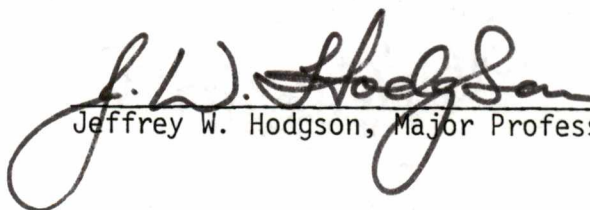


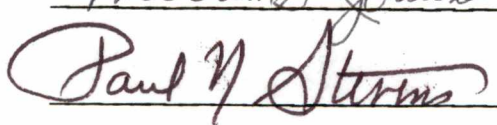
To the Graduate Council:

I am submitting herewith a thesis written by Stanley A. Posey entitled "A Finite Element Solution to the Steady State Potential and Thermal Field Distributions in the Joint Region of a Water-Cooled Resistive Conductor." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mechanical Engineering.


Jeffrey W. Hodgson, Major Professor

We have read this thesis
and recommend its acceptance:





Accepted for the Council:


Vice Provost and
Dean of the Graduate School

A FINITE ELEMENT SOLUTION TO THE STEADY STATE POTENTIAL
AND THERMAL FIELD DISTRIBUTIONS IN THE JOINT REGION
OF A WATER-COOLED RESISTIVE CONDUCTOR

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Stanley A. Posey

March 1985

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ABSTRACT

To aid in the design and development of resistive magnetic field coils for nuclear fusion applications, a finite element solution procedure has been developed. The computations include solution to the electric potential distribution, the spatial variation of the Joule heat generation, and the temperature distribution in a water-cooled resistive conductor over a joint region.

Sufficient assumptions are made that permit the thermal uncoupling of the governing electrical and thermal differential equations, but, extensions of the analysis technique appear feasible for problems where temperature dependency of conductor properties is significant.

The electrical and thermal effects of conductor surface contacts are included through use of experimental information. Results developed in this study agree favorably with the results of conductor joint tests performed at The Oak Ridge National Laboratory.

TABLE OF CONTENTS

SECTION	PAGE
1. INTRODUCTION.	1
Helical Coil Joint Concept.	4
2. REVIEW OF PREVIOUS WORK	8
3. MATHEMATICAL MODELING	12
Assumptions	12
Analytical Development.	13
4. NUMERICAL TECHNIQUE	21
Choice of Approximate Method.	21
Finite Element Application.	23
Variational Formulation of Field Problems	24
5. DEMONSTRATION PROBLEM	32
Problem Statement	32
Analytical Solution	33
Finite Element Approximation.	34
6. FINITE ELEMENT MODELING OF TEST JOINTS.	46
Finite Element Models	47
Input Parameters.	49
7. RESULTS AND DISCUSSION.	54
General Test Information.	54
Reference Joint Results	56
Resistive Effects of Contact Surfaces	61
Double Side Bar Joint Results	68
Results at Design Conditions.	81
8. CONCLUSIONS	88
REFERENCES.	89
VITA.	92

LIST OF TABLES

TABLE	PAGE
I. NASTRAN Results of Test Problem	48
II. Ref Input Parameters for Numerical Approximation at 20 Kamp	57
III. DSB Joint Test Results.	72
IV. DSB Input Parameters for Numerical Approximation at 14 Kamp	79
V. DSB Input Parameters for Numerical Approximation at 31.25 Kamp.	84

LIST OF FIGURES

FIGURE	PAGE
1. HF Coil Set with a Removed Segment	2
2. Typical Conductor Joint Geometry	6
3. Schematic of Joint Transport and Conduction Processes. . . .	7
4. Differential Volume Element of Conductor	17
5. One Dimensional Solid Conductor.	32
6. Two Element Finite Element Model	34
7. Voltage Distribution	45
8. Temperature Distribution	45
9. Finite Element Model of Reference Joint.	50
10. Finite Element Model of Double Side Bar Joint.	50
11. Location of Voltage Taps and Thermocouples for the Reference Joint.	55
12. Voltage Solution of Reference Joint at 20,000 Amperes. . . .	57
13. Constant Voltage Conductor Length.	58
14. Temperature Solution of Reference Joint at 20,000 Amperes. .	60
15. Behavior of Resistance with Increasing Current	62
16. Location of Voltage Taps and the Thermocouples for the Double Side Bar Joint.	64
17. Voltage Drop as a Function of Current.	69
18. Resistance as a Function of Current.	70
19. Average Temperature as a Function of Current	71
20. Linear Electrical Analysis of Double Side Bar Joint.	75
21. Resistance as a Function of Average Temperature.	76

FIGURE	PAGE
22. Resistance as a Function of Current.	77
23. Voltage Solution of Double Side Bar Joint at 14,000 Amperes.	80
24. Temperature Solution of Double Side Bar Joint at 14,000 Amperes.	82
25. Voltage Solution of Double Side Bar Joint at Design Con- ditions (31,250 Amperes)	85
26. Temperature Solution of Double Side Bar Joint at Design Conditions (31,250 Amperes).	86

LIST OF SYMBOLS

A	Area (in^2)
A_c	Contact area (in^2)
h	Convective film coefficient ($\text{Watts/m}^2\text{-K}$)
i	Current (Kamperes)
J	Current density (Kamp/in^2)
k	Thermal conductivity (Watts/m-K)
L_c	Contact conduction length (in)
q	Power (Watts)
Q	Power per unit volume (Watts/m^3)
R	Resistance (μohm)
ρ	Electrical resistivity ($\mu\text{ohm-cm}$)
T_b	Water bulk temperature ($^{\circ}\text{C}$)
v	Water flow velocity (cm/s^2)
v	Electric potential (mVolts)
V	Electric potential (mVolts)
∇V	Electric potential gradient (mVolts/in)
{ }	Column matrix
{ } ^T	Row matrix
[]	Square matrix

1. INTRODUCTION

In the design of thermonuclear fusion reactors, containment of the plasma requires the production of a composite magnetic field that is induced through arrangements of resistive conducting coils. The arrangement and geometry of the magnetic field coils depends on the desired particle paths of the plasma ions. The present fusion design project at the Oak Ridge National Laboratory, the Advanced Toroidal Facility (ATF) is a test reactor that requires a coil combination such that the particle paths are contained on helical magnetic flux surfaces as they accelerate around the torus.

The ATF is a current-free toroidal device; that is, no current will conduct in the plasma. This means that the helical magnetic flux surfaces must be produced by the magnetic coils alone. The magnetic coils providing the primary field influence on the helical surfaces consist of a set of helically wound copper (bus) conductor coils. Due to restrictions in the assembly and production schedules, the helical field coils (HF) will not be wound in a continuous helix but instead will be fabricated in 180 degree segments to be joined at the ends. During assembly, joints will be machined on the ends of each segment in order to form the continuous helix. The HF coil set is illustrated in Figure 1 along with a removed 180 degree segment and a schematic of a typical coil cross-section.

The HF coil design concept requires fabricating 24 identical segments that will each consist of two stacks of seven water-cooled

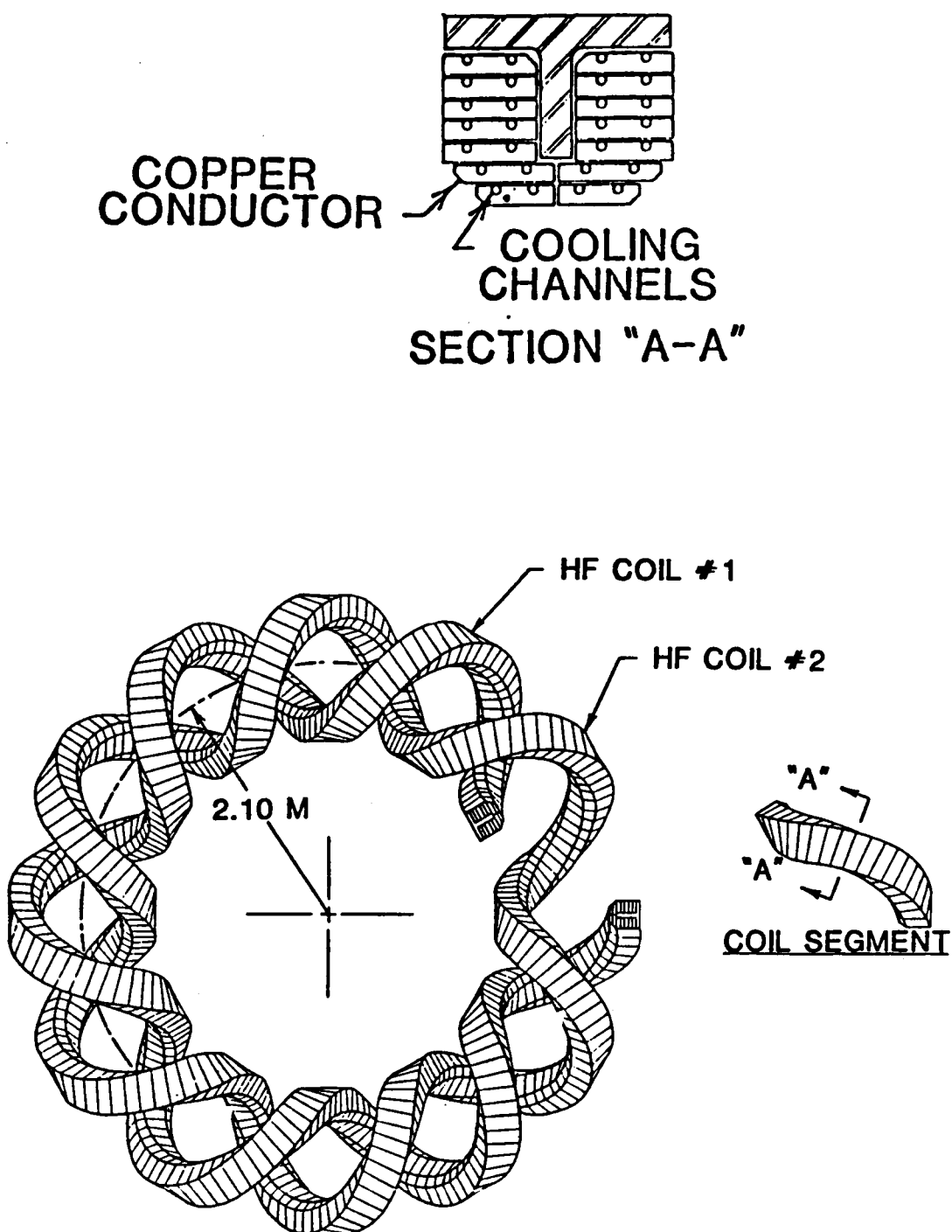


Figure 1. HF coil set with a removed segment.

copper bus conductors mounted to a stainless steel structural member. A major uncertainty in the ATF design is the ability to build and install the HF coils to very close positional tolerances required by the path followed by the coil's current center. This influences the fabrication of the conductor parts, the assembly of these parts into coil segments, the machining of joint details and the final integration of the HF coil segments into a complete, continuous coil system.

Geometric and mechanical constraints require the joint region of a segment to be of the same cross-sectional area as in the basic coil region. Since the joint cross-sectional area cannot be solid copper, the amount of copper for electrical conduction is reduced. This increases the resistance beyond its nominal value due to the increased current density in the joint region created by the decreased cross sectional conduction area.

The design objectives for the joint include the following:

- no change in coil cross-sectional area,
- minimal increase in nominal resistance,
- minimal temperature rise in the bulk copper,
- accommodation of magnetic loads and thermal stresses,
- accessible for maintenance and adjustments.

The problem being addressed in this thesis involves the development of electrical and thermal analyses procedures to support the HF coil joint design.

Helical Coil Joint Concept

The ideal joint design from a thermal viewpoint would minimize the i^2R losses while providing sufficient cooling to keep the temperatures in the joint within design specifications. This would require minimizing the electrical resistance, and machining the cooling path as close to the joint as possible without increasing the joint resistance. The resistance will be at a minimum when the area of the joint contact surface is as large as possible. There are, however, some mechanical restrictions in maximizing the contact surface area due to structural and magnetic loads on the joint. Therefore, a method of quantifying the electrical and thermal distributions throughout the coil joint region is necessary for comparing the various proposed designs.

Many joint designs were proposed and tests have been performed on several of the designs. The tests measure local voltage potentials and local temperatures throughout the joint region. The test power supply, however, is capable of delivering no more than half of the current needed to simulate design operating conditions. Joint performance at actual operating conditions can only be extrapolated from information obtained with the low current tests.

Thus, it was proposed that a mathematical model be developed for analysis of candidate joint designs that would provide (1) the electric potential distribution and (2) the temperature distribution throughout the joint region during steady state electrical conduction.

It was proposed that the model would reproduce joint test results at low current conditions and would then be applied to analyze the actual operating conditions.

Due to schedule and cost constraints it was not feasible to fabricate and test an entire coil cross section of a conductor joint (see Figure 1, section "A-A"). Although each conductor in the coil stack is geometrically different (i.e., conductor cross-section, conductor radius of curvature, etc.), the joint configuration would be the same for each conductor. Therefore, the fabrication, testing, and analysis of a single (typical) conductor joint should provide the necessary information on the electrical and thermal performance of the entire coil cross-section at the joint.

Figure 2 illustrates a typical single conductor joint geometry that might be fabricated for testing. Note that a typical conductor also offers symmetry about a plane through its half-width. The analysis, then, should simulate the conduction and transport processes shown schematically in Figure 3.

The presence of the cooling tube presents a 3-dimensional solution domain. Also, geometrical discontinuities appear in the problem due to the conductor surface contacts. Depending on the bolt clamping pressure, the resistive effects at the contact surfaces can vary strongly. The mathematical and numerical formulations of the solution procedure include these and other considerations which are discussed throughout the following sections.

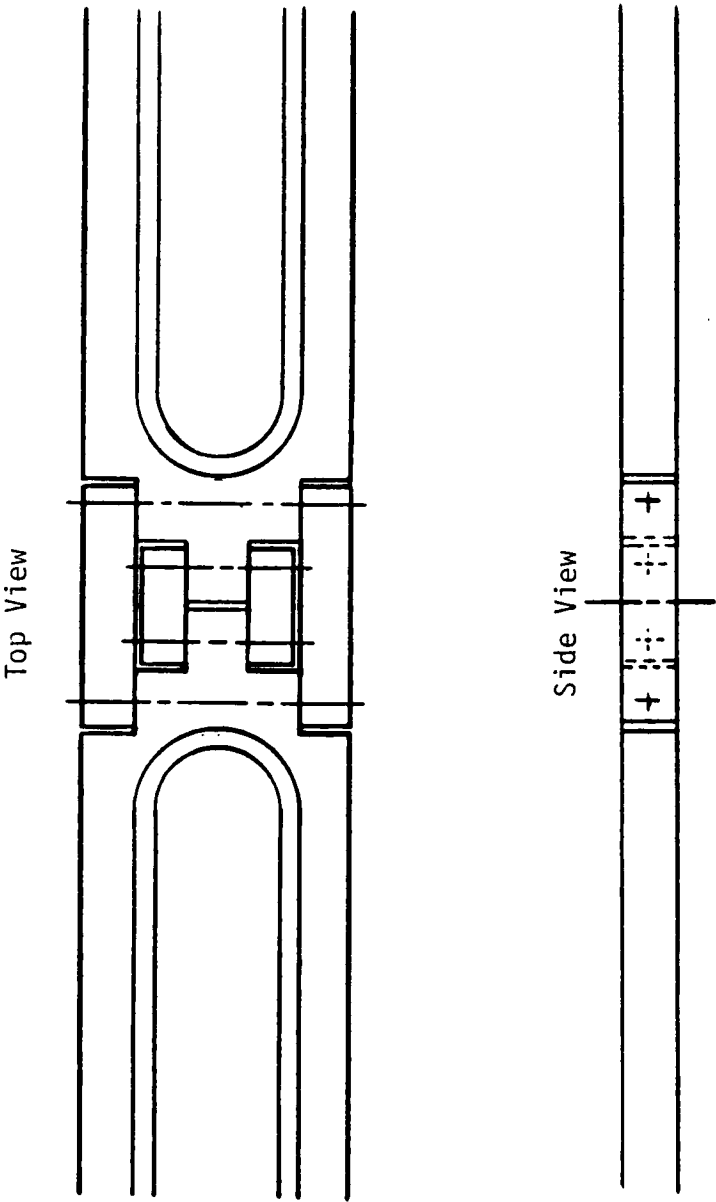


Figure 2. Typical conductor joint geometry.

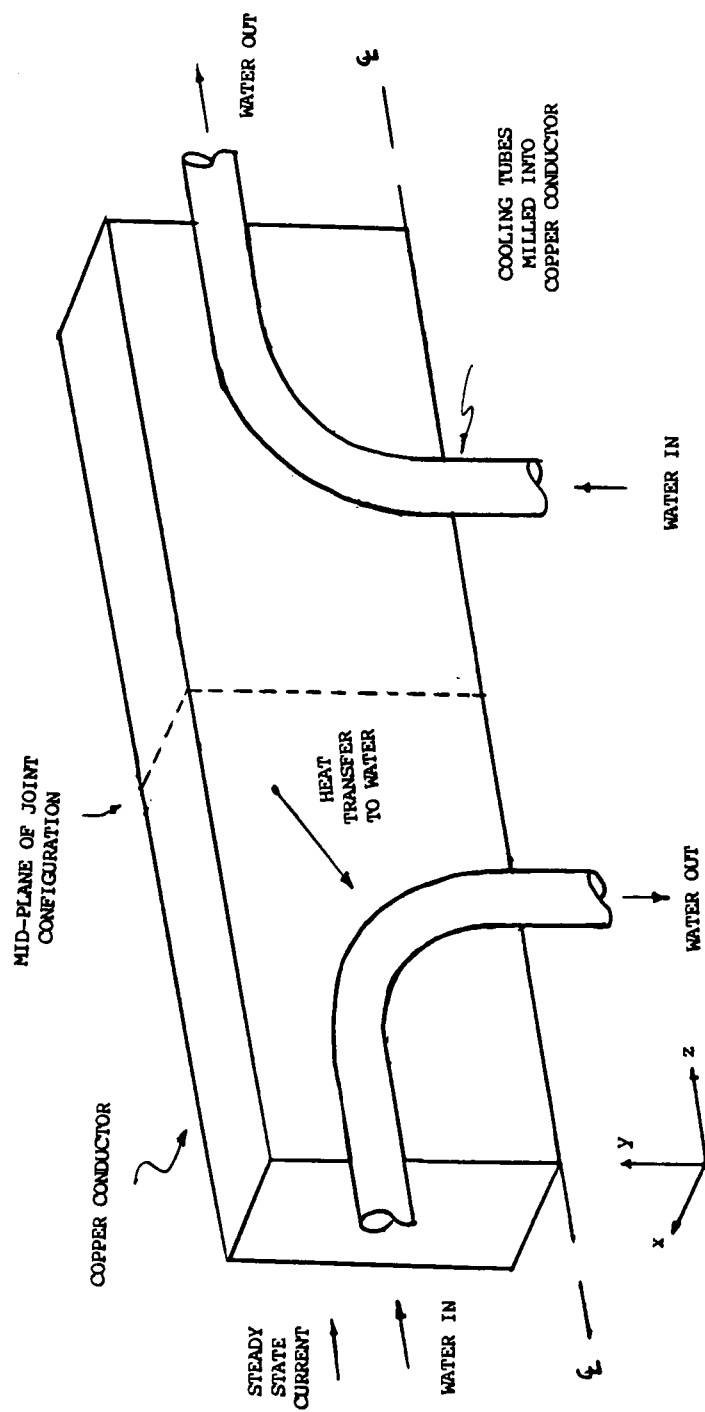


Figure 3. Schematic of joint transport and conduction processes.

2. REVIEW OF PREVIOUS WORK

Research into the literature was necessary for information on three primary topics:

- mathematical assumptions,
- resistive effects of contacts,
- finite element application.

The mathematical assumptions for the solution procedure were collected as a result of the available experimental information. Because of the experimental nature of the problem, the literature does not directly address the associated mathematics. However, many papers and texts have provided the necessary insight required to formulate the analytical approach.

At the beginning of this study, it was recognized from the work of Miller, Dresner and Lue [1] with nonresistive conductors that thermal uncoupling of the problem would greatly simplify the mathematics without greatly affecting the physics. Thermal problems arise in nonresistive conductors when current must transfer from the nonresistive material into a normal resistive material. Derby [2] also describes a similar problem for nonresistive conductors where thermal uncoupling is assumed. This idea was incorporated into the present analysis on resistive conductors, and consequently, linear solution techniques could be used in the numerical approximation. Krieth's [3] derivation of the heat conduction equation along with Meaden's [4] and Williams' [5] descriptions of electrical resistance were

helpful in developing a mechanical/electrical analogy in the derivation of the governing differential equation of the electric field.

Due to the many possible joint designs, the analytical development is derived in its most general form. That is, the development is valid for geometrical nonuniformity of conductor properties. The numerical technique emphasizes the geometrical details, while the analytical development simply provides the mathematical framework necessary to satisfy physical arguments.

The surface contacts introduce discontinuities and geometrical nonlinearities in the solution domain. Although the governing differential equations account for the spatial variation of the conductor properties in their derived form, the specific geometrical variation of the properties must be known in order to solve the equations. The electrical effects of contact surfaces have been investigated in recent papers. Noterdaeme, Montgomery, and Lidsky [6] report on an experimental study of a demountable resistive joint wherein data of contact resistance vs. clamping pressure were collected. The results apply to the specific surface finish and geometry of the design, thus providing only qualitative information. Giesen, Goring, Kohlhaas, and Stickelmann [7] also provide only qualitative, but very useful information of the electrical effects at contacts. Their experimental investigations include water-cooled contacts. A paper presented by Hartwig and Van Sciver [19] describes the results of testing aluminum and copper solder and pressure joints. Each of these experimental investigations describe similar qualitative findings but

fail to provide any useful correlations. It was recognized at this point that the analysis would have to depend on the experimental information of a specific joint for quantitative information of the contact effects. This restricts the application of the analysis to those joint geometries that are actually fabricated and tested.

The numerical treatment of the problem required an extensive survey of the finite element literature. The finite element method is suggested for problems having extreme geometrical complexities. Kellog [8] and Babuska [9] both report on the mathematical foundations supporting the finite element's treatment of material discontinuities. Because the finite element method is based on integral formulations, the discontinuities can be approximated with piecewise, smooth functions. The specific choice of the integral representation, however, varies from one analyst to another. Zienkiewicz and Cheung [10] report that the finite element method applied through a variational approach is possible for all elliptic field problems. Many authors describe the variational method for finite element application [11], [12],[13],[14]. Baker [18] and Huebner [16] provide examples along with their treatment of the variational methodology in finite elements. The use of isoparametric elements in irregular solution domains is described by Zienkiewicz and Pareich [17] and Huebner [16]. Use of isoparametric elements will be necessary for discretization of curved boundaries created by the conductor cooling path. The well known finite element computer code NASTRAN [15] was found

to employ a variational technique in the solution to elliptic field problems. NASTRAN also includes provisions for discontinuities in the solution domain and employs three-dimensional isoparametric elements.

3. MATHEMATICAL MODELING

The analytical development of the governing equations and their boundary conditions are presented in this section. Also, the assumptions made in obtaining the mathematical formulations are discussed.

Assumptions

The main assumptions made in the analytical and computational modeling are presented and discussed in this section. The following is a listing of these assumptions:

1. The bus conductors in the coil stack are assumed to be electrically and thermally insulated from each other.
2. Geometrical symmetry is assumed about the mid-plane of the conductor width. Curvature in the conductor is neglected throughout the joint region.
3. The temperature dependencies of the conductor properties (i.e., the electrical resistivity and the thermal conductivity) are neglected. Included instead are temperature compensations for the conductor properties based on predicted average temperature values throughout the joint region. This permits independent solutions of the electrical and thermal problems, and avoids a simultaneous solution scheme. Typical predicted temperature ranges do not exceed 180°F which creates a maximum change of only 2.5% in the properties from their mean values.

4. The bulk temperature and the flow rate of the cooling water are input parameters and are assumed to be constant based on previous test measurements.

5. The flow characteristics of the cooling water produce a range of Reynolds number well into the turbulent regime. Based on the measured bulk water temperature rise (typically 1 to 2°C), the cooling tube wall surface temperature should not exceed 190°F. The wall temperature variation is predicted to be near linear because the applied rate of heat flux is near constant. Also, there is sufficient coolant entry length for the test piece to assume that the velocity profile is fully developed at the cooling tube entrance of the finite element model. Thus, the film coefficient is assumed to be constant along the cooling tube inner surface, and is computed from an empirical correlation.

6. The copper conductor is assumed to consist of a uniform material (typically OFHC copper).

7. The thermal effects of the brazing material (ASTM B-Ag 8) around the copper cooling tube are neglected. The differences in the material properties are small when considering the conduction thickness of the braze material.

Analytical Development

The general differential equation governing the temperature distribution of the conductor joint is the steady state, 3-dimensional nonhomogeneous, and spatially nonlinear heat conduction equation written as:

$$\nabla[k\nabla T(x,y,z)] + Q(x,y,z) = 0 \quad (1)$$

The heat being generated by the electric field appearing in the $Q(x,y,z)$ term, is swept away by the cooling water in the tubes that are milled into the conductor. From assumption 1, all heat transfer is to the water only--this will impose a nonhomogeneous boundary condition of the third kind along the inner surface of the cooling tube. Also from assumption 1, all other boundaries of the conductor are assumed to be insulated. These conditions may be expressed mathematically as,

$$\begin{aligned} \frac{\partial T(0,y,z)}{\partial x} &= 0 & \frac{\partial T(L_x,y,z)}{\partial x} &= 0 \\ \frac{\partial T(x,0,z)}{\partial y} &= 0 & \frac{\partial T(x,L_y,z)}{\partial y} &= 0 \\ \frac{\partial T(x,y,0)}{\partial z} &= 0 & \frac{\partial T(x,y,L_z)}{\partial z} &= 0 \\ k \frac{\partial T(x,y,z)}{\partial n} \bigg|_{TS} + hT_{TS} &= hT_b \end{aligned} \quad (2)$$

where the subscript TS represents "tube surface" and $\underline{n}$ is an outward unit vector everywhere normal to the inner tube surface.

Generally, the differential equations governing the heat conduction and the electrical conduction are coupled through the joule heat generation term; however, according to assumption 3, the electrical and thermal problems are uncoupled and can be solved

independently of one another. If the temperature ranges encountered are so great that this effect should be considered, a temperature solution could always yield an update in the value of the electrical resistivity for an iterative solution process rather than performing a simultaneous solution.

Before the solution to the temperature distribution can be obtained, the spatial variation of the heat generation term must be determined. Therefore, the electrical problem must be addressed before the thermal problem can be solved. An expression for the heat generation per unit volume can be derived through the basic laws of electrical conduction. The heating will be nonuniform due to the distribution of the current paths throughout the conductor width. Paths through small cross-sectional areas experience a higher resistance and will generate greater amounts of heat. In general, the heat energy is expressed as

$$q = i^2 R \quad \left(R = \frac{\rho L}{A} \right) .$$

By substituting for R and dividing both sides by the volume LxA , an expression for watts per unit volume can be obtained

$$Q = \frac{i^2 \rho}{A^2} , \quad (3)$$

where ρ is in units of Ω -inches. The current density J will be defined as $J = \frac{i}{A}$. This can be substituted into equation (3) yielding

$$Q = J^2 \rho . \quad (4)$$

The current density will be a function of each spatial variable and can be written from Ohm's law as

$$J = \frac{1}{\rho} \nabla V , \quad (5)$$

where ∇V denotes the electric field gradient. Substituting equation (5) into equation (4) yields an expression for Q in terms of the square of the voltage gradient

$$Q = \left(\frac{1}{\rho} \nabla V \right)^2 \rho . \quad (6)$$

Dividing through equation (6) with the resistivity and expanding the differential operator results in the following expression for the joule heat distribution

$$Q = \frac{1}{\rho} \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right] . \quad (7)$$

Therefore, a solution for the temperature distribution in the joint region requires first a solution for the voltage gradients (or potentials) throughout the same region. The voltage gradients can be obtained from a solution of the voltage distribution by developing an electrical/thermal analogy of conduction through a solid. To derive such a relationship, consider a small differential volume element $dx dy dz$ of the conductor as shown in Figure 4.

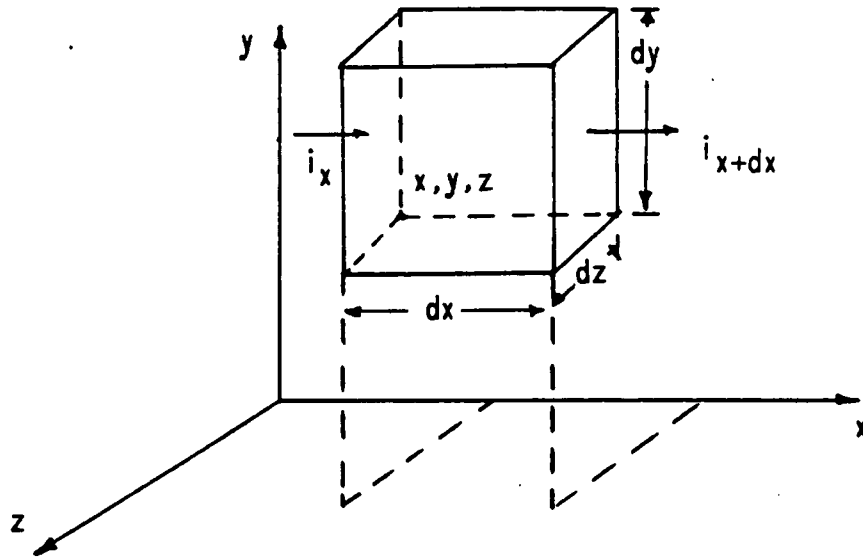


Figure 4. Differential volume element of conductor.

The conservation of charge for the element in the steady state is written as

$$i_x + i_y + i_z = i_{x+dx} + i_{y+dy} + i_{z+dz}$$

The rate of charge conduction passing through the left face at x can, according to Ohm's law, be written as

$$i_x = \frac{1}{\rho} \left(\frac{\partial V}{\partial x} \right) dydz .$$

The rate of charge conduction passing out of the element through the right face at $x+dx$ is

$$i_{x+dx} = \left[\frac{1}{\rho} \left(\frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial V}{\partial x} \right) dx \right] dydz .$$

Subtracting the rate of conduction out of the element from the rate of conduction into the element yields

$$i_x - i_{x+dx} = -\frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial V}{\partial x} \right) dx dy dz ,$$

and similarly for the y and z directions

$$i_y - i_{y+dy} = -\frac{\partial}{\partial y} \left(\frac{1}{\rho} \frac{\partial V}{\partial y} \right) dx dy dz ,$$

$$i_z - i_{z+dz} = -\frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial V}{\partial z} \right) dx dy dz ,$$

Substituting these terms into the expression for conservation of charge and dividing through by $dx dy dz$ gives

$$\frac{\partial}{\partial x} \left(\frac{1}{\rho_x} \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\rho_y} \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho_z} \frac{\partial V}{\partial z} \right) = 0 \quad (8)$$

If ρ is assumed to be spatially uniform the solution must satisfy the Laplacian

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 . \quad (9)$$

The resulting differential equation for the voltage distribution represents a potential field problem and by comparison with the heat conduction equation, clearly demonstrates a thermal/electrical analogy. The voltage field would be analogous to a temperature field and the current flow paths correspond to heat flow paths in a potential flow field. This will present a clear advantage in the numerical modeling because a general solution for linear, elliptic, steady state field problems can be developed for the solution of both kinds of physical problems.

The boundary conditions for the electric field equation are chosen to simulate the nature of steady state current conduction for a particular joint geometry. Current paths will travel perpendicular to planes of symmetry that traverse the joint region and planes of constant conductor cross-section in the basic conductor region. These planes are constant voltage planes in a potential flow field. The boundary conditions would simply be an applied potential in the direction of the current flow with all other conductor boundaries remaining insulated; the mathematical statements of which are given by

$$\begin{aligned}
 \frac{\partial V(0,y,z)}{\partial x} &= 0 & \frac{\partial V(L_x,y,z)}{\partial x} &= 0 \\
 \frac{\partial V(x,0,z)}{\partial y} &= 0 & \frac{\partial V(x,L_y,z)}{\partial y} &= 0 \\
 V(x,y,0) &= 0 & V(x,y,L_z) &= V_{app} .
 \end{aligned} \tag{10}$$

The applied voltage V_{app} is an input parameter whose value depends on the value of the steady state current being conducted through the joint test piece. Solution to the voltage distribution and consequently the voltage gradients will provide the heat source distribution term required to solve for the temperature distribution.

A summary of the complete solution procedure is given below.

1. Determine the voltage distribution from the solution to equation (8) with the boundary conditions given in equation (10).
2. Compute the voltage gradients and substitute the values into equation (7) for computation of the joule heat distribution.

3. Determine the temperature distribution from the solution to equation (1) with the boundary conditions given in equation (2) including the heat source term described by equation (7).

4. NUMERICAL TECHNIQUE

Choice of Approximate Method

The solution to the field problems lend themselves to numerical approximations due to the irregular geometry imposed by the solution domain. Before the selection of an appropriate numerical technique is presented, some additional comments concerning the restrictions already placed on the solution procedure should be stated. First and foremost, the temperature dependence of electrical resistivity is neglected. This uncouples the differential equations allowing them to be solved independently rather than simultaneously.

The differential equations and their boundary conditions do, however, include the spatial variation of the electrical resistivity, equation (8). This will become important when electrical conduction across copper contacts is investigated. The inclusion of a spatially dependent resistivity still permits independent solutions of the electric and thermal field problems.

Restricting conductor properties to functions of position only (assumption 3) will allow for the application of linear approximation principles when solving the differential equations. With this consideration in mind, several numerical techniques become obvious candidates. However, considering the extreme irregular geometrical constraints imposed by the joint designs (e.g., the curvature in the cooling path and the presence of bolted surfaces), immediately

discourages the use of finite difference schemes. The cooling tube geometry with its convective boundary condition would require a fourth order difference approximation to achieve sufficient solution accuracy and, consequently, would require a very fine mesh. The result would be an extremely large, sparse, and asymmetric matrix system that would be costly to solve in terms of machine CPU and disk memory. A further consideration is the eventual need for a stress analysis of the joint subjected to magnetic loads and thermal stresses. These considerations suggest the choice to be the finite element method.

The solution to partial differential equations and their boundary conditions using the finite element method (like most numerical techniques) ultimately reduces the problem to the solution of a set of linear algebraic equations that approximates the behavior of the differential equation over the solution domain. But within the finite element framework, several possibilities exist for the technique used in assembling the system matrix. For linear steady state field problems, a variational principle is generally applicable [10] whereas the stress analyst might use a weighted residual technique. Nevertheless, the discretization of the solution domain is independent of the system matrix assembly operation which permits a general application of the finite element numerical technique.

Finite Element Application

Application of the finite element method requires integral formulations that approximate the behavior of the dependent field variable over the solution domain. This is done through a discretization of the solution domain into a finite number of unknown values of the field variable at specified locations called nodes. The nodes coincide with the boundaries of small regions where the action of the differential equation is approximated through the integral formulation over the subregion or finite element. The finite element equations are then assembled into a system of algebraic equations that approximate the action of the differential equation and boundary conditions over the entire discretized solution domain.

The development of the integral formulation that represents the action of the differential equation is (most often) problem dependent. For steady state field problems, finite element integral formulations can be obtained by either a variational principle or a more general weighted residual technique. The variational approach is the most convenient for the problem being studied due to the elliptic nature of the differential equations.

In the classical variational formulation, a field problem is reduced to finding the unknown function which extremizes an integral involving a functional and satisfies a set of constraints placed on the functional. In the classical analytical approach, the solution involves integration of the governing differential equation and the

satisfaction of the accompanying boundary conditions. The flexibility in choosing the form of the functional allows the two problem formulations to be equivalent. That is, by choice of the appropriate functional, the analytical and variational problem statements are equivalent because the solution that satisfies the differential equation and its boundary conditions is also the function that extremizes the functional subject to its constraints.

The method of weighted residuals (MWR) starts with the governing differential equation and avoids the search for the mathematically equivalent variational statement. The classical variational formulation for field problems has many advantages over the MRW's differential formulation from the standpoint of obtaining approximate solutions. The greatest advantage is that of the functional always containing lower order derivatives than that of the differential operator in the governing equation. Consequently, lower order interpolation polynomials can be used in the approximate solution. Use of the weighted residual technique is usually reserved for field problems where a variational statement is not known or does not exist.

Another important and practical advantage of the variational formulation is its rigorous mathematical support [18] of error bounds and solution convergence.

Variational Formulation of Field Problems

The general form of a steady state field problem for an unknown dependent field variable $U(x,y,z)$ can be written as,

$$\frac{\partial}{\partial x} \left(p_x \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left(p_y \frac{\partial U}{\partial y} \right) + \frac{\partial}{\partial z} \left(p_z \frac{\partial U}{\partial z} \right) + w = 0 , \quad (11)$$

where p represents the property of the medium and w is some unknown specified function of x , y , and z . A general set of boundary conditions can be written as

$$U(x,y,z) = U_B$$

$$p_x \frac{\partial U}{\partial x} + p_y \frac{\partial U}{\partial y} + p_z \frac{\partial U}{\partial z} + \alpha U + w = 0 . \quad (12)$$

This classical problem of differential calculus will be recast in variational form through the use of the "calculus of variations." The variational form will provide the integral formulations that are necessary for application of the finite element method. The classical problem presented in the theory of the calculus of variations is to determine a function $U(x,y,z)$ that minimizes the integral

$$I(U) = \int f(x,y,z,U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}) \, dx dy dz . \quad (13)$$

It can be formally shown that $U(x,y,z)$ must also be a solution to the following differential equation known as the Euler-Lagrange equation,

$$\frac{\partial}{\partial x} \left[\frac{\partial f}{\partial U'} \right] + \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial U'} \right] + \frac{\partial}{\partial z} \left[\frac{\partial f}{\partial U'} \right] - \frac{\partial f}{\partial U} = 0 \quad (14)$$

if $U(x,y,z)$ is to minimize the integral $I(U)$. In order to develop a variational equivalent to the presented field problem, the form of the functional f must be chosen so that the differential equation

involving the dependent field variable $U(x,y,z)$ is produced upon substitution of f into the Euler-Lagrange equation.

If such a functional f exists for the differential equation involving the field problem, then the function $U(x,y,z)$ that minimizes the functional integral is the solution to the governing differential equation (Euler-Lagrange equation). The functionals for the electrical and thermal conduction differential equations are known to exist and they indeed involve lower order derivatives than their differential operators. The electrical conduction integral formulation will be presented first followed by the formulation of the heat conduction equation.

The electric field problem involving the voltage potential V is written as

$$\frac{\partial}{\partial x} \left(\frac{1}{\rho_x} \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\rho_y} \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho_z} \frac{\partial V}{\partial z} \right) = 0 ,$$

with the following boundary conditions:

$$\frac{\partial V(0,y,z)}{\partial x} = 0$$

$$\frac{\partial V(L_x,y,z)}{\partial x} = 0$$

$$\frac{\partial V(x,0,z)}{\partial y} = 0$$

$$\frac{\partial V(x,L_y,z)}{\partial y} = 0$$

$$V(x,y,0) = 0$$

$$V(x,y,L_z) = V_{app} .$$

As shown by Zienkiewicz [11], the functional form of f that will produce the above differential equation upon substitution of f into the Euler-Lagrange equation is

$$f = \frac{1}{2} \left[\frac{1}{\rho_x} \left(\frac{\partial V}{\partial x} \right)^2 + \frac{1}{\rho_y} \left(\frac{\partial V}{\partial y} \right)^2 + \frac{1}{\rho_z} \left(\frac{\partial V}{\partial z} \right)^2 \right] . \quad (15)$$

In the formal development of the extremum problem that leads to the Euler-Lagrange equation, it is shown that the natural boundary conditions of $\frac{\partial V}{\partial n} = 0$ and known boundary values, $V = V_{app}$, are satisfied in the functional. Thus, the fixed boundary conditions on z as well as the insulated boundary conditions on x and y are satisfied in equation (15). This results in a variational formulation equivalent to solution of the electric field problem. The procedure consists of determining the function $V(x,y,z)$ that will minimize the integral

$$I_v = \frac{1}{2} \int \left[\frac{1}{\rho_x} \left(\frac{\partial V}{\partial x} \right)^2 + \frac{1}{\rho_y} \left(\frac{\partial V}{\partial y} \right)^2 + \frac{1}{\rho_z} \left(\frac{\partial V}{\partial z} \right)^2 \right] d(\text{Vol}) . \quad (16)$$

The thermal field differential equation involving the temperature distribution is written as

$$\nabla[k\nabla T(x,y,z)] + Q(x,y,z) = 0 ,$$

with boundary conditions

$$\begin{aligned}
\frac{\partial T(0,y,z)}{\partial x} &= 0 & \frac{\partial T(L_x,y,z)}{\partial x} &= 0 \\
\frac{\partial T(x,0,z)}{\partial y} &= 0 & \frac{\partial T(x,L_y,z)}{\partial y} &= 0 \\
\frac{\partial T(x,y,0)}{\partial z} &= 0 & \frac{\partial T(x,y,L_z)}{\partial z} &= 0 \\
k \frac{\partial T(x,y,z)}{\partial n} TS + hT_{TS} &= hT_b .
\end{aligned}$$

The functional form needed to produce the above differential equation upon substitution into the E-L equation appears as

$$f = \frac{1}{2} \left[k_x \left(\frac{\partial T}{\partial x} \right)^2 + k_y \left(\frac{\partial T}{\partial y} \right)^2 + k_z \left(\frac{\partial T}{\partial z} \right)^2 - 2qT \right] . \quad (17)$$

As stated for the electric field problem, insulated and fixed boundary conditions are satisfied in the functional; however, the convective boundary condition must be imposed as a constraint on the functional through use of a Lagrange Multiplier [18]. The constraint must also have the appropriate functional form that will produce the convective boundary condition upon substituting the constraint functional into the E-L equation. Thus, the constraint will appear as

$$g = \frac{1}{2} hT^2 - hT_b T \quad (18)$$

Therefore, the variational equivalent to solution of the thermal field problem can be obtained by determining the function $T(x,y,z)$ that will minimize the integral

$$\begin{aligned}
I_t = & \frac{1}{2} \int \left[k_x \left(\frac{\partial T}{\partial x} \right)^2 + k_y \left(\frac{\partial T}{\partial y} \right)^2 + k_z \left(\frac{\partial T}{\partial z} \right)^2 - 2QT \right] d(\text{Vol}) \\
& + \frac{1}{2} \int \left[hT^2 - 2hT_b T \right] d(\text{Area}) .
\end{aligned} \tag{19}$$

The differential equations and their boundary conditions have been transformed into integral formulations through a variational principle. The finite element method will be used to approximate these integrals for each element, form the appropriate elemental matrix from the approximation, and sum up the elemental contributions to form the global system of equations that describe the behavior of the dependent field variable over the solution domain.

Through approximation theory [13], the dependent field variable in each of the integrals will be replaced with an interpolation function based on a series polynomial. Polynomial approximations are convenient because they are easy to integrate and differentiate. The truncation of the series expansion, however, is very sensitive to the error residual. For most field problems, quadratic interpolation functions are recommended [16]. The quadratic approximating function will converge an order of magnitude more rapidly than that of a linear function, and with a significantly more accurate solution [18]. Also, the 2nd order quadratic function is the lowest order interpolation function available for use of curve shaped or isoparametric elements. These elements are essential in approximating the boundary curvature of the cooling tube.

With the solution domain subdivided into elements, the unknown field variable can be represented in each element as:

$$U(x,y,z) = \{N(x,y,z)\}^T \{u\}$$

where $\{N\}^T$ is a row matrix of quadratic interpolation functions which depend on the nodal x, y , and z coordinates of the element and $\{u\}$ is a column matrix of the unknown values of the field variable at the nodes of the element. Making these substitutions into the integral statements, the elemental integral functionals become

$$I_v = \frac{1}{2} \int \left[\frac{1}{\rho} \{v\}^T \nabla \{N\} \nabla \{N\}^T \{v\} \right] d(\text{Vol}) \quad (20)$$

$$\begin{aligned} I_t = & \frac{1}{2} \int \left[k \{t\}^T \nabla \{N\} \nabla \{N\}^T \{t\} - 2Q \{t\}^T \{N\} \right] d(\text{Vol}) \\ & + \frac{1}{2} \int \left[h \{t\}^T \{N\} \{N\}^T \{t\} - 2hT_b \{t\}^T \{N\} \right] d(\text{Area}) \end{aligned} \quad (21)$$

Using the finite element approximation, the integral has become solely dependent upon the unknown nodal field variable distribution. Therefore the minimum of I_v and I_t can only be established by differentiation with respect to $\{v\}$ and $\{t\}$ respectively. That is,

$$\frac{\partial I_v}{\partial \{v\}} = 0$$

and

$$\frac{\partial I_t}{\partial \{t\}} = 0$$

This operation provides integral statements for equations (20) and (21) with only one unknown appearing in the integrand--the

dependent field variable. By performing the integrations, the problems reduce to solution of the linear systems:

$$[P(\rho)]\{v\} = \{0\} \quad (22)$$

$$[P(k,h)]\{t\} = \{D(Q,h)\} \quad (23)$$

where $[P(\rho)]$ and $[P(k,h)]$ are square matrices that contain element property information and $\{D(Q,h)\}$ is a row matrix that contains input "data" information. The details of this solution procedure are given in section 5.

5. DEMONSTRATION PROBLEM

In order to better illustrate the proposed solution technique, a simple one-dimensional problem is solved through (1) an analytical treatment of the set of differential equations and their boundary conditions, and (2) a two-element quadratic finite element approximation to the same problem. This simple problem will clearly demonstrate the proposed solution procedure that will be applied to the complex geometry of the joint region.

Problem Statement

Consider the one-dimensional case of steady state current flow through a solid conductor having dimensions as shown in Figure 5,

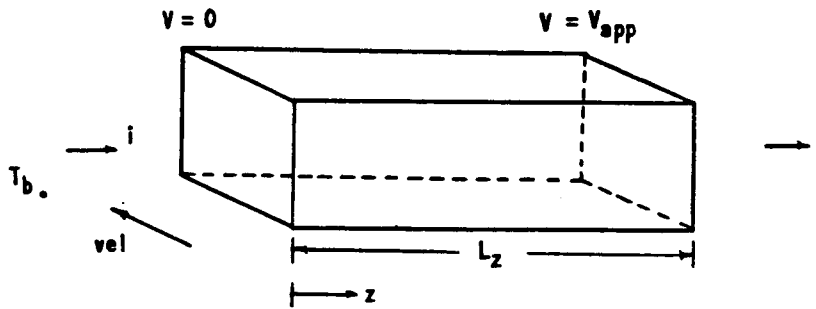


Figure 5. One dimensional solid conductor.

with conductor properties ρ and k . The differential equation governing the electrical conduction in one-dimension is simply

$$\frac{d^2V}{dz^2} = 0 \quad (24)$$

with boundary conditions

$$V(z=0) = 0 \quad V(z=L_z) = V_{app} \quad (25)$$

The solution to equation (24) permits calculation of the volumetric heat generation which is given by

$$Q = \frac{1}{\rho} \left(\frac{dV}{dz} \right)^2 . \quad (26)$$

The temperature distribution can then be determined from solution of the governing differential equation

$$k \frac{d^2 T}{dz^2} + Q = 0 , \quad (27)$$

with boundary conditions

$$\left. \frac{dT}{dz} \right|_{z=L_z} = 0 \quad (28) \quad -k \left. \frac{dT}{dz} \right|_{z=0} + hT|_{z=0} = hT_b \quad (29)$$

Analytical Solution

The analytical solution to the voltage distribution is given by

$$V(z) = \frac{V_{app}}{L_z} z . \quad (30)$$

Thus the heat generation is computed as

$$Q = \left(\frac{V_{app}}{L_z} \right)^2 \frac{1}{\rho_{av}} . \quad (31)$$

Finally the analytical solution to the temperature distribution is given as

$$T = T_b + Q \left[\frac{L_z z}{k} - \frac{z^2}{2k} + \frac{L_z}{h} \right] . \quad (32)$$

Finite Element Approximation

In the discretization of the conductor geometry, line elements are required for a one-dimensional finite element approximation. The use of quadratic interpolation functions will require three nodes for each element in order to satisfy the three unknown coefficients in a quadratic series. Therefore, the two element model of the one-dimensional conductor will appear as shown in Figure 6.

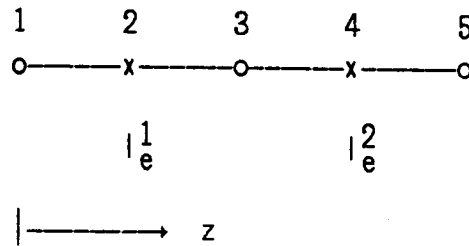


Figure 6. Two element finite element model.

The finite element approximation proceeds (as described in section 4) with developing the functional integral for an arbitrary element, performing the elemental integrations, and differentiating with respect to the nodal parameters to obtain the global matrix system that describes the behavior of the differential equation and its boundary conditions over the entire solution domain.

The finite element integral functions would appear in one dimension as:

$$I_v = \frac{1}{2} \int \left[\frac{1}{\rho} \left(\frac{dV}{dz} \right)^2 \right] dz \quad (33)$$

$$I_t = \frac{1}{2} \int \left[k \left(\frac{dT}{dz} \right)^2 - 2QT \right] dz + \frac{1}{2} \int \left[hT^2 - 2hT_b T \right] dz \quad (34)$$

Through approximation theory, the field variables $V(z)$ and $T(z)$ will be replaced by the product of a set of quadratic interpolation functions and the unknown values of the field variable at the nodal coordinates:

$$V(z) = \{N(z)\}^T \{v\} \quad , \quad \frac{dV}{dz} = \nabla \{N\}^T \{v\}$$

$$T(z) = \{N(z)\}^T \{t\} \quad , \quad \frac{dT}{dz} = \nabla \{N\}^T \{t\}$$

Making these substitutions into the functions of (33) and (34), the elemental integrals become:

$$I_v = \frac{1}{2} \int \left[\frac{1}{\rho} \{v\}^T \nabla \{N\} \nabla \{N\}^T \{v\} \right] dz \quad (35)$$

$$\begin{aligned} I_t = & \frac{1}{2} \int \left[k \{t\}^T \nabla \{N\} \nabla \{N\}^T \{t\} - 2Q \{t\}^T \{N\} \right] dz \\ & + \frac{1}{2} \int \left[h \{t\}^T \{N\} \{N\}^T \{t\} - 2hT_b \{t\}^T \{N\} \right] dz \end{aligned} \quad (36)$$

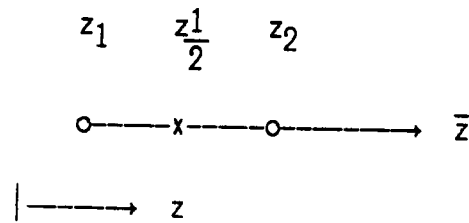
Note that $\{N(z)\}$ will be the same for each problem because it is strictly a geometry dependent parameter (i.e., the same discretization is used for both solutions to $V(z)$ and to $T(z)$). Therefore $\{N(z)\}$ can be developed independently of the field problems and then the appropriate substitutions can be made before extremizing the integrals.

The quadratic basis $\{N(z)\}$ can be determined for a general unknown field variable $U(z)$ from the solution of the matrix statement:

$$U(z) = \{N(z)\}^T \{u\} ,$$

where $U(z)$ is approximated with a quadratic power series.

For a quadratic interpolation, three nodes are required to define a one-dimensional element in order to satisfy the three unknown coefficients resulting from the series polynomial expansion. The elemental matrix system can be developed by establishing a local coordinate system $\bar{z}$.



The translational relationship between the local coordinate system $\bar{z}$ and the global coordinate system z can be written as

$$z = z_1 + \bar{z}$$

The quadratic approximation over an arbitrary element can be constructed from the power series

$$U(z) = \sum_{i=0}^2 a_i z^i$$

The expansion coefficients (a_i 's) must have the dimensions of the dependent variable. This requires the independent variable to appear in the nondimensional form $(\frac{\bar{z}}{l_e})$ based on the elemental length $l_e = z_1 - z_2$. This is written in expanded form as

$$U(\bar{z}) = a_0 + a_1 \left(\frac{\bar{z}}{l_e} \right) + a_2 \left(\frac{\bar{z}}{l_e} \right)^2 \quad (37)$$

The expansion coefficients can be determined by evaluating $U(z)$ at the three nodal coordinates of the element:

$$\begin{aligned} U(\bar{z}=0) &= u_1 = a_0 \\ U(\bar{z}=\frac{l_e}{2}) &= u_{1/2} = a_0 + a_1\left(\frac{1}{2}\right) + a_2\left(\frac{1}{4}\right) \\ U(\bar{z}=l_e) &= u_2 = a_0 + a_1 + a_2 \end{aligned} \quad (38)$$

Solving this system of equations for a_0 , a_1 , and a_2 in terms of u_1 , $u_{1/2}$, and u_2 , the expression for the set of interpolation functions $\{N\}$, is formed. It can be shown that solution to equations (38) yields

$$\begin{aligned} a_0 &= u_1 \\ a_1 &= -u_2 + 4u_{1/2} - 3u_1 \\ a_2 &= 2u_2 + 2u_1 - 4u_{1/2} \end{aligned}$$

The expressions for the coefficients are substituted back into equation (37) to give

$$u_1 + (-u_2 + 4u_{1/2} - 3u_1)\left(\frac{\bar{z}}{l_e}\right) + (2u_2 + 2u_1 - 4u_{1/2})\left(\frac{\bar{z}}{l_e}\right)^2 = \{N(\bar{z})\}^T \{u\} \quad (39)$$

Equation (39) is simplified to obtain the desired form of $\{N\}\{u\}$ as

$$\{N\}\{u\} = \begin{Bmatrix} 1 - 3 \left(\frac{\bar{z}}{l_e} \right) + 2 \left(\frac{\bar{z}}{l_e} \right)^2 \\ 4 \left(\frac{\bar{z}}{l_e} - \frac{\bar{z}^2}{l_e^2} \right) \\ - \left(\frac{\bar{z}}{l_e} \right) + 2 \left(\frac{\bar{z}}{l_e} \right)^2 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_{1/2} \\ u_2 \end{Bmatrix} \quad (40)$$

and,

$$\nabla\{N\} = \begin{Bmatrix} -\frac{3}{l_e} + \frac{4\bar{z}}{l_e} \\ \frac{4}{l_e} - \frac{2\bar{z}}{l_e^2} \\ -\frac{1}{l_e} + \frac{4\bar{z}}{l_e^2} \end{Bmatrix} \quad (41)$$

Returning to the field problems, the integrations of the functionals can be performed with the appropriate substitutions of $\{N\}$ and $\nabla\{N\}$. Beginning with the functional involving the electric field problem, the integral I_v simplifies to

$$I_v = \frac{1}{2} \{v\}^T \frac{1}{\rho} \int_0^{l_e} \nabla\{N\} \nabla\{N\}^T dz \{v\}$$

Forming the product $\nabla\{N\} \nabla\{N\}^T$, the integral becomes

$$I_v = \frac{1}{2} \{v\}^T \frac{1}{\rho} \int_0^{l_e} [M(z)] dz \{v\}$$

where

$$[M(z)] = \frac{1}{l_e^2} \begin{bmatrix} 9-24\xi+16\xi^2, & -12+40\xi-32\xi^2, & 3-16\xi+16\xi^2 \\ -12+40\xi-32\xi^2, & 16-64\xi+64\xi^2, & -4+24\xi-32\xi^2 \\ 3-16\xi+16\xi^2, & -4+24\xi-32\xi^2, & 1-8\xi+16\xi^2 \end{bmatrix}$$

and

$$\xi = \frac{\bar{z}}{l_e}$$

After performing the integration, I_v reduces to

$$I_v = \frac{1}{2} \{v\}^T \frac{1}{\rho} \frac{1}{3l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \{v\}$$

The minimum of I_v is established by differentiating with respect to $\{v\}$,

$$\frac{\partial I_v}{\partial \{v\}} = 0 = \frac{1}{3\rho l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_{1/2} \\ v_2 \end{Bmatrix}$$

which simplifies to

$$\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_{1/2} \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This represents the global contribution of a single arbitrary element. For the two element geometry that was established for this problem, the global matrix will appear as

$$\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The boundary conditions are applied before the solution is obtained. This reduces the system to three equations in three unknowns.

$$v_1 = 0 \quad , \quad v_5 = V_{app}$$

or

$$\begin{bmatrix} -8 & 16 & -8 \\ -8 & 14 & -8+V_{app} \\ 0 & -8 & 16-8V_{app} \end{bmatrix} \begin{Bmatrix} v2 \\ v3 \\ v4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The finite element temperature approximation is obtained in a similar fashion. The integral $\int \nabla N \nabla N^T$ has already been determined, allowing I_t to be directly written as

$$I_t = \frac{1}{2} \{t\}^T \left\{ k \frac{1}{3l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \{t\} - \int_0^{l_e} 2Q\{N\} dz \right. \\ \left. + h \int_0^{z_1} \{N\} \{N\}^T dz \{t\} - 2hT_b \int_0^{z_1} \{N\} dz \right\}$$

The integral involving $\{N\}$ is evaluated as

$$\int \{N(z)\} dz = \frac{l_e}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

Note that for the integral involving the convective boundary conditions, both limits are zero. This is due to node #1 coinciding

with the convective boundary, thus leaving only one term after evaluating the integral.

$$\int \{N\} \{N\}^T dz = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and similarly,

$$\int \{N\} dz = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

The expression for I_t simplifies to

$$I_t = \frac{1}{2} \{t\}^T \left\{ k \frac{1}{3l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \{t\} - Q \frac{l_e}{3} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} \right\}$$

$$+ h \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 2hT_b \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \left. \right\}$$

As before, the minimum of I_t is established by differentiating with respect to $\{t\}$,

$$\frac{\partial I_t}{\partial \{t\}} = 0 = \left\{ \frac{k}{3l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + h \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \{t\}$$

$$- Q \frac{l_e}{3} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} - 2hT_b \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

which simplifies to

$$\frac{k}{3l_e} \begin{bmatrix} 7+h(\frac{3l_e}{k}) & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_{1/2} \\ t_2 \end{Bmatrix} = Q \frac{l_e}{3} \begin{Bmatrix} hT_b \frac{6}{Ql_e} + 1 \\ 4 \\ 1 \end{Bmatrix}$$

This represents the global contribution of a single arbitrary element subject to a convective boundary condition. For the two element geometry that was established for this problem, the global matrix system will become

$$\frac{k}{Q|_e^2} \begin{bmatrix} 7+h\left(\frac{3|_e}{k}\right) & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} t1 \\ t2 \\ t3 \\ t4 \\ t5 \end{Bmatrix} = \begin{Bmatrix} hT_b \frac{6}{Q|_e} + 1 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

This requires the solution to a system of five equations in five unknowns.

In order to compare the results of the finite element approximation with the exact solution, the following numerical values were chosen for input parameters:

$$V_{app} = 4 \text{ mV}$$

$$\rho = 1.724 \times 10^{-6} \text{ } \Omega\text{-cm}$$

$$k = 2.5 \frac{W}{m-k}$$

$$h = 15.4667 \frac{W}{m^2-k}$$

$$L_z = 20 \text{ m}$$

$$T_b = 27^\circ\text{C}.$$

The electrical problem is evaluated first by solving equation (24) for the exact solution, and (33) for the finite element approximation. Results are shown in Figure 7 for each node location.

EXACT (mV)	0.0	1.0	2.0	3.0	4.0
FEA (mV)	0.0	1.0	2.0	3.0	4.0
	0-----X-----0-----X-----0				

Figure 7. Voltage distribution.

The heat generation is computed from

$$Q = \left(\frac{V_{app}}{L_z} \right)^2 \frac{1}{\rho_{av}} = \left(\frac{4}{20} \right)^2 \frac{10^2}{1.724} = 2.32 \frac{W}{m^3}$$

Solution for the temperature distribution is obtained at the same node locations from solution to equation (30) for the exact solution and equation (35) for the finite element approximation. These results are shown in Figure 8.

EXACT (°C)	30.0	111.2	169.2	204.0	215.6
FEA (°C)	30.0	111.2	169.2	204.0	215.6
	0-----X-----0-----X-----0				

Figure 8. Temperature distribution.

The solution procedure proves very effective for the simple one-dimensional, linear constant property problem. The extension of this technique to the actual joint geometry will employ the same solution procedure; however, use of numerical quadrature will be required for evaluation of the elemental integrals. That is, the mathematics of the problem will remain the same. The finite element application, on the other hand, must be completely generalized to include 3-dimensional isoparametric elements, spatially varying properties, and nonuniform boundary conditions.

6. FINITE ELEMENT MODELING OF TEST JOINTS

The extension of the finite element solution procedure to the actual 3-dimensional joint geometry is accomplished through implementation of the large scale, multi-purpose computer code NASTRAN. The programming complexities associated with the quadrature of irregular shaped (isoparametric) elements, accounting for the spatial variations of the elemental properties at conductor surface contacts, and finally, the numerical solution to the global system of element equations are all eliminated by the application of NASTRAN. In addition, NASTRAN offers other conveniences like mesh generation capabilities, graphic representations of the finite element model, double precision solutions, post-processing of solution results, and most of all, calculation of the elemental gradients of the dependent field variable. The elemental voltage gradient calculations are especially helpful in computing the local heat generation associated with each conduction element. NASTRAN is supported and maintained on the CRAY (class-6) computer at The National Magnetic and Fusion Energy Computing Center, Lawrence Livermore California. The computer connections made from the ORNL DEC-10 are via satellite.

NASTRAN solves steady state field problems through the same variational technique presented in section 5, but with a much more general finite element treatment [15]. For a 3-dimensional solution domain, 20-node "brick" elements are required to obtain a quadratic interpolation of the approximation, and many of the elements will

require an isoparametric mapping of their boundaries. Through implementation of NASTRAN, the problem is reduced to the discretization of the joint geometry and preparation of the appropriate boundary conditions depending on the field problem being solved. As previously stated in section 4, the same finite element model can be used for the electrical, thermal, and (with minor modifications) stress analyses. In fact, the electrical problem is solved with the NASTRAN heat conduction solution by the substitution of electrical material properties in place of the thermal properties and treating the temperature solution as a voltage solution.

In order to become familiar with the code and to validate the general solution procedure, the same one-dimensional problem considered in section 5 was solved with NASTRAN. The solution required two executions of NASTRAN with an intermediate step where a special algorithm was included to calculate the local volumetric heat generation of each element. The results are identical to the exact solution and are shown in Table I.

Finite Element Models

Two different conductor joints were chosen to illustrate the application of the described analysis. The first joint involved a very simple conductor that contained the joint cooling tube geometry but no actual joint. This reference "joint" would represent the limiting case of the contact surface area being an absolute maximum and with the joint being under an infinite contact pressure. It

TABLE I
NASTRAN RESULTS OF TEST PROBLEM

Distance (m)	Analytical	NASTRAN
	Voltage Distribution (mV)	
0.0	6.0	6.0
5.0	7.0	7.0
10.0	8.0	8.0
15.0	9.0	9.0
20.0	10.0	10.0
Temperature Distribution (°C)		
0.0	30.0	30.0
5.0	111.2	111.2
10.0	169.2	169.2
15.0	204.0	204.0
20.0	215.6	215.6

would be safe to expect the conductor temperature rise to be small enough so that the average properties could be assumed to remain constant (assumption 3).

The finite element model of the reference joint fits the conductor boundaries with 165 20-node brick elements and has a total of 1023 nodes. Because of midplane symmetry, modeling only half of the test joint was required for the solution. The reference joint finite element model is shown in Figure 9.

The second test joint that was chosen for the analysis (a design candidate) is a double side bar joint that consists of four contact surfaces for the half width conductor. Again symmetry was applied to reduce the finite element model to represent half of the joint where only two contact surfaces are present. The finite element model of the double side bar joint, appearing in Figure 10, was constructed with 257 20-node brick elements and has a total of 1737 nodes.

Input Parameters

The input parameters for the finite element electrical and thermal conduction models include data from the joint tests, tabulated material properties, and the use of appropriate empirical correlations. Below is a list of the various input parameters for each step of the analysis along with a description of how they were obtained.

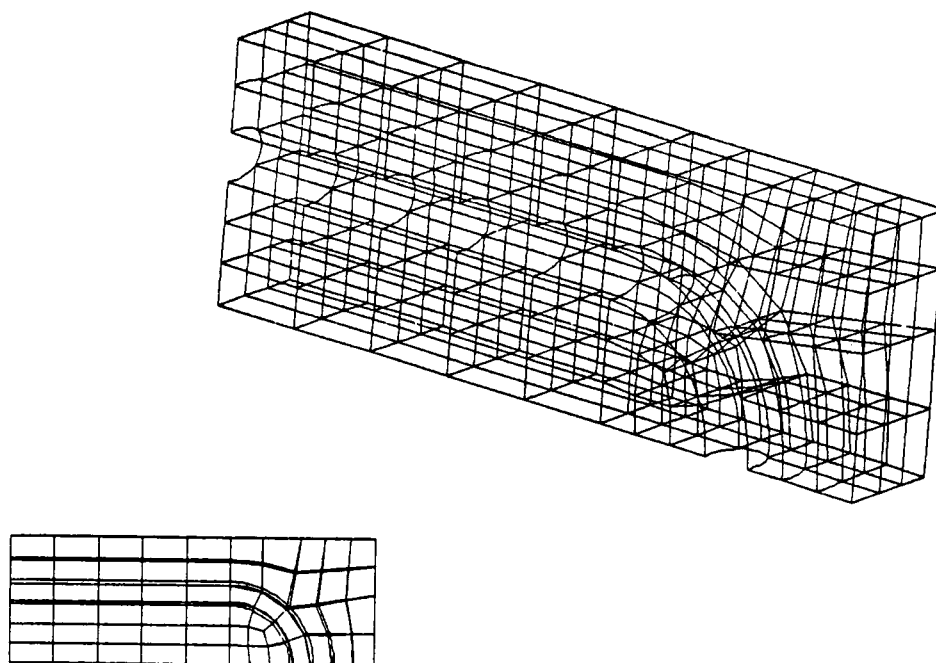


Figure 9. Finite element model of reference joint.

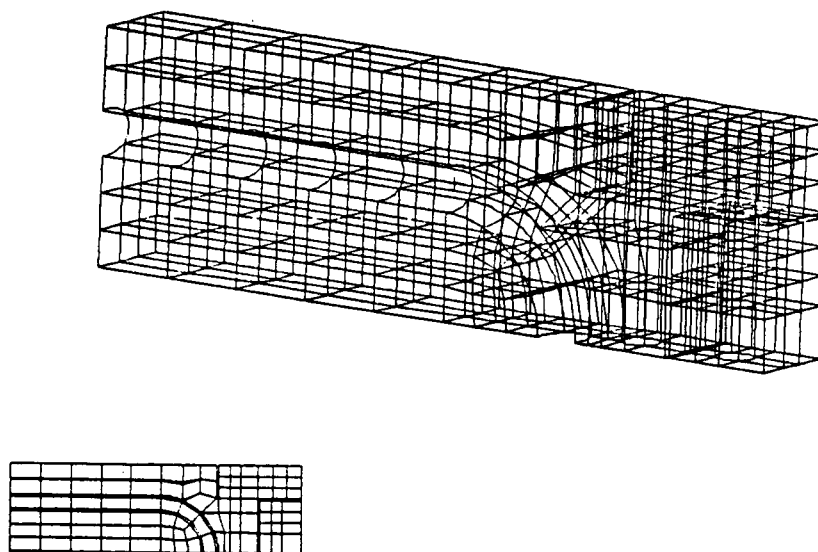


Figure 10. Finite element model of double side bar joint.

I. Potential Distribution

V_{app} --applied voltage in direction of current flow (mV).

ρ --electrical resistivity ($\mu\Omega$ -cm).

2. Calculation of Volumetric Heat Generation (Intermediate Step)

$\nabla V(x,y,z)$ --voltage gradients for each element as computed from the potential solution, ($\frac{mV}{m}$).

ρ --electrical resistivity ($\mu\Omega$ -cm).

3. Temperature Distribution

Q_e --volumetric heat generation for each element based on intermediate step computations, ($\frac{W}{m^2}$).

k --thermal conductivity, ($\frac{W}{m-K}$).

h --convective film coefficient, ($\frac{W}{m^2-K}$).

T_b --water bulk temperature ($^{\circ}C$).

$h(T_b - T_w)$ --convection load on cooling tube inner wall, ($\frac{W}{m^2}$).

Applied Voltage--measured test value based on the desired test value of the steady state current.

Electrical Resistivity--average values are computed from an empirical correlation given by Montgomery [24], valid for all temperatures above that of liquid nitrogen,

$$\rho_{av} = \rho_{20} [1 + .0041 (T_{av} - 20^{\circ}C)] \quad (42)$$

where $\rho_{20} = 1.72 \times 10^{-6}$ ohm-cm and T_{av} is the average temperature rise between the conductor and the coolant. The predicted value of T_{av} will be based on test performance. Note that a value for ρ is not

required in the solution to the potential distribution for the reference joint because the governing differential equation is the Laplace equation (i.e., no contacts to give rise to a spatial variation of ρ). Its value is input at the intermediate step only.

For the case of the double side bar joint, the governing differential equation is equation (8) and ρ must be included in steps 1 and 2. The variation of ρ due to the finite resistance of the surface contact is computed from empirical information. The details are given in the appropriate portion of section 7.

Thermal Conductivity--taken from tables of [25] at the temperature T_{cu} , the predicted average temperature rise between the cold and hot spots in the copper conductor over the joint region. The predicted values of T_{cu} are based on test performance.

Convective Film Coefficient--computed from an empirical correlation given by Montgomery [24] that depends on the bulk water temperature (T_b), the flow velocity (v), and the cooling tube diameter (d) all based on cgs units,

$$h = 9 \times 10^{-3} (1 + 1.5 \times 10^{-2} T_b) \frac{v^{.8}}{d^{.2}} . \quad (43)$$

Montgomery's correlation is nothing more than a form of W. H. McAdam's correlation that includes the Prandtl number as a function of the fluid bulk temperature. Also, the appropriate conversion factors are included to obtain the convective coefficient in units of $W/(cm^2-K)$. The correlation should not be used for $Pr > 1.0$ and $Re < 2,100$. It is predicted from measured T_b values that Pr will

range from 2.0 to 7.0 and Re will range from 65,000 to 300,000.

Water Bulk Temperature--based on values obtained in joint test (inlet and outlet temperature varied only 1 to 2°C).

7. RESULTS AND DISCUSSION

Comparisons of the analytical results with test results of the single half-width conductor joint are presented in this section. Analysis and test results for both the reference joint and the double side bar joint are included.

General Test Information

The general details and specifications of the test assembly are presented in this section. A direct current (DC) power supply was used to provide the steady state current. The cooling water passages were in series with the inlet water entering one side of a joint, passing through the joint region (see Figure 11) and exiting on the other side of the joint. Instrumentation was located on the test assembly to monitor:

1. temperature rise in bulk copper,
2. temperature rise close to the joint,
3. bulk temperature at the inlet and outlet of the cooling tube,
4. water flow rate,
5. voltage drop along the conductor,
6. applied steady state current.

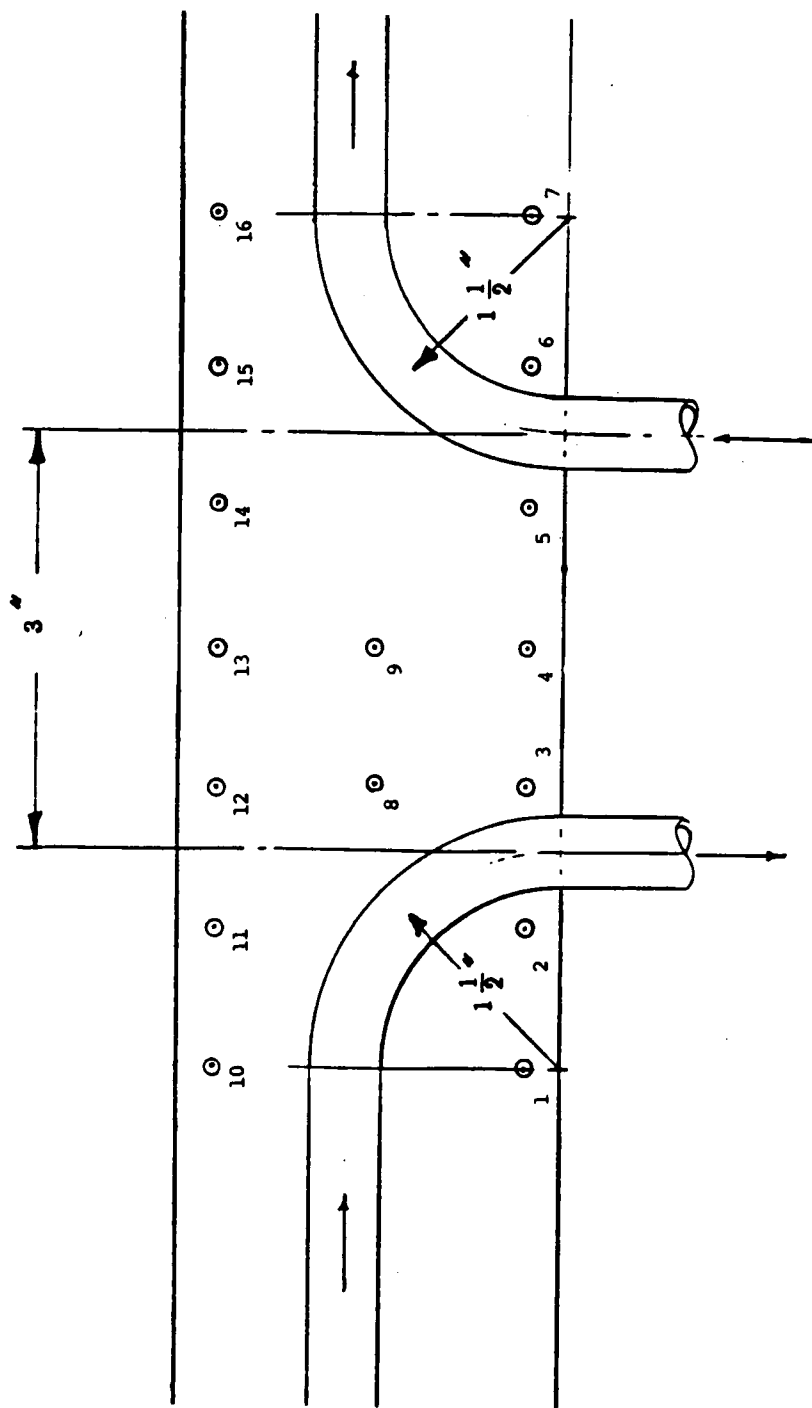


Figure 11. Location of voltage taps and thermocouples for the reference joint.

Reference Joint Results

The importance of testing a "no joint" reference sample was noted because of the limited knowledge and information available on the current distribution and resistive effects at contact surfaces. The reference joint was used as a standard for calibrating the measurements of the actual joint concepts and to verify conductor property behavior.

By measuring the current and voltage drop over a conductor length that includes a joint, the increase in resistance due to the joint can be determined from reference joint measurements at the same current rating. This will provide a means of normalizing joint performance in order to compare joints of different lengths. A schematic of the reference joint test arrangement (locations of voltage taps and thermocouples) is illustrated in Figure 11. Test information was collected at a steady state current of 20 Kamp with a water flow rate of 6.00 gal/min. From the collected data, values for the input parameters were obtained as described in section 6. Table II contains a list of the input parameter values used in the 20 Kamp analysis.

The electrical analysis was initially performed and the results agree favorably with the potential test values collected at 20,000 amperes. The results are presented on a 2-dimensional contour plot of the conductor surface with the test values spotted at the locations of the voltage taps. Results of the electrical analysis are presented in Figure 12. Note that the voltage distribution is nearly constant

TABLE II
REF INPUT PARAMETERS FOR NUMERICAL APPROXIMATION AT 20 KAMP

Parameter	Value
Thermal Conductivity	3.860 W/m-K
Electrical Resistivity	1.724 $\mu\text{ohm-cm}$
Film Coefficient	15,780 W/m ² -k
Flow Rate	6.0 gpm
Bulk Temperature	70.0°F
Total Heat Input	9.50x10 ⁶ W/m ³
Voltage Potential	35.5 mV

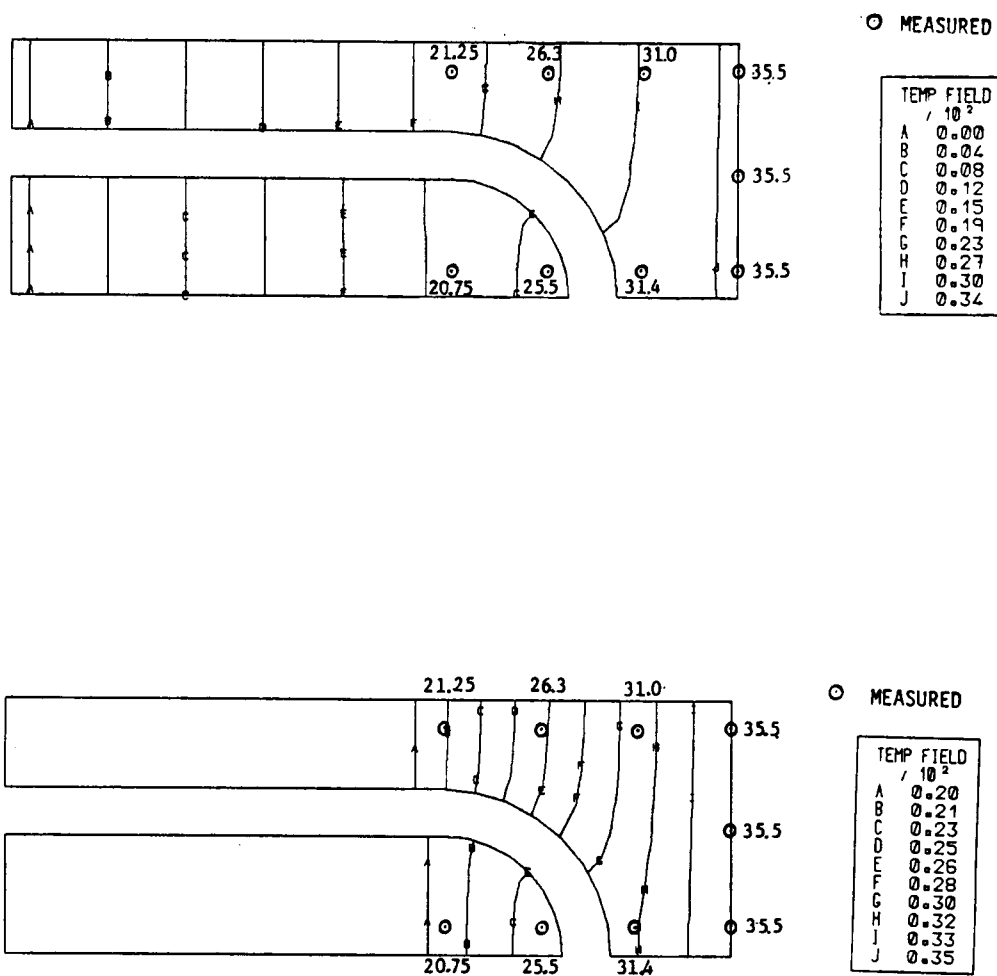


Figure 12. Voltage solution of reference joint at 20,000 amperes (mv).

at conductor cross sections just before the cooling tube makes its turn. This provides a check on the actual current value imposed on the analysis. By choosing a conductor length over which the computed voltage is nearly constant throughout the cross section, the steady state current can be computed from

$$i = \frac{V_z A_z}{\rho L_z},$$

where A_z , L_z , and V_z are obtained as shown in the schematic of Figure 13.

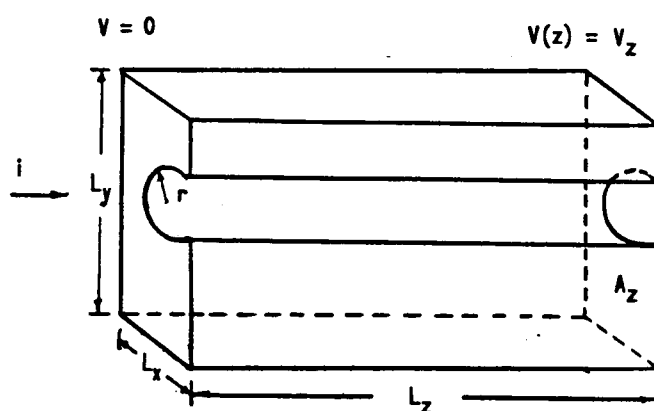


Figure 13. Constant voltage conductor length.

For the analysis at 20 Kamp, the check on the steady state current produced a value of 19.82 Kamp with the value of resistivity given in Table II. The volumetric heat generation for each element was computed from the resulting voltage gradients and the thermal analysis was performed on the same finite element geometry. The results of the thermal analysis are illustrated in Figure 14 along with

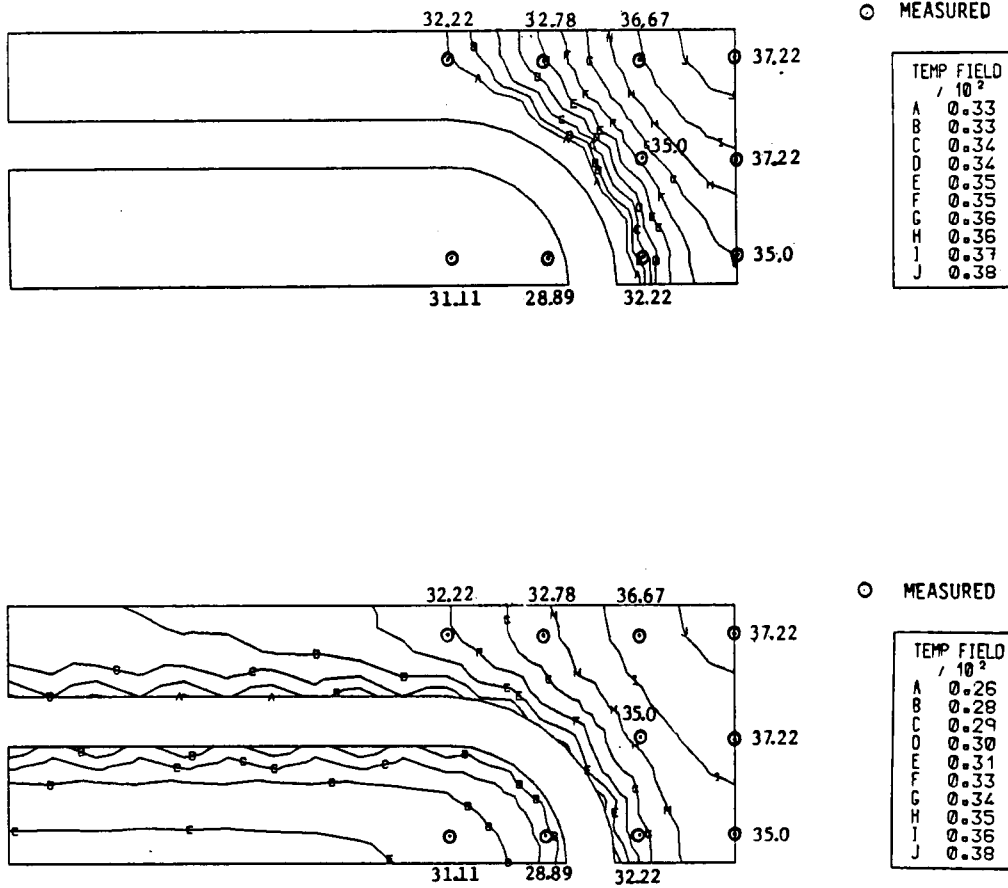


Figure 14. Temperature solution of reference joint at 20,000 amperes ($^{\circ}\text{C}$).

the temperature values collected from the 20 Kamp test. These results also agree with the collected test values. The fluctuation of the isotherms near the cooling tube is not the behavior of the NASTRAN solution; it is a characteristic of the plotting routine.

The results of the electrical and thermal analysis for the relatively simple geometry of the reference joint compare well with the test results. The assumption of average conductor properties (assumption 3) appears to be a good approximation for the linear behavior of the voltage and temperature solutions. Because results of the reference joint were so accurate, the finite element solution procedure only required a slight modification to include the surface contact effects in order to apply the solution technique to a candidate joint design. The analysis of the double side bar would include two contact surfaces, which requires electrical resistivity to be input as a spatially varied parameter. This is accomplished through an evaluation of the joint test data and an initial electrical analysis that ignores the contact surfaces. The details of this approach are given in the following section.

Resistive Effects of Contact Surfaces

Very little is known about the quantitative effects of electrical surface contacts. Many papers [6], [7], [19], [21] include experimental results in design studies of resistive joints; however no correlations were developed or are known to exist. The quality of a contact depends largely on the surface finish and the amount of clamping pressure being applied. These two effects provide qualitative

information about the specific test assembly only, and no general results are available.

Joint resistance is the sum of three components: (1) increasing temperature (R_t), (2) geometry that may distort current paths (R_g), and (3) contact resistance which varies with the quality of the contact (R_c). To further complicate matters, all components can affect each other. Nevertheless, having measured the overall voltage drop across the joint, the test resistance (R_{test}) can be determined at a particular current value. Also, joint resistance due to increasing temperatures can be effectively determined from the measured temperature values at a particular current value. For joints without excessive temperatures, test data indicates that the characteristic increase in resistance for any joint configuration remains relatively constant regardless of the current value. This behavior is shown schematically in Figure 15.

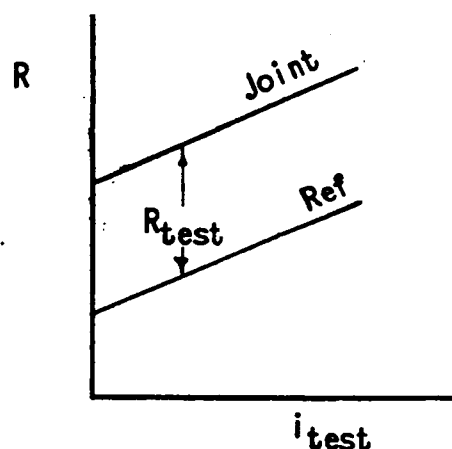


Figure 15. Behavior of resistance with increasing current.

Therefore, increases in resistance due to temperature increases are assumed independent of joint geometry and are determined by temperature measurements only.

In regard to the analysis, two unknowns exist--the resistance of the joint due to its geometrical configuration (path resistance) and the resistance due to the presence of contact surfaces. One of the unknowns is eliminated through evaluation of test information and the other is eliminated through a linear analysis neglecting surface contact effects. The linear analysis would address the geometrical resistance only and is merely a solution to the Laplace equation, equation (9). From this information, the effects of increasing temperature effects on the total resistance can be included and the remaining resistance (the difference between the test resistance and the resistance due to geometry and temperature effects) is due to the contact surfaces. The general solution procedure is outlined as follows:

1. Determine R_t , the total resistance of the joint from information based on test results. The joint voltage drop (V_{11-12}) is measured between voltage taps 11 and 12 (see Figure 16) and R_{test} can be computed for any monitored steady state current value as

$$R_{test} = \frac{V_{11-12}}{i_{test}}$$

2. Develop the solution to equation (9) with the boundary conditions given in equation (10) for the finite element model of the

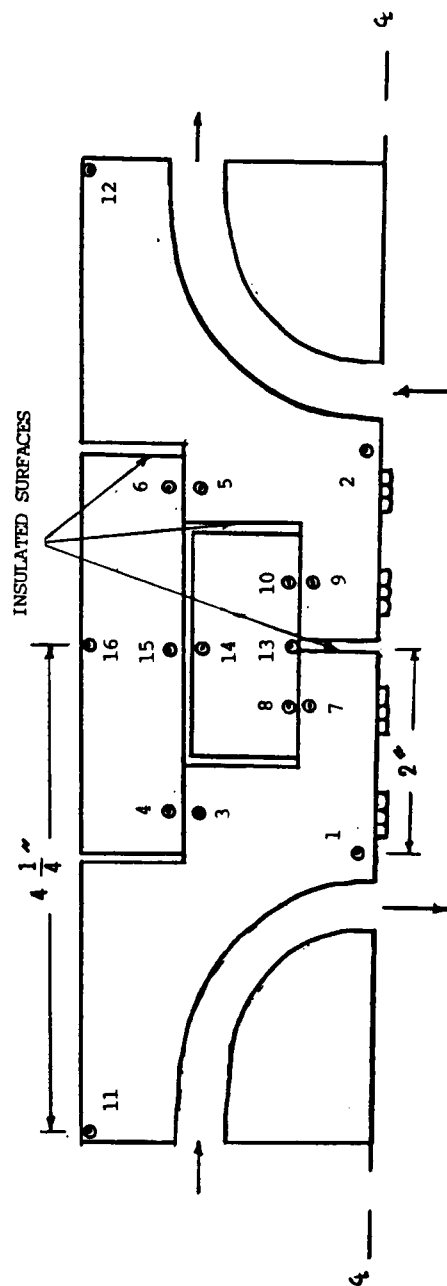


Figure 16. Location of voltage taps and the thermocouples for the double side bar joint.

joint. Compute the resistance due to the geometrical configuration and increased temperature effects (for some i) from

$$R_g + R_t = \frac{V_{app} - V_j}{i} \quad (44)$$

where

$$i = \frac{V_z A_z}{\rho_{av} L_z} \quad (45)$$

$$\rho_{av} = \rho_{20} [1 + .0041 (T_{av} - 20^\circ \text{C})] \quad (46)$$

The voltage drop ($V_{app} - V_j$) occurs between locations on the finite element model that correspond to the same locations of voltage taps 11 and 12 on the test joint. The linear electrical analysis parameters V_z , A_z , and L_z are obtained as described in Figure 13, page 59, and T_{av} is computed as the average temperature between the joint conductor hot spot and the water temperature as described in section 6. T_{av} is a measured parameter whose value depends on the particular value of test current. The resulting expression can be written as,

$$R_g + R_t = \left(\frac{V_{app} - V_j}{V_z} \right) \frac{L_z}{A_z} \rho_{20} [1 + .0041 (T_{av} - 20^\circ \text{C})]$$

3. The results of step 2 provide information on the resistance of the joint due to its geometrical configuration and increasing temperature effects. The test results of step 1 provide information on the resistance due to the geometry, temperature increase, and the contact surfaces. Thus, the added resistance of the contacts is computed from

$$R_c = R_{\text{test}} - R_g - R_t$$

4. Determine the electrical resistivity of the contacts as a function of the contact geometry from

$$\rho_c = \frac{R_c A_c}{L_c} \quad (47)$$

where A_c is the total cross sectional area of all contact surfaces, and L_c is the "thickness" of the space between the surfaces in contact ($\ll 1$). This "thickness," or contact spacing in the actual joint is meaningless; however, the finite element model accounts for the contact resistance by incorporating a planar region of highly resistive membrane elements with a small but finite conduction thickness L_c .

5. Determine the solution to equation (8) with the boundary conditions given in (10) where ρ_{av} will be placed in the solid conductor elements and ρ_c will be placed in very thin ($\approx 10^{-4}$ in.) membrane elements at the contacts.

6. From the solution obtained in step 5, compute the volumetric heat generation as before in preparation for the thermal analysis from

$$Q = \frac{1}{\rho} (\nabla V)^2$$

7. Determine the temperature distribution from equation (1) with the boundary conditions given in equation (2) and including for the thermal resistive effects of the contacts. The thermal conductivity of the contact will be a function of the electrical resistance and geometry of the contact

$$k_c = \frac{\rho_c}{T_c} \left(\frac{L_c i}{A_c} \right)^2, \quad (48)$$

where T_c is a function of the average measured temperature difference across the contact surface. The conductor thermal conductivity k_{av} is obtained as described in section 6.

Despite the simplicity of the solution procedure, there is actually little error involved. The flexibility in computing ρ_{av} from measured temperatures and choosing a finite conduction thickness of the contact spacing L_c , are compensated for in the computation of ρ_c . That is, the resistive effects of the contact surfaces are exact in the overall model (based on test information), only the intensity of the distribution of the contact resistive effects may introduce error. This follows from the general nature of elliptic differential equations.

From the appropriate substitutions, an expanded expression for ρ_c can be obtained

$$\rho_c = \left[R_{test} - \left(\frac{V_{app} - V_j}{V_z} \right) \frac{\rho_{av} L_z}{A_z} \right] \frac{A_c}{L_c} \quad (49)$$

where V_{app} , L_z , V_z , A_z and V_j are all a result of the linear, uniform property analysis (solution of equation (9)), R_{test} and ρ_{av} are given from the test results, and A_c and L_c characterize the geometry of the contact surface.

Double Side Bar Joint Results

The double side bar joint (DSB) was fabricated with the same cross sectional area as the reference joint. Again, only a half-width conductor was required for testing due to symmetry and four insulated bolts were inserted with a torque of 80 ft-lb to mate the conducting surfaces. Also, each contact was coated with a liquid silver plating to enhance the quality of the contact. A schematic of the test assembly is given in Figure 16.

Through a series of tests with increasing steady state current values, a joint resistance curve was developed from collected voltage drop measurements between voltage taps 11 and 12. The resistance R_{test} was computed as,

$$R_{\text{test}} = \frac{V_{11-12}}{i_{\text{test}}} .$$

Temperatures were also measured at each steady state current value and the average temperature at a particular current value was computed from,

$$T_{\text{av}} = \frac{T_{\text{max}} - T_{\text{b}}}{2}$$

The test was controlled by the applied current load allowing i_{test} to act as the independent variable. Characteristic curves of V_{11-12} , R_{test} , and T_{av} as functions of the increasing current value i_{test} , were developed for the DSB joint. The curves of voltage drop, resistance and temperature appear in Figures 17, 18, and 19. The collected data are given in Table III.

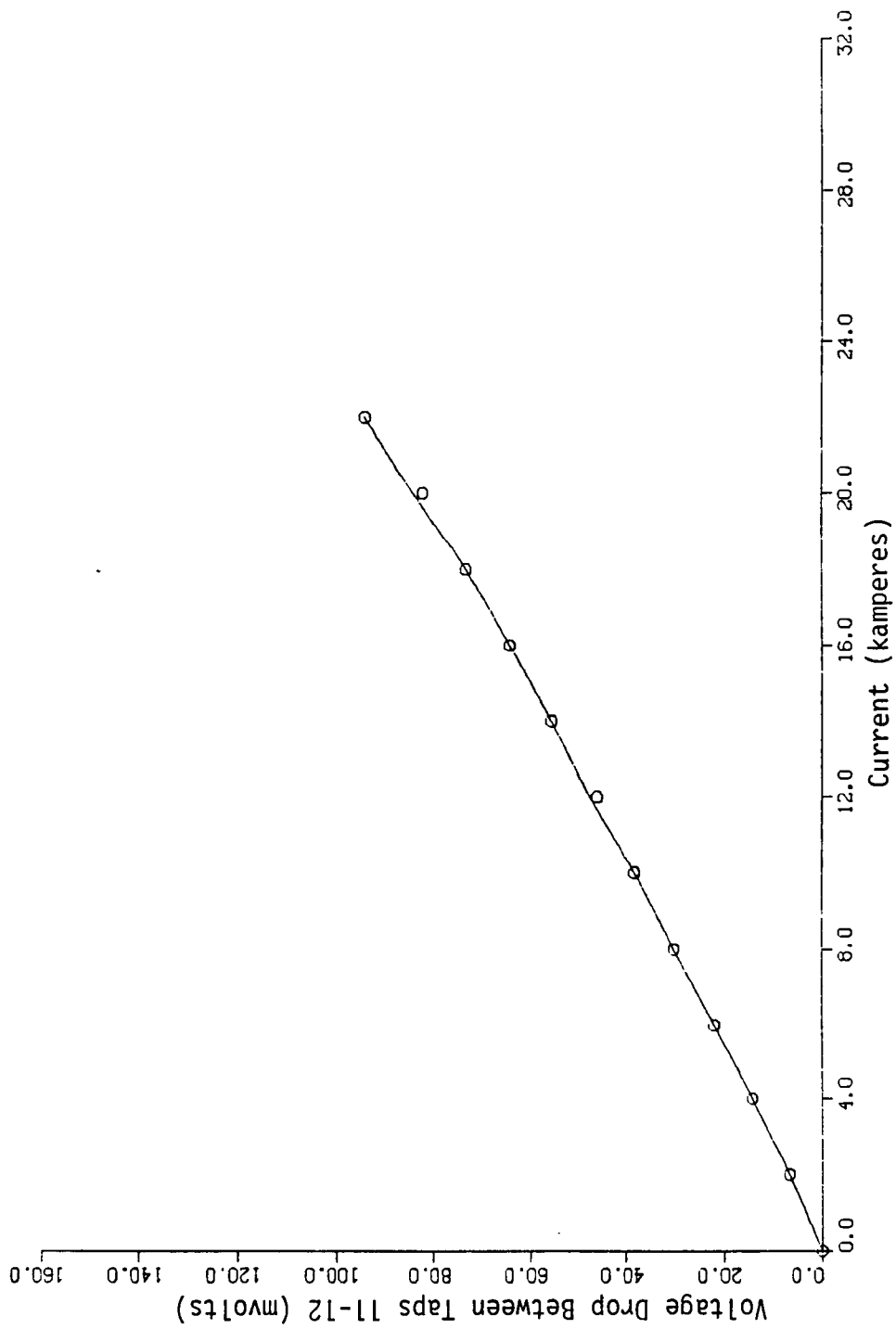


Figure 17. Voltage drop as a function of current.

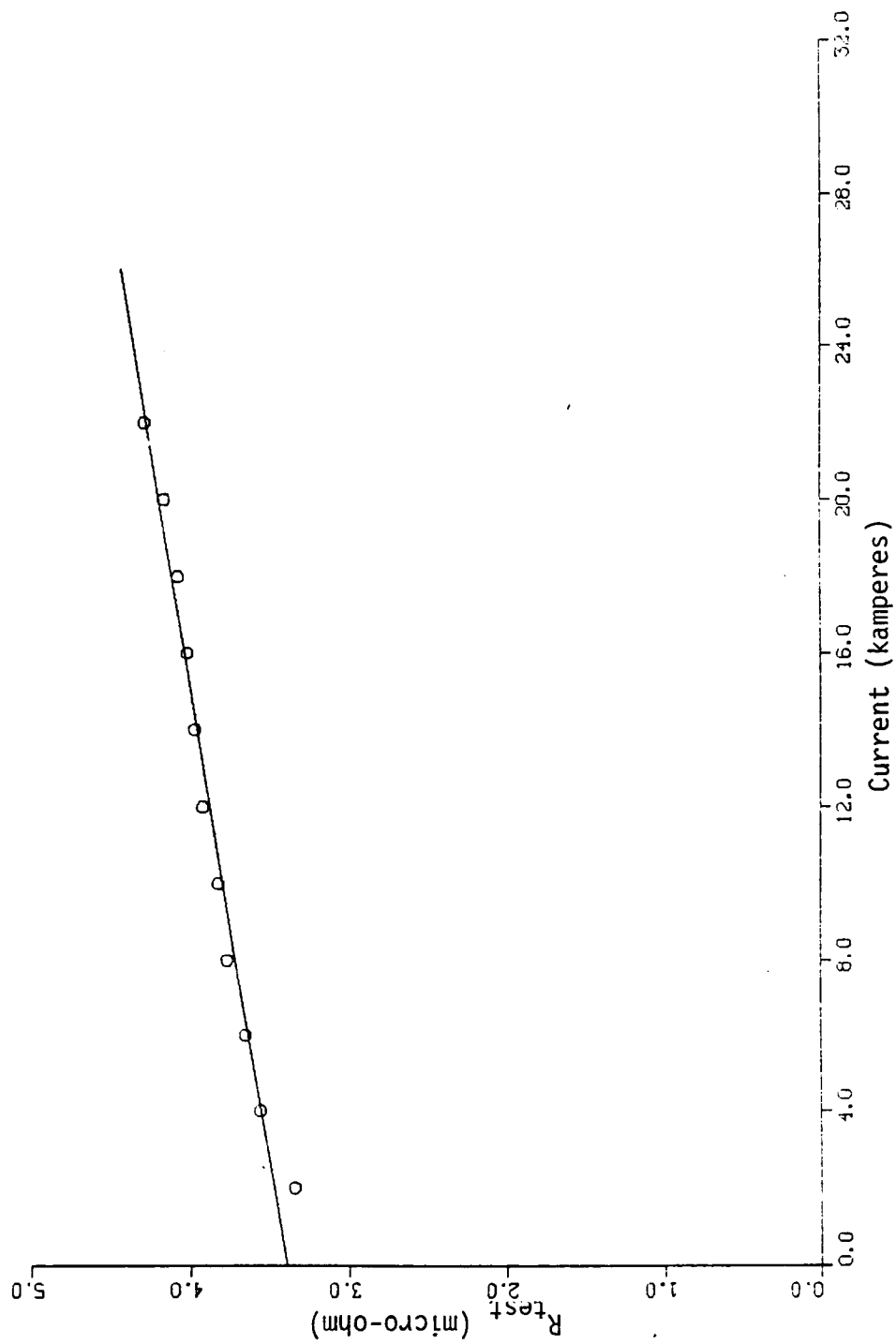


Figure 18. Resistance as a function of current.

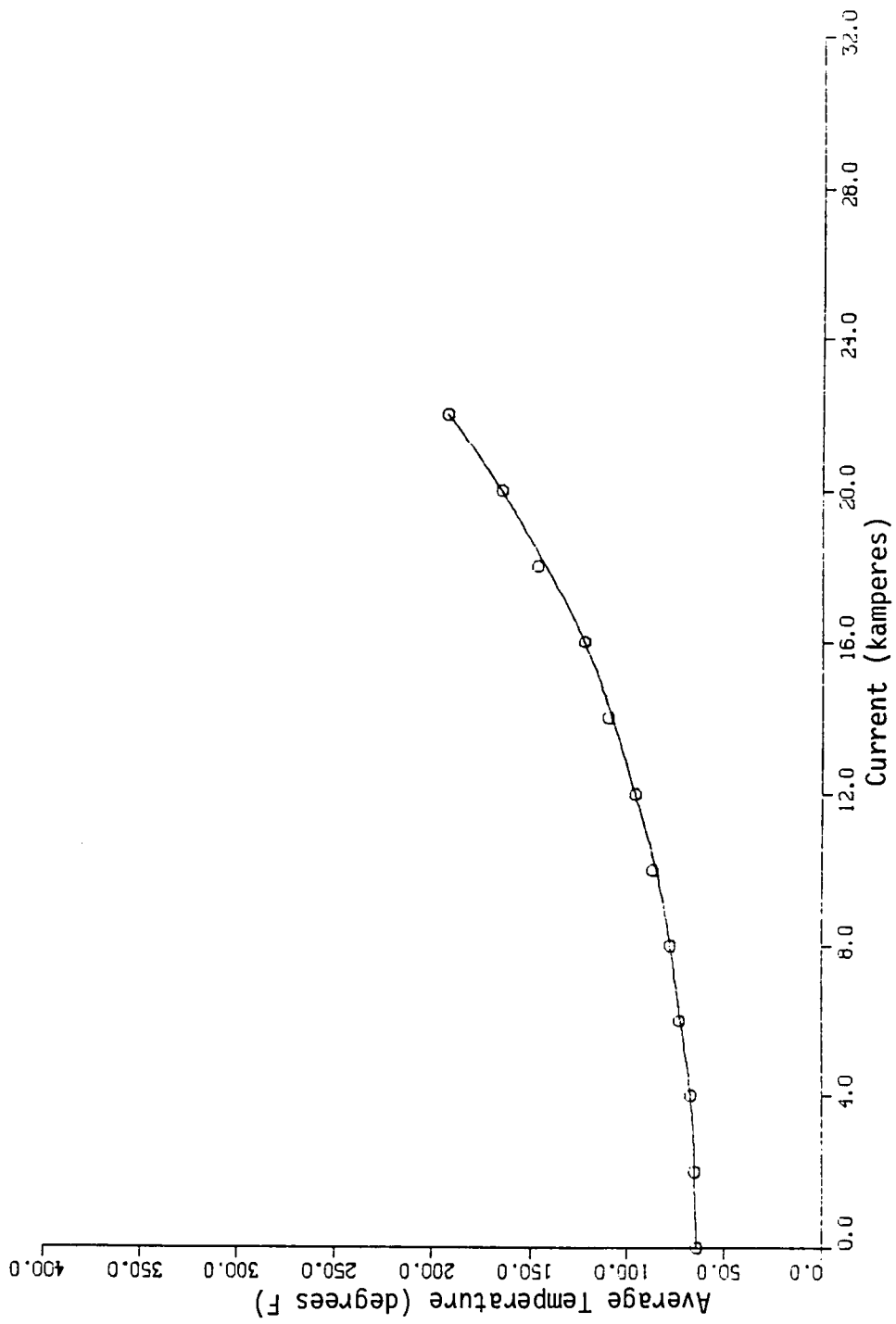


Figure 19. Average temperature as a function of current.

TABLE III
DSB JOINT TEST RESULTS

Current, KA	Voltage Drop, mV	Resistance, $\mu\Omega$	Average Temp, °F
0.00	0.00	--	64.00
2.00	6.70	3.35000	65.00
4.00	14.25	3.56250	67.00
6.00	21.90	3.65000	72.50
8.00	30.10	3.76250	77.50
10.00	38.20	3.82000	87.00
12.00	46.00	3.83333	96.00
14.00	55.50	3.96428	110.00
16.00	64.20	4.01250	122.50
18.00	73.25	4.06944	146.50
20.00	82.20	4.11000	165.00
22.00	94.00	4.27273	192.50

The resistance of the joint due to its geometrical configuration, R_g , can be computed for any V_{app} in the linear electrical analysis of the solid conductor (analysis ignoring surface contacts) because the resistance will always be constant. The resistance R_g is computed from,

$$R_g = \frac{V_{app} - V_j}{V_z} \frac{L_z}{A_z} \rho_{20} = \text{constant}$$

Since the average temperature increase as a function of current has been provided from the test, the geometrical resistance and the resistance due to increasing temperature effects can be lumped together to be expressed as a function of T_{av} and a constant K obtained from the linear electrical analysis

$$R_g + R_t = K \rho_{av} ,$$

where

$$K = \frac{V_{app} - V_j}{V_z} \frac{L_z}{A_z} = \text{constant}$$

and

$$\rho_{av} = \rho_{20} [1 + .0041 (T_{av} - 20^\circ\text{C})] .$$

Below is a list of the input parameters along with the resulting output parameters used to compute K in the linear analysis of the DSB joint:

$$V_{app} = 22 \text{ mV}$$

$$V_j = 9.55 \text{ mV} , V_z = 1.866 \text{ mV}$$

$$L_z = .9125 \text{ in}$$

$$A_z = 2.89262 \text{ in}^2$$

and

$$K = \frac{V_{app} - V_j}{V_z} \frac{L_z}{A_z} = 2.1047 \text{ in}^{-1}.$$

Solution contours of the linear electrical analysis are shown in Figure 20. The lumped resistance $R_g + R_t$ was computed and Figure 21 shows curves of $R_g + R_t$ and R_g as functions of the average temperatures in Table III. Because T_{av} is given as a function of i , $R_g + R_t$ can also be expressed as a function of i . Figure 22 contains curves of R_{test} and $R_g + R_t$ as functions of current i .

The resistance due to the DSB joint surface contacts, R_c , is the difference between the two curves in Figure 22. That is

$$R_c = R_{test} - R_t - R_g$$

The contact resistance for the DSB joint is the result of two contact surfaces. The conduction length, L_c , will be the same for each contact, but the cross sectional area A_c that appears in equation (47) is total contact area. The contact resistivity is computed from equation (47) as

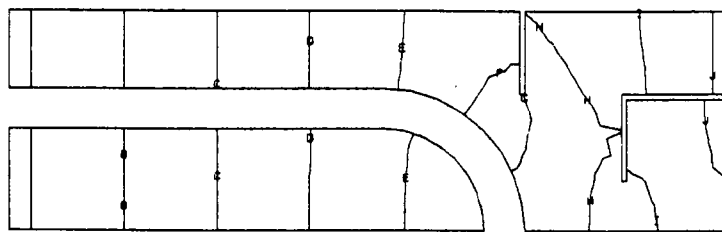
$$\rho_c = \frac{R_c A_c}{L_c},$$

where

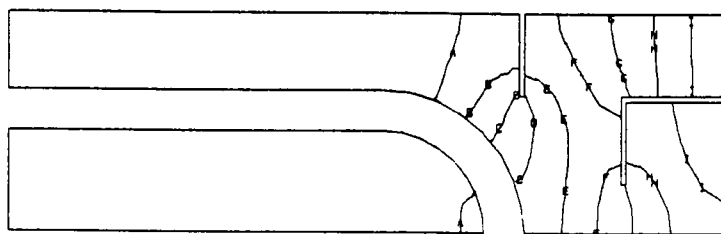
$$A_c = 2.5118 \text{ in}^2$$

$$L_c = .0001 \text{ in.}$$

This completes step 4 of the general solution procedure given in the previous section and provides the spatial variation of the electrical resistivity needed for the solution of equation (8). The



TEMP FIELD	
	/ 10 ¹
A	0.05
B	0.28
C	0.51
D	0.75
E	0.98
F	1.21
G	1.44
H	1.68
I	1.91
J	2.14



TEMP FIELD	
	/ 10 ¹
A	1.10
B	1.22
C	1.34
D	1.46
E	1.58
F	1.71
G	1.83
H	1.95
I	2.07
J	2.20

Figure 20. Linear electrical analysis of double side bar joint (mv).

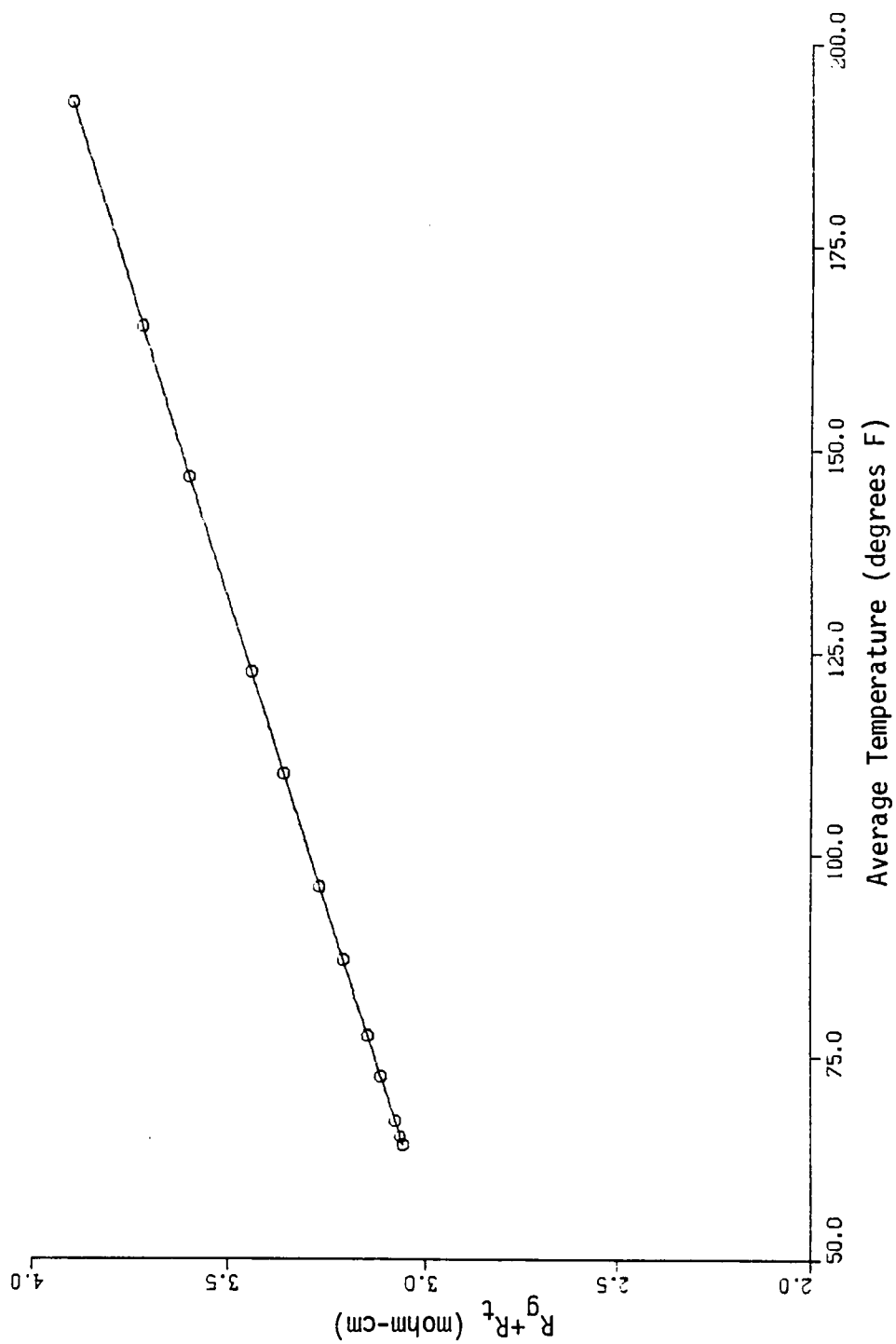


Figure 21. Resistance as a function of average temperature.

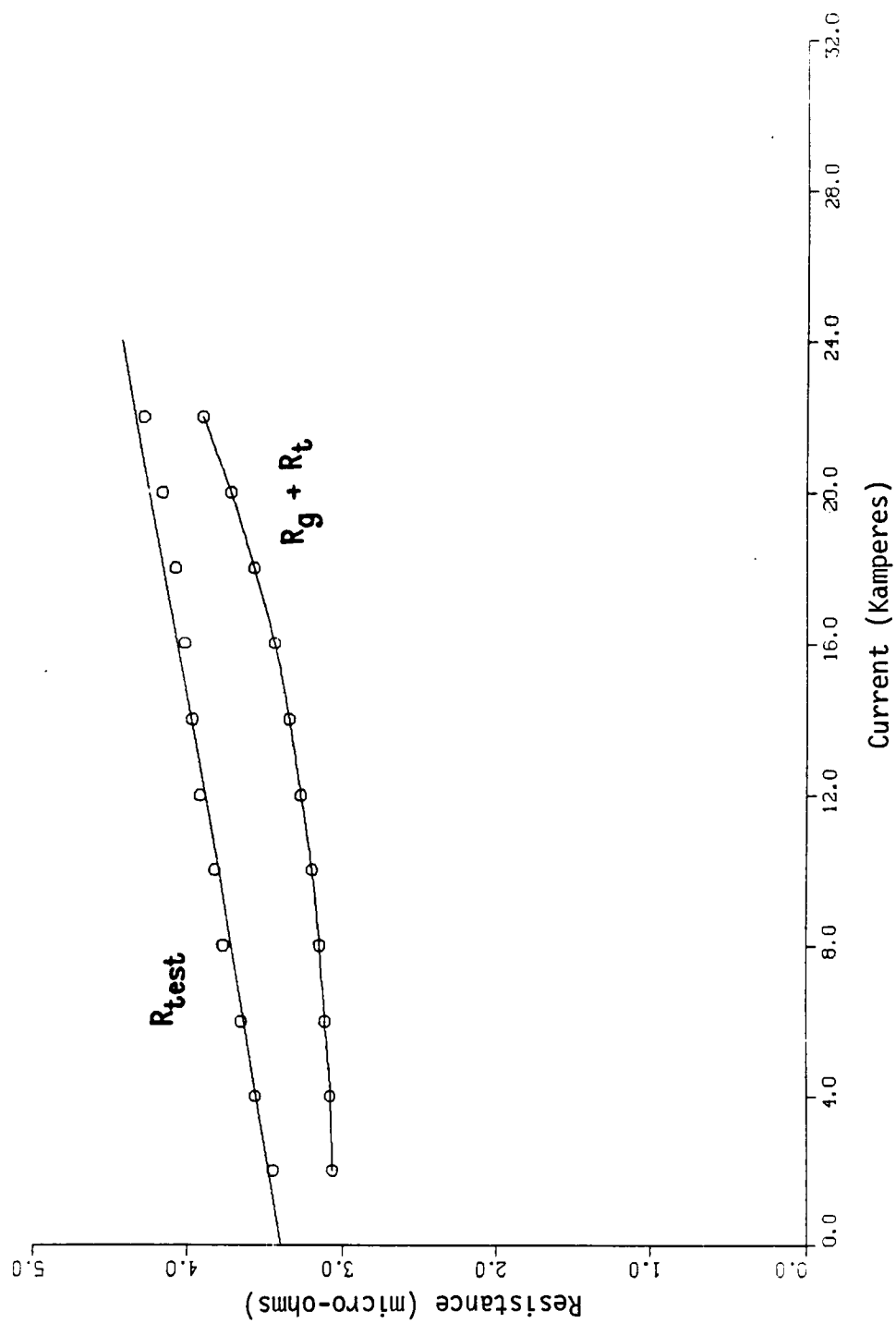


Figure 22. Resistance as a function of current.

analysis proceeds as before with the value of V_{app} being an input parameter based on the DSB test results of Figure 17, page 69, and the current value checked as presented in Figure 13, page 59. The results of the DSB joint test at 14 Kamp produced the input parameters given in Table IV.

The results of the electrical analysis at 14 Kamp are presented in Figure 23 with the test data spotted at the locations of the voltage taps. The results generally agree with only a slight discrepancy of ± 2 mV appearing at locations near the contacts. This is due in part to the flexibility in choosing the conduction thickness of the surface contact membrane elements.

The volumetric heat generation for each element was computed from

$$Q = \frac{1}{\rho} (\nabla V)^2 ,$$

where the value of ρ is ρ_{av} for the conduction elements and ρ_c for the contact elements. The thermal resistive effects at the contacts are a function of the additional heat generated due to the electrical resistance and the geometry of the contact. The thermal conductivity of the contact elements k_c is determined from equation (48). This expression was derived by recognizing that the difference between the amount of heat conducted into the contact area and the heat conducted out of the contact area is equal to the heat generated at the contact; that is,

TABLE IV

DSB INPUT PARAMETERS FOR NUMERICAL APPROXIMATION AT 14 KAMP

Parameter	Value
Thermal Conductivity	
Average	382.5 W/m-K
Contact	.00219 W/m-K
Electrical Resistivity	
Average	1.885 $\mu\text{ohm-cm}$
Contact	53,667 $\mu\text{ohm-cm}$
Film Coefficient	15,835 W/m ² -K
Flow Rate	6.0 gpm
Bulk Temperature	73.0°F
Average Temperature	110.0°F
Total Heat Input	12.80x10 ⁶ W/m ³
Voltage Potential	42.6 mV

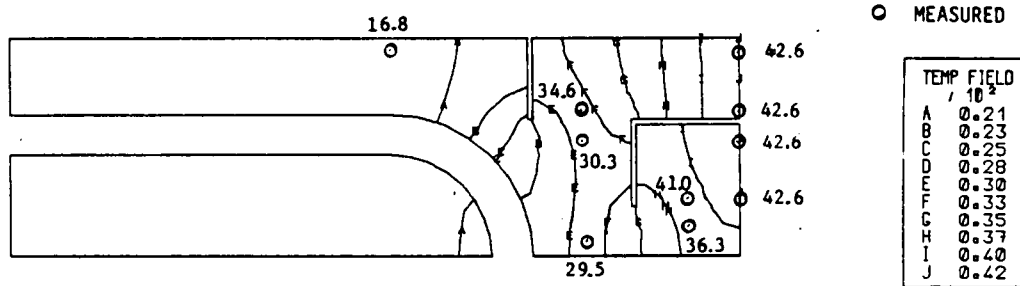
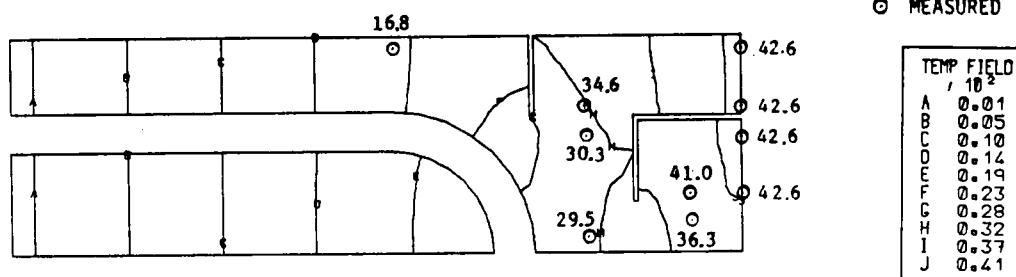
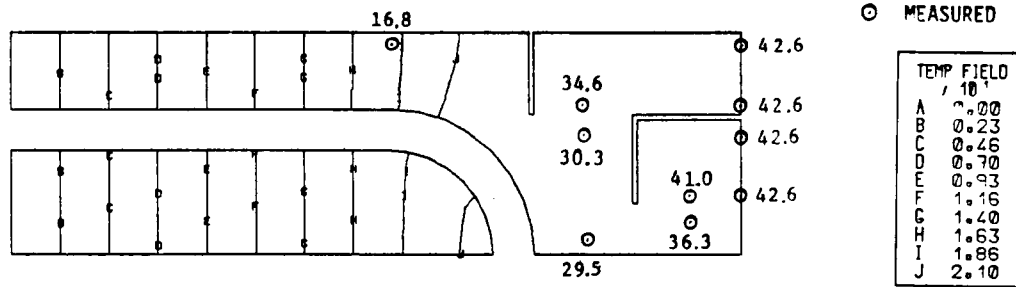


Figure 23. Voltage solution of double side bar joint at 14,000 amperes (mv).

$$i^2 R_c = \frac{d}{dy} \left(-k_c A_c \frac{dT}{dy} \Big|_c \right) dy \quad (49)$$

If R_c is written in terms of a differential conduction length (contact conduction length) equation (49) becomes

$$i^2 \frac{\rho_c}{A_c} = -k_c A_c \frac{d^2 T}{dy^2} \quad (50)$$

The thermal contact resistive effects must rely on measured values of the temperature difference across the contacts (thermocouples 4-3, etc.) at a particular current value. A second order temperature difference approximation is substituted into equation (50) whose value depends on the measured temperatures across the contact to obtain the expression for k_c , equation (48). The magnitude of the temperature gradients across the contacts were about the same magnitude as the temperature gradients in the solid conductor region ($\pm 1^\circ\text{C}/\text{in}$). Therefore, the thermal resistive effects of the contacts are small and the second order difference approximation is on the same order as the average temperature difference (T_{3-4} , etc.)

Table IV contains the input parameters of k_c and k_{av} as they were computed for the 14-Kamp thermal analysis. The results of the thermal analysis at 14 Kamps appear in Figure 24 with the test values spotted at the locations of the thermocouples.

Results at Design Conditions

In order to apply the present analysis procedure to the design operating conditions for the DSB joint, it must be assumed that the

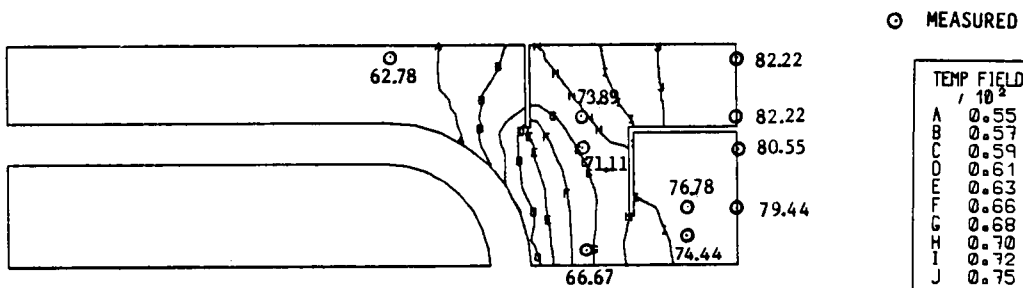
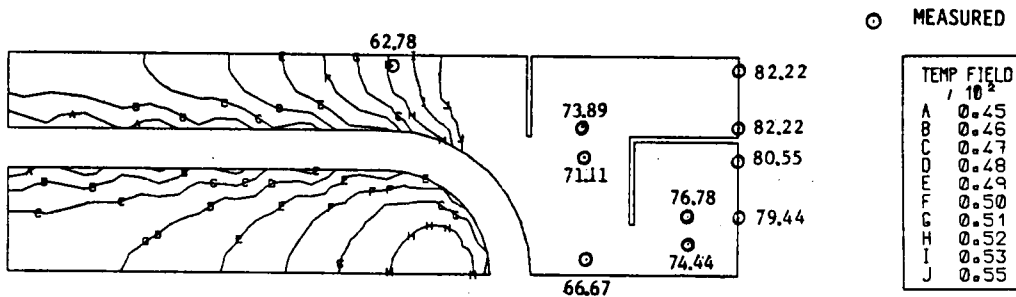
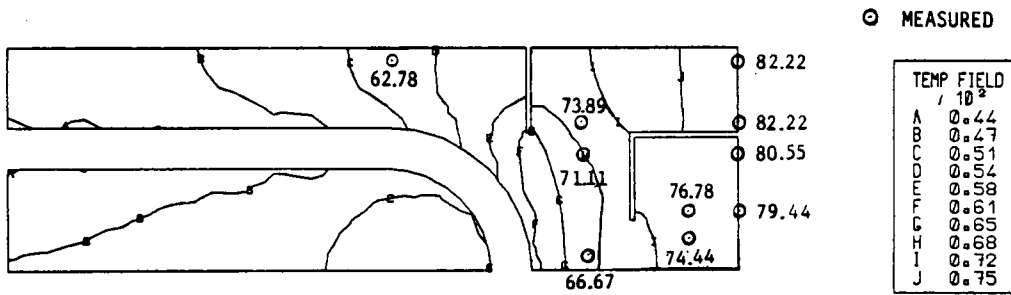


Figure 24. Temperature solution of double side bar joint at 14,000 amperes ($^{\circ}\text{C}$).

joint resistance R_{test} , average temperature T_{av} , and consequently the contact resistance R_c can be extrapolated to meaningful values at the design current load. The extrapolation for each parameter is accomplished with a cubic spline curve fit used in a subroutine of the DISSPLA Graphics Software System. An extrapolation of the curve in Figure 17, page 69, provided the necessary value of V_{app} to insure that the design current of 31.25 Kamp was imposed on the joint.

The steady state design current value (31.25 Kamp) produces the 1 Tesla peak magnetic field required for plasma containment in the ATF torus design. Based on the operational constraints of the helical field coils, the HF coil joints must perform in accordance with the following steady state design conditions at 1 Tesla operation:

Joint Resistance	4.25 $\mu\Omega$ max
Conductor Temperature	160°C max
Conductor Potential	300 mV max
Coolant Flow Rate	6 gpm

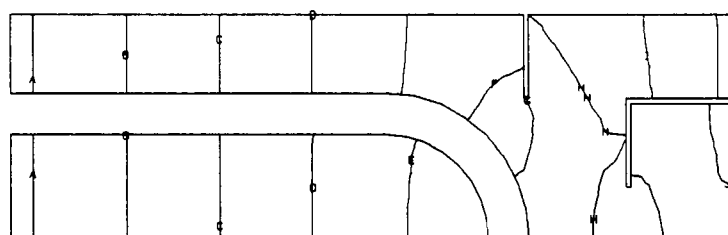
The input parameters for the DSB analysis at the design current load are given in Table V.

The electrical analysis provided the volumetric heat generation at design conditions and the thermal analysis was completed as previously described. The predicted potential solution at 31.25 Kamp appears in Figure 25 and the resulting temperature solution appears in Figure 26.

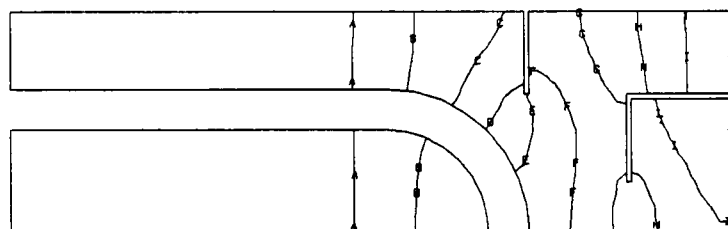
TABLE V

DSB INPUT PARAMETERS FOR NUMERICAL APPROXIMATION AT 31.25 KAMP

Parameter	Value
Thermal Conductivity	
Average	372.2 W/m-K
Contact	.00644 W/m-K
Electrical Resistivity	
Average	3.076 $\mu\text{ohm-cm}$
Contact	42,256 $\mu\text{ohm-cm}$
Film Coefficient	16,356 W/m ² -K
Flow Rate	6.0 gpm
Bulk Temperature	79.0°F
Average Temperature	396.0°F
Total Heat Input	22.30x10 ⁶ W/m ³
Voltage Potential	90.4 mV



TEMP FIELD	
/ 10 ²	
A	0.02
B	0.11
C	0.21
D	0.30
E	0.40
F	0.49
G	0.59
H	0.69
I	0.78
J	0.88



TEMP FIELD	
/ 10 ²	
A	0.35
B	0.41
C	0.47
D	0.53
E	0.59
F	0.65
G	0.71
H	0.77
I	0.83
J	0.90

Figure 25. Voltage solution of double side bar joint at design conditions (31,250 amperes). (mv).

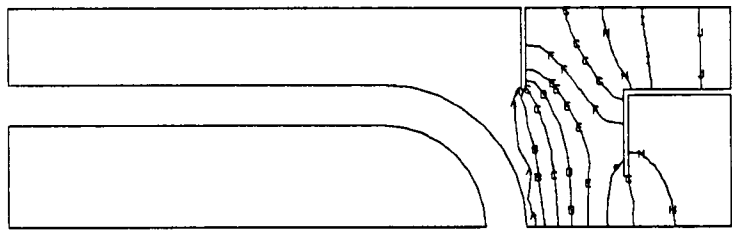
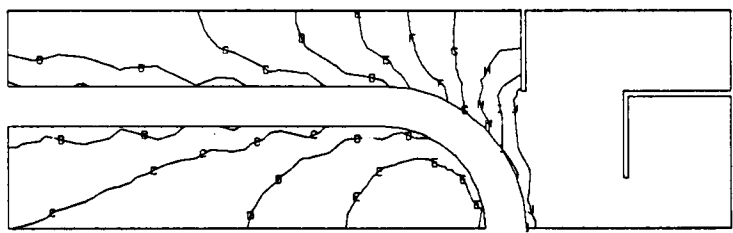


Figure 26. Temperature solution of double side bar joint at design conditions (31,250 amperes) (°C).

From the computed results for the DSB joint at design conditions, it is predicted that the joint will exceed the specified operational design limits. Both the resistance and maximum temperature for the DSB are beyond their design values. The HF coil joint design will require more conduction area than that offered by the DSB joint design and may possibly require separate cooling at the joint cross-section. Nonetheless, the proposed solution procedure has produced an effective means of comparing the performance of resistive conductor joint designs. Other candidate designs will be considered and the present analysis technique should provide an important service in the selection process for ATF and possibly for future designs of separable magnetic coils for fusion energy test reactors.

8. CONCLUSIONS

A finite element solution procedure has been established for the computation of the potential and temperature distributions in water-cooled resistive conductor joints. The assumptions made that led to the application of linear solution techniques appear to have been valid based on experimental comparisons with the analysis results.

The candidate joint design investigated, the double side bar joint, is predicted to exceed the electrical and thermal design limits at steady state operating conditions (based on analysis results). Other candidate designs are to be fabricated for testing, and the analysis procedure presented here will be able to predict the performance of these joints at operating conditions. The analysis procedure has proven very useful in the complete design process for selection of an acceptable joint design.

One restriction with the solution procedure is that the analysis must be accompanied by joint test results of the particular joint geometry being analyzed. A possible area of future research is the development of normalized results of electrical and thermal effects at contact surfaces that would apply to any geometrical configuration.

The proposed finite element approach here is well-suited for predictions of joint performance even when large temperature gradients are present causing the conductor properties to become strong functions of temperature. The analysis could be extended to include this non-linear coupling effect of the differential equations and is recommended as an area for future study.

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VITA

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