To the Graduate Council:

I am submitting herewith a thesis written by Jeremy S. Higdon entitled “A Note on Hamel Bases.” I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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A Note on Hamel Bases

A Thesis
Presented for the
Master of Science
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Jeremy S. Higdon
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Dedication

This thesis is dedicated to my family, which has been a wonderful source of support and encouragement. They love me greatly, as I love them.

Abstract

The purpose of this paper is to discuss certain properties of Hamel bases. In particular, we reprove and generalize a theorem of R. Mabry (Aequations Mathematicae 71 (2006) p. 294-299) on the non-existence of nontrivial Hamel bases closed under multiplication.

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Chapter I
Introduction and Preliminaries

A subset H of an R-module S is called a Hamel basis for S over R if for every nonzero $s \in S$, there exist unique nonzero elements $r_1, \ldots, r_n \in R$, and $h_1, \ldots, h_n \in H$ such that $s = r_1 \circ h_1 + \ldots + r_n \circ h_n$, or equivalently, if H is linearly independent over R and $\text{span}_R(H) = S$. Hamel basis is used as opposed to just basis to distinguish them from Schauder bases, where the sum $s = \sum_{i=1}^{n} r_i h_i$ only has to be countable, not finite.

For arbitrary ring R and R-module S, there may not be a Hamel basis for S over R. Hereafter, we assume R is unitary.

Theorem 1.1: Let R be a ring with one, and S an R-module. Then S has a basis over R if and only if S is a free R-module, that is, $S \simeq R^{|J|}$ for some set J.

Proof: This is clear from the definition.

In particular, since every vector space is free over it's base field, every vector space has a basis.

Hamel bases were named for Georg Hamel, who in 1905 used them to prove there were infinitely many nontrivial solutions to Cauchy's functional equation f(x + y) = f(x) + f(y). His proof was along these lines [1]: Suppose f satisfies the Cauchy functional equation f(x + y) = f(x) + f(y).

Then f(x) = f(x + 0) = f(x) + f(0), so f(0) = 0. Hence 0 = f(0) = f(x -x) = f(x) + f(-x), so f(-x) = -f(x). Also, for $n\in\mathbb{N}$, we have f(nx) = $f(\underbrace{x+...+x}_{n\,\text{terms}})$ = $\underbrace{f(x)+...+f(x)}_{n\,\text{terms}}$ =n f(x). So f(x) =f($n\frac{x}{n}$) = n f($\frac{x}{n}$), so f($\frac{x}{n}$)= $\frac{1}{n}$ f(x). Therefore for all $q\in\mathbb{Q},r\in\mathbb{R}$, we have f(rq) = q f(r).

Let H be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Then for all $r\in\mathbb{R}, r=\sum_{i=1}^{n} q_i h_i$ we have f(r) = $\sum_{i=1}^{n} q_i f(h_i)$. Thus the value of f on H completely determines the value of f on $\mathbb{R}$. Any arbitrary assignment of values on the irrational elements of H yields a valid solution f, which is only linear on $\mathbb{R}$ if it is linear on H. Since H can only contain at most one rational, and has cardinality continuum, we have uncountably many nontrivial solutions.

We note in passing that if f is not linear, then it's graph must be dense in $\mathbb{R}^2$. For if f is not linear, then there are x_1 and x_2 such that $x_2 f(x_1) \neq x_1 f(x_2)$, so $v_1 = (x_1, f(x_1))$ and $v_2 = (x_2, f(x_2))$ are linearly independent and span $\mathbb{R}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, for any $\epsilon>0$, and $v\in\mathbb{R}^2$, there are q_1, q_2 such that $\|v-(q_1 v_1+q_2 v_2)\|<\epsilon$. Thus we have

$\epsilon>\|v-(q_1 v_1+q_2 v_2)\|=\|v-(q_1 x_1+q2\, x_2, q_1 f(x_1)+q_2 f(x_2)\ )\|$

$=\|v-(q_1 x_1+q2\, x_2, f(q_1 x_1+q_2 x_2)\ )\|$. So the graph of f is dense in $\mathbb{R}^2$.

Hamel bases are also used in Dehn's [2] proof of Hilbert's third problem, showing that there are tetrahedron with equal area that are neither equidecomposable or equicomplementable.

There is also the surprising result due to Erdös and Kakutani [3] that the continuum hypothesis is equivalent to the existence of a countable partition of $\mathbb{R} \setminus \{0\}$ into Hamel basis.

Hamel bases are also used to display "pathological" examples, such as the fact that there are subsets of the reals without the Baire property and non-Lebesgue measurable.

Thus we can see that Hamel bases are encountered in many areas, and have some importance. In the remainder of this paper, we discuss some known results about Hamel bases, as well as introduce some new ones.

Chapter II

$\mathbb{R}$ over $\mathbb{Q}$

A very common instance of a Hamel basis is the case of $\mathbb{R}$ over $\mathbb{Q}$, or $\mathbb{R}$ over $\mathbb{Q}_{\mathbb{R}}$, where $\mathbb{Q}_{\mathbb{R}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. A natural question that arises is where one might be able to find a Hamel basis in either of these cases.

Since $\mathbb{R}$ is a field and contains $\mathbb{Q}$ and $\mathbb{Q}_{\mathbb{R}}$, to guarantee the existence of a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ or $\mathbb{Q}_{\mathbb{R}}$ inside a subset T of $\mathbb{R}$, we just need show that $\mathrm{span}_{\mathbb{Q}}(T) = \mathbb{R}$, or $\mathrm{span}_{\mathbb{Q}_{\mathbb{R}}}(T) = \mathbb{R}$, respectively. It turns out that such subsets abound, since every nonempty open subset of $\mathbb{R}$ qualifies, as indeed any set containing an interval does, which our next theorem will show. This theorem and the next are so natural that they surely have been proved before, but the author was unable to find any references.

Theorem 2.1: Let K be $\mathbb{Q}$ or $\mathbb{Q}_{\mathbb{R}}$. Then every nonempty open subset of $\mathbb{R}$ contains a basis for $\mathbb{R}$ over K.

Proof: Let U be a nonempty open subset of $\mathbb{R}$, and H a Hamel basis of $\mathbb{R}$ over K. (Such a basis exists by Theorem 1.1) Then there are $s, t \in \mathbb{Q}$, $s<t$ such that $(s,t) \subset U$. Let $r \in \mathbb{Q}^+$ be such that $r < \frac{t-s}{3}$. Then for every $h \in H$, there is a $z_h \in \mathbb{Z}$ such that $z_h \circ r \leqslant h < (z_h+1) \circ r$. Let $j_h = h - (z_h+1) \circ r$, and define J = $\{j_h: h\in H\} \cup \{\frac{-r}{2}\}$. Then for each $h\in H$ we have $h = j_h - 2(z_h+1)(\frac{-r}{2})$. Since $j_h, \frac{-r}{2} \in J$ and $-2(z_h+1) \in K$, we have $h\in \mathrm{span}_K(J)$, hence $H\subseteq \mathrm{span}_K(J)$, thus $\mathbb{R}=\mathrm{span}_K(H)\subseteq \mathrm{span}_K(J)$. Therefore J contains a Hamel basis G of $\mathbb{R}$ over K.

Suppose the Hamel representation of $\frac{s+t}{2}$ with respect to G is $\frac{s+t}{2}=k_1\circ g_1+\ldots+k_n\circ g_n$ for some $k_1,\ldots,k_n\in K$, and $g_1,\ldots,g_n\in G$. Let $\mathrm{I}=\{\frac{s+t}{2}\} \cup \{g+\frac{s+t}{2} : g\in G, g\neq g_1\}$. Let $g \in G$. If $g\neq g_1$, then $g=(g+\frac{s+t}{2})-\frac{s+t}{2}$. If $g = g_1$, then $g = g_1 =$

$$\frac{1}{k_1}\circ\left(\frac{s+t}{2}-k_2\circ g_2-k_3\circ g_3-\ldots-k_n\circ g_n\right)=\frac{1}{k_1}\circ\frac{s+t}{2}-\frac{k_2}{k_1}\circ g_2-\frac{k_3}{k_1}\circ g_3-\ldots-\frac{k_n}{k_1}\circ g_n =$$

$$\frac{1}{k_1}\circ\frac{s+t}{2}-\frac{k_2}{k_1}\circ\left(g_2+\frac{s+t}{2}-\frac{s+t}{2}\right)-\frac{k_3}{k_1}\circ\left(g_3+\frac{s+t}{2}-\frac{s+t}{2}\right)-\ldots-\frac{k_n}{k_1}\circ\left(g_n+\frac{s+t}{2}-\frac{s+t}{2}\right) =$$

$$\frac{1}{k_1}\circ\frac{s+t}{2}-\frac{k_2}{k_1}\circ\left(g_2+\frac{s+t}{2}\right)+\frac{k_2}{k_1}\circ\frac{s+t}{2}-\frac{k_3}{k_1}\circ\left(g_3+\frac{s+t}{2}\right)+\frac{k_3}{k_1}\circ\frac{s+t}{2}-\ldots-\frac{k_n}{k_1}\circ\left(g_n+\frac{s+t}{2}\right)+\frac{k_n}{k_1}\circ\frac{s+t}{2} =$$

$$\frac{1+k_2+k_3+\ldots+k_n}{k_1}\circ\frac{s+t}{2}-\frac{k_2}{k_1}\circ\left(g_2+\frac{s+t}{2}\right)-\frac{k_3}{k_1}\circ\left(g_3+\frac{s+t}{2}\right)-\ldots-\frac{k_n}{k_1}\circ\left(g_n+\frac{s+t}{2}\right)$$

. Thus in either case, $g \in \mathrm{span}_K(\mathrm{I})$. Thus $G\subseteq \mathrm{span}_K(\mathrm{I})$, and hence $\mathbb{R}=\mathrm{span}_K(G)\subseteq \mathrm{span}_K(\mathrm{I})$. Therefore I contains a basis L of $\mathbb{R}$ over K.

Now, for all $h\in H$, we have $z_h\circ r\leqslant h<(z_h+1)\circ r$ so $z_h\circ r-(z_h+1)\circ r\leqslant h-(z_h+1)\circ r<0$ thus $-r\leqslant j_h<0$, so $J\subseteq[-r,0)$, and hence $G\subseteq[-r,0)$. Now $\frac{s+t}{2} \in (s,t)$, and for every $g\in G$, we have $g+\frac{s+t}{2}\in[-r+\frac{s+t}{2}, \frac{s+t}{2})\subseteq[-\frac{t-s}{3}+\frac{s+t}{2}, \frac{s+t}{2})=[\frac{5s+t}{6}, \frac{s+t}{2})\subseteq(s,t)$. Hence

$I \subseteq (s,t)$, so $L \subseteq (s,t) \subseteq U$ Thus U contains the Hamel basis L of $\mathbb{R}$ over K ■

We see that Hamel basis are not rare creatures at all. In fact, even sets with empty interior may contain one, as we will prove next.

Proposition 2.2: The Cantor set contains a Hamel basis for $\mathbb{R}$ over K.

Proof: Let C denote the Cantor set. Let all numbers in the remainder of this proof be tertiary. Then C consists of 0, 1, and all decimals in (0,1) whose tertiary representation does not include a 1 or end in an infinite sequence of 2's. Let $x \in [0,1]$, and the tertiary representation of x not ending in an infinite sequence of 2's be $x = d_0.d_1d_2d_3...$. Define $e_i = \begin{cases} d_i & \text{if } d_i \neq 1 \\ 0 & \text{if } d_i = 1 \end{cases}$ and $f_i = \begin{cases} 0 & \text{if } d_i \neq 1 \\ 2 & \text{if } d_i = 1 \end{cases}$ and define $e = e_0.e_1e_2e_3...$ and $f = f_0.f_1f_2f_3...$. Then $e, f \in C$ and $x = e + \frac{1}{2} f$, so x has a representation as a linear combination (over K) of elements of C. Let $y \in \mathbb{R}$. Then there exists $n \in \mathbb{Z} - \{0\}$ such that $\frac{y}{n} \in [0,1]$, so by the above there are $e, f \in C$ such that $\frac{y}{n} = e + \frac{1}{2} f$ so $y = n * e + \frac{n}{2} f$ is a representation of y as a linear combination (over K) of elements of C. Therefore $\mathbb{R} \subseteq \text{span}_K(C)$, so C contains a Hamel basis of $\mathbb{R}$ over K. ■

Of course, any countable set has empty interior, but can not contain a basis, as the set has only countable span, while $\mathbb{R}$ is of course uncountable. Of course, there are uncountable sets that do not contain a Hamel basis, such any Hamel basis with an element removed.

Chapter III

A closure property of $\mathbb{R}$ over $\mathbb{Q}$

In 1969, Aczél and Erdös [4] asked whether or not there was a Hamel basis H for $\mathbb{R}$ over $\mathbb{Q}$ that is closed under the taking of powers, that is, for every $h \in H$, is $h^n \in H$ for every $n \in \mathbb{N}$. In 1980, Smítal [5] proved that the answer was yes. We present his proof here:

Theorem 3.1 (Smítal): There is a Hamel basis H for $\mathbb{R}$ over $\mathbb{Q}$ that is closed under the taking of powers, that is,for all $h \in H$, we have that $h^n \in H$ for every $n \in \mathbb{N}$.

Proof: Following standard notation, for a subset M of the reals, we let F(M) denote the field generated by M, A(M) to be the set of numbers algebraic over M, c to represent the cardinality of the reals, and Ω the first ordinal of cardinality c. Now, let X = $\{x_\alpha\}_{\alpha<\Omega}$ be a well ordering of the reals, with $x_0 = 1$.

By induction, we will define transfinite sequences Y = $\{y_\alpha\}_{\alpha<\Omega}$ and Z = $\{z_\alpha\}_{\alpha<\Omega}$ such that if $H_\alpha = \{y_\omega^k : \omega \leq \alpha, k \in \mathbb{Z}\} \cup \{z_\omega^k : \omega \leq \alpha, k \in \mathbb{Z}\}$, then the elements of H_α are linearly independent, and $X_\alpha = \{x_\omega\}_{\omega \leq \alpha} \subseteq \mathrm{span}_{\mathbb{Q}}(H_\alpha)$. First, set $y_0=z_0=1$, so that $H_0=\{1\}$ and clearly $X_\alpha = \{1\} \subseteq \mathrm{span}_{\mathbb{Q}}(H_\alpha)$.

Assume by induction that we have defined H_α for all $\alpha < \beta$. Define $J_\beta = \bigcup_{\alpha<\beta} H_\alpha$ Let x_λ be the first element of X that is not in $K_\beta = \mathrm{span}_{\mathbb{Q}}(J_\beta)$.

Now, since $\lambda < \Omega$, the cardinality of $K_\beta \cup \{x_\lambda\}$ is less than c, and hence so are the cardinalities of F($K_\beta \cup \{x_\lambda\}$) and A($K_\beta \cup \{x_\lambda\}$), so there is a transcendental number over F($K_\beta \cup \{x_\lambda\}$). Denote this number by y_β and set $z_\beta = x_\lambda - y_\beta$. Since $X_\alpha = \underset{\omega \leqslant \alpha}{\{x_\omega\}} \subseteq \mathrm{span}_{\mathbb{Q}}(H_\alpha)$ for $\alpha < \beta$, we must have $\lambda \geqslant \beta$. Thus, since x_λ is the first element of X that is not in K_β, either x_β is in K_β or $x_\lambda = x_\beta$. In either case, x_β is in the span of H_β.

Suppose that H_β is not linearly independent. Then there are polynomials over $\mathbb{Q}$ f_i, g_i, h_i, and j_i with i = 1, ..., s and ordinals $\alpha_1 < ... < \alpha_s = \beta$ such that $f_1(y_{\alpha_1}) + ... + f_s(y_{\alpha_s}) + g_1(1/y_{\alpha_1}) + ... + g_s(1/y_{\alpha_s}) +$ $h_1(z_{\alpha_1}) + ... + h_s(z_{\alpha_s}) + j_1(1/z_{\alpha_1}) + ... + j_s(1/z_{\alpha_s}) = 0$ Denote the degree of g_s and j_s by d and e, respectively. If we then multiply both sides of our polynomial equation by $y_\beta^d\ (x_\lambda - y_\beta)^e$, we get

$$((w + f_s(y_\beta) + h_s(x_\lambda - y_\beta))y_\beta^d\ (x_\lambda - y_\beta)^e +$$

$$q(y_\beta) \circ (x_\lambda - y_\beta)^e + r(x_\lambda - y_\beta) \circ y_\beta^d = 0$$, where $w \in K_\beta$, q and r are polynomials over $\mathbb{Q}$, and the degrees of q,r are less than or equal to d,e respectively. Since y_β is transcendental over F($K_\beta \cup \{x_\lambda\}$), this implies that y_β^d divides $q(y_\beta)$ over $F(K_\beta)$, and since the degree of q is less than or equal to d, we have q(x) = $a * x^d$ where a is a rational. Then, since

$g_s(1/y_\beta) \circ y_\beta^d = a \circ y_\beta^d$, we have $g_s(x) = a$ is a constant. Similarly, $j_s(x)$ is a constant. We thus have $w' + f_s(y_\beta) + h_s(x_\lambda - y_\beta) = 0$, with $w' \in K_\beta$. Since w_β is transcendental over $F(K_\beta \cup \{x_\lambda\})$, we have the degrees of f_s and h_s are the same, denote it by m. Suppose m > 0. We can then expand our previous equation as $a_0 + a_1 y_\beta + ... + a_m y_\beta^m = 0$, with $a_i \in F(K_\beta \cup \{x_\lambda\})$. Since y_β is transcendental over $F(K_\beta \cup \{x_\lambda\})$, we must have all the a_i's to be zero. In particular, $a_{m-1} = 0$. Expand $f_s(x)$ and $h_s(x)$ as

$f_s(x) = b_0 + b_1 x + ... + b_m x^m$ and $h_s(x) = c_0 + c_1 y_\beta + ... + c_m x^m$.

If m > 1, we then have $a_{m-1} = b_{m-1} + (-1)^{m-1} (m\ x_\lambda\ c_m + c_{m-1})$. This means that $x_\lambda \in \mathbb{Q}$, a contradiction.

If m = 1, then we have $a_0 = w' + b_0 + c_0 + c_1 \circ x_\lambda$, so that $x_\lambda \in K_\beta$, a contradiction.

Therefore, m = 0, and hence the elements of J_β are linearly dependent, a contradiction.

Thus it must be that the elements of H_β are linearly independent, and hence $X_\beta = \{x_\omega\}_{\omega \leqslant \beta} \subseteq \mathrm{span}_{\mathbb{Q}}(H_\beta)$, so J_Ω is a basis for $X_\Omega = \mathbb{R}$ that has the desired property. ■

A Closure Property of Fields over a Proper Subfield

In the preceding chapter we discovered there was a Hamel basis of the reals over the rationals that was closed under the taking of powers. In this chapter, we find a negative result for whether there is one that is closed under multiplication. In fact, it is negative for any field over a proper subfield. This result is due to Mabry [6]. In the rest of this chapter, K is a field, F a proper subfield of K, and until we get a contradiction, we assume H is multiplicatively closed Hamel basis of K over F. We proceed via the following lemmas:

Lemma 4.1: Suppose $\sum_{i=1}^{n} f_i \circ h_i = \sum_{s=1}^{m} g_s \circ j_s$ with the f_i, g_s in F, and the h_i, j_s in H, and further that all the f_i are nonzero, and the h_i distinct. Then we have $\{h_1, h_2, .., h_n\} \subseteq \{j_1, j_2, .., j_m\}$.

Proof: Suppose there exists $h_k \in \{h_1, h_2, .., h_n\}$ such that $h_k \notin \{j_1, j_2, .., j_m\}$.

Since F is a field, and f_k is nonzero, we have f_i / f_k and g_s / f_k are in F for all defined i and s. Then $\sum_{i=1}^{n} f_i \circ h_i = \sum_{s=1}^{m} g_s \circ h_s$ implies $h_k \sum_{i=1}^{n} (f_i \circ h_i) - \sum_{\substack{s=1 \\ s \neq k}}^{m} (g_s \circ h_s)$.

Since the h_i are all distinct and $h_k \notin \{j_1, j_2, .., j_m\}$, this contradicts the linear independence over F of the elements of H ∎

The reverse of Lemma 4.1 need not be true, as the k_s are not assumed to be distinct, and could thus cancel out among themselves.

Lemma 4.2: H is a group under multiplication.

Proof: H is clearly nonempty, and associative under multiplication. By assumption, it is also closed under multiplication. Thus we only have to check if 1 / a is in H for every a in H.

Let $a, b \in H$. Then $a, b \in K$, so $a / b \in K$. Let a / b = $\sum_{i=1}^{n} f_i \circ h_i$ be the Hamel representation of a / b. Then a = $\sum_{i=1}^{n} (f_i \circ h_i) \circ b$ = $\sum_{i=1}^{n} f_i \circ (h_i \circ b)$ is the Hamel representation of a, since H being multiplicatively closed implies $h_i \circ b \in H \text{ for all } i$, and $b \circ h_i = b \circ h_J$ implies $h_i = h_J$ implies i = j. Since a is in H, and representations are unique, we must have n = 1, and so a = $f_1 \circ b \circ h_1$. Since $a, b \circ h_1 \in H$, and elements of H are linearly independent, we must have a = $b \circ h_1$, so a / b = $h_1 \in H$. So a / b is in H for all a and b in H. In particular, 1 = a / a is in H, and thus so is 1 / a. Therefore, H is a group under multiplication. ■

Lemma 4.3: If for some $h \in H, n \in \mathbb{N}$ we have $h^n = 1$, then h = 1.

Proof: Suppose $h \in H, n \in \mathbb{N}$ such that $h^n = 1$. Let m be the smallest positive integer such that $h^m = 1$. If $h \neq 1$, this means m > 1, so

$0 = h^m - 1 = (h-1)\sum_{i=0}^{m-1} h^i$, and again since $h \neq 1$, we have

$0 = \sum_{i=0}^{m-1} h^i = 1 + \sum_{i=1}^{m-1} h^i$, so $1 = \sum_{i=1}^{m-1} -h^i$. By lemma 4.2, $1 \in H$, so by Lemma 4.1, $1 \in \{ h, h^{2,} h^3, \ldots, h^m - 1 \}$, contradicting m's minimality. Thus h = 1. ■

Lemma 4.4: If for $h, k \in H$ there is a nonzero integer n such that $h^n = k^n$, then h = k.

Proof: Suppose for $h, k \in H$ there is a nonzero integer n such that $h^n = k^n$. Then $\left(\frac{h}{k}\right)^n = 1$. By Lemma 4.2, $\frac{h}{k} \in H$, so by Lemma 4.3 we have

$\frac{h}{k} = 1$, so h = k. ■

We are now ready to proceed to the main theorem of this chapter, as promised:

Theorem 4.5 (Mabry): Let K be a field, F a proper subfield of K, and H a Hamel basis of K over F. Then H is not multiplicatively closed.

Proof: Suppose H is multiplicatively closed, and let $a, b \in H$, $a \neq b$. Since the elements of H are linearly independent over F, we have $a + b \neq 0$. Let the Hamel representation of $\frac{1}{a+b}$ be $\frac{1}{a+b} = \sum_{i=1}^{n} k_i h_i$. Then letting

$g_i = h_i \circ a \in H$, and $j_i = h_i \circ b \in H$ we have $1 = \sum_{i=1}^{n} k_i g_i + \sum_{i=1}^{n} k_i j_i$. Note that

$g_i = h_i \Rightarrow h_i a = h_i b \Rightarrow a = b$, a contradiction, and that

$g_i = g_m \Rightarrow h_i a = h_m a \Rightarrow h_i = h_m, \Rightarrow i = m$, so the g_i's are all distinct, as the j_i's are similarly. By Lemma 4.2, 1 is in H, and so by Lemma 4.1, we have $1 \in \{g_1, g_2, .., g_n, h_1, h_2, .., h_n\}$. Without loss of generality, assume $g_1 = 1$.

Suppose $k_1 = 1$. Then $1 = 1 + \sum_{i=2}^{n} k_i g_i + \sum_{i=1}^{n} k_i j_i$. By the linear independence of the elements of H , and the fact that each of the g_i's and j_i's are distinct, each remaining g_i must equal exactly one of the j_i's, and vice versa. However, there are more j_i's than there are g_i's left. Thus $k_i \neq 1$.

If n = 1, then $1 = k_1 g_1 + k_1 j_1 = k_1 + k_1 j_1$. Thus $0 = (k_1 - 1)1 + k_1 j_1$. By the linear independence of the elements of H, this forces $k_1 - 1 = 0$. so $k_1 = 1$, a contradiction. Thus n > 1.

`If we rewrite our equation as $(1 - k_1)g_1 + \sum_{i=2}^{n-k_i} g_i = \sum_{s=1}^{n} k_s j_s$, we see since each of the g_i's and j_i's are all distinct, that by lemma 4.1 we must have each g_i equal to exactly 1 j_L. Since g_1 cannot equal j_1, we have $g_1 = j_m$ for some m > 1.

Suppose n = 2. Then $g_1 = j_2$, and so $h_1 a = h_2 b$, so $\frac{a}{b} = \frac{h_2}{h_1}$. Also, we have $g_2 = j_1$, so that $h_2 a = h_1 b$, so $\frac{a}{b} = \frac{h_1}{h_2}$. Then $\left(\frac{a}{b}\right)^2 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{h_2}{h_1}\frac{h_1}{h_2} = 1$, so $a^2 = b^2$ by lemma 4.4 we have a = b, a contradiction. Hence n > 2.

In general, suppose n = N, and relabel as necessary to get $g_1 = j_1$, $g_2 = j_3$, ..., $g_{N-1} = j_N$, $g_N = j_1$. As before, this means $\frac{a}{b} = \frac{h_2}{h_1} = \frac{h_3}{h_2} = \ldots = \frac{h_N}{h_{N-1}} = \frac{h_1}{h_N}$

Then $\left(\frac{a}{b}\right)^N = \frac{h_2}{h_1} \circ \frac{h_3}{h_2} \circ \ldots \circ \frac{h_N}{h_{N-1}} \circ \frac{h_1}{h_N} = 1$, so $\frac{a}{b} = 1$, so a =b, a contradiction.

Hence there is no natural number n, so $\frac{1}{a+b}$ is not in the span of H, thus H cannot be a Hamel basis for K over F if H is multiplicatively closed. ■

A Generalization of a Closure Property

We now give a shorter proof of a generalization of Mabry's results.

Theorem 5.1: Let S be a free R-module, and H a Hamel basis of S over R. If there exists an $s \in S$ such that s - 1 is invertible, then $Hs \not\subseteq H$.

Proof: Let $g: S \rightarrow R$ be the Hamel function, defined by $g(t) = \sum_{i=1}^{n} r_i$, where $t = \sum_{i=1}^{n} r_i h_i$ is the Hamel representation of t. For all $h \in H$, we see g(h) = 1.

Suppose $Hs \subseteq H$. Then for all $h \in H$, we have $hs \in H$, so for all $t \in S$, $t = \sum_{i=1}^{n} r_i h_i$, we have $g(ts) = g\left(\left[\sum_{i=1}^{n} r_i h_i\right] s\right) = g\left(\sum_{i=1}^{n} r_i [h_i s]\right) = \sum_{i=1}^{n} r_i = g(t)$. Also, g is clearly additive, with $g(t_1 + t_2) = g(t_1) + g(t_2)$.

Let $h \in H$, and let $j = h(s - 1)^{-1}$. Then h = j(s - 1) = js - j. We then have 1 = g(h) = g (js - j) = g(js) - g(j) = g(j) - g(j) = 0, a contradiction. Thus $Hs \not\subseteq H$. ■

Now we see Mabry's result is a special case of our Theorem 5.1:

Corollary 5.2: Let K be a field, F a proper subfield of K, and H a Hamel basis of K over F. Then H is not multiplicatively closed.

Proof: Let $h \in H$, $h \neq 1$. (Such a h exists since F is proper.) Then h-1 is invertible, so by Theorem 5.1, $Hh \not\subseteq H$. Hence H is not multiplicatively closed. ■

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Vita

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