove (A5.8), but rather it gives us an identity about subsets of a set that we can verify by other means.

A6. New Results

The following three propositions are new results about concepts described in this appendix.

Proposition A6.1. *For $n, r \in \mathbf{N}$ with $r \geq 2$, if $\mathcal{B}$ is an r-minimal cover of an n-set S, then $|\mathcal{B}| \leq n + r - 2$.*

Proof. We already know this for two-minimal covers, so assume $r \geq 3$. Further, recalling that s-minimal implies r-minimal for $1 \leq s \leq r$, assume $\mathcal{B}$ is "strictly" r-minimal—that is, $\mathcal{B}$ is not $(r-1)$-minimal.

There exist distinct $B_1, B_2, \ldots, B_{r-2} \in \mathcal{B}$ such that $\mathcal{B} - \{B_1, B_2, \ldots, B_{r-2}\}$ covers S. Removal of any two distinct blocks from $\mathcal{B} - \{B_1, B_2, \ldots, B_{r-2}\}$ uncovers some element of S since $\mathcal{B}$ is r-minimal. Thus $\mathcal{B} - \{B_1, B_2, \ldots, B_{r-2}\}$ is a two-minimal cover of S and has at most n blocks. Therefore $|\mathcal{B}| \leq n + r - 2$.

If $\mathcal{B}$ is not strictly r-minimal , then there exists $s \in \mathbf{P}$ with $1 \leq s < r$ such that $\mathcal{B}$ is strictly s-minimal. Thus, by the above argument $|\mathcal{B}| \leq n + s - 2 < n + r - 2$.

□

If $\mathcal{B}$ is a cover of an n-set S, then $\mathcal{B} \subseteq 2^S - \emptyset$ and thus has at most $2^n - 1$ blocks. So the previous result is not sharp for $r > 2^n - n + 1$ as in that case $n + r - 2 > 2^n - 1$.

On the other hand, Wagner [19] has demonstrated that this result is sharp for $2 \leq r \leq n + 1$ by constructing the cover

$$\mathcal{B} = \big\{\{k\} \mid k \in [n]\big\} \cup \big\{\{1, k\} \mid k \in [r-1]\big\}$$

of the n-set $[n]$. This cover has $n + r - 2$ blocks and is easily seen to be r-minimal.

Proposition A6.2. *Let $\mathcal{B}$ be a minimal cover of the doubles of $[n]$ by triples, for $n \in \mathbf{N}$. Then $|\mathcal{B}| \leq \binom{n-1}{2}$. Further there is such a cover $\mathcal{B}$ with $|\mathcal{B}| = \binom{n-1}{2}$.*

Proof. This is obvious for $0 \leq n \leq 3$, so assume $n \geq 4$.

Since $\mathcal{B}$ is minimal, each triple in $\mathcal{B}$ covers some double of $[n]$ uniquely. Bound the cardinality of $\mathcal{B}$ by observing how quickly it must "use up" the doubles of $[n]$.

If $\mathcal{C} \subseteq \mathcal{B}$, define

$$\phi(\mathcal{C}) = \min\left\{ i \in [n] \;\middle|\; i \notin \bigcup_{B \in \mathcal{C}} B \right\}.$$

Construct a sequence of blocks from $\mathcal{B}$ as follows: Choose $B_1 \in \mathcal{B}$ such that $\{1, 2\} \subseteq B_1$. Next choose $B_2 \in \mathcal{B}$ such that $\{1, \phi(B_1)\} \subseteq B_2$. in general, choose $B_{i+1} \in \mathcal{B}$ so that $\{1, \phi(B_1 \cup B_2 \cup \cdots \cup B_i)\} \subseteq B_{i+1}$. Stop when $B_1 \cup B_2 \cup \cdots \cup B_k = [n]$ for some $k \in \mathbf{N}$. The fact that $\mathcal{B}$ is a cover of doubles by triples guarantees the success of this construction.

The triples in $\mathcal{C} = \{B_1, B_2, \ldots, B_k\}$ are all of the form $\{1, a, b\}$ for distinct $a, b \in \{2, 3, \ldots, n\}$, so $\mathcal{C}$ covers k doubles which do not contain the element 1 as well as all $n - 1$ doubles of the form $\{1, a\}$ for $a \in \{2, 3, \ldots, n\}$. Altogether $\mathcal{C}$ covers $n + k - 1$ doubles.

As the set $[n]$ has $\binom{n}{2}$ doubles and each triple of $\mathcal{B}$ covers some double uniquely, $|\mathcal{B} - \mathcal{C}| \leq \binom{n}{2} - (n + k - 1)$. In other words

$$|\mathcal{B}| \leq \binom{n}{2} - (n + k - 1) + k = \frac{n(n-1)}{2} - (n-1) = \binom{n-1}{2}.$$

Now define $\mathcal{B} = \{\{1, a, b\} \mid a, b \in \{2, 3, \ldots, n\} \text{ and } a \neq b\}$. Clearly $\mathcal{B}$ is a minimal cover of the doubles of $[n]$ by triples and $|\mathcal{B}| = \binom{n-1}{2}$.

□

At the other extreme we would like to know the minimum number of blocks in a cover $\mathcal{B}$ of the doubles of $[n]$ by triples for $n \in \mathbf{N}$. Since there are $\binom{n}{2}$ doubles and each triple covers three doubles, an obvious lower bound is $\lceil \binom{n}{2}/3 \rceil$. This is not always attainable, the simplest counterexample being the set $\{1, 2, 3, 4\}$ whose six doubles cannot be covered by two triples.

The bound $\lceil \binom{n}{2}/3 \rceil$ is attainable, however, in an infinite number of cases.

Proposition A6.3. *For $n \in \mathbf{N}$, there is a minimal cover of the doubles of $[2^n - 1]$ by triples that has $\binom{2^n-1}{2}/3$ blocks.*

(Note that $2^n - 1 = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1$ and that $\binom{2^n-1}{2}$ is divisible by 3).

Proof. Let V be an n-dimensional vector space over the finite field with two elements. Then V has $\binom{n}{1}_2 = 2^{n-1} + 2^{n-2} + \cdots + 1$ one-subspaces. Label these $V_1, V_2, \ldots, V_{2^{n-1}+2^{n-2}+\cdots+1}$. Further, V has $\binom{n}{2}_2$ two-subspaces. Give these the labels $W_1, W_2, \ldots, W_{\binom{n}{2}_2}$.

For $k = 1, 2, \ldots, \binom{n}{2}_2$ define

$$B_k = \{i \in [2^n - 1] \mid V_i \subseteq W_k\}.$$

Clearly $B_k \subseteq [2^n - 1] = [2^{n-1} + 2^{n-2} + \cdots + 1]$ for all k. Also each two-subspace has $\binom{2}{1}_2 = 3$ one-subspaces, so for all k we have $|B_k| = 3$. Finally for $i, j \in [2^n - 1]$ with $i \neq j$ there is precisely one two-subspace which contains V_i and V_j, namely $\text{span}\{V_i \cup V_j\}$. Thus the double $\{i, j\}$ is covered uniquely by B_k, where $W_k = \text{span}\{V_i \cup V_j\}$. In short $\mathcal{B} = \{B_1, B_2, \ldots, B_{\binom{n}{2}_2}\}$ is a minimal cover of the doubles of $[2^n - 1]$ by triples in which each double is covered uniquely.

This being the case, we know that $|\mathcal{B}| = \binom{2^n-1}{2}/3$, but more directly

$$\begin{aligned} |\mathcal{B}| = \binom{n}{2}_2 &= \frac{(2^n - 1)(2^n - 2)}{(2^2 - 1)(2^2 - 2)} \\ &= \frac{(2^n - 1)(2^n - 2)}{6} \\ &= \frac{1}{3}\binom{2^n - 1}{2}. \end{aligned}$$

□

This proof should, of course, generalize to covers of the doubles of the set $[\binom{n}{1}_q]$ by $\binom{2}{1}_q$-tuples for q a power of a prime.

APPENDIX B

Generating Functions for "Partitions"

Long and Wagner [11] note the following: Let $(c_n)_{n\geq 0}$ with $c_0 = 0$ be a non-negative sequence of real numbers and $(g_n)_{n\geq 0}$ with $g_0 = 1$ a positive sequence of real numbers. Define the generating function (i.e., formal power series—see [13])

$$F(z) = \sum_{n\geq 0} c_n \frac{z^n}{g_n} \tag{B1}$$

and a set of generalized k-nomial coefficients

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{g_n}{g_{n_1} g_{n_2} \cdots g_{n_k}},$$

where $k, n, n_1, n_2, \ldots, n_k \in \mathbf{N}$ with $k \geq 2$ and $n_1 + n_2 + \cdots + n_k = n$. Then the sequence of generating functions

$$F^k(z) = \sum_{n\geq k} d(n,k) \frac{z^n}{g_n} \qquad \text{for all } k \in \mathbf{N}$$

and the explicit formulas

$$d(n,0) = \delta_{n,0}, \qquad \text{(the Kronecker delta)}$$
$$d(n,1) = c_n,$$

and

$$d(n,k) = \sum_{\substack{n_i>0 \\ i\in[k]}} \binom{n}{n_1, n_2, \ldots, n_k} c_{n_1} c_{n_2} \ldots c_{n_k} \qquad \text{for } k \geq 2$$

define the same triangular array $\left(d(n,k)\right)_{0\leq k\leq n<\infty}$.

Long and Wagner mention, further, that certain choices of c_n and g_n give $d(n,k)$ a known combinatorial meaning:

If $c_n, g_n = 1$ for all $n \in \mathbf{P}$, then $d(n,k)$ is the number of ordered partitions of the integer n into k summands. That is, $d(n,k)$ is the number of k-tuples of positive integers with sum n.

If $c_n = 1$ and $g_n = n!$ for all $n \in \mathbf{P}$, then $d(n,k)$ is the number of ordered partitions of an n-set into k blocks.

Let $c_n = 1$ and

$$g_n = (q^{n-1} + q^{n-2} + \cdots + 1)(q^{n-2} + q^{n-3} + \cdots + 1)\ldots(q+1)(1)$$

for all $n \in \mathbf{P}$, where q is a power of a prime. If V is an n-space over F_q, then $d(n,k)$ is the number of chains of length k that include $\{0\}$ and V in the lattice of subspaces of V. This is a special case of an example given by Stanley [16, p.145].

If $c_n = 1$ and

$$g_n = (q^n - 1)(q^n - q)\ldots(q^n - q^{n-1})$$

for all $n \in \mathbf{P}$, where q is a power of a prime, then $d(n,k)$ is the number of direct sum decompositions of an n-space over F_q into k subspaces [1, pp.762–764].

All of these are known results about ordered partitions or analogues thereof, and they all involve similar generating functions. For the sake of completeness Appendix B consists of direct proofs that the coefficients $d(n,k)$ take on the values specified in the preceeding paragraph. We restrict our attention to the case $k \geq 2$ since the results are trivially correct otherwise. The first two results are standard. The third and fourth are proved by far more elegant but less direct means in the references listed above.

We use the standard rules for multiplying power series: If $a_n, b_n \in \mathbf{C}$ for all $n \in \mathbf{N}$ then

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n. \qquad (B2)$$

In general if $a_{n,m} \in \mathbf{C}$ for all $n \in \mathbf{N}$ and $m \in [k]$, then

$$\prod_{m=1}^{k}\left(\sum_{n\geq 0} a_{n,m}x^n\right) = \sum_{n\geq 0}\left(\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i\geq 0 \text{ for all } i}} a_{n_1,1}a_{n_2,2}\ldots a_{n_k,k}\right)x^n. \qquad (B3)$$

These rules together with a little algebraic manipulation and the following propositions suffice to show all four results.

Proposition B1. *Let $2 \leq k \leq n$. There are $\binom{n-1}{k-1}$ ordered sums of the form $n_1 + n_2 + \cdots + n_k = n$ where $n_i \in \mathbf{P}$ for all $i \in [k]$.*

Proof. Let $m_1, m_2, \ldots, m_{k-1} \in [n-1]$ be distinct with $m_1 < m_2 < \cdots < m_{k-1}$. Further, define $m_0 = 0$ and $m_k = n$. The equation $n_i = m_i - m_{i-1}$ defines a one-to-one correspondence between such collections of m_i and ordered sums of positive integers $n_1 + n_2 + \cdots + n_k = n$. Every $(k-1)$-subset of $[n-1]$ yields exactly one

such collection of m_i. Therefore the number of ordered sums $n_1 + n_2 + \cdots + n_k = n$, being the same as the number of such collections of m_i, is $\binom{n-1}{k-1}$.

□

Proposition B2. *Let $2 \leq k \leq n$ and let $n_1 + n_2 + \cdots + n_k = n$ with $n_k \in \mathbf{P}$ for all $i \in [k]$. Then the multinomial coefficient*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!} \tag{B4}$$

counts the number of ordered partitions of $[n]$ in which the ith block has cardinality n_i for $i \in [k]$..

Proof. Let $d(n; n_1, n_2, \ldots, n_k)$ be the number of ordered partitions of $[n]$ in which the ith block has cardinality n_i for $i \in [k]$. Show

$$d(n; n_1, n_2, \ldots, n_k) n_1! n_2! \ldots n_k! = n! \tag{B5}$$

by showing that both sides of equation (B5) count the number of bijections from $[n]$ to itself.

This is well-known for the right-hand side of (B5).

To get the left-hand side define

$$A_1 = \{1, 2, \ldots, n_1\},$$

and for $i = 2, 3, \ldots, k$ define

$$A_i = \{n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_{i-1} + 2, \ldots, n_1 + \cdots + n_i\}.$$

In other words, the A_i partition $[n]$ in such a fashion that the first n_1 positive integers are in A_1, the next n_2 are in A_2, etc. Let $\mathcal{B} = (B_1, B_2, \ldots, B_k)$ be one of the ordered partitions $d(n; n_1, n_2, \ldots, n_k)$ counts. For $i \in [k]$ choose bijections $f_i : B_i \to A_i$, and then define the function $f : [n] \to [n]$ by $f(r) = f_i(r)$ where $r \in B_i$. Clearly f is a bijection on $[n]$.

Each f_i can be chosen in $n_i!$ ways, so there are $n_1! n_2! \ldots n_k!$ ways to define f. Further, distinct choices of $\mathcal{B}$ (for fixed $n_1, n_2, \ldots, n_k$) produce distinct bijections and every bijection is attained for some $\mathcal{B}$ (If $f : [n] \to [n]$ is a bijection, define $B_i = f^{-1}(A_i)$ for $i = 1, 2, \ldots, k$). The number of ways to choose $\mathcal{B}$ is $d(n; n_1, n_2, \ldots, n_k)$, so the total number of bijections from $[n]$ to itself is $d(n; n_1, n_2, \ldots, n_k) n_1! n_2! \ldots n_k!$.

Thus $d(n; n_1, n_2, \ldots, n_k) n_1! n_2! \ldots n_k! = n!$. In other words

$$d(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \ldots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}.$$

□

Proposition B3. *Let $g_0 = 1$ and for all $r \in \mathbf{P}$ let*

$$g_r = \prod_{j=1}^{r} \left(\sum_{i=0}^{j-1} q^i \right)$$
$$= (q^{r-1} + q^{r-2} + \cdots + 1)(q^{r-2} + q^{r-3} + \cdots + 1) \ldots (q+1)(1),$$

where q is a power of a prime. Let $2 \leq k \leq n$ and $n_1 + n_2 + \cdots + n_k = n$ with $n_i \in \mathbf{P}$ for all $i \in [k]$. If V is an n-space over F_q and L is the lattice of subspaces of V, then the number of chains $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$ in L with $\dim(V_i) = n_1 + n_2 + \cdots + n_i$ for $i \in [k]$ is the generalized k-nomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{g_n}{g_{n_1} g_{n_2} \cdots g_{n_k}}. \tag{B6}$$

Proof. Define $d(n; n_1, n_2, \ldots, n_k)$ to be the number of such chains. In a fashion similar to that of the previous proposition, show

$$d(n; n_1, n_2, \ldots, n_k) g_{n_1} g_{n_2} \cdots g_{n_k} = g_n \tag{B7}$$

by showing that both sides of the equation count maximal chains in L.

Proposition A5.4 establishes the right-hand side of (B7).

Let $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$ be a chain in L for which $\dim(V_i) = n_1 + n_2 + \cdots + n_i$ for all $i \in [k]$. We wish to count the maximal chains in L that include $V_0 \subset V_1 \subset \cdots \subset V_k$ as a subchain—i.e., to count the chains $\{0\} = U_0 \subset U_1 \subset \cdots \subset U_n = V$ in which $U_{n_1 + \cdots + n_i} = V_i$ for all $i \in [k]$.

First choose a collection of subspaces $U_1, U_2, \ldots, U_{n_1 - 1}$ in such a fashion that $U_0 \subset U_1 \subset \cdots \subset U_{n_1} = V_1$ is a maximal chain in the interval $[\{0\}, V_1]$. By proposition A5.4 there are

$$g_{n_1} = (q^{n_1 - 1} + q^{n_1 - 2} + \cdots + 1)(q^{n_1 - 2} + q^{n_1 - 3} + \cdots + 1) \ldots (q+1)(1)$$

ways to do this. Next choose $U_{n_1+1}, U_{n_1+2}, \ldots, U_{n_1+n_2-1}$ so as to make the chain $V_1 = U_{n_1} \subset U_{n_2} \subset \cdots \subset U_{n_1+n_2} = V_2$ maximal in the interval $[V_1, V_2]$. The intervals $[V_1, V_2]$, $[V_1/V_1, V_2/V_1]$, and $[\{0\}, W]$, where W is an n_2-space over F_q, are isomorphic as lattices, so there are g_{n_2} ways to do this. Continuing in this fashion there are g_{n_i} ways to form a maximal chain in the interval $[V_{n_i - 1}, V_{n_i}]$ for $i \in [k]$. Altogether, then, $g_{n_1} g_{n_2} \ldots g_{n_k}$ maximal chains of L include the subchain $V_0 \subset V_1 \subset \cdots \subset V_k$.

On the other hand every maximal chain in L has a unique subchain in which the ith member has dimension $n_1 + n_2 + \cdots + n_i$.

Therefore the total number of maximal chains in L is

$$d(n; n_1, n_2, \ldots, n_k) g_{n_1} g_{n_2} \cdots g_{n_k},$$

which is the left-hand side of $(B7)$

Thus

$$d(n; n_1, n_2, \ldots, n_k) = \frac{g_n}{g_{n_1} g_{n_2} \cdots g_{n_k}} = \binom{n}{n_1, n_2, \ldots, n_k}.$$

□

Note that there is a one-to-one correspondence between ordered partitions of an n-set S and chains in the lattice of subsets of S. This correspondence maps the ordered partition $(B_1, B_2, \ldots, B_k)$ into the chain $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S$, where $S_i = \cup_{j=1}^{i} B_j$ for all $i \in [k]$. Thus chains of subspaces of a finite n-space are indeed an analog of ordered partitions of an n-set.

Proposition B4. *Let $g_0 = 1$ and for all $r \in \mathbf{P}$ let*

$$g_r = (q^r - 1)(q^r - q) \ldots (q^r - q^{r-1}).$$

Further let $2 \leq k \leq n$ and $n_1 + n_2 + \cdots + n_k = n$ with $n_i \in \mathbf{P}$ for all $i \in [k]$. If V is an n-space over F_q, then the number of ordered direct sums $V_1 \oplus V_2 \oplus \cdots \oplus V_k = V$ in which $\dim(V_i) = n - i$ for all $i \in [k]$ is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{g_n}{g_{n_1} g_{n_2} \cdots g_{n_k}}. \qquad (B8)$$

Proof. Proceed as in the previous propositions by letting $d(n; n_1, n_2, \ldots, n_k)$ be the number of ordered direct sums of the sort described and then demonstrating that

$$d(n; n_1, n_2, \ldots, n_k) g_{n_1} g_{n_2} \cdots g_{n_k} = g_n \qquad (B9)$$

by showing that both sides of equation (B9) count the ordered bases of V.

Recall that an ordered basis $(v_1, v_2, \ldots, v_n)$ of V is just an n-tuple of linearly independent vectors. The proof of Proposition A5.2 establishes the right-hand side of (B9).

Let $(v_1, v_2, \ldots, v_n)$ be an ordered basis of V. This basis uniquely determines an ordered direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_k = V$ with $\dim(V_i) = n_i$ for $i \in [k]$ by

$$V_i = \text{span}\{v_{n_1+n_2+\cdots+n_{i-1}+1}, v_{n_1+n_2+\cdots+n_{i-1}+2}, \ldots, v_{n_1+n_2+\cdots+n_i}\}$$

How many ordered bases of V produce this same direct sum? Clearly another ordered basis will do so only if for $i \in [k]$ the subsequence

$$(v_{n_1+n_2+\cdots+n_{i-1}+1}, v_{n_1+n_2+\cdots+n_{i-1}+2}, \ldots, v_{n_1+n_2+\cdots+n_i})$$

is replaced by another ordered basis of V_i. Again by the proof of A5.2, for $i \in [k]$ the subspace V_i has

$$(q^{n_i}-1)(q^{n_i}-q)\ldots(q^{n_i}-q^{n_i-1}) = g_{n_i}$$

ordered bases. Thus $g_{n_1}g_{n_2}\ldots g_{n_k}$ ordered bases of V produce this same direct sum.

Therefore there are $d(n;n_1,n_2,\ldots,n_k)g_{n_1}g_{n_2}\ldots g_{n_k}$ ordered bases of V. This is the left-hand side of equation (B9).

Thus

$$d(n;n_1,n_2,\ldots,n_k) = \frac{g_n}{g_{n_1}g_{n_2}\cdots g_{n_k}} = \binom{n}{n_1,n_2,\ldots,n_k}.$$

□

Returning to the consideration of power series, let $c_0 = 0$ and $c_n = 1$ for $n \in \mathbf{P}$. If $k \geq 2$, define $a_{0,i} = 0$ and $a_{n,i} = 1$ for all $i \in [k]$ and $n \in \mathbf{P}$. Further let $g_0 = 1$ and for all $n \in \mathbf{P}$ let g_n be a positive real number. Then

$$\begin{aligned}
\left(\sum_{n\geq 0} c_n \frac{x^n}{g_n}\right)^k &= \prod_{i=1}^{k}\left(\sum_{n\geq 0}\frac{a_{n,i}}{g_n}x^n\right) \\
&= \sum_{n\geq 0}\left(\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i\geq 0 \text{ for all } i\in[k]}} \frac{a_{n_1,1}}{g_{n_1}}\frac{a_{n_2,2}}{g_{n_2}}\cdots\frac{a_{n_k,k}}{g_{n_k}}\right)x^n \\
&= \sum_{n\geq 0}\left(\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i> 0 \text{ for all } i\in[k]}} \frac{1}{g_{n_1}}\frac{1}{g_{n_2}}\cdots\frac{1}{g_{n_k}}\right)x^n \\
&= \sum_{n\geq 0}\left(\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i> 0 \text{ for all } i\in[k]}} \frac{g_n}{g_{n_1}g_{n_2}\cdots g_{n_k}}\right)\frac{x^n}{g_n} \\
&= \sum_{n\geq 0}\left(\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i> 0 \text{ for all } i\in[k]}} \binom{n}{n_1,n_2,\ldots,n_k}\right)\frac{x^n}{g_n} \\
&= \sum_{n\geq 0} d(n,k)\frac{x^n}{g_n}
\end{aligned}$$

as Long and Wagner [11] note.

Propositions B1–B4 now yield the following interpretations of $d(n,k)$.

First interpretation: If $g_n = 1$ for all $n \in \mathbf{P}$, then

$$\begin{aligned} d(n,k) &= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0 \text{ for all } i\in[k]}} 1 \\ &= \text{the number of ordered partitions of } n \\ &= \binom{n-1}{k-1}. \end{aligned}$$

Second interpretation: If $g_n = n!$ for all $n \in \mathbf{P}$ then

$$\begin{aligned} d(n,k) &= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0 \text{ for all } i\in[k]}} \binom{n}{n_1,n_2,\ldots,n_k} \\ &= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0 \text{ for all } i\in[k]}} \begin{array}{l}\text{(the number of ordered partitions of an} \\ n\text{-set into } k \text{ blocks in which the } i\text{th block} \\ \text{has } n_i \text{ elements)}\end{array} \\ &= \text{the number of ordered partitions of an } n\text{-set into } k \text{ blocks.} \end{aligned}$$

Third interpretation: Let

$$g_n = (q^{n-1} + q^{n-2} + \cdots + 1)(q^{n-2} + q^{n-3} + \cdots + 1)\ldots(q+1)(1)$$

for all $n \in \mathbf{P}$. If $n \in \mathbf{P}$ and L is the lattice of subspaces of an n-space V over F_q, then

$$\begin{aligned} d(n,k) &= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0 \text{ for all } i\in[k]}} \binom{n}{n_1,n_2,\ldots,n_k} \\ &= \text{the number of chains of length } k \text{ in } L \text{ that.} \\ &\quad \text{include } \{0\} \text{ and } V \end{aligned}$$

Fourth interpretation: If

$$g_n = (q^n - 1)(q^n - q)\ldots(q^n - q^{n-1})$$

for all $n \in \mathbf{P}$, then

$$\begin{aligned} d(n,k) &= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i>0 \text{ for all } i\in[k]}} \binom{n}{n_1,n_2,\ldots,n_k} \\ &= \text{the number of ordered direct sum decom-} \\ &\quad \text{positions of a finite } n\text{-space over } F_q \text{ into} \\ &\quad k \text{ summands.} \end{aligned}$$

VITA

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