T125
S5

To the Graduate Council:

I am submitting herewith a thesis written by Michael Joseph Petkac entitled "A Compilation and Some Numerical Results of Numerical Methods for the Solution of a Special Class of Second-Order Differential Equations." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

Steven M. Serbin

Steven M. Serbin, Major Professor

We have read this thesis
and recommend its acceptance:

Don Hinton

John J. Fay Jr.

Accepted for the Council:

Lee Minkel

Vice Provost
and Dean of The Graduate School

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a Master's of Art degree at The University of Tennessee, Knoxville, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, by the Head of Interlibrary Services when, in the opinion of either, the proposed use of the material is for scholarly purposes. Any copying or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature Michael Joseph Pettac

Date August 6, 1986

A COMPILATION AND SOME NUMERICAL RESULTS OF NUMERICAL METHODS FOR THE SOLUTION OF A SPECIAL CLASS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

A Thesis

Presented for the

Master of Arts

Degree

The University of Tennessee, Knoxville

Michael Joseph Petkac

August 1986

ACKNOWLEDGEMENTS

I would like to thank my committee members, Professors Don Hinton and John Long, for their suggestions which enabled me to attain the final form of this thesis. Special thanks to Professor Steven M. Serbin, who served as my thesis director, for the many hours of assistance and insight that he gave from the very start of my research. I would also like to thank Jessica Daugherty for typing this final version.

ABSTRACT

Recent method developments for the numerical solution of initial value problems of the form $y'' = f(x, y)$ are investigated in this thesis. Since text books on the numerical solution of differential equations contain little or no information on methods produced in the last ten years, the content of major journal articles is surveyed. The approaches that authors have used to develop new methods are discussed. Since a large portion of the literature pertains to P-stable methods, results of numerical experiments conducted for many P-stable methods are reported and some conclusions are drawn. A proposed stability criterion for $y'' = f(x, y)$ is also briefly examined.

TABLE OF CONTENTS

CHAPTER		PAGE
INTRODUCTION		1
1.	NUMERICAL METHODS FOR SOLUTION OF $Y'' = F(X, Y)$; $Y(X_0) = Y_0$, $Y'(X_0) = Y'_0$	2
	1.1. Notation	2
	1.2. P-Stability and Hybrid Two-Step Methods	2
	1.3. A P-Stable Four-Step Method and P-Stable One-Step Methods	9
	1.4. P-Stable Methods Dependent on a Parameter	12
	1.5. A Phase Property of Two-Step Methods	15
	1.6. Modifications of Numerov's Method	16
	1.7. Nystrom and Runge-Kutta Type One-Step Methods	17
	1.8. Other Methods	20
2.	TWO-STEP P-STABLE METHODS	25
	2.1. Introduction	25
	2.2. P-Stability for Two-Step Methods	26
	2.3. Numerical Results	29
	2.4. Conclusions	65
3.	A NEW STABILITY CRITERION FOR $\ddot{Y} = F(X, Y)$	70
	BIBLIOGRAPHY	72
	VITA	77

LIST OF TABLES

TABLE PAGE

1. P-Stable Methods Numerically Tested 30
2. |Error| in the Solution of $x = 20\pi$ for the Integration of (2.3.1) 45
3. |Error| in the Solution at $x = 2\pi$ and $x = 20\pi$ for the Integration of (2.3.2) 56
4. |Error| in the Solution of $x = 6$ for the Integration of (2.3.3) 61
5. |Error| in the Solution at $x = 40\pi$ for the Integration of (2.3.4) 66

INTRODUCTION

Much work recently has been directed to the numerical solution of the initial value problem

$$(I.1) \qquad y'' = f(x, y) \; ; \; y(x_0) = y_0 \; , \; y'(x_0) = y_0' \; .$$

However, most books directed to the numerical solution of differential equations contain little of no information on this special problem, especially the work done recently.

Chapter 1, of this thesis deals with advances produced in the last ten years in methods designed to solve (I.1). While this treatment is not claimed to encompass every work produced since the mid-seventies nor highly detail those articles which are explored, it is an attempt to highlight some of the major method developments and method modifications.

In Chapter 2, P-stable methods for problems of the form (I.1) whose solutions are periodic are explored. The analysis and most of the numerical results given are for two-step (hybrid) methods, although some one-step methods and a four-step (hybrid) method are considered.

The final chapter, Chapter 3, is a brief discussion of a proposed stability criterion for one-step methods applied to (I.1).

CHAPTER 1

NUMERICAL METHODS FOR SOLUTION OF

$$Y'' = F(X, Y) \; ; \; Y(X_0) = Y_0 \; , \; Y'(X_0) = Y_0'$$

1.1. NOTATION

In considering numerical methods for the solution of

$$(1.1.1) \quad y'' = f(x, y) \; ; \; y(x_0) = y_0 \; , \; y'(x_0) = y_0'$$

some common notation will be used.

(a) The points at which the solution is approximated are denoted x_n where $n \in (0, 1, \ldots, k)$. The corresponding approximate solution will be denoted y_n .

(b) The function $f(x_n, y_n)$ will be referred to as f_n .

(c) The step size, h , is the value given by $x_{n+1} - x_n$, and will be assumed to be constant.

1.2. P-STABILITY AND HYBRID TWO-STEP METHODS

Lambert and Watson [38] investigate linear multistep methods of the form

$$(1.2.1) \quad \alpha_k y_{n+k} + \ldots + \alpha_1 y_{n+1} + \alpha_0 y_n = h^2(\beta_k f_{n+k} + \ldots + \beta_0 f_n)$$

or, equivalently

$$(1.2.2) \quad \sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \; .$$

The polynomials ρ and σ given by

$$(1.2.3) \quad \rho(\tau) = \sum_{j=0}^{k} \alpha_j \tau^j \; ; \; \sigma(\tau) = \sum_{j=0}^{k} \beta_j \tau^j$$

characterize the multistep methods when applied to (1.1.1). Thus, the authors refer to (1.2.2) as method (ρ, σ). The order of (ρ, σ) is defined to be P with error constant C_{P+2} if for an adequately smooth test function $z(x)$

$$(1.2.4) \quad \sum_{j=0}^{k} \alpha_j z(x + jh) - h^2 \sum_{j=0}^{k} \beta_j z''(x + jh) = C_{P+2} h^{P+2} z_{(x)}^{(P+2)} + O(h^{P+3}) .$$

The method (ρ, σ) is said to be symmetric if $\alpha_j = \alpha_{k-j}$ and $\beta_j = \beta_{k-j}$ for $j = 0, 1, \ldots, k$, and it is shown that a symmetric method has even order.

Applying (ρ, σ) to the test problem

$$(1.2.5) \quad y'' = -\lambda^2 y \quad , \qquad \lambda , y \in \mathbb{R}$$

Lambert and Watson obtain the difference equation

$$(1.2.6) \quad \sum_{j=0}^{k} (\alpha_j + H^2 \beta_j) y_{n+j} = 0 \quad , \quad H = h\lambda \quad .$$

The general solution of this difference equation is

$$(1.2.7) \quad y_n = \sum_{s=1}^{k} A_s r_s^n$$

where r_s , $s = 1, 2, \ldots, k$, are the roots of the polynomial, assumed distinct, given by

$$(1.2.8) \quad \Omega(r; H^2) = \rho(r) + H^2\sigma(r) .$$

The authors say the method (ρ, σ) has interval of periodicity $(0, H_0^2)$ if $H^2 \in (0, H_0^2)$, the roots, r_s , of $\Omega(r; H^2)$ defined by (1.2.8) satisfy

$$(1.2.9) \quad r_1 = e^{i\theta(H)} , \; r_2 = e^{-i\theta(H)} , \; |r_s| \leq 1 , \; s = 3, 4, \ldots, k .$$

where $\theta(H) \in \mathbf{R}$. Thus, if H^2 lies within the interval of periodicity, (ρ, σ) will have a period approximating the theoretical solution of (1.2.5).

Lambert and Watson define a method (ρ, σ) which has an interval of periodicity $(0, \infty)$ as P-stable. This stability criterion is best applied for problems of the form (1.1.1) whose solution is periodic.

It is also stated that a method (ρ, σ) which is P-stable must be symmetric and has an order no greater than two. The authors list a two-step step method which is P-stable along with some higher order methods that have a finite interval of periodicity.

Dahlquist [19] discusses unconditional stability, an idea similar to that of P-stability. When (ρ, σ) is applied to (1.2.5) and all solutions of the difference equations are bounded the method is referred to as unconditionally stable.

Because of Lambert and Watson's work, many attempts have been made to develop P-stable (hybrid) multistep methods which have order greater than two.

In his 1979 article [28], Hairer began the idea of "successive substitution" to develop a hybrid two-step method of $O(h^4)$ which is P-stable with the exception of $h\lambda = \sqrt{12}$. The development of the method begins by considering the two-step formula

$$(1.2.10)\quad y_{n+1} - 2y_n + y_{n-1} = h^2[\beta_0(f_{n+1} + f_{n-1}) + \gamma f_n + \beta_1(f_{n+\alpha_1} + f_{n-\alpha_1})]$$

where $0 \leq \alpha_1 < 1$. Note that (1.2.10) is symmetric about the point (x_n, y_n), a condition Lambert and Watson found necessary for a linear multistep method to be P-stable. Since α_1 is an off-step point, a formulation for $y_{n\pm\alpha_1}$ must be found. Successive substitution involves the approximation of the quantities $y_{n\pm\alpha_1}$ by expressions which only involve y_{n-1}, y_n and y_{n+1}.

The general symmetric (hybrid) two-step fourth order method as given by Cash [3] is

$$(1.2.11)\quad y_{n+1} - 2y_n + y_{n-1} = h^2\{(\frac{1}{12} - \beta_1\alpha_1^2)(y''_{n+1} + y''_{n-1}) + [\frac{5}{6} + 2\beta_1(\alpha_1^2 - 1)]y''_n + \beta_1(\bar{y}''_{n+\alpha_1} + \bar{y}''_{n-\alpha_1})\}$$

$$(1.2.12)\quad \bar{y}_{n+\alpha_1} = Ay_{n+1} + By_n + Cy_{n-1} + h^2[s\, y''_{n+1} + ty''_n + uy''_{n-1}]$$

$$(1.2.13)\quad \bar{y}''_{n-\alpha_1} = Cy_{n+1} = By_n + Ay_{n-1} + h^2[uy''_{n+1} + ty''_n + sy''_{n-1}]$$

where $C = A - \alpha_1$, $B = 1 + \alpha_1 - 2A$, $s = u + \alpha_1^3/6 - \alpha_1/6$, $t = -A + 2\alpha_1/3 + \alpha_1^2/2 - \alpha_1^3/6 - 2u$, and where $\bar{y}''_{n+\alpha_1} \equiv f(x_{n+\alpha_1}, \bar{y}_{n+\alpha_1})$. To obtain his method, which involves two function evaluations per step, Hairer uses the constants $\alpha_1 = 0$, $A = 0$, $\beta_1 = \frac{R}{24}$, $u = -\frac{1}{12R}$ where R is a free parameter. Hairer notes that in the limit case $R = 0$ (which is not possible because of division by zero), his method resembles the Numerov method

$$(1.2.14)\quad y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}(f_{n+1} + 10f_n + f_{n-1})$$

and may thus be considered as a stabilized modification of (1.2.14).

In 1981, Cash [2] and Chawla [6] independently developed P-stable two-step methods that require three function evaluations per step. In both author's methods, the formulae for $\bar{y}_{n+\alpha_1}$ and $\bar{y}_{n-\alpha_1}$ are slightly different than (1.2.12) and (1.2.13) respectively.

Later, to return to fourth order methods requiring two function evaluations per step, a slightly different approach than that used by Hairer was implemented by Costabile and Costabile [17]. They chose $\alpha_1 = \frac{1}{2}$, $A = \frac{1}{2}$, $\beta_1 = \frac{2}{3}$ and $u = 0$ in (1.2.11) - (1.2.13) so the value of $\bar{y}''_{n+\alpha_1}$ at one step can be used for $\bar{y}''_{n-\alpha_1}$ at the next step. The authors final statement of their method contains an error that was corrected by Cash [3].

Chawla [7] modified Numerov's method (1.2.14) by replacing y_n in f_n by

$$(1.2.15)\quad \bar{y}_n = y_n - \alpha h^2(f_{n+1} - 2f_n + f_{n-1}) .$$

The resulting fourth order formula is P-stable for $\alpha > \frac{1}{120}$ and requires two function evaluations per step.

Cash [3] also proposed an $O(h^4)$ method that requires two function evaluations per step. To obtain his method, Cash chose $\alpha_1 = 0$, $A = 0$, $u = -\frac{1}{94}$ and $\beta_1 = \frac{1}{3}$ in (1.2.11) - (1.2.13). Again, the original article contains an error that was corrected in [4].

Jain, Jain and Krishnaiah [32] derive five P-stable methods that have the form

$$(1.2.16)\quad y_{n+1} - 2y_n + y_{n-1} = h^2[W_0 f_n + W_1(f_{n-1} + f_{n+1}) + \sum_{i=1}^{m} W_{r_i}(f_{n-r_i} + f_{n+r_i})]$$

where W_0, W_1, W_{r_i} are the weights and $r_i \neq 0$ or 1, are the off-step points. To find the weights and off-step points, both sides of (1.2.16) are expanded into the Taylor series about the point $x = x_n$ and like powers of h are equated to their required order. From this system of linear equations the values of W can be solved in terms of r_i which are chosen as modes of a known quadrature rule. The methods given include r_i's corresponding to four point Lobatto quadratures, Radau formula with 1 as a node and Gauss three point quadrature. One

of these symmetric methods requires three function evaluations per step, two require five and two require seven function evaluations per step respectively. Note that $r = \frac{5 - \sqrt{5}}{10}$ and $s = \frac{5 + \sqrt{5}}{10}$ in the author's equation (10) should be $r = \frac{5 - \sqrt{15}}{10}$ and $s = \frac{5 + \sqrt{15}}{10}$.

Using the same general form (1.2.16), Jain, Jain and Krishnaiah also derive sixth order symmetric P-stable methods. The difference in order results from a change to higher order approximations for $y_{n\pm r_i}$. This change in approximations also results in two additional function evaluations per step for each method.

To develop a sixth order method involving five function evaluations per step, Cash [2] looks at the general form.

$$(1.2.17)\quad y_{n+1} - 2y_n + y_{n-1} = h^2[\beta_0(f_{n+1} + f_{n-1}) + \beta_1(f_{n+\alpha_2} + f_{n-\alpha_2}) + \gamma f_n]$$

where $y_{n\pm\alpha_2}$ depend on $y_{n\pm\alpha_1}$. Interestingly, Cash uses $O(h^5)$ approximations for $y_{n\pm\alpha_1}$ and uses over approximations of order seven to approximate $y_{n\pm\alpha_2}$. The high order approximations used for $y_{n\pm\alpha_2}$ show up favorably in numerical results.

Cash [3] reduces the function evaluations required to four per step for a sixth order method by working with

$$(1.2.18)\quad y_{n+1} - 2y_n + y_{n-1} = h^2[\beta_0(f_{n+1} + f_{n-1}) + \beta_1(f_{n+\alpha_1} + f_{n-\alpha_1}) + \beta_2(f_{n+\alpha_2} + f_{n-\alpha_2}) + \gamma f_n]$$

and taking $\alpha_2 = 0$. While $y_{n\pm\alpha_2}$ again depend on $y_{n\pm\alpha_1}$, Cash lowers the approximations to $y_{n\pm\alpha_2}$ to order six.

Using the form (1.2.17) with $\alpha_1 = \alpha_2 = \frac{1}{2}$, Chawla and Rao [11], [12] also derive a sixth order P-stable method. While the authors use an $O(h^6)$ approximation to $y_{n\pm\alpha_2}$, they note that the symmetry involved in their formula makes it sufficient to use an $O(h^3)$ approximation to $y_{n\pm\alpha_1}$. In general, this method requires five function evaluations per step. However, the number of function evaluations is reduced for nonlinear problems solved by using a modified Newton iterative procedure given by the authors (see section 2.4).

1.3. A P-STABLE FOUR-STEP METHOD AND P-STABLE ONE-STEP METHODS

Jain, Kambo and Goel [35] produced a P-stable four-step method by applying successive substitution to

$$(1.3.1)\quad y_n - (2 + a)y_{n+1} + (2 + 2a)y_{n+2} - (2 + a)y_{n+3} + y_{n+4}$$

$$= h^2\left(\frac{18 + a}{240} f_n + \frac{26 - 3a}{30} f_{n+1} + \frac{14 - 97a}{120} f_{n+2}\right.$$

$$\left. + \frac{26 - 3a}{30} f_{n+3} + \frac{18 + a}{240} f_{n+4}\right), \quad -2 < a < 2 .$$

The formula (1.3.1), which was given by Lambert and Watson [38] has a finite interval of periodicity of $(0, \frac{60(2 + a)}{22 + 9a})$. Working with $a = \frac{1}{2}$, Jain, Kambo and Goel obtained their method by replacing f_n in (1.3.1) by $f(x_n, \bar{y}_n)$ where

$$(1.3.2)\quad \bar{y}_n = y_n - \frac{7h^2}{144}(f_n - 4f_{n+1} + 6\bar{f}_{n+2} - 4f_{n+3} + f_{n+4})$$

and

$$(1.3.3)\quad \bar{y}_{n+2} = y_{n+2} - \frac{h^2}{101}(f_n - 2f_{n+2} + f_{n+4}) .$$

The problem with this method and all K-step methods where $K > 1$ is that they require a starting procedure to produce initial approximations of $y_1, y_2, \ldots, y_{K-1}$.

On the other hand, the one-step P-stable methods require only the initial conditions of (1.1.1) to produce a numerical solution. Hairer [28] proposed a P-stable class of one-step Nystrom type formulas based on Gaussian quadrature. A listing of the general method as well as the (s, s) matrices required to produce methods of order $2s$ for $s = 2$ and $s = 3$ are contained in Table 1, Methods 19-20.

Another class of one-step P-stable methods of orders six and eight were developed by Jain, Kambo and Goel [34]. Integrating (1.1.1) from x_0 to $x_0 + h$, the authors obtain the integral equations

$$(1.3.4)\quad y(x_0 + h) = y(x_0) + hy'(x_0) + \int_{x_0}^{x_0+h} (x_0 + h - \tau)f(\tau, y(\tau))d\tau$$

$$(1.3.5)\quad y(x_0 + h) = y'(x_0) + \int_{x_0}^{x_0+h} f(\tau, y(\tau))d\tau .$$

Labatto K-point quadrative formulas are used to approximate the integrals in (1.3.4) and (1.3.5), which results in the formulas

$$(1.3.6)\quad y_{n+1} = y_n + hy'_n + \frac{h^2}{K(K-1)}f_n + h^2 \sum_{i=1}^{K-2} W_i(1 - t_i)f(x_{n+t_i}, y_{n+t_i})$$

$$(1.3.7)\quad hy'_{n+1} = hy'_n + \frac{h^2}{K(K-1)}(f_n + f_{n+1}) + h^2 \sum_{i=1}^{K-2} W_i f(x_{n+t_i}, y_{n+t_i})$$

where t_i's are Lobatto modes and W_i's the corresponding weights. The y_{n+t_i}'s in (1.3.5) and (1.3.7) are given by the approximation

$$(1.3.8)\quad y_{n+t_i} = y_n + t_i h + h^2 \sum_{j=1}^{K-2} \alpha_{ij} f(x_{n+t_i}, y_{n+t_i}) ,$$

$$i = 2, 3, \ldots, K-1$$

where the α_{ij} are chosen to yeild formulas of orders six and eight.

Jain, Kambo and Goel [33] consider the hybrid one-step method of the form

$$(1.3.9)\quad y_{n+1} = y_n + hy'_n + h^2 \sum_{i=0}^{\ell} \alpha_i f_{n+i} + h^2 \alpha_2 f_{n+P}$$

$$(1.3.10)\quad y'_{n+1} = y'_n + h \sum_{i=0}^{\ell} \beta_i f_{n+i} + h^2 \beta_2 f_{n+P}$$

$$(1.3.11)\quad y_{n+P} = \sum_{i=0}^{\ell} \gamma_i y_{n+i} + h \sum_{i=0}^{\ell} \gamma_{i+2} y'_{n+i} + h^2 \sum_{i=0}^{\ell} \gamma_{i+4} f_{n+i} .$$

The authors determine α_i, β_i and γ_i so that the resulting method is P-stable and has order four. An interesting aspect of the final formula is that it requires only the calculation of y'_{2n+2} to produce the y_{2n+1} and y_{2n+2} approximations.

1.4. P-STABLE METHODS DEPENDENT ON A PARAMETER

While the P-stable methods discussed in sections 1.2 and 1.3 were not dependent on any parameters, the P-stability property of the methods in this section is dependent on a paramter, P, which must be chosen as the square of the frequency of the solution to (1.1.1). These methods therefore require an a priori knowledge of the solution's period. Early work in this area was conducted by Gautschi [24] and Salyer [42] who developed methods based on Fourier polynomials, Steifel and Bettis [45] who discussed a stabilization of the Cowell method for integrating periodic initial value problems and Scheifele [43] who developed a one-step method based on the series expansions of certain entire functions.

More recently, Jain, Jain and Krishnaiah [31] have developed P-stable single-step methods. They begin by writing (1.1.1) in the form

$$y'' + Py = \phi(x, y) \tag{1.4.1}$$

where

$$\phi(x, y) = f(x, y) + Py \tag{1.4.2}$$

where P is a parameter to be determined. The equation (1.4.1) is then written as

$$(1.4.3) \quad y'' + Py = g(x)$$

where $g(x)$ is an approximation of $\phi(x, y)$. The general solution of (1.4.3) and its derivative consists of a complementary function and a particular integral involving $g(\tau)$. The general solution and its derivative are evaluated at the points x_{n+1} and x_n to obtain.

$$(1.4.4) \quad y(x_{n+1}) = y(x_n)\cos(\sqrt{P}\,h) + y'(x_n)\sin(\sqrt{P}\,h) + \frac{1}{\sqrt{P}} \int_{x_n}^{x_{n+1}} g(\tau)\sin[\sqrt{P}(x_{n+1} - \tau)]d\tau$$

$$(1.4.5) \quad y'(x_{n+1}) = -\sqrt{P}\, y(x_n)\sin(\sqrt{P}\,h) + y'(x_n)\cos(\sqrt{P}\,h) + \int_{x_n}^{x_{n-1}} g(\tau)\cos[\sqrt{P}(x_{n+1} - \tau)]d\tau \quad .$$

The authors approximate $g(\tau)$ beginning with a Taylor interpolating polynomial. However, to avoid calculation of higher order derivatives, they write the resulting approximations in a modified Runge-Kutta form to obtain a method which is $0(h^2)$ $(0(h^4)$ as $P \to 0)$. When P is chosen as the square of the frequency, this method is shown to be P-stable.

Jain, Jain and Krishnaiah derive a second one-step method by writing $g(\tau)$ as

$$(1.4.6) \quad g(\tau) = P(y_n) + A + B(\tau - x_n) + \frac{C}{2}(\tau - x_n)^2 \quad .$$

To determine P, A, B and C, the authors evalute (1.4.6) at the points x_n, $x_i = x_n + a_i h$, $i = 2, 3, 4$, where the a_i's are given by the modes of the classical Runge-Kutta four-stage method. The approximation (1.4.5) is then used in (1.4.4) and (1.4.5) to obtain a formula with order $O(h)$ ($O(h^3)$ as $P \to 0$). The method is again P-stable if P is given as the square of the frequency.

In [30], Jain, Jain and Krishnaiah begin with similar analysis as above to derive the solution of (1.1.1) as a complementary function and an integral dependent upon $g(\tau)$. Evaluating the solution at the points x_{n+1}, x_n and x_{n-1} gives the formula

$$(1.4.7) \quad y(x_{n+1}) - 2\cos(\sqrt{P}\,h)y(x_n) + y(x_{n-1}) = \frac{1}{\sqrt{P}} \int_{x_n}^{x_{n+1}} \sin[\sqrt{P}\,(x_{n+1} - \tau)][g(\tau) + g(2x_n + \tau)]d\tau .$$

To obtain an explicit method, the authors use the Newton backward difference interpolation polynomial at the points x_n, x_{n-1}, $\cdots$, x_{n-K+1} to approximate $g(\tau)$. An implicit method is given by using the Newton backward difference polynomial at the points x_{n+1}, x_n, $\cdots$, x_{n-K+1} to approximate $g(\tau)$. Both methods are P-stable when P is chosen as the square of the frequency and have $O(h^2)$ accuracy ($O(h^4)$ as $P \to 0$).

Krishnaiah [37] approximates $g(\tau)$ in (1.4.7) with Tayor polynomials of order $2q$ to obtain an explicit method of order $2q$ ($2q + 2$ as $P \to 0$). The author also modifies the second order implicit method of [30] by using the method of undetermined coefficients to obtain an

$O(h^4)$ implicit method ($O(h^6)$ as $P \to 0$). Both methods are P-stable when P is taken as the square of the frequency.

1.5. A PHASE PROPERTY OF TWO-STEP METHODS

Thomas [46] discusses phase lags of two-step methods, an idea discussed earlier by Brusa and Nigro [1] and Gladwell and Thomas [25]. Thomas looks at methods applied to (1.2.5) to obtain the difference equation

$$(1.5.1) \quad q_2(\lambda h)y_{n+1} + q_1(\lambda h)y_n + q_0(\lambda h)y_{n-1} = 0 .$$

where q_0, q_1 and q_2 are polynomials, even in λh. The general solution of (1.5.1) is

$$(1.5.2) \quad y_n = B_1 r_1^n + B_2 r_2^n$$

where r_1 and r_2 are the zeros of the stability polynomial

$$(1.5.3) \quad q_2 r^2 + q_1 r + q_0 = 0 .$$

The author writes the zeros of the stability polynomial (1.5.3) in the form

$$(1.5.4) \quad r_1 = \exp([-a(\lambda h) + ib(\lambda h)]\lambda h)$$

$$(1.5.5) \quad r_2 = \exp([-a(\lambda h) - ib(\lambda h)]\lambda h)$$

Since the solution, $e^{i\lambda x}$, of (1.2.5) is exactly reproduced if $a \equiv 0$ and $b \equiv 1$, the frequency distortion depends on the magnitude of $|b - 1|$. The modulus of the leading term of the expansion of $b - 1$ in powers of λh is defined to be the phase lag of the method.

Thomas examines Cash's fourth and sixth order methods [2] and shows they have the same order phase lags as their respective accuracy orders. The author then gives a fourth order two-step method having a sixth order phase lag as well as a sixth order two-step method with a phase lag of order eight. These formulas are not P-stable.

1.6. MODIFICATIONS OF NUMEROV'S METHODS

By making Numerov's implict two-step method (1.2.14) explicit in a way discribed below, Chawla [8] observed that the interval of periodicity increases in size from $\sqrt{6}$ to $\sqrt{12}$. The modification of (1.2.14) is produced by replacing f_{n+1} by $f(x_{n+1}, \bar{y}_{n+1})$ where

$$\bar{y}_{n+1} = 2y_n - y_{n-1} + h^2 f_n \quad . \tag{1.6.1}$$

Another modification of Numerov's method is addressed by Chawla and Rao in [10]. The author's intent was to produce a Numerov-type method with minimal phase lag. By replacing f_n by $f(x_n, \bar{y}_n)$ in (1.2.14) where

$$\bar{y}_n = y_n - \alpha h^2 (f_{n+1} - 2f_n + f_{n-1}) \text{ , } \alpha \text{ a free paramter.} \tag{1.6.2}$$

Chawla and Rao reduce the phase lag from $\frac{1}{480} \lambda^4 h^4$ to $\frac{1}{12096} \lambda^6 h^6$.

This modification also produces an increase in the size of periodicity from 2.449 ($\sqrt{6}$) to 2.71.

1.7. NYSTROM AND RUNGE-KUTTA TYPE ONE-STEP METHODS

P.J. van der Houwen [47] analysed a class of explicit Runge-Kutta-Nystrom methods of orders one and two for the integration of large systems of second order differential equations of the form (1.1.1). The systems considered arise from the semi discretization of certain classes of hyperbolic differential equations. The author's Runge-Kutta type formulas are given by

$$
\begin{aligned}
& y_{n+1}^{(0)} = y_n \\
(1.7.1) \quad & y_{n+1}^{(j)} = y_n + \mu_j h y'_n + h^2 \sum_{\ell=0}^{j-1} \lambda_{j,\ell} f(x_n + \mu_\ell h, y_{n+1}^{(\ell)}), \quad j = 1, 2, \ldots, m \\
& y_{n+1} = y_{n+1}^{(m)} \qquad y'_{n+1} = y'_n + h \sum_{\ell=0}^{m-1} \beta_\ell f(x_n + \mu_\ell h, y_{n+1}^{(\ell)}), \quad \mu_0 = 0
\end{aligned}
$$

where

$$(1.7.2) \qquad \mu_m = \sum_{\ell=0}^{m-1} \beta_\ell = 1$$

$$(1.7.3) \qquad \sum_{\ell=0}^{m-1} \lambda_{m,\ell} = \sum_{\ell=1}^{m-1} \beta_\ell \mu_\ell = \frac{1}{2} .$$

When (1.7.1) is applied to

$$(1.7.4) \qquad y'' = Jy$$

wnere J is a matrix, the relation

$$(1.7.5)\quad \begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} = R(h^2 J) \begin{pmatrix} y_n \\ hy'_n \end{pmatrix}$$

is obtained. Here, R is a matrix valued function defined by

$$R_0(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(1.7.6)\quad R_j(z) = \begin{pmatrix} 1 & \mu_j \\ 0 & 1 \end{pmatrix} + z \sum_{\ell=0}^{j-1} \lambda_{j,\ell} R_\ell(z) \ , \ j = 1, 2, \ldots, m-1$$

$$R(z) = R_m(z) = \begin{pmatrix} 1 & \mu_m \\ 0 & 1 \end{pmatrix} + z \sum_{\ell=0}^{m-1} \begin{pmatrix} \lambda_{m,\ell} & 0 \\ \beta_\ell & 0 \end{pmatrix} R_\ell(z) \ .$$

The elements of $R(z)$ are polynomials of degree m in z. Letting z be a complex number and $\alpha_j(z)$ denote the eigenvalues of $R(z)$, the stability region of (1.7.1) is defined as

$$(1.7.7)\quad \{z| \ |\alpha_j(z)| < 1, \ j = 1, 2\} \ .$$

The values of m for these m-stage Runge-Kutta type methods considered by van der Houwen are small $(m \leq 4)$ and the parameters in (1.7.1) are chosen so the stability region along the negative axis is as large as possible. The stability region is shown to be $[-\beta, 0]$ where $\beta \cong 4m^2$.

Using the same stability condition, van der Houwen [48] continues the investigation of large second order systems resulting from the same discretization of partial differential equations of hyperbolic type.

However, the author considers m-stage methods for large values of m. Since internal instability is often observed when using Runge-Kutta type schemes with many stages, van der Houwen derives modified second order Nystrom formulas which are stable in all stages of the integration step. The formulae, which are based on properties of Chebyshev polynomials, are referred to by the author as Nystrom-Chebyshev methods. As before, the new methods have a stability interval of $4m^2$. These methods allow for adjustable damping of unwanted higher harmonics in the numerical solution.

Hairer [29] developed a tenth order Nystrom method requiring eleven function evaluations per step. The author points out the significance of this development; all known tenth order Runge-Kutta methods require at least seventeen function evaluations per step.

Hairer [27], Chawla and Sharma [13], [14] and Chawla [9] have all studied Nystrom methods. Hairer developed all methods of order two, three and four while Chawla and Sharma developed all methods of order five. The function evaluations required for these methods are one less than their orders. The remainder of [13] and the two articles [9] and [14] are devoted to a study of intervals of periodicity and absolute stability of the methods.

In addition, three German articles concerning methods of the type considered in this section have recently appeared. The authors' summaries are as follows:

1) Fehlberg [21]

> A 9-th order Runge-Kutta-Nystrom formula for $\ddot{x} = f(t, x)$ with step size control in x is presented. The complete equations of condition for such a formula are listed. The reduction of these equations is discussed. The computation of the coefficients of a

formula with small error terms are presented in fraction form. The application of the formula to quadrature problems $\ddot{x} = f(t)$ is explained. In two numerical examples a comparison with respect to accuracy and time consumption is made between the 9-th order formula and known formulas of the 8-th and 10-th orders.

2) Fehlberg [22]

For differential equations in the form $\ddot{x} = f(t, x)$, R-K-T-formulas of the $(m+5)$-th and $(m+6)$-th order in x and of the $(m+6)$-th order in x are presented. The formulas provide for a stepsize control in x. They require five evaluations and m total differentiations per step of the differential equations. The error coefficients of the formulas are compared with those of a formula of Hebsaker.

3) Filippi and Graf [23]

A new explicit, direct Runge-Kutta-Nystrom formula-pair of order 11(12) with 20 function evalutions per step for initial value problems in ordinary differential equations of second order of the special form $y'' = f(x, y)$ is derived. Three numerical examples demonstrate the efficiency of the new formula-pair.

1.8. OTHER METHODS

Despite the frequent use of the essential one-step method of de Vogelaere [20], Coleman and Mohamed [15] point out the unavailability of error and stability analysis. The authors provide the missing analysis as well as a derivation of de Vogelaere's method based on Taylor expansions. They also give truncation error estimates to provide inexpensive error control in the implementation of the formula, and show the method is of order four.

Costabile [16] gives a two-step (hybrid) fourth order method which requires two function evaluations per step. The method involves an off step point, u. In [18], Costabile and Varano study the method's convergence and stability properties. It is shown that a necessary con-

dition for convergence is $u \in (0, \frac{1}{2}]$. The intervals of periodicity are also given for various values of u, the largest being $(0, 10.1)$ for $u = .4$.

Chang and Gnepp [5] recently developed an explicit one-step method of order four requiring three function evaluations per step. Integrating (1.1.1) from x_n to x twice gives the formulae

$$(1.8.1) \quad y(x) = y(x_n) = (x - x_n)y'(x_n) + \int_{x_n}^{x} (x - s)f(s, y(s))ds$$

$$(1.8.2) \quad y'(x) = y'(x_n) + \int_{x_n}^{x} f(s, y(s))ds$$

Letting $x = x_n + h$ and applying Simpson's rule to the integrals in (1.8.1) and (1.8.2), the authors obtain formulae involving functions of the form $f(x_{n+1/2}, y_{n+1/2})$. $y_{n+1/2}$ is approximated with terms involving y_n and $\bar{y}_{n+1}$, a fourth order approximation to y_{n+1}, with the aid of a Hermite interpolating polynomial. The method has interval of periodicity $(0, 8.0722)$.

Neta and Lee [41] consider linear K-step methods of the form

$$(1.8.3) \quad \sum_{j=0}^{K} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{K'} \beta_j f_{n+j}$$

where $K' = K$ or $K' = K - 1$. The first and second characteristic polynomials are respectively given by

$$(1.8.4) \quad \rho(\eta) = \sum_{j=0}^{K} \alpha_j \eta^j$$

$$(1.8.5) \quad \sigma(\eta) = \sum_{j=0}^{K'} \beta_j \eta^j \quad .$$

The authors define the method as zero stable if no root of the characteristic polynomial (1.8.4) has modulus greater than one and if every root of modulus one has multiplicity not greater than two. It is known that the order, P, of a zero stable method cannot exceed $K + 2$ and if $P = K + 2$, K must be even. Neta and Lee try to improve the order results through the use of the function f_{n+r} where r is an off step point.

If the characteristic polynomials are expressed as

$$(1.8.6) \quad \rho(\eta) = \sum_{i=2}^{K} a_i(\eta - 1)^i$$

$$(1.8.7) \quad \sigma(\eta) = \sum_{i=0}^{K'} b_i(\eta - 1)^i$$

where the coefficients a_i are given, then r, β_r and b_i are given by

$$(1.8.8) \quad r = K' + 1 + \frac{dK+2}{dK'+1} (K' + 2)$$

$$(1.8.9) \quad \beta_r = \frac{dK'+1}{r \choose (K'+1)}$$

$$(1.8.10) \quad b_i = d_i - \beta_r \binom{r}{i} \quad i = 0, 1, \ldots, K'$$

where

$$(1.8.11) \quad d_i = \sum_{j=0}^{\min(K-2,i)} a_{j+1}\gamma_{i-j}$$

and a table for γ_ℓ is given. The authors give general modified three-step methods with orders five and six. In addition, Neta and Lee show that methods of maximal order $(P = K + K' + 1)$ exist.

Neta [39] continues work with hybrid K-step methods, this time choosing $a_i = \binom{K-2}{i-2}$ $i = 2, \ldots, K$ in (1.8.6) so that the methods obtained are Stormer-Cowell type. Consideration was given for methods with $K \leq 10$. In addition, Neta [40] constructs maximal order methods for $K \leq 10$ which were shown to exist in [41].

Finally, Krishnaiah, [36] considers the second order initial value problem

$$(1.8.12) \quad F(x, y, y'') = 0 \; ; \; y(x_0) = y_0, \; y'(x_0) = y_0' \; .$$

Instead of working with the case (1.1.1), the author considers problems of the form

$$(1.8.13) \quad y = g(x, y'') \; ; \; y(x_0) = y_0 \; , \; y'(x_0) = y_0' \; .$$

Krishnaiah notes that sometimes (1.1.1) can be transformed into (1.8.13). The methods considered are inverse linear multistep methods of the form

$$(1.8.14) \quad h^{-2} \sum_{j=0}^{K-1} \alpha_j y_{n+j} = \sum_{j=0}^{K} \beta_j f_{n+j}$$

where $\beta_K = 1$ and $y_{n+j} = g(t_{n+j}, f_{n+j})$.

Letting $\sigma(\eta) = \sum_{j=0}^{K} \beta_j \eta^j$, the author defines a method as infinitely stable if $\sigma(\eta)$ possesses only roots of modulus less than one. Infinitely stable methods of orders one, two and three are given. According to the author, the advantage of these methods is their yield of good numerical results for large step sizes when applied to problems they are designed for.

CHAPTER 2

TWO-STEP P-STABLE METHODS

2.1. INTRODUCTION

In recent years there has been much interest in numerical methods to approximate the second-order initial value problem

$$(2.1.1) \quad y'' = f(x, y) \; ; \; y(x_0) = y_0 \; , \; y'(x_0) = y'_0$$

for cases where the required solution is periodic. The desired stability criterion for such problems, as shown by Lambert and Watson [38], is that of P-stability.

Periodic solutions for problems of the form (2.1.1) can be categorized by:

(a) problems whose periods are initially known and

(b) problems whose periods are initially unknown.

Numerical methods for problems of class (a) have been proposed by authors including Gautschi [24], Salzer [42], Steifel and Bettis [45], and more recently Jain, Jain and Krishnaiah [30], [31] and Krishnaiah [37]. However, the majority of literature on P-stable numerical methods produced in the last ten years applies to the more general problems of class (b).

In 1976, Lambert and Watson [38] showed that a P-stable K-step method of the form

$$(2.1.2) \quad \sum_{j=0}^{K} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{K} \beta_j f_{n+j} \; , \quad K > 1$$

cannot obtain an order of accuracy greater than two. Since that time many P-stable (hybrid) multistep methods have been introduced to break the second order accuracy barrier. Although numerical results for a four-step method and three one-step methods are given, all of which are P-stable, the focus of this chapter is P-stable (hybrid) two-step methods. For a general discussion of P-stability and P-stable methods of class (b), see sections 1.2 and 1.3.

2.2. P-STABILITY FOR TWO-STEP METHODS

P-stability is based upon the application of numerical methods to the problem

$$(2.2.1) \quad y''(x) = -\lambda^2 y(x) \qquad y,\ \lambda \in \mathbf{R}\ .$$

If h is the step size, $x_{n+1} - x_n$, and $z = \lambda h$, then applying a proposed (hybrid) two-step method to (2.2.1) may yield an expression of the form

$$(2.2.2) \quad Q(z)y_{n+1} - 2P(z)y_n + Q(z)y_{n-1} = 0\ .$$

whose characteristic equation is

$$(2.2.3) \quad Q(z)\rho^2 - 2P(z)\rho + Q(z) = 0\ .$$

It is noted that all methods considered in this chapter give the expression (2.2.2) when applied to (2.2.1). Since $Q(z)$ is assumed to

be non-zero, (2.2.2) may be expressed as

$$(2.2.4) \quad y_{n+1} + y_{n-1} = 2 \frac{P(z)}{Q(z)} y_n .$$

It can be shown (Serbin [44]) that $\frac{P(z)}{Q(z)}$ is an approximation to $\cos(z)$. Letting $R(z) = \frac{P(z)}{Q(z)}$ and using the power series expansion for $\cos(z)$ to approximate $R(z)$, we obtain the expression

$$(2.2.5) \quad Q(z)R(z) - P(z) = C_{\gamma+2}z^{\gamma+2} + C_{\gamma+4}z^{\gamma+4} + \ldots$$

where γ is the order of the method and $C_{\gamma+2}$ is the error constant of the method.

The approximation $R(z)$ determines the stability of the method as follows:

(2.2.6) If $|R(z)| \leq 1 \quad \forall z \in (0, T)$ and $|R(z)| > 1$.

$\forall z \geq T$, the method is conditionally stable.

(2.2.7) If $|R(z)| \leq 1 \quad \forall z \in (0, \infty)$ the method is unconditionally stable.

(2.2.8) If the roots, ρ_i , of the characteristic equation (2.2.3) are complex and $|\rho_i| = 1$ for $i = 1, 2 \quad \forall z \in (0, H)$, the method has interval of periodicity $(0, H)$.

(2.2.9) If the interval of periodicity of a method is $(0, \infty)$, the method is P-stable.

Note that (2.2.6) $\Rightarrow$ (2.2.9) since:

$$|R(z)| \leq 1 \Rightarrow \left|\frac{P(z)}{Q(z)}\right| \leq 1$$

$$\Rightarrow P^2_{(z)} \leq Q^2_{(z)}$$

$$\therefore \rho = \frac{P(z) \pm \sqrt{P^2(z) - Q^2(z)}}{Q(z)}$$

$$= \frac{P(z)}{Q(z)} \pm \sqrt{\frac{P^2(z)}{Q^2(z)} - 1}$$

$$= \frac{P(z)}{Q(z)} \pm i\sqrt{1 - \frac{P^2(z)}{Q^2(z)}}$$

Thus, $|\rho|^2 = \left(\frac{P(z)}{Q(z)}\right)^2 + \left(1 - \frac{P^2(z)}{Q^2(z)}\right) = 1$. Also, (2.2.9) $\Rightarrow$ (2.2.6) since:

$$\rho_i \notin \mathbf{R} \Rightarrow P^2(z) \leq Q^2(z)$$

$$\Rightarrow \left|\frac{P(z)}{Q(z)}\right| \leq 1 \quad (\text{or} \quad |R(z)| \leq 1) .$$

Therefore, for the two-step methods considered here, and all symmetric two-step methods, P-stability is equivelent to unconditional stability. A listing of the P-stable two-step methods, as well as the P-stable one-step methods and the P-stable four-step method tested numerically are

given in Table 1.

The minimum absolute error constant, $C_{\gamma+2}$, attainable by these methods was given by Hairer [28] as

$$(2.2.10) \qquad |C_{\gamma+2}| = \frac{(j!)^2}{(2j)!(2j+1)!} \quad , \quad j = \frac{\gamma}{2} \quad .$$

Of the methods discussed in sections 1.2 and 1.3, Hairer's methods (Methods 1, 19 and 20, Table 1) are the only schemes which obtain mimimum error constants. Method 15 of Table 1 is a slight modification of the four function evaluation $O(h^6)$ method proposed by Cash [3] (Method 14, Table 1). The modification results in a two-step sixth order method that has minimum absolute error constant.

2.3. NUMERICAL RESULTS

Example 1. The first numerical example considered is Lambert and Watson's [8] Example 2, a problem frequently tested in other articles. The problem is given by

$$(2.3.1) \qquad y_1'' + \lambda^2 y_1 = f''(x) + \lambda^2 f(x) \; ; \; y_1(0) = a + f(0), \; y_1'(x) = f'(0)$$

$$y_2'' + \lambda^2 y_2 = f''(x) + \lambda^2 f(x) \; : \; y_2(0) = f(0), \; y_2'(0) = \lambda a + f'(0).$$

The solution of this system, with $f(x) \equiv e^{-0.05x}$, is

$$(2.3.2) \qquad y_1(x) = a \cos\lambda x + e^{-0.05x}, \; y_2(x) = a \sin\lambda x + e^{-0.05x} \quad .$$

Table 1. P-Stable Methods Numerically Tested

Method 1

Hairer's $O(h^4)$ method [28] (P-stable except $\lambda h = \sqrt{12}$)

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}[f_{n+1} + (10-R)f_n + f_{n-1} + R\bar{f}_n]$$

$$\bar{y}_n = y_n - \frac{h^2}{12R}(f_{n+1} - 2f_n + f_{n-1})$$

R a free parameter

Rational approximation, $R(z) = \dfrac{1 - \frac{5}{12} z^2 + \frac{1}{144} z^4}{1 + \frac{1}{12} z^2 + \frac{1}{144} z^4}$

|Error constant| = 0.00139

Method 2

Cash's $O(h^4)$ method [2]

$$y_{n+1} - 2y_n + y_{n-1} = h^2[-\frac{11}{360}(f_{n+1} + f_{n-1}) + \frac{3}{20} f_n + \frac{41}{90}(f_{n+1/2} + f_{n-1/2})]$$

$$\bar{y}_{n+1/2} = \frac{1}{4} y_{n+1} + y_n - \frac{1}{4} y_{n-1} + h^2(-\frac{3}{48} f_{n+1} + \frac{9}{48} f_n)$$

$$\bar{y}_{n-1/2} = \frac{1}{2} y_n + \frac{1}{2} y_{n-1} + h^2(\frac{9}{192} f_{n+1} - \frac{15}{96} f_n - \frac{1}{64} f_{n-1})$$

Rational approximation, $R(z) = \dfrac{1 - \frac{5}{12} z^2 + \frac{41}{5760} z^4}{1 + \frac{1}{12} z^2 + \frac{41}{5760} z^4}$

|Error constant| = 0.00148

Table 1 (continued)

Method 3

Chawla's $O(h^4)$ method [6] (with off-step point $s = 1/2$)

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[-\frac{1}{18}(f_{n+1} + f_{n-1}) + \frac{5}{9}(f_{n+1/2} + f_{n-1/2})\right]$$

$$\bar{y}_{n+1/2} = \frac{3}{4} y_{n+1} + \frac{1}{4} y_{n-1} + h^2\left[Bf_{n+1} - (\frac{3}{8} + B)f_{n-1}\right]$$

$$\bar{y}_{n-1/2} = \frac{1}{4} y_{b+1} + \frac{3}{4} y_{n-1} + h^2\left[-(\frac{3}{8} + B)f_{n+1} + Bf_{n-1}\right]$$

B a free parameter

Rational approximation, $R(z) = \dfrac{1}{1 + \frac{1}{2} z^2 + \frac{15}{72} z^4}$

|Error constant| = 0.01181

Method 4

Costabile, Costabile's $O(h^4)$ method [17]

$$y_{n+1} - 2y_n + y_{n-1} = -\frac{h^2}{12}\left[f_{n+1} + f_{n-1} + 2f_n - 8(f_{n+1/2} + f_{n-1/2})\right]$$

$$\bar{y}_{n+1/2} = \frac{1}{2}(y_{n+1} + y_n) = \frac{h^2}{16}(f_{n+1} + f_n)$$

$$\bar{y}_{n-1/2} = \frac{1}{2}(y_n + y_{n-1}) - \frac{h^2}{16}(f_n + f_{n-1})$$

Rational approximation, $R(z) = \dfrac{1 - \frac{1}{4} z^2 - \frac{1}{24} z^4}{1 + \frac{1}{4} z^2 + \frac{1}{24} z^4}$

|Error constant| = 0.01181

Table 1 (continued)

Method 5

Chawla's $O(h^4)$ method [7] (P-stable for $\alpha > \frac{1}{120}$)

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}(f_{n+1} + 10\bar{f}_n + f_{n-1})$$

$$\bar{y}_n = y_n - \alpha h^2(f_{n+1} - 2f_n + f_{n-1})$$

$\alpha = \frac{1}{119}$ used in calculations

Rational approximation, $R(z) = \dfrac{1 - \frac{5}{12}z^2 + \frac{5\alpha}{6}z^4}{1 + \frac{1}{12}z^2 + \frac{5\alpha}{6}z^4}$

Minimum |error constant| = 0.00139 (as $\alpha \to \frac{1}{120}$)

Method 6

Cash's $O(h^4)$ method [3]

$$y_{n+1} - 2y_n + y_{n-1} = h^2[\tfrac{1}{12}(f_{n+1} + f_{n-1}) + \tfrac{1}{6}f_n + \tfrac{2}{3}\bar{f}_n]$$

$$\bar{y}_n = y_n - \frac{h^2}{94}(f_{n+1} - 2f_n + f_{n-1})$$

Rational approximation, $R(z) = \dfrac{1 - \frac{5}{12}z^2 + \frac{1}{141}z^4}{1 + \frac{1}{12}z^2 + \frac{1}{141}z^4}$

|Error constant| = 0.00146

Table 1 (continued)

Method 7

Jain, Jain, Krishnaiah's $O(h^4)$ method [32] (eq. 6)

$r = \sqrt{\frac{2}{5}}$ (P-stable for $\alpha \leq -\frac{1}{5}$)

$$y_{n+1} - 2y_n + y_{n-1} = h^2[\tfrac{7}{12} f_n + \tfrac{5}{24}(f_{n+r} + f_{n-r})]$$

$$\bar{y}_{n+q} = \tfrac{1}{2}(1 - \alpha + q)y_{n+1} + \alpha y_n + \tfrac{1}{2}(1 - \alpha - q)y_{n-1}$$
$$+ \tfrac{h^2}{12}[(3\alpha - 3 - q + 3q^2 + q^3)f_{n+1} + (3\alpha - 3 + q + 3q^2 - q^3)f_{n-1}]$$

$\bar{y}_{n-q} = \bar{y}_n + q$ with (q) replaced by $(-q)$

$q = r$

$\alpha = -\frac{1}{5}$ used in calculations

Rational approximation, $R(z) = \dfrac{1 - \frac{7+5\alpha}{24} z^4}{1 + \frac{5-5\alpha}{24} z^2 + \frac{3-5\alpha}{48} z^4}$

Minimum |error constant| = 0.03264 (with $\alpha = -\frac{1}{5}$)

Table 1 (continued)

Method 8

Jain, Jain, Krishnaiah's $O(h^4)$ method [32] (eq. 8)

$r = \frac{5 - \sqrt{5}}{10}$, $s = \frac{5 + \sqrt{5}}{10}$ (P-stable for $\alpha \leq 0$)

$$y_{n+1} - 2y_n + y_{n-1} = h^2[\frac{1}{6} f_n + \frac{5 + \sqrt{5}}{24}(f_{n+r} + f_{n-r}) + \frac{5 - \sqrt{5}}{24}(f_{n+s} + f_{n-s})]$$

For $\bar{y}_{n+q}$, $\bar{y}_{n-q}$ ($q = r, s$) see method 7

$\alpha = 0$ used in calculations

Rational approximation, $R(z) = \dfrac{1 - \frac{1 + 5\alpha}{12} z^2}{1 + \frac{5 - 5\alpha}{12} z^2 + \frac{4 - 5\alpha}{24} z^4}$

Minimum |error constant| = 0.06736 (with $\alpha = 0$)

Method 9

Jain, Jain, Krishnaiah's $O(h^4)$ method [32] (eq. 10)

$r = \frac{5 - \sqrt{15}}{10}$, $s = \frac{5 + \sqrt{15}}{10}$, $t = \frac{1}{2}$ (P-stable for $\alpha \leq 0$)

$$y_{n+1} - 2y_n + y_{n-1} = h^2[\frac{5 + \sqrt{15}}{36} (f_{n+r} + f_{n-r}) + \frac{5 - \sqrt{15}}{36}(f_{n+s} + f_{n-s}) + \frac{2}{9}(f_{n+t} + f_{n-t})]$$

For $\bar{y}_{n+q}$, $\bar{y}_{n-q}$ ($q = r, s, t$) see method 7

$\alpha = 0$ used in calculations

Rational approximation, $R(z) = \dfrac{1 - \frac{\alpha}{2} z^2}{1 + \frac{1 - \alpha}{2} z^2 + \frac{5 - 6\alpha}{24} z^4}$

Minimum |error constant| = 0.08472 (with $\alpha = 0$)

Table 1 (continued)

Method 10

Jain, Jain, Krishnaiah's $O(h^6)$ method [32] (eq. 6)

$r = \sqrt{\frac{2}{5}}$, $q = \frac{1}{2}$

$$y_{n+1} - 2y_n + y_{n-1} = h^2[\frac{7}{12} f_n + \frac{5}{24} (f_{n+r} + f_{n-r})]$$

$$\bar{y}_{n+r} = \frac{1}{2} (1 + r)y_{n+1} + \frac{1}{2} (1 - r)y_{n-1} + h^2[(\frac{1}{300} - \frac{17}{1500} r)f_{n+1}$$

$$+ (\frac{1}{300} + \frac{17}{1500} r)f_{n-1} - (\frac{23}{150} + \frac{29}{375} r)f_{n+q} - (\frac{23}{150} - \frac{29}{375} r)f_{n-q}]$$

$\bar{y}_n = r = y_{n+r}$ with (r) replaced by (-r)

For $\bar{y}_{n+q}$, $\bar{y}_{n-q}$ see method 7 with $\alpha = 0$

Rational approximation, $R(z) = \dfrac{1 - \frac{7}{24} z^2}{1 + \frac{5}{24} z^2 + \frac{1}{16} z^4 + \frac{23}{960} z^6}$

|Error constant| = 0.00959

Table 1 (continued)

Method 11

Jain, Jain, Krishnaiah's $O(h^6)$ method [32] (eq. 8)

$$r = \frac{5 - \sqrt{5}}{10}\ ,\ s = \frac{5 + \sqrt{5}}{10}\ ,\ q = \frac{1}{2}$$

$$y_{n+1} - 2y_n + y_{n-1} = [\frac{1}{6} f_n + \frac{5 + \sqrt{5}}{24}(f_{n+r} + f_{n-r}) + \frac{5 - \sqrt{5}}{24}(f_{n+s} + f_{n-s})]$$

$$\bar{y}_{n+r} = \frac{15 - \sqrt{5}}{20} y_{n+1} + \frac{5 + \sqrt{5}}{20} y_{n-1} + \frac{h^2}{22,500} [(125 + 142\sqrt{5}) f_{n+1}$$

$$+ (350 + 83\sqrt{5}) f_{n-1} - (5075 + 659\sqrt{5}) f_{n+q} - (3275 + 691\sqrt{5}) f_{n-q}]$$

$$\bar{y}_{n-r} = \frac{5 + \sqrt{5}}{20} y_{n+1} + \frac{15 - \sqrt{5}}{20} y_{n-1} + \frac{h^2}{22,500} [(350 + 83\sqrt{5}) f_{n+1}$$

$$+ (125 + 142\sqrt{5}) f_{n-1} - (3275 + 691\sqrt{5}) f_{n+q} = (5075 + 659\sqrt{5}) f_{n-q}]$$

$$\bar{y}_{n+s} = \frac{15 - \sqrt{5}}{20} y_{n+1} + \frac{5 - \sqrt{5}}{20} y_{n-1} + \frac{h^2}{22,500} [(125 - 142\sqrt{5}) f_{n+1}$$

$$+ (350 - 83\sqrt{5}) f_{n-1} - (5075 - 659\sqrt{5}) f_{n+q} - (3275 - 691\sqrt{5}) f_{n-q})]$$

$$\bar{y}_{n-s} = \frac{5 - \sqrt{5}}{20} y_{n+1} + \frac{15 - \sqrt{5}}{20} y_{n-1} + \frac{h^2}{22,500} [(350 - 83\sqrt{5}) f_{n+1}$$

$$+ (125 - 142\sqrt{5}) f_{n-1} - (3275 - 691\sqrt{5}) f_{n+q} - (5075 - 659\sqrt{5}) f_{n-q}]$$

For $\bar{y}_{n+q}$, $\bar{y}_{n-q}$ see method 7 with $\alpha = 0$

Rational approximation, $R(z) = \dfrac{1 - \frac{1}{12} z^2}{1 + \frac{5}{12} z^2 + \frac{1}{6} z^4 + \frac{97}{1440} z^6}$

|Error constant| = 0.02729

Table 1 (continued)

Method 12

Jain, Jain, Krishnaiah's $O(h^6)$ method [32] (eq. 10)

$$r = \frac{5 - \sqrt{15}}{10}, \quad s = \frac{5 + \sqrt{15}}{10}, \quad t = \frac{1}{2}, \quad q = \frac{1}{2}$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[\frac{5 + \sqrt{15}}{36}(f_{n+r} + f_{n-r}) + \frac{5 - \sqrt{15}}{36}(f_{n+s} + f_{n-s}) + \frac{2}{9}(f_{n+t} + f_{n-t})\right]$$

$$\bar{y}_{n+r} = \frac{15 - \sqrt{15}}{20} y_{n+1} + \frac{5 + \sqrt{15}}{20} y_{n-1} + \frac{h^2}{45,000}[(375 + 206\sqrt{15})f_{n+1} + (675 + 144\sqrt{5})f_{n-1} + (8100 + 1287\sqrt{15})f_{n+q} - (6450 + 1313\sqrt{15})f_{n-q}]$$

$$\bar{y}_{n-r} = \frac{5 + \sqrt{15}}{20} y_{n+1} + \frac{15 - \sqrt{15}}{20} y_{n-1} + \frac{h^2}{45,000}[(675 + 144\sqrt{5})f_{n+1} + (375 + 206\sqrt{15})f_{n-1} - (6450 + 1313\sqrt{15})f_{n+q} - (8100 + 1287\sqrt{15})f_{n-q}]$$

$$\bar{y}_{n+s} = \frac{15 + \sqrt{15}}{20} y_{n+1} + \frac{5 - \sqrt{15}}{20} y_{n-1} + \frac{h^2}{45,000}[(375 - 206\sqrt{15})f_{n+1} + (675 - 144\sqrt{15})f_{n-1} - (8100 - 1287\sqrt{15})f_{n+q} - (6450 - 1313\sqrt{15})f_{n-q}]$$

$$\bar{y}_{n-s} = \frac{5 - \sqrt{15}}{20} y_{n+1} + \frac{5 + \sqrt{15}}{20} y_{n-1} + \frac{h^2}{45,000}[(675 - 144\sqrt{15})f_{n+1} + (375 - 206\sqrt{15})f_{n-1} - (6450 \quad 1313\sqrt{15})f_{n+q} - (8100 - 1287\sqrt{15})f_{n-q}]$$

$$\bar{y}_{n+t} = \frac{3}{4} y_{n+1} + \frac{1}{4} y_{n-1} + h^2\left[\frac{1}{192} f_{n+1} + \frac{1}{64} f_{n-1} - \frac{1}{4} f_{n+q} - \frac{7}{48} f_{n-q}\right]$$

Table 1 (continued)

Method 12 (continued)

$$\bar{y}_{n-t} = \frac{1}{4} y_{n+1} + \frac{3}{4} y_{n-1} + h^2\left[\frac{1}{64} f_{n+1} + \frac{1}{192} f_{n-1} - \frac{7}{48} f_{n+q} - \frac{1}{4} f_{n-q}\right]$$

For $\bar{y}_{n+q}$, $\bar{y}_{n-q}$ see method 7 with $\alpha = 0$

Rational approximation, $R(z) = \dfrac{1}{1 + \frac{1}{2} z^2 + \frac{5}{24} z^4 + \frac{61}{720} z^6}$

|Error constant| = 0.03435

Method 13

Cash's $O(h^6)$ method [2]

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[\frac{13}{480}(f_{n+1} + f_{n-1}) - \frac{1}{15} f_n + \frac{81}{160}(f_{n+1/3} + f_{n-1/3})\right]$$

$$\bar{y}_{n+1/3} = \frac{11}{243} y_{n+1} + \frac{302}{243} y_n - \frac{70}{243} y_{n-1} + h^2\left[-\frac{113}{131220} f_{n+1} + \frac{2257}{21870} f_n + \frac{607}{131220} f_{n-1} + \frac{2588}{32805} f_{n-1/2} - \frac{292}{32805} f_{n+1/2}\right]$$

$$\bar{y}_{n-1/3} = \frac{2}{3} y_n + \frac{1}{3} y_{n-1} + h^2\left[-\frac{23}{131220} f_{n+1} - \frac{473}{21870} f_n - \frac{743}{131220} f_{n-1} - \frac{2812}{32805} f_{n-1/2} + \frac{68}{32805} f_{n+1/2}\right]$$

$$\bar{y}_{n+1/2} = \frac{11}{32} y_{n+1} + \frac{13}{16} y_n - \frac{5}{32} y_{n-1} + h^2\left[-\frac{7}{192} f_{n+1} + \frac{1}{24} f_n + \frac{5}{192} f_{n-1}\right]$$

$$\bar{y}_{n-1/2} = \frac{1}{2} y_n + \frac{1}{2} y_{n-1} + h^2\left[\frac{5}{384} f_{n+1} - \frac{17}{192} f_n - \frac{19}{384} f_{n-1}\right]$$

Rational approximation, $R(z) = \dfrac{1 - \frac{9}{20} z^2 + \frac{397}{21600} z^4 - \frac{7}{86400} z^6}{1 + \frac{1}{20} z^2 + \frac{37}{21600} z^4 + \frac{7}{86400} z^6}$

|Error constant| = 0.000014

Table 1 (continued)

Method 14

Cash's $O(h^6)$ method [3]

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[\frac{1}{60}(f_{n+1} + f_{n-1}) + \frac{4}{15}(f_{n+1/2} + f_{n-1/2}) + \frac{1}{15}\bar{f}_n + \frac{11}{30} f_n\right]$$

$$\bar{y}_{n+1/2} = \frac{11}{32} y_{n+1} + \frac{13}{16} y_n - \frac{5}{32} y_{n-1} + h^2\left[-\frac{7}{192} f_{n+1} + \frac{1}{24} f_n + \frac{5}{192} f_{n-}\right.;$$

$$\bar{y}_{n-1/2} = \frac{1}{2} y_n + \frac{1}{2} y_{n-1} + h^2\left[\frac{5}{384} f_{n+1} - \frac{17}{192} f_n - \frac{19}{384} f_{n-1}\right]$$

$$\bar{y}_n = -\frac{7}{8}(y_{n+1} + y_{n-1}) + \frac{11}{4} y_n + h^2\left[\frac{371}{4320}(f_{n+1} + f_{n-1})\right.$$

$$\left. + \frac{1743}{2160} f_n - \frac{7}{135}(f_{n+1/2} + f_{n-1/2})\right]$$

Rational approximation, $R(z) = \dfrac{1 - \frac{9}{20} z^2 + \frac{397}{21600} z^4 - \frac{7}{86400} z^6}{1 + \frac{1}{20} z^2 + \frac{37}{21600} z^4 + \frac{7}{86400} z^6}$

|Error constant| = 0.000014

Method 15

A modification of Method 15

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[\frac{1}{60}(f_{n+1} + f_{n-1}) + \frac{4}{15}(f_{n+1/2} + f_{n-1/2}) + \frac{1}{15}\bar{f}_n + \frac{11}{30} f_n\right]$$

$$\bar{y}_{n+1/2} = \frac{11}{32} y_{n+1} + \frac{13}{16} y_n - \frac{5}{32} y_{n-1} + h^2\left[-\frac{7}{192} f_{n+1} + \frac{1}{24} f_n + \frac{5}{192} f_{n-1}\right]$$

$$\bar{y}_{n-1/2} = \frac{1}{2} y_n + \frac{1}{2} y_{n-1} + h^2\left[\frac{5}{384} f_{n+1} - \frac{17}{192} f_n - \frac{19}{384} f_{n-1}\right]$$

$$\bar{y}_n = -\frac{7}{8}(y_{n+1} + y_{n-1}) + \frac{11}{4} y_n + h^2\left[\frac{121}{1440}(f_{n+1} + f_{n-1}) + \frac{1719}{2160} f_n - \frac{2}{45}(f_{n+1/2} + f_{n-1/2})\right]$$

Table 1 (continued)

Method 15 (continued)

Rational approximation, $R(z) = \dfrac{1 - \frac{9}{20}z^2 + \frac{11}{600}z^4 - \frac{1}{14400}z^6}{1 + \frac{1}{20}z^2 + \frac{1}{600}z^4 + \frac{1}{14400}z^6}$

$|\text{Error constant}| = 0.0000099$

Method 16

Chawla, Rao's $O(h^6)$ method [11] (P-stable for $\alpha < \frac{343}{6000}$)

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{60}[f_{n+1} + 26f_n + f_{n-1} + 16(\bar{\bar{f}}_{n+1/2} + \bar{\bar{f}}_{n-1/2})]$$

$$\bar{\bar{y}}_{n+1/2} = \frac{1}{2}(y_n + y_{n+1}) - \frac{h^2}{96}(f_{n+1} + 10\,\bar{f}_{n+1/2} + f_n)$$

$$\bar{\bar{y}}_{n-1/2} = \frac{1}{2}(y_n + y_{n-1}) - \frac{h^2}{96}(f_{n-1} + 10\bar{f}_{n-1/2} + f_n)$$

$$\bar{y}_{n+1/2} = \frac{1}{2}(y_n + y_{n+1}) - h^2(\alpha f_n + (\tfrac{1}{8} - \alpha)f_{n+1})$$

$$y_{n-1/2} = \frac{1}{2}(y_n + y_{n-1}) - h^2(\alpha f_n + (\tfrac{1}{8} - \alpha)f_{n-1})$$

$\alpha = \frac{343}{6001}$ used in calculations

Rational approximation, $R(z) = \dfrac{1 - \frac{7}{20}z^2 + \frac{1}{60}z^4 - \frac{\alpha}{36}z^6}{1 + \frac{3}{20}z^2 + \frac{1}{60}z^4 + \frac{1}{36}(\frac{1}{8} - \alpha)z^6}$

Minimum |error constant| = 0.00043 (as $\alpha \to \frac{343}{6000}$)

Table 1 (continued)

Method 17 — Jain, Kambo, Goel's $O(h^6)$ method [35]

$$y_n - \frac{3}{2} y_{n+1} + y_{n+2} - \frac{3}{2} y_{n+3} + y_{n+4}$$
$$= \frac{h^2}{96} (7f_n + 88f_{n+1} + 50f_{n+2} + 88f_{n+3} + 7f_{n+4})$$
$$\bar{y}_n = y_n - \frac{7h^2}{144} (f_n - 4f_{n+1} + 6\bar{f}_{n+2} - 4f_{n+3} + f_{n+4})$$
$$\bar{y}_{n+2} = y_{n+2} - \frac{h^2}{101} (f_n - 2f_{n+2} + f_{n+4})$$

|Error constant| = 0.00018

Method 18 — Jain, Kambo, Goel's $O(h^4)$ method [33]

$$y_{2n+1} = \frac{1}{2}[y_{2n} + y_{2n+2}] + \frac{1}{4} h[y'_{2n} - y'_{2n+2}]$$
$$y_{2n+2} = y_{2n} + 2hy'_{2n} + \frac{1}{3} h^2[4f_{2n+1} + 2f_{2n}]$$
$$y'_{2n+2} = y'_{2n} + \frac{1}{3} h(f_{2n} + 4f_{2n+1} + f_{2n+2})$$

|Error constant| = 0.08889

Table 1 (continued)

Method 19-20

Hairer [28] use one-step methods of the form

$$y_1 = y_0 + hy_0' + h^2 \sum_{i=1}^{s} b_i k_i$$

$$y_1' = y_0' + h \sum_{i=1}^{s} \bar{b}_i k_i$$

$$k_i = f(x_0 + \alpha_i h,\ y_0 + \alpha_i h y_0' + h^2 \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1, \ldots, s$$

where $A = G^2$, $\alpha = G1$, $\bar{b} = g$, $b^T = g^T G$, $(1 = (1, \ldots, 1)^T)$

Method 19

an $O(h^4)$ method given by

$$g = \begin{bmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{bmatrix} \qquad G = \begin{bmatrix} \frac{1}{4} & \frac{3-2\sqrt{3}}{12} \\ \\ \frac{3+2\sqrt{3}}{12} & \frac{1}{4} \end{bmatrix}$$

|Error constant| = 0.00139

Table 1 (continued)

Method 20 — an $O(h^6)$ method given by

$$g = \begin{bmatrix} \frac{5}{18} \\ \frac{4}{9} \\ \frac{5}{18} \end{bmatrix} \qquad G = \begin{bmatrix} \frac{5}{36} & \frac{80 - 24\sqrt{15}}{360} & \frac{50 - 12\sqrt{15}}{360} \\ \frac{50 - 15\sqrt{15}}{360} & \frac{2}{9} & \frac{50 - 15\sqrt{15}}{360} \\ \frac{50 + 12\sqrt{15}}{360} & \frac{80 + 24\sqrt{15}}{360} & \frac{5}{36} \end{bmatrix}$$

|Error constant| = 0.0000099

As mentioned by Cash [2], as well as Lambert and Watson [38], P-stability is particularly relevant when high frequency oscillations are not present in the solution (or in problems whose solutions contain high frequency oscillations of small amplitude). Thus, following the lead of those authors who included results for this problem, the case $a = 0$ was tested with $\lambda = 5, 10, 15, \ldots, 40$ at $x = 20\pi$ for various step sizes. The measure of the error, E, at $x = 20\pi$ was taken as $E = [(\text{actual } y_1\text{-calculated } y_1)^2 + (\text{actual } y_2\text{-calculated } y_2)^2]^{1/2}$. Also, since this problem is linear in y, no interative procedure was employed to produce the results given in Table 2.

Although a general discussion of results for Example 1 as well as the three examples to follow is included in the next section, some comments on the results for this problem are needed at this time. First, for the stepsizes listed, the majority of the fourth and sixth order methods do not show convergence rates of order four or six respectively. However, as the value of h was taken smaller (in general from $\frac{\pi}{64}$ to $\frac{\pi}{128}$) the methods displayed expected convergence rates. A sample of convergence rates for $\lambda = 5$, 20 and 40 is included in Table 2. Second, the accuracy of the results given by Cash [2] are far inferior to those reported here. It is assumed that Cash used an iterative procedure, perhaps with only one iteration. Third, Cash's fourth order method (Method 2, Table 1) produced much more accurate results than the other fourth order methods tested. In comparing the truncation error of Cash's method to Chawla's fourth order method (Method 5, Table 1) for this problem, it was found that Cash's error involved even powers of λ less than or equal to four while Chawla's error involved even powers

Table 2. |Error| in the Solution of $x = 20\pi$ for the Integration of (2.3.1).

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 1							
5	5.42(-12)	4.40	2.40(-9)	3.83	4.88(-7)	3.35	4.98(-6)
10	1.78(-12)		9.15(-9)		1.39(-7)		1.14(-6)
15	7.28(-12)		2.21(-10)		4.10(-8)		3.23(-6)
20	7.58(-12)	4.03	2.01(-9)	3.71	3.43(-7)	2.06	1.43(-6)
25	2.78(-12)		1.59(-9)		3.26(-7)		3.24(-6)
30	2.29(-11)		6.76(-10)		2.21(-7)		3.50(-6)
35	3.36(-11)		8.78(-10)		3.19(-7)		3.24(-6)
40	3.20(-11)	2.64	1.25(-9)	2.50	4.01(-8)	6.15	2.85(-6)
Method 2							
5	2.25(-16)	4.07	6.35(-14)	5.32	1.02(-10)	5.41	4.33(-9)
10	1.41(-17)		1.21(-14)		3.68(-13)		3.52(-10)
15	2.67(-17)		2.33(-14)		3.28(-11)		3.17(-9)
20	1.02(-17)	5.62	2.48(-14)	5.84	8.10(-11)	4.36	1.66(-9)
25	7.62(-18)		2.49(-14)		8.11(-11)		3.45(-9)
30	2.94(-18)		2.61(-14)		6.63(-11)		3.67(-9)
35	1.34(-18)		1.68(-14)		7.92(-11)		3.37(-9)
40	5.72(-18)	5.60	1.34(-14)	3.85	2.80(-12)	10.05	2.96(-9)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 3							
5	1.13(-11)	4.98	1.13(-8)	2.37	3.02(-7)	2.09	1.29(-6)
10	1.61(-11)		6.24(-9)		8.14(-8)		3.26(-7)
15	8.15(-11)		5.24(-10)		3.64(-8)		1.45(-7)
20	5.31(-11)	2.10	9.73(-10)	2.19	2.05(-8)	1.99	8.17(-8)
25	4.57(-11)		7.37(-10)		1.32(-8)		5.32(-8)
30	3.85(-11)		5.41(-10)		9.14(-9)		3.63(-8)
35	2.97(-11)		4.07(-10)		6.72(-9)		2.67 (-8)
40	1.69(-11)	2.11	3.15(-10)	2.02	5.15(-9)	1.99	2.04(-8)
Method 4							
5	9.96(-12)	3.92	2.27(-9)	2.52	7.45(-8)	1.67	2.37(-7)
10	1.09(-11)		4.09(-10)		1.80(-9)		7.95(-8)
15	7.49(-12)		4.66 (-10)		7.54(-9)		3.59(-8)
20	6.87(-12)	2.30	1.66(-10)	2.47	5.07(-9)	2.00	2.03(-8)
25	8.86(-13)		2.00(-10)		3.40(-9)		1.30(-8)
30	2.30(-12)		4.95(-12)		2.40(-9)		9.03(-9)
35	4.63(-13)		2.38(-11)		1.78(-9)		6.64(-9)
40	3.59(-12)	2.31	8.86(-11)	1.97	1.36(-9)	1.90	5.08(-9)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 5							
5	5.46(-12)	4.41	2.48(-9)	3.84	5.12(-7)	3.23	4.80(-6)
10	1.70(-11)		4.89(-9)		9.44(-8)		4.09(-8)
15	7.45(-11)		2.64(-9)		7.16(-8)		3.19(-6)
20	1.02(-11)	3.57	1.44(-9)	3.93	3.36(-7)	2.15	1.50(-6)
25	1.48(-11)		6.24(-10)		3.24(-7)		3.28(-6)
30	1.65(-11)		4.20(-10)		2.34(-7)		3.53(-6)
35	8.89(-11)		1.44(-9)		3.16(-7)		3.25(-6)
40	3.61(-11)	2.04	6.13(-10)	2.81	3.03(-8)	6.56	2.86(-6)
Method 6							
5	5.53(-12)	4.41	2.51(-9)	3.81	4.91(-7)	3.19	4.49(-6)
10	1.55(-12)		8.52(-9)		2.27(-8)		2.78(-7)
15	7.47(-12)		6.99(-10)		1.17(-7)		3.12(-6)
20	8.21(-12)	3.69	1.36(-9)	3.94	3.22(-7)	2.30	1.59(-6)
25	1.55(-11)		1.39(-9)		3.19(-7)		3.35(-6)
30	1.01(-11)		1.66(-9)		2.54(-7)		3.57(-6)
35	3.00(-11)		1.35(-9)		3.11(-7)		3.28(-6)
40	5.31(-12)	3.37	5.67(-10)	2.38	1.53(-8)	7.56	2.88(-6)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 7							
5	5.11(-11)	3.97	1.27(-8)	2.17	2.58(-7)	4.65	6.49(-6)
10	5.09(-11)		2.87(-9)		8.91(-7)		8.36(-6)
15	5.01(-11)		1.35(-8)		1.09(-6)		4.52(-6)
20	4.72(-11)	2.77	2.21(-9)	2.94	1.31(-7)	4.22	2.45(-6)
25	3.45(-11)		1.01(-8)		2.34(-7)		1.50(-6)
30	7.54(-12)		6.58(-9)		2.62(-7)		1.00(-6)
35	5.43(-11)		4.31(-9)		2.18(-7)		7.19(-7)
40	4.67(-11)	3.51	6.05(-9)	2.41	1.71(-7)	1.66	5.41(-7)
Method 8							
5	1.73(-11)	4.96	1.67(-8)	2.27	3.88(-7)	1.81	1.36(-6)
10	9.51(-11)		6.16(-9)		8.90(-8)		3.36(-7)
15	4.74(-11)		2.48(-9)		3.84(-8)		1.49(-7)
20	4.67(-12)	4.27	1.74(-9)	1.81	2.14(-8)	1.97	8.36(-8)
25	4.74(-11)		1.16(-9)		1.36(-8)		5.35(-8)
30	3.47(-12)		7.86(-10)		9.41(-9)		3.71(-8)
35	1.23(-11)		5.56(-10)		6.90(-9)		2.73(-8)
40	2.62(-11)	1.98	4.10(-10)	1.84	5.28(-9)	1.98	2.09(-8)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 9							
5	1.26(-11)	4.92	1.16(-8)	2.37	3.10(-7)	2.10	1.33(-6)
10	1.65(-11)		6.41(-9)		8.36(-8)		3.32(-7)
15	8.37(-11)		5.49(-10)		3.74(-8)		1.49(-7)
20	5.46(-11)	2.10	9.99(-10)	2.20	2.11(-8)	1.99	8.39(-8)
25	4.70(-11)		7.57(-10)		1.35(-9)		5.37(-8)
30	3.95(-11)		5.55(-10)		9.39(-9)		3.73(-8)
35	3.05(-11)		4.18(-10)		6.90(-9)		2.74(-8)
40	1.73(-11)	2.11	3.24(-10)	2.01	5.29(-9)	1.99	2.10(-8)
Method 10							
5	3.44(-12)	5.24	6.32(-9)	2.89	3.48(-7)	1.90	1.30(-6)
10	4.28(-12)		1.82(-9)		8.21(-8)		3.26(-7)
15	2.78(-11)		2.51(-9)		3.65(-8)		1.45(-7)
20	3.69(-11)	2.75	1.66(-9)	1.82	2.06(-8)	1.99	8.17(-8)
25	5.54(-12)		9.47(-10)		1.32(-8)		5.23(-8)
30	2.80(-11)		6.15(-10)		9.14(-9)		3.63(-8)
35	1.04(-11)		4.37(-10)		6.72(-9)		2.70(-8)
40	5.98(-12)	2.89	3.29(-10)	1.98	5.15(-9)	1.99	2.04(-8)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 11							
5	8.96(-12)	5.22	1.25(-8)	2.34	3.19(-7)	2.03	1.30(-6)
10	9.06(-12)		4.96(-9)		8.16(-8)		3.26(-7)
15	4.21(-11)		2.34(-9)		3.65(-8)		1.45(-7)
20	3.14(-11)	2.68	1.29(-9)	2.00	2.05(-8)	1.99	8.17(-8)
25	9.56(-12)		8.19(-10)		1.32(-8)		5.23(-8)
30	3.38(-11)		8.19(-10)		9.14(-9)		3.63(-8)
35	1.92(-11)		4.18(-10)		6.72(-9)		2.67(-8)
40	2.39(-11)	1.87	3.20(-10)	2.00	5.15(-9)	1.99	2.04(-8)
Method 12							
5	1.09(-11)	4.29	1.42(-8)	3.12	3.16(-7)	2.04	1.30(-6)
10	3.65(-11)		2.07(-9)		8.16(-8)		3.26(-7)
15	2.67(-11)		2.04(-9)		3.65(-8)		1.45(-7)
20	5.77(-11)	2.21	1.23(-9)	2.03	2.05(-8)	1.99	8.17(-8)
25	4.91(-11)		8.03(-10)		1.32(-8)		5.23(-8)
30	3.84(-11)		5.63(-10)		9.14(-9)		3.63(-8)
35	9.61(-12)		4.16(-10)		6.72(-9)		2.67(-8)
40	2.11(-11)	1.96	3.19(-10)	2.01	5.15(-9)	1.99	2.04(-8)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 13							
5	1.66(-20)	7.75	7.64(-16)	9.33	3.15(-10)	2.29	1.54(-9)
10	6.76(-20)		2.20(-14)		2.20(-11)		6.77(-10)
15	1.39(-19)		4.87(-14)		6.18(-12)		1.04(-10)
20	1.94(-19)	9.12	5.97(-14)	2.92	3.44(-12)	5.77	1.88(-10)
25	1.21(-18)		8.14(-15)		3.31(-12)		1.74(-10)
30	7.16(-18)		3.57(-15)		1.28(-12)		2.08(-10)
35	7.94(-18)		4.75(-15)		1.64(-12)		1.28(-10)
40	1.27(-19)	7.52	4.94(-15)	5.02	5.22(-12)	3.16	4.68(-11)
Method 14							
5	3.51(-19)	4.86	2.98(-16)	9.94	2.88(-10)	2.39	1.51(-9)
10	3.08(-19)		1.44(-14)		2.15(-11)		6.74(-10)
15	2.04(-19)		4.12(-14)		6.12(-12)		1.04(-10)
20	7.57(-20)	9.74	5.45(-14)	2.98	3.43(-12)	5.78	1.88(-10)
25	1.34(-19)		7.69(-15)		3.30(-12)		1.74(-10)
30	2.74(-18)		3.44(-15)		1.27(-12)		2.08(-10)
35	4.34(-18)		4.62(-15)		1.64(-12)		1.28(-10)
40	8.31(-20)	7.93	4.83(-15)	5.04	5.21(-12)	3.17	4.68(-11)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 15							
5	2.06(-19)	3.36	2.17(-17)	15.25	3.28(-8)	----	2.10(-9)
10	1.68(-19)		6.07(-15)		2.97(-11)		4.96(-10)
15	9.09(-20		2.78(-14)		8.04(-12)		1.90(-10)
20	5.99(-22)	16.90	8.91(-12)	.08	9.92(-12)	4.80	2.76(-10)
25	3.77(-19)		3.67(-14)		6.70(-12)		7.59(-11)
30	3.31(-18)		2.28(-14)		6.45(-12)		1.89(-10)
35	6.32(-20)		1.22(-14)		5.57(-12)		1.48(-10)
40	1.71(-18)	5.52	3.60(-15)	4.82	2.87(-12)	1.46	7.87(-11)
Method 16							
5	1.62(-13)	6.01	6.77(-10)	3.62	1.02(-7)	3.76	1.38(-6)
10	5.29(-13)		4.41(-8)		7.43(-9)		3.54(-7)
15	1.38(-12)		2.02(-9)		1.01(-8)		1.57(-7)
20	2.75(-12)	4.52	1.44(-9)	1.23	7.87(-9)	3.49	8.87(-8)
25	6.08(-12)		9.33(-10)		5.63(-9)		5.68(-8)
30	5.19(-12)		6.40(-10)		4.12(-9)		3.94(-8)
35	1.01(-11)		1.14(-10)		3.12(-9)		2.90(-8)
40	2.17(-11)	1.65	2.14(-10)	1.75	2.43(-9)	3.19	2.22(-8)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 17							
5	6.11(-14)	8.49	7.92(-9)	1.29	4.73(-8)	6.50	4.27(-6)
10	2.14(-13)		1.09(-8)		3.31(-6)		4.77(-5)
15	3.05(-12)		4.70(-9)		2.14(-6)		2.40(-5)
20	2.54(-12)	5.48	5.05(-9)	5.43	9.37(-6)	0.60	1.41(-5)
25	1.85(-12)		5.89(-9)		8.51(-6)		9.29(-6)
30	7.94(-12)		4.74(-9)		6.57(-6)		6.56(-6)
35	4.47(-11)		4.67(-10)		5.02(-6)		4.87(-6)
40	3.60(-11)	4.68	1.72(-8)	3.91	3.91(-6)	----	3.75(-6)
Method 18							
5	2.03(-11)	3.60	2.98(-9)	4.48	1.48(-6)	1.66	4.69(-6)
10	1.83(-11)		3.04(-9)		2.84(-7)		1.24(-5)
15	2.05(-11)		5.36(-9)		9.70(-7)		7.07(-6)
20	9.68(-12)	3.34	9.88(-10)	4.07	2.78(-7)	3.96	4.32(-6)
25	8.99(-12)		1.18(-9)		8.22(-7)		2.87(-6)
30	1.25(-11)		6.07(-9)		9.27(-7)		2.03(-6)
35	2.17(-11)		5.48(-9)		8.73(-7)		1.51(-6)
40	8.92(-12)	4.40	3.96(-9)	3.81	7.77(-7)	0.59	1.17(-6)

Table 2 (continued)

λ	$h = \frac{\pi}{32}$	convergence rate	$h = \frac{\pi}{8}$	convergence rate	$h = \frac{\pi}{2}$	convergence rate	$h = \pi$
Method 19							
5	2.05(-12)	7.22	4.58(-8)	3.95	1.09(-5)	2.09	4.64(-5)
10	2.61(-10)		8.08(-8)		1.75(-5)		3.57(-4)
15	4.51(-10)		6.56(-8)		3.65(-5)		3.16(-4)
20	7.53(-10)	3.87	1.61(-7)	3.02	1.07(-5)	5.85	6.19(-4)
25	2.04(-10)		6.03(-8)		6.35(-6)		4.74(-4)
30	7.06(-10)		1.65(-7)		6.31(-5)		2.41(-4)
35	7.28(-10)		3.34(-7)		2.70(-5)		3.02(-5)
40	1.07(-9)	3.42	1.23(-7)	4.75	8.93(-5)	0.35	1.41(-4)
Method 20							
5	3.30(-15)	6.04	1.42(-11)	8.00	9.37(-7)	----	8.35(-7)
10	1.34(-14)		5.18(-11)		1.51(-8)		9.05(-7)
15	3.06(-14)		4.37(-11)		6.55(-8)		7.13(-7)
20	4.56(-14)	8.17	3.79(-9)	1.67	3.84(-8)	2.10	1.65(-7)
25	8.90(-14)		4.54(-10)		4.32(-8)		8.49(-7)
30	9.67(-14)		3.84(-10)		1.12(-8)		2.88(-7)
35	2.16(-13)		1.88(-10)		8.04(-9)		3.51(-7)
40	3.07(-13)	4.20	1.05(-10)	4.50	5.41(-8)	3.78	7.42(-7)

of λ less than or equal to six. The higher power of λ involved in the truncation errors may also explain why the remaining fourth order methods were less accurate than Cash's method. Next, the errors given by Jain, Kambo and Goel [35] are below the corresponding errors in Table 2 by a factor of $\sqrt{2}$. It is assumed they have use a different norm to compute their error. Finally, while Jain, Kambo and Goel [33] did not specify what they considered to be their step size, it is apparent from their numerical results that they have taken $h = x_{2n+2} - x_{2n}$. However, the results reported here are based on $h = x_{2n+1} - x_{2n}$.

Example 2. The second problem considered is a version of the test problem (2.2.1) given by

(2.3.2) $y'' = -\lambda^2 y \quad ; \quad y(0) = 1 \quad y'(x) = \lambda$. (See Table 3)

The numerical solution was compared to the exact solution, $y = \cos\lambda x + \sin\lambda x$, at $x = 2\pi$ and $x = 20\pi$ for $\lambda = 1$, $\lambda = 10$, $\lambda = 20$ and $\lambda = 40$. As expected, the error constants predict how the methods will perform in comparison to one another. The smaller the method's error constant, the smaller the error obtained.

The results obtained for this problem reflect the fact that P-stable methods may not be particularly effective when high frequency oscillations without small amplitude are present in the solution. The amplitude, $a = \sqrt{2}$, is constant for this problem, while the frequency grows with λ . With $\lambda = 1$, the frequency, $f = \frac{1}{2\pi}$, is relatively

Table 3. |Error| in the Solution at $x = 2\pi$ and $x = 20\pi$ for the Integration of (2.3.2).

x	$h = \frac{\pi}{64}$	$\frac{\pi}{32}$	$\frac{\pi}{8}$	$\frac{\pi}{2}$
Method 1				
2π	5.07(-8)	8.10(-7)	2.06(-4)	4.60(-2)
20π	4.30(-8)	6.88(-6)	1.71(-3)	2.93(-1)
Method 2				
2π	5.38(-8)	8.61(-7)	2.19(-4)	4.94(-2)
20π	5.38(-7)	8.61(-6)	2.19(-3)	5.81(-1)
Method 3				
2π	3.09(-6)	4.92(-5)	1.19(-2)	1.28(0)
20π	2.09(-5)	4.92(-4)	1.25(-1)	1.75(-1)
Method 4				
2π	5.07(-7)	8.10(-6)	2.06(-3)	5.36(-1)
20π	4.30(-6)	6.88(-5)	1.72(-2)	2.27(0)
Method 5				
2π	5.17(-8)	8.27(-7)	2.10(-4)	4.72(-2)
20π	5.17(-7)	8.27(-6)	2.10(-3)	5.51(-1)

Table 3 (continued)

x	$h = \frac{\pi}{64}$	$\frac{\pi}{32}$	$\frac{\pi}{8}$	$\frac{\pi}{2}$
Method 6				
2π	5.34(-8)	8.53(-7)	2.17(-4)	4.80(-2)
20π	5.34(-7)	8.53(-6)	2.17(-3)	5.73(-1)
Method 7				
2π	1.19(-6)	1.90(-5)	4.75(-3)	8.05(-1)
20π	1.19(-5)	1.90(-4)	4.85(-2)	1.12(0)
Method 8				
2π	2,46(-6)	3.92(-5)	9.59(-3)	1.17(0)
20π	2.45(-5)	3.92(-4)	9.98(-2)	7.21(-1)
Method 9				
2π	3.09(-6)	4.92(-5)	1.19(-2)	1.28(0)
20π	3.09(-5)	4.93(-5)	1.25(-1)	1.75(-1)
Method 10				
2π	8.47(-10)	5.42(-8)	2.17(-4)	5.44(-1)
20π	8.47(-9)	5.42(-7)	2.18(-3)	1.21(-1)

Table 3 (continued)

x	$h = \frac{\pi}{64}$	$\frac{\pi}{32}$	$\frac{\pi}{8}$	$\frac{\pi}{2}$
Method 11				
2π	2.40(-9)	1.53(-7)	5.94(-4)	8.77(-1)
20π	2.40(-8)	1.53(-6)	5.97(-3)	1.85(0)
Method 12				
2π	3.02(-9)	1.92(-7)	7.40(-4)	9.39(-1)
20π	3.02(-8)	1.92(-6)	7.42(-3)	2.23(0)
Method 13				
2π	1.21(-12)	7.75(-11)	3.16(-7)	1.23(-3)
20π	1.21(-11)	7.75(-10)	3.16(-6)	1.24(-2)
Method 14				
2π	1.21(-12)	7.75(-11)	3.16(-7)	1.23(-3)
20π	1.21(-11)	7.75(-10)	3.15(-6)	1.24(-2)
Method 15				
2π	8.72(-13)	5.58(-11)	2.27(-7)	8.50(-4)
20π	8.72(-12)	5.58(-10)	2.27(-6)	8.53(-3)

Table 3 (continued)

x	$h = \frac{\pi}{64}$	$\frac{\pi}{32}$	$\frac{\pi}{8}$	$\frac{\pi}{2}$
Method 16				
2π	3.79(-11)	2.42(-9)	9.76(-6)	2.98(-2)
20π	3.79(-10)	2.42(-8)	7.76(-5)	3.33(-1)
Method 17				
2π	3.21(-12)	2.05(-10)	7.91(-7)	8.79(-4)
20π	3.22(-11)	2.06(-9)	8.29(-6)	3.53(-2)
Method 18				
2π	8.10(-7)	1.29(-5)	3.20(-3)	5.35(-1)
20π	8.10(-6)	1.29(-4)	3.25(-2)	2.31(-1)
Method 19				
2π	5.07(-8)	8.10(-7)	2.06(-4)	4.65(-2)
20π	5.07(-7)	8.10(-6)	2.06(-3)	5.41(-1)
Method 20				
2π	8.72(-13)	5.58(-11)	2.27(-7)	8.50(-4)
20π	8.72(-12)	5.58(-10)	2.26(-6)	8.53(-3)

small and each method shows the correct convergence for rather large step sizes at both $x = 2\pi$ and $x = 20\pi$. While the stability of the methods was evident as λ grows, it was also clear that the higher frequencies made it necessary to use small step sizes to see proper convergence taking place. For $f = \frac{20}{\pi}$, corresponding to $\lambda = 40$, it was common to see convergence beginning near $\frac{\pi}{64}$ or $\frac{\pi}{128}$ for $x = 2\pi$ and near $\frac{\pi}{128}$ or $\frac{\pi}{256}$ for $x = 20\pi$. Thus, the only results reported are for $\lambda = 1$. Again, no iterative procedure was implemented since the problem is linear in y.

Example 3. The third example is given by

$$(2.3.3) \quad y'' + (100 + \frac{1}{4x^2})y = 0 . \qquad \text{(See Table 4)}$$

The general solution can be expressed in terms of Bessel functions, $y = c_1 \sqrt{x}\ J_0(10x) + c_2 \sqrt{x}\ Y_0(10x)$. The initial values were chosen at $x = \frac{1}{2}$ to single out the particular solution $\sqrt{x}\ J_0(10x)$. As in the previous examples, no iterative procedure was used to solve this problem.

The final problem tested is the Duffing equation forced by a harmonic function:

$$(2.3.4) \quad y'' + y + y^3 = B \cos\Omega x \ ; \ y(0) = \beta \ , \ y'(0) = 0$$

where $\beta = 0.200426728067$, $B = 0.002$ and $\Omega = 1.01$. The analytic solution computed by the Galerkin method with a 10^{-12} precision of the coefficients is given by

Table 4. |Error| in the Solution at $x = 6$ for the Integration of (2.3.3).

$h =$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$
Method 1					
	5.09(-6)	8.10(-5)	1.28(-3)	2.21(-2)	4.17(-1)
Method 2					
	5.39(-6)	8.58(-5)	1.36(-3)	2.37(-2)	4.42(-1)
Method 3					
	3.08(-4)	4.96(-3)	1.02(-1)	5.32(-2)	4.32(-1)
Method 4					
	4.31(-5)	6.81(-4)	1.11(-2)	2.38(-1)	4.22(-1)
Method 5					
	5.20(-6)	8.27(-5)	1.31(-3)	2.26(-2)	4.27(-1)
Method 6					
	5.36(-6)	8.53(-5)	1.35(-3)	2.35(-2)	4.40(-1)
Method 7					
	1.19(-4)	1.91(-3)	3.52(-2)	4.62(-1)	2.96(-2)

Table 4 (continued)

h =	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$
Method 8					
	2.42(-4)	3.87(-3)	6.58(-2)	7.59(-1)	5.72(-2)
Method 9					
	3.02(-4)	4.83(-3)	7.93(-2)	1.08(0)	7.23(-2)
Method 10					
	8.60(-7)	5.43(-5)	3.41(-3)	3.16(-1)	1.37(-2)
Method 11					
	2.51(-6)	1.60(-4)	1.01(-2)	1.41(0)	4.32(-2)
Method 12					
	3.08(-6)	1.96(-4)	1.25(-2)	1.49(0)	4.98(-2)
Method 13					
	1.23(-9)	7.84(-8)	4.98(-6)	3.12(-4)	2.48(-2)
Method 14					
	1.26(-9)	8.03(-3)	5.09(-6)	3.17(-4)	2.49(-2)

Table 4 (continued)

h =	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$
Method 15					
	9.17(-10)	5.83(-8)	3.67(-6)	2.24(-4)	1.42(-2)
Method 16					
	3.85(-8)	2.44(-6)	1.50(-4)	8.92(-3)	4.59(-1)
Method 17					
	3.21(-9)	2.04(-7)	1.32(-5)	7.79(-4)	5.08(-2)
Method 18					
	8.42(-5)	1.34(-3)	2.28(-2)	4.22(-1)	1.92(-2)
Method 19					
	5.29(-6)	8.43(-5)	1.34(-3)	2.28(-2)	4.22(-1)
Method 20					
	9.23(-10)	5.89(-8)	3.73(-6)	2.28(-4)	1.32(-2)

$$(2.3.5) \qquad y(t) = A_1\cos\Omega x + A_3\cos 3\Omega x + A\cos 5\Omega x + A_7\cos 7\Omega x + A_9\cos 9\Omega x$$

where

$$\begin{aligned} A_1 &= 0.200179477536 \\ A_3 &= 0.000246945143 \\ A_5 &= 0.000000304014 \\ A_7 &= 0.000000000374 \\ A_9 &= 0.000000000000. \end{aligned} \qquad (2.3.6)$$

Since this problem is nonlinear, an iterative procedure is required is for solution. The modified Newton iteration scheme given by Cash [2] and Chawla and Rao [11] was implemented. Note that equation (3.4) in [4] should be $F(y_{n+1}^{(P-1)})$ instead of $-F(y_{n+1}^{(P-1)})$.

Using Chawla's notation, the methods were written in the form

$$(2.3.7) \qquad G(y_{n+1}) = y_{n+1} - \phi(y_{n+1}) = 0 \ .$$

Letting $y_{n+1}^{(0)}$ denote the initial approximation to y_{n+1} defined by

$$(2.3.8) \qquad y_{n+1}^{(0)} = 2y_n - y_{n-1} + h^2 f_n \ ,$$

the modified Newton scheme used is given by

$$(2.3.9) \qquad G(y_{n+1}^{(i)}) + G'(y_{n+1}^{(0)})(y_{n+1}^{(i+1)} - y_{n+1}^{(i)}) = 0 \ , \ i = 0, 1, 2, \ldots$$

The two-step methods were the only methods tested for this problem. The fourth order schemes converged well with just one iteration per step to $x = 40\pi$ while the sixth order methods required two iterations. The results reported by Jain, Jain and Krishnaiah [32] for Method 8, a fourth order method, differ from those in Table 5 since the authors used two Newton iterations instead of one.

2.4. CONCLUSIONS

In the four examples considered, most of the two-step methods are seen to compare favorably to the three one-step methods and the four-step method tested. However the main concern here is how the two-step methods performed in relation to one another. This will involve not only the comparison of absolute errors, but also the number of function evaluations required to obtain those errors.

Despite limited cases in Example 1 where Cash's [3] sixth order method (Method 14, Table 1) gave smaller errors than the modification of the method (Method 15, Table 1), Method 15, which has the smallest absolute error constant attainable by a sixth order unconditionally stable method, generally produced more accurate results than Method 14 for all the examples considered. Also, for the linear problems in Examples 1, 2 and 3, Methods 14 and 15 were the most efficient $O(h^6)$ methods for obtaining a given error tolerance in the least number of function evaluations.

Chawla and Rao [12] point out that in their sixth order method (Method 16, Table 1), $\bar{f}_{n-1/2}$ and $\bar{\bar{f}}_{n-1/2}$ are independent of y_{n+1}.

Table 5. |Error| in the Solution at $x = 40\pi$ for the Integration of (2.3.4).

h =	$\frac{\pi}{40}$	$\frac{\pi}{20}$	$\frac{\pi}{10}$	$\frac{\pi}{5}$
Method 1				
	1.08(-6)	1.72(-5)	2.75(-4)	4.35(-3)
Method 2				
	9.36(-7)	1.45(-5)	2.17(-4)	2.90(-3)
Method 3				
	6.43(-5)	1.02(-3)	1.59(-2)	1.95(-1)
Method 4				
	1.31(-6)	2.09(-4)	3.35(-3)	5.42(-2)
Method 5				
	1.10(-6)	1.76(-5)	2.80(-4)	4.41(-3)
Method 6				
	1.13(-6)	1.81(-5)	2.88(-4)	4.57(-3)
Method 7				
	2.30(-5)	3.67(-4)	5.82(-3)	8.88(-2)

Table 5 (continued)

h =	$\frac{\pi}{40}$	$\frac{\pi}{20}$	$\frac{\pi}{10}$	$\frac{\pi}{5}$
Method 8				
	4.74(-5)	7.55(-4)	1.19(-2)	1.63(-1)
Method 9				
	5.96(-5)	9.48(-4)	1.48(-2)	1.88(-1)
Method 10				
	4.57(-8)	2.92(-6)	1.84(-4)	1.11(-2)
Method 11				
	1.29(-7)	8.21(-6)	5.09(-4)	2.89(-2)
Method 12				
	1.63(-7)	1.03(-5)	6.35(-4)	3.52(-2)
Method 13				
	6.41(-11)	1.65(-9)	2.43(-7)	1.80(-5)
Method 14				
	3.44(-10)	2.15(-8)	1.36(-6)	8.27(-5)

Table 5 (continued)

h =	$\frac{\pi}{40}$	$\frac{\pi}{20}$	$\frac{\pi}{10}$	$\frac{\pi}{5}$
Method 15				
	3.27(-10)	2.03(-8)	1.28(-6)	7.81(-5)
Method 16				
	2.02(-9)	1.28(-7)	8.12(-6)	4.99(-4)

Thus, when the modified Newton's method discussed in Example 4 is implemented, the vectors $f_{n+1}^{(i)}$, $\bar{f}_{n+1/2}^{(i)}$ and $\bar{\bar{f}}_{n+1/2}^{(i)}$ are the only new terms that depend on y_{n+1}. This results in a total of $3I + 6$ functions evaluations, where I is the number of iterations of the modified Newton's method, to calcualte y_{n+1} to sixth order accuracy at each step. However, for the non-linear problem in Example 4, Cash's sixth order method, Method 14 and its modification, Method 15, give more accurate results for the same number of function evaluations. In terms of the smallest number of function evaluations required by the sixth order methods to reach a given error tolerance in Example 4, Method 13 (Cash [2]) gave the best results. This in part can be explained by the over-approximation of $O(h^7)$ used by Cash to compute $y_{n\pm\alpha_2}$ in Method 13.

Because of the superior results given Cash's $O(h^4)$ method (Method 2, Table 1) in comparison to the other fourth order methods for Example 1, it is also hard to analyse which fourth order method is "best" for the problems considered. However, it is easily seen that Cash's method, which requires three function evaluations per step, is not as effective for Example 2, 3 and 4 as Hairer's $O(h^4)$ method (Method 1, Table 1), Chawla's $O(h^4)$ method (Method 5, Table 1) nor Cash's $O(h^4)$ method (Method 6, Table 1) when considering these methods require only two function evaluations per step. Because of the simplicity of the previously mentioned methods of Hairer and Chawla, it seems reasonable to choose the implementation of these methods over the the other fourth order methods considered.

CHAPTER 3

A NEW STABILITY CRITERION FOR $\ddot{Y} = F(X, Y)$

A stability criterion for one-step methods applied to the special problem

$$\ddot{y} = f(x, y) \; ; \; y(x_0) = y_0 \quad , \; \dot{y}(x_0) = \dot{y}_0 \tag{3.1}$$

has recently been proposed by Serbin and received some attention by Goyal [26]. The stability criterion, which is similar to AN-stability for first order systems, is derived from the problem

$$\ddot{y} + a(x)y = 0 \; . \tag{3.2}$$

We remark that in contrast to an analogous situation for first order systems, mere positiveity of the coefficient $a(x)$ does not imply boundedness of the solution of (3.2). Thus, to illustrate the origin of the stability criterion, consider the functional

$$J(y) = ay^2 + \dot{y}^2 \tag{3.3}$$

where a and y depend on x. Differentiating (3.3) with respect to x gives

$$\begin{aligned} \frac{d}{dx} J(y) &= \dot{a}y^2 + 2ay\dot{y} + 2\dot{y}\ddot{y} \\ &= \dot{a}y^2 + 2ay\dot{y} + 2\dot{y}(-ay) \\ &= \dot{a}y^2 \; . \end{aligned} \tag{3.4}$$

Thus, if $\dot{a} \leq 0$, $J(y)$ is a non-increasing function. To say a method satisfies the stability criterion requires

$$(3.5) \qquad a(x_{n+1})y_{n+1}^2 + \dot{y}_{n+1}^2 \leq a(x_n)y_n^2 + \dot{y}_n^2 , \quad \forall_n .$$

That is, the numerical scheme provides the same behavior descretely with regard to the functional J (with similar results for $\dot{a} \geq 0$ for a modified functional). Goyal [26] has proven that a particular one-stage member of a class of second order Rosenbrock-type methods satisfy this criterion.

The one-step methods proposed by Hairer [28] (Methods 19-20, Table 1) as well as the corresponding $O(h^2)$ method given by $G = \frac{1}{2}$, $g = 1$ were tested numerically using Example 3 of Section 2.3. It was found that each method satisfied the stability (3.5) for all stepsizes. However, the $O(h^4)$ method of Jain, Kambo and Goel [33] (Method 18, Table 1) did not satisfy (3.5) for any step size used for Example 3.

While no conclusions can be drawn from this limited information, the success of Hairer's methods seems to indicate promise for the further investigation of the stability criterion.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. L. Brusa and L. Nigro, A one-step method for direct integration of structural dynamic equations, Int. J. Num. Meth. in Engng. 15: 685-699 (1980).

2. J.R. Cash, High order P-stable formulae for the numerical integration of periodic initial value problems, Numer. Math. 37: 355-370 (1981).

3. J.R. Cash, Efficient P-stable methods for periodic initial value problems, BIT 24: 248-252 (1984).

4. J.R. Cash, Erratum for efficient P-stable methods for periodic initial value problems, BIT 24: 400 (1984).

5. S.H. Chang and P.C. Gnepp, A direct method for the numerical solution of $y'' = (f, y)$, Calcolo 21: 369-377 (1984).

6. M.M. Chawla, Two-step fourth order P-stable methods for second order differential equations, BIT 21: 190-193 (1981).

7. M.M. Chawla, Unconditionally stable Noumerov-type methods for second order differential equations, BIT 23: 541-542 (1983).

8. M.M. Chawla, Numerov made explicit has better stability, BIT 24: 117-118 (1984).

9. M.M. Chawla, On the order and attainable intervals of periodicity of explicit Nystrom methods for $y'' = f(t,y)$, SIAM J. Numer. Anal. 22: 127-133 (1985).

10. M.M. Chawla and P.S. Rao, A Noumerov-type method with minimal phase-lag for the integration of second order periodic initial-value problems, J. Comput. Appl. Math. 11: 277-281 (1984).

11. M.M. Chawla and P.S. Rao, High-accuracy P-stable methods for $y'' = f(t, y)$, IMA J. Numer. Anal. 5: 215-220 (1985).

12. M.M. Chawla and P.S. Rao, High-accuracy P-stable methods for $y'' = f(t, y)$: corrigendum, IMA J. Numer. Anal. 6: 252 (1986).

13. M.M. Chawla and S.R. Sharma, Families of fifth order Nystrom methods for $y'' = f(x, y)$ and intervals of periodicity Computing 26: 247-256 (1981).

14. M.M. Chawla and S.R. Sharma, Intervals of periodicity and absolute stability of explicit Nystrom methods for $y'' = f(x, y)$, BIT 21: 455-464 (1981).

15. J.P. Coleman and J. Mohamed, On de Vogelaere's method for $y'' = f(x, y)$, Math. Comp. 32: 751-762 (1978).

16. F. Costabile, Un metado per l'integrazione numerica della equazione differenziale ordinaria $y'' = f(x, y)$ can condizioni iniziali, Calcolo 12: 249-258 (19795).

17. F. Costabile and C. Costabile, Two-step fourth order P-stable methods for second order differential equations, BIT 22: 384-386 (1982).

18. F. Costabile and A. Varano, Convergence, stability and truncation error estimation of a method for the numerical integration of the initial value problem $y'' = f(x, y)$, Calcolo 18: 371-384 (1982).

19. G. Dahlquist, On accuracy and unconditional stability of linear multistep methods for second order differential equations, BIT 18: 233-136 (1978).

20. R. de Vogelaere, A method for the numerical integration of differential equations of second order without explicit first derivatives, J. Res. Nat. Bur. Standards 54: 119-125 (1955).

21. E. Fehlberg, Eine Runge-Kutta-Nyström-Formel 9-ter Ordnung mit schrittweitenkontrolle fur differentialgleichungen $\ddot{x} = f(t, x)$, ZAMM 61: 477-485 (1981).

22. E. Fehlberg, Eine RKT-formel für differentialgleichungen der form $\ddot{x} = f(t, x)$, ZAMM 63: 261-268 (1983).

23. S. Filippi and J. Gräf, Ein Runge-Kutta-Nyström-Formelpaar der ordnung "(12) fur differentialgleichungen def form $y'' = f(x, y)$, Computing 34: 271-282 (1985).

24. W. Gautschi, Numerical integration of ordinary differential equations based on trigonometric polynomials, Num. Math. 3: 381-397 (1961).

25. I. Gladwell and R.M. Thomas, Damping and phase analysis for some methods for solving second-order differential equations, Int. J. Num. Meth. in Engng. 19: 495-503 (1983).

26. S. Goyal, A class of Rosenbrock-type schemes for second-order non-linear systems of ordinary differential equations, Ph.D. Dissertation, University of Tennessee, 1983.

27. E. Hairer, Methodes de Nyström pour l'equation differentielle $y'' = f(x, y)$, Numer. Math. 27: 283-300 (1977).

28. E. Hairer, Unconditionally stable methods for second order differential equations, Numer. Math. 32: 373-379 (1979).

29. E. Hairer, A one-step method of order 10 for $y'' = f(x, y)$, IMA J. Numer. Anal. 2: 83-94 (1982).

30. M.K. Jain, R.K. Jain and U.A. Krishnaiah, P-stable methods for periodic initial value problems of second order differential equations, BIT 19: 347-355 (1979).

31. M.K. Jain, R.K. Jain and U.A. Krishnaiah, P-stable singlestep methods for periodic initial-value problems involving second-order differential equations, J. Engng. Math. 13: 317-326 (1979).

32. M.K. Jain, R.K. Jain and U.A. Krishnaiah, Hybrid numerical methods for periodic initial value problems involving second-order differential equations, Appl. Math. Mod. 5: 153-156 (1981).

33. R.K. Jain, N.S. Kambo and R. Goel, A fourth order method for $y'' = f(x, y, y')$, J. Comput. Appl. Math. 9: 81-90 (1983).

34. R.K. Jain, N.S. Kambo and R. Goel, High-order one-step P-stable methods for the numerical integration of periodic initial value problems, J. Comput. Appl. Math. 9: 361-367 (1983).

35. R.K. Jain, N.S. Kambo and R. Goel, A sixth-order P-stable symmetric multistep method for periodic initial-value problems for second-order differential equations. IMA J. Numer. Anal. 4: 117-125 (1984).

36. U.A. Krishnaiah, Inverse linear multistep methods for the numerical solution of initial value problems of second order differential equations, J. Comput. Appl. Math. 7: 111-114 (1981).

37. U.A. Krishnaiah, Adaptive methods for periodic initial value problems of second order differential equations, J. Comput. Appl. Math. 8: 101-104 (1982).

38. J.D. Lambert and I.A. Watson, Symmetric multistep methods for periodic initial value problems, J. Inst. Math. Applics. 18: 189-202 (1976).

39. B. Neta, Higher order hybrid Störmer-Cowell methods for ordinary differential equations, Congr. Numer. 42: 251-264 (1984).

40. B. Neta, Hybrid predictors and correctors for solving a special class of second order differential equations, Congr. Numer. 46: 241-248 (1985).

41. B. Neta and S.C. Lee, Hybrid methods for a special class of second-order differential equations, Congr. Numer. 38: 203-225 (1983).

42. H.E. Salzer, Trigonometric interpolation and predictor-corrector formulas for numerical integration, ZAMM 42: 403-412 (1962).

43. G. Scheifele, On numerical integration of perturbed linear oscillating systems, ZAMP 22: 186-210 (1971).

44. S.M. Serbin, Rational approximations of trigonometric matrices with applications to second-order systems of differential equations, Appl. Math. Comput. 5: 75-92 (1979).

45. E. Stiefel and D.G. Bettis, Stabilization of Cowell's method, Numer. Math. 13: 154-175 (1969).

46. R.M. Thomas, Phase properites of high order almost P-stable formulae, BIT 24: 225-238 (1984).

47. P.J. van der Houwen, Stabilized Runge-Kutta methods for second order differential equations without first derivatives, SIAM J. Numer. Anal. 16: 523-537 (1979).

48. P.J. van der Houwen, Modified Nyström methods for semi-discrete hyperbolic differential equations, SIAM J. Numer. Anal. 18: 1081-1097 (1981).

VITA

Michael Joseph Petkac of Grassflat, Pennsylvania was born in Philipsburg, Pennsylvania on August 3, 1962. He attended the West Branch Area School District in Morrisdale, Pennsylvania until his graduation in May 1980. In the fall of 1980 he entered The Pennsylvania State University and received a Bachelor of Science degree in Mathematics in May 1984. The following September he accepted a teaching assistantship at The University of Tennessee, Knoxville and began work on a Master of Arts degree, with a major in Mathematics. This degree was conferred in August 1986.