To the Graduate Council:

I am submitting herewith a dissertation written by Jacob Honeycutt entitled "Geometric rectifiability in metric measure spaces." I have examined the final paper copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Vyron Vellis, Major Professor

We have read this dissertation
and recommend its acceptance:

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Vyron Vellis

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Joan Lind

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Conrad Plaut

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Scott Zimmerman

Accepted for the Council:

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Dixie L. Thompson

Vice Provost and Dean of the Graduate School

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(Original signatures are on file with official student records.)

Geometric rectifiability in metric measure spaces

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Jacob Honeycutt

May 2025

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To my family.

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Abstract

We extend several results on the geometric rectifiability of subsets in Euclidean spaces to metric measure spaces with controlled geometry (Ahlfors regularity and supporting a Poincaré inequality). Specifically, we provide a strengthened version of the classical Denjoy-Riesz theorem, generalize MacManus' result on the quasisymmetric rectifiability of uniformly disconnected subsets, and extend the work of David and Semmes on bi-Lipschitz rectifiability to metric measure spaces.

For the Denjoy-Riesz theorem, we prove that compact, totally disconnected subsets of metric measure spaces with controlled geometry can be captured by an arc that is locally bi-Lipschitz away from the totally disconnected set. In the case of MacManus' result, we show that compact, uniformly disconnected subsets in these spaces can be captured by the image of a quasisymmetric embedding, quantitatively. For David and Semmes' work, we provide sharp conditions on a metric measure space such that a subset which is bi-Lipschitz homeomorphic to a subset of the real line is captured by a bi-Lipschitz arc, quantitatively.

Finally, we establish several results concerning the co-uniformity of subsets in these spaces under various conditions, including bounds on Assouad dimension and uniform disconnectedness.

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Chapter 1

Introduction

In the early 1900s, Denjoy [Den10] and Riesz [Rie05] independently proved what has come to be known as the Denjoy-Riesz theorem, which states that a compact, totally disconnected subset of the plane is contained in an arc. This builds upon work by Zoretti [Zor05], which instead uses a more general class of curves. Recall that a totally disconnected set is one whose only connected components are singletons and an arc is the homeomorphic image of an interval.

Moore and Kline [MK19] provided a complete characterization of subsets of the plane which are contained in arcs, and this includes the Denjoy-Riesz theorem as a special case. Later, Zippin [Zip32] provided a complete classification of metric spaces for which the Moore-Kline theorem holds. A corollary of Zippin's work is that if (X, d) is a compact, connected, locally connected, separable metric space with no local cut points, then every compact, totally disconnected set is contained in an arc; see also Whyburn's work [Why31]. Recall that a locally connected space is a topological space in which every point has a neighborhood which is connected. Separable means that there is a countable subset whose closure is the whole space, akin to the rational numbers in the real line. A local cut point is a point which has a connected neighborhood that becomes disconnected when the point is removed.

A useful application of the (generalized) Denjoy-Riesz theorem is that for every $n \geq 2$, there is an arc in $\mathbb{R}^n$ which has positive Lebesgue n-measure, and for every $n \geq 3$ there is a wild arc in $\mathbb{R}^n$, that is, an arc which is not ambiently homeomorphic to a line segment (it is knotted in a way that is impossible to untangle). It is natural to ask whether one can obtain more regularity on the arc which passes through the entire set, but notions of regularity necessarily require geometric restrictions in addition to the previous purely topological ones.

1.1 Metric spaces with controlled geometry

In this section, we provide the definitions of what we mean when we say that a metric measure space has "controlled geometry." The metric measure spaces we consider must have some sort of connection between the metric and measure, akin to the translation invariance and scaling property of the Euclidean distance and Lebesgue measure in Euclidean spaces. It is also necessary there are many curves in our spaces, and we must be able to quantify what "many" means. These conditions are satisfied when we consider metric measure spaces which are complete, are Ahlfors Q-regular, and support a p-Poincaré inequality for some p sufficiently small compared to Q.

Ahlfors regularity is a condition on a metric measure space which guarantees the metric and measure interact together in a nice way: the measure of a ball is comparable to some power of its radius. This mimics the scaling and translation properties of the Lebesgue measure in Euclidean space, while the constant of comparability allows enough wiggle room to not disallow spaces we consider important.

The property of the Poincaré inequality is more involved, but, roughly speaking, the Poincaré inequality (along with Ahlfors regularity) guarantees there are a lot of rectifiable curves in a metric measure space, in a quantitative sense, as Heinonen and Koskela [HK98] proved, building upon work by Loewner [Loe59].

A large variety of metric measure spaces satisfy the assumptions of Theorem 2.3.2, including Carnot groups [Jer86, Var86]; orientable, Ahlfors n-regular, linearly locally contractible n-manifolds with $n \geq 3$ [Sem96]; certain hyperbolic buildings [BP99]; Laakso spaces [Laa00]; and certain Menger sponges [MTW13, EBG22].

One of the most important properties of Euclidean space is the connection between the Euclidean metric and the Lebesgue measure. The Lebesgue measure is translation invariant, and scales with the metric in a predictable way. The notion of Ahlfors regularity originated from a paper by Ahlfors [Ahl35], and the name was coined by David and Semmes in the 1990s. Ahlfors regularity ensures that the metric and measure interact somewhat similar to the way that the Euclidean metric and Lebesgue measure interact. More formally, a metric measure space (X, d, μ) is *Ahlfors Q-regular* for some $Q \geq 0$ if there is a constant $C \geq 1$ so that for all points $x \in X$ and all scales $0 < r \leq \operatorname{diam} X$,

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q.$$

The Poincaré inequality originates from a paper of Poincaré [Poi90] in the late 1800s. It is influential in many areas, but most notably in Sobolev theory. The Poincaré inequality is a method of controlling the oscillation of a function by the size of its derivative, and the notion generalizes to less rigid forms of derivatives, like the following introduced by Heinonen and Koskela [HK96, HK98].

For a metric space (X, d), a nonnegative function $g : X \to [0, \infty)$ is called an *upper gradient* of a locally Lipschitz function $f : X \to \mathbb{R}$ if, for all points $x, y \in X$ and all rectifiable curves $\gamma : [a, b] \to X$ joining x to y,

$$|f(x) - f(y)| \leq \int_\gamma g \, \mathrm{d}s.$$

This notion of derivative is a natural extension of the fundamental theorem of calculus to metric spaces.

A metric measure space (X, d, μ) supports a *p-Poincaré inequality* for some $p \geq 1$ if there are constants $C \geq 1$ and $\lambda \geq 1$ so that for all locally Lipschitz functions

$f : X \to \mathbb{R}$, all upper gradients $g : X \to [0, \infty)$ of f, all points $x \in X$, and all scales $0 < r \leq \operatorname{diam} X$,

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, \mathrm{d}\mu \leq C\lambda r \left(\fint_{B(x,\lambda r)} g^p \, \mathrm{d}\mu \right)^{1/p},$$

where

$$f_{B(x,r)} = \fint_{B(x,r)} f \, \mathrm{d}\mu \quad \text{and} \quad \fint_A h \, \mathrm{d}\mu = \frac{1}{\mu(A)} \int_A h \, \mathrm{d}\mu$$

for all measurable sets $A \subseteq X$ and all measurable functions $h : X \to \mathbb{R}$.

1.2 Bi-Lipschitz rectifiability

The most natural notion of regularity in metric spaces is an isometry, a function which preserves distances. Unfortunately, this is far too rigid to allow useful study of metric spaces. The immediate successor of isometries are bi-Lipschitz maps. A function between two metric spaces is *bi-Lipschitz* if it is Lipschitz and it has an inverse which is also Lipschitz. While Lipschitz rectifiability has been completely resolved in Euclidean spaces [Jon90, Oki92] (and even in all Hilbert spaces [Sch07]), the problem of bi-Lipschitz rectifiability remains open mainly due to topological constraints. The key tool in constructing a bi-Lipschitz curve which contains a set is a bi-Lipschitz extension theorem. David and Semmes [DS91, Proposition 17.1] proved that a bi-Lipschitz embedding of a subset of the real line into $\mathbb{R}^n$ extends to a bi-Lipschitz embedding of the entire real line, provided $n \geq 3$. MacManus [Mac95] extended David and Semmes' result to the case $n = 2$, which is much more difficult since intersecting lines in $\mathbb{R}^3$ may be easily modified so they no longer intersect, but this is not the case in $\mathbb{R}^2$. This allows for the following corollary.

Corollary. *A subset of $\mathbb{R}^n$ is bi-Lipschitz rectifiable if and only if it is bi-Lipschitz homeomorphic to a subset of the real line.*

The result of David and Semmes is a special case of a more general result from the same work [DS91], where instead they consider bi-Lipschitz embeddings of subsets of $\mathbb{R}^d$ into $\mathbb{R}^n$, which they prove extend to bi-Lipschitz embeddings of all of $\mathbb{R}^d$ provided $n \geq 2d + 1$. The main motivation of their result is to establish the equivalence of the boundedness of certain singular operators on Euclidean space via quantitative rectifiability. This extension result can be seen as a rougher version of the classical Whitney extension theorem [Whi34]; while the maps considered here are analytically weaker (as they are bi-Lipschitz instead of differentiable), they are metrically and topologically stronger.

The next theorem generalizes David and Semmes' result to the setting in which Euclidean spaces are replaced by metric measure spaces with controlled geometry. The proof of David and Semmes' theorem relies on two key facts about Euclidean spaces: there are many rectifiable curves and there are differentiable bump functions $\phi : \mathbb{R} \to \mathbb{R}^n$ with controlled derivatives. The second fact is not (in general) recoverable in a general metric measure space, but the first is: those metric measure spaces which are Ahlfors regular and support a Poincaré inequality have quantifiably many rectifiable curves.

Theorem (Theorem 2.3.2)**.** *Let (X, d, μ) be a complete, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality for some $Q > 2$. If A is a subset of the real line and $f : A \to X$ is a bi-Lipschitz embedding, then f has a bi-Lipschitz extension $F : I \to X$, where I is the smallest closed interval containing A.*

In Theorem 2.8.1, we prove a stronger quantitative version of this result in the sense that the constant of the bi-Lipschitz extension depends only on the constant of the bi-Lipschitz embedding and the data of Ahlfors regularity and the Poincaré inequality. This allows for the following corollary regarding bi-Lipschitz rectifiability in metric measure spaces with controlled geometry.

Corollary. *Let (X, d, μ) be a complete, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality. A subset of X is bi-Lipschitz rectifiable if and only if it is bi-Lipschitz homeomorphic to a subset of the real line.*

The assumptions of the theorem are sharp in that none of completeness, Ahlfors regularity, nor the Poincaré inequality can be removed from the statement (see Section 2.3). A useful corollary of Theorem 2.3.2 gives a sufficient condition for bi-Lipschitz rectifiability based on a bound on Assouad dimension (sometimes called the metric dimension). The Assouad dimension was introduced by Assouad [Ass83] to prove the Assouad Embedding Theorem.

The *Assouad dimension* of a metric space (X, d) is the infimum of the values $\alpha > 0$ such that there exists a constant $C \geq 1$ such that for any scales $0 < r < R \leq \operatorname{diam} X$ and any point $x \in X$, the ball $B(x, R)$ can be covered by at most $C(R/r)^\alpha$-many balls of radius r. The Assouad dimension is larger than the other commonly studied fractal dimensions; for example, the Hausdorff, Minkowski, and packing dimensions.

Corollary (Corollary 2.3.3)**.** *Let (X, d, μ) be a complete, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality. A subset of X with Assouad dimension less than one is bi-Lipschitz rectifiable.*

Assuming Theorem 2.3.2, the proof of the corollary is short and follows the same ideas as in the Euclidean case. A subset E of X with Assouad dimension less than one must be uniformly disconnected [DS97, Lemma 15.2], hence it is bi-Lipschitz homeomorphic to an ultrametric space [DS97, Proposition 15.7]. There is a bi-Lipschitz embedding $g : E \to \mathbb{R}$ [LML94, Theorem 3.8], and, by Theorem 2.3.2, there exists a bi-Lipschitz extension $f : I \to X$ of the bi-Lipschitz map $g^{-1} : g(E) \to X$. The image of f is the desired bi-Lipschitz arc containing E.

The proof of Theorem 2.3.2 has two main ingredients. The first is the construction of short curves in $X \setminus E$ that stay quantitatively far from E. This is the notion of uniformity, and it is interesting in its own right, see Section 1.4. Here, we use the

Poincaré inequality to find bounds on the modulus of curves joining two balls, which allows us to choose a short curve that avoids the small set E.

The second ingredient in the proof of Theorem 2.3.2 is a standard "straightening" argument for paths. In particular, Lytchak and Wenger [LW20, Lemma 4.2] proved that, given any topological arc in a geodesic space, there exists a bi-Lipschitz arc with the same endpoints that is close to the original; see also [MW21, Lemma 4.2] for a similar result for topological circles. In Section 2.6, we prove a quantitative version of their result, Lemma 2.6.1. Moreover, under the additional assumptions of Ahlfors regularity and a Poincaré inequality, in Proposition 2.3.5, we show as a corollary of Theorem 2.3.2 that every continuum (that is, every compact, connected set) can be approximated by a bi-Lipschitz curve (in fact, a quasisimilarity) in the Hausdorff distance.

Corollary (Proposition 2.3.5)**.** *Let (X, d, μ) be a complete, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality, let K be a continuum in X, and let $\epsilon \in (0,1)$. For any $x, y \in K$ with $d(x,y) \geq \epsilon \operatorname{diam} K$, there exists a quasisimilarity $\gamma : [0,1] \to X$ with $\operatorname{dist}_H(K, \gamma([0,1])) \leq \epsilon \operatorname{diam} K$.*

In particular, every compact, Ahlfors regular metric measure space satisfying a Poincaré inequality contains "almost space filling" bi-Lipschitz curves. A function $f : (X, d_X) \to (Y, d_Y)$ is a (L, Λ)-*quasisimilarity* for $L \geq 1$ and $\Lambda > 0$ if

$$\frac{\Lambda}{L} d_X(x,y) \leq d_Y(f(x), f(y)) \leq L\Lambda d_X(x,y).$$

Roughly speaking, a quasisimilarity is a generalization of a similarity in the way that a bi-Lipschitz map is a generalization of an isometry.

1.3 Quasisymmetric rectifiability

Our next goal is to provide a quasisymmetric version of the Denjoy-Riesz theorem in abstract metric spaces. A heuristic interpretation of quasisymmetric maps is that

they send equilateral triangles to triangles which are roughly equilateral. The precise definition is the following: a map $f : (X, d_X) \to (Y, d_Y)$ is *η-quasisymmetric* for a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ if for all points $x, a, b \in X$ with $x \neq b$,

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta\left(\frac{d_X(x, a)}{d_X(x, b)}\right).$$

Quasisymmetric maps arose as the boundary values of quasiconformal maps (a generalization of conformal maps) of the upper half-space [BA56]. Later on, Tukia and Väisälä studied quasisymmetric maps in arbitrary metric measure spaces [TV80]. In fact, it has been shown that the notion of a quasisymmetric map is equivalent to quasiconformal maps in all Euclidean spaces aside from the real line [Väi71, HK98]. The relaxation of conformality is necessary for the study of conformal analysis in higher dimensions due to the Liouville rigidity theorem, which states that the only conformal maps in Euclidean spaces with dimension higher than two are Möbius transformations (compositions of inversions, rotations, scalings, and translations).

Quasisymmetric arcs have appeared in geometric function theory, geometric group theory (limits of quasi-Fuchsian groups), complex dynamics (Julia sets of certain rational functions), function spaces (domains of extension for Sobolev and BMO maps), Loewner evolution (traces of driving functions λ with Hölder norm less than 4), potential theory (Dirichlet integrals and harmonic bending), and hyperbolic geometry (bi-Lipschitz equivalence of hyperbolic and distance-ratio metrics). See [GH12] for a detailed exposition.

To replace "homeomorphic" with "quasisymmetric" in the Denjoy-Riesz theorem, it is necessary to make some geometric assumptions about the totally disconnected set one is trying to capture. For example, if $E \subset \mathbb{R}^n$ is a Cantor set with positive n-dimensional Lebesgue measure, then any arc which contains E has positive n-dimensional Lebesgue measure, so it cannot be the quasisymmetric image of $[0, 1]$ [Väi81, Theorem 4.1]. A quantitative version of total disconnectedness was introduced by David and Semmes [DS97, Definition 15.1]: in simple words, a metric space is

uniformly disconnected if it has isolated islands at every location and at every scale. More rigorously, a metric space (X, d) is *c-uniformly disconnected* for some $c \geq 1$ if for all points $x \in X$ and all scales $0 < r \leq \operatorname{diam} X$, there is a subset A of X containing x with $\operatorname{diam} A \leq r$ and $\operatorname{dist}(A, X \setminus A) \geq c^{-1} r$.

Using the notion of uniform disconnectedness, MacManus [Mac99] proved a quasisymmetric version of the Denjoy-Riesz theorem in Euclidean spaces.

Theorem ([Mac99, Theorem 1]). *A compact, uniformly disconnected subset of $\mathbb{R}^n$ is contained in the image of the unit interval under a quasisymmetric map.*

The next question one may ask is "Which metric measure spaces allow for a generalization of MacManus' theorem?" Similar to Zippin's no local cut point condition, a topological condition should be assumed on the metric measure space. In contrast to a cut set, we call a subset E of a metric space (X, d) a *full set* if $X \setminus E$ is connected. We say a subset E is *locally full* if for all connected neighborhoods U of E, $U \setminus E$ is connected. A metric space (X, d) is *fully connected* if it is path connected, locally path connected, and every full set is locally full. For example, every Euclidean space $\mathbb{R}^n$ is fully connected but no torus $(\mathbb{S}^1)^n$ is fully connected.

Theorem (Theorem 3.3.2). *Let (X, d, μ) be a complete, fully connected, unbounded, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality. A uniformly disconnected subset of X is quasisymmetrically rectifiable.*

One canonical example of a class of metric measure spaces satisfying the above properties are the Carnot groups. While we do not define the Carnot groups, we make note to mention that Carnot groups include the Euclidean spaces and the Heisenberg groups, and are typical examples of subRiemannian manifolds (not to be confused with Riemannian submanifolds), where the tangent bundle has a subbundle of "admissible directions." A curve which does not adhere to the admissible directions is fractal in the sense that it is locally unrectifiable. In short, Carnot groups are metric measure spaces which are topologically identical to Euclidean space, but geometrically

can behave much worse: they may have fewer rectifiable curves and surfaces. On the other hand, by definition, Carnot groups satisfy the Hörmander condition, which guarantees they are a geodesic space (a metric space where every two points can be joined by a curve with length equal to their distance).

The techniques and tools used in the proof of Theorem 3.3.2 (as well as modern proofs of the Denjoy-Riesz theorem) involve several classical tools from Euclidean space which are not available here. First, Euclidean spaces have a dyadic decomposition by cubes. This decomposition can be used to create a defining sequence for any Cantor set which consists of piecewise linear n-manifolds with boundary; see for example [Mac99, Section 2] and [BV19, Corollary 5.2]. Second, in Euclidean spaces, arcs can be approximated in the Hausdorff distance by bi-Lipschitz arcs quantitatively, but this is less obvious in metric measure spaces with fractal directions such as Carnot groups. We resolve the first issue by creating "good" defining sequences for all compact, totally disconnected sets, and we resolve the second issue by using Ahlfors regularity and the Poincaré inequality to generate nice curves. In fact, these techniques allow us to establish the following strengthening of the Denjoy-Riesz theorem.

Theorem (Theorem 3.3.3)**.** *Let (X, d, μ) be a complete, fully connected, unbounded, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality. If E is a compact, totally disconnected set, then there exists an embedding $\phi : [0,1] \to X$ such that $E \subset \phi([0,1])$ and ϕ is locally bi-Lipschitz away from $\phi^{-1}(E)$. In particular, the Hausdorff dimension $\dim_H \phi([0,1]) \leq \max\{1, \dim_H E\}$.*

It is natural to ask whether a complete classification of quasisymmetrically rectifiable sets exists, in the spirit of Moore-Kline. As in the bi-Lipschitz case, the problem could be solved by the existence of a quasisymmetric extension theorem. We leave the following as an open question, and note that it is unknown even if (X, d, μ) is a Euclidean space, although partial results exist if one allows the dimension of the target of the extension to increase [Vel18].

Question. *Let (X, d, μ) be a complete, fully connected, Ahlfors Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality. If E is a subset of the real line and $f : E \to X$ is a quasisymmetric embedding, is there a quasisymmetric extension $F : \mathbb{R} \to X$?*

1.4 Uniformity

Roughly speaking, a subset U of a metric space (X, d) is uniform if any two points can be joined by a curve in U which is not too long compared to the distance of its endpoints, and which stays quantitatively far from the exterior of U, aside from possibly at its endpoints. More precisely, U is *c-uniform* for some $c \geq 1$ if for all points $x, y \in U$, there is a curve $\gamma : [a, b] \to X$ with $\gamma(a) = x$, $\gamma(b) = y$,

(a) $\operatorname{length}(\gamma) \leq c d(x, y)$, and

(b) $\operatorname{dist}(\gamma(t), \partial U) \geq c^{-1} \operatorname{dist}(\gamma(t), \{x, y\})$ for each $t \in [a, b]$.

If only condition (a) is satisfied, we say U is *c-quasiconvex*.

Uniformity plays a crucial role in geometric function theory and the study of Sobolev and BMO (bounded mean oscillation) functions [GO79, Jon80, Jon81]. Furthermore, Newton-Sobolev functions (a generalization of Sobolev functions to metric measure spaces) also have extension properties on uniform domains, provided sufficiently nice geometry [BS07, FOS15, OS17]. Quasiconvex spaces are important themselves for two main reasons: every quasiconvex space is bi-Lipschitz homeomorphic to a geodesic space, and a weakly quasisymmetric map (a weakening of quasisymmetric maps which is, in general, much simpler to work with) between two quasiconvex metric spaces is in fact quasisymmetric [Väi99].

The study of uniform and quasiconvex sets has been prevalent in literature since the inception of the notion originally by John [Joh61], and later Mario and Sarvas [MS79]. It is an open problem to classify the subsets Y of a metric space X such that Y is co-uniform (has uniform complement) or such that $X \setminus Y$ is quasiconvex.

Uniform domains have been studied in more general Carnot groups as well, for example see [CT95, CGN00, Gre01]. Co-uniformity of a subset Y of a metric space (X, d) has been studied in various other ways, including uniform disconnectedness of Y [Mac99], quasihyperbolicity [Her87, HVW17, Her22], and Gromov hyperbolicity [BHK01, HSX08]. Quasihyperbolicity and Gromov hyperbolicity roughly mean that the space has "negative curvature." Hakobyan and Herron [HH08] showed that if Y is a subset of $\mathbb{R}^n$ with zero $(n-1)$-dimensional Hausdorff measure, then $\mathbb{R}^n \setminus Y$ is quasiconvex and this assumption is sharp. Herron, Lukyanenko, and Tyson [HLT18] proved the same result in the Heisenberg group, where, in this setting, it is assumed that the 3-dimensional Hausdorff measure is zero. This assumption is natural, since while the Heisenberg group is homeomorphic to $\mathbb{R}^3$, it is Ahlfors 4-regular. Väisälä [Väi88] showed that if a subset Y of $\mathbb{R}^n$ is co-uniform, then the topological dimension of Y is at most $n-2$. The following theorem works in the opposite direction: if X is an Ahlfors regular metric measure space supporting a Poincaré inequality, then any subset Y which has small Assouad dimension is co-uniform.

Theorem (Proposition 2.3.4)**.** *Let (X, d, μ) be a complete, Ahlfors Q-regular metric measure space supporting a p-Poincaré inequality for some $p \in (1, Q]$. A closed subset of X with Assouad dimension less than $Q - p$ is co-uniform.*

Note that if Y is a closed subset of X and has Assouad dimension less than $Q - p$, then the $(Q-p)$-dimensional Hausdorff measure of Y is zero, so on one hand, Proposition 2.3.4 can be seen as a slightly weaker version of the aforementioned results of Hakobyan and Herron [HH08], and Herron, Lukyanenko, and Tyson [HLT18], while on the other hand, Proposition 2.3.4 holds in more general spaces. The assumption on the Assouad dimension in Proposition 2.3.4 is sharp.

Another result of David and Semmes [DS97, Proposition 15.11] states that uniformly disconnected subsets of Euclidean space are quasisymmetrically homeomorphic to a porous subset of the real line. Roughly speaking, a subset Y of a metric space (X, d) is porous if at every point of Y and at every scale, there are quantitatively large

holes in Y. Furthermore, MacManus [Mac99] proved that a uniformly disconnected subset of $\mathbb{R}^n$ is co-uniform. MacManus' result generalizes to Ahlfors regular metric measure spaces with a Poincaré inequality: uniformly disconnected subsets are co-uniform.

Theorem (Theorem 3.3.5)**.** *Let (X, d, μ) be a complete, fully connected, Ahlfors Q-regular metric measure space supporting a Q-Poincaré inequality. A uniformly disconnected subset of X is co-uniform.*

The theorem is quantitative in the sense that the constant of uniformity depends only on the data of Ahlfors regularity, the data of the Poincaré inequality, and the constant of uniform disconnectedness. Note also that in Theorem 3.3.5, a Q-Poincaré inequality is assumed, which is weaker than the $(Q-1)$-Poincaré inequality assumed in Theorem 3.3.2.

1.5 Outline

Chapter 2 is, up to formatting and a disclosure in Section 2.1, a direct copy of the article of the same name [HVZ24]. In Section 2.4, we define several terms used throughout the rest of the chapter, and provide a few important lemmas which are used as tools in the proofs of the results of the remaining sections of the chapter. Particularly, in Section 2.5, we prove Proposition 2.3.4, which states that sets of small Assouad dimension are co-uniform. The main result of Section 2.6 is Proposition 2.3.5, which shows that in metric measure spaces with controlled geometry, it is possible to approximate continua by bi-Lipschitz arcs in Hausdorff distance. Sections 2.7 and 2.8 are dedicated to the proof of the main result of the chapter, Theorem 2.8.1, which is the quantitative version of Theorem 2.3.2.

Chapter 3 is, up to formatting and a disclosure in Section 3.1, the same as the preprint of the same title [HV25]. In Section 3.4, we give multiple definitions and lemmas used as tools throughout the rest of Chapter 3. The main goal of Section 3.5

is to define a "good" decomposition of a compact totally disconnected set in a metric measure space with controlled geometry. In Section 3.6, we give a proof of Theorem 3.3.5, which heavily relies on the decomposition given in Section 3.5. Sections 3.7 through 3.10 are all devoted to the proof of Theorem 3.3.2. Specifically, in Section 3.7, we list some notation that is used in the rest of the chapter and in Section 3.8, we define bi-Lipschitz arcs which are used as the building blocks of the arc in the conclusion of Theorem 3.3.2. In Section 3.9, we provide a "cutting and pasting" argument to locally modify the bi-Lipschitz arcs defined in the previous section to ensure their concatenation is locally bi-Lipschitz, and finally in Section 3.10, we prove Theorem 3.3.2.

Chapter 2

Bi-Lipschitz arcs in metric spaces with controlled geometry

2.1 Disclosure

This chapter is, up to formatting, a direct copy of the article of the same name [HVZ24]. The paper is a collaborative work with Vyron Vellis and Scott Zimmerman, is published in *Revista Matemática Iberoamericana*, and is available at https://doi.org/10.4171/rmi/1484. The ideas are mostly an adaptation of results of David and Semmes [DS91] to metric measure spaces. The author, Vyron Vellis, and Scott Zimmerman wrote and edited the article together.

2.2 Abstract

We generalize a bi-Lipschitz extension result of David and Semmes from Euclidean spaces to complete metric measure spaces with controlled geometry (Ahlfors regularity and supporting a Poincaré inequality). In particular, we find sharp conditions on metric measure spaces X so that any bi-Lipschitz embedding of a subset of the real line into X extends to a bi-Lipschitz embedding of the whole line. Along the way, we prove that if the complement of an open subset Y of X has small Assouad dimension, then it is a uniform domain. Finally, we prove a quantitative approximation of continua in X by bi-Lipschitz curves.

2.3 Introduction

Given metric spaces (X, d_X) and (Y, d_Y), a map $f : X \to Y$ is said to be *an L-bi-Lipschitz embedding* (or simply *L-bi-Lipschitz* or just *bi-Lipschitz*) if there is a constant $L \geq 1$ such that

$$L^{-1} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. A *bi-Lipschitz arc* in a metric space X is the image of an interval in the real line $\mathbb{R}$ under a bi-Lipschitz map.

We will consider the following question: given a set $E \subset X$ which is the image of a subset of $\mathbb{R}$ under a bi-Lipschitz map, is E contained in a bi-Lipschitz arc? If E is any finite subset of $\mathbb{R}^n$, the answer is trivially "yes". For general sets $E \subset \mathbb{R}^n$, the question was answered in the positive when $n \geq 3$ by the following extension theorem of David and Semmes [DS91]:

Theorem 2.3.1 ([DS91, Proposition 17.1])**.** *Let $n \geq 3$ be an integer, let $A \subset \mathbb{R}$, and let $f : A \to \mathbb{R}^n$ be a bi-Lipschitz embedding. Then there exists a bi-Lipschitz extension $F : \mathbb{R} \to \mathbb{R}^n$.*

MacManus [Mac95] extended the result of David and Semmes to the case $n = 2$, which is much more difficult since intersecting lines in $\mathbb{R}^3$ may be easily modified so that they no longer intersect, but this is not the case in $\mathbb{R}^2$. One may view these extension results as rougher versions of the classical Whitney Extension Theorem [Whi34]; while the maps considered here are analytically weaker (as they are bi-Lipschitz rather than differentiable), they are metrically and topologically stronger.

Theorem 2.3.1 is a special case of a more general result in [DS91] where $A \subset \mathbb{R}^d$ and $n \geq 2d+1$. The main motivation behind that result was to establish the equivalence of the boundedness of certain singular operators on $\mathbb{R}^n$ via quantitative rectifiability. More precisely, Theorem 2.3.1 was used in [DS91] to show that, when $n \geq 3$, every Ahlfors 1-regular set $A \subset \mathbb{R}^n$ (see (2.4.1) for the definition of Ahlfors regularity) which admits a corona decomposition (roughly speaking, A can be decomposed into a collection of subsets which are well-approximated by Lipschitz graphs and a collection of subsets which are not, and both of these collections have controlled measure) contains "big pieces" of bi-Lipschitz arcs i.e. for any $\epsilon > 0$, there exists an $M > 0$ such that, for any $x \in A$ and any $R > 0$, there is an M-bi-Lipschitz embedding $\rho : \mathbb{R} \to \mathbb{R}^n$ such that

$$|E \cap (B(x, R) \setminus \rho(\mathbb{R}))| \leq \epsilon R.$$

Another application of Theorem 2.3.1 is in the problem of the *bi-Lipschitz rectifiability* of sets in Euclidean spaces. In other words, one hopes to classify

those subsets of $\mathbb{R}^n$ that are contained in a bi-Lipschitz arc. While the classical characterization of the Lipschitz rectifiability of sets in Euclidean spaces has been completely resolved [Jon90, Oki92], the problem of bi-Lipschitz rectifiability remains open mainly due to topological constraints. Theorem 2.3.1 can be used to show that, if a set $E \subset \mathbb{R}^n$ has Assouad dimension less than 1, then E is bi-Lipschitz rectifiable; see [BV19, Corollary 3.5] for a different approach. See Section 2.4 for the definition of the Assouad dimension.

In this article, we generalize Theorem 2.3.1 to the setting in which Euclidean spaces $\mathbb{R}^n$ are replaced by a large class of metric measure spaces. There are two main difficulties in this generalization. Firstly, the target metric space X must contain many of rectifiable curves, and this notion of "many" must be understood quantitatively. A notable example (and, in fact, the initial motivation for this project) is the *Heisenberg group* $\mathbb{H}$ in which the classical Whitney Extension Theorem for curves has been well-studied recently; see [Zim18, PSZ19, Zim23, SZ23]. We will not define the Heisenberg group here but only recall that it is a geodesic space homeomorphic to $\mathbb{R}^3$, and there exists a distribution $H : \mathbb{R}^3 \to \mathbf{Gr}(2, \mathbb{R}^3)$ such that if a curve $\gamma : [0, 1] \to \mathbb{H}$ is rectifiable, then it is differentiable almost everywhere and $\dot{\gamma}(t) \in H_{\gamma(t)}$ for almost every t. This fact implies that there must be many fewer rectifiable curves in $\mathbb{H}$ than in $\mathbb{R}^3$. Secondly, the proof in the Euclidean case relies on the existence of differentiable bump functions $\phi : \mathbb{R} \to \mathbb{R}^n$ with controlled derivatives, and we cannot hope to recover this idea in a general metric space.

The class of metric measure spaces to which the bi-Lipschitz extension result will be generalized will have two properties. The first is *Ahlfors regularity*: we say that a metric measure space (X, d, μ) is Ahlfors Q-regular (or simply Q-regular) if the measure of any ball of radius r is comparable to r^Q. The second property is the existence of a *Poincaré inequality*. Such an inequality roughly states that, if we use u_B to denote the average value of a function $u : X \to \mathbb{R}$ on a ball B, then the average of the variation $|u - u_B|$ is controlled by the average of a "weak derivative" of u on B. See Section 2.4 for all relevant definitions. It is known that Ahlfors regular spaces

supporting a Poincaré inequality must contain quantitatively many rectifiable curves. Moreover, such spaces admit a notion of differentiation [Che99].

The following is the main result of this paper:

Theorem 2.3.2. *Let (X, d, μ) be a Q-regular, complete metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q - 1$. If $A \subset \mathbb{R}$ and $f : A \to X$ is a bi-Lipschitz embedding, then f extends to a bi-Lipschitz embedding $F : I \to X$ where I is the smallest closed interval containing A.*

In Theorem 2.8.1 we prove a stronger quantitative version of this result in the sense that the bi-Lipschitz constant of F depends only the bi-Lipschitz constant of f and on the data of Ahlfors Q-regularity and the Poincaré inequality. Moreover, if X is unbounded, then we can choose $I = \mathbb{R}$.

A large variety of metric spaces satisfy the assumptions of Theorem 2.3.2, including orientable, n-regular, linearly locally contractible n-manifolds with $n \geq 3$ [Sem96], Carnot groups [Var86, Jer86] (which include Euclidean spaces and the Heisenberg group), certain hyperbolic buildings [BP99], Laakso spaces [Laa00], and certain Menger sponges [MTW13, EBG22].

The assumptions of the theorem are sharp in that neither Ahlfors regularity nor the Poincaré inequality can be removed from the statement. For Ahlfors regularity, let $X = \mathbb{S}^2 \times \mathbb{R}$ with the length metric and the induced Hausdorff 3-measure. Then X is complete, has Ricci curvature bounded from below so it satisfies the 1-Poincaré inequality [Cha01, Chapter VI.5], but is not Ahlfors regular. Define $f : \{2^n : n \in \mathbb{N}\} \to X$ by $f(2^n) = (p_0, (-2)^n)$ where $p_0 \in \mathbb{S}^2$. The map f is bi-Lipschitz and if $F : \mathbb{R} \to X$ is any homeomorphic extension of f, then for any $n \in \mathbb{N}$, $F([2^n, 2^{n+1}])$ intersects with $(\mathbb{S}^2 \times \{0\})$ so F can not be bi-Lipschitz.

Since the Poincaré inequality is an open ended condition [KZ08], we may assume that $p < Q - 1$ for the proof of the theorem. However, the bound $Q - 1$ is sharp. To see this, let $n \geq 2$, let P_1, P_2 be two n-dimensional planes in $\mathbb{R}^{2n-1}$ intersecting on a line ℓ, and let $p_0 \in \ell$. The metric space $X = (P_1 \cup P_2) \setminus B(p_0, 1)$ with the induced

Euclidean metric and n-dimensional Lebesgue measure is complete, n-regular, and satisfies the p-Poincaré inequality for all $p > n-1$ [HK98, Theorem 6.15]. Let $f : (-\infty,-1]\cup\{-\frac{1}{2},\frac{1}{2}\}\cup[1,\infty) \to X$ be a map such that $f(-\frac{1}{2}) \in P_1 \setminus (\ell \cup B(p_0,1))$, $f(\frac{1}{2}) \in P_2 \setminus (\ell \cup B(p_0,1))$, and f maps $\mathbb{R}\setminus(-1,1)$ isometrically onto $\ell \setminus B(p_0,1)$. Then f is bi-Lipschitz but admits no homeomorphic (let alone bi-Lipschitz) extension $F : \mathbb{R} \to X$.

2.3.1 Related results

The first corollary of Theorem 2.3.2 gives a sufficient condition for bi-Lipschitz rectifiability in Ahlfors regular spaces satisfying a Poincaré inequality.

Corollary 2.3.3. *Let X be a complete Q-regular metric measure space supporting a p-Poincaré inequality for some $1 < p \le Q-1$. If $E \subset X$ has Assouad dimension less than 1, then E is bi-Lipschitz rectifiable.*

The proof of the corollary follows the same ideas as in the Euclidean case. Since the Assouad dimension of E is less than 1, [DS97, Lemma 15.2] implies that E must be uniformly disconnected, and hence it is bi-Lipschitz equivalent to an ultrametric space Z of Assouad dimension less than 1 [DS97, Proposition 15.7]. By [LML94, Theorem 3.8], there exists a bi-Lipschitz embedding $g : E \to \mathbb{R}$, and, by Theorem 2.3.2, there exists a closed interval I and a bi-Lipschitz extension $f : I \to X$ of the map $g^{-1} : g(E) \to X$. Thus $E \subset f(I)$, so E is contained in a bi-Lipschitz arc.

The proof of Theorem 2.3.2 has two main ingredients. The first is the construction of short curves in $X \setminus f(A)$ that stay quantitatively far from $f(A)$. To build such curves, we will use the notion of the uniformity of a set. Given a set $U \subset X$, we say that U is *c-uniform* if, for every $x, y \in U$, there exists a path $\gamma : [0,1] \to U$ joining x to y such that

1. the length of γ is at most $cd(x,y)$, and
2. $\operatorname{dist}(\gamma(t), X\setminus U) \ge c^{-1}\operatorname{dist}(\gamma(t),\{x,y\})$ for all $t \in [0,1]$.

In other words, U is uniform if, for any $x, y \in U$, there exists a curve connecting them which is short compared to $d(x, y)$ and stays far from $X \setminus U$ quantitatively. If U satisfies only property (1) in this definition, then we say that U is *c-quasiconvex*.

It is an open problem to classify the closed sets $Y \subset X$ for which $X \setminus Y$ is quasiconvex or uniform. Hakobyan and Herron [HH08] showed that, if $Y \subset \mathbb{R}^n$ has Hausdorff $(n-1)$-measure $\mathcal{H}^{n-1}(Y) = 0$, then $\mathbb{R}^n \setminus Y$ is quasiconvex. Moreover, this assumption is sharp. Herron, Lukyanenko, and Tyson [HLT18] proved the same result in the Heisenberg group $\mathbb{H}$ where, in this setting, it is assumed that $\mathcal{H}^3(Y) = 0$. The dimension 3 is natural as $\mathbb{H}$ is 4-regular while $\mathbb{R}^n$ is n-regular. It is unknown if a similar result exists in all Carnot groups.

The question of whether $X \setminus Y$ is uniform has been studied in terms of uniform disconnectedness of Y [Mac99] and quasihyperbolicity of X and Y [Her87, HVW17, Her22]. Väisälä [Väi88] showed that, if $\mathbb{R}^n \setminus Y$ is uniform, then the topological dimension of Y is at most $n-2$. The following proposition, which we prove in Section 2.5, works in the opposite direction: if X is Ahlfors regular and supports a Poincaré inequality and if the Assouad dimension of Y is small, then $X \setminus Y$ is uniform.

Proposition 2.3.4. *Let (X, d, μ) be a complete Q-Ahlfors regular metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q$. If $Y \subset X$ is a closed set with Assouad dimension less than $Q - p$, then $X \setminus Y$ is a uniform domain.*

Note that if $Y \subset X$ and has Assouad dimension less than $Q-p$, then $\mathcal{H}^{Q-p}(Y) = 0$. The assumption on the Assouad dimension is sharp. For example, let $X = \mathbb{R}^n$, let P be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, and let Y be a maximal 1-separated subset of P. Then it is easy to see that $\dim_A(Y) = n-1$, $\mathcal{H}^{n-1}(Y) = 0$, and $\mathbb{R}^n \setminus Y$ is not uniform.

The second ingredient in the proof of Theorem 2.3.2 is a standard "straightening" argument for paths. In particular, Lytchak and Wenger [LW20, Lemma 4.2] proved that, given any topological arc in a geodesic space, there exists a bi-Lipschitz arc with the same endpoints that is close to the original one; see also [MW21, Lemma 4.2] for

a similar result for topological circles. In Section 2.6, we prove a quantitative version of their result. Moreover, under the additional assumptions of Q-regularity and a Poincaré inequality, we show as a corollary of Theorem 2.3.2 that every continuum (i.e., every compact connected set) can be approximated by a bi-Lipschitz curve in the Hausdorff distance.

Proposition 2.3.5. *Let (X, d, μ) be a complete Q-regular metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q - 1$, let $K \subset X$ be a continuum, and let $\epsilon \in (0, 1)$. For any $x, y \in K$ with $d(x, y) \geq \epsilon \operatorname{diam} K$, there exists a curve $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, and there exists a constant $L \geq 1$ depending only on ϵ, the constants of Q-regularity, and the data of the Poincaré inequality such that*

$$\frac{1}{L}|s - t| \operatorname{diam} K \leq d(\gamma(t), \gamma(s)) \leq L|s - t| \operatorname{diam} K$$

for all $s, t \in [0, 1]$, and the Hausdorff distance $\operatorname{dist}_H(K, \gamma([0, 1])) \leq \epsilon \operatorname{diam} K$.

In particular, every compact Ahlfors regular metric measure space supporting a Poincaré inequality contains "almost space-filling" bi-Lipschitz curves.

2.3.2 Outline of the proof of Theorem 2.3.2

We start with two simple reductions. First, since bi-Lipschitz maps extend on the completion of their domain, we may assume that A is a closed set. Second, it is well known that the Poincaré inequality, completeness, and Ahlfors regularity imply that X is quasiconvex [Che99, Theorem 17.1]. Every complete quasiconvex space is bi-Lipschitz equivalent to a geodesic metric space and since the properties of Ahlfors Q-regularity and the p-Poincaré inequality are preserved under bi-Lipschitz mappings [HKST15, Lemma 8.3.18], we may assume for the rest that X is geodesic.

For the proof of Theorem 2.3.2, similar to the proof of Theorem 2.3.1 and the Whitney Extension Theorem, we construct a *Whitney decomposition* $\{\mathcal{Q}_i\}_{i \in \mathbb{N}}$ of $I \setminus A$,

i.e., a collection of closed intervals in $I \setminus A$ with mutually disjoint interiors such that their union is $I \setminus A$ and the length of each interval is comparable to its distance from A.

In Section 2.7 we define two auxiliary embeddings. Specifically, in §2.7.1 we construct a bi-Lipschitz embedding π of E into X, where E is the set of endpoints of the Whitney intervals $\mathcal{Q}_i$. The final map F will map elements of E very close to their image under π. In §2.7.2, we use the results of Sections 2.5 and 2.6 to define a second bi-Lipschitz embedding

$$g : A \cup \bigcup_{i \in \mathbb{N}} \hat{\mathcal{Q}}_i \to X$$

of f. Here, $\hat{\mathcal{Q}}_i$ denotes the middle third closed interval in $\mathcal{Q}_i$. If we write $\mathcal{Q}_i = [x, y]$, then the image $g(\hat{\mathcal{Q}}_i)$ is a bi-Lipschitz curve that has endpoints very close to $\pi(x)$ and $\pi(y)$.

In Section 2.8 we describe a method to modify and extend the map g near the points $\pi(x)$ to build a curve on the entire interval I, and we verify that this curve is indeed bi-Lipschitz to complete the proof of Theorem 2.3.2.

2.4 Preliminaries

Given quantities $x, y \geq 0$ and constants $a_1, \ldots, a_n > 0$ we write $x \lesssim_{a_1,\ldots,a_n} y$ if there exists a constant C depending at most on $a_1, \ldots, a_n$ such that $x \leq Cy$. If C is universal, we write $x \lesssim y$. We write $x \simeq_{a_1,\ldots,a_n} y$ if $x \lesssim_{a_1,\ldots,a_n} y$ and $y \lesssim_{a_1,\ldots,a_n} x$.

Given a metric space (X, d) and two points $x, y \in X$, we say that γ is a *path joining x with y* if there exists some continuous $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Given a set $Y \subset X$ and $r > 0$, we write $B(Y, r) := \{x \in X : \operatorname{dist}(x, Y) < r\}$.

2.4.1 Porosity and regularity

For a constant $C > 1$, a metric space X is called *C-doubling* if every ball of radius r can be covered by at most C balls of radii at most $r/2$. Given another constant $\alpha \geq 0$, X is called *(C, α)-homogeneous* if every ball of radius R can be covered by at most $C(R/r)^\alpha$ balls of radii at most r. We will occasionally refer to such a metric space as *α-homogeneous* when the constant C is not important. Clearly, a (C, α)-homogeneous space is $(C2^\alpha)$-doubling. Conversely, given $C > 0$ there exists $C' > 0$ and $\alpha > 0$ such that a C-doubling space is (C', α)-homogeneous.

The *Assouad dimension* of a metric space X (denoted $\dim_A(X)$) is the infimum of all $\alpha \geq 0$ such that X is α-homogeneous.

A metric measure space (X, d, μ) is said to be *Q-Ahlfors regular* (or *Q-regular*) for $Q \geq 0$ if there exists $C \geq 1$ such that, for all $x \in X$ and all $r \in (0, \operatorname{diam} X)$,

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q. \tag{2.4.1}$$

It is easy to see that if (X, d, μ) is Q-regular, then X is Q-homogeneous and $\dim_A(X) = Q$. If we want to emphasize the constant C in (2.4.1), then we say that (X, d, μ) is *(C, Q)-regular*.

Given $Y \subset X$ we say that Y is *p-porous* for some $p \geq 1$ if, for all $y \in Y$ and all $r \in (0, \operatorname{diam} X)$, there exists some $x \in B(y, r)$ such that $B(x, r/p) \subset B(y, r) \setminus Y$. In other words, Y contains relatively large "holes" near every point.

Lemma 2.4.1 ([BHK01, Lemma 3.12])**.** *Let $(X, d, \mathcal{H}^Q)$ be (C, Q)-regular, where $\mathcal{H}^Q$ is the Q-dimensional Hausdorff measure. A set $Y \subset X$ is p-porous for some $p \geq 1$ if and only if $\dim_A(Y) \leq Q - \epsilon$ for some $\epsilon > 0$. Here, ϵ and p depend only on each other, Q, and C.*

2.4.2 Poincaré inequality

Given a locally Lipschitz function u defined on a metric space (X, d), we say that a function $g : X \to [0, \infty)$ is an *upper gradient* of u if

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s$$

for all $x, y \in X$ and all paths γ in X joining x with y.

We say that a metric measure space (X, d, μ) supports a $(1, p)$-*Poincaré inequality* (or simply a p-*Poincaré inequality*) for some $1 \leq p < \infty$ if there exist $\lambda \geq 1$ and $C > 1$ with the following property: if $u : X \to \mathbb{R}$ is locally Lipschitz and $g : X \to [0, \infty)$ is an upper gradient of u, then, for all $x \in X$ and $r > 0$,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq C \operatorname{diam}(B(x, \lambda r)) \left(\fint_{B(x,\lambda r)} g^p \, \mathrm{d}\mu \right)^{1/p}, \tag{2.4.2}$$

where

$$\fint_A f \, \mathrm{d}\mu = \frac{1}{\mu(A)} \int_A f \, \mathrm{d}\mu$$

and

$$u_{B(x,r)} = \fint_{B(x,r)} u \, \mathrm{d}\mu.$$

It follows from Hölder's inequality that if $1 \leq p \leq q$ and (X, d, μ) satisfies a p-Poincaré inequality, then it satisfies a q-Poincaré inequality. Moreover, if the space is geodesic and doubling, then one can choose $\lambda = 1$; see for example [HKST15, Remark 9.1.19]. Henceforth, given a geodesic doubling space X that satisfies the p-Poincaré inequality, we will assume that $\lambda = 1$ in (2.4.2) and the constant C will be called the *data* of the Poincaré inequality.

For a detailed exposition on the Poincaré inequality on metric measure spaces, the reader is referred to [HKST15].

2.4.3 Modulus of curve families

The basic tool in the proof of Theorem 2.3.2 and Proposition 2.3.4 is the notion of the modulus of curves. In a sense, the modulus is a measurement of "how many" rectifiable curves are contained in a curve family.

Given a family Γ of rectifiable curves in a metric measure space (X, d, μ), we say that a Borel function $\rho : X \to [0, \infty)$ is admissible for Γ if

$$\int_\gamma \rho \, \mathrm{d}s \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

For $p \geq 1$, we define the *p-modulus* of Γ by

$$\mathrm{Mod}_p(\Gamma) := \inf \left\{ \int_X \rho^p \, \mathrm{d}\mu : \rho \text{ is admissible for } \Gamma \right\}.$$

It is well known that Mod_p is an outer measure on the space of all curve families in X.

The next lemma relates the modulus of curve families with the *locally Lipschitz capacity* between compact sets. Given two sets E and F in a metric space, we say that a curve γ joins E with F if there are points $x \in E$ and $y \in F$ such that γ joins x with y.

Lemma 2.4.2 ([KS01, Theorem 1.1])**.** *Suppose that (X, d, μ) is a geodesic metric measure space equipped with a doubling measure μ and supporting a p-Poincaré inequality with $p > 1$, and suppose that Ω is a domain in X. Let E, F be disjoint, compact, non-empty subsets of Ω, and let Γ be the collection of curves in Ω that join E with F. Then the p-modulus of Γ is equal to the p-capacity of E and F:*

$$\mathrm{Mod}_p(\Gamma) = \mathrm{Cap}_p(E, F) := \inf \int_\Omega g^p \, \mathrm{d}\mu,$$

where the infimum is taken over all Borel functions $g : \Omega \to [0, \infty)$ such that each g is an upper gradient of some locally Lipschitz function $u : \Omega \to \mathbb{R}$ satisfying $u|_E \geq 1$ and $u|_F \leq 0$.

2.5 Uniformity in metric measure spaces

The goal of this section is the proof of Proposition 2.3.4. The next lemmas are the crux of the proof.

Lemma 2.5.1. *Let (X, d, μ) be a (C_1, Q)-Ahlfors regular geodesic metric measure space supporting a p-Poincaré inequality with data C, for some $C, C_1 \geq 1$, and $1 < p < Q$. Let $x, y \in X$, let $r \in (0, \frac{1}{3}d(x, y))$, and let Γ be the collection of paths in $B(x, 2d(x, y))$ that connect $\overline{B}(x, r)$ with $\overline{B}(y, r)$. Then*

$$\operatorname{Mod}_p(\Gamma) \gtrsim_{p,C,C_1,Q} (d(x, y))^{Q-p} \left(\frac{d(x, y)}{r}\right)^{-Qp}.$$

Proof. Set $D = d(x, y)$. Let $u : B(x, 2D) \to \mathbb{R}$ be a locally Lipschitz function satisfying $u|_{B(x,r)} \geq 1$ and $u|_{B(y,r)} \leq 0$. Let also $g : B(x, 2D) \to [0, \infty)$ be an upper gradient of u. By the p-Poincaré inequality,

$$\begin{aligned}
&\int_{B(x,2D)} g^p \, \mathrm{d}\mu \\
&\geq \frac{\mu(B(x, 2D))}{C^p[\operatorname{diam}(B(x, 2D))\mu(B(x, 2D))]^p} \left(\int_{B(x,2D)} |u - u_{B(x,2D)}| \, \mathrm{d}\mu\right)^p \\
&\gtrsim_{p,C,C_1,Q} D^{Q-p-Qp} \left(\int_{B(x,r)\cup B(y,r)} |u - u_{B(x,2D)}| \, \mathrm{d}\mu\right)^p \\
&\geq D^{Q-p-Qp} \left(\tfrac{1}{2} \min\{\mu(B(x, r)), \mu(B(y, r))\}\right)^p \gtrsim_{p,C_1} D^{Q-p} \left(\frac{D}{r}\right)^{-Qp}.
\end{aligned}$$

Denote by Γ the collection of curves joining $\overline{B}(x,r)$ with $\overline{B}(y,r)$. By Lemma 2.4.2,

$$\mathrm{Mod}_p(\Gamma) \gtrsim_{p,C,C_1,Q} D^{Q-p}\left(\frac{D}{r}\right)^{-Qp}. \qquad \square$$

Lemma 2.5.2. *Let (X,d,μ) be a (C_1,Q)-Ahlfors regular metric measure space, let $R>0$, let $\ell>0$, and let Γ be the collection of paths in $B(x,R)$ that have length at least ℓR. Then,*

$$\mathrm{Mod}_p(\Gamma) \lesssim_{C_1} \ell^{-p} R^{Q-p}.$$

Proof. Note that the function $\rho = (\ell R)^{-1}\chi_{B(x,R)}$ is admissible for Γ. Therefore

$$\mathrm{Mod}_p(\Gamma) \leq \int_X \rho^p \,\mathrm{d}\mu \leq C_1 \ell^{-p} R^{Q-p}. \qquad \square$$

Lemma 2.5.3. *Let (X,d,μ) be a (C_1,Q)-Ahlfors regular metric measure space, let $Y\subset X$ be a (C_2,α)-homogeneous set, let $R>0$, let $\delta>0$, and let Γ be the collection of paths in $B(x,R)$ with an endpoint outside of $B(Y,2\delta R)$ and which intersect $B(Y,\delta R)$. Then*

$$\mathrm{Mod}_p(\Gamma) \lesssim_{Q,C_1,C_2} \delta^{Q-p-\alpha} R^{Q-p}.$$

Proof. Define the function

$$\rho := (\delta R)^{-1}\chi_{B(Y,2\delta R)\cap B(x,R)}$$

and note that ρ is admissible for Γ. Indeed, if $\gamma\in\Gamma$, then the the total length of the part of γ that is inside $B(Y,2\delta R)$ must be at least δR.

If V is a (δR)-net of $Y\cap B(x,R)$, then

$$B(Y,2\delta R)\cap B(x,R) \subset \bigcup_{v\in V} B(v,3\delta R),$$

and, by the homogeneity of X, it follows that $\operatorname{card}(V) \lesssim_{C_2} \delta^{-\alpha}$. Therefore

$$\operatorname{Mod}_p(\Gamma) \leq \int_X \rho^p \, \mathrm{d}\mu \lesssim_{Q,C_1,C_2} \delta^{Q-p-\alpha} R^{Q-p}. \qquad \square$$

Corollary 2.5.4. *Let (X, d, μ) be a (C_1, Q)-regular, geodesic metric measure space supporting a p-Poincaré inequality with $1 < p < Q$ and data C. Let $Y \subset X$ be a (C_2, α)-homogeneous set with $0 < \alpha < Q - p$. Given $x, y \in X \setminus Y$, there exists a path $\gamma : [0,1] \to X \setminus Y$ such that $\gamma(0) = x$, $\gamma(1) = y$,*

1. *$\gamma([0,1]) \subset B(x, 2d(x,y))$,*
2. *$\operatorname{length}(\gamma) \lesssim_{p,C,C_1,Q} d(x,y) \max\left\{1, \left(\dfrac{d(x,y)}{\operatorname{dist}(\{x,y\}, Y)}\right)^Q\right\}$*
3. *for all z in the image of γ,*

$$\operatorname{dist}(z, Y) \gtrsim_{p,\alpha,Q,C,C_1,C_2} d(x,y) \min\left\{1, \left(\frac{\operatorname{dist}(\{x,y\}, Y)}{d(x,y)}\right)^{\frac{Qp+Q-p-\alpha}{Q-p-\alpha}}\right\}.$$

Proof. Set $D := d(x,y)$ and

$$r := \tfrac{1}{4} \min\left\{D, \operatorname{dist}(\{x,y\}, Y)\right\}.$$

Let Γ_1 be the collection of all curves in $B(x, 2D)$ that join $\overline{B}(x,r)$ to $\overline{B}(y,r)$. Let Γ_ℓ be the collection of all curves in $B(x,2D)$ that have length at least $2D\ell$. Let Γ'_δ be the collection of all curves in $B(x,2D)$ that intersect a $(2D\delta)$-neighborhood of Y and have length at least $2D\delta$.

By Lemma 2.5.1, Lemma 2.5.2, and Lemma 2.5.3, there exist

$$\ell \simeq_{p,C,C_1,Q} \left(\frac{D}{r}\right)^Q \quad \text{and} \quad \delta \simeq_{p,\alpha,Q,C,C_1,C_2} \left(\frac{r}{D}\right)^{\frac{Qp}{Q-p-\alpha}}$$

such that

$$\operatorname{Mod}_p(\Gamma \setminus (\Gamma_l \cup \Gamma'_\delta)) > 0.$$

It follows that $\Gamma \setminus (\Gamma_l \cup \Gamma'_\delta)$ is non-empty. Fix now $\gamma \in \Gamma \setminus (\Gamma_l \cup \Gamma'_\delta)$ and concatenate γ with geodesic segments $[x, \gamma(0)]$ and $[\gamma(1), y]$. The resulting curve satisfies the conclusions of the corollary. □

Proof of Proposition 2.3.4. By Lemma 2.4.1, the regularity of X, and the homogeneity of Y, there exists $p_0 > 1$ such that Y is p_0-porous.

Fix now $x, y \in X \setminus Y$ and denote $r := d(x, y)$. There exists $z_0 \in B(x, r) \setminus (B(x, 2^{-1}r) \cup B(y, 2^{-1}r))$ such that

$$B(z_0, 2^{-1}r/p_0) \subset B(x, r) \setminus (B(x, 2^{-1}r) \cup B(y, 2^{-1}r) \cup Y)$$

by applying porosity of Y to a ball of radius $2^{-1}r$ contained in

$$B(x, r) \setminus (B(x, 2^{-1}r) \cup B(y, 2^{-1}r)).$$

Moreover, for each $n \in \mathbb{N}$, there exist points $z_n \in B(x, 2^{-n}r) \setminus B(x, 2^{-n-1}r)$ and $z_{-n} \in B(y, 2^{-n}r) \setminus B(y, 2^{-n-1}r)$ such that

$$B(z_n, 2^{-n-1}r/p_0) \subset B(x, 2^{-n}r) \setminus (B(x, 2^{-n-1}r) \cup Y)$$

and

$$B(z_{-n}, 2^{-n-1}r/p_0) \subset B(y, 2^{-n}r) \setminus (B(y, 2^{-n-1}r) \cup Y),$$

again by applying the porosity of Y to balls in the annuli $B(x, 2^{-n}r) \setminus B(x, 2^{-n-1}r)$ and $B(y, 2^{-n}r) \setminus B(y, 2^{-n-1}r)$.

Applying Corollary 2.5.4, there exists $c > 1$ depending only on p_0, p, Q, C, and C_1 such that, for each $n \in \mathbb{Z}$, there exists a path $\gamma_n : [0, 1] \to X \setminus Y$ with

1. $\gamma_n(0) = z_n$, $\gamma_n(1) = z_{n+1}$,
2. $\operatorname{length}(\gamma_n) \leq c d(z_n, z_{n+1}) \leq 2^{3-|n|} cr$, and
3. for all $t \in [0, 1]$, $\operatorname{dist}(\gamma_n(t), Y) \geq c^{-1} 2^{-2-|n|} r$.

Concatenating all the paths $\{\gamma_n\}_{n\in\mathbb{Z}}$ and adding the points x, y we obtain a path $\gamma : [0,1] \to X \setminus Y$. Note that

$$\text{length}(\gamma) = \sum_{n\in\mathbb{Z}} \text{length}(\gamma_n) \leq \sum_{n\in\mathbb{Z}} 2^{3-|n|} cr = 24cr = 24cd(x,y).$$

Let now $z \in \gamma([0,1])$. If z is either of x or y, then there is nothing to show. Otherwise, there exists $n \in \mathbb{Z}$ such that z is in the image of γ_n. Assume as we may that $n \geq 0$. Then

$$d(x,z) \leq d(z_n,x) + d(z_n,z) \leq (8c+1)2^{-n} r \leq 4c(8c+1)\,\text{dist}(z,Y),$$

which completes the proof. □

2.6 Bi-Lipschitz approximation of curves

In this section we show how paths in geodesic spaces can be approximated by bi-Lipschitz arcs with the same endpoints. The main goal will be the proof of Proposition 2.3.5.

The next lemma is important in the proof of Theorem 2.3.2 and is almost identical to [LW20, Lemma 4.2]. The difference here is the quantitative control on the bi-Lipschitz constant L.

Lemma 2.6.1. *Given $C \geq 1$ and $\epsilon > 0$, there exist $L = L(C,\epsilon) \geq 1$ with the following property. Let (X,d) be a C-doubling geodesic metric space and let $\sigma : [0,1] \to X$ be a curve with $\sigma(0) \neq \sigma(1)$. There exists a curve $\gamma : [0,1] \to X$ such that $\gamma(0) = \sigma(0)$, $\gamma(1) = \sigma(1)$, for all $s,t \in [0,1]$*

$$\frac{1}{L}|s-t|\,\text{diam}\,\sigma([0,1]) \leq d(\gamma(s),\gamma(t)) \leq L|s-t|\,\text{diam}\,\sigma([0,1]),$$

and

$$\operatorname{dist}(\gamma(t), \sigma([0,1])) \leq \epsilon \operatorname{diam} \sigma([0,1]).$$

The doubling property is not necessary to guarantee the existence of the bi-Lipschitz map γ; see [LW20, Lemma 4.2]. It is, however, necessary to control the constant L. For example, let $X = \ell_2$, let $e_1, e_2, \ldots$ be an orthonormal basis of ℓ_2, and let $n \in \mathbb{N}$. Define $\sigma : [0,1] \to \ell_2$ so that $\sigma(0) = e_0 := 0$, $\sigma(i/n) = e_i$ for $i \in \{1, \ldots, n\}$, and $\sigma|_{[(i-1)/n, i/n]}$ is linear for each $i \in \{1, \ldots, n\}$. Note that $\operatorname{diam} \sigma([0,1]) = \sqrt{2}$. It is easy to see that, if $\epsilon < 1/6$, then for each $i \in \{1, \ldots, n-1\}$ the set

$$B(\sigma([0,1]), \sqrt{2}\epsilon) \setminus B(\sigma(i/n), 3\sqrt{2}\epsilon)$$

is disconnected. Therefore, if γ is a path in ℓ_2 joining 0 with e_n and satisfying $\gamma([0,1]) \subset B(\sigma([0,1]), \sqrt{2}\epsilon)$, then $\gamma([0,1])$ must intersect each ball $B\left(\sigma(i/n), 3\sqrt{2}\epsilon\right)$ for $i = 1, \ldots, n$. In particular, the length of γ is at least a fixed multiple of n, while $|\gamma(0) - \gamma(1)| = 1$. It follows that, if γ is L-bi-Lipschitz, then L must depend on n and not just on ϵ.

For the proof of Lemma 2.6.1, we require a simple lemma. Here and for the rest of this section, all geodesic curves are parameterized by arc-length.

Lemma 2.6.2. *Let X be a geodesic metric space, let $a \geq b > 0$, let $f : [0,a] \to X$ be L-bi-Lipschitz, let $p \in X$, and suppose $f(b)$ is the closest point in $f([0,a])$ to p, i.e.,*

$$c := \operatorname{dist}(f([0,a]), p) = d(f(b), p).$$

If $g : [b, b+c] \to X$ is the geodesic from $f(b)$ to p, then the concatenation of $f|_{[0,b]}$ and g is $(2L)$-bi-Lipschitz.

Proof. Let $h : [0, b+c] \to X$ be the concatenation of $f|_{[0,b]}$ and g. Clearly $h|_{[0,b]}$ is L-bi-Lipschitz and $h|_{[b,b+c]}$ is 1-bi-Lipschitz. Fix now $s \in [0,b]$ and $t \in [b,c]$. Then

$$d(h(s), h(t)) \leq d(f(s), f(b)) + d(g(b), g(t)) \leq L(b-s) + t - b \leq L(t-s).$$

For the lower bound, we claim that $d(h(t), h(s)) \geq d(h(t), h(b))$. Indeed, if this was not the case, then

$$\begin{aligned}\operatorname{dist}(f([0,b]), p) &\leq d(h(s), h(t)) + d(h(t), p) < d(h(b), h(t)) + d(h(t), p) \\ &= d(f(b), p) = \operatorname{dist}(f([0,b]), p)\end{aligned}$$

which is impossible. Similarly, $d(h(s), h(t)) \geq d(h(s), h(b))$. Therefore,

$$\begin{aligned}d(h(s), h(t)) &\geq \tfrac{1}{2} d(h(s), h(b)) + \tfrac{1}{2} d(h(t), h(b)) \\ &\geq (2L)^{-1}|s-b| + \tfrac{1}{2}|b-t| \geq (2L)^{-1}|s-t|.\end{aligned}$$

□

We are now ready to show Lemma 2.3.5.

Proof of Lemma 2.3.5. Without loss of generality, assume that $\operatorname{diam} \sigma([0,1]) = 1$. Since X is doubling, it is (C', α)-homogeneous for some $C' > 0$ and $\alpha > 0$.

Fix $\epsilon > 0$. If $d(\sigma(0), \sigma(1)) < 2\epsilon$, then we can simply define γ to be the geodesic from $\sigma(0)$ to $\sigma(1)$ which is 1-bi-Lipschitz. Assume now that $d(\sigma(0), \sigma(1)) \geq 2\epsilon$.

Let $Y \subset \sigma([0,1])$ be a maximal $(\epsilon/4)$-separated set that contains $\sigma(0)$ and $\sigma(1)$. Since $\sigma([0,1])$ is connected, there exists a finite sequence of distinct points $x_0, \ldots, x_n$ in Y such that $x_0 = \sigma(0)$, $x_n = \sigma(1)$, and $d(x_{i-1}, x_i) < \epsilon/2$ for all $i \in \{1, \ldots, n\}$. By the homogeneity of X, we have that $n \leq C'(\epsilon/4)^{-\alpha}$.

We define a curve γ inductively. Let $\gamma_1 : [0, s_1] \to X$ be a geodesic with $\gamma_1(0) = \sigma(0)$ and $\gamma_1(s_1) = x_1$. Clearly, γ_1 is 1-bi-Lipschitz and for all $t \in [0, s_1]$

$$\operatorname{dist}(\gamma_1(t), \sigma([0,1]) \leq \operatorname{length}(\gamma_1) \leq \epsilon/2.$$

Suppose that for some $k \in \{1, \ldots, n-1\}$ we have defined $s_k > 0$ and a 2^{k-1}-bi-Lipschitz curve $\gamma_k : [0, s_k] \to X$ parameterized by arc-length, such that $\gamma_k(0) = \sigma(0)$,

$\gamma_k(s_k) = x_k$, and $\gamma_k([0, s_k]) \subset \overline{B}(\sigma([0,1]), \epsilon/2)$. Let $r_k \in [0, s_k]$ be such that

$$c_k := \operatorname{dist}(\gamma_k([0, s_k]), x_{k+1}) = d(\gamma_k(r_k), x_{k+1}).$$

Define $s_{k+1} = r_k + c_k$ and let $\gamma_{k+1} : [0, s_{k+1}] \to X$ be the concatenation of $\gamma_k|_{[0,r_k]}$ with a geodesic joining $\gamma_k(r_k)$ to $x_{k+1} \in \sigma([0,1])$. Note that

$$d(\gamma_k(r_k), x_{k+1}) \leq d(x_k, x_{k+1}) < \epsilon/2,$$

so for each $t \in [0, s_{k+1}]$, we have

$$\operatorname{dist}(\gamma_{k+1}(t), \sigma([0,1])) < \epsilon/2.$$

Moreover, by Lemma 2.6.2, the curve γ_{k+1} is 2^k-bi-Lipschitz.

By induction, we have defined a number $0 < s_n \leq n(\epsilon/2)$ and a 2^{n-1}-bi-Lipschitz curve $\gamma_n : [0, s_n] \to X$ such that $\gamma_n(0) = \sigma(0)$, $\gamma_n(s_n) = \sigma(1)$, and

$$\gamma_n([0, s_n]) \subset B(\sigma([0,1]), \epsilon).$$

The desired curve $\gamma : [0,1] \to X$ is the reparameterization $\gamma(t) = \gamma_n(s_n t)$. □

2.6.1 Proof of Proposition 2.3.5

The proof of Proposition 2.3.5 will rely on the quantitative version of Theorem 2.3.2; see Theorem 2.8.1.

We first review some elementary notions from graph theory. A *(combinatorial) graph* is a pair $G = (V, E)$ of a finite vertex set V and an edge set E which contains elements of the form $\{v, v'\}$ where $v, v' \in V$ and $v \neq v'$. A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$, $E' \subset E$, and $E' \subset V' \times V'$. A *simple path* joining $x, y \in V$ in G is a set $\gamma = \{v_0, \ldots, , v_n\} \subset V$ of distinct points such that $v_0 = x$,

$v_n = y$, and $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \dots, n\}$. A graph G is *connected* if any two distinct vertices can be joined by a simple path in G.

Lemma 2.6.3. *Given a graph $G = (V, E)$ and two distinct $v, v' \in V$, there exists a finite sequence $(v_i)_{i=1}^N$ in V such that $\{v_1, \dots, v_n\} = V$, $v_1 = v$, $v_n = v'$, for each $i \in \{1, \dots, n-1\}$, we have $\{v_i, v_{i+1}\} \in E$, and for each $e \in E$ there exists at most two $i \in \{1, \dots, n-1\}$ such that $e = \{v_i, v_{i+1}\}$.*

Proof. We will use the fact that every connected graph admits a 2-to-1 Euler tour along its edges, that is, for each vertex z there exists a finite sequence $(z_j)_{j=1}^m$ of vertices in G such that $z_1 = z_m = z$, $\{z_j, z_{j+1}\}$ is an edge for all j, and for each edge e there exists exactly two j such that $e = \{z_j, z_{j+1}\}$. See for example the Euler tour technique introduced in [TV84].

Now let G, v, v' be as in the statement. Deleting some edges from E, we may assume that G is a (combinatorial) tree, that is, for any two distinct vertices there exists unique simple path in G that connects them. Let $\tilde{V} = \{v_1, \dots, v_k\}$ be the unique such path with $v_1 = v$ and $v_k = v'$. For each $i \in \{1, \dots, k\}$, let $G_i = (V_i, E_i)$ be the maximal subgraph of G with the property that any simple path connecting a vertex of G_i with a vertex of $\tilde{V}$ must contain v_i. Since G is connected, it follows that each G_i is connected. Moreover, since G is a tree, for any $i \neq j$ the graphs G_i and G_j are trees with mutually disjoint vertices (and hence edges).

The construction of the finite sequence $(v_i)_i$ is as follows. Firstly, do a 2-to-1 tour of G_1 starting and ending on v_1. Then proceed to v_2 and do a 2-to-1 tour of G_2 starting and ending on v_2. Continue in this way until reaching v_k where we do a 2-to-1 tour of G_k starting and ending on v_k. □

Proof of Proposition 2.3.5. If $\operatorname{diam} K = 0$, then there is nothing to prove. Assume now that $\operatorname{diam} K > 0$ and, rescaling, we may further assume that $\operatorname{diam} K = 1$.

Let Y be a maximal $(\epsilon/4)$-separated subset of K that contains x and y. By the regularity of X, the cardinality of Y is at most $C'\epsilon^{-Q}$ for some $C' > 0$ depending only on the constants of Q-regularity. Define a graph G with vertex set Y such that

two points $z, z' \in G$ are connected by an edge if and only if $d(z, z') < \epsilon/2$. Since K is connected, it follows that G is connected. By Lemma 2.6.3 there exists a tour $x = v_0, \ldots, v_n = y$ of the vertices Y such that each edge is visited at most twice.

For each $z \in Y$, denote by m_z the number of indices i such that $v_i = z$. There exists $C'' > 0$ depending only on the constants of Q-regularity such that each vertex of G is contained in at most C'' edges. Therefore, for each $z \in Y$, $m_z \leq C''$ and it follows that $n \leq C''C'\epsilon^{-Q}$. Moreover, there exists $c > 4$ depending only on the constants of Q-regularity such that for each $z \in Y$ there exist points $v_{z,1}, \ldots, v_{z,m_z} \in B(z, \epsilon/16)$ such that

$$d(v_{z,i}, v_{z,j}) \geq c^{-1}\epsilon, \quad \text{for all } z \in Y \text{ and } i \neq j.$$

We may also assume that $v_{x,1} = x$ and $v_{y,m_y} = y$.

Given $i \in \{0, \ldots, n\}$ let $j(i)$ be the number of indices $l \in \{0, \ldots, i\}$ such that $v_l = v_i$. Define now $\tilde{v}_i = v_{v_i, j(i)}$. Note that the new sequence $\tilde{v}_0, \ldots, \tilde{v}_n$ satisfies

1. $\tilde{v}_0 = x$, $\tilde{v}_n = y$,
2. for each distinct $i, j \in \{0, \ldots, n\}$ we have $d(\tilde{v}_i, \tilde{v}_j) \geq c^{-1}\epsilon$,
3. for each $z \in K$ there exists $i \in \{0, \ldots, n\}$ such that $d(z, \tilde{v}_i) \leq \epsilon/2$,
4. for each $i \in \{0, \ldots, n\}$, $\operatorname{dist}(\tilde{v}_i, K) \leq \epsilon/16$.

Define a map $f : \{i\epsilon : i = 0, \ldots, n\} \to X$ by $f(i\epsilon) = \tilde{v}_i$ and note that f is L'-bi-Lipschitz with $L' = \max\{nc, 2/\epsilon\}$. Indeed, for any distinct i, j we have $\epsilon \leq |i\epsilon - j\epsilon| \leq n\epsilon$ and $c^{-1}\epsilon \leq d(f(i\epsilon), f(j\epsilon)) \leq 1 + 2c^{-1}\epsilon < 2$.

By Theorem 2.8.1, there exists a constant L depending on ϵ, the constants of Q-regularity, and the data of the Poincaré inequality, and there exists a bi-Lipschitz arc $F : [0, n\epsilon] \to X$ that extends f and

$$\operatorname{dist}_H(K, F([0, n\epsilon])) \lesssim \epsilon.$$

The arc $\gamma : [0, 1] \to X$ in question is obtained by reparameterizing F. □

2.7 Whitney intervals and a preliminary extension

Here and for the rest of this section we assume that X is a complete geodesic (C_1, Q)-Ahlfors regular metric measure space supporting a p-Poincaré inequality with data C where $p \in (1, Q-1)$ and $C_1, C > 1$. We also assume that $A \subset \mathbb{R}$ is a closed set and that $f : A \to X$ is an L-bi-Lipschitz embedding.

Let I be the smallest closed interval with $A \subset I$ (possibly $\mathbb{R}$). We need a Whitney decomposition of $I \setminus A$ as in Whitney's classical proof of his extension theorem [Whi34]. We may assume that A is not a closed interval itself, as then there is no extension to be made.

Lemma 2.7.1 ([Ste70, Theorem VI.1.1,Proposition VI.1.1])**.** *There exists a collection of closed intervals $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$ such that*

(i) $\bigcup_{i=1}^{\infty} \mathcal{Q}_i = I \setminus A$,

(ii) the intervals $\{\mathcal{Q}_i\}$ have disjoint interiors, and

(iii) $\operatorname{diam} \mathcal{Q}_i \leq \operatorname{dist}(\mathcal{Q}_i, A) \leq 4 \operatorname{diam} \mathcal{Q}_i$ *for all* $i \in \mathbb{N}$.

Moreover, if the intervals $\mathcal{Q}_i$ and $\mathcal{Q}_j$ share an endpoint, then

$$\tfrac{1}{4} \operatorname{diam} \mathcal{Q}_j \leq \operatorname{diam} \mathcal{Q}_i \leq 4 \operatorname{diam} \mathcal{Q}_j. \tag{2.7.1}$$

Henceforth, the intervals $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$ will be called *Whitney intervals*.

2.7.1 Reference points

Let E denote the collection of endpoints of $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$. For each $x \in E$, fix a point $a_x \in A$ that is a closest point of A to x, that is, $|x - a_x| = \operatorname{dist}(x, A)$.

Proposition 2.7.2. *There exists $\xi \in (0,1)$ and $\tilde{L} > 1$ depending only on L, C_1, and Q, and there exists an $\tilde{L}$-bi-Lipschitz map $\pi : E \to X$ such that, for all distinct $x, y \in E$,*

1. $\frac{1}{4}|x - a_x| \leq d(\pi(x), f(a_x)) \leq 4|x - a_x|$

2. $\operatorname{dist}(\pi(x), f(A)) \geq \xi|x - a_x|$,

3. $d(\pi(x), \pi(y)) \geq \xi\,(|x - a_x| + |y - a_y|)$.

Moreover, if $\mathcal{Q}_i = [x, y]$, *then*

$$d(\pi(x), \pi(y)) \leq d(f(a_x), f(a_y)) + 36 \operatorname{diam} \mathcal{Q}_i \leq 46L \operatorname{diam} \mathcal{Q}_i. \tag{2.7.2}$$

We start with a result that allows us to partition E into a finite number of subsets such that elements of the same subset are far apart quantitatively. Recall that by Lemma 2.4.1, there exists $p_0 \geq 1$ depending only on L, C_1, and Q such that $f(A)$ is p_0-porous.

Lemma 2.7.3. *There exists* $n \in \mathbb{N}$ *depending only on* L, C_1, *and* Q, *and there exists a partition of* E *into mutually disjoint sets* $E_1, \ldots, E_n$ *such that for any* $i \in \{1, \ldots, n\}$ *and for any* $x, y \in E_i$, *either*

(F1) $|x - y| > (12L) \max\{|x - a_x|, |y - a_y|\}$, *or*

(F2) $\max\{|x - a_x|, |y - a_y|\} > (8p_0) \min\{|x - a_x|, |y - a_y|\}$.

Proof. Enumerate $E = \{x_1, x_2, \ldots\}$, and for each $i \in \mathbb{N}$, define V_i be the set of all indices $j \in \mathbb{N}$ such that

$$|x_i - x_j| \leq (12L) \max\{|x_i - a_{x_i}|, |x_j - a_{x_j}|\}$$

and

$$(8p_0)^{-1}|x_i - a_{x_i}| \leq |x_j - a_{x_j}| \leq (8p_0)|x_i - a_{x_i}|.$$

Note that $i \in V_j$ if and only if $j \in V_i$.

We claim that there exists $n \in \mathbb{N}$ depending only on L, C_1, and Q such that $\operatorname{card}(V_i) \leq n$ for each $i \in \mathbb{N}$. To this end, fix $i \in \mathbb{N}$ and note that for any $j, k \in V_i$

with $j \neq k$,

$$|x_j - x_k| \leq (24L)\max\{|x_j - a_{x_j}|, |x_k - a_{x_k}|, |x_i - a_{x_i}|\} \leq (192p_0L)|x_i - a_{x_i}|. \quad (2.7.3)$$

Moreover, let $j, k \in V_i$ with $j \neq k$, and let $\mathcal{Q}_{j_1}$ and $\mathcal{Q}_{j_2}$ be the two Whitney intervals which share the endpoint x_j and $\mathcal{Q}_{k_1}$ and $\mathcal{Q}_{k_2}$ be the two Whitney intervals which share the endpoint x_k. We have

$$\operatorname{dist}(\mathcal{Q}_{j_1}, A) \geq |x_j - a_{x_j}| - \operatorname{diam}\mathcal{Q}_{j_1} \geq |x_j - a_{x_j}| - \operatorname{dist}(\mathcal{Q}_{j_1}, A),$$

so $|x_j - a_{x_j}| \leq 2\operatorname{dist}(\mathcal{Q}_{j_1}, A)$. By Lemma 2.7.1(iii),

$$\operatorname{diam}\mathcal{Q}_{j_1} \geq \tfrac{1}{4}\operatorname{dist}(\mathcal{Q}_{j_1}, A) \geq \tfrac{1}{8}|x_j - a_{x_j}|,$$

and a similar estimate holds for $\mathcal{Q}_{j_2}$, $\mathcal{Q}_{k_1}$, and $\mathcal{Q}_{k_2}$. Since $x_j \neq x_k$, one of the intervals for which they are endpoints lies between them. That is,

$$\begin{aligned} |x_j - x_k| &\geq \min\{\operatorname{diam}\mathcal{Q}_{j_1}, \operatorname{diam}\mathcal{Q}_{j_2}, \operatorname{diam}\mathcal{Q}_{k_1}, \operatorname{diam}\mathcal{Q}_{k_2}\} \\ &\geq \tfrac{1}{8}\min\{|x_j - a_{x_j}|, |x_k - a_{x_k}|\} \\ &\geq (64p_0)^{-1}|x_i - a_{x_i}|. \end{aligned}$$

Combining this with (2.7.3), we conclude that $\operatorname{card}(V_i) \leq 192L(8p_0)^2 =: n$.

Define now a map $\mathbf{c} : \mathbb{N} \to \{1, \ldots, n\}$ such that $\mathbf{c}(1) = 1$, and for each $i \geq 2$,

$$\mathbf{c}(i) := \min\{\ell \in \mathbb{N} : \ell \neq \mathbf{c}(k) \text{ for all } k \in V_i \cap \{1, \ldots, i-1\}\}.$$

It is clear that if $i \in V_j$ and $i \neq j$ then $\mathbf{c}(i) \neq \mathbf{c}(j)$. For each $i \in \{1, \ldots, n\}$, define $E_i := \{x_j : \mathbf{c}(j) = i\}$. Given $x_j, x_k \in E_i$, $\mathbf{c}(i) = \mathbf{c}(j)$, so $j \notin V_k$ (equivalently, $k \notin V_j$). Properties (F1) and (F2) follow. □

We now turn to the proof of Proposition 2.7.2.

Proof of Proposition 2.7.2. Let $n \in \mathbb{N}$ and $E_1, \ldots, E_n$ be the integer and partition, respectively, from Lemma 2.7.3. For each $k \in \{1, \ldots, n\}$, define $E^{(k)} = E_1 \cup \cdots \cup E_k$.

Let $i \in \{1, \ldots, n\}$, $x \in E_i$, and $x' \in \partial B(f(a_x), |x - a_x|)$. By the porosity of $f(A)$ there exists a point $\tilde{x} \in X$ such that

$$B(\tilde{x}, (2p_0)^{-1}|x - a_x|) \subset B\left(x', \tfrac{1}{2}|x - a_x|\right) \setminus f(A).$$

Then

$$\tfrac{1}{2}|x - a_x| \leq d(\tilde{x}, f(a_x)) \leq \tfrac{3}{2}|x - a_x| \tag{2.7.4}$$

and

$$\operatorname{dist}(\tilde{x}, f(A)) \geq (2p_0)^{-1}|x - a_x|. \tag{2.7.5}$$

For any $i \in \{1, \ldots, n\}$, and for any $x, y \in E_i$, we will show that

$$d(\tilde{x}, \tilde{y}) \geq (8p_0)^{-1} \left(|x - a_x| + |y - a_y|\right). \tag{2.7.6}$$

Fix such i, x, and y, and assume without loss of generality that $|x - a_x| \geq |y - a_y|$. If (F1) holds, then by (2.7.4),

$$\begin{aligned} d(\tilde{x}, \tilde{y}) &\geq d(f(a_x), f(a_y)) - d(f(a_x), \tilde{x}) - d(f(a_y), \tilde{y}) \\ &\geq L^{-1}|a_x - a_y| - \tfrac{3}{2}(|x - a_x| + |y - a_y|) \geq L^{-1}|x - y| - 6|x - a_x| \\ &> 6|x - a_x| \geq 3(|x - a_x| + |y - a_y|). \end{aligned}$$

Assume now that (F2) holds and (F1) fails. By (2.7.4) and (2.7.5),

$$\begin{aligned} d(\tilde{x}, \tilde{y}) &\geq d(\tilde{x}, f(a_y)) - d(f(a_y), \tilde{y}) \geq (2p_0)^{-1}|x - a_x| - \tfrac{3}{2}|y - a_y| \\ &\geq (8p_0)^{-1}(|x - a_x| + |y - a_y|). \end{aligned}$$

We define the map π on each $E^{(k)}$ in an inductive manner. Define $\pi : E_1 \to X$ by $\pi(x) = \tilde{x}$. Properties (1)–(3) of the proposition for E_1 follow from (2.7.4), (2.7.5), and (2.7.6) with $\xi_1 = (8p_0)^{-1}$.

Assume now that for some $k \in \{1, \dots, n-1\}$ we have defined a constant $\xi_k \in (0, 1)$ and a function $\pi : E^{(k)} \to X$ such that for all distinct $x, y \in E^{(k)}$,

$$\tfrac{1}{4}|x - a_x| \le d(\pi(x), f(a_x)) \le 4|x - a_x|, \tag{2.7.7}$$

$$\operatorname{dist}(\pi(x), f(A)) \ge (4p_0)^{-1}|x - a_x|, \tag{2.7.8}$$

and

$$d(\pi(x), \pi(y)) \ge \xi_k(|x - a_x| + |y - a_y|). \tag{2.7.9}$$

Fix $x \in E_{k+1}$, and assume that there exist $y_1, \dots, y_N \in E^{(k)}$ such that

$$d(\tilde{x}, \pi(y_j)) < (8p_0)^{-1}(|x - a_x| + |y_j - a_{y_j}|) \tag{2.7.10}$$

for each $j \in \{1, \dots, N\}$. First, for each such j, by (2.7.7), (2.7.5), and (2.7.10),

$$\begin{aligned} |y_j - a_{y_j}| &\ge \tfrac{1}{4} d(\pi(y_j), f(a_{y_j})) \ge \tfrac{1}{4}\left(d(\tilde{x}, f(a_{y_j})) - d(\tilde{x}, \pi(y_j))\right) \\ &\ge \tfrac{1}{4}\left((2p_0)^{-1}|x - a_x| - (8p_0)^{-1}(|x - a_x| + |y_j - a_{y_j}|)\right). \end{aligned}$$

This gives

$$|y_j - a_{y_j}| \ge (12p_0)^{-1}|x - a_x|. \tag{2.7.11}$$

Next, for any $j, \ell \in \{1, \dots, N\}$, (2.7.9) and (2.7.11) yield

$$d(\pi(y_j), \pi(y_\ell)) \ge (12p_0)^{-1}\xi_k|x - a_x|, \tag{2.7.12}$$

so $\{\pi(y_1), \dots, \pi(y_N)\}$ is a $((12p_0)^{-1}\xi_k|x-a_x|)$-separated set. By (2.7.4), (2.7.8), and (2.7.10), for any $j \in \{1, \dots, N\}$,

$$\begin{aligned} |x-a_x| &\geq \tfrac{2}{3} d(\tilde{x}, f(a_x)) \geq \tfrac{2}{3} \left(d(\pi(y_j), f(a_x)) - d(\tilde{x}, \pi(y_j)) \right) \\ &\geq (6p_0)^{-1}|y_j - a_{y_j}| - (12p_0)^{-1}(|x-a_x| + |y_j - a_{y_j}|). \end{aligned}$$

Therefore,

$$|y_j - a_{y_j}| \leq 24p_0|x-a_x|. \tag{2.7.13}$$

By Ahlfors regularity, (2.7.10), (2.7.13), and (2.7.12), we obtain $N \lesssim_{L,C_1,Q} \xi_k^{-Q}$.

By Lemma 2.4.1, $\{\pi(y_1), \dots, \pi(y_N)\}$ is p_k-porous for some $p_k \geq 1$ depending only on C_1, Q, L, and k. Hence, there exists a point $\pi(x) \in B(\tilde{x}, (32p_0)^{-1}|x-a_x|)$ such that

$$B(\pi(x), (32p_0p_k)^{-1}|x-a_x|) \subset X \setminus \{\pi(y_1), \dots, \pi(y_N)\}. \tag{2.7.14}$$

To complete the inductive step, we show that π defined on $E^{(k+1)}$ satisfies properties (1)–(3) of the proposition for some appropriate $\xi_{k+1} \in (0,1)$.

For the first property, fix $x \in E_{k+1}$. By (2.7.4),

$$d(\pi(x), f(a_x)) \leq d(\pi(x), \tilde{x}) + d(f(a_x), \tilde{x}) \leq 2|x-a_x|$$

and

$$d(\pi(x), f(a_x)) \geq d(f(a_x), \tilde{x}) - d(\pi(x), \tilde{x}) \geq \tfrac{1}{4}|x-a_x|.$$

For the second property, fix $x \in E_{k+1}$. By (2.7.5)

$$\operatorname{dist}(\pi(x), f(A)) \geq \operatorname{dist}(\tilde{x}, f(A)) - d(\pi(x), \tilde{x}) \geq (4p_0)^{-1}|x-a_x|.$$

For the third property, fix distinct $x, y \in E^{(k+1)}$ and assume that $x \in E_{k+1}$. If $y \in E_{k+1}$, then by (2.7.6)

$$d(\pi(x), \pi(y)) \geq d(\tilde{x}, \tilde{y}) - d(\pi(x), \tilde{x}) - d(\pi(y), \tilde{y}) \geq (16p_0)^{-1}(|x - a_x| + |y - a_y|).$$

Assume now that $y \in E^{(k)}$. If

$$d(\tilde{x}, \tilde{y}) \geq (8p_0)^{-1} \left(|x - a_x| + |y - a_y|\right),$$

then we work as in the preceding case. If

$$d(\tilde{x}, \tilde{y}) < (8p_0)^{-1} \left(|x - a_x| + |y - a_y|\right),$$

then by (2.7.13) and (2.7.14),

$$d(\pi(x), \pi(y)) \geq (32p_0p_k)^{-1}|x - a_x| \geq (2^9 3p_0^2 p_k)^{-1} \left(|x - a_x| + |y - a_y|\right).$$

After n steps, we have defined $\xi := \xi_n$ and the map $\pi : E \to X$ that satisfies properties (1)–(3) in the statement of the lemma.

To show that π is bi-Lipschitz, fix distinct $x, y \in E$ and assume without loss of generality that $|x - a_x| \geq |y - a_y|$. By Lemma 2.7.1(iii), $|x - a_x| \leq 4|x - y|$. By property (3),

$$\begin{aligned} d(\pi(x), \pi(y)) &\leq d(\pi(x), f(a_x)) + d(f(a_x), f(a_y)) + d(\pi(y), f(a_y)) \\ &\leq 4|x - a_x| + L|a_x - a_y| + 4|y - a_y| \leq 41L|x - y|. \end{aligned}$$

For the lower bound, suppose first that $|a_x - a_y| \geq 16L|x - a_x|$. Then, $|x - y| \leq 2|a_x - a_y|$ and by property (1)

$$d(\pi(x), \pi(y)) \geq d(f(a_x), f(a_y)) - d(\pi(x), f(a_x)) - d(\pi(y), f(a_y)) \geq (2L)^{-1}|a_x - a_y|.$$

Suppose now that $|a_x - a_y| \leq 16L|x - a_x|$. Then, $|x - y| \leq (2 + 16L)|x - a_x|$ and by property (3), $d(\pi(x), \pi(y)) \geq \xi|x - a_x|$.

For (2.7.2), fix a Whitney interval $\mathcal{Q}_i = [x, y]$ and assume, without loss of generality, that $|x - a_x| \leq |y - a_y|$. There are two cases to consider. Assume first that $a_x = a_y$. By property (1),

$$d(\pi(x), \pi(y)) \leq 4|x - a_x| + 4|y - a_x| \leq 8|x - a_x| + 4 \operatorname{diam} \mathcal{Q}_i \leq 36 \operatorname{diam} \mathcal{Q}_i.$$

Assume now that $a_x \neq a_y$. Then

$$|y - a_y| \leq |y - a_x| \leq |x - y| + |x - a_x| \leq 5 \operatorname{diam} \mathcal{Q}_i$$

which yields that $|a_x - a_y| \leq 11 \operatorname{diam} \mathcal{Q}_i$. By property (1),

$$\begin{aligned} d(\pi(x), \pi(y)) &\leq 4|x - a_x| + d(f(a_x), f(a_y)) + 4|y - a_y| \\ &\leq d(f(a_x), f(a_y)) + 40 \operatorname{diam} \mathcal{Q}_i \\ &\leq L|a_x - a_y| + 40 \operatorname{diam} \mathcal{Q}_i \leq 51L \operatorname{diam} \mathcal{Q}_i. \end{aligned}$$

□

2.7.2 The middle third of each Whitney interval

The goal of this subsection is to extend f to the union of the middle-thirds of all Whitney intervals $\{\mathcal{Q}_i\}_{i \in \mathbb{N}}$ in a bi-Lipschitz way. From here on, for each Whitney interval $\mathcal{Q}_i$, we denote by $\hat{\mathcal{Q}}_i$ the middle third interval of $\mathcal{Q}_i$. Recall the constants $\xi \in (0, 1)$ and $\tilde{L}$ from Proposition 2.7.2 depending only on L, C_1, and Q.

Proposition 2.7.4. *There exists a constant $\hat{L} \geq 1$ depending only on p, C, C_1, L, and Q, and there exists an $\hat{L}$-bi-Lipschitz extension of f*

$$g : A \cup \bigcup_{i \in \mathbb{N}} \hat{\mathcal{Q}}_i \to X$$

such that for each $i \in \mathbb{N}$, if $\mathcal{Q}_i = [w, z]$, and $\hat{\mathcal{Q}}_i = [\hat{w}, \hat{z}]$, then

1. $d(g(\hat{w}), \pi(w)) \leq (2^8\tilde{L})^{-1}\xi \operatorname{diam} \mathcal{Q}_i$,

2. $d(g(\hat{z}), \pi(z)) \leq (2^8\tilde{L})^{-1}\xi \operatorname{diam} \mathcal{Q}_i$, *and*

3. $g(\hat{\mathcal{Q}}_i) \subset B(\pi(w), 4R_i) \cap B(\pi(z), 4R_i)$, *where* $R_i = d(\pi(w), \pi(z))$.

Recall that, since f is bi-Lipschitz, the set $f(A)$ is 1-homogeneous in X.

Lemma 2.7.5. *There exist constants $\beta_0, \ell_0, \delta_0 > 0$ depending only on p, C, C_1, L, and Q with the following property. Let $\mathcal{Q}_i = [w, z]$ be a Whitney interval and Γ_i be the collection of curves $\gamma : [0, 1] \to X$ such that*

1. $\gamma([0, 1]) \subset B(\pi(w), 3R_i) \cap B(\pi(z), 3R_i)$ *where* $R_i = d(\pi(w), \pi(z))$,

2. $\max\{d(\gamma(0), \pi(w)), d(\gamma(1), \pi(z))\} < (2^8\tilde{L})^{-1}\xi \operatorname{diam} \mathcal{Q}_i$,

3. $\operatorname{length}(\gamma) \leq \ell_0 \operatorname{diam} \mathcal{Q}_i$,

4. $\operatorname{dist}(\gamma(t), f(A)) \geq \delta_0 \operatorname{diam} \mathcal{Q}_i$ *for all* $t \in [0, 1]$.

Then,

$$\operatorname{Mod}_p(\Gamma_i) \geq \beta_0(\operatorname{diam} \mathcal{Q}_i)^{Q-p}.$$

Proof. Since $B(\pi(w), 2R_i) \subset B(\pi(w), 3R_i) \cap B(\pi(z), 3R_i)$, we may apply Lemma 2.5.1, Proposition 2.7.2(3), and (2.7.2) to conclude that the family $\Gamma_i^{(1)}$ of curves

$$\gamma : [0, 1] \to B(\pi(w), 3R_i) \cap B(\pi(z), 3R_i)$$

such that $\gamma(0)$ lies in the closed ball $\overline{B}(\pi(w), (2^8\tilde{L})^{-1}\xi \operatorname{diam} \mathcal{Q}_i)$ and $\gamma(1)$ lies in the closed ball $\overline{B}(\pi(z), (2^8\tilde{L})^{-1}\xi \operatorname{diam} \mathcal{Q}_i)$ has p-modulus

$$\operatorname{Mod}_p(\Gamma_i^{(1)}) \geq \alpha(\operatorname{diam} \mathcal{Q}_i)^{Q-p}$$

where $\alpha > 0$ is some constant depending only on p, C, C_1, Q, and L.

By Lemma 2.5.2, there exists $\ell_0 > 0$ depending only on p, C, C_1, Q, and L such that the subfamily

$$\Gamma_i^{(2)} := \left\{ \gamma \in \Gamma_i^{(1)} : \operatorname{length}(\gamma) \leq \ell_0 \operatorname{diam} \mathcal{Q}_i \right\}$$

satisfies

$$\operatorname{Mod}_p(\Gamma_i^{(2)}) \geq \tfrac{1}{2}\alpha(\operatorname{diam} \mathcal{Q}_i)^{Q-p}.$$

By Lemma 2.5.3, there exists $\delta_0 > 0$ depending only on Q, p, C, C_1, and L such that the subfamily

$$\Gamma_i := \left\{ \gamma \in \Gamma_i^{(2)} : \operatorname{dist}(\gamma(t), f(A)) \geq \delta_0 \operatorname{diam} \mathcal{Q}_i \text{ for each } t \in [0,1] \right\},$$

satisfies

$$\operatorname{Mod}_p(\Gamma_i) \geq \tfrac{1}{4}\alpha(\operatorname{diam} \mathcal{Q}_i)^{Q-p}. \qquad \square$$

We now need a filtration of the Whitney decomposition, in the vein of the following result of David and Semmes. The proof of the lemma is almost identical to that of Lemma 2.7.3 and is left to the reader.

Lemma 2.7.6 ([DS91, Proposition 17.4])**.** *There exists an integer N depending only on L, C_1, and Q, and there exists a partition of $\mathbb{N}$ into sets $\{\mathscr{I}_1, \ldots, \mathscr{I}_N\}$ such that for any $k \in \{1, \ldots, N\}$ and for any $i, j \in \mathscr{I}_k$, either*

(i) $\operatorname{dist}(\mathcal{Q}_i, \mathcal{Q}_j) > 800L^2 \max\{\operatorname{diam} \mathcal{Q}_i, \operatorname{diam} \mathcal{Q}_j\}$*, or*

(ii) $\max\{\operatorname{diam} \mathcal{Q}_i, \operatorname{diam} \mathcal{Q}_j\} > 800L\delta_0^{-1} \min\{\operatorname{diam} \mathcal{Q}_i, \operatorname{diam} \mathcal{Q}_j\}$.

We are now ready to prove Proposition 2.7.4.

Proof of Proposition 2.7.4. The construction is in an inductive fashion. Let N and $\mathscr{I}_1, \ldots, \mathscr{I}_N$ be the integer and sets of indices from Lemma 2.7.6. Denote $A_0 := A$

and for each $k \in \{1, \dots, N\}$ denote

$$A_k := A_0 \cup \bigcup_{j=1}^{k} \bigcup_{i \in \mathscr{I}_j} \hat{\mathcal{Q}}_i.$$

For each $k \in \{0, \dots, N\}$, we find some $L_k \geq 1$ depending only on p, C, C_1, L, Q, and k, and we find an L_k-bi-Lipschitz embedding $f_k : A_k \to X$ such that for all $k \in \{1, \dots, N\}$, $f_k|_{A_{k-1}} = f_{k-1}$ and such that, if $i \in \mathscr{I}_k$, $\mathcal{Q}_i = [w, z]$, and $\hat{\mathcal{Q}}_i = [\hat{w}, \hat{z}]$, then

(a) $d(f_k(\hat{w}), \pi(w)) \leq (2^8 \tilde{L})^{-1} \xi \operatorname{diam} \mathcal{Q}_i$,

(b) $d(f_k(\hat{z}), \pi(z)) \leq (2^8 \tilde{L})^{-1} \xi \operatorname{diam} \mathcal{Q}_i$,

(c) $f_k(\hat{\mathcal{Q}}_i) \subset B(\pi(w), 4R_i) \cap B(\pi(z), 4R_i)$ where $R_i = d(\pi(w), \pi(z))$.

The map g of Proposition 2.7.4 will then be the map f_N.

For $k = 0$, set $L_0 = L$ and $f_0 = f$. Properties (a)–(c) are vacuous.

Assume now that for some $k \in \{0, \dots, N-1\}$, there exists a constant L_k and an L_k-bi-Lipschitz map $f_k : A_k \to X$ satisfying (a)–(c).

Fix $i \in \mathscr{I}_{k+1}$ and write $\mathcal{Q}_i = [w, z]$ and $\hat{\mathcal{Q}}_i = [\hat{w}, \hat{z}]$. Recall the family of curves Γ_i from Lemma 2.7.5. By Lemma 2.5.3, there exists $\delta_{k+1} \in (0, \xi)$ depending only on Q, p, C, C_1, L, and k (in particular on the homogeneity constant of $f(A_k)$) such that the subfamily

$$\Gamma'_{k,i} := \{\gamma \in \Gamma_i : \operatorname{dist}(\gamma(t), f_k(A_k)) \geq \delta_{k+1} \operatorname{diam} \mathcal{Q}_i \text{ for each } t \in [0,1]\},$$

satisfies

$$\operatorname{Mod}_p(\Gamma'_{k,i}) \geq \tfrac{1}{2} \beta_0 (\operatorname{diam} \mathcal{Q}_i)^{Q-p} > 0.$$

In particular, $\Gamma'_{k,i}$ is nonempty, so we can pick a curve $\sigma_i \in \Gamma'_{k,i}$. Applying Lemma 2.3.5 to σ_i with a suitable reparameterization, we find a constant L'_{k+1} depending only on Q, p, C, C_1, L, and k, and we find an L'_{k+1}-bi-Lipschitz curve $\gamma_i : \hat{\mathcal{Q}}_i \to X$ such that

$\gamma_i(\hat{w}) = \sigma_i(0)$, $\gamma_i(\hat{z}) = \sigma_i(1)$, and inductive hypothesis (c) for f_k gives

$$\begin{aligned}\gamma_i(\hat{\mathcal{Q}}_i) &\subset B\left(\sigma_i([0,1]), \tfrac{1}{2}\delta_{k+1}\operatorname{diam}\mathcal{Q}_i\right) \\ &\subset B(\pi(w), 3R_i + \tfrac{1}{2}\delta_{k+1}\operatorname{diam}\mathcal{Q}_i) \cap B(\pi(z), 3R_i + \tfrac{1}{2}\delta_{k+1}\operatorname{diam}\mathcal{Q}_i) \\ &\subset B(\pi(w), 4R_i) \cap B(\pi(z), 4R_i).\end{aligned} \tag{2.7.15}$$

In particular, we have that $\operatorname{dist}(\gamma_i(\hat{\mathcal{Q}}_i), f_k(A_k)) \geq \frac{1}{2}\delta_{k+1}\operatorname{diam}\mathcal{Q}_i$.

Define now $f_{k+1} : A_{k+1} \to X$ by setting $f_{k+1}|A_k = f_k$ and $f_{k+1}|\hat{\mathcal{Q}}_i = \gamma_i$ for each $i \in \mathscr{I}_{k+1}$. By (2.7.2) we have for all $i \in \mathscr{I}_{k+1}$

$$\operatorname{diam} f_{k+1}(\hat{\mathcal{Q}}_i) \leq 9R_i \leq 414L\operatorname{diam}(\mathcal{Q}_i). \tag{2.7.16}$$

Clearly, $f_{k+1}|A_k = f_k$. Properties (a)–(c) are clear from the design of f_{k+1} and Lemma 2.7.5. To complete the inductive step, we claim that f_{k+1} is L_{k+1}-bi-Lipschitz for some $L_{k+1} \geq 1$ depending only on Q, p, C, C_1, L, and k. Fix $x, y \in A_{k+1}$.

Firstly, if $x, y \in A_k$, then the claim follows by the fact that $f_{k+1}|A_k = f_k$ and the inductive hypothesis that f_k is L_k-bi-Lipschitz.

Secondly, assume that $x \in \hat{\mathcal{Q}}_i$ for some $i \in \mathscr{I}_{k+1}$ and $y \in A$. Let w be the endpoint of $\mathcal{Q}_i$ closest to A, let $\hat{w}$ be the endpoint of $\hat{\mathcal{Q}}_i$ between x and w, and note that $|w - x| \leq |x - a_w| \leq |x - y|$. By (2.7.16), Proposition 2.7.2(1), the fact $\operatorname{diam}\mathcal{Q}_i \leq |w - a_w|$, and properties (a), (b) for f_{k+1}.

$$\begin{aligned}&d(f_{k+1}(x), f_{k+1}(y)) \\ &\quad\leq d(f_{k+1}(x), f_{k+1}(\hat{w})) + d(f_{k+1}(\hat{w}), \pi(w)) + d(\pi(w), f(a_w)) + d(f(a_w), f(y)) \\ &\quad\leq (414L + 5)|w - a_w| + L|a_w - y| \\ &\quad\leq (414L + 5)|x - a_w| + L|a_w - y| \\ &\quad\leq (416L + 5)|x - y|.\end{aligned}$$

For the lower bound, we have by Lemma 2.7.5(4) and the design of γ_i

$$d(f_{k+1}(x), f_{k+1}(y)) \geq \operatorname{dist}(f_{k+1}(x), f(A)) \geq \tfrac{1}{2}\delta_0 \operatorname{diam} \mathcal{Q}_i \geq \tfrac{1}{8}\delta_0 |w - a_w|,$$

and, by (2.7.16), property (c) for f_{k+1}, and Proposition 2.7.2(2)

$$\begin{aligned} d(f_{k+1}(x), f(a_w)) &\leq d(f_{k+1}(x), f_{k+1}(\hat{w})) + d(f_{k+1}(\hat{w}), \pi(w)) + d(f(a_w), \pi(w)) \\ &\leq 414L \operatorname{diam} \mathcal{Q}_i + (2^8\tilde{L})^{-1}\xi \operatorname{diam} \mathcal{Q}_i + 4|w - a_w| \\ &\leq 419L|w - a_w|. \end{aligned}$$

Therefore, since $|x - a_w| \leq 2|w - a_w|$,

$$\begin{aligned} |x - y| &\leq |x - a_w| + |a_w - y| \\ &\leq 2|w - a_w| + L[d(f(a_w), f_{k+1}(x)) + d(f_{k+1}(x), f(y))] \\ &\leq 419L^2|w - a_w| + Ld(f_{k+1}(x), f(y)) \\ &\leq 3352L^2\delta_0^{-1} d(f_{k+1}(x), f_{k+1}(y)) \end{aligned}$$

Thirdly, assume that $x \in \hat{\mathcal{Q}}_i$ and $y \in \hat{\mathcal{Q}}_j$, for some $i, j \in \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_{k+1}$. Assume that $\operatorname{diam} \mathcal{Q}_i \geq \operatorname{diam} \mathcal{Q}_j$. For the upper bound, note that

$$|x - y| \geq \operatorname{dist}(\hat{\mathcal{Q}}_i, \hat{\mathcal{Q}}_j) \geq \operatorname{diam} \hat{\mathcal{Q}}_i + \operatorname{diam} \hat{\mathcal{Q}}_j = \tfrac{1}{3}\left(\operatorname{diam} \mathcal{Q}_i + \operatorname{diam} \mathcal{Q}_j\right). \tag{2.7.17}$$

Let a_i be the closest point of A to $\mathcal{Q}_i$, let a_j be the closest point of A to $\mathcal{Q}_j$, let e_i be the endpoint of $\mathcal{Q}_i$ that lies between x and a_i, and let e_j be the endpoint of $\mathcal{Q}_j$ that lies between y and a_j. By Proposition 2.7.2(1), (2.7.15), (2.7.2), Lemma 2.7.1(iii), and (2.7.17),

$$\begin{aligned} d(f_{k+1}(x), f_{k+1}(y)) \leq\ & d(f_{k+1}(x), \pi(e_i)) + d(\pi(e_i), f(a_i)) + d(f(a_i), f(a_j)) \\ & + d(f(a_j), \pi(e_j)) + d(\pi(e_j), f_{k+1}(y)) \end{aligned}$$

$$\begin{aligned}
&\le (16+184L)(\operatorname{diam}\mathcal{Q}_i+\operatorname{diam}\mathcal{Q}_j)+L|a_i-a_j|\\
&\le (16+189L)(\operatorname{diam}\mathcal{Q}_i+\operatorname{diam}\mathcal{Q}_j)+L|x-y|\\
&\le 616L|x-y|
\end{aligned}$$

since $|x-a_i|\le \operatorname{diam}\mathcal{Q}_i+|e_i-a_i|\le 5\operatorname{diam}\mathcal{Q}_i$ and, similarly, $|y-a_j|\le 5\operatorname{diam}\mathcal{Q}_j$.

For the lower bound, there are two cases to consider.

Case 1: $\operatorname{dist}(\mathcal{Q}_i,\mathcal{Q}_j) > 800L^2\operatorname{diam}\mathcal{Q}_i$. By Proposition 2.7.2(1), (2.7.15), and Lemma 2.7.1(iii),

$$\begin{aligned}
d(f_{k+1}(x),f_{k+1}(y)) &\ge d(f(a_i),f(a_j))-d(f(a_i),\pi(e_i))-d(\pi(e_i),f_{k+1}(x))\\
&\qquad -d(f(a_j),\pi(e_j))-d(\pi(e_j),f_{k+1}(y))\\
&\ge L^{-1}|a_i-a_j|-(184L+16)(\operatorname{diam}\mathcal{Q}_i+\operatorname{diam}\mathcal{Q}_j)\\
&\ge L^{-1}|x-y|-L^{-1}(|x-a_i|+|a_j-y|)-400L\operatorname{diam}\mathcal{Q}_i\\
&\ge L^{-1}|x-y|-(10L^{-1}+400L)\operatorname{diam}\mathcal{Q}_i\\
&> L^{-1}|x-y|-410L(800L^2)^{-1}\operatorname{dist}(\mathcal{Q}_i,\mathcal{Q}_j)\\
&\ge (3L)^{-1}|x-y|.
\end{aligned}$$

Case 2: $\operatorname{dist}(\mathcal{Q}_i,\mathcal{Q}_j)\le 800L^2\operatorname{diam}\mathcal{Q}_i$. In this case, we have

$$|x-y|\le \operatorname{diam}\mathcal{Q}_i+\operatorname{dist}(\mathcal{Q}_i,\mathcal{Q}_j)+\operatorname{diam}\mathcal{Q}_j\le 802L^2\operatorname{diam}\mathcal{Q}_i.$$

Case 2 splits now into two subcases.

Case 2.1: $i\in\mathscr{I}_{k+1}$ and $j\in\mathscr{I}_1\cup\cdots\cup\mathscr{I}_k$. According to the line following (2.7.15),

$$\begin{aligned}
d(f_{k+1}(x),f_{k+1}(y))\ge d(f_{k+1}(x),f_k(A_k)) &\ge \tfrac{1}{2}\delta_{k+1}\operatorname{diam}\mathcal{Q}_i\\
&\ge \delta_{k+1}(1604L^2)^{-1}|x-y|.
\end{aligned}$$

Case 2.2: $i, j \in \mathscr{I}_{k+1}$. By Lemma 2.7.6 we have that $\operatorname{diam} \mathcal{Q}_i > 800L\delta_0^{-1} \operatorname{diam} \mathcal{Q}_j$. By Lemma 2.7.5(4), the design of γ_i, Proposition 2.7.2(1), and (2.7.15),

$$\begin{aligned} d(f_{k+1}(x), f_{k+1}(y)) &\geq \operatorname{dist}(f_{k+1}(\hat{\mathcal{Q}}_i), f_{k+1}(y)) \\ &\geq \operatorname{dist}(f_{k+1}(\hat{\mathcal{Q}}_i), f(a_j)) - d(\pi(e_j), f(a_j)) - d(\pi(e_j), f_{k+1}(y)) \\ &\geq \tfrac{1}{2}\delta_0 \operatorname{diam} \mathcal{Q}_i - (16 + 184L) \operatorname{diam} \mathcal{Q}_j \\ &\geq \tfrac{1}{4}\delta_0 \operatorname{diam} \mathcal{Q}_i \geq \tfrac{1}{4}\delta_0 (802L^2)^{-1} |x - y|. \end{aligned}$$

□

2.8 Proof of Theorem 2.3.2

In this section, we will give the proof of the following quantitative version of Theorem 2.3.2.

Theorem 2.8.1. *Given $C, C_1 > 0$, $Q > 2$, $p \in (1, Q-1)$, and $L \geq 1$, there exists $L' \geq 1$ with the following property.*

Let (X, d, μ) be a complete geodesic (C_1, Q)-Ahlfors regular metric measure space supporting a p-Poincaré inequality with data C. Let $A \subset \mathbb{R}$ be a closed set, let I be the smallest closed interval of $\mathbb{R}$ containing A, and let $f : A \to X$ be an L-bi-Lipschitz embedding. Then there exists an L'-bi-Lipschitz extension $F : I \to X$ of f.

Moreover, if (x, y) is a component of $I \setminus A$, then

$$\operatorname{diam} F([x, y]) \leq 75 \max\{|x - y|, d(f(x), f(y))\}. \tag{2.8.1}$$

The remainder of this section is devoted to the proof of this theorem. Let $\{\mathcal{Q}_i\}_{i \in \mathbb{N}}$ be the Whitney decomposition of $I \setminus A$ from Lemma 2.7.1 and let

$$\hat{A} := A \cup \bigcup_{i \in \mathbb{N}} \hat{\mathcal{Q}}_i.$$

Recall that $\hat{\mathcal{Q}}_i$ denotes the middle third of the Whitney interval $\mathcal{Q}_i$ and that E denotes the set of endpoints of Whitney intervals $\{\mathcal{Q}_i\}_{i \in \mathbb{N}}$.

There is a map $\pi : E \to X$ satisfying the properties of Proposition 2.7.2, and there exist a constant $\hat{L} \geq 1$ depending only on C, C_1, Q, p, and L, and there exists an $\hat{L}$-bi-Lipschitz extension

$$g : \hat{A} \to X$$

of f satisfying the properties outlined in Proposition 2.7.4. In particular, if (x, y) is a component of $I \setminus A$, if $\mathcal{Q}_i \subset (x, y)$, and if x is the closest point of A to $\mathcal{Q}_i$, then by (2.7.2) and (2.7.15),

$$\max_{z \in \hat{\mathcal{Q}}_i} d(f(x), g(z)) \leq 2d(f(x), f(y)) + 73 \operatorname{diam} \mathcal{Q}_i. \tag{2.8.2}$$

We introduce several pieces of notation. Given $x \in E$, we denote by $\mathcal{L}_x$ (resp. $\mathcal{R}_x$) the Whitney interval for which x is the right (resp. left) endpoint. As above, $\hat{\mathcal{L}}_x$ and $\hat{\mathcal{R}}_x$ are the middle thirds of intervals $\mathcal{L}_x$ and $\mathcal{R}_x$. By (2.7.1), for any $x \in E$ we have

$$\tfrac{1}{4} \operatorname{diam} \mathcal{L}_x \leq \operatorname{diam} \mathcal{R}_x \leq 4 \operatorname{diam} \mathcal{L}_x.$$

Further, for any $x \in E$ we write

$$\mathcal{L}_x = [x_L, x], \quad \hat{\mathcal{L}}_x = [\tau_x^1, \tau_x^2], \quad \mathcal{R}_x = [x, x_R], \quad \hat{\mathcal{R}}_x = [\tau_x^3, \tau_x^4].$$

See Figure 2.1 below.

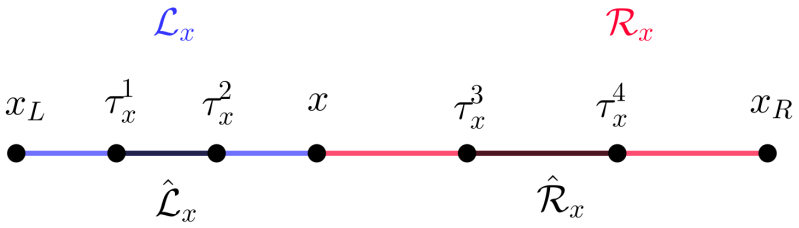

Figure 2.1: The relative positions of τ_x^1, τ_x^2, τ_x^3, and τ_x^4.

Since g is $\hat{L}$-bi-Lipschitz, there exists $C_2 > 0$ depending only on C, C_1, Q, p, and L such that the set $g(\hat{A})$ (and each of its subsets) is $(C_2, 1)$-homogeneous.

2.8.1 Local modifications around points in E

We divide E into two sets E' and E'' such that for any two points in E' there exists a point in E'' between them and vice-versa. That is, for any $x \in E'$ we have $x_L, x_R \in E''$, and for any $x \in E''$ we have $x_L, x_R \in E'$.

We perform local modifications around points in E starting with points in E'.

Local modifications around points in E'

Fix a point $x \in E'$. By the $(C_2, 1)$-homogeneity of $g(\hat{A} \setminus (\hat{\mathcal{L}}_x \cup \hat{\mathcal{R}}_x))$, by Corollary 2.5.4 and Proposition 2.7.4(1,2), there exists a constant $C' \geq 1$ depending only on C, C_1, Q, p, L, and there exists a curve $\sigma_x : [0,1] \to X$ such that

1. $\sigma_x(0) = g(\tau_x^2)$, $\sigma_x(1) = g(\tau_x^3)$,

2. $\sigma_x([0,1]) \subset B(g(\tau_x^2), 2d(g(\tau_x^2), g(\tau_x^3)))$, so for each $t \in [0,1]$

$$\begin{aligned} d(\sigma_x(t), \pi(x)) &\leq 2d(g(\tau_x^2), g(\tau_x^3)) + d(g(\tau_2^x), \pi(x)) \\ &\leq 5(2^8\tilde{L})^{-1}\xi \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\} \end{aligned}$$

3. $\operatorname{length}(\sigma_x) \leq C' \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}$

4. $\operatorname{dist}(\sigma_x([0,1]), g(\hat{A} \setminus (\hat{\mathcal{L}}_x \cup \hat{\mathcal{R}}_x))) \geq (C')^{-1} \min\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}$.

By Lemma 2.3.5, there exists $L^* > 1$ depending only on C, C_1, Q, p, and L, and there exists an L^*-bi-Lipschitz map

$$\gamma_x : [\tau_x^2, \tau_x^3] \to B\left(\pi(x), 6(2^8\tilde{L})^{-1}\xi \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}\right)$$

such that $\gamma_x(\tau_x^2) = \sigma_x(0) = g(\tau_x^2)$, $\gamma_x(\tau_x^3) = \sigma_x(1) = g(\tau_x^3)$, and for all $t \in [\tau_x^2, \tau_x^3]$,

$$\operatorname{dist}(\gamma_x(t), \sigma_x([0,1])) \leq (2^{11} C' \tilde{L})^{-1} \xi \max\{\operatorname{diam} \mathcal{L}_x, \operatorname{diam} \mathcal{R}_x\}.$$

In particular,

$$\operatorname{dist}\left(\gamma_x([\tau_x^2, \tau_x^3]), g(\hat{A} \setminus (\mathcal{L}_x \cup \mathcal{R}_x))\right) \geq (2C')^{-1} \max\{\operatorname{diam} \mathcal{L}_x, \operatorname{diam} \mathcal{R}_x\}. \tag{2.8.3}$$

Set $\epsilon = (2^{50} \tilde{L} \hat{L} L^* C')^{-2} \xi$. Define

$$t_x^1 = \min\{t \in [\tau_x^1, \tau_x^2] : \operatorname{dist}(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \epsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\}$$

and

$$t_x^2 = \max\{t \in [\tau_x^2, \tau_x^3] : d(g(t_x^1), \gamma_x(t)) = \epsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\}.$$

By (2.7.1), Proposition 2.7.2(3), and Proposition 2.7.4(1,2),

$$d(g(\tau_x^1), g(\tau_x^2)) \geq d(\pi(x_L), \pi(x)) - d(\pi(x_L), g(\tau_x^1)) - d(\pi(x), g(\tau_x^2)) \geq \tfrac{1}{2} \xi \operatorname{diam} \mathcal{L}_x,$$

so we have that

$$\begin{aligned} & d(g(t_x^1), g(\tau_x^1)) \qquad\qquad (2.8.4) \\ & \quad \geq d(g(\tau_x^1), g(\tau_x^2)) - \max_{t \in [\tau_x^2, \tau_x^3]} d(\gamma_x(t), g(\tau_x^2)) - \operatorname{dist}(g(t_x^1), \gamma_x([\tau_x^2, \tau_x^3])) \\ & \quad \geq \tfrac{1}{2} \xi \operatorname{diam} \mathcal{L}_x - \max_{t \in [0,1]} d(\sigma_x(t), g(\tau_x^2)) \\ & \qquad - (2^{11} C' \tilde{L})^{-1} \xi \max\{\operatorname{diam} \mathcal{L}_x, \operatorname{diam} \mathcal{R}_x\} - \epsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \\ & \quad \geq \left(\tfrac{1}{2} \xi - 2^{-5} \xi - 2^{-9} \xi - 5\epsilon\right) \operatorname{diam} \mathcal{L}_x \geq \tfrac{1}{4} \xi \operatorname{diam} \mathcal{L}_x \end{aligned}$$

and

$$d(g(t_x^1), g(\tau_x^2)) \geq \operatorname{dist}(g(t_x^1), \gamma_x([\tau_x^2, \tau_x^3])) = \epsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \tag{2.8.5}$$

Moreover,

$$\begin{aligned} d(\gamma_x(t_x^2), \gamma_x(\tau_x^3)) &\geq \operatorname{dist}(g(\tau_x^3), g([\tau_x^1, \tau_x^2])) - \operatorname{dist}(\gamma_x(t_x^2), g([\tau_x^1, \tau_x^2])) \\ &\geq \tfrac{1}{3}\hat{L}^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) - \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\geq \tfrac{1}{4}\hat{L}^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

Define now

$$t_x^4 = \max\{t \in [\tau_x^3, \tau_x^4] : d(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\}$$

and

$$t_x^3 = \min\{t \in [t_x^2, \tau_x^3] : d(\gamma_x(t), g(t_x^4)) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\}.$$

As in (2.8.4), we have that

$$d(g(t_x^4), g(\tau_x^4)) \geq \tfrac{1}{4}\xi \operatorname{diam}\mathcal{R}_x \tag{2.8.6}$$

and

$$d(g(t_x^4), g(\tau_x^3)) \geq \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x).$$

Moreover, if $t \in [\tau_x^2, \tau_x^3]$ satisfies $d(\gamma_x(t), g(t_x^4)) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)$, then

$$\begin{aligned} d(\gamma_x(t_x^2), \gamma_x(t)) &\geq \operatorname{dist}(g([\tau_x^1, \tau_x^2]), g([\tau_x^3, \tau_x^4])) - 2\epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\geq (\tfrac{1}{3}\hat{L}^{-1} - 2\epsilon)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\geq \tfrac{1}{4}\hat{L}^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

Therefore, t_x^3 is well defined and

$$t_x^3 - t_x^2 \geq (4\hat{L}L^*)^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \tag{2.8.7}$$

See Figure 2.2 for reference.

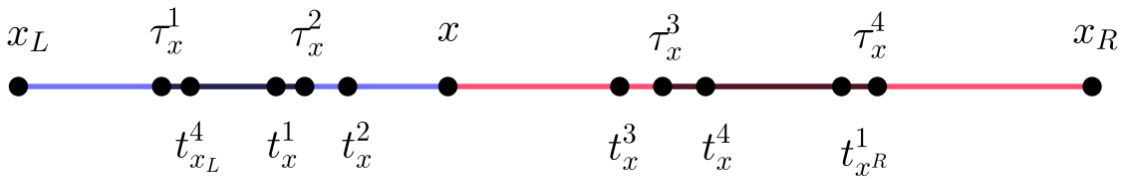

Figure 2.2: The relative positions of t_x^1, t_x^2, t_x^3, and t_x^4.

Local modifications around points in E''

Fix $x \in E''$. We proceed to define γ_x and points $t_x^1, \dots, t_x^4$ as in §2.8.1. The only difference is that we take into account the modifications done for points in $x_L, x_R \in E'$. In particular, we define

$$\begin{aligned}
t_x^1 &= \min\{t \in [t_{x_L}^4, \tau_x^2] : \operatorname{dist}(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\} \\
t_x^2 &= \max\{t \in [\tau_x^2, \tau_x^3] : d(g(t_x^1), \gamma_x(t)) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\} \\
t_x^4 &= \max\{t \in [\tau_x^3, t_{x_R}^1] : d(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\} \\
t_x^3 &= \min\{t \in [t_x^2, \tau_x^3] : d(\gamma_x(t), g(t_x^4)) = \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\}.
\end{aligned}$$

Equations (2.8.4), (2.8.6), (2.8.7) are still valid for $x \in E''$ as well.

Furthermore, suppose that $x < y$ are consecutive points in E; that is, $x = y_L$ (or equivalently $y = x_R$). Then,

$$\begin{aligned}
d(g(t_x^4),g(t_y^1)) &\geq d(\pi(x),\pi(y)) - d(\pi(x),g(t_x^4)) - d(\pi(y),g(t_y^1)) \qquad (2.8.8)\\
&\geq \xi \operatorname{diam}\mathcal{L}_y - \epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) - 6(2^8\tilde{L})^{-1}\xi(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\
&\quad - \epsilon(\operatorname{diam}\mathcal{L}_y + \operatorname{diam}\mathcal{R}_y) - 6(2^8\tilde{L})^{-1}\xi(\operatorname{diam}\mathcal{L}_y + \operatorname{diam}\mathcal{R}_y) \\
&\geq (\xi - 10\epsilon - 60(2^8\tilde{L})^{-1}\xi)\operatorname{diam}\mathcal{L}_y \geq \tfrac{1}{2}\xi \operatorname{diam}\mathcal{L}_y.
\end{aligned}$$

2.8.2 Definition of the extension F and proof of Theorem 2.8.1

Set

$$\tilde{A} = \hat{A} \setminus \bigcup_{x \in E} [t_x^1, t_x^4].$$

Define the map $F : I \to X$ so that

1. $F|\tilde{A} = g|\tilde{A}$,
2. for each $x \in E$, $F|[t_x^2, t_x^3] = \gamma_x|[t_x^2, t_x^3]$,
3. for each $x \in E$, $F|[t_x^1, t_x^2]$ is the geodesic from $g(t_x^1)$ to $\gamma_x(t_x^2)$ of constant speed,
4. for each $x \in E$, $F|[t_x^3, t_x^4]$ is the geodesic from $\gamma_x(t_x^3)$ to $g(t_x^4)$ of constant speed.

Clearly, F is an extension of f. In view of (2.8.2), the following proposition completes the proof of Theorem 2.8.1.

Proposition 2.8.2. *The map F is an L'-bi-Lipschitz embedding for some $L' \geq 1$ depending only on C, C_1, Q, p, and L.*

Proof. Fix $s, t \in I$ with $s < t$. We may assume that one of s, t is in $[t_x^1, t_x^4]$ for some $x \in E$, since, otherwise $F = g$, which is $\hat{L}$-bi-Lipschitz. Assume without loss of generality that $t \in [t_x^1, t_x^4]$ for some $x \in E$. The proof is a case study.

Case 1. Assume that $s \in [t_x^1, t_x^4]$. There are a few subcases to consider.

Case 1.1. Assume that $s, t \in [t_x^1, t_x^2]$ or $s, t \in [t_x^3, t_x^4]$. Without loss of generality, assume the former. In this case, $F(s)$ and $F(t)$ lie on a geodesic of unit speed joining $g(t_x^1)$ and $\gamma_x(t_x^2)$, and by (2.8.5),

$$\hat{L}^{-1}\epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \leq |t_x^1 - \tau_x^2| \leq |t_x^1 - t_x^2| \leq \operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x,$$

so

$$\frac{d(F(s), F(t))}{|s - t|} = \frac{d(g(t_x^1), \gamma_x(t_x^2))}{|t_x^1 - t_x^2|} \in [\epsilon, \hat{L}].$$

Case 1.2. Assume that $s, t \in [t_x^2, t_x^3]$. Here $F|[t_x^2, t_x^3] = \gamma_x|[t_x^2, t_x^3]$, and γ_x is L^*-bi-Lipschitz.

Case 1.3. Assume that $s \in [t_x^1, t_x^2]$ and $t \in [t_x^2, t_x^3]$ or $s \in [t_x^2, t_x^3]$ and $t \in [t_x^3, t_x^4]$. Without loss of generality, we assume the former. Then $F(s)$ lies on a geodesic of unit speed joining $g(t_x^1)$ and $\gamma_x(t_x^2)$, and $F(t) = \gamma_x(t)$. Since $\gamma_x(t_x^2)$ is a closest point of $\gamma_x([t_x^2, t_x^3])$ to $g(t_x^1)$, Lemma 2.6.2 implies that the gluing $F([t_x^1, t_x^3]) = g([t_x^1, t_x^2]) \cup \gamma_x([t_x^2, t_x^3])$ is bi-Lipschitz with a constant depending only on that of γ_x, which itself depends only on C, C_1, Q, p, and L.

Case 1.4. Assume that $s \in [t_x^1, t_x^2]$ and $t \in [t_x^3, t_x^4]$. By (2.8.7),

$$(4\hat{L}L^*)^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \leq |t_x^2 - t_x^3| \leq |s - t| \leq \operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x.$$

On one hand, using the fact that $F(s)$ and $F(t_x^2)$, and $F(t_x^3)$ and $F(t)$ lie on unit speed geodesics joining $g(t_x^1)$ to $\gamma_x(t_x^2)$ and $\gamma_x(t_x^3)$ to $g(t_x^4)$ respectively and

$$\begin{aligned} d(F(s), F(t)) &\leq d(F(s), F(t_x^2)) + d(\gamma_x(t_x^2), \gamma_x(t_x^3)) + d(F(t_x^3), F(t)) \\ &\leq d(g(t_x^1), \gamma_x(t_x^2)) + d(\gamma_x(t_x^2), \gamma_x(t_x^3)) + d(\gamma_x(t_x^3), g(t_x^4)) \\ &\leq (2\epsilon + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

On the other hand, arguing similarly gives

$$\begin{aligned} d(F(s), F(t)) &\geq d(\gamma_x(t_x^2), \gamma_x(t_x^3)) - d(F(s), F(t_x^2)) - d(F(t), F(t_x^3)) \\ &\geq (L^*)^{-1}|t_x^2 - t_x^3| - 2\epsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\geq ((4\hat{L}(L^*)^2)^{-1} - 2\epsilon)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\geq (8\hat{L}(L^*)^2)^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

Case 2. Assume that $s \in [t_y^1, t_y^4]$ for some $y \in E$ with $y < x$. First, using (2.8.8),

$$(10\hat{L})^{-1}\xi(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) \leq |t_y^4 - t_x^1| \leq |s - t|.$$

As with Case 1.4,

$$\begin{aligned} d(F(s), F(t_y^4)) &\le d(F(t_y^1), F(t_y^2)) + d(\gamma_y(t_y^2), \gamma_y(t_y^3)) + d(F(t_y^3), F(t_y^4)) \\ &\le (2\epsilon + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_y + \operatorname{diam}\mathcal{R}_y) \end{aligned}$$

and similarly

$$d(F(t_x^1), F(t)) \le (2\epsilon + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x).$$

Thus

$$\begin{aligned} d(F(s), F(t)) &\le d(F(s), F(t_y^4)) + d(g(t_y^4), g(t_x^1)) + d(F(t_x^1), F(t)) \\ &\le 5(2\epsilon + 1)(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) + \hat{L}|t_y^4 - t_x^1| \\ &\le 51\hat{L}\xi^{-1}(2\epsilon + 1)|s - t|. \end{aligned}$$

For the lower bound, if $y = x_L$, then $|s - t| \le 9|x - y|$ and (2.8.8) gives

$$\begin{aligned} d(F(s), F(t)) &\ge d(g(t_y^4), g(t_x^1)) - d(F(s), F(t_y^4)) - d(F(t_x^1), F(t)) \\ &\ge \tfrac{1}{2}\xi \operatorname{diam}\mathcal{L}_x - 10(2\epsilon + 12(2^8\tilde{L})^{-1}\xi) \operatorname{diam}\mathcal{L}_x \\ &\ge \tfrac{1}{50}\xi \operatorname{diam}\mathcal{L}_x \ge \tfrac{1}{450}|s - t|. \end{aligned}$$

If instead $y < x_L$, then

$$\begin{aligned} d(F(s), F(t)) &\ge d(\pi(x), \pi(y)) - d(\pi(y), F(s)) - d(\pi(x), F(t)) \\ &\ge \tilde{L}^{-1}|x - y| - 5(12(2^8\tilde{L})^{-1}\xi + 2\epsilon)(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) \\ &\ge (\tilde{L}^{-1} - 10(12(2^8\tilde{L})^{-1}\xi + 2\epsilon))|x - y| \ge (18\tilde{L})^{-1}|s - t|. \end{aligned}$$

Case 3. Assume that $s \in \tilde{A}$. Then, $y \in [t_{y_L}^4, t_y^1]$ for some $y \in E$. There are two subcases to consider.

Case 3.1. Assume that $y = x$. There are further subcases here.

Case 3.1.1 Assume that $t \in [t_x^1, t_x^2]$. As in Case 1.3, $g(t_x^1)$ is a closest point of $g([t_{x_L}^4, t_x^1])$ to $\gamma_x(t_x^2)$, so Lemma 2.6.2 tells us that $F([t_{x_L}^4, t_x^1])$ is bi-Lipschitz with a constant depending only on C, C_1, Q, p, and L.

Case 3.1.2. Assume that $t \in [t_x^2, t_x^3]$. By (2.8.5),

$$\epsilon \hat{L}^{-1}(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \le |t_x^1 - \tau_x^2| \le |s - t| \le \operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x,$$

so our desired bounds come from

$$d(F(s), F(t)) \ge \operatorname{dist}(g([\tau_x^1, t_x^1]), \gamma_x([t_x^2, t_x^3])) = \epsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)$$

and

$$\begin{aligned} d(F(s), F(t)) &\le \operatorname{diam} g(\hat{\mathcal{L}}_x) + \operatorname{diam} \gamma_x([\tau_2, \tau_3]) \\ &\le (\hat{L} + 12(2^8 \tilde{L})^{-1}\xi)(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \end{aligned}$$

Case 3.1.3. Assume that $t \in [t_x^3, t_x^4]$. By (2.8.7),

$$(4\hat{L}L^*)^{-1}(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \le |t_x^2 - t_x^3| \le |t - s| \le \operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x.$$

Now on one hand,

$$\begin{aligned} d(F(s), F(t)) &\le \operatorname{diam} g(\hat{\mathcal{L}}_x) + \operatorname{diam} \gamma_x([\tau_x^2, \tau_x^3]) + \operatorname{diam} g(\hat{\mathcal{R}}_x) \\ &\le (\hat{L} + 12(2^8 \tilde{L})^{-1}\xi)(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(F(s), F(t)) &\ge \operatorname{dist}(g(\hat{\mathcal{L}}_x), g(t_x^4)) - \operatorname{diam} F([t_x^3, t_x^4]) \\ &\ge ((3\hat{L})^{-1} - \epsilon)(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \end{aligned}$$

Case 3.2. Assume that $y < x$. Then

$$3^{-1}(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) \le |\tau_y^2 - \tau_x^1| \le |s-t|.$$

As in Case 2, we have

$$\begin{aligned} d(F(s),F(t)) &\le d(g(s),g(t_y^1)) + d(g(t_y^1),g(t_x^1)) + d(F(t_x^1),F(t)) \\ &\le \hat{L}|s-t_y^1| + \hat{L}|t_y^1 - t_x^1| + (2\epsilon + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\le 3(\hat{L} + L^* + 2\epsilon + 1)|s-t|. \end{aligned}$$

For the lower bound, set $M := (2\epsilon + 6(2^8\tilde{L})^{-1}\xi)$. If $|s-t| \le M \operatorname{diam}\mathcal{L}_x$, then the desired bound is a result of the following application of (2.8.3):

$$\begin{aligned} d(F(s),F(t)) &\ge \operatorname{dist}\Big(F([t_x^1,t_x^4]), g(\hat{A}\setminus(\mathcal{L}_x\cup\mathcal{R}_x))\Big) \\ &\ge ((2C')^{-1} - 4\epsilon)\max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\} \\ &\ge (16C')^{-1}\operatorname{diam}\mathcal{L}_x. \end{aligned}$$

If $|s-t| > M \operatorname{diam}\mathcal{L}_x$, then

$$\begin{aligned} d(F(s),F(t)) &\ge d(g(s),g(t_x^1)) - d(F(t_x^1),F(t)) \\ &\ge \hat{L}^{-1}|s-t_x^1| - \operatorname{diam}F([t_x^1,t_x^4]) \\ &\ge \tfrac{1}{16}\hat{L}^{-1}|s-t| - (2\epsilon + 6(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\ge \tfrac{1}{16}\hat{L}^{-1}|s-t| - 5(2\epsilon + 6(2^8\tilde{L})^{-1}\xi)\operatorname{diam}\mathcal{L}_x \ge \tfrac{1}{32}\hat{L}^{-1}|s-t|. \end{aligned}$$ □

2.8.3 The unbounded case

Assuming that X is unbounded, one can replace I in Theorem 2.8.1 by $\mathbb{R}$. The difference here is that we consider a Whitney decomposition of $\mathbb{R}\setminus A$. The

unboundedness of X guarantees the existence of function $\pi : E \to X$ as in Proposition 2.7.2. The rest of the proof is verbatim.

Chapter 3

Quasisymmetric rectifiability of uniformly disconnected sets

3.1 Disclosure

This chapter is, up to formatting, a direct copy of the preprint of the same name [HV25]. The manuscript is a collaborative work with Vyron Vellis. The ideas are mostly an adaptation of results of MacManus [Mac99] to metric measure spaces. The author and Vyron Vellis wrote and edited the paper together.

3.2 Abstract

We prove that uniformly disconnected subsets of metric measure spaces with controlled geometry (complete, Ahlfors regular, supporting a Poincaré inequality, and a mild topological condition) are contained in a quasisymmetric arc. This generalizes a result of MacManus [Mac99] from Euclidean spaces to abstract metric setting. Along the way, we prove a geometric strengthening of the classical Denjoy-Riesz theorem in metric measure spaces. Finally, we prove that the complement of a uniformly disconnected set in such a metric space is uniform, quantitatively.

3.3 Introduction

In the early 1900s, Denjoy [Den10] and Riesz [Rie05] independently proved what has come to be known as the Denjoy-Riesz Theorem, which states that *a compact, totally disconnected subset of the plane is contained in an arc.* Recall that a totally disconnected set is one whose connected components are singletons (or equivalently, one that has topological dimension equal to zero) and an arc is the homeomorphic image of the unit interval. This builds upon work by Zoretti [Zor05], which instead used a more general class of arcs than just Jordan arcs. More generally, Moore and Kline [MK19] came up with a complete characterization of subsets of the plane which are contained in arcs, and this includes the Denjoy-Riesz Theorem as a special case. Later, Zippin [Zip32] provided a complete classification of metric spaces for which

the Moore-Kline Theorem holds. A corollary of Zippin's work is that in a compact, connected, locally connected metric space with no local cut-points, every compact totally disconnected set is contained in an arc; see also [Why31]. A useful application of the (generalized) Denjoy-Riesz theorem is that for every $n \geq 2$, there is an arc in $\mathbb{R}^n$ that has positive Lebesgue measure, and for every $n \geq 3$ there is a *wild* arc in $\mathbb{R}^n$, that is, an arc that is not ambiently homeomorphic to a line segment.

It is natural to ask whether one can obtain more regularity on the arc which passes through the entire set. David and Semmes [DS91] proved a bi-Lipschitz version of the Moore-Kline Theorem: if $n \geq 3$, then a bounded subset of $\mathbb{R}^n$ is contained in a bi-Lipschitz arc in $\mathbb{R}^n$ if and only if it is bi-Lipschitz homeomorphic to a bounded subset of $\mathbb{R}$. MacManus [Mac95] extended the result of David and Semmes to the case $n = 2$, which is more difficult since intersecting lines in dimensions greater than two can be modified so that they no longer intersect, but this is not the case in $\mathbb{R}^2$. On one hand, these "bi-Lipschitz rectifiability" theorems can be viewed as a topological strengthening of the classical Lipschitz rectifiability theory which has been the focus of the Analyst's Traveling Salesman Theorem [Jon90, Oki92]. On the other hand, they can be viewed as rough versions of the classical Whitney extension theorem [Whi34].

To generalize the bi-Lipschitz rectifiability results to an abstract metric space X, in addition to the conditions of Zippin, it is necessary to assume certain geometric properties which ensure that X has plenty of rectifiable curves. Two such conditions are *Ahlfors regularity* and a *Poincaré inequality*. A metric space is Ahlfors Q-regular (or simply Q-regular) if there exists a measure μ on X so that the measure of every ball $B(x, r)$ is comparable to r^Q. A metric measure space has a Poincaré inequality if for every Lipschitz function $u : X \to \mathbb{R}$, the oscillation of u on each ball B is controlled by the average of a "weak derivative" of u on B. See Section 3.4 for all relevant definitions. It is known that Ahlfors regular spaces supporting a Poincaré inequality must contain quantitatively many rectifiable curves and admit a notion of differentiation [Che99]. The two authors along with Zimmerman [HVZ24] showed

that in Ahlfors regular metric measure spaces with a suitable Poincaré inequality, the bi-Lipschitz rectifiability classification of David and Semmes holds.

The goal of this paper is to provide a *quasisymmetric* version of the Denjoy-Riesz theorem in abstract metric spaces. Roughly speaking, a homeomorphism between metric spaces is quasisymmetric if any three points with comparable mutual distances are mapped to three points with comparable mutual distances in a quantitative fashion. Due to Buerling and Ahlfors [BA56], quasisymmetric maps are a generalization of conformal maps to the abstract metric setting which have been instrumental in the development of analysis on metric spaces. Quasisymmetric arcs (quasisymmetric images of $[0,1]$) have appeared in geometric function theory, geometric group theory (limits of quasi-Fuchsian groups), complex dynamics (Julia sets of certain rational functions), function spaces (domains of extension for Sobolev and BMO maps), Loewner evolution (traces of driving functions λ with Hölder norm less than 4), potential theory (Dirichlet integrals and harmonic bending), and hyperbolic geometry (bi-Lipschitz equivalence of hyperbolic and distance-ratio metrics). See [GH12] for a detailed exposition.

To replace "arc" by "quasisymmetric arc" in the Denjoy-Riesz theorem, necessarily some geometric assumptions should be made about the set E. For example, if $E \subset \mathbb{R}^n$ is a Cantor set with positive Lebesgue n-measure, then any arc that contains E will have positive Lebesgue n-measure so it can not be a quasisymmetric arc [Väi81, Theorem 4.1]. A quantitative version of total disconnectedness was introduced by David and Semmes [DS97]: a metric space E is *uniformly disconnected* if for all $x \in E$ and all $r \in (0, \operatorname{diam} E)$ there exists a set $E_{x,r} \subset E$ such that $x \in E_{x,r}$, $\operatorname{diam} E_{x,r} \leq r$, and $\operatorname{dist}(E_{x,r}, E \setminus E_{x,r})$ is at least a fixed multiple of r (i.e., E has isolated islands at every scale and at every point). Using this notion, MacManus [Mac99] proved a quasisymmetric version of the Denjoy-Riesz theorem in Euclidean spaces.

Theorem 3.3.1 ([Mac99, Theorem 1])**.** *A compact, uniformly disconnected subset of $\mathbb{R}^n$ is contained in a quasisymmetric arc.*

Our main focus is to generalize Theorem 3.3.1 to a larger class of metric measure spaces. Towards this end, a topological condition in the spirit of Zippin's no-local-cut-point condition is assumed on the space X. We say that a set $E \subset X$ is a *cut-set* if $X \setminus E$ is not connected; otherwise we call E a *full set*. The latter term is inspired by [NS19] where a planar set E is called full if $\mathbb{R}^2 \setminus E$ has no bounded components. The assumptions of Q-regularity and a Q-Poincaré inequality on X yield that if E is a compact set such that $X \setminus E$ has no bounded components, then E is a full set; see Lemma 3.5.3. Singletons in such a space X can not be local cut-sets but the same is not true with totally disconnected (or even uniformly disconnected) sets; see the discussion after Question 3.3.4.

The next topological condition is designed to make sure that closed totally disconnected sets do not separate the space. A metric space X is *fully connected* if it is path connected, locally path connected, and every full set is locally full. That is, if $X \setminus E$ is connected and if U is an open, connected set such that $U \cap E \neq \varnothing$ and $E \cap \partial U = \varnothing$, then $U \setminus E$ is connected. For example, all Euclidean spaces $\mathbb{R}^n$ are fully connected, while tori $(\mathbb{S}^1)^n$ are not. Our main theorem is as follows.

Theorem 3.3.2. *Let X be a complete, fully connected, unbounded, Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality with $Q > 2$. If $E \subset X$ is a compact, uniformly disconnected set, then E is contained in a quasisymmetric arc.*

This theorem is quantitative in the sense that the quasisymmetric control function depends only on the constants of Ahlfors regularity, the constants of the Poincaré inequality, and the constant of uniform disconnectedness. Canonical examples of of metric measure spaces satisfying the assumptions of Theorem 3.3.2 are Carnot groups, which include all Euclidean spaces and all Heisenberg groups. We will not define Carnot groups here but only mention that they are topologically identical to Euclidean space, but geometrically can behave much worse; although they are geodesic spaces, they have much fewer rectifiable curves and surfaces.

The techniques and tools used in the proof of Theorem 3.3.1, as well as modern proofs of the Denjoy-Riesz theorem, involve several classical tools from Euclidean spaces which are not available here. First, Euclidean spaces have a *dyadic decomposition* of cubes. This decomposition can be used to create a defining sequence for any Cantor set which consists of piecewise linear n-manifolds with boundary; see for example [Mac99, Section 2] and [BV19, Corollary 5.2]. While generalizations of dyadic decompositions exist in general metric spaces [HK12, KRS12], here we take a more simple approach based on open neighborhoods and create "good" defining sequences for all compact, totally disconnected sets. Second, in Euclidean spaces, arcs can be approximated in the Hausdorff distance by bi-Lipschitz arcs quantitatively, but this is much less obvious in metric spaces with fractal directions such as the Heisenberg group. We resolve this issue by using the Poincaré inequality and Ahlfors regularity in order to generate nice curves; see §3.3.2 for further discussion.

In fact, the techniques of this paper allow us to establish the following strengthening of the Denjoy-Riesz Theorem. Here, $\dim_H X$ denotes the Hausdorff dimension of a metric space X.

Theorem 3.3.3. *Let X be a complete, fully connected, unbounded, Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality with $Q > 2$. If $E \subset X$ is a compact, totally disconnected set, then there exists an embedding $f : [0,1] \to X$ such that $E \subset f([0,1])$ and f is locally bi-Lipschitz away from $f^{-1}(E)$. In particular,* $\dim_H f([0,1]) = \max\{1, \dim_H E\}$.

One can also ask whether a complete classification of quasisymmetrically rectifiable sets exists, in the spirit of Moore-Kline (for arcs) and of David-Semmes and Honeycutt-Vellis-Zimmerman (for bi-Lipschitz arcs). In the bi-Lipschitz classification, the key ingredient is the existence of bi-Lipschitz extension theorems [DS91, HVZ24]. In contrast, quasisymmetric maps are in general much harder to extend and we leave the following as an open question.

Question 3.3.4. *Let X be a complete, fully connected, unbounded, Q-regular metric measure space supporting a $(Q-1)$-Poincaré inequality for some $Q > 2$. If $E \subset \mathbb{R}$ and $f : E \to X$ is a quasisymmetric embedding, is there a quasisymmetric extension $F : \mathbb{R} \to X$?*

Question 3.3.4 is unknown even in the case where X is a Euclidean space although partial results exist if one increases the dimension of the target of the extension [Vel18].

Although we do not know whether full connectedness can be removed from Theorem 3.3.2 and Theorem 3.3.3, it can not be removed from Question 3.3.4. To see that, let $s \in (2,3)$, let $A \subset \mathbb{R}^3$ be an unbounded closed s-regular uniformly disconnected set (e.g. a blow-up tangent of an s-regular self-similar Cantor set), and let $A' \subset [0,\infty)$ be an unbounded closed Ahlfors regular uniformly disconnected set (e.g. a blow-up tangent of the standard Cantor set at 0). If $X = \mathbb{R}^3 \cup_A \mathbb{R}^3$ is the geodesic gluing of two copies of $\mathbb{R}^3$ along A equipped with the Hausdorff 3-measure, then X is 3-regular and it supports a 2-Poincaré inequality [HK98, Theorem 6.15]. Let now $y_1, y_2 \in X \setminus A$ be two points in different copies of $\mathbb{R}^3$, and let $E = A' \cup \{-2,-1\}$. Then there is a quasisymmetric embedding $f : E \to X$ such that $f(A') = A$, $f(-1) = y_1$ and $f(-2) = y_2$ (a simple variation of [DS97, Proposition 15.11]) which can not be extended homeomorphically (let alone quasisymmetrically) on $\mathbb{R}$.

We also remark that in Euclidean spaces there exist higher dimensional bi-Lipschitz and quasisymmetric rectifiability results; in particular, if $E \subset \mathbb{R}^n$ is compact and uniformly disconnected, then for any $m \in \{1,\dots,n-1\}$ there exists a quasisymmetric embedding f of $[0,1]^m$ into $\mathbb{R}^n$ that contains E in its image [BV19, Theorem 3.8]. If, in addition, the Assouad dimension of E is less than m, then f can be assumed bi-Lipschitz [BV19, Theorem 3.4]. While Ahlfors regular metric spaces with Poincaré inequalities contain a lot of rectifiable curves, the same is not true with surfaces; for example, the Heisenberg group contains no bi-Lipschitz images of the unit square. It would be very interesting if analogous results were true for certain Carnot groups.

3.3.1 Co-uniformity of uniformly disconnected sets

The notion of local separation of a uniformly disconnected set E naturally lends itself to the idea that there are lots of holes in it; in particular, the complement of E contains a lot of curves. A quantification of this idea was given by MacManus who proved that if $E \subset \mathbb{R}^n$ is uniformly disconnected, then $\mathbb{R}^n \setminus E$ is a uniform set [Mac99, Theorem 1]. Recall that a set U is uniform if for any two points in U there is a curve joining them which is not too long compared to the distance of the points and does not get too close to the boundary, aside from possibly near the endpoints. Since their inception by Martio and Sarvas [MS79], uniform domains have played a crucial role in geometric function theory, especially in the extendability of Sobolev and BMO functions [GO79, Jon80, Jon81].

The study of uniform domains is prevalent in literature. In particular, uniformity has been explored through concepts like quasihyperbolicity [Her87, HVW17, Her22], Gromov hyperbolicity [BHK01, HSX08], uniform disconnectedness [Mac99], bounds on topological dimension [Väi88], and bounds on Assouad dimension [HVZ24]. We show that MacManus' result generalizes to a large class of metric measure spaces, as given by the following theorem.

Theorem 3.3.5. *Let (X, d, μ) be a complete, fully connected, unbounded, Q-regular metric measure space supporting a Q-Poincaré inequality. If E is a uniformly disconnected subset of X, then $X \setminus E$ is uniform.*

The theorem is quantitative in the sense that the constant of uniformity depends only on on the constants of Ahlfors regularity, the constants of the Poincaré inequality, and the constant of uniform disconnectedness. Note also that in Theorem 3.3.5, we assume a Q-Poincaré inequality which is weaker than the $(Q-1)$-Poincaré inequality which is assumed in Theorem 3.3.2.

3.3.2 Outline of the paper

In Section 3.4, we review some important definitions from analysis on metric spaces and present some preliminary results.

The common ingredient in the proof of all three main theorems is the construction of a "thickening" of compact totally disconnected sets in Section 3.5. This thickening is a replacement for the cubic meshes used in MacManus' [Mac99] proof to give a natural decomposition of uniformly disconnected sets in Euclidean space. In particular, Proposition 3.5.1 provides us with a defining sequence $\{D_n\}_{n\in\mathbb{N}}$ of open sets in X such that

1. for all $n \in \mathbb{N}$, $D_n = D_{n,1} \sqcup \cdots \sqcup D_{n,k_n}$ where $D_{n,1}, \ldots, D_{n,k_n}$ are mutually disjoint full domains with diameters at most a fixed multiple of 2^{-n},

2. for all $n \in \mathbb{N}$, each $D_{n,i}$ contains some $D_{n+1,j}$, and each $D_{n+1,i}$ is contained in some $D_{n,j}$

3. for all $n \in \mathbb{N}$, $\operatorname{dist}(D_{n+1}, \partial D_n) > 0$, and

4. $E = \bigcap_{n\in\mathbb{N}} \bigcup_{i=1}^{k_n} \overline{D_{n,i}}$.

Moreover, in the case that E is uniformly disconnected, we have quantitative control on the size of k_n, diameters of D_n, and distances $\operatorname{dist}(D_{n+1}, \partial D_n)$. Using this decomposition we prove Theorem 3.3.5 in Section 3.6.

In Section 3.7, we set up notation and some preliminary lemmas for the proof of Theorems 3.3.2 and 3.3.3. The crux of the proof of Theorem 3.3.2 and Theorem 3.3.3 is the construction of bi-Lipschitz arcs in each $D_{n,i} \setminus D_{n+1}$ with the first arc going from the boundary of $D_{n,i}$ to the boundary of the first component D_{n+1,i_1} of D_{n+1} inside D_n, the second curve going from the boundary of D_{n+1,i_1} to the boundary of the second component D_{n+1,i_2} of D_{n+1} inside D_n and so on, so forth. The last curve, goes from the boundary of the last component of D_{n+1} inside D_n to the boundary of D_n. We take extra care so that these curves do not intersect with one another or

with curves in other domains $D_{n,j}$. This construction relies heavily on the Poincaré inequality and the Ahlfors regularity of the space and is given in Section 3.8.

In Section 3.9, we perform modifications around the endpoints of the curves previously defined to ensure that the concatenation is locally bi-Lipschitz, (and globally quasisymmetric when E is uniformly disconnected) and in Section 3.10 we give the proofs of Theorem 3.3.2 and Theorem 3.3.3.

3.4 Preliminaries

Here and for the rest of the paper, when we write $C(c_1, \ldots, c_n)$, we mean that the constant C depends only on the constants $c_1, \ldots, c_n$. When we write $x \lesssim_{c_1,\ldots,c_n} y$, we mean there is a constant $C(c_1, \ldots, c_n) > 0$ such that $x \leq Cy$. If the constant $C > 0$ is universal or the constants are understood, then we write $x \lesssim y$.

If X is a metric space, $x \in X$, $r > 0$, and $E, F \subset X$, then we denote the distance of x to E by $\operatorname{dist}(x, E)$, the distance of E to F by $\operatorname{dist}(E, F)$, the diameter of E by $\operatorname{diam}(E)$, and the open r-neighborhood of the set E by

$$N(E, r) = \bigcup_{y \in E} B(y, r) = \{z \in X : \operatorname{dist}(z, E) < r\}.$$

We also denote the closed r-neighborhood of the set E by $\overline{N}(E, r)$. If $B = B(x, r)$ is a ball in X and $c > 0$, we denote by cB the ball $B(x, cr)$. Provided E and F are nondegenerate continua (compact and connected sets with more than one point), we denote the *relative distance* of E and F by

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}.$$

Given a set $A \subset \mathbb{R}$ and a map $f : A \to X$, we denote by $\operatorname{Im}(f)$ the image of f. If A is a closed interval and f is continuous (that is, f is a curve), then we conflate f with its image $\operatorname{Im}(f)$ as a subset of X.

Define the *length* of a curve $\gamma : [a, b] \to X$ by

$$\operatorname{length}(\gamma) = \sup \left\{ \sum_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1})) : a < t_0 < \cdots < t_n < b \right\}.$$

3.4.1 Words

An *alphabet* is either a set of the form $A = \{1, \ldots, n\}$ (in which case we say that we have a finite alphabet) or $A = \mathbb{N}$ (in which case we say that we have an infinite alphabet). Given an alphabet A, we denote a typical element (a word) of A^k by $a_1 \cdots a_k$ instead of $(a_1, \ldots, a_k)$. The number k is the *length* of any word $a \in A^k$, and we write $|a|$ to denote this value. We use the convention that $A^0 = \{\varepsilon\}$, where ε denotes the *empty word*, which satisfies $|\varepsilon| = 0$. We also denote $A^* = \bigcup_{k \geq 0} A^k$, the collection of all words of finite length. A *dictionary* is any subset of A^*. Lastly, given $w \in A^* \setminus \{\varepsilon\}$, denote by $w^{\uparrow}$ the unique word $u \in A^*$ for which there exists $i \in A$ such that $w = ui$ (sometimes called the *parent* of w).

3.4.2 Homogeneously totally bounded spaces and Ahlfors regularity

Inspired by [TV80, §2.7], we say that a metric space (X, d) is (C, α)-*homogeneously totally bounded* (abbv. (C, α)-HTB) for some $C, \alpha > 0$ if for each $x \in X$, and each $0 < r < R$, the ball $B(x, R)$ can be covered by at most $C(R/r)^{\alpha}$ many balls of radii r. Note that if X is (C, α)-HTB, then X is (C, β)-HTB for all $\beta \geq \alpha$. If we do not care for the constant C, we say that X is α-HTB. The image of a (C, α)-HTB space under a bi-Lipschitz map is (C', α)-HTB quantitatively.

A metric measure space (X, d, μ) is called *Ahlfors* (C_0, Q)*-regular* (or simply (C_0, Q)*-regular*) for constants $C_0 \geq 1$ and $Q \geq 0$ if for each point $x \in X$ and $r \in (0, \operatorname{diam} X)$,

$$C_0^{-1} r^Q \leq \mu(B(x, r)) \leq C_0 r^Q. \tag{3.4.1}$$

If the right inequality in (3.4.1) holds for all $x \in X$ and $r \in (0, \operatorname{diam} X)$, then we say that (X, d, μ) is *upper* (C_0, Q)*-regular*. Note that if (X, d, μ) is (C_0, Q)-regular, then $(X, d, \mathcal{H}^Q)$ is (C_0', Q)-regular for some C_0' depending only on C_0 [MT10, §1.4.3]. Here and for the rest of the paper, $\mathcal{H}^Q$ denotes the Q-dimensional Hausdorff measure. It is not hard to see that if (X, d, μ) is Q-regular, then (X, d) is (C, Q)-HTB for some $C > 0$ depending only on C_0 and Q.

3.4.3 Mappings

Let (X, d) and (Y, d') be two metric spaces. Given $L \geq 1$ and $\lambda > 0$, a map $f : X \to Y$ is said to be an (L, λ)*-quasisimilarity* if for all $x_1, x_2 \in X$,

$$\frac{\lambda}{L} d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L\lambda d(x_1, x_2).$$

If $\lambda = 1$, then we say that f is L*-bi-Lipschitz*. If $L = 1$, then f is a λ*-similarity*. If $L = \lambda = 1$, then f is an *isometry*.

Given a homeomorphism $\eta : [0, \infty) \to [0, \infty)$, a map $f : X \to Y$ is η*-quasisymmetric* if for all $x, a, b \in X$ with $x \neq b$,

$$\frac{d'(f(x), f(a))}{d'(f(x), f(b))} \leq \eta\left(\frac{d(x, a)}{d(x, b)}\right). \tag{3.4.2}$$

Since their inception by Beurling and Ahlfors [BA56], quasisymmetric maps have played an important role in the study of analysis on metric spaces. While bi-Lipschitz maps roughly preserve absolute distances, quasisymmetric maps roughly preserve relative distances.

It is well-known that under certain conditions on X and Y, quasisymmetry is equivalent to a weaker condition, aptly named *weak quasisymmetry*.

Lemma 3.4.1 ([Hei01, Theorem 10.19])**.** *Let X be a connected, HTB metric space and let Y be a HTB metric space. An embedding $f : X \to Y$ is η-quasisymmetric if and only if there exists $H \geq 1$ such that for any triple $x, y, z \in X$ with $d(x, y) \leq d(x, z)$*

we have $d'(f(x), f(y)) \leq Hd'(f(x), f(z))$. *The function* η, *the constant* H, *and the HTB constants of* X *and* Y *are quantitatively related.*

See [Hei01] for a detailed exposition on the theory of quasisymmetric maps.

3.4.4 Porosity and uniform disconnectedness

We say that a set $E \subset X$ is *p-porous* for some $p \geq 1$ if for all $x \in E$ and all $r \in (0, \operatorname{diam} X)$ there exists a ball $B(y, p^{-1}r) \subset B(x, r) \backslash E$. The following lemma gives a classification of porous sets in Ahlfors regular spaces in terms of their dimension.

Lemma 3.4.2 ([BHR01, Lemma 3.12])**.** *Let* $(X, d, \mathcal{H}^Q)$ *be a* (C_0, Q)*-regular metric space. A set* $E \subset X$ *is p-porous if and only if it is* (C, α)*-HTB for some* $C > 0$ *and* $\alpha \in (0, Q)$. *Constants* C, Q, C_0, *and* α *are quantitatively related.*

In light of the discussion in §3.4.2, the measure $\mathcal{H}^Q$ in Lemma 3.4.2 can be replaced by any other Q-regular measure.

We say that a metric space (X, d) is *c-uniformly disconnected* for some $c \geq 1$ if for all $x \in X$ and $r \in (0, \operatorname{diam} X)$, there exists a set $X_{x,r} \subset X$ such that $x \in X_{x,r}$, $\operatorname{diam} X_{x,r} \leq r$, and $\operatorname{dist}(X_{x,r}, X \setminus X_{x,r}) \geq c^{-1}r$.

We say that a collection $\{x_0, \ldots, x_n\}$ of points in a metric space is a *δ-chain* if $d(x_{i-1}, x_i) \leq \delta d(x_0, x_n)$ for each $i \in \{1, \ldots, n\}$. We say that the chain *joins* x_0 and x_n. The following lemma relates the definition of uniform disconnectedness to the nonexistence of δ-chains.

Lemma 3.4.3 ([Hei01, Exercise 14.26])**.** *A metric space* X *is c-uniformly disconnected for some* $c \geq 1$ *if and only if there exists* $\delta \in (0, 1)$ *such that for any two points* $x, y \in X$, *there is no* δ*-chain joining* x *and* y. *Here,* δ *and* c *depend only on each other.*

3.4.5 Poincaré inequality and modulus of curve families

Let (X, d) be a metric space. We say that a function $g : X \to [0, \infty]$ is an *upper gradient* of a locally Lipschitz function $u : X \to \mathbb{R}$ if for all points x and y of X and all rectifiable curves $\gamma \subset X$ joining x with y,

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s.$$

A metric measure space (X, d, μ) is said to support a *p-Poincaré inequality* if there exist constants $C, \lambda > 1$ such that for every locally Lipschitz function $u : X \to \mathbb{R}$, every upper gradient $g : X \to [0, \infty]$ of u, every point $x \in X$, and every $r > 0$,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq C \operatorname{diam}(B(x, \lambda r)) \left(\fint_{B(x,\lambda r)} g^p \, \mathrm{d}\mu \right)^{1/p},$$

where, for an integrable function $f : X \to \mathbb{R}$,

$$f_{B(x,r)} = \fint_{B(x,r)} f \, \mathrm{d}\mu = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f \, \mathrm{d}\mu.$$

It follows from Hölder's inequality that if $p < q$ and (X, d, μ) satisfies a p-Poincaré inequality, then it also satisfies a q-Poincaré inequality. Moreover, if (X, d, μ) is geodesic, then one can take $\lambda = 1$ [HKST15, Remark 9.1.19]. If we want to emphasize the constants, we say that (X, d, μ) satisfies an (p, C)-Poincaré inequality.

Let (X, d, μ) be a metric measure space and let Γ be a collection of rectifiable curves in X. A function $f : X \to [0, \infty)$ is said to be *admissible* for Γ if

$$\int_\gamma f \, \mathrm{d}s \geq 1$$

for each $\gamma \in \Gamma$. We define the *p-modulus* of Γ by

$$\operatorname{Mod}_p(\Gamma) = \inf \left\{ \int_X f^p \, \mathrm{d}\mu : \ f : X \to [0, \infty) \text{ is admissible for } \Gamma \right\}.$$

The following proposition is useful for calculating upper bounds on the modulus of families of curves in upper Ahlfors regular spaces.

Lemma 3.4.4 ([Hei01, Lemma 7.18]). *Suppose that (X, d, μ) is a metric measure space that is upper (C_0, Q)-regular. There exists $C_1(C_0, Q) \geq 1$ such that for $x \in X$ and $2r < R$,*

$$\operatorname{Mod}_Q(\Gamma) \leq C_1 \left(\log \left(\frac{R}{r} \right) \right)^{1-Q},$$

where Γ is the family of curves joining $\overline{B}(x, r)$ to $X \setminus B(x, R)$.

A metric space (X, d) is said to be *Q-Loewner* if there exists a decreasing homeomorphism $\Phi : (0, \infty) \to (0, \infty)$ such that $\operatorname{Mod}_Q(E, F) \geq \Phi(\Delta(E, F))$ for each pair of nondegenerate continua E and F in X. Ahlfors regularity and a Poincaré inequality imply the Loewner property.

Lemma 3.4.5 ([HK98, Theorem 5.7]). *Let (X, d, μ) be a complete, (C_0, Q)-regular metric measure space supporting a (Q, C)-Poincaré inequality. Then X is Q-Loewner and the Loewner function Φ depends only on C_0, Q, and C.*

3.4.6 Uniformity

A subset U of a metric space (X, d) is *c-uniform* for some $c \geq 1$ if for all $x, y \in \overline{U}$, there exists a curve $\gamma : [0, 1] \to \overline{U}$ with $\gamma(0) = x$, $\gamma(1) = y$,

1. $\operatorname{length}(\gamma) \leq c d(x, y)$ and
2. $\operatorname{dist}(\gamma(t), X \setminus U) \geq c^{-1} \operatorname{dist}(\gamma(t), \{x, y\})$ for all $t \in [0, 1]$.

If for each pair of points $x, y \in U$, there is a curve γ that satisfies condition (1), then we say that U is *c-quasiconvex*.

The next two lemmas allow us to assume for the rest of the paper that X is geodesic, since any quasiconvex metric space is bi-Lipschitz equivalent to a geodesic metric space through the identity map.

Lemma 3.4.6 ([Che99, Theorem 17.1]). *If (X, d, μ) is a complete (C_0, Q)-regular metric measure space and satisfies a (p, C)-Poincaré inequality, then there is a constant $c(C_0, Q, p, C) \geq 1$ such that X is c-quasiconvex.*

Lemma 3.4.7 ([HKST15, Lemma 8.3.18]). *Let d_1 and d_2 be two bi-Lipschitz equivalent metrics on X. If μ is Ahlfors regular with respect to d_1, then it is Ahlfors regular with respect to d_2, quantitatively. Furthermore, if (X, d_1, μ) supports a p-Poincaré inequality, then (X, d_2, μ) supports a p-Poincaré inequality, quantitatively.*

3.4.7 Bi-Lipschitz approximation of curves and avoiding small sets.

We end this section with a lemma that allows us—inside a good metric measure space—to approximate any curve by a bi-Lipschitz arc which also avoids a "small set" quantitatively.

Lemma 3.4.8. *Given $C_0, C, C_1 > 1$, $Q > 2$, $p \in [1, Q-1)$, $c \in (0,1)$, $\lambda \in (0, c)$, and $\alpha \in (0, Q-p)$, there exist constants $L(C_0, Q, p, C, C_1, \alpha, \lambda, c) > 1$ and $\beta(C_0, Q, p, C, C_1, \alpha, c) \in (0, c)$ with the following property.*

Let (X, d, μ) be a complete, (C_0, Q)-regular metric measure space supporting a (p, C)-Poincaré inequality, let $Y \subset X$ be (C_1, α)-HTB, and let $\sigma : [0,1] \to X$ be a curve satisfying $d(\sigma(0), \sigma(1)) \geq c \operatorname{diam} \sigma([0,1])$ and $\operatorname{dist}(\{\sigma(0), \sigma(1)\}, Y) \geq c \operatorname{diam} \sigma([0,1])$. There exists an $(L, \operatorname{diam} \sigma([0,1]))$-quasisimilarity $\gamma : [0,1] \to X$ such that $\gamma(0) = \sigma(0)$, $\gamma(1) = \sigma(1)$,

$$\sup_{t \in [0,1]} \operatorname{dist}(\gamma(t), \sigma([0,1])) \leq 4\lambda \operatorname{diam} \sigma([0,1]),$$

and

$$\inf_{t \in [0,1]} \operatorname{dist}(\gamma(t), Y) \geq \beta\lambda \operatorname{diam} \sigma([0,1]).$$

Proof. Rescaling the metric, we may assume for simplicity that $\operatorname{diam} \sigma([0,1]) = 1$.

Let V be a maximal λ-separated subset of σ that contains $\sigma(0)$ and $\sigma(1)$. Define a graph $G = (V, E)$ with vertex set V and edges $E = \{\{v, v'\} : v, v' \in V, d(v, v') \leq \lambda\}$. We claim that G is connected, i.e., for any $v, v' \in V$, there exist $v_1, \dots, v_n \in V$ such that $v_1 = v$, $v_n = v'$, and $\{v_{i-1}, v_i\} \in E$ for all $i \in \{2, \dots, n\}$. Towards a contradiction, suppose that $G' = (V', E')$ is a nonempty connected component of G with $V' \subsetneq V$. Then, $\operatorname{dist}(V', V \setminus V') > \lambda$. But then σ is contained in the union of the disjoint closed sets $G_1 = \overline{N}(V', \lambda/2)$ and $G_2 = \overline{N}(V \setminus V', \lambda/2)$, which contradicts the fact that σ is connected.

Since G is connected, there is a shortest path of vertices $\{v_1, \dots, v_n\}$ in G with $v_1 = \sigma(0)$ and $v_n = \sigma(1)$. Since $B(v_i, 2^{-1}\lambda)$ are disjoint and contained in $B(\sigma(0), 1)$, by the (C_0, Q)-regularity of X,

$$2^{-Q} C_0^{-1} \lambda^Q n \leq \sum_{i=1}^{n} \mu(B(v_i, 2^{-1}\lambda)) \leq \mu(B(\sigma(0), 1)) \leq C_0,$$

so $n \leq 2^Q C_0^2 \lambda^{-Q}$. By Lemma 3.4.2, Y is p_0-porous for some $p_0(C_0, Q, C_1, \alpha) \geq 1$ and for each $i \in \{2, \dots, n-1\}$, there exists $v_i' \in B(v_i, 3^{-1}\lambda)$ such that $B(v_i', (3p_0)^{-1}\lambda) \subset B(v_i, 3^{-1}\lambda) \setminus Y$. Note that since $\operatorname{dist}(\{\sigma(0), \sigma(1)\}, Y) \geq c$, it suffices to choose $v_1' = v_1 = \sigma(0)$ and $v_n' = v_n = \sigma(1)$.

Fix $i \in \{1, \dots, n-1\}$. By [HVZ24, Corollary 3.4], there is a curve S_i lying in $B(v_i', 2\lambda)$ joining v_i' and v_{i+1}' such that $\operatorname{length}(S_i) \leq \ell\lambda$ and for each $z \in S_i$, $\operatorname{dist}(z, Y) \geq \beta'\lambda$ for some $\ell(C_0, Q, p, C, c) \geq 1$ and $\beta'(C_0, Q, p, C, C_1, \alpha, c) \in (0, 1/2)$. Let $S : [0, 1] \to X$ denote the concatenation of curves S_i. Then S satisfies $S(0) = \sigma(0)$, $S(1) = \sigma(1)$,

$$\operatorname{length}(S) \leq (n-1)\ell\lambda \leq 2^Q C_0^2 \ell \lambda^{1-Q},$$

and S stays at least a distance $\beta'\lambda$ from Y. By [HVZ24, Proposition 1.5] there exists a constant $L(C, C_0, Q, p, C_1, \alpha, \lambda, c) \geq 1$ and there exists an L-bi-Lipschitz curve

$\gamma : [0,1] \to X$, joining $S(0)$ to $S(1)$ and satisfying

$$\begin{aligned}\operatorname{dist}_H(\gamma([0,1]), S([0,1])) &< \beta' \lambda^Q (2^{Q+1} C_0^2 \ell)^{-1} \operatorname{diam} S([0,1]) \\ &\leq \beta' \lambda^Q (2^{Q+1} C_0^2 \ell)^{-1} \operatorname{length}(S) \\ &\leq 2^{-1} \beta' \lambda.\end{aligned}$$

The curve γ has length at most L and

$$\operatorname{dist}(\gamma([0,1]), Y) \geq \operatorname{dist}(S([0,1]), Y) - \operatorname{dist}_H(\gamma([0,1]), S([0,1])) \geq 2^{-1}\beta'\lambda.$$

Lastly, for each $t \in [0,1]$,

$$\operatorname{dist}(\gamma(t), \sigma([0,1])) \leq \operatorname{dist}(\gamma(t), S([0,1])) + \operatorname{dist}(S([0,1]), \sigma([0,1])) \leq 2^{-1}\beta'\lambda + 3\lambda \leq 4\lambda.$$

□

3.5 A decomposition of compact totally disconnected sets

One natural property of compact, totally disconnected sets in Euclidean spaces is that they admit a *defining sequence*, which is, roughly speaking, a way to coherently encapsulate clusters of the set; see for example Armentrout's work in $\mathbb{R}^3$ [Arm66]. The main result of this section is Proposition 3.5.1, which shows that the same idea works in metric spaces with nice geometry.

Proposition 3.5.1. *Let (X, d, μ) be a complete, fully connected, unbounded, (C_0, Q)-regular, geodesic metric measure space supporting a (Q, C)-Poincaré inequality. Let E be a compact, totally disconnected subset of X. There exists a dictionary $\mathcal{W} \subset \mathbb{N}^*$, and there exists a collection $\{U_w : w \in \mathcal{W}\}$ of bounded full sets such that the following conditions hold.*

(D1) *The empty word* $\varepsilon \in \mathcal{W}$. *If* $w \in \mathcal{W} \setminus \{\varepsilon\}$, *then* $w^{\uparrow} \in \mathcal{W}$. *For each* $w \in \mathcal{W}$, *there exists* $n_w \in \mathbb{N}$ *such that* $wi \in \mathcal{W}$ *if and only if* $i \in \{1, \ldots, n_w\}$.

(D2) *For each* $n \in \mathbb{N} \cup \{0\}$, $E \subset \bigcup_{\substack{w \in \mathcal{W} \\ |w|=n}} U_w$. *Conversely, if* $w \in \mathcal{W}$, *then* $U_w \cap E \neq \varnothing$.

(D3) *If* $wi, wj \in \mathcal{W}$ *with* $i \neq j$, *then* $U_{wi} \cap U_{wj} = \varnothing$.

(D4) *If* $w \in \mathcal{W} \setminus \{\varepsilon\}$, *then* $U_w \subset U_{w^{\uparrow}}$, $\operatorname{diam} U_w \leq \frac{1}{2} \operatorname{diam} U_{w^{\uparrow}}$, *and* $\operatorname{dist}(U_w, \partial U_{w^{\uparrow}}) > 0$.

(D5) *If* $w \in \mathcal{W}$, *then* $\operatorname{dist}(\partial U_w, E) > 0$.

(D6) *If* $w \in \mathcal{W}$, *then the set*

$$\hat{U}_w := \overline{U_w} \setminus \bigcup_{i=1}^{n_w} U_{wi}$$

is path connected.

Moreover, if E *is* c*-uniformly disconnected for some* $c > 1$, *then there exists* $\xi(C_0, Q, C, c) \geq 1$ *such that for all* $\delta \in (0, (2\xi)^{-1})$, *there exists* $N(C_0, \delta, Q) \in \mathbb{N}$ *such that* $\mathcal{W} \subset \{1, \ldots, N\}^*$ *and the following quantitative versions of (D4) and (D5) hold.*

(D4') *If* $w \in \mathcal{W} \setminus \{\varepsilon\}$, *then* $U_w \subset U_{w^{\uparrow}}$,

$$\delta \operatorname{diam} U_{w^{\uparrow}} \leq \operatorname{diam} U_w \leq \xi \delta \operatorname{diam} U_{w^{\uparrow}}, \tag{3.5.1}$$

and

$$\operatorname{dist}(U_w, \partial U_{w^{\uparrow}}) \geq \xi^{-1} \operatorname{diam} U_w. \tag{3.5.2}$$

(D5') *If* $w \in \mathcal{W}$, *then* $\operatorname{dist}(\partial U_w, E) \geq \xi^{-1} \operatorname{diam} U_w$.

For the rest of this section, we fix a complete, fully connected, unbounded, (C_0, Q)-regular, geodesic metric measure space (X, d, μ) supporting a (Q, C)-Poincaré inequality.

Lemma 3.5.2. *Given $r > 0$ there exists $c_0(C_0, Q, r) \geq 1$ with the following property. If U is a bounded, path connected subset of X and $x, y \in U$, then there is a curve γ lying in $N(U, r \operatorname{diam} U)$ joining x with y such that* $\operatorname{length}(\gamma) \leq c_0 d(x, y)$.

Proof. Rescaling the metric, we may assume that $\operatorname{diam} U = 1$. Set $N_r = N(U, r)$.

If $d(x, y) < r/6$, then we can join x and y by a geodesic γ, and this satisfies, for each $z \in \gamma$,

$$\operatorname{dist}(z, X \setminus N_r) \geq \operatorname{dist}(\{x, y\}, X \setminus N_r) - \tfrac{1}{2} \operatorname{length}(\gamma) \geq \tfrac{11}{12} r$$

and $\operatorname{dist}(z, \{x, y\}) \leq \operatorname{length}(\gamma) = d(x, y) < r/6$.

If $d(x, y) \geq r/6$, then let V be a $(r/6)$-net of U containing x and y. For each $u, v \in V$, fix a geodesic $\gamma_{u,v} = \gamma_{v,u}$ joining them and define

$$G = \bigcup_{\substack{u,v \in V \\ d(u,v) \leq r}} \gamma_{u,v}.$$

We show that the set G is connected. Towards a contradiction, suppose that Z is a nonempty path connected component of G with $G \setminus Z \neq \varnothing$. Let $W = N(Z, 2r)$. Note that ∂W cannot be empty, since otherwise W and $X \setminus W$ separate X. Thus, there is a point $\xi \in X$ with $\operatorname{dist}(\xi, Z) = 2r$. As V is an r-net of X, there is a point $v \in V$ such that $d(\xi, v) \leq r$. Necessarily $v \in G \setminus U$, since $\operatorname{dist}(v, Z) \geq \operatorname{dist}(\xi, Z) - d(\xi, v) \geq r$. We claim that there is a point $u \in V \cap Z$ such that $d(u, \xi) \leq 5r$. To see this, choose $\zeta \in U$ such that $d(\xi, \zeta) = \operatorname{dist}(\xi, Z) = 2r$. If $\zeta \in V$, then taking $u = \zeta$, we have $d(u, \xi) = 2r$. If $\zeta \notin V$, then ζ lies on a geodesic joining two points $v_1, v_2 \in V$ for which $d(v_1, v_2) \leq r$. Then either $d(\zeta, v_1) \leq 3r$ or $d(\zeta, v_2) \leq 3r$, say $d(\zeta, v_1) \leq 3r$. Taking $u = v_1$, $d(u, \xi) \leq d(u, \zeta) + d(\zeta, \xi) \leq 5r$. Finally, $d(u, v) \leq d(u, \xi) + d(\xi, v) \leq 6r$, so in fact $\gamma_{u,v}$ is a path in G joining Z and $G \setminus Z$, a contradiction, which proves that G is connected.

Since G is connected there is a finite sequence $v_0, \ldots, v_k$ of points in V with $v_0 = x$, $v_k = y$, $d(v_{i-1}, v_i) \leq r$ for each $i \in \{1, \ldots, k\}$, and $v_i \neq v_j$ for distinct $i, j \in \{0, \ldots, k\}$. Let γ be the concatenation of the geodesic edges joining these vertices. Since V is $(r/6)$-separated and contained in U, by Q-regularity of X,

$$C_0^{-1}(r/12)^Q \operatorname{card} V \leq \sum_{v \in V} \mu(B(v, r/12)) \leq \mu(B(x, 1)) \leq C_0,$$

so $\operatorname{card} V \leq 12^Q C_0^2 r^{-Q}$. Thus

$$\operatorname{length}(\gamma) \leq kr \leq r \operatorname{card} V \leq 12^Q C_0^2 r^{1-Q} \leq 12^{Q+1} C_0^2 r^{-Q} d(x, y). \qquad \square$$

Lemma 3.5.3. *There exists $K(C_0, Q, C) \geq 1$ such that if U is a bounded and connected subset of X, then*

(a) $X \setminus U$ has exactly one component V with $\operatorname{diam} V \geq K \operatorname{diam} U$, and

(b) $\operatorname{diam} \partial U \leq \operatorname{diam} U \leq K \operatorname{diam} \partial U$.

Proof. Let $C_1(C_0, Q) \geq 1$ be the constant from Lemma 3.4.4, and let $\Phi : (0, \infty) \to (0, \infty)$ be the Q-Loewner function of X given by Lemma 3.4.5, which depends only on C_0, Q, and C. Set

$$K = \exp\left(\left(C_1^{-1}\Phi(4)\right)^{1/(1-Q)}\right).$$

For (a), we first claim that there exists at least one component of $X \setminus U$ of diameter at least $K \operatorname{diam} U$. Indeed, if all components of $X \setminus U$ had diameter at most $K \operatorname{diam} U$, then it can be easily seen that $\operatorname{diam} X \leq (2K + 1) \operatorname{diam} U < \infty$ which is false since X is unbounded.

Now suppose for a contradiction that V_1 and V_2 are two different components of $X \setminus U$ with $\operatorname{diam} V_i \geq K \operatorname{diam} U$ for $i = 1, 2$. Fix $u \in U$ and note that $U \subset \overline{B}(u, \operatorname{diam} U)$. There also exist points $v_i \in V_i$ for $i = 1, 2$ such that $d(u, v_i) = K \operatorname{diam} U$. Let σ_i be a curve in V_i joining v_i to ∂U for $i = 1, 2$. Let x_i be the first point that σ_i intersects $\overline{B}(u, \frac{1}{2} K \operatorname{diam} U)$ for $i = 1, 2$. Denote by γ_i a subcurve of σ_i joining

v_i to x_i for $i = 1, 2$. Note that $\Delta(\gamma_1, \gamma_2) \leq 2$, so $\operatorname{Mod}_Q(\gamma_1, \gamma_2) \geq \Phi(\Delta(\gamma_1, \gamma_2)) \geq \Phi(2)$. On the other hand, since γ_1 and γ_2 lie in different components of $X \setminus U$, any curve joining them must pass through U, and hence also $\overline{B}(u, \operatorname{diam} U)$. Thus $\operatorname{Mod}_Q(\gamma_1, \gamma_2) \leq \operatorname{Mod}_Q(\overline{B}(u, \operatorname{diam} U), \gamma_1)$. Let Γ be the collection of curves joining $X \setminus B(u, 2K \operatorname{diam} U)$ to $\overline{B}(u, \operatorname{diam} U)$. By Lemma 3.4.4, $\operatorname{Mod}_Q(\Gamma) \leq C_1(\log(2K))^{1-Q}$. Clearly $\operatorname{Mod}_Q(\overline{B}(u, \operatorname{diam} U), \gamma_1) \leq \operatorname{Mod}_Q(\Gamma)$. By choice of K,

$$\begin{aligned}\Phi(2) \leq \operatorname{Mod}_Q(\gamma_1, \gamma_2) \leq \operatorname{Mod}_Q(\overline{B}(u, \operatorname{diam} U), \gamma_1) \leq \operatorname{Mod}_Q(\Gamma) &\leq C_1(\log(2K))^{1-Q} \\ &< \Phi(4),\end{aligned}$$

which is a contradiction since Φ is decreasing.

For (b), since $\partial U \subset \overline{U}$ and $\operatorname{diam} U = \operatorname{diam} \overline{U}$, we have $\operatorname{diam} \partial U \leq \operatorname{diam} \overline{U} = \operatorname{diam} U$. Next, we prove $\operatorname{diam} U \leq K \operatorname{diam} \partial U$. Suppose for a contradiction that $\operatorname{diam} U > K \operatorname{diam} \partial U$. Choose $x_1, x_2 \in U$ such that $d(x_1, x_2) \geq \frac{1}{2} \operatorname{diam} U$ and let γ_1 be a curve in U joining x_1 and x_2. Note that there exists $y' \in X$ such that $\operatorname{dist}(y', U) \geq 2 \operatorname{diam} U$ as otherwise we would have $\operatorname{diam} X < 5 \operatorname{diam} U$, which is false. Since X is connected, there is a point $y \in X$ such that $\operatorname{dist}(y, U) = 2 \operatorname{diam} U$. Let σ be a curve joining y to a point on the boundary of U, and let y_0 be the first point σ intersects $\overline{N}(U, \operatorname{diam} U)$. Set γ_2 to be a subcurve of σ joining y to y_0. Noting $\Delta(\gamma_1, \gamma_2) \leq 4$, we have $\operatorname{Mod}_Q(\gamma_1, \gamma_2) \geq \Phi(4)$.

Fix now $z \in \partial U$. Set $B_1 = \overline{B}(z, \partial U)$ and $B_2 = B(z, \operatorname{diam} U)$. Note $\gamma_2 \subset X \setminus B_2$, as $\operatorname{dist}(\gamma_2, U) \geq \operatorname{diam} U$. Let γ be a curve joining γ_1 to γ_2. Since $\gamma_1 \subset U$ and $\gamma_2 \subset X \setminus U$, γ necessarily intersects ∂U, and hence B_1. On the other hand, γ intersects γ_2, hence it intersects $X \setminus B_2$. Thus $\operatorname{Mod}_Q(\gamma_1, \gamma_2) \leq \operatorname{Mod}_Q(B_1, X \setminus B_2)$. By Lemma 3.4.4,

$$\operatorname{Mod}_Q(B_1, X \setminus B_2) \leq C_1 \left(\log \left(\frac{\operatorname{diam} U}{\operatorname{diam} \partial U} \right) \right)^{1-Q} < C_1(\log(K))^{1-Q}.$$

Combining the previous inequalities and using the definition of K,

$$\Phi(4) \leq \operatorname{Mod}_Q(\gamma_1, \gamma_2) \leq \operatorname{Mod}_Q(B_1, X \setminus B_2) < C_1(\log(K))^{1-Q} = \Phi(4)$$

which gives a contradiction. □

Lemma 3.5.4. *Let E be a compact, totally disconnected metric space. Given $\delta \in (0,1)$, there exist mutually disjoint closed sets $E_1, \ldots, E_k \subset E$ such that $E = E_1 \cup \cdots \cup E_k$ and for all $i = 1, \ldots, k$, we have $\operatorname{diam} E_i \leq \delta \operatorname{diam} E$.*

Proof. By Brouwer's theorem (see [Kec95, Theorem 7.8]), there exists an embedding $f : E \to \mathcal{C}$ of E into the standard Cantor set $\mathcal{C}$.

We create a dyadic subdivision $\{\mathcal{C}_u : u \in \{1,2\}^*\}$ as follows. Let $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ be the similarities $\phi_1(x) = \frac{1}{3}x$ and $\phi_2(x) = \frac{1}{3}x + \frac{2}{3}$. Define $\mathcal{C}_\varepsilon = \mathcal{C}$. Assuming that we have defined $\mathcal{C}_u$ for some $u \in \{1,2\}^*$, define $\mathcal{C}_{iu} = \phi_i(\mathcal{C}_u)$ for $i \in \{1,2\}$.

By continuity of f, there exists $n \in \mathbb{N}$ such that $\sup_{w \in \{1,2\}^n} \operatorname{diam} f^{-1}(\mathcal{C}_w) \leq \delta \operatorname{diam} E$. To complete the proof, set $\{E_1, \ldots, E_k\} = \{f^{-1}(\mathcal{C}_w) : w \in \{1,2\}^n\} \setminus \{\varnothing\}$. □

We are now ready to prove Proposition 3.5.1.

Proof of Proposition 3.5.1. Rescaling the metric, we may assume that $\operatorname{diam} E \leq 1/2$.

Fix a point $o \in E$, define $V_\varepsilon = B(o, 1)$, and note that $\operatorname{dist}(\partial V_\varepsilon, E) \geq \frac{1}{2}$. By Lemma 3.5.3, $X \setminus V_\varepsilon$ has a unique unbounded component; denote the complement of this unbounded component by U_ε. Clearly U_ε is connected and has connected complement, so it is full. Moreover, $\operatorname{diam} U_\varepsilon \leq K \operatorname{diam} V_\varepsilon \leq 2K$ and

$$\operatorname{dist}(\partial U_\varepsilon, E) \geq \operatorname{dist}(\partial V_\varepsilon, E) \geq 1/2 \geq (4K)^{-1} \operatorname{diam} U_\varepsilon.$$

Assume now that for some $w \in \mathbb{N}^*$ we have defined a full set U_w such that $E \cap U_w \neq \varnothing$ and $\operatorname{dist}(\partial U_w, E) > 0$. Note that $E \cap U_w$ is compact and totally

disconnected. By Lemma 3.5.4 we can partition $E \cap U_w = E_1 \cup \cdots \cup E_k$ such that $\operatorname{diam} E_i \leq (6K)^{-1} \operatorname{diam} U_w$. Let

$$\epsilon = \min \left\{ \tfrac{1}{2} \operatorname{dist}(\partial U_w, E), \tfrac{1}{3} \min_{i \neq j} \operatorname{dist}(E_i, E_j), (6K)^{-1} \operatorname{diam} U_w \right\}.$$

Let V_w be the ϵ-neighborhood of $E \cap U_w$ and denote by $\{V_{w,i} : i \in \mathcal{J}_w\}$ the components of V_w. Since each $V_{w,i}$ contains a ball of radius ϵ, by Ahlfors regularity of X, we have that $\mathcal{J}_w$ is a finite set. Moreover,

$$\epsilon \leq \operatorname{diam} V_{w,i} \leq 2\epsilon + (6K)^{-1} \operatorname{diam} U_w \leq (2K)^{-1} \operatorname{diam} U_w. \tag{3.5.3}$$

Given $i \in \mathcal{J}_w$, by Lemma 3.5.3, the set $X \setminus V_{w,i}$ has a unique unbounded component; denote the complement of this unbounded component by $\tilde{V}_{w,i}$. Clearly $V_{w,i} \subseteq \tilde{V}_{w,i}$. We claim that for any distinct $i, j \in \mathcal{J}_w$, either $\tilde{V}_{w,i} \subset \tilde{V}_{w,j}$, $\tilde{V}_{w,j} \subset \tilde{V}_{w,i}$, or $\tilde{V}_{w,i} \cap \tilde{V}_{w,j} = \varnothing$. If $V_{w,i} = V_{w,j}$, then $\tilde{V}_{w,i} = \tilde{V}_{w,j}$ and there is nothing to show. If $V_{w,i} \neq V_{w,j}$, then it is enough to show that either $(\tilde{V}_{w,i} \cup \tilde{V}_{w,j}) \setminus V_{w,i} \neq \varnothing$ or $(\tilde{V}_{w,i} \cup \tilde{V}_{w,j}) \setminus V_{w,j} \neq \varnothing$. For example, if $(\tilde{V}_{w,i} \cup \tilde{V}_{w,j}) \setminus V_{w,i} \neq \varnothing$, then $\tilde{V}_{w,i} \cup \tilde{V}_{w,j}$ is contained in a bounded component of $X \setminus V_{w,i}$, hence $\tilde{V}_{w,j} \subseteq \tilde{V}_{w,i}$. Since the union of two connected sets with nonempty intersection is connected, we need just show that, either $(\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus V_{w,i} \neq \varnothing$, or $(\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus V_{w,j} \neq \varnothing$. Suppose for contradiction that $(\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus V_{w,i} = \varnothing$ and $(\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus V_{w,j} = \varnothing$. Then

$$\varnothing = ((\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus V_{w,i}) \cup ((\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus V_{w,j}) = (\tilde{V}_{w,i} \cap \tilde{V}_{w,j}) \setminus (V_{w,i} \cap V_{w,j}) = \tilde{V}_{w,i} \cap \tilde{V}_{w,j}$$

since $V_{w,i} \cap V_{w,j} = \varnothing$ as $\{V_{w,i} : i \in \mathcal{J}_w\}$ are disjoint.

By (3.5.3) and Lemma 3.5.3, for each $i \in \mathcal{J}_w$

$$\begin{aligned} \epsilon \leq \operatorname{diam} V_{w,i} \leq \operatorname{diam} \tilde{V}_{w,i} \leq K \operatorname{diam} \partial \tilde{V}_{w,i} \leq K \operatorname{diam} \partial V_{w,i} \leq K \operatorname{diam} V_{w,i} \\ \leq \tfrac{1}{2} \operatorname{diam} U_w \end{aligned} \tag{3.5.4}$$

Removing some indices from $\mathcal{J}_w$ if necessary, we obtain $\tilde{\mathcal{J}}_w \subset \mathcal{J}_w$ such that $\bigcup_{i\in\mathcal{J}_w} \tilde{V}_{w,i} \subset \bigcup_{i\in\tilde{\mathcal{J}}_w} \tilde{V}_{w,i}$ and distinct $i, j \in \tilde{\mathcal{J}}_w$ satisfy $\tilde{V}_{w,i} \neq \tilde{V}_{w,j}$. Let $\{U_{wi} : i \in \{1,\dots,n_w\}\}$ denote the maximal elements (with respect to inclusion) of $\{\tilde{V}_{w,i} : i \in \tilde{\mathcal{J}}_w\}$. For each $i \in \{1,\dots,n_w\}$, there exists $j \in \tilde{\mathcal{J}}_w$ such that $U_{wi} = \tilde{V}_{w,j}$. Since the complement of $\tilde{V}_{w,j}$ is connected, it is clear that U_{wi} is connected and has connected complement, so U_{wi} is full.

Let $\mathcal{W}$ be the set of all finite words $v \in \mathbb{N}^*$ such that U_v has been defined. Property (D1) follows easily by induction on $|w|$. Properties (D2), (D3), and (D5) follow immediately from the construction of the domains U_w and induction on $|w|$. Property (D4) follows from the construction of the domains U_w, induction on $|w|$, and (3.5.4).

Finally, for (D6), fix $w \in \mathcal{W}$. Since X is fully connected and U_w is a connected set containing the full set U_{w1}, $\overline{U_w} \setminus U_{w1}$ is path connected. Assume for some $k \in \{1,\dots,n_w\}$ that $\overline{U_w} \setminus \bigcup_{i=1}^{k-1} U_{wi}$ is path connected. Again since X is fully connected and $\overline{U_w}\setminus\bigcup_{i=1}^{k-1} U_{wi}$ is a connected set containing the full set U_{wk}, $\overline{U_w}\setminus\bigcup_{i=1}^{k} U_{wi}$ is path connected. Proceeding inductively, we obtain (D6).

Assume now that X is c-uniformly disconnected. We prove the proposition for $\xi = 12cK$ where $K > 1$ is the constant from Lemma 3.5.3. Fix $\delta \in (0, (2\xi)^{-1})$ and let $N = \lceil C_0^2 \delta^{-Q} \rceil$.

Define U_ε as before. Assume that for some $w \in \{1,\dots,N\}^*$ we have defined a full set U_w such that $\operatorname{dist}(\partial U_w, E) \geq (12c)^{-1} \operatorname{diam} U_w$. Let V_w be a $(\delta \operatorname{diam} U_w)$-neighborhood of $E \cap U_w$ and denote by $\{V_{w,i} : i \in \mathcal{J}_w\}$ the components of V_w. We claim that

$$\delta \operatorname{diam} U_w \leq \operatorname{diam} V_{w,i} \leq 12c\delta \operatorname{diam} U_w, \qquad \text{for all } i \in \mathcal{J}_w. \tag{3.5.5}$$

The lower bound of $\operatorname{diam} V_{w,i}$ is clear. Suppose for a contradiction that $\operatorname{diam} V_{w,i} > 12c\delta \operatorname{diam} V_w$ for some $i \in \mathcal{J}_w$. Choose $x, y \in V_{w,i}$ such that $d(x,y) \geq 2^{-1} \operatorname{diam} V_{w,i}$ and choose $x', y' \in V_{w,i} \cap E$ such that $d(x,x') < \delta \operatorname{diam} U_w$ and $d(y,y') < \delta \operatorname{diam} U_w$.

Then

$$d(x', y') \geq 2^{-1} \operatorname{diam} V_{w,i} - 2\delta \operatorname{diam} U_w > 2(3c-1)\delta \operatorname{diam} U_w.$$

Since $V_{w,i}$ is connected and each point in $V_{w,i}$ is distance at most $\delta \operatorname{diam} U_w$ from a point in E, we can find a finite chain of points $\{x_0, \ldots, x_k\} \subset E$, where $x_0 = x'$, $x_k = y'$, and $d(x_{j-1}, x_j) \leq 2\delta \operatorname{diam} U_w$ for each $j \in \{1, \ldots, k\}$. But this contradicts the equivalent definition of uniform disconnectedness given in Lemma 3.4.3. Thus, (3.5.5) follows. Moreover, for all $i \in \mathcal{J}_w$,

$$\operatorname{dist}(\partial V_{w,i}, E) \geq \delta \operatorname{diam} U_w \geq (12c)^{-1} \operatorname{diam} V_{w,i}. \tag{3.5.6}$$

For each $i \in \mathcal{J}_w$, let $B_i \subset V_{w,i}$ be a ball of radius $\delta \operatorname{diam} U_w$. The balls B_i are mutually disjoint and contained in U_w. By Ahlfors regularity,

$$C_0^{-1} \delta^Q (\operatorname{diam} U_w)^Q \operatorname{card} \mathcal{J}_w \leq \sum_{i \in \mathcal{J}_w} \mu(B_i) \leq \mu(U_w) \leq C_0 (\operatorname{diam} U_w)^Q.$$

Therefore, $\operatorname{card} \mathcal{J}_w \leq C_0^2 \delta^{-Q} \leq N$.

Define $\tilde{V}_{w,i}$ as before and recall that $V_{w,i} \subseteq \tilde{V}_{w,i}$ and that for any distinct $i, j \in \mathcal{J}_w$, either $\tilde{V}_{w,i} \subset \tilde{V}_{w,j}$, $\tilde{V}_{w,j} \subset \tilde{V}_{w,i}$, or $\tilde{V}_{w,i} \cap \tilde{V}_{w,j} = \varnothing$. By (3.5.5) and Lemma 3.5.3, for each $i \in \mathcal{J}_w$

$$\begin{aligned} \delta \operatorname{diam} U_w \leq \operatorname{diam} V_{w,i} \leq \operatorname{diam} \tilde{V}_{w,i} \leq K \operatorname{diam} \partial \tilde{V}_{w,i} \leq K \operatorname{diam} \partial V_{w,i} \leq K \operatorname{diam} V_{w,i} \\ \leq 12cK \operatorname{diam} U_w. \end{aligned} \tag{3.5.7}$$

Define now $\{U_{wi} : i \in \tilde{\mathcal{J}}_w\}$ as before and recall that each U_{wi} is full. Let $\mathcal{W}$ be the set of all finite words $v \in \mathbb{N}^*$ such that U_v has been defined. The fact that $\mathcal{W} \subset \{1, \ldots, N\}^*$ follows by an induction on $|w|$. Inequality (3.5.1) follows from (3.5.7) while (D5') follows from (3.5.6) and the fact that for all $w \in \{1, \ldots, n_w\}$ there exists $j \in \mathcal{J}_w$ such that $\partial U_{w,i} \subset \partial V_{w,j}$.

For (3.5.2), recall that $\xi = 12cK$ and that $\delta \in (0, (2\xi)^{-1})$. By (D5') and (3.5.1),

$$\begin{aligned}\operatorname{dist}(U_w, \partial U_{w\uparrow}) &\geq \operatorname{dist}(\partial U_{w\uparrow}, E) - \delta \operatorname{diam} U_{w\uparrow} \\ &\geq (\xi^{-1} - \delta) \operatorname{diam} U_{w\uparrow} \\ &\geq (2\xi)^{-1} \operatorname{diam} U_{w\uparrow} \\ &\geq \xi^{-1} \operatorname{diam} U_w.\end{aligned}$$

This completes the proof of the proposition. □

3.6 Co-uniformity of uniformly disconnected sets

The goal of this section is to prove Theorem 3.3.5. For this section, fix a complete, fully connected, unbounded, (C_0, Q)-regular metric measure space X supporting a (Q, C)-Poincaré inequality. By Lemmas 3.4.6 and 3.4.7 we may assume that (X, d) is geodesic. Fix also a c-uniformly disconnected set $E \subset X$. Without loss of generality we may assume that $\operatorname{diam} E \leq 1/2$.

Let $\xi \geq 1$ and $N \in \mathbb{N}$ be the constants of Proposition 3.5.1. Fix $\delta = (4\xi)^{-1}$ and let $\mathcal{W} \subseteq \{1, \ldots, N\}^*$ and $\{U_w : w \in \mathcal{W}\}$ be the dictionary and decomposition of Proposition 3.5.1. While the "top set" U_ε is only a bounded neighborhood of E, it is useful for the proof of Theorem 3.3.5 to have a decomposition of the entirety of X.

Lemma 3.6.1. *There exists a collection of full, open sets $U_\varepsilon = U_0 \subset U_{-1} \subset U_{-2} \subset \cdots$ such that the following conditions hold.*

(E1) For each $n \in \mathbb{N}$, $(2\xi)^{-2} \operatorname{diam} U_{-n} \leq \operatorname{diam} U_{1-n} \leq \frac{1}{4} \operatorname{diam} U_{-n}$, and

$$\operatorname{dist}(U_{1-n}, \partial U_{-n}) \geq \xi^{-1} \operatorname{diam} U_{1-n}.$$

(E2) For each $n \in \mathbb{N}$, $\operatorname{dist}(\partial U_{1-n}, E) \geq \xi^{-1} \operatorname{diam} U_{1-n}$.

(E3) For each $n \in \mathbb{N}$, $\hat{U}_{-n} := \overline{U}_{-n} \setminus U_{1-n}$ is path connected.

(E4) We have $X = \bigcup_{n\in\mathbb{N}} U_{-n}$.

Proof. Set $U_0 = U_\varepsilon$. Properties (E1)–(E3) are vacuously true. Assume that for some integer $n \geq 0$, we have defined a full set U_{-n} such that (E1)–(E3) hold for n. Let V_{-n-1} be the $(4\xi \operatorname{diam} U_{-n})$-neighborhood of U_{-n}. By Lemma 3.5.3, there is a unique unbounded component of $X \setminus V_{-n-1}$; denote the complement of this unbounded component by U_{-n-1}. Clearly U_{-n-1} is connected and has connected complement, so it is full. Moreover, recalling that $\xi = 12cK$ in the proof of Proposition 3.5.1,

$$(2\xi)^{-2} \operatorname{diam} U_{-n-1} \leq \operatorname{diam} U_{-n} \leq 2^{-2} \operatorname{diam} U_{-n-1}$$

and

$$\operatorname{dist}(\partial U_{-n-1}, E) \geq \operatorname{dist}(\partial U_{-n-1}, U_{-n}) \geq 4\xi \operatorname{diam} U_{-n} \geq \xi^{-1} \operatorname{diam} U_{-n-1}.$$

Lastly, since X is fully connected and U_{-n-1} is a connected set containing a neighborhood of the full set U_{-n}, we have $\hat{U}_{-n-1}$ is connected and we obtain (E1)–(E3) for $n+1$.

Proceeding inductively, we define sets $\{U_{-n}\}_{n\in\mathbb{N}}$. Given an integer $n \geq 0$, by (E1),

$$\operatorname{diam} U_0 + \cdots + \operatorname{diam} U_{-n} \geq \operatorname{diam} U_\varepsilon (1 + \cdots + 4^n) \geq 4^n \operatorname{diam} U_\varepsilon.$$

Fix $o \in U_\varepsilon$. Then for all integers $n \geq 0$, $B(o, 4^n \operatorname{diam} U_\varepsilon) \subset U_{-n-1}$ and (E4) follows. □

We now proceed to the proof of Theorem 3.3.5. To simplify the exposition, when we write `data`, we mean the constants C_0, Q, C, and c.

We also extend the word-parent function by setting $\varepsilon^\uparrow = -1$ and by setting $(-n)^\uparrow = -n - 1$ for each $n \in \mathbb{N}$. We also extend the word-length function by setting $|-n| = -n$ for all $n \in \mathbb{N}$. Finally, we set $\mathcal{W}' = \{-n : n \in \mathbb{N}\} \cup \mathcal{W}$.

Proof of Theorem 3.3.5. Fix $x, y \in X \setminus E$. There exists a maximal $u \in \mathcal{W}'$ (with respect to $|\cdot|$) such that $x, y \in U_u$. Further, there exist $v, w \in \mathcal{W}'$ such that $x \in \hat{U}_v$ and $y \in \hat{U}_w$. Clearly $|u| \le \min\{|v|, |w|\}$. Without loss of generality, we may assume $|v| \le |w|$. There are a few cases to consider.

Case 1. Suppose that $d(x,y) < (2\xi)^{-5} \operatorname{diam} U_u$. We claim that $|v| \le |u| + 1$ and $|w| \le |u| + 1$. To see this, suppose for a contradiction that $|w| - |u| > 1$. Let $u_1, u_2 \in \mathcal{W}'$ such that $u_1^{\uparrow} = u$, $u_2^{\uparrow} = u_1$, and $y \in U_{u_2}$. By Proposition 3.5.1(D4') and Lemma 3.6.1(E1),

$$d(x,y) \ge \operatorname{dist}(\partial U_{u_1}, U_{u_2}) \ge \xi^{-1} \operatorname{diam} U_{u_2} \ge 2^{-2}\xi^{-3} \operatorname{diam} U_{u_1} \ge 2^{-4}\xi^{-5} \operatorname{diam} U_u,$$

which contradicts the assumption $d(x,y) < (2\xi)^{-5} \operatorname{diam} U_u$.

Let Γ be a geodesic joining x to y. Clearly $\operatorname{length}(\Gamma) = d(x,y)$. By Proposition 3.5.1(D5'), Lemma 3.6.1(E2), and since $|w| \le |u| + 1$, for any $z \in \Gamma$ and $w_0 \in \mathcal{W}'$ with $w_0^{\uparrow} = w$,

$$\begin{aligned}
\operatorname{dist}(z, E) &\ge \operatorname{dist}(\partial U_{w_0}, E) - \operatorname{length}(\Gamma) \\
&\ge \xi^{-1} \operatorname{diam} U_{w_0} - d(x,y) \\
&> 2^{-2}\xi^{-3} \operatorname{diam} U_w - (2\xi)^{-5} \operatorname{diam} U_u \\
&\ge (2\xi)^{-5} \operatorname{diam} U_u \\
&> d(x,y) \\
&= \operatorname{length}(\Gamma) \\
&\ge \operatorname{dist}(z, \{x, y\}).
\end{aligned}$$

Case 2. Suppose that $d(x,y) \ge (2\xi)^{-5} \operatorname{diam} U_u$ and $u = v = w$. By Lemma 3.5.2, there is a curve Γ joining x and y lying in the neighborhood $N(\hat{U}_u, (2\xi)^{-3} \operatorname{diam} U_u)$ with length at most $c_0 d(x,y)$, where $c_0(\texttt{data}) \ge 1$. Then

$$d(x,y) \le \operatorname{length}(\Gamma) \le c_0 d(x,y) \le c_0 \operatorname{diam} U_u$$

and for any $z \in \Gamma$ and $u_0 \in \mathcal{W}'$ such that $u_0^{\uparrow} = u$, by Proposition 3.5.1(D4',D5') and Lemma 3.6.1(E1,E2),

$$\begin{aligned}\operatorname{dist}(z, E) &\geq \operatorname{dist}(\partial U_{u_0}, E) - (2\xi)^{-3} \operatorname{diam} U_u \geq \xi^{-1} \operatorname{diam} U_{u_0} - (2\xi)^{-3} \operatorname{diam} U_u \\ &\geq (2\xi)^{-3} \operatorname{diam} U_u.\end{aligned}$$

Case 3. Suppose that $d(x, y) \geq (2\xi)^{-5} \operatorname{diam} U_u$ and $u = v \neq w$. For each $i \in \{1, \ldots, |w| - |u|\}$, let $w_i \in \mathcal{W}'$ such that $w_1 = w$, $w_i^{\uparrow} = w_{i+1}$, and $w_{|w|-|u|} = u$. Let $z_i \in \partial U_{w_i}$ for each $i \in \{1, \ldots, |w| - |u| - 1\}$. By Case 2, there exists a curve γ_i in $N(\hat{U}_{w_i}, (2\xi)^{-3} \operatorname{diam} U_{w_i})$ joining z_{i-1} and z_i with length at most $c_0 d(z_{i-1}, z_i)$ for each $i \in \{2, \ldots, |w| - |u| - 2\}$, there exists a curve γ_1 in $N(\hat{U}_w, (2\xi)^{-3} \operatorname{diam} U_w)$ joining x and z_1 with length at most $c_0 d(x, z_1)$, and there exists a curve $\gamma_{|w|-|u|}$ in $N(\hat{U}_u, (2\xi)^{-3} \operatorname{diam} U_u)$ joining $z_{|w|-|u|-1}$ to y with length at most $c_0 d(z_{|w|-|u|-1}, y)$. Define Γ to be the concatenation of these curves, which joins x to y. By Proposition 3.5.1(D4') and Lemma 3.6.1(E1),

$$\begin{aligned}\operatorname{length}(\Gamma) &= \sum_{i=1}^{|w|-|u|} \operatorname{length}(\gamma_i) \\ &\lesssim_{c_0} d(x, z_1) + d(z_{|w|-|u|-1}, y) + \sum_{i=2}^{|w|-|u|-1} d(z_{i-1}, z_i) \\ &\leq \operatorname{diam} U_w + \operatorname{diam} U_u + \sum_{i=2}^{|w|-|u|-1} \operatorname{diam} U_{w_i} \\ &< 2 \operatorname{diam} U_u \\ &\leq 2^6 \xi^5 d(x, y).\end{aligned}$$

Let now $z \in \Gamma$ and suppose $z \in \gamma_i$ for some $i \in \{2, \ldots, |w| - |u| - 1\}$ (the cases where $z \in \gamma_1$ or $z \in \gamma_{|w|-|u|}$ are similar). Since γ_i lies in $N(\hat{U}_{w_i}, (2\xi)^{-3} \operatorname{diam} U_{w_i})$, by Proposition 3.5.1(D4',D5') and Lemma 3.6.1(E1,E2), for any $v_0 \in \mathcal{W}'$ such that

$v_0^{\uparrow} = w_i$,

$$\begin{aligned}\operatorname{dist}(z, E) &\geq \operatorname{dist}(\partial U_{v_0}, E) - (2\xi)^{-3} \operatorname{diam} U_{w_i} \\ &\geq \xi^{-1} \operatorname{diam} U_{v_0} - (2\xi)^{-3} \operatorname{diam} U_{w_i} \\ &\geq (2\xi)^{-3} \operatorname{diam} U_{w_i},\end{aligned}$$

while

$$\begin{aligned}\operatorname{dist}(z, \{x, y\}) \leq \operatorname{dist}(\gamma_i, \{x, y\}) + \operatorname{length}(\gamma_i) &\leq \operatorname{diam} U_{w_i} + c_0 d(z_{i-1}, z_i) \\ &\leq (1 + c_0) \operatorname{diam} U_{w_i}.\end{aligned}$$

Case 4. Suppose that $d(x, y) \geq (2\xi)^{-5} \operatorname{diam} U_u$ and $u \neq v$. The proof is a two-sided argument similar to that in Case 3. Let $u_1, u_2 \in \mathcal{W}'$ such that $u_1^{\uparrow} = u_2^{\uparrow} = u$ and $x \in U_{u_1}$ and $y \in U_{u_2}$. Fix $x' \in \partial U_{u_1}$ and $y' \in \partial U_{u_2}$. Let γ_1 be the curve from Case 3 joining x to x', let γ_2 be the curve from Case 3 joining y' to y, and let γ_0 be the curve in $N(\hat{U}_u, (2\xi)^{-3} \operatorname{diam} U_u)$ from Case 2 joining x' to y'. Let Γ be the concatenation of the curves γ_0, γ_1, and γ_2. We have

$$\operatorname{length}(\Gamma) = \operatorname{length}(\gamma_0) + \operatorname{length}(\gamma_1) + \operatorname{length}(\gamma_2) \lesssim_{c_0, \xi} \operatorname{diam} U_u \lesssim_{\xi} d(x, y).$$

For a point $z \in \Gamma$, we have $\operatorname{dist}(z, \{x, y\}) \lesssim_{c_0, \xi} \operatorname{dist}(z, E)$ from Cases 2 and 3. □

3.7 Preparation for the proofs of Theorem 3.3.2 and Theorem 3.3.3

Fix, for the rest of the paper, a metric measure space (X, d, μ) which is complete, fully connected, (C_0, Q)-regular, and supports a $(Q - 1, C)$-Poincaré inequality for some $C_0, C > 1$, and $Q > 2$. By Lemmas 3.4.6 and 3.4.7 we may assume that (X, d, μ) is geodesic. Fix also a totally disconnected set $E \subset X$ and a number $c \geq 1$. Let $\xi \geq 1$

be the constant from Proposition 3.5.1 associated to the number c. Set $\delta = (4\xi)^{-1}$, and let $N \in \mathbb{N}$, $\mathcal{W} \subset \mathbb{N}^*$, and $\{U_w : w \in \mathcal{W}\}$ be as in Proposition 3.5.1 associated to the set E and the number δ.

To simplify the exposition, we make the assumption that the only word in $\mathcal{W}$ with length one is 1 (so E is completely contained in U_1). This can easily be arranged by rescaling the metric. The use of this assumption is that given a word $w \in \mathcal{W}$, we construct curves that lie in a neighborhood of the children of U_w, and join the boundaries of the grandchildren of U_w. Since we look two generations ahead, it is useful to have U_ε and U_1 be "extraneous" so that we do not have to consider these cases separately.

3.7.1 Basic notation

We establish some notation which we use throughout the rest of the paper.

- For each $w \in \mathcal{W}$, by Proposition 3.5.1, there exists $n_w \in \mathbb{N}$ such that $wi \in \mathcal{W}$ if and only if $i \in \{1, \dots, n_w\}$.
- For each $w \in \mathcal{W}$, define $\hat{U}_w = U_w \setminus \bigcup_{i=1}^{n_w} U_{wi}$.
- For each $w \in \mathcal{W}$, define

$$\epsilon_w = (4 \operatorname{diam} U_w)^{-1} \min \left\{ \operatorname{dist}(U_w, \partial U_{w^\uparrow}), \min_{i=1,\dots,n_w} \operatorname{dist}(\partial U_w, U_{wi}), \right.$$
$$\left. \min_{\substack{i=1,\dots,n_w \\ j=1,\dots,n_{wi}}} \operatorname{dist}(\partial U_{wi}, U_{wij}), \min_{i=1,\dots,n_w} \operatorname{diam} U_{wi} \right\}$$

  with the understanding that $\operatorname{dist}(\partial U_w, \partial U_{w^\uparrow})$ is dropped if $w = \varepsilon$. If E is c-uniformly disconnected, then we set $\epsilon_w = (4\xi)^{-3}$.

- For each $w \in \mathcal{W}$, define $\mathcal{N}_w = N(\hat{U}_w, \epsilon_w \operatorname{diam} U_w)$.

- For each $w \in \mathcal{W} \setminus \{\varepsilon\}$, define

$$\lambda_w = \min\left\{ \max_{i,j=1,\dots,n_w} \frac{\operatorname{diam} U_{wi}}{\operatorname{diam} U_{wj}}, \max_{i=1,\dots,n_w} \frac{\operatorname{diam} U_{wi}}{\operatorname{dist}(\partial U_{wi}, E)} \right\}.$$

  If E is c-uniformly disconnected, then set $\lambda_w = \xi$.

Note that if E is c-uniformly disconnected, and if $w \neq \varepsilon$, then a simple computation involving (3.5.1) and (3.5.2) shows that $\epsilon_w \geq (4\xi)^{-3}$ and $\lambda_w \geq \xi$, which justifies the choices $\epsilon_w = (4\xi)^{-3}$ and $\lambda_w = \xi$.

Lemma 3.7.1. *Let $w, u \in \mathcal{W}$ be two distinct words that do not have a common parent and one is not the parent of the other. Then*

$$\operatorname{dist}(\mathcal{N}_w, \mathcal{N}_u) \geq \epsilon_w \operatorname{diam} U_w + \epsilon_u \operatorname{diam} U_u.$$

Proof. The assumptions on w, u imply that

$$\operatorname{dist}(\hat{U}_w, \hat{U}_u) \geq \operatorname{dist}\left(\hat{U}_w, \partial U_{w^\uparrow} \cup \bigcup_{i=1}^{n_w} \bigcup_{j=1}^{n_{wi}} U_{wij}\right) \geq 4\epsilon_w \operatorname{diam} U_w.$$

Therefore,

$$\operatorname{dist}(\mathcal{N}_w, \mathcal{N}_u) \geq \operatorname{dist}(\hat{U}_w, \hat{U}_u) - \epsilon_w \operatorname{diam} U_w - \epsilon_u \operatorname{diam} U_u \geq \epsilon_w \operatorname{diam} U_w + \epsilon_u \operatorname{diam} U_u. \square$$

3.7.2 Entries and exits

We define here for each $w \in \mathcal{W}$ an "entry" to U_w and an "exit". The following lemma allows us to do so.

Lemma 3.7.2. *Given $n \in \mathbb{N}$ and $\lambda > 1$, there is a constant $\rho(n, C_0, Q, C, \lambda) \in (0,1)$ satisfying the following property. Let $W_1, \dots, W_n \subset X$ be mutually disjoint domains such that for each distinct $i, j \in \{1, \dots, n\}$, $\operatorname{diam} W_i \leq \lambda \operatorname{diam} W_j$ and for each $i \in \{1, \dots, n\}$ there exists a ball $B_i \subset W_i$ of radius $\lambda^{-1} \operatorname{diam} W_i$. Then for each*

$i \in \{1, \dots, n\}$ there exists $x_i \in \partial W_i$ such that if $i, j \in \{1, \dots, n\}$ are distinct, then $d(x_i, x_j) \geq \rho \operatorname{diam} W_i$.

Proof. The construction of constant ρ and points $x_1, \dots, x_n$ is done in an inductive fashion.

Fix $x_1 \in \partial W_1$ and let $\rho_1 = 1/2$. Suppose that for some $k \in \{1, \dots, n-1\}$, we have defined $\rho_k \in (0, 1)$ and points $x_i \in \partial W_i$ for $i \in \{1, \dots, k\}$ such that $d(x_i, x_j) \geq \rho_k \operatorname{diam} W_i$ for distinct $i, j \in \{1, \dots, k\}$. By the assumptions of the lemma, there exists $u \in W_{k+1}$ such that $B(u, \lambda^{-1} \operatorname{diam} W_{k+1}) \subset W_{k+1}$. Fix $v \in X$ such that $\operatorname{dist}(v, W_{k+1}) = \operatorname{diam} W_{k+1}$ and set $B = \overline{B}(u, (3\lambda)^{-1} \operatorname{diam} W_{k+1})$ and $B' = \overline{B}(v, (3\lambda)^{-1} \operatorname{diam} W_{k+1})$. Note that

$$\Delta(B, B') \leq \frac{d(u, v)}{\operatorname{diam} B} \leq \frac{\operatorname{diam} W_{k+1} + \operatorname{dist}(v, W_{k+1})}{\operatorname{diam} B} \leq 6\lambda,$$

so by Lemma 3.4.5 we have $\operatorname{Mod}_Q(B, B') \geq \Phi(6\lambda)$ where Φ is a decreasing homeomorphism of $(0, \infty)$ depending only on C_0, Q, and C.

Let $C_1(C_0, Q) \geq 1$ be the constant from Lemma 3.4.4. Let $a_{k+1} \in (0, (3\lambda)^{-1})$ so that

$$C_1 \left(\log \left(\frac{\lambda^{-1}}{3a_{k+1}} \right) \right)^{1-Q} \leq \frac{\Phi(6\lambda)}{2k}.$$

For $i \in \{1, \dots, k\}$ define the balls $B_i = \overline{B}(x_i, a_{k+1} \operatorname{diam} W_{k+1})$ and $B_i' = B(x_i, (3\lambda)^{-1} \operatorname{diam} W_{k+1})$. Let Γ_0 denote the family of curves joining B and B', and for each $i \in \{1, \dots, k\}$, let Γ_i denote the family of curves in Γ_0 that intersect B_i. Note that for each $i \in \{1, \dots, k\}$, $d(x_i, u) \geq \lambda^{-1} \operatorname{diam} W_{k+1}$. Hence each curve $\gamma \in \Gamma_i$ has a subcurve which joins B_i and $X \setminus B_i'$, so by Lemma 3.4.4,

$$\operatorname{Mod}_Q(\Gamma_i) \leq \operatorname{Mod}_Q(B_i, X \setminus B_i') \leq C_1 \left(\log \left(\frac{\lambda^{-1}}{3a_{k+1}} \right) \right)^{1-Q} \leq \frac{\Phi(6\lambda)}{2k}.$$

Therefore,

$$\operatorname{Mod}_Q(\Gamma_0 \setminus (\Gamma_1 \cup \cdots \cup \Gamma_k)) \geq \operatorname{Mod}_Q(\Gamma_0) - \sum_{i=1}^{k} \operatorname{Mod}_Q(\Gamma_i) \geq \tfrac{1}{2}\Phi(6\lambda).$$

Namely, $\operatorname{Mod}_Q(\Gamma_0 \setminus (\Gamma_1 \cup \cdots \cup \Gamma_k)) > 0$, so there is a curve γ which joins B and B' and does not intersect B_i for any $i \in \{1, \ldots, k\}$. Since $B \subset W_{k+1}$ and $B' \cap W_{k+1} = \varnothing$, there exists a point x_{k+1} where the curve γ intersects ∂W_{k+1}. Then $d(x_{k+1}, x_i) \geq a_{k+1} \operatorname{diam} W_{k+1}$ for each $i \in \{1, \ldots, k\}$. To complete the inductive step, set $\rho_{k+1} = \min\{\rho_k, \lambda^{-1} a_{k+1}\}$ and note that $d(x_i, x_j) \geq \rho_{k+1} \operatorname{diam} W_i$ for all distinct $i, j \in \{1, \ldots, k+1\}$.

Proceeding inductively, we find $\rho_n(C_0, Q, C, \lambda, n) \in (0, 1)$ and points $x_i \in \partial W_i$ for each $i \in \{1, \ldots, n\}$ such that $d(x_i, x_j) \geq \rho_n \operatorname{diam} W_i$ when $i \neq j$. □

If $x \in U_{wij}$ for some $i \in \{1, \ldots, n_w\}$ and $j \in \{1, \ldots, n_{wi}\}$, then the ball $B(x, \lambda_w \operatorname{diam} U_{wi}) \subset U_{wi}$. Therefore, for each $w \in \mathcal{W}$, we can apply Lemma 3.7.2 with $\lambda = \lambda_w$ and $W_i = U_{wi}$, $i = 1, \ldots, n_w$, and we obtain a constant $\rho_w \in (0, 1)$ and points $x_{wi} \in \partial U_{wi}$, $i \in \{1, \ldots, n_w\}$, such that

$$d(x_{wi}, x_{wj}) \geq \rho_w \max\{\operatorname{diam} U_{wi}, \operatorname{diam} U_{wj}\}, \qquad \text{for distinct } i, j \in \{1, \ldots, n_w\}.$$

For each $i \in \{1, \ldots, n_w\}$, fix a point $y_{wi} \in \partial U_{wi}$ such that $d(x_{wi}, y_{wi}) = \frac{1}{3}\rho_w \operatorname{diam} U_{wi}$ and note that this implies $d(x_{wi}, y_{wj}) \geq \frac{1}{3}\rho_w \operatorname{diam} U_{wi}$ and $d(y_{wi}, y_{wj}) \geq \frac{1}{3}\rho_w \operatorname{diam} U_{wi}$ for distinct $i, j \in \{1, \ldots, n_w\}$.

Remark 3.7.3. If E is c-uniformly disconnected, then ρ_w is independent of w and only depends on C_0, Q, C, and c.

Lastly, for each $w \in \mathcal{W}$ and $i \in \{1, \ldots, n_w\}$, define

$$X_{wi} = B(x_{wi}, \tfrac{1}{12}\epsilon_{wi}\rho_w \operatorname{diam} U_{wi}) \qquad \text{and} \qquad Y_{wi} = B(y_{wi}, \tfrac{1}{12}\epsilon_{wi}\rho_w \operatorname{diam} U_{wi})$$

We view X_{wi} as the "entry" of the set U_{wi} and we view Y_{wi} as the "exit" of the set U_{wi}.

Lemma 3.7.4. *Let $w, u \in \mathcal{W}$, $i \in \{1, \dots, n_w\}$, and $j \in \{1, \dots, n_u\}$. Then*

$$\operatorname{dist}(X_{wi}, Y_{uj}) \geq \tfrac{1}{12}(\rho_w \epsilon_{wi} \operatorname{diam} U_{wi} + \rho_u \epsilon_{uj} \operatorname{diam} U_{uj}). \tag{3.7.1}$$

Moreover, if $wi \neq uj$, then

$$\min\{\operatorname{dist}(X_{wi}, X_{uj}), \operatorname{dist}(Y_{wi}, Y_{uj})\} \geq \tfrac{1}{12}(\rho_w \epsilon_{wi} \operatorname{diam} U_{wi} + \rho_u \epsilon_{uj} \operatorname{diam} U_{uj}). \tag{3.7.2}$$

Proof. We distinguish two cases. If $w \neq u$, then by Lemma 3.7.1 and the fact that $\rho_w, \rho_u \in (0, 1)$,

$$\begin{aligned}
\operatorname{dist}(X_{wi}, X_{uj}) &\geq d(x_{wi}, x_{uj}) - \tfrac{1}{12}\left(\epsilon_{wi}\rho_w \operatorname{diam} U_{wi} + \epsilon_{uj}\rho_u \operatorname{diam} U_{uj}\right) \\
&\geq \operatorname{dist}(\partial U_{wi}, \partial U_{uj}) - \tfrac{1}{12}\left(\epsilon_{wi}\rho_w \operatorname{diam} U_{wi} + \epsilon_{uj}\rho_u \operatorname{diam} U_{uj}\right) \\
&\geq \tfrac{1}{2}(\epsilon_{wi} \operatorname{diam} U_{wi} + \epsilon_{uj} \operatorname{diam} U_{uj}).
\end{aligned}$$

The same argument works for $\operatorname{dist}(Y_{wi}, Y_{uj})$ and for $\operatorname{dist}(X_{wi}, Y_{uj})$.

If $w = u$, then recalling that $\epsilon_{wi}, \epsilon_{wj} \in (0, 1)$,

$$\begin{aligned}
\operatorname{dist}(X_{wi}, X_{wj}) &\geq d(x_{wi}, x_{wj}) - \tfrac{1}{12}\left(\epsilon_{wi}\rho_w \operatorname{diam} U_{wi} + \epsilon_{wj}\rho_w \operatorname{diam} U_{wj}\right) \\
&\geq \tfrac{1}{2}\left(\rho_w \operatorname{diam} U_{wi} + \rho_w \operatorname{diam} U_{wj}\right) \\
&\qquad - \tfrac{1}{12}\left(\epsilon_{wi}\rho_w \operatorname{diam} U_{wi} + \epsilon_{wj}\rho_w \operatorname{diam} U_{wj}\right) \\
&\geq \tfrac{1}{4}\left(\rho_w \operatorname{diam} U_{wi} + \rho_w \operatorname{diam} U_w\right).
\end{aligned}$$

A similar argument works for $\operatorname{dist}(Y_{wi}, Y_{uj})$ and for $\operatorname{dist}(X_{wi}, Y_{uj})$. □

3.7.3 Decomposition of the unit interval

In the next lemma we define a collection of closed intervals $\{J_w : w \in \mathcal{W} \setminus \{\varepsilon\}\}$ in $[0,1]$ which form a defining sequence for some compact totally disconnected subset of $[0,1]$. Recall our assumption that the only word in $\mathcal{W}$ of length one is 1.

Lemma 3.7.5. *There exists a family of closed intervals $\{J_w : w \in \mathcal{W} \setminus \{\varepsilon\}\}$ which satisfy the following properties.*

(a) We have $J_1 = [0,1]$.

(b) For each $w \in \mathcal{W}\setminus\{\varepsilon\}$ and $i \in \{1,\dots,n_w\}$, $J_{wi} \subset J_w$ and $|J_{wi}| = (2n_w+1)^{-1}|J_w|$.

(c) For each $w \in \mathcal{W}\setminus\{\varepsilon\}$ and $i \in \{1,\dots,n_w-1\}$, $\operatorname{dist}(J_{wi}, J_{w(i+1)}) = (2n_w+1)^{-1}|J_w|$ with the right endpoint of J_{wi} lying to the left of the left endpoint of $J_{w(i+1)}$.

(d) For each $w \in \mathcal{W} \setminus \{\varepsilon\}$ and $i \in \{1, n_w\}$, $\operatorname{dist}(J_{wi}, \mathbb{R} \setminus J_w) = (2n_w+1)^{-1}|J_w|$.

Proof. Set $J_1 = [0,1]$. Assume now that for some $w \in \mathcal{W} \setminus \{\varepsilon\}$ we have defined a nondegenerate closed interval $J_w = [a,b]$. For each $i \in \{1,\dots,n_w\}$ define

$$J_{wi} = \left[a + (2i-1)(2n_w+1)^{-1}|J_w|, a + 2i(2n_w+1)^{-1}|J_w|\right].$$

A simple induction argument shows now that the intervals $\{J_w : w \in \mathcal{W}\}$ satisfy (a)–(d). □

For a given $w \in \mathcal{W} \setminus \{\varepsilon\}$, the set $J_w \setminus \bigcup_{i=1}^{n_w} J_{wi}$ is a union of $n_w + 1$ open intervals. Denote the closures of these intervals by

$$\{I_{w,i} : i \in \{0,\dots,n_w\}\},$$

ordered so that the right endpoint of $I_{w,i}$ lies to the left of the left endpoint of $I_{w,i+1}$ for each $i \in \{1,\dots,n_w-1\}$.

Lemma 3.7.6. *The collection* $\{I_{w,i} : w \in \mathcal{W} \setminus \{\varepsilon\}, i \in \{0, \ldots, n_w\}\}$ *satisfies the following properties.*

(a) For each $w \in \mathcal{W}\setminus\{\varepsilon\}$ *and* $i \in \{0, \ldots, n_w\}$, $I_{w,i} \subset J_w$ *and* $|I_{w,i}| = (2n_w+1)^{-1}|J_w|$.

(b) For each $w \in \mathcal{W}\setminus\{\varepsilon\}$ *and* $i \in \{0, \ldots, n_w-1\}$, $\operatorname{dist}(I_{w,i}, I_{w,i+1}) = (2n_w+1)^{-1}|J_w|$ *with the right endpoint of* $I_{w,i}$ *lying to the left of the left endpoint of* $I_{w,i+1}$.

(c) If $I_{w,i}$ *is adjacent to* $I_{u,j}$, *then either* $w = u^{\uparrow}$ *or* $u = w^{\uparrow}$. *In either case,*

$$|I_{w,i}| \geq (2n_w + 1)^{-1}|I_{u,j}|.$$

Proof. The first two claims are immediate. For the third, assume first that $|w| \leq |u|$. By design, $w \neq u$. Therefore, $I_{w,i}$ is adjacent to J_u. But then, $w = u^{\uparrow}$. In this case,

$$|I_{w,i}| = |J_u| \geq |I_{u,j}|.$$

Suppose now that $|w| \geq |u|$. Working as in the previous case, $u = w^{\uparrow}$ and

$$|I_{w,i}| = (2n_w + 1)^{-1}|J_w| = |I_{u,j}|. \qquad \square$$

For each $w \in \mathcal{W} \setminus \{\varepsilon\}$ and $i \in \{0, \ldots, n_w\}$, denote the middle third of $I_{w,i}$ by $\hat{I}_{w,i}$ and set

$$\Lambda_w = \frac{\operatorname{diam} U_w}{|I_{w,1}|} = (2n_w + 1)\frac{\operatorname{diam} U_w}{|J_w|}. \tag{3.7.3}$$

Note that for all $i \in \{0, \ldots, n_w\}$ we have $|I_{w,i}| = \Lambda_w^{-1} \operatorname{diam} U_w$ and $|\hat{I}_{w,i}| = (3\Lambda_w)^{-1} \operatorname{diam} U_w$.

Remark 3.7.7. The limit set

$$J_\infty = \bigcap_{n=1}^{\infty} \bigcup_{\substack{w\in\mathcal{W}\\|w|=n}} J_w \subset J_1 \tag{3.7.4}$$

is compact and totally disconnected. Moreover, if E is c-uniformly disconnected, then J_∞ is c'-uniformly disconnected for some $c'(c) > 1$ and the collection of intervals

$$\{I_{w,i} : w \in \mathcal{W} \setminus \{\varepsilon\},\ i = 0, \dots, n_w\}$$

is a *Whitney decomposition* of $J_1 \setminus J_\infty$; see [Ste70, Theorem VI.1.1,Proposition VI.1.1].

3.8 Bi-Lipschitz arcs in X

The crux of the proofs of Theorem 3.3.2 and Theorem 3.3.3 is the the following proposition which is the main result of this section.

Proposition 3.8.1. *For each $w \in \mathcal{W}$ there exists a bi-Lipschitz embedding*

$$h_w : \bigcup_{i=1}^{n_w} \bigcup_{j=0}^{n_{wi}} \hat{I}_{wi,j} \to \bigcup_{i=1}^{n_w} \mathcal{N}_{wi}$$

with the following properties.

1. *For each $w \in \mathcal{W}$ and each $i \in \{1, \dots, n_w\}$,*
   - *(a) the arc $h_w|\hat{I}_{wi,0}$ joins X_{wi} to X_{wi1} in $\mathcal{N}_{wi}$,*
   - *(b) the arc $h_w|\hat{I}_{wi,j}$ joins Y_{wij} to $X_{wi(j+1)}$ in $\mathcal{N}_{wi}$ for $j \in \{1, \dots, n_{wi} - 1\}$,*
   - *(c) the arc $h_w|\hat{I}_{wi,n_{wi}}$ joins $Y_{win_{wi}}$ to Y_{wi} in $\mathcal{N}_{wi}$.*
2. *For each $w \in \mathcal{W}$, $\operatorname{dist}(\operatorname{Im}(h_w), \bigcup_{u \in \mathcal{W}\setminus\{w\}} \operatorname{Im}(h_u)) > 0$.*

Moreover, if E is c-uniformly disconnected, then, in addition to (1), there exists a constant $L(C_0, Q, C, c) \geq 1$ such that for each $w \in \mathcal{W}$,

(3) h_w is a (L, Λ_w)-quasisimilarity, and

(4) $\operatorname{dist}(\operatorname{Im}(h_w), \bigcup_{u \in \mathcal{W}\setminus\{w\}} \operatorname{Im}(h_u)) \geq L^{-1} \operatorname{diam} U_w$.

The construction of embeddings $\{h_w : w \in \mathcal{W}\}$ is done in two steps. In Section 3.8.1 we construct these embeddings for those words $w \in \mathcal{W}$ that have even number of letters, and in Section 3.8.2 we construct them for those words $w \in \mathcal{W}$ that have odd number of letters; the extra complication is that curves defined in the second subsection must avoid those that have been defined in the first. Recall that when we refer to `data`, we mean the constants C_0, Q, C, and c.

3.8.1 Even generations

Fix for the remainder of this subsection a word $w \in \mathcal{W}$ that has an even number of letters. The construction of the embedding h_w is a double induction on the indices i and j. For each $i \in \{1, \dots, n_w\}$, define

$$\mathcal{I}_{wi} = \bigcup_{k=1}^{i} \bigcup_{j=0}^{n_{wk}} \hat{I}_{wk,j}.$$

Base case

Here we construct a bi-Lipschitz embedding $h_{w,1} : \mathcal{I}_{w1} \to \mathcal{N}_{w1}$ in an inductive manner. For purposes of induction, set $Y_{w10} = X_{w1}$ and $X_{w1(n_{w1}+1)} = Y_{w1}$.

By Proposition 3.5.1(D6), there is a curve $\sigma_{w1,0}$ in $\hat{U}_{w1}$ joining Y_{w10} (i.e., X_{w1}) to X_{w11}. Using Lemma 3.4.8 with $Y = \varnothing$ and $\lambda = \epsilon_{w1}/8$, we obtain a bi-Lipschitz embedding $\gamma_{w1,0} : \hat{I}_{w1,0} \to X$ joining Y_{w10} to X_{w11} with

$$\sup_{z \in \gamma_{w1,0}} \operatorname{dist}(z, \sigma_{w1,0}) < \tfrac{1}{2}\epsilon_{w1} \operatorname{diam} U_{w1}. \tag{3.8.1}$$

If E is c-uniformly disconnected, then $\lambda = 2^{-9}\xi^{-3}$, and Lemma 3.4.8 implies that $\gamma_{w1,0} : \hat{I}_{w1,0} \to X$ is a $(L'_{1,0}, \Lambda_w)$-quasisimilarity for some constant $L'_{1,0}(\mathtt{data}) \geq 1$. The curve $\sigma_{w1,0}$ lies in $\hat{U}_{w1}$, so by (3.8.1), the curve $\gamma_{w1,0}$ lies in $\mathcal{N}_{w1}$. Set now $h_{w,1,0} = \gamma_{w1,0}$.

For the inductive step, suppose that for some $j_0 \in \{0, \dots, n_{w1}-1\}$, we have defined a bi-Lipschitz embedding $h_{w,1,j_0} : \bigcup_{j=0}^{j_0} \hat{I}_{w1,j} \to \mathcal{N}_{w1}$ such that the arc $h_{w,1,j_0}|\hat{I}_{w1,j}$ joins Y_{w1j} to $X_{w1(j+1)}$ for $j \in \{0, \dots, j_0\}$. Assume, moreover, that if E is c-uniformly disconnected, then $h_{w,1,j_0}$ is a (L'_{1,j_0}, Λ_w)-quasisimilarity for some $L'_{1,j_0} \geq 1$.

By Proposition 3.5.1(D6), there exists a curve $\sigma_{w1,j_0+1} : [0,1] \to \hat{U}_{w1}$ joining $Y_{w1(j_0+1)}$ to $X_{w1(j_0+2)}$. Since $h_{w,1,j_0}$ is bi-Lipschitz, we have that $h_{w,1,j_0}(\hat{I}_{w1,0} \cup \cdots \cup \hat{I}_{w1,j_0})$ is 1-HTB. Using Lemma 3.4.8 with $\lambda = \epsilon_{w1}/8$ and $Y = h_{w,1,j_0}(\hat{I}_{w1,0} \cup \cdots \cup \hat{I}_{w1,j_0})$ we obtain a bi-Lipschitz embedding $\gamma_{w1,j_0+1} : \hat{I}_{w1,j_0+1} \to X$ joining $Y_{w1(j_0+1)}$ to $X_{w1(j_0+2)}$ disjoint from Y such that

$$\sup_{z \in \gamma_{w1,j_0+1}} \operatorname{dist}(z, \sigma_{w1,j_0+1}) < \tfrac{1}{2}\epsilon_{w1} \operatorname{diam} U_{w1}.$$

If E is c-uniformly disconnected, then $\lambda = 2^{-9}\xi^{-3}$ and Y is $(C_{1,j_0}, 1)$-HTB for some constant $C_{1,j_0}(\mathtt{data}, L'_{1,j_0}) \geq 1$. Hence, γ_{w1,j_0+1} is an $(L'_{1,j_0+1}, \Lambda_w)$-quasisimilarity for some constant $L'_{1,j_0+1}(\mathtt{data}, L'_{1,j_0}) \geq 1$ and there exists $\beta_{1,j_0+1}(\mathtt{data}, L'_{1,j_0}) \geq 1$ such that

$$\begin{aligned}
\inf_{z \in \gamma_{w1,j_0+1}} \operatorname{dist}(z, Y) &\geq 2^{-9}\beta_{1,j_0+1}\xi^{-3} \operatorname{diam}(\sigma_{w1,j_0+1}) \\
&\geq 2^{-9}\beta_{1,j_0+1}\xi^{-3} \operatorname{dist}(Y_{w1(j_0+1)}, X_{w1(j_0+2)}) \\
&\geq 2^{-19}\beta_{1,j_0+1}\xi^{-6}\rho(\operatorname{diam} U_{w1(j_0+1)} + \operatorname{diam} U_{w1(j_0+2)}) \\
&\geq 2^{-20}\beta_{1,j_0+1}\xi^{-7}\rho \operatorname{diam} U_{w1} \\
&=: \eta_{1,j_0+1} \operatorname{diam} U_{w1}
\end{aligned} \tag{3.8.2}$$

by (3.7.1) and (3.5.1) (recalling that $\delta = (4\xi)^{-1}$, $\varepsilon_{w1} = (4\xi)^{-3}$, and that $\rho_w = \rho$ is independent of w by Remark 3.7.3). As in the case $j = 0$, we have that γ_{w1,j_0+1} lies in $\mathcal{N}_{w1}$. The next lemma completes the inductive step.

Lemma 3.8.2. *Let* $h_{w,1,j_0+1} : \hat{I}_{w1,0} \cup \cdots \cup \hat{I}_{w1,j_0+1} \to \mathcal{N}_{w1}$ *be defined by*

$$h_{w,1,j_0+1}|(\hat{I}_{w1,0} \cup \cdots \cup \hat{I}_{w1,j_0}) = h_{w,1,j_0} \quad \textit{and} \quad h_{w,1,j_0+1}|\hat{I}_{w1,j_0+1} = \gamma_{w1,j_0+1}.$$

Then $h_{w,1,j_0+1}$ *is bi-Lipschitz. Moreover, if* E *is* c*-uniformly disconnected, then* $h_{w,1,j_0+1}$ *is an* (L_{1,j_0+1}, Λ_w)*-quasisimilarity for some constant* $L_{1,j_0+1}(\texttt{data}, L_{1,j_0}) \geq 1$.

Proof. Since $h_{w,1,j_0}$ and γ_{w1,j_0+1} are bi-Lipschitz and have disjoint images, it follows that $h_{w,1,j_0+1}$ is bi-Lipschitz.

Assume now that E is c-uniformly disconnected. By the inductive step and by design $h_{w,1,j_0}$ and γ_{w1,j_0+1} are quasisimilarities. Let now $s \in \hat{I}_{w1,0} \cup \cdots \cup \hat{I}_{w1,j_0}$ and $t \in \hat{I}_{w1,j_0+1}$. On the one hand,

$$(2N+1)^{-2} \leq (2N+1)^{-1} \frac{|J_{w1}|}{|J_w|} \leq \frac{\operatorname{dist}(I_{w1,j_0}, I_{w1,j_0+1})}{|J_w|} \leq \frac{|s-t|}{|J_w|} \leq \frac{|J_{w1}|}{|J_w|} \leq 1,$$

which implies that

$$(2N+1)^{-2} \Lambda_w^{-1} \operatorname{diam} U_w \leq |s-t| \leq \Lambda_w^{-1} \operatorname{diam} U_w.$$

On the other hand, by (3.8.2) and (3.5.1),

$$(4\xi)^{-1} \eta_{1,j_0+1} \operatorname{diam} U_w \leq \eta_{1,j_0+1} \operatorname{diam} U_{w1} \leq d(h_{w1,j_0+1}(s), h_{w1,j_0+1}(t)) \leq \operatorname{diam} \mathcal{N}_{w1}$$
$$\leq \operatorname{diam} U_w. \ \square$$

Proceeding inductively, we obtain a bi-Lipschitz embedding $h_{w,1} := h_{w,1,n_{w1}}$ such that property (1) of Proposition 3.8.1 holds for $i = 1$. Moreover, if E is c-uniformly disconnected, then $h_{w,1}$ is a (L_1, Λ_w)-quasisimilarity for some constant $L_1(\texttt{data}) \geq 1$.

Inductive step

Suppose that for some $i_0 \in \{1, \ldots, n_w - 1\}$, we have constructed a bi-Lipschitz embedding $h_{w,i_0} : \mathcal{I}_{wi_0} \to \bigcup_{i=1}^{i_0} \mathcal{N}_{wi}$ such that for each $i \in \{1, \ldots, i_0\}$, the arc $h_{w,i_0}|\hat{I}_{wi,0}$

satisfies property (1) of Proposition 3.8.1. Moreover, suppose that if E is c-uniformly disconnected, then h_{w,i_0} is an (L_{i_0}, Λ_w)-quasisimilarity for some constant $L_{i_0} \geq 1$.

As with $h_{w,1}$, the construction of $h_{w,i_0+1} : \mathcal{I}_{w(i_0+1)} \to \bigcup_{i=1}^{i_0+1} \mathcal{N}_{wi}$ is in an inductive fashion and we only outline the steps. Set $Y_{w(i_0+1)0} = X_{w(i_0+1)}$ and $X_{w(i_0+1)(n_{w(i_0+1)}+1)} = Y_{w(i_0+1)}$. By Proposition 3.5.1(D6), we find a curve $\sigma_{w(i_0+1),0}$ joining $Y_{w(i_0+1)0}$ to $X_{w(i_0+1)1}$ in $\hat{U}_{w(i_0+1)}$. Since h_{w,i_0} is bi-Lipschitz, its image Y is 1-HTB and applying Lemma 3.4.8 with $\lambda = \epsilon_{w(i_0+1)}/8$, we obtain a bi-Lipschitz curve $\gamma_{w(i_0+1),0} : \hat{I}_{w(i_0+1),0} \to \mathcal{N}_{w(i_0+1)}$ joining $Y_{w(i_0+1)0}$ to $X_{w(i_0+1)1}$ whose image is disjoint from Y. If E is c-uniformly disconnected, then the image of h_{w,i_0} is $(C_{i_0}, 1)$-HTB, which implies that $\gamma_{w(i_0+1),0}$ is an $(L'_{i_0+1,0}, \Lambda_w)$-quasisimilarity. Now define $h_{w,i_0+1,0} : \mathcal{I}_{wi_0} \cup \hat{I}_{w(i_0+1),0} \to \bigcup_{i=1}^{i_0+1} \mathcal{N}_{wi}$ by

$$h_{w,i_0+1,0}|\mathcal{I}_{wi_0} = h_{w,i_0} \quad \text{and} \quad h_{w,i_0+1,0}|\hat{I}_{w(i_0+1),0} = \gamma_{w(i_0+1),0}.$$

Working as in Lemma 3.8.2, we can show that $h_{w,i_0+1,0}$ is bi-Lipschitz, and if E is c-uniformly disconnected, then $h_{w,i_0+1,0}$ is an $(L_{i_0+1,0}, \Lambda_w)$-quasisimilarity.

Assuming we have defined h_{w,i_0+1,j_0} for some $j_0 \in \{0, \ldots, n_{w(i_0+1)} - 1\}$, we use Proposition 3.5.1(D6) to find a curve $\sigma_{w(i_0+1),j_0+1}$ joining $Y_{w(i_0+1)(j_0+1)}$ to $Y_{w(i_0+1)(j_0+2)}$ in $\hat{U}_{w(i_0+1)}$. The image Y of h_{w,i_0+1,j_0} is 1-HTB since h_{w,i_0+1,j_0} is bi-Lipschitz, so by Lemma 3.4.8 (with $\lambda = \epsilon_{w(i_0+1)}/8$), we obtain a bi-Lipschitz embedding $\gamma_{w(i_0+1),j_0+1} : \hat{I}_{w(i_0+1),j_0+1} \to \mathcal{N}_{w(i_0+1)}$ which is disjoint from Y. Moreover, if E is c-uniformly disconnected, then $\gamma_{w(i_0+1),j_0+1}$ is an $(L'_{i_0+1,j_0+1}, \Lambda_w)$-quasisimilarity. Working as in Lemma 3.8.2, we get that the map

$$h_{w,i_0+1,j_0+1} : \mathcal{I}_{wi_0} \cup \bigcup_{j=0}^{j_0+1} \hat{I}_{w(i_0+1),j} \to \bigcup_{i=1}^{i_0+1} \mathcal{N}_{wi}$$

defined by

$$h_{w,i_0+1,j_0+1}|\mathcal{I}_{wi_0} \cup \bigcup_{j=0}^{j_0} \hat{I}_{w(i_0+1),j} = h_{w,i_0+1,j_0}$$

and

$$h_{w,i_0+1,j_0+1}|\hat{I}_{w(i_0+1),j_0+1} = \gamma_{w(i_0+1),j_0+1}$$

is bi-Lipschitz and, if E is c-uniformly disconnected, it is an $(L_{i_0+1,j_0+1}, \Lambda_w)$-quasisimilarity.

Proceeding inductively, we obtain a bi-Lipschitz embedding

$$h_{w,i_0+1} = h_{w,i_0+1,n_{w(i_0+1)}} : \mathcal{I}_{w(i_0+1)} \to \bigcup_{i=1}^{i_0+1} \mathcal{N}_{wi}$$

such that property (1) of Proposition 3.8.1 holds for $i \in \{1, \dots, i_0 + 1\}$.

End of induction

After n_w many steps we obtain a bi-Lipschitz embedding

$$h_w := h_{w,n_w} : \mathcal{I}_{wn_w} \to \bigcup_{i=1}^{n_w} \mathcal{N}_{wi}$$

that satisfies property (1) of Proposition 3.8.1. Moreover, if E is c-uniformly disconnected, then $n_w \leq N(\texttt{data})$ and h_w is a (L', Λ_w)-quasisimilarity for some $L'(\texttt{data}) \geq 1$.

3.8.2 Odd generations

Fix for this subsection a word $w \in \mathcal{W}$ that has an odd number of letters. By §3.8.1, we have constructed bi-Lipschitz embeddings $h_{w^\uparrow}, h_{w1}, \dots, h_{wn_w}$ such that, if E is c-uniformly disconnected, then h_w is an $(L', \Lambda_{w^\uparrow})$-quasisimilarity and each h_{wi} is an (L', Λ_{wi})-quasisimilarity. The construction of the embedding h_w mirrors the previous subsection, but here we also avoid the images of $h_{w^\uparrow}, h_{w1}, \dots, h_{wn_w}$.

Denote by H the union of the images of $h_{w^\uparrow}, h_{w1}, \dots, h_{wn_w},$. Since these maps are bi-Lipschitz, H is 1-HTB and, by Lemma 3.4.2, H is porous. Further, if E is

c-uniformly disconnected, then H is $(C', 1)$-HTB for some $C'(\texttt{data}) \geq 1$ and H is p_0-porous for some $p_0(\texttt{data}) \geq 1$. By porosity, there exist balls $X'_{wi} \subset X_{wi}$ and $Y'_{wi} \subset Y_{wi}$ which are disjoint from H; if E is c-uniformly disconnected, these balls can be taken to have diameters $(2p_0)^{-1} \operatorname{diam} X_{wi}$ and $(2p_0)^{-1} \operatorname{diam} Y_{wi}$, respectively and have distance from H at least $(2p_0)^{-1} \operatorname{diam} X_{wi}$ and $(2p_0)^{-1} \operatorname{diam} Y_{wi}$, respectively. Similarly, for each $i \in \{1, \dots, n_w\}$ and $j \in \{1, \dots, n_{wi}\}$, there exist balls $X'_{wij} \subset X_{wij}$ and $Y'_{wij} \subset Y_{wij}$ which are disjoint from H and if E is c-uniformly disconnected, the diameters of these balls are $(2p_0)^{-1} \operatorname{diam} X_{wij}$ and $(2p_0)^{-1} \operatorname{diam} Y_{wij}$, respectively, and they have distance from H at least $(2p_0)^{-1} \operatorname{diam} X_{wij}$ and $(2p_0)^{-1} \operatorname{diam} Y_{wij}$, respectively.

We give a sketch of how to define h_w. We apply the same methods as in §3.8.1, but now also avoid H. Specifically, the only difference is when applying Lemma 3.4.8; instead of taking Y to be only the union H' of the images of the bi-Lipschitz embedding constructed prior to the inductive step, we take $Y = H \cup H'$, which allows the new bi-Lipschitz curve γ to be disjoint from H as well. Indeed, $H \cup H'$ is 1-HTB, as the union of two 1-HTB sets.

If E is c-uniformly disconnected, then $H \cup H'$ is $(C'', 1)$-HTB for some $C''(\texttt{data}) \geq 1$ and the curve γ is an (L'', Λ_w)-quasisimilarity for some $L''(\texttt{data}) \geq 1$. We proceed by extending the definition of h_w using the curve γ, as before.

In the end, after finitely many steps, we obtain a bi-Lipschitz embedding h_w and, provided E is c-uniformly disconnected, h_w is an (L, Λ_w)-quasisimilarity for some $L(\texttt{data}) \geq 1$ and there exists a constant $\eta(\texttt{data}) \in (0, 1)$ such that

$$\operatorname{dist}\left(\operatorname{Im}(h_w), \operatorname{Im}(h_{w\uparrow}) \cup \bigcup_{i=1}^{n_w} \operatorname{Im}(h_{wi})\right) \geq \eta \operatorname{diam} U_w. \tag{3.8.3}$$

In Figure 3.1, w has an even number of letters, $n_w = 2$ and $n_{w1} = n_{w2} = 2$. The blue curves are the image of h_w defined in §3.8.1 while the red curves are part of the image of $h_{w\uparrow}$ defined in §3.8.2.

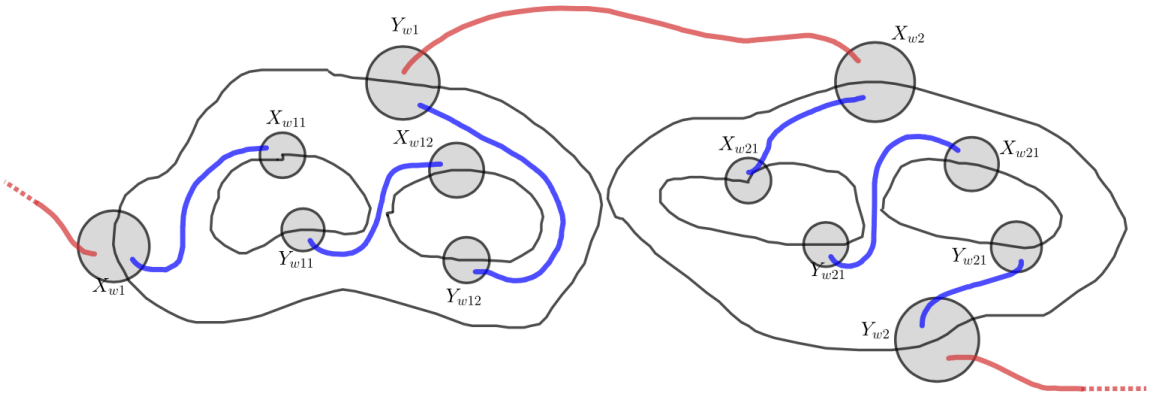

Figure 3.1: An example of the construction.

3.8.3 Proof of Proposition 3.8.1

It is immediate from design that embeddings $\{h_w : w \in \mathcal{W}\}$ satisfy properties (1) and (3) of Proposition 3.8.1.

Fix now distinct $w, u \in \mathcal{W}$. Suppose first that one is not the parent of the other. Fix also a point $p \in \mathrm{Im}(h_w)$ and a point $q \in \mathrm{Im}(h_u)$. Then $p \in \mathcal{N}_{wi}$ and $q \in \mathcal{N}_{uj}$ for some $i \in \{1, \ldots, n_w\}$ and $j \in \{1, \ldots, n_u\}$. By Lemma 3.7.1,

$$d(p, q) \geq \mathrm{dist}(\mathcal{N}_{wi}, \mathcal{N}_{uj}) \geq \epsilon_{wi} \operatorname{diam} U_{wi} + \epsilon_{uj} \operatorname{diam} U_{uj},$$

which gives properties (2) and (4).

Suppose now that $u = wi$ for some $i \in \{1, \ldots, n_w\}$. By design (see §3.8.2), the images of h_w and h_{wi} are disjoint, which gives property (2). For property (4), suppose E is c-uniformly disconnected. If u has an odd number of letters, then w has an even number of letters, while if u has an even number of letters, then w has an odd number

of letters. In either case, by (3.8.3),

$$d(p,q) \geq \operatorname{dist}(\operatorname{Im}(h_w), \operatorname{Im}(h_u)) \geq \eta \min\{\operatorname{diam} U_w, \operatorname{diam} U_u\}.$$

3.9 Local modifications

In this section, within each ball X_w and Y_w, with $|w| \geq 2$, we patch curves defined in Proposition 3.8.1 together with a bi-Lipschitz curve using Lemma 3.4.8, then delete overlapping portions and join with geodesics. These local modifications are identical to those in [HVZ24, Section 6.1], aside from notational changes.

Here and for the rest of this section, given a non-degenerate interval $[a,b]$ and two points $p,q \in X$, we say that a curve $g : [a,b] \to X$ is *a geodesic joining p to q* if $g(a) = p$, $g(b) = q$ and for all $t,s \in [a,b]$ with $t < s$ we have that

$$\frac{d(g(s),g(t))}{s-t} = \frac{d(g(a),g(b))}{b-a}.$$

The constant on the right-hand side is called the *scaling factor* of the geodesic g.

Recall the definition of the totally disconnected set $J_\infty \subset [0,1]$ from (3.7.4). Given our assumption that the only word with length one is 1, we have that $J_\infty \subset J_1$. It is clear from Lemma 3.7.5 and Lemma 3.7.6 that the collection $\{I_{w,i} : w \in \mathcal{W} \setminus \{\varepsilon\}, i \in \{0,\dots,n_w\}\}$ is a collection of closed subintervals of $J_1 \setminus J_\infty$ with disjoint interiors such that

$$J_1 = J_\infty \cup \bigcap_{w \in \mathcal{W}\setminus\{\varepsilon\}} \bigcup_{i=0}^{n_w} I_{w,i}.$$

Write $I_{w,i} = [a_{w,i}, b_{w,i}]$ and $\hat{I}_{w,i} = [\hat{a}_{w,i}, \hat{b}_{w,i}]$. Set

$$\kappa_w = (\operatorname{diam} U_w)^{-1} \min\left\{ \operatorname{dist}\left(\operatorname{Im}(h_w), \bigcup_{v \in \mathcal{W}\setminus\{w\}} \operatorname{Im}(h_v)\right), \min_{\substack{i\in\{1,\dots,n_w\}\\ j\in\{0,\dots,n_{wi}\}}} d(h_w(\hat{a}_{wi,j}), h_w(\hat{b}_{wi,j}))\right\}, \tag{3.9.1}$$

which is positive by Proposition 3.8.1. If E is c-uniformly disconnected, then the right hand side of (3.9.1) is bounded from below by $(3L(2N+1))^{-1}$ (where N is the constant of Proposition 3.5.1 and L is the constant of Proposition 3.8.1) and we can set

$$\kappa_w = (3L(2N+1))^{-1}.$$

With the exception of $I_{1,0}$ and I_{1,n_1}, each interval $I_{w,i}$ has an adjacent interval to its left and an adjacent interval to its right. In particular, given $w \in \mathcal{W}$, $i \in \{1, \dots, n_w\}$, and $j \in \{0, \dots, n_w\}$,

- if $j \neq n_w$, then the interval $I_{wi1,0}$ is to the right of $I_{wi,j}$, while if $j = n_w$ and $wi \neq 1$, then the interval $I_{w,i+1}$ is to the right of $I_{wi,j}$;
- if $j \neq 0$, then the interval $I_{wij,n_{wij}}$ is to the left of $I_{wi,j}$, while if $j = 0$ and $wi \neq 1$, then the interval $I_{w,i-1}$ is to the left of $I_{wi,j}$;
- $I_{1,0}$ has no interval to its left and I_{1,n_1} has no interval to its right.

Let $\mathscr{I} = \{I_{w,i} : w \in \mathcal{W}, i \in \{0, \dots, n_w\}\}$, and let $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$ be a partition of $\mathscr{I}$ into two disjoint subsets such that no two intervals in $\mathscr{I}_1$ are adjacent and no two intervals of $\mathscr{I}_2$ are adjacent. We may assume that $I_{1,0}$ and I_{1,n_1} are intervals of $\mathscr{I}_1$. Note that each interval of $\mathscr{I}_2$ lies between two intervals of $\mathscr{I}_1$, and each interval of $\mathscr{I}_1 \setminus \{I_{1,0}, I_{1,n_1}\}$ lies between two intervals of $\mathscr{I}_2$. In the next subsection, we perform local modifications on intervals of $\mathscr{I}_1$, and then on intervals of $\mathscr{I}_2$ in the section following that. The only difference is that we take care to consider the local modifications done on intervals of $\mathscr{I}_1$ when performing them on intervals of $\mathscr{I}_2$. Recall that `data` refers to the constants C_0, Q, C, and c, and recall the constant L from Proposition 3.8.1.

3.9.1 Local modifications on intervals of $\mathscr{I}_1$

Let $I_{wi,j} \in \mathscr{I}_1 \setminus \{I_{1,0}\}$. We assume that $I_{ui',j'}$ is the interval to the left of $I_{wi,j}$; see Figure 3.2.

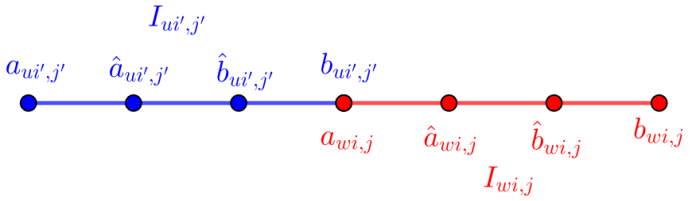

Figure 3.2: The positions of $\hat{a}_{ui',j'}$, $\hat{b}_{ui',j'}$, $\hat{a}_{wi,j}$, and $\hat{b}_{wi,j}$.

The points $h_u(\hat{b}_{ui',j'})$ and $h_w(\hat{a}_{wi,j})$ lie in a ball $B_{wij} \in \{X_{wij}, Y_{wij}\}$ by Proposition 3.8.1(1). By Lemma 3.7.6, the embedding h_u, in addition to being an (L, Λ_u)-quasisimilarity, is a $(\bar{L}, \Lambda_w)$-quasisimilarity for some $\bar{L}(L, \texttt{data}) \geq 1$ since one of u or w is the parent of the other.

Set

$$H_{(ui',j'),(wi,j)} = \left(\operatorname{Im}(h_{w\uparrow}) \cup \operatorname{Im}(h_w) \cup \bigcup_{i=1}^{n_w} \operatorname{Im}(h_{wi}) \right) \setminus (h_u(\hat{I}_{ui',j'}) \cup h_w(\hat{I}_{wi,j})).$$

The ball B_{wij} is path connected, so there is a curve $\sigma_{(ui',j'),(wi,j)} : [0,1] \to B_{wij}$ joining $h_u(\hat{b}_{ui',j'})$ to $h_w(\hat{a}_{wi,j})$. Using Lemma 3.4.8 with $\lambda = \frac{1}{8}$ and $Y = H_{(ui',j'),(wi,j)}$, we find a bi-Lipschitz arc

$$\Gamma_{(ui',j'),(wi,j)} : [\hat{b}_{ui',j'}, \hat{a}_{wi,j}] \to X$$

such that $\Gamma_{(ui',j'),(wi,j)}(\hat{b}_{ui',j'}) = h_u(\hat{b}_{ui',j'})$, $\Gamma_{(ui',j'),(wi,j)}(\hat{a}_{wi,j}) = h_w(\hat{a}_{wi,j})$,

$$\operatorname{dist}(\Gamma_{(ui',j'),(wi,j)}, H_{(ui',j'),(wi,j)}) > 0 \tag{3.9.2}$$

and, for all $z \in \Gamma_{(ui',j'),(wi,j)}$,

$$\operatorname{dist}(z, \sigma_{(ui',j'),(wi,j)}) \leq \tfrac{1}{2} \operatorname{diam}(\sigma_{(ui',j'),(wi,j)}) \leq \tfrac{1}{2} \operatorname{diam} B_{wij} \leq \frac{1}{24} \epsilon_{wij} \rho_{wi} \operatorname{diam} U_{wij}. \tag{3.9.3}$$

Furthermore, if E is c-uniformly disconnected, then there exist $L^*(\texttt{data}) \geq 1$ and $\beta^*(\texttt{data}) \in (0,1)$ such that $\Gamma_{(ui',j'),(wi,j)}$ is an (L^*, Λ_w)-quasisimilarity and, by Proposition 3.8.1(4), for each $z \in \Gamma_{(ui',j'),(wi,j)}$,

$$\begin{aligned} \operatorname{dist}(z, H_{(ui',j'),(wi,j)}) \geq \beta^* \operatorname{diam}(\sigma_{(ui',j'),(wi,j)}) &\geq \beta^* d(h_u(\hat{b}_{ui',j'}), h_w(\hat{a}_{wi,j})) \\ &\geq \frac{\beta^*}{L} \operatorname{diam} U_w. \end{aligned} \tag{3.9.4}$$

Inequality (3.9.3) ensures that the arc $\Gamma_{(ui',j'),(wi,j)}$ does not exit the ball $\frac{3}{2}B_{wij}$, hence it does not leave $\mathcal{N}_{wi}$ and therefore cannot intersect with any arc $\operatorname{Im}(h_v)$ for any $v \in \mathcal{W}$ with $||v| - |w|| > 1$. Inequality (3.9.2) ensures that $\Gamma_{(ui',j'),(wi,j)}$ does not intersect with any arc $\operatorname{Im}(h_v)$ such that $v \in \mathcal{W}$ with $||v|-|w|| \leq 1$, aside from possibly in the arcs $h_u(\hat{I}_{ui',j'})$ and $h_w(\hat{I}_{wi,j})$. In the case where E is c-uniformly disconnected, by (3.9.4), $\Gamma_{(ui',j'),(wi,j)}$ has quantitative distance from any arc $\operatorname{Im}(h_v)$ such that $v \in \mathcal{W}$ with $||v| - |w|| \leq 1$, aside from the arcs $h_u(\hat{I}_{ui',j'})$ and $h_w(\hat{I}_{wi,j})$.

We now define a few points in $[\hat{a}_{ui',j'}, \hat{a}_{wi,j}]$. Set

$$\theta_{wi,j} = (2L)^{-1}\beta^*\epsilon_w\epsilon_{wi}\epsilon_{wij}\rho_{wi}.$$

Remark 3.9.1. If E is uniformly disconnected, since the quantities $\epsilon_w, \epsilon_{wi}, \epsilon_{wij}$ and ρ_{wi} are chosen independent of (wi, j), so is $\theta_{wi,j}$ chosen independent of (wi, j) and we simply denote it by θ.

Now define

$$\begin{aligned} \tilde{b}_{ui',j'} = \min\{t \in [\hat{a}_{ui',j'}, \hat{b}_{ui',j'}] :& \\ \operatorname{dist}(h_u(t), \Gamma_{(ui',j'),(wi,j)}([\hat{b}_{ui',j'}, \hat{a}_{wi,j}])) &= \theta_{wi,j} \operatorname{diam} U_w\}, \\ t^R_{ui',j'} = \max\{t \in [\hat{b}_{ui',j'}, \hat{a}_{wi,j}] : \operatorname{dist}(h_u(\tilde{b}_{ui',j'}), \Gamma_{(ui',j'),(wi,j)}(t)) &= \theta_{wi,j} \operatorname{diam} U_w\}. \end{aligned}$$

Note that

$$d(h_u(\tilde{b}_{ui',j'}), h_u(\hat{b}_{ui',j'})) \geq \operatorname{dist}(h_u(\tilde{b}_{ui',j'}), \Gamma_{(ui',j'),(wi,j)}([\hat{b}_{ui',j'}, \hat{a}_{wi,j}])) \geq \theta_{wi,j} \operatorname{diam} U_w,$$

so $\hat{b}_{ui',j'} - \tilde{b}_{ui',j'} > 0$ and, if E is c-uniformly disconnected, then

$$\hat{b}_{ui',j'} - \tilde{b}_{ui',j'} \geq (L\Lambda_w)^{-1}\theta \operatorname{diam} U_w. \tag{3.9.5}$$

Moreover, by choice of $\theta_{wi,j}$,

$$\begin{aligned}
&d(\Gamma_{(ui',j'),(wi,j)}(t^R_{ui',j'}), \Gamma_{(ui',j'),(wi,j)}(\hat{a}_{wi,j})) \\
&\quad \geq \operatorname{dist}(h_w(\hat{a}_{wi,j}), h_u([\hat{a}_{ui',j'}, \hat{b}_{ui',j'}])) - \operatorname{dist}(\Gamma_{(ui',j'),(wi,j)}(t^R_{ui',j'}), h_u([\hat{a}_{ui',j'}, \hat{b}_{ui',j'}])) \\
&\quad \geq \kappa_w \operatorname{diam} U_w - \theta_{wi,j} \operatorname{diam} U_w \\
&\quad \geq \theta_{wi,j} \operatorname{diam} U_w,
\end{aligned}$$

so $\hat{a}_{wi,j} - t^R_{ui',j'} > 0$. If E is c-uniformly disconnected, then

$$\hat{a}_{wi,j} - t^R_{ui',j'} \geq (L^*\Lambda_w)^{-1}\theta \operatorname{diam} U_w. \tag{3.9.6}$$

Further, (3.9.5) and Lemma 3.7.6 imply that

$$\frac{\theta \operatorname{diam} U_w}{L\Lambda_w} \leq \hat{b}_{ui',j'} - \tilde{b}_{ui',j'} \leq t^R_{ui',j'} - \tilde{b}_{ui',j'} \leq 2(|\hat{I}_{ui',j'}| + |\hat{I}_{wi,j}|) \leq \frac{(4N+2)\operatorname{diam} U_w}{\Lambda_w}. \tag{3.9.7}$$

Fix a geodesic $g^R_{ui',j'} : [\tilde{b}_{ui',j'}, t^R_{ui',j'}] \to X$ joining $h_u(\tilde{b}_{ui',j'})$ to $\Gamma_{(ui',j'),(wi,j)}(t^R_{ui',j'})$ and note that by choice of $\theta_{wi,j}$, $g^R_{ui',j'}$ does not leave the ball $2B_{wij}$, and by extension, does not exit $\mathcal{N}_{wi}$. The superscript "R" here is to denote that this geodesic and the point $t^R_{ui',j'}$ are defined on the right side of the interval $I_{ui',j'}$. By (3.9.7), the scaling

factor of $g^R_{ui',j'}$ satisfies

$$\frac{\theta_{wi,j}\Lambda_w}{4N+2} \leq \frac{d(h_u(\tilde{b}_{ui',j'}), \Gamma_{(ui',j'),(wi,j)}(t^R_{ui',j'}))}{t^R_{ui',j'} - \tilde{b}_{ui',j'}} \leq L\Lambda_w. \tag{3.9.8}$$

Now set

$$\tilde{a}_{wi,j} = \max\{t \in [\hat{a}_{wi,j}, \hat{b}_{wi,j}] : \operatorname{dist}(h_w(t), \Gamma_{(ui',j'),(wi,j)}([\hat{b}_{ui',j'}, \hat{a}_{wi,j}])) = \theta_{wi,j} \operatorname{diam} U_w\},$$
$$t^L_{wi,j} = \min\{t \in [t^R_{ui',j'}, \hat{a}_{wi,j}] : d(\Gamma_{(ui',j'),(wi,j)}(t), h_w(t^R_{wi,j})) = \theta_{wi,j} \operatorname{diam} U_w\}.$$

As with (3.9.6), $\hat{a}_{wi,j} - t^L_{wi,j} > 0$, and, if E is c-uniformly disconnected,

$$\hat{a}_{wi,j} - t^L_{wi,j} \geq (L^*\Lambda_w)^{-1}\theta \operatorname{diam} U_w. \tag{3.9.9}$$

Moreover, if $t \in [\hat{b}_{ui',j'}, \hat{a}_{wi,j}]$ satisfies $d(\Gamma_{(ui',j'),(wi,j)}(t), h_w(t^R_{wi,j})) = \theta_{wi,j} \operatorname{diam} U_w$,

$$\begin{aligned}
&d(\Gamma_{(ui',j'),(wi,j)}(t^R_{ui',j'}), \Gamma_{(ui',j'),(wi,j)}(t)) \\
&\quad \geq \operatorname{dist}(h_u([\hat{a}_{ui',j'}, \hat{b}_{ui',j'}]), h_w([\hat{a}_{wi,j}, \hat{b}_{wi,j}])) - 2\theta_{wi,j} \operatorname{diam} U_w \\
&\quad \geq \kappa_w \operatorname{diam} U_w - 2\theta_{wi,j} \operatorname{diam} U_w \\
&\quad \geq \theta_{wi,j} \operatorname{diam} U_w
\end{aligned}$$

by choice of $\theta_{wi,j}$. Thus $t^L_{wi,j}$ is well-defined, $t^L_{wi,j} - t^R_{ui',j'} > 0$ and, if E is c-uniformly disconnected,

$$t^L_{wi,j} - t^R_{ui',j'} \geq (L^*\Lambda_w)^{-1}\theta \operatorname{diam} U_w. \tag{3.9.10}$$

See Figure 3.3 for a visual.

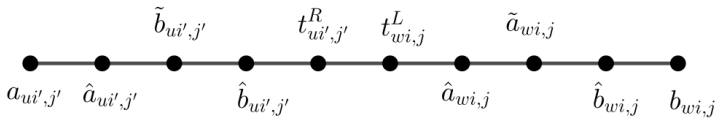

Figure 3.3: The positions of $\tilde{b}_{ui',j'}$, $t^R_{ui',j'}$, $t^L_{wi,j}$, and $\tilde{a}_{wi,j}$.

Fix a geodesic $g^L_{wi,j} : [t^L_{wi,j}, \tilde{a}_{wi,j}] \to X$ joining $\Gamma_{(ui',j'),(wi,j)}(t^L_{wi,j})$ to $h_w(\tilde{a}_{wi,j})$. By choice of $\theta_{wi,j}$, $g^L_{wi,j}$ does not leave the ball $2B_{wij}$, hence does not exit $\mathcal{N}_{wi}$. The superscript "L" here is to denote that this geodesic and $t^L_{wi,j}$ are defined on the left side of the interval $I_{wi,j}$. A computation similar to (3.9.8) gives the same bounds for the scaling factor of $g^L_{wi,j}$. See Figure 3.4, where the red curves are the images of $h_u|\hat{I}_{ui',j'}$ (left) and $h_w|\hat{I}_{wi,j}$ (right), the blue curve is the curve $\Gamma_{(ui',j'),(wi,j)}$ and the green curves are the geodesics $g^R_{ui',j'}$ (left) and $g^L_{wi,j}$ (right).

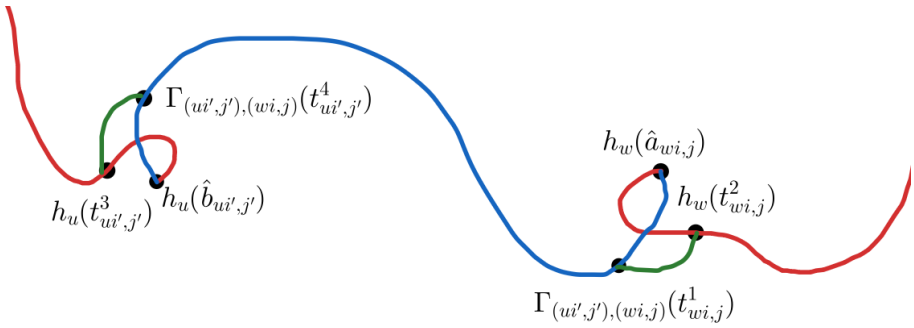

Figure 3.4: An example of the local modifications.

3.9.2 Local modifications on intervals of $\mathscr{I}_2$

Let $I_{wi,j} \in \mathscr{I}_2 \setminus \{I_{1,0}\}$. Again we assume that $I_{ui',j'}$ is the interval to the left of $I_{wi,j}$. From the prior subsection, we have already defined points $t^L_{ui',j'}$, $\tilde{a}_{ui',j'}$, $\tilde{b}_{wi,j}$, and $t^R_{wi,j}$, along with geodesics $g^L_{ui',j'} : [t^L_{ui',j'}, \tilde{a}_{ui',j'}] \to X$ and $g^R_{wi,j} : [\tilde{b}_{wi,j}, t^R_{wi,j}] \to X$. We proceed to define an arc $\Gamma_{(ui',j'),(wi,j)}$, points $\tilde{b}_{ui',j'}$, $t^R_{ui',j'}$, $t^L_{wi,j}$, and $\tilde{a}_{wi,j}$, and geodesics $g^R_{ui',j'} : [\tilde{b}_{ui',j'}, t^R_{ui',j'}] \to X$ and $g^L_{wi,j} : [t^L_{wi,j}, \tilde{a}_{wi,j}] \to X$ as in the last subsection. The only difference is that we take into account the modifications performed in §3.9.1. Set

$$\begin{aligned}
\tilde{b}_{ui',j'} &= \min\{t \in [\tilde{a}_{ui',j'}, \hat{b}_{ui',j'}] : \\
&\qquad \operatorname{dist}(h_u(t), \Gamma_{(ui',j'),(wi,j)}([\hat{b}_{ui',j'}, \hat{a}_{wi,j}])) = \theta_{wi,j} \operatorname{diam} U_w\}, \\
t^R_{ui',j'} &= \max\{t \in [\hat{b}_{ui',j'}, \hat{a}_{wi,j}] : d(h_u(\tilde{b}_{ui',j'}), \Gamma_{(ui',j'),(wi,j)}(t)) = \theta_{wi,j} \operatorname{diam} U_w\}, \\
\tilde{a}_{wi,j} &= \max\{t \in [\hat{a}_{wi,j}, \tilde{b}_{wi,j}] : \\
&\qquad \operatorname{dist}(h_w(t), \Gamma_{(ui',j'),(wi,j)}([\hat{b}_{ui',j'}, \hat{a}_{wi,j}])) = \theta_{wi,j} \operatorname{diam} U_w\}, \\
t^L_{wi,j} &= \min\{t \in [t^R_{ui',j'}, \hat{a}_{wi,j}] : d(\Gamma_{(ui',j'),(wi,j)}(t), h_w(\tilde{a}_{wi,j})) = \theta_{wi,j} \operatorname{diam} U_w\}.
\end{aligned}$$

Inequalities analogous to (3.9.5), (3.9.6), (3.9.9), and (3.9.10) hold for these points as well.

Fix a geodesic $g^R_{ui',j'} : [\tilde{b}_{ui',j'}, t^R_{ui',j'}] \to X$ joining $h_u(\tilde{b}_{ui',j'})$ to $\Gamma_{(ui',j'),(wi,j)}(t^R_{ui',j'})$ and a geodesic $g^L_{wi,j} : [t^L_{wi,j}, \tilde{a}_{wi,j}] \to X$ joining $\Gamma_{(ui',j'),(wi,j)}(t^L_{wi,j})$ to $h_w(\tilde{a}_{wi,j})$. Similar to §3.9.1, the choice of $\theta_{wi,j}$ implies $g^R_{ui',j'}$ and $g_{wi,j}$ do not leave $2B_{wij}$, thus do not exit $\mathcal{N}_{wi}$. The same bounds as in (3.9.8) hold for the scaling factors of $g^R_{ui',j'}$ and $g^L_{wi,j}$. Lastly,

$$\begin{aligned}
&d(h_w(\tilde{a}_{wi,j}),h_w(\tilde{b}_{wi,j})) \\
&\qquad \geq d(h_w(\hat{a}_{wi,j}), h_w(\hat{b}_{wi,j})) - d(h_w(\hat{a}_{wi,j}), h_w(\tilde{b}_{wi,j})) - d(h_w(\hat{b}_{wi,j}), h_w(\tilde{a}_{wi,j})) \\
&\qquad \geq \kappa_w \operatorname{diam} U_w - \theta_{wi,j} \operatorname{diam} U_w - \theta_{wi,j} \operatorname{diam} U_w \\
&\qquad \geq \theta_{wi,j} \operatorname{diam} U_w,
\end{aligned}$$

hence $\tilde{b}_{wi,j} - \tilde{a}_{wi,j} > 0$ and, if E is c-uniformly disconnected,

$$\tilde{b}_{wi,j} - \tilde{a}_{wi,j} \geq (L\Lambda_w)^{-1}\theta \operatorname{diam} U_w. \tag{3.9.11}$$

Define $\gamma_{(ui',j'),(wi,j)} : [\tilde{b}_{ui',j'}, \tilde{a}_{wi,j}] \to X$ by

$$\gamma_{(ui',j'),(wi,j)}(t) = \begin{cases} g^R_{ui',j'}(t), & t \in [\tilde{b}_{ui',j'}, t^R_{ui',j'}], \\ \Gamma_{(ui',j'),(wi,j)}(t), & t \in [t^R_{ui',j'}, t^L_{wi,j}], \\ g^L_{wi,j}(t), & t \in [t^L_{wi,j}, \tilde{a}_{wi,j}], \end{cases} \tag{3.9.12}$$

and note that $\gamma_{(ui',j'),(wi,j)}$ does not leave the ball $2B_{wij}$, since none of its components do. By (3.7.2), note that if $\gamma_{(ui',j'),(wi,j)}$ and $\gamma_{(vi'',j''),(\omega\iota,\zeta)}$ are any two such curves, then

$$\begin{aligned} \operatorname{dist}(\gamma_{(ui',j'),(wi,j)}, \gamma_{(vi'',j''),(\omega\iota,\zeta)}) &\geq \operatorname{dist}(B_{wij}, B_{vi''j''}) \\ &\geq \tfrac{1}{12}(\rho_{wi}\epsilon_{wij} \operatorname{diam} U_{wij} + \rho_{vi''}\epsilon_{vi''j''} \operatorname{diam} U_{vi''j''}). \end{aligned} \tag{3.9.13}$$

Further, by (3.9.4) and the choice of $\theta_{wi,j}$,

$$\operatorname{dist}(z, H_{(ui',j'),(wi,j)}) \geq \frac{\beta^*}{L} \operatorname{diam} U_w - \theta_{wi,j} \operatorname{diam} U_w \geq \theta_{wi,j} \operatorname{diam} U_w \tag{3.9.14}$$

for each $z \in \gamma_{(ui',j'),(wi,j)}$.

3.10 Proof of Theorem 3.3.2 and Theorem 3.3.3

In this section we give the proof of Theorems 3.3.2 and 3.3.3. We start by defining a map

$$\Phi : [\hat{a}_{1,0}, \hat{b}_{1,n_1}] \to U_1.$$

The definition of Φ is done in four steps.

- Recall that $I_{1,0}$ is adjacent only to $I_{11,0}$, and that $I_{11,0}$ lies to the right of $I_{1,0}$. Moreover, a point $\tilde{b}_{1,0}$ has been defined but no point $\tilde{a}_{1,0}$. Define

$$\Phi|[\hat{a}_{1,0}, b_{1,0}](t) = \begin{cases} h_\varepsilon(t), & t \in [\hat{a}_{1,0}, \tilde{b}_{1,0}], \\ \gamma_{(1,0),(11,0)}(t), & t \in [\tilde{b}_{1,0}, b_{1,0}]. \end{cases}$$

- Recall that I_{1,n_1} is adjacent only to $I_{1n_1,n_{1n_1}}$, and that $I_{1n_1,n_{1n_1}}$ lies to the right of $I_{1,0}$. Moreover, a point $\tilde{a}_{1,n_1}$ has been defined but no point $\tilde{b}_{1,n_1}$. Define

$$\Phi|[a_{1,n_1}, \hat{b}_{1,n_1}](t) = \begin{cases} \gamma_{(1n_1,n_{1n_1}),(1,n_1)}(t), & t \in [a_{1,n_1}, \tilde{a}_{1,n_1}], \\ h_\varepsilon(t), & t \in [\tilde{a}_{1,n_1}, \hat{b}_{1,n_1}]. \end{cases}$$

- Let $I_{wi,j} \in \mathscr{I} \setminus \{I_{1,0}, I_{1,n_1}\}$. Then there exists an interval $I_{ui',j'}$ adjacent to $I_{wi,j}$ on the left and there exists an interval $I_{vi'',j''}$ adjacent to $I_{wi,j}$ on the right. Define

$$\Phi|I_{wi,j}(t) = \begin{cases} \gamma_{(ui',j'),(wi,j)}(t), & t \in [a_{wi,j}, \tilde{a}_{wi,j}], \\ h_w(t), & t \in [\tilde{a}_{wi,j}, \tilde{b}_{wi,j}], \\ \gamma_{(wi,j),(vi'',j'')}(t), & t \in [\tilde{b}_{wi,j}, b_{wi,j}]. \end{cases}$$

- It remains to define Φ on J_∞. Define $\mathbf{W}$ to be the set of all sequences $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$ such that $\mathbf{a}(m) \in \mathcal{W}$ for all $m \in \mathbb{N}$. (Here, if $\mathbf{a} = i_1 i_2 \cdots \in \mathbb{N}^{\mathbb{N}}$, then $\mathbf{a}(m) = i_1 \cdots i_m \in \mathbb{N}^m$ is the truncation of $\mathbf{a}$.) If $\mathbf{w} \in \mathbf{W}$, then by Lemma 3.7.5, $|J_{\mathbf{w}(m)}| \leq 3^{1-|\mathbf{w}(m)|}|J_1| = 3^{1-|\mathbf{w}(m)|}$ and, by Proposition 3.5.1, $\operatorname{diam} U_{\mathbf{w}(m)} \leq 2^{-|\mathbf{w}(m)|} \operatorname{diam} U_\varepsilon$. Therefore, since X is complete, elements of J_∞ and elements of E are uniquely determined by elements of $\mathbf{W}$. In particular, there exist bijections $\pi_1 : \mathbf{W} \to J_\infty$ and $\pi_2 : \mathbf{W} \to E$ such that for each $\mathbf{w} \in \mathbf{W}$, $\pi_1(\mathbf{w})$ is the unique element of $\bigcap_{m=1}^{\infty} J_{\mathbf{w}(m)}$ and $\pi_2(\mathbf{w})$ is the unique element of

$\bigcap_{m=1}^{\infty} U_{\mathbf{w}(m)}$. Define now Φ on $[\hat{a}_{1,0}, \hat{b}_{1,n_1}]$ by setting

$$\Phi|J_\infty = \pi_2 \circ (\pi_1)^{-1}.$$

The following proposition, which is the goal of this section, completes the proof of Theorem 3.3.2 and Theorem 3.3.3.

Proposition 3.10.1. *The map Φ is an embedding of $[\hat{a}_{1,0}, \hat{b}_{1,n_1}]$ into X such that $\Phi(J_\infty) = E$ and Φ is locally bi-Lipschitz away from J_∞. Furthermore, if E is c-uniformly disconnected, there is a constant $H(\texttt{data}) \geq 1$ such that Φ is an H-weak quasisymmetry.*

For the proof of the proposition, we require some lemmas.

Lemma 3.10.2. *For all $w \in \mathcal{W} \setminus \{\varepsilon, 1\}$ we have $\Phi(J_w) \subset \mathcal{N}_w$.*

Proof. We have that

$$J_w = (J_\infty \cap J_w) \cup \bigcup_{wu \in \mathcal{W}} \bigcup_{j=0}^{n_{wu}} I_{wu,j}$$

for all $w, wu \in \mathcal{W}$. For a fixed $wu \in \mathcal{W}$ and $j \in \{0, \dots, n_{wu}\}$, there exists $v \in \mathcal{W} \setminus \{\varepsilon\}$ and $i \in \{1, \dots, n_v\}$ such that $wu = vi$. Note that v is either $w^\uparrow$ or a descendant of $w^\uparrow$. There exists an interval $I_{v'i',j'}$ adjacent to $I_{vi,j}$ on the left, and there exists an interval $I_{v''i'',j''}$ adjacent to $I_{vi,j}$ on the right. Then,

$$\Phi(I_{vi,j}) \subset h_v(I_{vi,j}) \cup \gamma_{(v'i',j'),(vi,j)} \cup \gamma_{(vi,j),(v''i'',j'')}. \tag{3.10.1}$$

By design, each of the sets in the right-hand side of (3.10.1) is contained in $\mathcal{N}_{vi}$ (see Proposition 3.8.1, the discussion after (3.9.4), the discussion after (3.9.7), and the discussion after (3.9.10)). Since $\mathcal{N}_{vi} \subset \mathcal{N}_w$, we conclude that $\Phi(J_w \setminus J_\infty) \subset \mathcal{N}_w$. Therefore,

$$\Phi(J_w) = \Phi(J_w \setminus J_\infty) \cup \pi_2 \circ (\pi_1)^{-1}(J_w \cap J_\infty) = \Phi(J_w \setminus J_\infty) \cup (E \cap U_w) \subset \mathcal{N}_w. \quad \square$$

Lemma 3.10.3. *Let $I_{wi,j} \in \mathscr{I}$. Suppose that $I_{ui',j'}$ is adjacent to $I_{wi,j}$ on the left and $I_{vi'',j''}$ is adjacent to $I_{wi,j}$ on the right and that Φ is defined on $I_{ui',j'} \cup I_{wi,j} \cup I_{vi'',j''}$. Then $\Phi|(I_{ui',j'} \cup I_{wi,j} \cup I_{vi'',j''})$ is a bi-Lipschitz embedding. Furthermore, if E is c-uniformly disconnected, then there is a constant $\tilde{L}(\texttt{data}) \geq 1$ so that $\Phi|(I_{ui',j'} \cup I_{wi,j} \cup I_{vi'',j''})$ is an $(\tilde{L}, \Lambda_w)$-quasisimilarity.*

Proof. Let $I_{v\iota,\zeta}$ be the interval to the left of $I_{ui',j'}$ and let $I_{\omega\iota',\zeta'}$ be the interval to the right of $I_{vi'',j''}$. For brevity, we write v in place of $(v\iota, \zeta)$, u in place of (ui', j'), w in place of (wi, j), v in place of (vi'', j''), and ω in place of $(\omega\iota', \zeta')$. We make the assumption that both $t_u^L, t_v^R \in [a_u, b_v]$, but it does not affect the proof if this is not the case. There are thirteen subintervals to be considered, which are given by adjacent endpoints of the following string of inequalities:

$$a_u < t_u^L < \tilde{a}_u < \tilde{b}_u < t_u^R < t_w^L < \tilde{a}_w < \tilde{b}_w < t_w^R < t_v^L < \tilde{a}_v < \tilde{b}_v < t_v^R < b_v.$$

Fix $s, t \in I_u \cup I_w \cup I_v$. By design, Φ restricted to each subinterval is bi-Lipschitz, and if E is uniformly disconnected, then Φ restricted to each subinterval is a quasisimilarity, quantitatively, so we need not worry about the case in which s and t lie in the same subinterval.

If s and t lie in adjacent subintervals then one lies in a subinterval on which Φ is defined by a geodesic and the other lies in a subinterval on which Φ is defined by a bi-Lipschitz arc. Say $\Phi(s)$ lies on a geodesic g and $\Phi(t)$ lies on a bi-Lipschitz arc f. Since this geodesic g is chosen to join closest points of two bi-Lipschitz arcs (with f being one of them), [HVZ24, Lemma 4.2] implies that the concatenation of g and f is still bi-Lipschitz. (The idea of the cited lemma is that a geodesic joining some point p to a nearest point of a bi-Lipschitz arc must intersect the bi-Lipschitz arc at a large angle, otherwise it is not a nearest point to p.) If E is uniformly disconnected, then the maps g and f are quasisimilarities with scaling factor Λ_w. Rescaling the metric by this factor, these quasisimilarities become bi-Lipschitz arcs, where again [HVZ24, Lemma 4.2] can be applied (the notion of closest points is preserved under this scaling

of the metric). Upon returning to the original metric, the bi-Lipschitz concatenation becomes a quasisimilarity with scaling factor Λ_w.

Assume now that s and t are separated by a subinterval above. Then $\Phi(s)$ and $\Phi(t)$ lie in disjoint bi-Lipschitz arcs, hence we obtain the desired bounds and the first claim of the lemma. We now assume for the rest of the proof that E is uniformly disconnected. Note that $s, t \in J_{w\uparrow}$. By Lemma 3.10.2, $\Phi(s), \Phi(t) \in \mathcal{N}_{w\uparrow}$, so by (3.5.1),

$$d(\Phi(s), \Phi(t)) \leq \operatorname{diam} \mathcal{N}_{w\uparrow} \leq 2 \operatorname{diam} U_{w\uparrow} \leq 8\xi \operatorname{diam} U_w.$$

On the other hand, s and t are separated by a subinterval. By (3.9.7), (3.9.9), (3.9.10), and (3.9.11), the diameter of the smallest subinterval above is still of length at least

$$\theta \min\{L^{-1}, (L^*)^{-1}\} \min\{\Lambda_u^{-1} \operatorname{diam} U_u, \Lambda_w^{-1} \operatorname{diam} U_w, \Lambda_v^{-1} \operatorname{diam} U_v\}.$$

By using (3.7.3), we obtain

$$|s - t| \geq (2N + 1)^{-1} \theta \min\{L^{-1}, (L^*)^{-1}\} \Lambda_w^{-1} \operatorname{diam} U_w.$$

It remains to provide lower bounds on $d(\Phi(s), \Phi(t))$ in terms of $|s - t|$, which we do by considering three cases.

Case 1. Suppose that s and t lie in distinct subintervals on which Φ is defined by h_u, h_w, or h_v. Namely, $s, t \in [\tilde{a}_u, \tilde{b}_u] \cup [\tilde{a}_w, \tilde{b}_w] \cup [\tilde{a}_v, \tilde{b}_v]$. For convenience, we assume that $s \in [\tilde{a}_u, \tilde{b}_u]$ and $t \in [\tilde{a}_w, \tilde{b}_w]$, but the other cases are identical. By Proposition 3.8.1(4), we have

$$d(\Phi(s), \Phi(t)) \geq \operatorname{dist}(\operatorname{Im}(h_u), \operatorname{Im}(h_w)) \geq L^{-1} \operatorname{diam} U_w,$$

while, since $s, t \in J_{w\uparrow}$, by (3.7.3),

$$|s-t| \le |J_{w\uparrow}| \le (2n_{w\uparrow}+1)(2n_w+1)|J_{wi}| \le (2N+1)^2|J_{wi}| = (2N+1)^2\Lambda_w^{-1}\operatorname{diam} U_w. \tag{3.10.2}$$

Case 2. Suppose that s and t lie in distinct subintervals on which Φ is defined by $\gamma_{v,u}$, $\gamma_{u,w}$, $\gamma_{w,v}$, or $\gamma_{v,\omega}$. Namely, $s, t \in [a_u, \tilde{a}_u] \cup [\tilde{b}_u, \tilde{a}_w] \cup [\tilde{b}_w, \tilde{a}_v] \cup [\tilde{b}_v, b_v]$. For convenience, we assume that $s \in [\tilde{b}_u, \tilde{a}_w]$ and $t \in [\tilde{b}_w, \tilde{a}_v]$, but the other cases are identical. By (3.9.13) and (3.5.1),

$$d(\Phi(s), \Phi(t)) \ge \operatorname{dist}(\gamma_{u,w}, \gamma_{w,v}) \ge \tfrac{1}{12}\rho_w\epsilon_{wi}\operatorname{diam} U_{wi} \ge \tfrac{1}{48}\xi^{-1}\rho_w\epsilon_{wi}\operatorname{diam} U_w,$$

while the bound of (3.10.2) still holds. Recall that since E is uniformly disconnected, the constants ρ_w and ϵ_{wi} are independent of w and wi, respectively, and depend only on the `data`.

Case 3. Suppose that s lies in a subinterval on which Φ is defined by h_u, h_w, or h_v and t lies in a subinterval on which Φ is defined by $\gamma_{v,u}$, $\gamma_{u,w}$, $\gamma_{w,v}$, or $\gamma_{v,\omega}$. Namely, s belongs to an interval considered in Case 1, and t belongs to an interval considered in Case 2. For convenience, we assume that $s \in [\tilde{a}_u, \tilde{b}_u]$ and $t \in [\tilde{b}_w, \tilde{a}_v]$, but the other cases are identical. By (3.9.14),

$$d(\Phi(s), \Phi(t)) \ge \operatorname{dist}(\gamma_{w,v}, \operatorname{Im}(h_v)) \ge \operatorname{dist}(\gamma_{w,v}, H_{w,v}) \ge \theta \operatorname{diam} U_w$$

and again the bound of (3.10.2) holds. □

Lemma 3.10.4. *Suppose that E is c-uniformly disconnected and that $I_{ui',j'}$ and $I_{wi,j}$ are not adjacent intervals. Then*

$$\operatorname{dist}(\Phi(I_{ui',j'}), \Phi(I_{wi,j})) \ge (4\xi)^{-2}\theta \min\{\operatorname{diam} U_u, \operatorname{diam} U_w\}.$$

Proof. Let $s \in I_{ui',j'}$ and $t \in I_{wi,j}$. There are three cases to consider.

Case 1. Assume $s \in [\tilde{a}_{ui',j'}, \tilde{b}_{ui',j'}]$ and $t \in [\tilde{a}_{wi,j}, \tilde{b}_{wi,j}]$. As $\Phi(s) = h_u(s)$ and $\Phi(t) = h_w(t)$, Proposition 3.8.1(4) implies $d(\Phi(s), \Phi(t)) \geq L^{-1} \max\{\operatorname{diam} U_u, \operatorname{diam} U_w\}$.

Case 2. Assume $s \in I_{ui',j'} \setminus [\tilde{a}_{ui',j'}, \tilde{b}_{ui',j'}]$ and $t \in [\tilde{a}_{wi,j}, \tilde{b}_{wi,j}]$. Note that $\Phi(t) = h_w(t)$ and $\Phi(s) = \gamma(s)$ for some curve γ defined as in (3.9.12). In particular, either $\Phi(s) = \gamma_{(vi'',j''),(ui',j')}(s)$ or $\Phi(s) = \gamma_{(ui',j'),(vi'',j'')}(s)$ for some interval $I_{vi'',j''}$ adjacent to $I_{ui',j'}$. In either case, by (3.9.14),

$$d(\Phi(s), \Phi(t)) \geq \theta \operatorname{diam} U_u.$$

Case 3. Assume $s \in I_{ui',j'} \setminus [\tilde{a}_{ui',j'}, \tilde{b}_{ui',j'}]$ and $t \in I_{wi,j} \setminus [\tilde{a}_{wi,j}, \tilde{b}_{wi,j}]$. Note that either $\Phi(s) = \gamma_{(vi'',j''),(ui',j')}(s)$ or $\Phi(s) = \gamma_{(ui',j'),(vi'',j'')}(s)$ and either $\Phi(t) = \gamma_{(\omega i''',j'''),(wi,j)}(t)$ or $\Phi(t) = \gamma_{(wi,j),(\omega i''',j''')}(t)$. By (3.9.13) and (3.5.1), we get

$$\begin{aligned} d(\Phi(s), \Phi(t)) &\geq \frac{1}{12}(\rho_{ui'}\epsilon_{ui'j'} \operatorname{diam} U_{ui'j'} + \rho_{wi}\epsilon_{wij} \operatorname{diam} U_{wij}) \\ &\geq (4\xi)^{-2}\theta \min\{\operatorname{diam} U_u, \operatorname{diam} U_w\}. \end{aligned}$$

□

Proof of Proposition 3.10.1. For the first claim, it suffices to prove that Φ is an embedding, as from Lemma 3.10.3 we have that Φ is locally bi-Lipschitz away from J_∞. To prove continuity of Φ on J_∞, fix $\mathbf{w} \in \mathbf{W}$ and fix $\epsilon > 0$. There exists $n \in \mathbb{N}$ such that $\operatorname{diam} \mathcal{N}_{\mathbf{w}(n)} \leq 2^{-n} \operatorname{diam} U_\varepsilon < \epsilon$. Then, the interior of $J_{\mathbf{w}(n)}$ is a neighborhood of $\pi_1(\mathbf{w})$ with a diameter of at most 3^{-n} and $\Phi(\pi_1(\mathbf{w})) \in \Phi(J_{\mathbf{w}(n)}) \subset \mathcal{N}_{\mathbf{w}(n)}$ by Lemma 3.10.2. Therefore, Φ is continuous.

By design, it is easy to see that $\Phi|[\hat{a}_{1,0}, \hat{b}_{1,n_1}] \setminus J_\infty$ is injective. Moreover, if $s \in J_\infty$ and $t \in [\hat{a}_{1,0}, \hat{b}_{1,n_1}] \setminus J_\infty$, then $\Phi(s) \in E$ while $\Phi(t) \in X \setminus E$. Finally, since π_1 and π_2 are bijections, it follows that $\Phi|J_\infty$ is bijective. This shows that Φ is injective which completes the first claim of the proposition.

For the second claim, assume for the rest of the proof that E is c-uniformly disconnected. Let $s, t, \tau \in [\hat{a}_{1,0}, \hat{b}_{1,n_1}]$ with $|s - t| \leq |s - \tau|$. The proof is a case study of the positions of s, t, and τ.

Case 1. Suppose that s and τ are separated by a set $I_{wi,j}$ for some $w \in \mathcal{W}$, $i \in \{1, \ldots, n_w\}$, and $j \in \{0, \ldots, n_{wi}\}$. We may assume that $I_{wi,j}$ is a largest such interval, that is, $|w|$ is minimal among all words $v \in \mathcal{W}$ for which $I_{vi',j'}$ separates s and τ for some $i' \in \{1, \ldots, n_v\}$ and $j' \in \{0, \ldots, n_{vi'}\}$.

Recall that the interval $I_{wi,j}$ is contained in the interval J_{wi}, which in turn is contained in the interval J_w. In addition to the subinterval J_{wi}, the interval J_w contains two other subintervals of note: the interval $I_{w,i-1}$ is adjacent to J_{wi} on the left and the interval $I_{w,i}$ is adjacent to J_{wi} on the right. The interval J_{wi} contains subintervals of the form J_{wik} for $k \in \{1, \ldots, n_{wi}\}$ and of the form $I_{wi,k}$ for $k \in \{0, \ldots, n_{wi}\}$. In particular, the interval J_{wij} is adjacent to $I_{wi,j}$ on the left (except if $j = 0$, where instead $I_{w,i-1}$ is adjacent to $I_{wi,j}$ on the left) and the interval $J_{wi(j+1)}$ is adjacent to $I_{wi,j}$ on the right (except if $j = n_{wi}$, where instead $I_{w,i}$ is adjacent to $I_{wi,j}$ on the right). Within the interval J_{wij}, there are subintervals of the form J_{wijk} for $k \in \{1, \ldots, n_{wij}\}$ and of the form $I_{wij,k}$ for $k \in \{0, \ldots, n_{wij}\}$, while in the interval $J_{wi(j+1)}$, there are subintervals of the form $J_{wi(j+1)k}$ for $k \in \{1, \ldots, n_{wij}\}$ and of the form $I_{wi(j+1)k}$ for $k \in \{0, \ldots, n_{wij}\}$.

This is the maximal extent to which we subdivide the intervals around $I_{wi,j}$: the furthest s and τ can be is when one is in $I_{w,i-1}$ and the other is in $I_{w,i}$, as otherwise one of these intervals would separate s and τ, which contradicts the maximality of $I_{wi,j}$. We consider intervals up to one generation after $I_{wi,j}$ since we can obtain the desired bounds in these cases, but this is not always possible when considering intervals of the same generation of $I_{wi,j}$.

We note that since $s, \tau \in J_w$, $|s - t| \leq |s - \tau| \leq |J_w|$ and because $s \in J_w$, it must be that $s, t \in J_{w^\uparrow}$. We now obtain the upper bound

$$d(\Phi(s), \Phi(t)) \leq \operatorname{diam} \mathcal{N}_{w^\uparrow} \leq 2 \operatorname{diam} U_{w^\uparrow} \leq 8\xi \operatorname{diam} U_w$$

by Lemma 3.10.2 and (3.5.1). The remainder of Case 1 is a calculation of a lower bound of $d(\Phi(s), \Phi(\tau))$ in terms of $\operatorname{diam} U_w$ based on the placement of s and τ within the intervals specified above.

Case 1.1. Suppose that s and τ lie in distinct intervals of the collection

$$\{I_{w,i-1}, I_{w,i}\} \cup \{I_{wij,k} : k \in \{0, \dots, n_{wij}\}\} \cup \{I_{wi(j+1),k} : k \in \{0, \dots, n_{wi(j+1)}\}\}.$$

By Lemma 3.10.4 and (3.5.1),

$$d(\Phi(s), \Phi(\tau)) \geq (4\xi)^{-2}\theta \operatorname{diam} U_{wi} \geq (4\xi)^{-3}\theta \operatorname{diam} U_w.$$

Case 1.2. Suppose that s and τ lie in distinct intervals of the collection

$$\{J_{wijk} : k \in \{1, \dots, n_{wij}\}\} \cup \{J_{wi(j+1)k} : k \in \{1, \dots, n_{wi(j+1)}\}\}.$$

Note that since s and τ are separated by the interval $I_{wi,j}$, it must be the case that one lies in an interval of the left collection and the other must lie in the interval of the right collection, say $s \in J_{wijk_1}$ and $\tau \in J_{wijk_2}$. By Lemma 3.10.2, Lemma 3.7.1, and (3.5.1),

$$\begin{aligned} d(\Phi(s), \Phi(\tau)) \geq \operatorname{dist}(\mathcal{N}_{wijk_1}, \mathcal{N}_{wi(j+1)k_2}) &\geq \frac{(\operatorname{diam} U_{wijk_1} + \operatorname{diam} U_{wi(j+1)k_2})}{(4\xi)^3} \\ &\geq \frac{\operatorname{diam} U_w}{2(4\xi)^6}. \end{aligned}$$

Case 1.3. Suppose that s lies in an interval in the collection considered in Case 1.1 and τ lies in an interval in the collection considered in Case 1.2. Again we note that one of s or τ lies to the left of $I_{wi,j}$ and the other lies to the right of $I_{wi,j}$. As in Case 1.2, by Lemma 3.10.2, Lemma 3.7.1, and (3.5.1),

$$d(\Phi(s), \Phi(\tau)) \geq 2(4\xi)^{-3} \operatorname{diam} U_w.$$

Case 2. Suppose that s and τ lie in adjacent intervals, say $s \in I_{wi,j}$ and $\tau \in I_{ui',j'}$. The proof again divides into two subcases, this time based on the size of $|s-\tau|$ in comparison to $\operatorname{diam} U_w$.

Case 2.1. Suppose that $|s-\tau| \le (2N+1)^{-2}\Lambda_w^{-1} \operatorname{diam} U_w$. Note that by (3.7.3),

$$|s-t| \le |s-\tau| \le (2N+1)^{-2}\Lambda_w^{-1} \operatorname{diam} U_w \le (2n_w+1)^{-1}(2n_{wi}+1)^{-1}|J_{wi}| \le |I_{vi'',j''}|,$$

where $I_{vi'',j''}$ is the other interval adjacent to $I_{wi,j}$. It follows that $t \in I_{ui',j'} \cup I_{wi,j} \cup I_{vi'',j''}$, and by Lemma 3.10.3, the map $\Phi|(I_{ui',j'} \cup I_{wi,j} \cup I_{vi'',j''})$ is a $(\tilde{L}, \Lambda_w)$-quasisimilarity. Therefore

$$d(\Phi(s), \Phi(t)) \le \tilde{L}\Lambda_w |s-t| \le \tilde{L}\Lambda_w |s-\tau|$$

and $d(\Phi(s), \Phi(\tau)) \ge (\tilde{L})^{-1}\Lambda_w |s-\tau|$.

Case 2.2. Suppose that $|s-\tau| > (2N+1)^{-2}\Lambda_w^{-1} \operatorname{diam} U_w$. As in Case 1, it follows that $s, t, \tau \in J_{w^\uparrow}$. By Lemma 3.10.2 and (3.5.1),

$$d(\Phi(s), \Phi(t)) \le \operatorname{diam} \mathcal{N}_{w^\uparrow} \le 2 \operatorname{diam} U_{w^\uparrow} \le 8\xi \operatorname{diam} U_w < 8(2N+1)^2 \Lambda_w |s-\tau|.$$

On the other hand, by Lemma 3.10.3, $d(\Phi(s), \Phi(\tau)) \ge (\tilde{L})^{-1}\Lambda_w |s-\tau|$. □

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Vita

I began my studies in mathematics at Clemson University in August 2015, where I worked with Dr. Leo Rebholz and Dr. Keri Sather-Wagstaff and received my Bachelor of Science degree in Mathematics in May 2019. I continued my studies at the University of Tennessee, Knoxville, starting in August 2019, where I worked with Dr. Vyron Vellis and received my Master of Science degree in Mathematics in May 2023. I am currently working with Dr. Vyron Vellis and aim to receive my Doctor of Philosophy degree in Mathematics in May 2025.

During my time at Clemson University, I received several honors, including being named to the Dean's List in December 2015, December 2016, May 2017, and May 2018, and to the President's List in May 2016, December 2017, and May 2019. I also received the John Charles Harden Award in April 2018 and the Samuel M. Martin Award in April 2019 from the School of Mathematical and Statistical Sciences. At the University of Tennessee, Knoxville, I received a Graduate Research Assistantship in Fall 2023, funded by NSF Grant No. DMS-2154918. In April 2024, I was honored with the Graduate Student Academic Achievement Award from the Department of Mathematics at the University of Tennessee, Knoxville, and I served as a referee for the *Bulletin of the Malaysian Mathematical Sciences Society*.

My current research interests lie in the fields of analysis and geometry on metric spaces, with particular focus on sub-Riemannian geometry and geometric function theory, including the study of quasiconformal, quasisymmetric, and bi-Lipschitz mappings. Previously I have focused on topics in the areas of partial differential

equations, numerical analysis, commutative algebra, and graph theory. While at Clemson University, I collaborated with Sarah Kelly and Hannah Johnson under the supervision of Dr. Leo Rebholz on a project that resulted in the publication "Using Data Assimilation to Better Predict Contaminant Transport" in *SIAM Undergraduate Research Online*. I also worked with Dr. Keri Sather-Wagstaff on the project "Closed Neighborhood Ideals of Finite Simple Graphs," published in *La Matematica*. At the University of Tennessee, Knoxville, I collaborated with Dr. Vyron Vellis and Dr. Scott Zimmerman on the project "Bi-Lipschitz Arcs in Metric Spaces with Controlled Geometry," which was published in *Rev. Mat. Iberoam.*. Dr. Vyron Vellis and I are currently working on a project titled "Quasisymmetric Rectifiability of Uniformly Disconnected Sets," which we aim to publish in 2025.

Throughout my academic career, I have had the opportunity to present my research at various conferences and seminars. While at Clemson University, I presented a poster "Contaminant Transport" at the MAA Southeastern Conference at Clemson University in March 2018 and another poster "Closed Neighborhood Ideals" at the MAA Southeastern Conference at Lee University in March 2019. At the University of Tennessee, Knoxville, I gave multiple talks in the Analysis Seminar, including "A Bi-Lipschitz Extension Theorem" on September 21, 2022, "Bi-Lipschitz Arcs in Metric Spaces" on September 27, 2023, and "Geometric Rectifiability and Co-Uniformity" on October 23, 2024. I also presented my work at several AMS meetings, including "Bi-Lipschitz Arcs in Metric Spaces" at the AMS 2024 Spring Central Sectional Meeting at the University of Wisconsin, Milwaukee, on April 21, 2024; "Geometric Rectifiability and Co-Uniformity" at the AMS 2024 Fall Eastern Sectional Meeting at the University at Albany on October 23, 2024; and "Geometric Rectifiability and Co-Uniformity" at JMM 2025 at the Seattle Convention Center on January 9, 2025. Additionally, I presented "Bi-Lipschitz Arcs in Metric Spaces" at the Ohio River Analysis Meeting 13 at the University of Kentucky on March 17, 2024, and I am scheduled to present "Geometric Rectifiability in Metric Measure Spaces"

at the Ohio River Analysis Meeting 14 at the University of Cincinnati on March 29, 2024.

I have attended several conferences and workshops, including the 51st John and Lida Barrett Memorial Lectures at the University of Tennessee, Knoxville, from June 9 to 11, 2022, the Ohio River Analysis Meeting 12 at the University of Cincinnati on March 18 and 19, 2023, and the Quasiworld Workshop at the University of Helsinki from August 14 to 18, 2023.

In addition to my research, I have also been involved in teaching and mentoring. At Clemson University, I was an undergraduate teaching assistant for Calculus of Several Variables in Fall 2018 and Introduction to Differential Equations in Spring 2019. At the University of Tennessee, Knoxville, I served as a graduate teaching associate for several courses, including College Algebra (Fall 2019), Basic Calculus (Spring 2020, Summer 2022), Mathematical Reasoning (Fall 2020, Spring 2021), Calculus 1B with Precalculus (Spring 2024), and Finite Mathematics (Fall 2021, Spring 2022, Fall 2022, Spring 2023, Fall 2024, Spring 2025). From Fall 2024 to Spring 2025, I was also an undergraduate research mentor in the Knoxville Math Lab, where I mentored two projects focused on the Analyst's Traveling Salesman Problem.