To the Graduate Council:

I am submitting herewith a dissertation written by Richard Douglas Chatham entitled "Going-down Pairs of Commutative Rings." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

David E. Dobbs, Major Professor

We have read this dissertation
and recommend its acceptance:

David F Anderson

Robert Hamilton Gilmer

Mark Mahowald

Accepted for the Council:

Associate Vice Chancellor and
Dean of The Graduate School

GOING-DOWN PAIRS OF COMMUTATIVE RINGS

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Richard Douglas Chatham

May 2000

ACKNOWLEDGEMENTS

First, I would like to thank my advisor, David Dobbs, for his sage advice, his encouragement and support, and his patient, careful examination and correction of innumerable drafts of this work. I also thank the other members of my dissertation committee – David Anderson, Robert McConnel and Mark Kot – for their helpful comments and support.

I would like to thank the Mathematics Department for their financial support and for the other ways they have helped me during my time as a graduate student here at the University of Tennessee.

Finally, but most importantly, I thank my family, especially my mother, Geraldine Chatham. Without their love and support, I would not have made it to this point.

ABSTRACT

This work develops criteria for determining whether an extension of rings $R \subseteq T$ satisfies the condition that $A \subseteq B$ satifies $\mathcal{P}$ for each ring extension such that $R \subseteq A \subseteq B \subseteq T$ (i.e., (R, T) is a "$\mathcal{P}$-pair"), for some properties $\mathcal{P}$ related to "going-down."

In Chapter I, we define many properties and associated pairs of rings related to "going-down" and study how these ring-theoretic properties behave under the formation of factor domains, finite products, and reduced rings.

In Chapter II, assuming the (Krull) dimension of the base ring R is zero, we find that (R, T) is a "going-down-ring pair" (resp., "open-ring pair") if and only if, for each minimal prime P of T, we have that the transcendence degree of T/P over $R/(P \cap R)$ is less than or equal to one (resp., is zero). We provide examples to show that transcendence degree criteria are not sufficent to characterize going-down-ring pairs or open-ring pairs in the case where the dimension of the base ring is nonzero, and then we provide some positive partial results involving INC-pairs.

In Chapter III, we generalize results of Dobbs and Papick on extensions of going-down domains and open domains and we use these generalizations to show that a going-down-ring pair (resp., open-ring pair) is a going-down-pair (resp., open-pair) if the top ring has a unique minimal prime ideal.

In Chapter IV, using results of Dobbs, Mulay, and Akiba, we show that if the base ring R is Prüfer or the top ring T is a field, then a pair of domains (R, T) is a flat pair if and only if $R \subseteq T$ is either an algebraic field extension or the inclusion of a Prüfer domain inside an overring. We also prove some results on i-pairs of arbitrary rings.

TABLE OF CONTENTS

INTRODUCTION 1

CHAPTER I. ON SEVERAL PROPERTIES OF PAIRS OF COMMUTATIVE RINGS 5

CHAPTER II. PAIRS AND TRANSCENDENCE DEGREE 15

CHAPTER III. CONNECTIONS BETWEEN $\mathcal{P}$-PAIRS AND $\mathcal{D}$-RING PAIRS 31

CHAPTER IV. FLAT-PAIRS AND ADDITIONAL REMARKS 39

BIBLIOGRAPHY 47

VITA 51

INTRODUCTION

Throughout this work, all rings are commutative with 1 and all ring homomorphisms $R \to T$ send the 1 of R to the 1 of T. We assume the reader is familiar with basic commutative algebra as described in [K], [G2] and [E]. Familiarity with Papick's thesis [P] and with the going-down survey article of Dobbs and Papick [DP2] would also be helpful to the reader.

We proceed to discuss background material. An *overring* of a domain R is a domain containing R and contained in the quotient field of R (which we denote by $qf(R)$). An *underring* of a domain R is a subring A of R such that $qf(A) = qf(R)$. More generally, an *overring* of an arbitrary ring S is a ring containing S and contained in the total quotient ring of S (which we denote by $T(S)$). The symbol $\dim(R)$ always represents the Krull dimension of a ring R. The *transcendence degree* of a domain T over a subring R, represented by $td_R(T)$, is defined to be the transcendence degree of the quotient field of T over the quotient field of R.

The set of prime ideals of a ring R will be denoted $Spec(R)$. Except where otherwise noted, $Spec(R)$ will have the Zariski topology, which is defined by declaring the basic open sets to be the sets of the form $\{P \in Spec(R) | g \notin P\}$, where g runs over the set of elements of R. On a few occasions, we impose on $Spec(R)$ the *flat topology*, which is defined by declaring as the closed sets the images of maps $Spec(S) \to Spec(R)$ induced by ring homomorphisms $R \to S$ that make S a flat R-module. The one fact we will use about the flat topology states that a ring extension $R \subseteq T$ satisfies going-down if and only if the induced contraction map $Spec(T) \to Spec(R)$ (that sends a prime ideal P of T to the prime ideal $P \cap R$ of R) is a closed map in the flat topology. (This is [DFP, Proposition 2.7].)

Let R be a ring. The set of minimal prime ideals of R will be denoted $Min(R)$. The set of maximal prime ideals of R will be denoted $Max(R)$. The set

of nilpotent elements of R will be denoted $Nil(R)$. Recall that R is *reduced* if it contains no nonzero nilpotent elements. We define the reduced ring of a ring R to be $R_{red} := R/Nil(R)$.

The lying-over and incomparability properties described in [K, p. 28] will be denoted LO and INC, respectively.

A ring extension $R \subseteq T$ is *open* (resp., an *LH-extension*, an *i-extension*, *unibranched*) if the contraction map $Spec(T) \to Spec(R)$ is an open map (resp., local homeomorphism, injection, bijection) in the Zariski topology. A ring extension $R \subseteq T$ is *flat* if T is a flat R-module. Flat and open extensions satisfy going-down (cf. [DP2, Theorem 3.5]).

A *going-down-domain* is a domain R such that $R \subseteq T$ satisfies going-down for all overrings T of R. The most familiar examples of going-down domains are Prüfer domains and domains of (Krull) dimension less than or equal to 1 (cf. [DP2, p. 273]). Among other facts, we will need the result that for classical $D + M$ constructions, $D + M$ is a going-down-domain if and only if D is a going-down-domain, by [DP1, Corollary].

Similarly, we can define *open-domain*, *LH-domain*, *i-domain*, and *flat-domain*. Since LH-extensions are open, LH-domains are open-domains. Also, open-domains are going-down-domains. An open-domain R must be semiquasilocal; that is, R must have only finitely many maximal ideals, but need not be Noetherian ([P, Corollary 3.4]). By [R, Theorem 4], a domain R is a flat-domain if and only if R is a Prüfer domain. Also, a domain R is a Prüfer domain if and only if each torsion-free R-module is R-flat (cf. [C, Theorem 4.2]).

A ring extension $R \subseteq T$ is *residually algebraic* if $td_{R/P\cap R}(T/P) = 0$ for all $P \in Spec(T)$ (cf. [AJ]). By [FHP, Theorem 6.5.6], a pair of rings (R,T) is a *residually algebraic pair* (i.e., $A \subseteq B$ is residually algebraic for all rings $R \subseteq A \subseteq$

$B \subseteq T$) if and only if (R, T) is an INC-pair (i.e., $A \subseteq B$ satisfies INC for all rings $R \subseteq A \subseteq B \subseteq T$).

In [D6], Dobbs introduced the concept of a "going-down-ring." A ring R is a *going-down ring* if R/P is a going-down-domain for all $P \in Spec(R)$. For example, any one-dimensional ring is a going-down-ring. In Chapter I, we will similarly define *open-rings*, *LH-rings*, and *i-rings*.

Let $R \subseteq T$ be an extension of rings. Then R is *integrally closed in* T if any element of T that is a root of a polynomial with coefficients in R must itself be an element of R. Also, if R is a domain, then R is *seminormal* if for each $t \in qf(R)$, we have that $t^2, t^3 \in R$ implies $t \in R$. Note that R integrally closed (in $qf(R)$) implies that R is seminormal.

We next proceed to quote a few theorems that we will find useful later. First we quote Evans' generalization of Zariski's Main Theorem.

[Ev, Theorem] Let $R \subseteq T$ be an extension of rings with R integrally closed in T such that there exist $t_1, \ldots t_n$ with T integral over $R[t_1, \ldots t_n]$. If a prime ideal P of T is isolated over $P \cap R$ (i.e., if P is maximal and minimal with respect to the primes of T whose intersection with R is $R \cap P$), then there exists an element $s \in R \backslash (P \cap R)$ such that $T_s = R_s$.

[AD, Theorem 3.10] For rings $R \subseteq T$, the following are equivlaent:

(1) $Spec(R) = Spec(T)$ (as sets);

(2) $Max(R) = Max(T)$;

(3) $Max(R) \subseteq Max(T)$;

(4) $Max(T) \subseteq Max(R)$;

(5) R and T have the same radical ideals.

Moreover, if (any of) the above conditions hold and $R \neq T$, then R is quasilocal.

[AD, Corollary 3.26] Let $R \subsetneq T$ be distinct rings such that $Spec(R) = Spec(T)$. Let M be the unique maximal ideal of R. Then $Spec(R) = Spec(S)$ for each ring contained between R and T if and only if the field extension $R/M \subseteq T/M$ is algebraic.

[CMZ, Corollary 4.15] Let D be a domain with quotient field K. Then $D + XK[X]$ is a Prüfer domain if and only if D is a Prüfer domain.

[DM, Theorem 2.2] Let A be a subring of a domain R, and assume that R is flat over each of its underrings that contains A. Assume also that R is not a field. Then R is algebraic over A.

Finally, we quote a result of Wadsworth giving necessary conditions for a pair of domains (R, T) to be a Noetherian pair of domains (i.e., every domain S such that $R \subseteq S \subseteq T$ is a Noetherian domain).

[W, Theorem 4] Suppose (R, T) is a Noetherian pair of domains. Then,

(1) if R is not a field, then $[qf(T) : qf(R)] < \infty$;

(2) if R is a field, then either $qf(T)$ is algebraic over $qf(R)$ or $qf(T)$ is an algebraic function field in one variable over $qf(R)$.

CHAPTER I

ON SEVERAL PROPERTIES OF PAIRS OF COMMUTATIVE RINGS

The going-down property is one of the fundamental properties of ring extensions that are described in [K, p. 28] (the others being incomparability, lying-over and going-up). At various times, authors have studied properties of pairs of rings related to incomparability, lying-over and going-up. (For example, see [FHP] and [D3].) Our purpose here is to study various kinds of pairs related to going-down.

In this paper, $\mathcal{P}$ always refers to a property of (some) ring extensions and $\mathcal{D}$ always refers to a property of (some) domains. Also, the notation (R, T) is used to indicate that $R \subseteq T$ is a ring extension. The following five definitions will ease the exposition.

Definition 1.1 Let $R \subseteq T$ be a ring extension. Then (R, T) is a *$\mathcal{P}$-pair* if $A \subseteq B$ satisfies $\mathcal{P}$ for all ring extensions $A \subseteq B$ such that $R \subseteq A \subseteq B \subseteq T$.

Definition 1.2 A domain R is a *$\mathcal{P}$-domain* if $R \subseteq S$ satisfies $\mathcal{P}$ for all overrings S of R.

Definition 1.3 Let $R \subseteq T$ be an extension of domains. Then (R, T) is a *$\mathcal{D}$-domain pair* if S satisfies $\mathcal{D}$ for all domains S such that $R \subseteq S \subseteq T$.

Consider applying the above definitions in case $\mathcal{P}$ is either going-down or open. The most obvious examples of going-down-domain pairs that need not be open-domain pairs are (R, T), where R is any Prüfer domain and T is an overring of R; and $(k, k(X))$, where k is any field and X is an indeterminate over k. Since "open" implies "going-down", we see that each open-pair is a going-down-pair and

each open-domain is a going-down-domain. Also, we note that each going-down-domain pair (resp., open-domain pair) is a going-down-pair (resp., open-pair), since R a going-down-domain (resp., open domain) implies that $R \subseteq T$ satisfies going-down (resp., is an open extension) for all domains T containing R, by [DP2, Theorem 4.1] (resp., [P, Proposition 3.10]). The converse is false: for instance, open-pairs of domains need not be going-down-domain pairs. An example of this is given in Example 3.1.

In extending certain domain-theoretic results to the context of arbitrary commutative rings, we generalize the approach given for $\mathcal{P}$ = going-down in [D6] by introducing the following definition.

Definition 1.4 A ring R is a *$\mathcal{D}$-ring* if R/P is a $\mathcal{D}$-domain for all $P \in Spec(R)$.

Note that if R is a domain, then R a $\mathcal{D}$-ring implies that R is a $\mathcal{D}$-domain, since (0) is a prime ideal of any domain and $R/(0) \cong R$. However, for some properties $\mathcal{D}$ such as "having Krull dimension 1", R a $\mathcal{D}$-domain does not imply that R is a $\mathcal{D}$-ring.

We similarly extend the concept of $\mathcal{D}$-domain pairs.

Definition 1.5 (R, T) is a *$\mathcal{D}$-ring pair* if S is a $\mathcal{D}$-ring for all rings S such that $R \subseteq S \subseteq T$.

As we see in the known results collected in the next Proposition, many domain properties $\mathcal{F}$ satisfy the condition that if $\mathcal{F}$ holds for a domain R, then $\mathcal{F}$ also holds for R/P where P is any prime ideal of R.

Proposition 1.6 For each of the following properties $\mathcal{D}$, the class of $\mathcal{D}$-domains is stable under formation of factor domains: $\mathcal{D}$ = going-down, open, INC-, LH-, i-, Prüfer.

Proof: For the statement for going-down-domains, use [D2, Remarks 2.11 and 3.2(a),(b)]. For the statement for open-domains, use [D2, pp. 360-1, paragraph after Remark 2.11]. For the statement for LH-domains, use [P, Corollary 4.18]. For the statement for INC-domains, combine [FHP, Proposition 6.5.1] with [FHP, Corollary 6.5.14]. For the statement for i-domains, use [P, p. 3, before Lemma 2.2]. Finally, the statement for Prüfer domains is well-known: cf. [G2, Proposition 22.5]. ■

For the rest of this chapter, we will let $\mathcal{F}$ represent any property of (some) domains such that when R satisfies $\mathcal{F}$, so does R/P for all prime ideals P of R and so does any domain isomorphic to R.

We next note some nice properties of $\mathcal{F}$-rings and $\mathcal{F}$-ring pairs.

Proposition 1.7 If R is an $\mathcal{F}$-ring and I is an ideal of R, then R/I is also an $\mathcal{F}$-ring.

Proof: Prime ideals in R/I have the form P/I where $P \in Spec(R)$ and $I \subseteq P$. Then $(R/I)/(P/I) \cong R/P$, which satisfies $\mathcal{F}$. ■

Proposition 1.8 R is an $\mathcal{F}$-ring if and only if R/P_0 is a domain satisfying $\mathcal{F}$ for all $P_0 \in Min(R)$.

Proof: The "only if" assertion is trivial. For the converse, let $P \in Spec(R)$. By Zorn's Lemma, $P_0 \subseteq P$ for some $P_0 \in Min(R)$. By hypothesis, R/P_0 is a domain satisfying $\mathcal{F}$. Therefore $R/P \cong (R/P_0)/(P/P_0)$ also satisfies $\mathcal{F}$. ■

Corollary 1.9 Let R be a domain. Then R is an $\mathcal{F}$-ring if and only if R is an $\mathcal{F}$-domain.

Proof: For the "only if" implication, note that since R is a domain, $(0) \in Spec(R)$; therefore, $R \cong R/(0)$ must be a $\mathcal{F}$-domain. Conversely, since R is a domain, $Min(R) = \{(0)\}$. Now, $R/(0) \cong R$ is assumed to be an $\mathcal{F}$-domain; hence by Proposition 1.8 , R is an $\mathcal{F}$-ring. ■

The next two propositions will reduce the study of $\mathcal{F}$-ring pairs to that of $\mathcal{F}$-domain pairs.

Proposition 1.10 Suppose (R,T) is an $\mathcal{F}$-ring pair. Let Q be an ideal of T and let $P = Q \cap R$. Then $(R/P, T/Q)$ is an $\mathcal{F}$-ring pair. If, in addition, $Q \in Spec(T)$, then $(R/P, T/Q)$ is an $\mathcal{F}$-domain pair.

Proof: Let D be any ring such that $R/(Q \cap R) \subseteq D \subseteq T/Q$. Note $R/(Q \cap R) \cong (R+Q)/Q$. Then $D \cong A/Q$ for some ring A such that $R + Q \subseteq A \subseteq T$. Since (R,T) is an $\mathcal{F}$-ring pair, A is an $\mathcal{F}$-ring. Therefore, by Proposition 1.7, $D \cong A/Q$ is also an $\mathcal{F}$-ring. The final assertion follows by combining the first assertion with Corollary 1.9. ■

Proposition 1.11 The following statements are equivalent for a ring extension $R \subseteq T$:

(1) (R,T) is an $\mathcal{F}$-ring pair;

(2) $(R/(Q \cap R), T/Q)$ is an $\mathcal{F}$-domain pair, for all Q in $Min(T)$;

(3) $(R/(Q \cap R), T/Q)$ is an $\mathcal{F}$-domain pair, for all Q in $Spec(T)$.

Proof: (1) $\Rightarrow$ (3): Apply Proposition 1.10.

(3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): For each ring A such that $R \subseteq A \subseteq T$ and for all Q in $Min(A)$, there is a Q_0 in $Min(T)$ such that $Q_0 \cap A = Q$ (cf. [K, p. 41, Exercise 1] and [K,

Theorem 10]). Now $A/Q \cong (A+Q_0)/Q_0$; therefore, $R/(R\cap Q_0) \subseteq A/Q \subseteq T/Q_0$. Hence, by (2), A/Q is a $\mathcal{F}$-domain. Since A/Q is an $\mathcal{F}$-domain for all Q in $Min(A)$, A is an $\mathcal{F}$-ring by Proposition 1.8. Therefore, (R,T) is an $\mathcal{F}$-ring pair. ■

The analogue of Proposition 1.11 for going-down-pairs is false, as the following example demonstrates.

Example 1.12 Let R be a one-dimensional quasilocal domain. Let $\mathfrak{P}$ be the unique nonzero prime ideal of R. Let K be the residue field $R/\mathfrak{P}$ and let $T = R\times K$. By [H, Example 73], we know that $R \subseteq T$ is a ring extension that does not satisfy going-down (where we view $R \hookrightarrow R\times(R/\mathfrak{P})$ via $r \mapsto (r,\overline{r})$). Therefore (R,T) is not a going-down-pair. Nevertheless, we show $(R/(P\cap R), T/P)$ is a going-down-pair for all $P \in Spec(T)$.

If $P \in Spec(T)$, then P must be $(0)\times K$, $\mathfrak{P}\times K$ or $R\times(0)$. If $P = (0)\times K$, then $P\cap R = (0)$; hence $(R/(P\cap R), T/P) = (R,R)$, which is a going-down pair. Moreover, if $P = \mathfrak{P}\times K$ or $P = R\times(0)$, then $P\cap R = \mathfrak{P}$, hence $(R/(P\cap R), T/P) = (K,K)$, which is a going-down pair.

We want results that apply to rings with zero divisors. Since making a finite product of rings is an easy way to construct a ring with zero divisors, we next consider finite products of rings.

Proposition 1.13 A finite product of rings $R_1 \times \ldots \times R_n$ is an $\mathcal{F}$-ring if and only if R_i is an $\mathcal{F}$-ring, for all $i = 1, \ldots, n$.

Proof: For the "only if" implication, use Proposition 1.7. For the other implication, let A=$R_1 \times \ldots \times R_n$. Any P in $Spec(A)$ has the form $P = R_1 \times \ldots \times R_{j-1} \times P_j \times R_{j+1} \times \ldots \times R_n$, for some $P_j \in Spec(R_j)$. For any such P, $A/P \cong R_j/P_j$, which is

a domain satisfying $\mathcal{F}$. ■

To prove an analogue of Proposition 1.13 for $\mathcal{F}$-ring pairs, we first need the following technical lemma, whose elementary proof is left to the reader.

Lemma 1.14 Let $R \subseteq S$ and $T \subseteq V$ be ring extensions. Let $P \in Spec(S \times V)$ be of the form $P_1 \times V$ for some $P_1 \in Spec(S)$. Then we have the following commutative diagram

$$\begin{array}{ccc} (R \times T)/((R \cap P_1) \times T) & \xrightarrow{h} & (S \times V)/(P_1 \times V) \\ \uparrow f & & \uparrow g \\ R/(R \cap P_1) & \xrightarrow{\quad j \quad} & S/P_1 \end{array}$$

where f and g are ring isomorphisms, h and j are ring monomorphisms,
$f(r + (R \cap P_1)) = (r, 0) + ((R \cap P_1) \times T)$ for all $r \in R$,
$g(s + P_1) = (s, 0) + (P_1 \times V)$ for all $s \in S$,
$h((r, t) + ((R \cap P_1) \times T)) = (r, t) + (P_1 \times V)$ for all $(r, t) \in R \times T$, and
$j(r + (R \cap P_1)) = r + P_1$ for all $r \in R$.

Using the above Lemma we can prove the following result.

Proposition 1.15 Let $R \subseteq S$ and $T \subseteq V$ be ring extensions. Then $(R \times T, S \times V)$ is an $\mathcal{F}$-ring pair if and only if (R, S) and (T, V) are $\mathcal{F}$-ring pairs.

Proof: For the "only if" implication, apply Proposition 1.10 to the ideals $(0) \times V$ and $S \times (0)$ of $S \times V$. Conversely, let P be a prime ideal of $S \times V$. Then either $P = P_1 \times V$ for some P_1 in $Spec(S)$ or $P = S \times P_2$ for some P_2 in $Spec(V)$. Suppose $P = P_1 \times V$ for some $P_1 \in Spec(S)$. Then $(S \times V)/P \cong S/P_1$ and $(R \times T)/((R \times T) \cap P) =$

$(R\times T)/((P_1\cap R)\times T)\cong R/(P_1\cap R)$. Since (R,S) is assumed to be an $\mathcal{F}$-ring pair, we have by Proposition 1.11 that $(R/(P_1\cap R), S/P_1)$ is an $\mathcal{F}$-domain pair. Using Lemma 1.14, we get that $((R\times T)/(R\times T)\cap P), (S\times V)/P)$ is an $\mathcal{F}$-domain pair. In case $P = S\times P_2$, then by a similar argument $((R\times T)/((R\times T)\cap P), (S\times V)/P)$ is an $\mathcal{F}$-domain pair. Hence, by Proposition 1.11, $(R\times T, S\times V)$ is an $\mathcal{F}$-ring pair. ■

The next several results will show that, for $\mathcal{F}$-rings and many of the pairs we are interested in analyzing, we can assume without loss of generality that the rings involved are reduced.

Proposition 1.16 A ring R is an $\mathcal{F}$-ring if and only if $R_{red}(= R/Nil(R))$ is an $\mathcal{F}$-ring.

Proof: Every prime in $Spec(R_{red})$ has the form $P/Nil(R)$ for some $P\in Spec(R)$. Also, every prime $Q\in Spec(R)$ satisfies $Q\supseteq Nil(R)$ and so gives rise to $Q/Nil(R)\in Spec(R_{red})$. Therefore, for all $P/Nil(R)\in Spec(R_{red})$ we have $R_{red}/(P/Nil(R)) = (R/Nil(R))/(P/Nil(R))\cong R/P$. So $R_{red}/(P/Nil(R))$ is an $\mathcal{F}$-domain if and only if R/P is an $\mathcal{F}$-domain. Hence R is an $\mathcal{F}$-ring if and only if R_{red} is a $\mathcal{F}$-ring. ■

Proposition 1.17 Let $R\subseteq T$ be an extension of rings. Then (R,T) is an $\mathcal{F}$-ring pair if and only if (R_{red}, T_{red}) is an $\mathcal{F}$-ring pair.

Proof: R_{red} is canonically a subring of T_{red}, since $Nil(T)\cap R = Nil(R)$. For the "only if" implication, let A be a ring such that $R_{red}\subseteq A\subseteq T_{red}$. Such A can be expressed in the form $S/Nil(T)$ for some ring S such that $R\subseteq R+Nil(T)\subseteq S\subseteq T$. Since (R,T) is assumed to be an $\mathcal{F}$-ring pair, S is an $\mathcal{F}$-ring. Hence by Proposition 1.16, $A = S/Nil(T) = S_{red}$ is a $\mathcal{F}$-ring. Conversely, let S be a ring such that $R\subseteq S\subseteq T$. Then as above, $S_{red} = S/Nil(S)\cong (S+Nil(T))/Nil(T)$ satisfies

$R_{red} \subseteq S_{red} \subseteq T_{red}$. Since (R_{red}, T_{red}) is assumed to be an $\mathcal{F}$-ring pair, S_{red} is an $\mathcal{F}$-ring. Therefore by Proposition 1.16, we have that S is an $\mathcal{F}$-ring. ■

Proposition 1.18 Let $\mathcal{T}$ be a property of (some) continuous maps between topological spaces such that

(1) homeomorphisms satisfy $\mathcal{T}$ and

(2) compositions of $\mathcal{T}$-maps satisfy $\mathcal{T}$.

Let $R \subseteq T$ be an extension of rings. Then the canonical map $Spec(T) \to Spec(R)$ under the Zariski (resp., flat) topology satisfies $\mathcal{T}$ if and only if the canonical map $Spec(T_{red}) \to Spec(R_{red})$ under the Zariski (resp., flat) topology satisfies $\mathcal{T}$.

Proof: Consider the commutative diagram

$$\begin{array}{ccc} Spec(T_{red}) & \xrightarrow{j} & Spec(R_{red}) \\ \Big\downarrow f & & \Big\downarrow h \\ Spec(T) & \xrightarrow{g} & Spec(R) \end{array}$$

where j and g are canonical, $f(P/Nil(T)) = P$ for all $P \in Spec(T)$, and $h(Q/Nil(R)) = Q$ for all $Q \in Spec(R)$. By [DFP, Remark 2.3(g)], the maps f and h are homeomorphisms under the Zariski (resp., flat) topology. Therefore, if g satisfies $\mathcal{T}$, so does $h^{-1}gf = j$. Conversely, if j satisfies $\mathcal{T}$, so does $hjf^{-1} = g$. ■

Using the above Proposition, we get the following equivalence whose proof we merely sketch below.

Corollary 1.19 For each of the following properties $\mathcal{P}$, an extension of rings $R \subseteq T$ satisfies $\mathcal{P}$ if and only if the extension $R_{red} \subseteq T_{red}$ satisfies $\mathcal{P}$: going-down, open, LH-, i-, INC-, lying-over, unibranched.

Proof (sketch): First we prove the statement for INC-extensions. $R_{red} \subseteq T_{red}$ is not an INC-extension $\Leftrightarrow$ there are $P \subsetneqq Q$ in $Spec(T)$ such that $P/Nil(T) \cap R_{red} = Q/Nil(T) \cap R_{red}$ $\Leftrightarrow$ there are $P \subsetneqq Q$ in $Spec(T)$ such that $(P \cap R)/Nil(R) = (Q \cap R)/Nil(R)$ $\Leftrightarrow$ there are $P \subsetneqq Q$ in $Spec(T)$ such that $P \cap R = Q \cap R \Leftrightarrow R \subseteq T$ is not an INC-extension.

Next we handle the other properties $\mathcal{P}$. (The statement for $\mathcal{P}$ = "going-down" is proved in [DF1, Lemma 3.2(a)], but the proof we sketch below is new.) By [DFP, Proposition 2.7], an extension of rings $R \subseteq T$ satisfies going-down if and only if the canonical map $Spec(T) \to Spec(R)$ is a closed map in the flat topology. The extension $R \subseteq T$ is open (resp. LH-, i-, lying-over, unibranched) if and only if the canonical map $Spec(T) \to Spec(R)$ is an open map (resp. local homeomorphism, injection, surjection, bijection) in the Zariski topology. The rest of the proof consists of routine checks that closed maps, open maps, local homeomorphisms, injections, surjections, and bijections satisfy the conditions listed in Proposition 1.18. ■

Proposition 1.20 Let $\mathcal{P}$ be one of the properties listed in the previous Corollary. Let $R \subseteq T$ be an extension of rings. Then (R, T) is a $\mathcal{P}$-pair if and only if (R_{red}, T_{red}) is a $\mathcal{P}$-pair.

Proof: For the "only if" implication, first we note that every intermediate ring extension in (R_{red}, T_{red}) can be expressed in the form $A/Nil(T) \subseteq B/Nil(T)$ for some rings A and B such that $R \subseteq R + Nil(T) \subseteq A \subseteq B \subseteq T$. Since (R, T) is

assumed to be a $\mathcal{P}$-pair, $A \subseteq B$ satisfies $\mathcal{P}$. By Corollary 1.19, $A_{red} \subseteq B_{red}$ satisfies $\mathcal{P}$. But, as above, $A_{red} = A/Nil(A) \cong (A + Nil(T))/Nil(T) = A/Nil(T)$ and (similarly and compatibly) $B_{red} \cong B/Nil(T)$. Therefore, $A/Nil(T) \subseteq B/Nil(T)$ satisfies $\mathcal{P}$ and we have that (R_{red}, T_{red}) is a $\mathcal{P}$-pair.

Conversely, let A and B be rings such that $R \subseteq A \subseteq B \subseteq T$. As above, we have canonical inclusions $R_{red} \subseteq A_{red} \subseteq B_{red} \subseteq T_{red}$. Since (R_{red}, T_{red}) is assumed to be a $\mathcal{P}$-pair, $A_{red} \subseteq B_{red}$ satisfies $\mathcal{P}$. So by Corollary 1.19, $A \subseteq B$ satisfies $\mathcal{P}$. Thus (R, T) is a $\mathcal{P}$-pair. ■

CHAPTER II

PAIRS AND TRANSCENDENCE DEGREE

At various places throughout the literature, authors have shown that for certain domain properties $\mathcal{D}$, all subrings of a domain R enjoy property $\mathcal{D}$ if and only if the transcendence degree of R over its prime subring has an appropriate value. For examples, see [D4, Theorem 2.1 and Corollary 2.2] and [G1, Theorems 4 and 5]. We intend to show that in certain cases, the $\mathcal{D}$-domain-pairedness of a pair of domains (R, T) can be characterized in terms of the transcendence degree of T over R, and by then using Proposition 1.11, we will be able to develop transcendence degree criteria for $\mathcal{D}$-ring-pairedness.

We start by considering going-down-domain pairs. In the special case of an extension of fields $k \subseteq K$, we get the following equivalences.

Proposition 2.1 For an extension of fields $k \subseteq K$, the following statements are equivalent:

(1) (k, K) is a going-down-domain pair;

(2) (k, K) is a going-down pair;

(3) $\dim(A) \leq 1$ for all rings A such that $k \subseteq A \subseteq K$;

(4) $td_k(K) \leq 1$.

Proof: (3) $\Rightarrow$ (1): Let A be an intermediate ring of the pair (k, K); i.e., A is a ring such that $k \subseteq A \subseteq K$. The ring A is actually a domain, since it is a subring of a field. Since $\dim(A) \leq 1$, the domain A is a going-down-domain.

(1) $\Rightarrow$ (2): As noted in Chapter I after Definition 1.3, this implication is a consequence of [DP2, Theorem 4.1].

(2) $\Rightarrow$ (4): Deny. Then we have two algebraically independent indeterminates X and Y such that $k \subseteq k[X,Y] \subseteq K$. Since K is a field, $qf(k[X,Y]) \subseteq K$; so all overrings of $k[X,Y]$ are intermediate rings of the pair (k,K). So $k[X,Y]$ is a going-down-domain, since overrings suffice as test algebras in determining going-down domains, by [D1, Theorem 4.1]. But $k[X,Y]$ is Noetherian, so by [D1, Corollary 2.3], $\dim k[X,Y] \leq 1$. We have a contradiction since, by [G2, Theorem 30.5], $\dim k[X,Y] = 2$.

(4) $\Rightarrow$ (3): Let A be an intermediate ring of (k,K). If $\dim(A) > 1$, then there is a chain $(0) \subsetneqq P_1 \subsetneqq P_2$ in $Spec(A)$. Pick $0 \neq \alpha \in P_1$ and $0 \neq \beta \in P_2 \backslash P_1$. Let $B = k[\alpha, \beta]$. Then $\dim B \geq 2$, since $(0) \subsetneqq P_1 \cap B \subsetneqq P_2 \cap B$. But by [E, Chapter 13, Theorem A], $\dim(B) = td_k(B) \leq td_k(K)$. Since it is assumed that $td_k(K) \leq 1$, we have $2 \leq \dim(B) \leq td_k(K) \leq 1$, a contradiction. Thus we must have $\dim(A) \leq 1$. ■

We next generalize to the case of $k \subseteq T$, where k is still a field but T is a domain.

Proposition 2.2 Let k be a field contained in a domain T. Then the following statements are equivalent:

(1) (k,T) is a going-down-domain pair;

(2) $(k, qf(T))$ is a going-down-domain pair;

(3) $td_k(T) \leq 1$.

Proof: (2) $\Leftrightarrow$ (3): Apply Proposition 2.1.

(2) $\Rightarrow$ (1): If A is an intermediate ring of (k,T), it is also an intermediate ring of $(k, qf(T))$.

(1) $\Rightarrow$ (3): Suppose $td_k(T) > 1$. Then $qf(T)$ contains two algebraically independent elements, X_0 and Y_0, transcendental over k. So, for some nonzero

$t_1, t_2, t_3, t_4 \in T$, we have $X_0 = t_1/t_2$ and $Y_0 = t_3/t_4$. Consider $A = k[t_1, t_2, t_3, t_4]$, a subdomain of T. The quotient field of A contains $k(X_0, Y_0)$; hence $td_k A \geq 2$. Since $\{t_1, t_2, t_3, t_4\}$ is a generating set for $qf(A)$ as a field over k, it contains a transcendence basis for $qf(A)$ over k. Therefore, A contains two algebraically independent indeterminates, X and Y, such that $k \subseteq k[X, Y] \subseteq T$. Since (k, T) is assumed to be a going-down-domain pair, $k[X, Y]$ is a going-down domain. However, $k[X, Y]$ is also Noetherian of Krull dimension 2, contradicting [D1, Corollary 2.3]. Thus $td_k(T) \leq 1$. ■

Combining Proposition 1.11 with Proposition 2.2, we obtain the following transcendence degree criteria for going-down-ring-pairedness when the base ring has dimension 0.

Proposition 2.3 Let (R, T) be a pair of rings with $\dim(R) = 0$. Then the following statements are equivalent:

(1) (R, T) is a going-down-ring pair;

(2) $td_{R/(P \cap R)}(T/P) \leq 1$ for all $P \in Spec(T)$;

(3) $td_{R/(P \cap R)}(T/P) \leq 1$ for all $P \in Min(T)$.

Proof: (1) $\Rightarrow$ (2): Let (R, T) be a going-down-ring pair. By Proposition 1.11, with $\mathcal{F}$ ="going-down", we have that $(R/(P \cap R), T/P)$ is a going-down-domain pair for each $P \in Spec(T)$. Since $\dim(R) = 0$, each such $R/(P \cap R)$ must be a field, and so we can apply Proposition 2.2 to obtain (2).

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Since $\dim(R) = 0$, we have that $R/(P \cap R)$ is a field for all $P \in Min(T)$. Thus by Proposition 2.2, (3) ensures that $(R/(P \cap R), T/P)$ is a going-down-domain pair for all $P \in Min(T)$. Hence by Proposition 1.11, the pair

(R, T) is a going-down-ring pair. ■

We note that it is possible to have a going-down-ring pair (R, T) for which $td_{R/(P\cap R)}(T/P) = 1$ for some $P \in Spec(T)$. For example, consider the pair $(k, k(X))$, where k is any field.

In many special cases we can eliminate the possibility that a pair of rings (R, T) satisifes $td_{R/(P\cap R)}(T/P) = 1$ for each $P \in Spec(T)$. We once again look at a field extension $k \subseteq K$ and obtain the following equivalences. For motivation, note the following immediate consequence of a result on lying-over-pairs [D3, Corollary 3.9(a)]: a field extension $k \subseteq K$ is algebraic (that is, has transcendence degree 0) if and only if (k, K) is an LO-pair.

Proposition 2.4 Let $k \subseteq K$ be an extension of fields. Then the following statements are equivalent:

(1) (k, K) is a unibranched-pair;

(2) (k, K) is an i-domain-pair;

(3) (k, K) is an i-pair;

(4) (k, K) is an LH-domain-pair;

(5) (k, K) is an LH-pair;

(6) (k, K) is an open-domain-pair;

(7) (k, K) is an open-pair;

(8) (k, K) is an INC-pair;

(9) T is a field for all rings T such that $k \subseteq T \subseteq K$;

(10) $td_k(K) = 0$.

Proof: (1) $\Rightarrow$ (3): Unibranched extensions are i-extensions.

(5) $\Rightarrow$ (7): Local homeomorphisms are open maps.

(4) $\Rightarrow$ (6): Since local homeomorphisms are open maps, LH-domains are open-domains.

(9) $\Rightarrow$ (2), (4), (6): Fields are i-domains, LH-domains and open-domains.

(3) $\Rightarrow$ (9): Since (k, K) is assumed to be an i-pair, the canonical map $Spec(T) \rightarrow Spec(k)$ is injective for all intermediate rings T. Since $Spec(k) = \{(0)\}$, we have $Spec(T) = \{(0)\}$. Therefore T is a field.

(5) $\Rightarrow$ (4): Let A be a ring such that $k \subseteq A \subseteq K$. Then $qf(A) \subseteq K$, and so every overring of A is an intermediate ring of the pair (k, K). Hence by (5), $A \subseteq T$ satisfies LH for every overring T of A; therefore A is an LH-domain.

(7) $\Rightarrow$ (6): We repeat the argument for (5) $\Rightarrow$ (4), except that we replace "LH" with "open" wherever "LH" occurs.

(6) $\Rightarrow$ (10): Deny. Suppose there is an element $u \in K$ transcendental over k. Since (k, K) is assumed to be an open-domain pair, $k[u]$ is an open-domain. By [P, Corollary 3.17], $k[u]$ is semiquasilocal, contradicting the fact that $k[u]$ has infinitely many maximal ideals. Hence $td_k(K) = 0$.

(8) $\Rightarrow$ (9): Since (k, K) is assumed to be an INC-pair, $k \subseteq T$ satisfies incomparability for all intermediate rings T. For any $P \in Spec(T)$, the intersection $k \cap P = (0)$. Since $(0) \subseteq P$, by incomparability we must have $P = (0)$. Hence T is a field.

(10) $\Rightarrow$ (8): Let $A \subseteq B$ be an intermediate extension of the pair (k, K). Since $k \subseteq K$ is an algebraic field extension, it is an integral extension. Thus $A \subseteq B$ is an integral extension, so $A \subseteq B$ satisfies incomparability.

(9) $\Rightarrow$ (1), (5): Let $A \subseteq B$ be an intermediate extension of the pair (k, K). Then A and B are fields, and so the canonical map $Spec(B) \rightarrow Spec(A)$ is the unique map from $\{(0)\}$ to $\{(0)\}$. Therefore $A \subseteq B$ is unibranched and LH.

(2) $\Rightarrow$ (10): (We thank S. B. Mulay for providing us the following argument.) Deny. Suppose $X \in K$ is transcendental over k. Let $X \neq f \in k[X]$ be an irreducible polynomial of degree at least 1. Consider $k[Xf, X^2f] \subseteq k[X]$. Then (X) and (f) are prime ideals of $k[X]$ such that $(X) \cap k[Xf, X^2f] = (Xf, X^2f) = (f) \cap k[Xf, X^2f] \in Max(k[Xf, X^2f])$. Since $qf(k[Xf, X^2f]) = k(X) = qf(k[X])$, we have that $k[Xf, X^2f]$ is not an i-domain. This contradicts the assumption that (k, K) is an i-domain pair. ■

Although going-down admitted an extension of a field-theoretic result (Proposition 2.1) to the context of domains containing a field (Proposition 2.2), the next result shows that there are no interesting non-field domain extensions (of Proposition 2.4) when "open" replaces "going-down."

Proposition 2.5 Let k be a field contained in a domain T. If (k, T) is an open-domain-pair, then T must be a field.

Proof: It suffices to show that T is algebraic over k. Suppose, on the contrary, that there is some $u \in T$ transcendental over k. Since $k[u]$ is then an open-domain, [P, Corollary 3.17] tells us that $k[u]$ is semiquasilocal. This contradicts the fact that $k[u]$ is not semiquasilocal, and so T is algebraic over k. ■

By the proof of (2) $\Rightarrow$ (10) in Proposition 2.4, we see that the analogue of Proposition 2.5 is valid when "open-domain pair" is replaced by "i-domain pair." Of course, the same is then true for "LH-domain-pair."

Using the previous two results, we present criteria for open-ring pairs and LH-ring pairs.

Proposition 2.6 Let $R \subseteq T$ be an extension of rings with $\dim(R) = 0$. Then the following statements are equivalent:

(1) (R, T) is an LH-ring pair;

(2) (R, T) is an open-ring pair;

(3) $\dim(A) = 0$ for all rings A such that $R \subseteq A \subseteq T$;

(4) (R, T) is an INC-pair;

(5) $R \subseteq T$ is a residually algebraic extension (i.e., $td_{R/(P\cap R)}(T/P) = 0$ for all $P \in Spec(T)$);

(6) $td_{R/(P\cap R)}(T/P) = 0$ for all $P \in Min(T)$.

Proof: (1) $\Rightarrow$ (2): Since LH-domains are open-domains, LH-rings are open-rings.

(2) $\Rightarrow$ (3): Let S be an intermediate ring. Let $P \in Spec(S)$ and $P_0 = P \cap R$. Then $(R/P_0, S/P)$ is an open-domain pair by (2) and Proposition 1.11. Since R/P_0 is a field, so is S/P by Proposition 2.5 . Since S/P is a field for all $P \in Spec(S)$, we conclude that $\dim(S) = 0$.

(3) $\Rightarrow$ (4): Let $A \subseteq B$ be an intermediate extension of rings. Since it is assumed that $\dim(B) = 0$, all prime ideals are B are pairwise incomparable. Hence $A \subseteq B$ satisfies the incomparability property vacuously.

(4) $\Rightarrow$ (5): By [FHP, Theorem 6.5.6], INC-pairs are residually algebraic pairs.

(5) $\Rightarrow$ (6): Trivial.

(6) $\Rightarrow$ (1): Since $\dim(R) = 0$, the domain $R/(P \cap R)$ is a field for all $P \in Min(T)$. So, by (6), T/P is also a field. By Proposition 2.4, $(R/(P \cap R), T/P)$ is an LH-domain pair for all $P \in Min(T)$. Therefore (R, T) is an LH-ring pair by Proposition 1.11 with $\mathcal{F}$ ="LH." ■

Note that the above result combined with Proposition 2.3 guarantees the

existence of going-down-ring pairs that are not open-ring pairs (and hence not LH-ring pairs). For example, consider again the pair $(k, k(X))$, where k is any field. Since $(k, k(X))$ is not an INC-pair, it is not an open-ring pair; however, since $td_k(k(X)) = 1$, the pair $(k, k(X))$ is a going-down-ring pair.

When the dimension of the base ring is not 0, the analogues of Propositions 2.3 and 2.6 do not hold in general, as the following examples show.

Examples 2.7 (a) A pair of domains (R, T) need not be a going-down-domain pair, even if (R, T) is an INC-pair (and hence $td_R(T) = 0$). For an example, consider $(\mathbb{Z}[X], \mathbb{Z}[X][\sqrt{2}])$. The extension $\mathbb{Z}[X] \subseteq \mathbb{Z}[X][\sqrt{2}]$ is integral and hence we have an INC-pair, but $\mathbb{Z}[X]$ is not a going-down-domain, since $\mathbb{Z}$ is not a field. This example also tells us that, in general, (R, T) being an INC-pair does not imply that (R, T) is an open-domain pair or a LH-domain pair.

(b) Even if R is a going-down-domain and $R \subseteq T$ is an integral extension of domains (and hence (R, T) is an INC-pair), the pair (R, T) need not be a going-down domain pair. In [DF2, Example 3.1], we have a integral domain extension $R \subseteq T$ where R is a 2-dimensional going-down-domain but T is not a going-down-domain. Therefore, (R, T) is not a going-down-domain pair.

(c) An LH-ring pair (or open-ring pair) need not be a zero-dimensional-ring-pair. For example, let R be a 1-dimensional semilocal Noetherian i-domain and let $K = qf(R)$. Clearly (R, K) is not a zero-dimensional-ring-pair. We show that (R, K) is an LH-domain pair: Let S be an intermediate ring. By [P, Corollary 4.12], S is semilocal. Also, by Krull-Akizuki, S is Noetherian and $\dim(S) \leq 1$. Therefore by [P, Proposition 3.20 and Theorem 4.16], S is an LH-domain. For a trivial example, consider (R, R), where R is a 1-dimensional valuation domain.

(d) An open-ring pair need not be a LH-ring pair. For example, consider the

pair $(\mathbb{Q}+X\mathbb{R}[[X]], \mathbb{Q}(\sqrt{2})+X\mathbb{R}[[X]])$. Every intermediate ring of this pair has the form $F + X\mathbb{R}[[X]]$, where $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{2})$. The domain $\mathbb{R}[[X]]$ is 1-dimensional and semilocal, so by [P, Proposition 3.20], $\mathbb{R}[[X]]$ is an open-domain; hence, by [P, Corollary 3.28], $\mathbb{Q}+X\mathbb{R}[[X]]$ and $\mathbb{Q}(\sqrt{2})+X\mathbb{R}[[X]]$ are open-domains. So our pair is an open-domain pair, hence an open-ring pair. However, since $\mathbb{R}$ is not algebraic over $\mathbb{Q}$, by Proposition 2.4 we have that $(\mathbb{Q}, \mathbb{R})$ is not an open-domain pair, and so by [P, Corollary 3.29 and Theorem 4.16], $\mathbb{Q} + X\mathbb{R}[[X]]$ is not an LH-domain. So $(\mathbb{Q} + X\mathbb{R}[[X]], \mathbb{Q}(\sqrt{2}) + X\mathbb{R}[[X]])$ is a pair of domains that is not an LH-domain pair, thus it is not an LH-ring pair. For a trivial example, consider the pair (R, R), where R is any open-domain that is not an LH-domain.

However, we can obtain many positive partial results in cases where the dimension of the base ring R is nonzero. The first such result will be that if $\dim(R) > 0$ and (R, T) is a going-down-ring pair, then $R \subseteq T$ must be an algebraic extension. (The result fails if $\dim(R) = 0$: consider $(k, k(X))$.)

We first need the following proposition. It will be used to extend to arbitrary rings the known result that for any domain R, the polynomial ring $R[X]$ is a going-down-domain if and only if $\dim(R) = 0$ (cf. [DP2, p. 274, first full paragraph]).

Proposition 2.8 Let $A_0 \subseteq \ldots \subseteq A_n$ be rings such that $\dim(A_0) \geq 1$. Suppose at least one of the following two conditions holds:

(1) $A_0 \subseteq A_n$ satisfies the going-up property;

(2) $A_0 \subseteq A_n$ satisifies the going-down and lying-over properties.

Then neither $A_0+A_1X+\ldots+A_{n-1}X^{n-1}+X^nA_n[X]$ nor $A_0+A_1X+\ldots+A_{n-1}X^{n-1}+X^nA_n[[X]]$ is a going-down-ring.

Proof: Since it is assumed that $\dim(A_0) \geq 1$, there exist $P \in Min(A_0)$ and $M \in Spec(A_0)$ such that $P \subsetneq M$.

We claim that there are $P_n \subsetneq M_n$ in $Spec(A_n)$ such that $P_n \cap A_0 = P$ and $M_n \cap A_0 = M$. In case condition (1) holds, then since minimal primes are lain over (cf. [K, Exercise 1, p. 41]), there exists $P_n \in Spec(A_n)$ such that $P_n \cap A_0 = P$. By going-up we obtain the M_n that satisfies our claim. In case condition (2) holds, lying-over gives us M_n and then going-down gives us the P_n we need for our claim.

Let $B = A_0 + A_1X + \ldots + A_{n-1}X^{n-1} + X^nA_n[X]$. Then $B \cap (M_n + XA_n[X]) \in Spec(B)$ contains two specific prime ideals, $B \cap (P_n + XA_n[X])$ and $B \cap M_n[X]$, both of which contain $B \cap P_n[X] \in Spec(B)$. Since $P \subsetneq M$, there is an element m in $M \backslash P$ and therefore in $B \cap M_n[X]$, but not in $B \cap (P_n + XA_n[X])$. Also, since $1 \notin M_n$, we have X in $B \cap (P_n + XA_n[X])$, but not in $B \cap M_n[X]$. Therefore $B \cap (P_n + XA_n[X])$ and $B \cap M_n[X]$ are incomparable. So $B/(B \cap P_n[X])$ is not a treed domain (i.e., as a partially ordered set, $Spec(B/(B \cap P_n[X]))$ is not a tree); hence by [D1, Theorem 2.2], $B/(B \cap P_n[X])$ is not a going-down-domain. So B is not a going-down-ring.

The proof of the statement for $A_0 + A_1X + \ldots + A_{n-1}X^{n-1} + X^nA_n[[X]]$ is the same as the above proof, except that every appearance of "$[X]$" is replaced by "$[[X]]$". ■

We note that the following example shows that the two conditions listed in Proposition 2.8 are necessary for the conclusion.

Example 2.9 Consider $\mathbb{Z} \subsetneq \mathbb{Q}$, an extension that does not satisfy lying-over (and hence does not satisfy going-up). Then since $\mathbb{Z}$ is Prüfer, so is $\mathbb{Z}+X\mathbb{Q}[X]$, by [CMZ, Corollary 4.15]. Hence $\mathbb{Z} + X\mathbb{Q}[X]$ is a going-down-ring. Also, $\mathbb{Z} + X\mathbb{Q}[[X]]$ is a $D+M$ construction; therefore, since $\mathbb{Z}$ is a going-down-domain, so is $\mathbb{Z}+X\mathbb{Q}[[X]]$, by [DP1, Corollary]. (For an alternate proof, since $\mathbb{Z}$ is a Prüfer domain with quotient

field $\mathbb{Q}$, note that $\mathbb{Z}+X\mathbb{Q}[[X]]$ is a Prüfer domain (cf. [G2, Exercise 13, pp. 286-7]) and hence a going-down-domain.)

As an easy corollary of Proposition 2.8, we get the promised extension of [DP2, p. 274, first full paragraph].

Corollary 2.10 For a ring R, the following statements are equivalent:

(1) $(R, R[X])$ is a going-down-ring pair;

(2) $R[X]$ is a going-down-ring;

(3) $\dim(R) = 0$.

Proof: (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Suppose $\dim(R) > 0$. Then by Proposition 2.8 with $n = 1$ and $A_0 = A_n = R$, the ring $R[X]$ cannot be a going-down-ring.

(3) $\Rightarrow$ (1): Note that each minimal prime ideal of $R[X]$ has the form $P[X]$ for some $P \in Spec(R)$. For each $P[X] \in Min(R[X])$, we have $td_{R/P[X]\cap R}(R[X]/P[X]) = td_{R/P}((R/P)[X]) = 1$. Hence by Proposition 2.3, $(R, R[X])$ is a going-down-ring pair. ■

As an immediate consequence of Corollary 2.10, we obtain the conclusion that a going-down-ring pair (R, T) must be an algebraic extension unless $\dim(R) = 0$.

Corollary 2.11 Let (R, T) be a going-down-ring pair such that $\dim(R) > 0$. Then $R \subseteq T$ must be an algebraic extension.

Proof: Deny. Then we can choose $u \in T$ transcendental over R. Then $R[X] \cong R[u]$ is an intermediate ring of the pair (R, T) and is therefore a going-down-ring. So by Corollary 2.10, $\dim(R) = 0$, which contradicts the assumption that $\dim(R) > 0$. ■

We also present an analogue of Corollary 2.10 for $R[[X]]$.

Corollary 2.12 The ring $R[[X]]$ is a going-down-ring if both $\dim(R) = 0$ and $\dim(R[[X]]) \neq \infty$. Also, the ring $R[[X]]$ is not a going-down-ring if $\dim(R) > 0$.

Proof: For the first statement, since $\dim(R) = 0$ and $\dim(R[[X]]) \neq \infty$, we have $\dim(R[[X]]) = 1$ by [Ar, Theorem 2]. Thus $R[[X]]$ is a going-down-ring. For the second statement, apply Proposition 2.8 with $n = 1$ and $A_0 = A_n = R$. ■

We next sharpen our focus from arbitrary going-down-ring pairs to the special case of open-ring pairs. We show that any open-ring pair (R, T) (and hence any LH-ring pair) must also be an INC-pair, regardless of the dimension of R. As we noted in Example 2.7(a), the converse fails in general.

Proposition 2.13 If (R, T) is an open-ring pair, then it is an INC-pair.

Proof 1: First we show that $R \subseteq T$ is residually algebraic, regardless of the dimension of R. Let $P \in Spec(T)$. Since (R, T) is assumed to be an open-ring pair, by Proposition 1.11 with $\mathcal{F}$ = "open," we have that $(R/(P \cap R), T/P)$ is an open-domain pair. In case $R/(P \cap R)$ is a field, then by Proposition 2.5 so is T/P, and hence by Proposition 2.4, $R/(P \cap R) \subseteq T/P$ is algebraic. In the remaining case, $R/(P \cap R)$ is not a field. Then, since open-domain pairs are going-down-domain pairs, Corollary 2.11 implies that $R/(P \cap R) \subseteq R/P$ is algebraic. Thus for all $P \in Spec(T)$, we have $td_{R/(P \cap R)}(R/P) = 0$; that is, $R \subseteq T$ is residually algebraic.

Since (R, T) is an open-ring pair, so is (A, B) for all intermediate extensions $A \subseteq B$ of the pair (R, T). Thus by the above argument, $A \subseteq B$ is a residually algebraic extension for all rings $R \subseteq A \subseteq B \subseteq T$. Hence (R, T) is a residually algebraic pair, and thus by [FHP, Theorem 6.5.6], (R, T) is an INC-pair. ■

Proof 2: Deny. Then there is an intermediate extension $A \subseteq B$ with $Q_1 \subsetneqq Q_2$ in $Spec(B)$ such that $Q_1 \cap A = Q_2 \cap A$. Let $P = Q_1 \cap A$. Without loss of generality, $B = A[u]$ for some $u \in Q_2 \backslash Q_1$. Then $(0) = Q_1/Q_1 \subsetneqq Q_2/Q_1$ are prime ideals in $A[u]/Q_1 = B/Q_1 = (A/P)[u + Q_1]$ that contract to (0) in $Spec(A/P)$. The element $u+Q_1$ is transcendental over A/P. (Otherwise, let $p(X)$ be a nonzero polynomial in $(A/P)[X]$ of minimal degree for which $u + Q_1$ is a root. Then the constant term is an element of $(A/P) \cap (Q_2/Q_1)$ and is therefore zero, contradicting the minimality of the degree of $p(X)$.) Therefore (cf. [K, p. 25]), there are infinitely many incomparable prime ideals of B/Q_1 that contract to (0) in $Spec(A/P)$. So B/Q_1 is not semiquasilocal. (Otherwise, some maximal ideal of the domain B/Q_1 contains at least two incomparable prime ideals, and hence B/Q_1 is not a treed domain. This contradicts the fact that all open-domains are going-down-domains and hence treed domains.) Therefore, B/Q_1 is not an open-domain, and so B is not an open-ring. Since (R, T) is assumed to be an open-ring pair, we have a contradiction, and so (R, T) must be an INC-pair. ■

We include the second proof of the above result because it suggests a relation between "finite fiber pairs" and INC-pairs. Let $R \subseteq T$ be a ring extension and let f be the canonical map $Spec(T) \to Spec(R)$. Then $R \subseteq T$ is said to have *finite fibers* if $f^{-1}(P)$ is a finite set for all $P \in Spec(R)$ (cf. [P, p. 21]). If (R, T) is a pair of rings where all intermediate extensions have finite fibers, we call (R, T) a *finite fiber pair*. Note that any i-pair is an example of a finite fiber pair.

Using a proof similar to Proof 2 of Proposition 2.13, we next show that each finite fiber pair is an INC-pair.

Proposition 2.14 If (R, T) is a finite fiber pair, then it is an INC-pair.

Proof: Repeat the second proof of Proposition 2.13 through the end of the sentence, " Therefore, there are infinitely many incomparable ideals of B/Q_1 that contract to (0) in $Spec(A/P)$." Hence, there are infinitely many prime ideals of B that contract to P in $Spec(A)$, which contradicts the assumption that $A \subseteq B$ has finite fibers. Thus (R, T) must be an INC-pair. ■

The converse of Proposition 2.14 is false. To see this, let K be any finite field, say with q elements, and let T be a product of infinitely many copies of K. Since each element $t \in T$ satisfies $t^q = t$, we see that T is integral (algebraic) over K, and so (R, T) is an INC-pair. However, (R, T) is not a finite fiber pair, since infinitely many prime ideals of T lie over the prime ideal (0) of K.

In Corollary 2.11 and Proposition 2.13 we stated some necessary conditions for going-down-ring pairs and open-ring pairs. We next prove some sufficient conditions for going-down-ring-pairedness.

We start with the following easy consequence of Kaplansky's Theorem 45.

Proposition 2.15 Let (R, T) be an INC-pair of rings with $\dim(R) \leq 1$. Then (R, T) is a going-down-ring pair.

Proof: Since (R, T) is assumed to be an INC-pair, [K, Theorem 45] implies that $\dim(A) \leq \dim(R) \leq 1$ for all intermediate rings A of (R, T). Thus all intermediate rings are going-down rings. ■

When (R, T) is not an INC-pair, it need not be a going-down-ring pair, even if $\dim(R) \leq 1$. For example, consider the pair $(\mathbb{Z}, \mathbb{Z}[X])$, which is not a going-down-ring pair since $\mathbb{Z}[X]$ is a two-dimensional Noetherian domain.

The next result shows that algebraic extensions of one-dimensional INC-domains form going-down-domain pairs.

Proposition 2.16 Let R be a one-dimensional INC-domain. Let $R \subseteq T$ be an algebraic extension of domains. Then (R, T) is a going-down-domain pair.

Proof: It suffices to show that $\dim(A) \leq 1$ for all intermediate domains A. Deny. So we have an intermediate domain A with a sequence of prime ideals $(0) \subsetneqq Q_1 \subsetneqq Q_2$. By [P, Proposition 2.26], R is an INC-domain if and only if the integral closure of R is a Prüfer domain. Hence, since $R \subseteq A$ is algebraic and R is an INC-domain, by [D5, Proposition 3], we have that $R \subseteq A$ is an INC-extension. By algebraicity, $Q_1 \cap R \neq (0)$. Since $\dim(R) = 1$, we have $Q_1 \cap R = Q_2 \cap R$, which contradicts the fact that $R \subset A$ is an INC-extension. ■

We next present two consequences of Propositions 2.15 and 2.16.

Proposition 2.17 Let (R, T) be a pair of domains for which all intermediate rings are Noetherian (i.e., (R, T) is a *Noetherian pair*). If $\dim(R) \leq 1$, then (R, T) is a going-down-domain pair.

Proof: In case R is a field, $td_R(T) \leq 1$ by [W, Theorem 4] and hence (R, T) is a going-down-domain pair by Proposition 2.2. If $\dim(R) = 1$, then T is algebraic over R by [W, Theorem 4(i)]. Since R is 1-dimensional Noetherian, the integral closure of R is Dedekind and hence Prüfer (cf. [DP2, pp. 281-2]). Thus by Proposition 2.16, (R, T) is a going-down-domain pair. ■

Proposition 2.18 Let (R, T) be an INC-pair of domains that is also a Noetherian pair. Then (R, T) is a going-down-domain pair if and only if $\dim(R) \leq 1$.

Proof: For the "only if" direction, note that R is a Noetherian going-down-domain. Hence by [D1, Corollary 2.3], $\dim(R) \leq 1$. Conversely, suppose $\dim(R) \leq 1$. Since (R, T) is an INC-pair, apply Proposition 2.15. ■

Judging from the results so far in this chapter, the reader may suspect that all going-down-ring pairs are either INC-pairs or pairs for which all intermediate rings have dimension less than 2. We close this chapter by showing two examples of going-down-ring pairs that are neither INC-pairs nor "one-or-zero-dimensional-ring pairs."

Examples 2.19 (a) Let $n > 1$ be either an integer or ∞. Let F be a field, and let $F(X) + M$ be a valuation domain with maximal ideal M and Krull dimension n. Let $R = F + M$ and $T = F[X] + M$. Each intermediate ring of (R, T) has the form $S + M$ for some intermediate ring S of the pair $(F, F[X])$. Since $(F, F[X])$ is a going-down domain pair, so is (R, T), by [DP1, Corollary]. Therefore, by Corollary 1.9, (R, T) is a going-down-ring pair. (We also note that by [D3, Remark 2.12], (R, T) is an LO-pair.) However, $R/M \subseteq T/M$ is not an algebraic extension, and so (R, T) is not a residually algebraic pair. Hence (R, T) is not an INC-pair. Also, note that $\dim(R) = n$.

(b) Let $k \subsetneq K$ be a field extension with $td_k(K) = 1$. Note, by Proposition 2.1, that (k, K) is a going-down-ring pair. Let (R, T) be a going-down-ring pair. Then by Proposition 1.15 with $\mathcal{F}$ ="going-down", $(k \times R, K \times T)$ is a going-down-ring pair. But $(k \times R, K \times T)$ is not an INC-pair, since $K \times T$ is not residually algebraic over $k \times R$. In this example, we can arrange $k \times R$ to have any desired Krull dimension > 1.

CHAPTER III

CONNECTIONS BETWEEN $\mathcal{P}$-PAIRS AND $\mathcal{D}$-RING PAIRS

As noted in Chapter I, all going-down-domain pairs are going-down-pairs and all open-domain pairs are open pairs. However, the converses of the above statements are false, as demonstrated by the following example of a open-pair of domains that is not a going-down-domain pair (and therefore not an open-domain pair).

Example 3.1 Let $L \subsetneq K$ be an algebraic field extension. Consider $T = K[[X,Y]]$ $= K + M$, where $M = XT + YT$. Let $R = L + M$. Since T is not treed, T is not a going-down-domain. Therefore (R,T) is not a going-down-domain pair.

Note that M, the unique maximal ideal of T, is a maximal ideal of R. Therefore, by [AD, Theorem 3.10], $Spec(R) = Spec(T)$ as sets. Since $L \cong R/M \subseteq T/M \cong K$ is an algebraic field extension, by [AD, Corollary 3.26] we have $Spec(R) = Spec(S)$ for all rings S such that $R \subseteq S \subseteq T$. Therefore, (R,T) is a going-down- (and open- and LH-) pair of domains that is *not* a going-down-domain pair.

Indeed, by [AD, Proposition B.2], since T is not a going-down domain, *no* intermediate ring of the pair (R,T) is a going-down domain.

Let $\mathcal{P}$ be a property of (some) ring extensions (e.g. going-down, open, incomparability) and let $\mathcal{D}$ be the property of being a $\mathcal{P}$-domain. Then in general, the concepts of $\mathcal{P}$-pairedness and $\mathcal{D}$-ring-pairedness (i.e., $\mathcal{P}$-domain-ring-pairedness) are independent, as the following three examples show.

For the first such example, we need the following technical lemma, whose elementary proof is left to the reader.

Lemma 3.2 Let R, T, V be rings such that $R \subseteq T$. Then all intermediate rings of the pair $(R \times V, T \times V)$ have the form $S \times V$, where S is a ring with $R \subseteq S \subseteq T$.

With the above lemma, we can present an example in the spirit of Example 3.1 with focus on rings with zero divisors.

Example 3.3 (a) Let $k \subseteq K$ be an algebraic extension of fields. Let T be a ring that is not a going-down-ring, and consider $(k \times T, K \times T)$. By Lemma 3.2, each intermediate ring has the form $F \times T$, for some field F such that $k \subseteq F \subseteq K$. Hence the canonical map $Spec(G \times T) \to Spec(F \times T)$ is a homeomorphism for all rings $k \times T \subseteq F \times T \subseteq G \times T \subseteq K \times T$. So $(k \times T, K \times T)$ is a going-down-pair, open-pair, LH-pair, and unibranched-pair which is not a going-down-ring pair. Indeed, in this example, no intermediate ring is a going-down-ring (cf. Proposition 1.11).

If we replace "going-down" in the above paragraph with "INC-" (resp., "i-"), we get an example of an INC-pair (resp., i-pair) of rings for which no intermediate ring is an INC-ring (resp., i-ring) (cf. Proposition 1.7).

(b) We also note that for any R that is not a going-down-ring, (R, R) gives us a trivial example of a going-down-, open-, i-pair of rings that is not a going-down-ring pair.

Our next example shows that INC-domain pairs need not be INC-pairs.

Example 3.4 Let $R = \mathbb{F}_p$, the finite field with p elements, and let $T = \mathbb{F}_q(X)$, where $q = p^r$ for some positive integer r. The pair (R, T) is a going-down-domain pair by Proposition 2.1. Also, by [G1, Theorem 4], (R, T) is a Noetherian pair. So all intermediate rings S of (R, T) have $\dim(S) \leq 1$. Hence, by [P, Corollary 2.27], (R, T) is an INC-domain pair. However, (R, T) is not an INC-pair by Proposition 2.4 since $td_R(T) = 1$.

The next result shows that any field extension (R, T) having the properties exhibited in Example 3.4 must be transcendental.

Proposition 3.5 Let D be an INC-domain and $D \subseteq T$ be an algebraic extension of domains. Then (D, T) is both an INC-pair and an INC-domain pair.

Proof: If $(D, qf(T))$ is an INC-pair and an INC-domain pair, so is (D, T). So without loss of generality, we can suppose T is a field algebraic over D.

By [D5, Proposition 3], $D \subseteq E$ satisfies incomparability for all intermediate rings E of (D, T). So (D, T) is an INC-pair. For all intermediate rings E, we see that $E \subseteq F$ satisfies incomparability for every overring F of E, since T is a field. So (D, T) is an INC-domain pair. ■

We next sharpen our focus to going-down-ring pairs and open-ring pairs. The proof that going-down-domain pairs are going-down pairs depends on the fact that $R \subseteq T$ satisfies going-down if R is a going-down-domain and T is a domain containing R. So, to obtain a relation between going-down-ring pairs and going-down-pairs, it would be useful to know under what conditions R a going-down-ring implies that $R \subseteq T$ satisfies going-down for (at least some) rings T containing R.

[D6, Example 1] demonstrates that $R \subseteq T$ need not satisfy going-down if R is a going-down-ring and T is an overring of R. (The R given in [D6, Example 1] is also an open-ring by [P, Theorem 3.16], since for each minimal prime ideal P of that R, we have that R/P is a quasisemilocal one-dimensional domain.) The offending ring R has exactly two minimal prime ideals. So, in search of a positive result, we consider the case where the base ring has only one minimal prime ideal.

If a ring R has a unique minimal prime ideal P_0, then we have $P_0 = Nil(R)$, and so $R/Nil(R)$ is a domain. Using the results in Chapter I on reduced rings and

the domainhood of $R/Nil(R)$ when R has a unique minimal prime ideal, we get the following extension of [DP2, Theorem 4.1] and [P, Proposition 3.10].

Proposition 3.6 Let R be a going-down-ring (resp., open-ring) with a unique minimal prime ideal. Then $R \subseteq T$ satisfies going-down (resp., is an open extension) for any ring T with a unique minimal prime ideal.

Proof: First we prove the statement for going-down-rings. Since R is a going-down-ring, $R/Nil(R) = R_{red}$ is a going-down-ring by Proposition 1.16. Since R and T each have a unique minimal prime ideal, R_{red} is a going-down-domain and T_{red} is also a domain. Therefore, $R_{red} \subseteq T_{red}$ is a going-down extension. Hence by Corollary 1.19, $R \subseteq T$ is a going-down extension.

The proof of the statement for open-rings is similar: replace every occurrence of "going-down" in the above argument by "open". ■

We can extend the "going-down" portion of the above Proposition to obtain the following result.

Corollary 3.7 Let $R \subseteq T$ be a ring extension with R a going-down-ring. Suppose R (resp., T) is such that each maximal ideal of R (resp., T) contains a unique minimal prime ideal of R (resp., T). Then $R \subseteq T$ satisfies going-down.

Proof: Suppose we have $P_2 \subseteq P_1 \in Spec(R)$ and a $Q_1 \in Spec(T)$ such that $Q_1 \cap R = P_1$. To show that $R \subseteq T$ satisfies going-down it suffices to show there exists some $Q_2 \in Spec(T)$ such that $Q_2 \subseteq Q_1$ and $Q_2 \cap R = P_2$.

Pick $M \in Max(R)$ such that $P_1 \subseteq M$. Consider $R_M \subseteq T_{R\setminus M}$: each of these rings has a unique minimal prime ideal, and R_M is a going-down-ring by [D6, Proposition 2.1(b)]. By Proposition 3.6, $R_M \subseteq T_{R\setminus M}$ satisfies going-down. Hence,

we can find $Q \in Spec(T_{R\setminus M})$ such that $Q \subseteq (Q_1)_{R\setminus M}$ and $Q \cap R_M = (P_2)_{R_M}$. Let $\pi : T \to T_{R\setminus M}$ be the canonical ring homomorphism. Then $Q_2 := \pi^{-1}(Q)$ has the desired properties. ■

As another corollary of Proposition 3.6, we get an extension of the result that every going-down-domain pair (resp., open-domain pair) is a going-down-pair (resp., an open-pair).

Corollary 3.8 Let (R, T) be a going-down-ring pair (resp., open-ring pair), where T has a unique minimal prime ideal. Then (R, T) is a going-down-pair (resp., open-pair).

Proof: Since T is assumed to have a unique minimal prime ideal, every subring A of T has a unique minimal prime ideal (since if A had two or more minimal prime ideals, they would be lain over by two or more minimal primes of T). Apply Proposition 3.6. ■

Proposition 3.6 and Corollary 3.8 are false in general without the hypothesis that $Min(T) = \{Nil(T)\}$. Recall Example 1.12, an extension $R \subseteq T$ in which R is a domain and T has two minimal prime ideals, $(0) \times K$ and $R \times (0)$. The extension $R \subseteq T$ is not a going-down extension and (R, T) is not a going-down-pair, even though, by Proposition 1.11, (R, T) is a going-down-ring pair.

We are also interested in conditions under which going-down-pairs (resp., open-pairs) (R, T) are going-down-ring pairs (resp., open-ring pairs). Let (0) be a primary ideal of the ring R (i.e., suppose all zero divisors of R are nilpotent). Then [D6, Corollary 2.6] says that R is a going-down ring if and only if $R \subseteq T$ satisfies going-down for all overrings T of R. We next prove the analogue of [D6, Corollary

2.6] for open-rings.

Proposition 3.9 Let R be a ring for which (0) is a primary ideal. Then R is an open-ring if and only if $R \subseteq T$ is an open extension for all overrings T of R.

Proof: First we prove the "only if" statement. Since (0) is assumed to be primary in R, we have that R has the unique minimal prime ideal $Nil(R)$. Moreover, (0) is primary in every overring T of R (cf. the proof of [D6, Theorem 2.4(b)]), and so T has the unique minimal prime $Nil(T)$. Apply Proposition 3.6.

Conversely, suppose $R \subseteq T$ is an open extension for all overrings T of R. Let $P = Nil(R)$. By Proposition 1.16, it suffices to show that R/P is an open-ring. Since $Nil(R)$ is the unique minimal prime ideal of R, we have that R/P is a domain, and so it suffices to show that $R/P \subseteq V$ is an open extension for all overrings V of R/P. Let V be an overring of R/P. Then $V \subseteq R_P/PR_P$. Let $T := \pi^{-1}(V)$, where $\pi : R_P \to R_P/PR_P$ is the canonical surjection. Then $V = T/PR_P$ and $R \subseteq R + PR_P \subseteq T \subseteq R_P = T(R)$, so T is an overring of R. Therefore $R/P \subseteq V$ is an open extension. ■

We next show a generalization of the fact that, for a domain R, the pair $(R, qf(R))$ is a going-down-domain pair (resp., open-domain pair) if and only if $(R, qf(R))$ is a going-down-pair (resp., open-pair).

Proposition 3.10 Let R be a ring for which (0) is a primary ideal. Then $(R, T(R))$ is a going-down-ring pair (resp., open-ring pair) if and only if $(R, T(R))$ is a going-down-pair(resp., open-pair).

Proof: We first prove the statements for "going-down". For the "only if" direction, note that since R has a unique minimal prime ideal, so does $T(R)$ (as before, cf. the proof of [D6, Theorem 2.4(b)]). Apply Corollary 3.8.

Conversely, suppose $(R, T(R))$ is a going-down-pair. Let S be an intermediate ring of the pair $(R, T(R))$. By Proposition 1.20, $(R_{red}, T(R)_{red})$ is a going-down-pair. But $T(R)_{red}$ is a field, since $T(R)$ has a unique minimal prime ideal. Therefore, since $R_{red} \subseteq S_{red} \subseteq T(R)_{red}$ and $(R_{red}, T(R)_{red})$ is a going-down-pair, we can conclude that $(S_{red}, T(R)_{red})$ is a going-down-domain pair; in particular, S_{red} is a going-down-domain. Hence by Proposition 1.16, S is a going-down-ring.

The proof for "open" is similar: replace every occurrence of "going-down" with"open." ■

We can improve the above result so that the top ring need not be the total quotient ring of the base ring, but instead just a larger ring S such that $S = T(S)$.

Proposition 3.11 Let $R \subseteq S$ be a ring extension with $S = T(S)$ and (0) primary in S. Then (R, S) is a going-down-ring pair (resp., open-ring pair) if and only if (R, S) is a going-down-pair (resp., open-ring pair).

Proof: We first prove the result for "going-down." For the "only if" direction, apply Corollary 3.8. Conversely, suppose (R, S) is a going-down-pair. Let A be an intermediate ring of (R, S). We show that A is a going-down-ring. Since each zero divisor in S is nilpotent, the same is true for A, whence (0) is primary in A. Then each non-zero divisor in A is non-nilpotent, hence a non-zero divisor in S, hence invertible in $T(S) = S$. Therefore, $T(A) \subseteq S$. Hence by Proposition 3.10, $(A, T(A))$ is a going-down-ring pair; in particular, A is a going-down-ring.

The proof for "open" is similar: replace every occurrence of "going-down" by "open." ■

The extension $R \subseteq T$ in Example 3.1 shows that one cannot delete the hypothesis "S=T(S)" from the "if" assertion in Proposition 3.11.

CHAPTER IV

FLAT-PAIRS AND ADDITIONAL REMARKS

Since "flat" implies "going-down," we next consider flat-pairs (that is, $\mathcal{P}$-pairs with $\mathcal{P}$ ="flat"). One of the most obvious examples of a flat-pair is given by (R, T) where R is a Prüfer domain and T is an overring of R (for instance, the quotient field of R). Indeed, given rings $R \subseteq A \subseteq B \subseteq T$, then by [G2, Theorem 26.1], A is a Prüfer domain and B, being torsion-free over A, must be A-flat. More generally, we see similarly that any Prüfer domain pair is a flat-pair. A partial converse is given in Lemma 4.6.

Our first result is a transcendence degree criterion in the spirit of Proposition 2.2 and Proposition 2.4.

Proposition 4.1 Let k be a field contained in a domain R. Then (k, R) is a flat-pair if and only if $k \subseteq R$ is an algebraic field extension.

Proof: For the "if" direction, since $k \subseteq R$ is assumed to be an algebraic field extension, all intermediate rings of (k, R) are fields. By [DM, Proposition 2.1(a)], any field is flat over each of its subrings, and so (k, R) is a flat-pair.

Suppose the converse fails. Then (k, R) is a flat-pair and, by Propositon 2.4, there is a nonfield S such that $k \subseteq S \subseteq R$. Then (k, S) is a flat-pair, and so by [DM, Theorem 2.2], $k \subseteq S$ is algebraic. Hence S must be a field, a contradiction. ■

Before we go further, we note that many of the so-called "flat pairs" in the literature are not the flat-pairs being discussed here. Akiba in [A2] defines two types of "flat pair" in the following manner.

Definition 4.2 A pair of rings (A, B) is a *flat pair of type 1* if for all intermediate rings C, we have that C is a flat A-module.

Definition 4.3 A pair of rings (A, B) is a *flat pair of type 2* if for all intermediate rings C, we have that B is a flat C-module.

Any pair of domains (R, T) for which R is a Prüfer domain is an example of a flat pair of type 1, since each intermediate ring is a torsion-free R-module and hence a flat R-module. Any pair of domains (R, K) for which K is a field is an example of a flat pair of type 2, since a field is flat over each of its subrings (cf. [DM, Proposition 2.1(a)]).

The next example show that a pair of domains (R, T) can be a flat pair of type 1 and a flat pair of type 2, but not a flat-pair.

Example 4.4 Consider the pair $(K, K(X, Y))$, where K is a field and X and Y are algeraically independent indeterminates. This pair is a flat pair of type 1 since K is Prüfer. This pair is a flat pair of type 2 since $K(X, Y)$ is a field. Suppose $(K, K(X, Y))$ were a flat-pair. Then $(K, K(X, Y))$ would be a going-down-pair, which contradicts Proposition 2.1 since $td_K(K(X, Y)) = 2$.

Flat pairs of types 1 and 2 are related to flat-pairs in the manner specified by the following Proposition, whose elementary proof is omitted.

Proposition 4.5 For an extension of rings $R \subseteq T$, the following conditions are equivalent:

(1) (R, T) is a flat-pair;

(2) (C, T) is a flat pair of type 1 for all rings C such that $R \subseteq C \subseteq T$;

(3) (R, B) is a flat pair of type 2 for all rings B such that $R \subseteq B \subseteq T$.

So, in particular, we recover the obvious fact that a flat-pair is a flat pair of type 1 and a flat pair of type 2.

We show in Proposition 4.9 that if the top domain K is a field, then the pair of domains (R, K) is a flat-pair if and only if either $R \subseteq K$ is an algebraic field extension or R is a Prüfer domain with quotient field K.

First we prove the following lemma, which shows that if the top domain is a field, then flat-pairs are the same as Prüfer domain pairs.

Lemma 4.6 Let K be a field containing a domain R. Then the pair (R, K) is a flat-pair if and only if (R, K) is a Prüfer domain pair.

Proof: The "if" direction was noted earlier. For completeness, we give its proof. Let A and B be rings such that $R \subseteq A \subseteq B \subseteq K$. Since A is assumed to be Prüfer and B is a torsion-free A-module, we have that B is a flat A-module.

Conversely, suppose that (R, K) is a flat-pair. Let A be an intermediate domain of (R, K). Since K is a field, we have that $(A, qf(A))$ is a flat-pair. Hence every overring T of A is a flat A-module, and so by [R, Theorem 4], A is a Prüfer domain. ■

Next we show that a Prüfer domain pair (R, T) consists of either an algebraic field extension or a Prüfer domain R inside its overring T.

Lemma 4.7 Let $R \subseteq T$ be domains. Then (R, T) is a Prüfer domain pair if and only if at least one of the following conditions holds:

(1) $R \subseteq T$ is an algebraic field extension;

(2) T is an overring of R and R is Prüfer.

Proof: First we show the "if" direction. In case condition (1) holds, every intermediate ring is a field and hence Prüfer. In the remaining case, condition (2) holds; therefore, every intermediate ring is an overring of a Prüfer domain and is hence Prüfer.

Conversely, since (R, T) is assumed to be a Prüfer domain pair, each intermediate ring is integrally closed and thus seminormal. Hence by [ADH, Lemma 2.2(c)], either R is a field or T is an overring of R. If T is an overring of R, then condition (2) holds and we are done. So, suppose next that R is a field. Then since (R, T) is a flat-pair, Proposition 4.1 implies that $R \subseteq T$ is an algebraic field extension. ■

We pause to note the following finitistic corollary of Lemma 4.7.

Corollary 4.8 Let $R \subseteq T$ be domains. Then (R, T) is a Dedekind domain pair if and only if at least one of the following conditions holds:

(1) $R \subseteq T$ is an algebraic field extension;

(2) T is an overring of R and R is Dedekind.

Next we combine the above two lemmas to obtain the following promised result.

Proposition 4.9 Let R be a domain and K a field containing R. Then (R, K) is a flat-pair if and only if at least one of the following conditions holds:

(1) $R \subseteq K$ is an algebraic field extension;

(2) R is a Prüfer domain and $T = qf(K)$.

Proof: Combine Lemma 4.7 and Lemma 4.6. ■

We next obtain a result similar to Proposition 4.9 if R is a Prüfer domain contained in a domain T.

Proposition 4.10 Let $R \subseteq T$ be an extension of domains with R Prüfer. Then (R, T) is a flat-pair if and only if at least one of the following two conditions holds:

(1) $R \subseteq T$ is an algebraic field extension;

(2) T is an overring of R.

Proof: The "if" direction follows from Lemma 4.7 and the fact that any Prüfer domain pair is a flat-pair. Conversely, supppose (R, T) is a flat-pair. By [A1, Theorem 2.7], either T is an overring of R or R is a field. If T is an overring of R, then we are done. So, next suppose that R is a field. Then apply Proposition 4.9. ■

Recall from (2.15)-(2.18) that INC-pairs occasionally exhibit going-down behavior. In this spirit, we next note a relation between INC-pairs and flat-pairs.

Proposition 4.11 Let (R, T) be an INC-pair of domains with R integrally closed in T. Then (R, T) is a flat-pair.

Proof: Since (R, T) is assumed to be an INC-pair, it is also a residually algebraic pair. Hence by [FHP, Theorem 6.5.18], (R, T) is a normal pair. Therefore, for all rings A such that $R \subseteq A \subseteq T$, the ring A is integrally closed in T. Also, since (R, T) is assumed to be an INC-pair, (A, T) is an INC-pair for all rings $R \subseteq A \subseteq T$. Hence by [FHP, Theorem 6.5.18], (A, T) is a flat pair of type 1 for all rings A such that $R \subseteq A \subseteq T$. Thus by Proposition 4.5, (R, T) is a flat-pair. ■

However, an INC-pair need not be a flat-pair, as the following example shows.

Example 4.12 Let R be a non-Prüfer i-domain. (Such domains exist by [P, Proposition 2.23], for instance $\mathbb{R} + X\mathbb{C}[[X]]$.) Since R is an i-domain, it is also an INC-domain. Therefore by Proposition 3.5, $(R, qf(R))$ is an INC-pair. However, since

R is not Prüfer, by [R, Theorem 4] there is an overring S which is not R-flat. So $(R, qf(R))$ is not a flat-pair. (Also, note that by [P, Corollary 2.13], $(R, qf(R))$ is a going-down-domain pair.)

We turn next to the "additional remarks" mentioned in the title of this chapter. Let (R, T) be an i-pair of domains. Then by [P, Proposition 2.12] (resp., [P, Corollary 3.11], (R, T) is a going-down-domain pair (resp., an open-domain pair) if and only if R is a going-down-domain (resp., an open-domain). In Corollary 4.14, we generalize the previous sentence to i-pairs of arbitrary rings. We first generalize [P, Proposition 2.12 and Corollary 3.11].

Proposition 4.13 Let $R \subseteq T$ be an i-extension of rings. If R is a going-down-ring (resp., open-ring), then T is also a going-down-ring (resp., open-ring).

Proof: We show that T/Q is a going-down-domain (resp., an open-domain) for each $Q \in Spec(T)$. Let $Q \in Spec(T)$ and $P := Q \cap R$. Then R/P is a going-down-domain (resp., open-domain). Hence, by [P, Proposition 2.12] (resp., [P, Corollary 3.11]), it suffices to show that $R/P \subseteq T/Q$ is an i-extension.

Each prime ideal of T/Q has the form A/Q for some $A \in Spec(T)$. Let $A, B \in Spec(T)$ be such that $A/Q \cap R/P = B/Q \cap R/P$. We show that $A/Q = B/Q$. We have $(A \cap R)/P = A/Q \cap R/P = B/Q \cap R/P = (B \cap R)/P$. Hence $A \cap R = B \cap R$, and so $A = B$ since $R \subseteq T$ is assumed to be an i-extension. Thus $A/Q = B/Q$. ■

Corollary 4.14 Let (R, T) be an i-pair of rings. Then (R, T) is a going-down-ring pair (resp., an open-ring pair) if and only if R is a going-down-ring (resp., an open-ring).

Proof: The "only if" statement is trivial. To show the converse, let A be an intermediate ring of (R,T). Then since (R,T) is assumed to be an i-pair, $R \subseteq A$ is an i-extension. By Proposition 4.13, A is a going-down-ring (resp., open-ring). ■

It is interesting to note that the INC-analogue of Corollary 4.14 fails: see Example 2.7(b) in regard to [DF2, Example 3.1].

Next we show that any residually algebraic pair of rings (R,S) with R integrally closed in S is an i-pair. (The special case of this statement for domains is part of [AJ, Theorem 2.10].)

Proposition 4.15 Let $R \subseteq S$ be rings, with R integrally closed in S. Then the following statements are equivalent:

(1) (R,S) is a residually algebraic pair;

(2) (R,S) is an INC-pair;

(3) (R,S) is an i-pair.

Proof: Recall that (1) $\Leftrightarrow$ (2) (cf. [FHP, Theorem 6.5.6]); and (3) $\Rightarrow$ (2) trivially. It remains to show that (2) $\Rightarrow$ (3). Deny. Then there is a ring A with $R \subseteq A \subseteq S$ and there are $Q_1 \neq Q_2$ in $Spec(A)$ such that $Q_1 \cap R = P := Q_2 \cap R$. Without loss of generality, there is some element $\alpha \in Q_1 \backslash Q_2$. Note that α is in $Q_1 \cap R[\alpha]$, but not in $Q_2 \cap R[\alpha]$. Replace A (and S) with $R[\alpha]$. Therefore, S is algebra-finite over R. By INC, we have that Q_1 is isolated in its fiber over P. Therefore, by Evans' generalization of Zariski's Main Theorem [Ev, Theorem], there is an $r \in R \backslash P$ such that $R_r \cong S_r$. Therefore, $Q_2 S_r = P R_r = Q_1 S_r$, and so $Q_1 = Q_2$, a contradiction. ■

Proposition 4.15 needs the hypothesis that R is integrally closed in S. For example, consider any integral extension $R \subseteq S$ which is not an i-extension, such

as $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{-5}]$. [P, Example 2.17] is another example that shows the necessity of the hypothesis in Proposition 4.15 that R is integrally closed in S. Note that in [P, Example 2.17], unlike the example given above, the top ring is an overring of the base ring.

Combining Proposition 4.15 with Corollary 4.14 gives us the following Corollary.

Corollary 4.16 Let (R, T) be a residually algebraic pair of rings with R integrally closed in T. Then (R, T) is a going-down-ring pair (resp., an open-ring pair) if and only if R is a going-down-ring (resp., an open-ring).

Finally, we note that the hypotheses in Corollary 4.16 are needed. Indeed, [DF2, Example 3.1] shows that we cannot delete the hypothesis that R is integrally closed in T. Moreover, by applying Proposition 2.1 to the purely transcendental field extension $k \subseteq k(X, Y)$, we see that we cannot delete the hypothesis in Corollary 4.16 that (R, T) is a residually algebraic pair.

BIBLIOGRAPHY

BIBLIOGRAPHY

[A1] T. Akiba, *Remarks on generalized rings of quotients. IV*, J. Math. Kyoto Univ., **25**(1985), 687-691.

[A2] T. Akiba, *Remarks on flat and relatively seminormal pairs*, Kobe J. Math., **6**(1989), 217-222.

[Ar] J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc., **7**(1973), 299-304.

[AD] D. F. Anderson and D. E. Dobbs, *Pairs of rings with the same prime ideals*, Canad. J. Math., **32**(1980), 362-384.

[ADH] D. F. Anderson, D. E. Dobbs and J. A. Huckuba, *On seminormal overrings*, Comm. Algebra, **10**(1982), 1421-1448.

[AJ] A. Ayache and A. Jaballah, *Residually algebraic pairs of rings*, Math. Z., **225**(1997), 49-65.

[C] S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc., **97**(1960), 457-473.

[CMZ] D. Costa, J. L. Mott and M. Zafrullah, *The construction* $D + XD_s[X]$, J. Algebra, **53**(1978), 423-439.

[D1] D. E. Dobbs, *On going down for simple overrings, II*, Comm. Alg., **1**(1974), 439-458.

[D2] D. E. Dobbs, *Divided rings and going-down*, Pac. J. Math., **67**(1976), 353-363.

[D3] D. E. Dobbs, *Lying-over pairs of commutative rings*, Canad. J. Math., **33**(1981), 454-475.

[D4] D. E. Dobbs, *Going-down underrings*, Bull. Austral. Math. Soc., **36**(1987), 503-513.

[D5] D. E. Dobbs, *Prüfer's ascent result using INC*, Comm. Algebra, **23**(1995), 5413-5417.

[D6] D. E. Dobbs, *Going-down rings with zero-divisors*, Houston J. Math., **23**(1997), 1-12.

[DF1] D. E. Dobbs and M. Fontana, *Universally going-down homomorphisms of commutative rings*, J. Algebra, **90**(1984), 410-429.

[DF2] D. E. Dobbs and M. Fontana, *Integral overrings of two-dimensional going-down domains*, Proc. Amer. Math. Soc., **115**(1992), 655-662.

[DFP] D. E. Dobbs, M. Fontana and I. J. Papick, *On the flat spectral topology*, Rend. Mat., **4**(1981), 559-578.

[DM] D. E. Dobbs and S. B. Mulay, *Flat underrings*, Arch. Math., **50**(1988), 337-341.

[DP1] D. E. Dobbs and I. J. Papick, *On going down for simple overrings*, III, Proc. Amer. Math. Soc., **54**(1976), 35-38.

[DP2] D. E. Dobbs and I. J. Papick, *Going-down: a survey*, Nieuw Arch. v. Wisk., **26**(1978), 255-291.

[E] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Springer, New York, 1995.

[Ev] E. G. Evans, *A generalization of Zariski's main theorem*, Proc. Amer. Math. Soc., **26**(1970), 45-48.

[FHP] M. Fontana, J. A. Huckuba and I. J. Papick, Prüfer Domains, Dekker, New York, 1997.

[G1] R. Gilmer, *Integral domains with Noetherian subrings*, Comment. Math. Helv., **45**(1970), 129-134.

[G2] R. Gilmer, Multiplicative Ideal Theory, Queen's Series in Pure and Applied Mathematics, Queen's University, Kingston, Ontario, 1992.

[H] H. C. Hutchins, Examples of Commutative Rings, Polygonal Publ. House, Washington, N.J., 1981.

[K] I. Kaplansky, Commutative Rings, rev. ed., Univ. of Chicago Press, Chicago, 1974.

[P] I. J. Papick, *Topologically defined classes of going-down domains*, Trans. Amer. Math. Soc., **219**(1976), 1-37.

[R] F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc., **16**(1965), 794-799.

[W] A. R. Wadsworth, *Pairs of domains where all intermediate domains are Noetherian*, Trans. Amer. Math. Soc., **195**(1974), 201-211.

VITA

Richard Douglas Chatham was born on June 5, 1968 in Statesville, North Carolina. He attended the public school system in Alexander County and graduated from Alexander Central High School in May 1986. In May 1990, Chatham graduated *summa cum laude* from Wake Forest University in Winston-Salem, North Carolina, with a Bachelor of Science degree in Mathematics with a minor in computer science. In August 1992, Chatham entered the graduate program in Mathematics at the University of Tennessee, Knoxville. The Ph.D. was awarded in May 2000.