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Covering maps in the uniform category

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Abstract

First, generalized uniform covering maps are classified in terms of subgroups of the uniform fundamental group. Hausdorff, locally uniform joinable, and chain connected covering spaces of a uniformly locally joinable chain connected space are classified in terms of closed subgroups of its uniform fundamental group. If the space is also semilocally simply uniform joinable, uniform covering spaces are classified in terms of all subgroups of its uniform fundamental group.

Next it is shown that the inverse limit of a strong Mittag-Leffler inverse system of Hausdorff uniform covering spaces is a generalized uniform covering space. The question of the converse is investigated. It is necessary for a generalized uniform covering map to have strong approximate uniqueness for it to be the inverse limit of uniform covering maps. It is shown that such a generalized uniform covering map is indeed the inverse limit of uniform covering maps. It is unknown whether all generalized uniform covering maps have strong approximate uniqueness.

The analog to the above characterization is investigated for regular generalized uniform covering maps, i.e., generalized uniform covering maps that are induced by group actions.

Finally, given a pointed 1-movable continuum represented as the inverse limit of its nerves, it is shown that connected pointed 1-movable generalized covering spaces are the same as the inverse limit of covering spaces of its nerves provided the inverse system of covering spaces satisfy certain properties. I thank Sergey Melikhov for suggesting to look at this issue.

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Chapter 1

Introduction

A covering map $p : C \to X$ is a continuous surjection such that for every $x \in X$ there is an evenly covered neighborhood U of x, i.e., $f^{-1}(U)$ is the disjoint union of open sets, each of which are mapped homeomorphically onto U. One can prove that covering maps have unique path lifting. This unique path lifting can then be used to prove the general lifting property of covering maps. A map $f : Z \to X$ from a path connected and locally path connected space Z has a lift $\tilde{f} : Z \to C$ if and only if $f_*(\pi_1(Z)) \subset p_*(\pi_1(C))$. Also, if the lift exists then it is unique.

Of particular interest are universal covering maps, i.e., covering maps $p : C \to X$ where C is simply connected. These are important because for such maps, $\pi_1(X, x_0)$ is in bijective correspondence with the subset $p^{-1}(x_0)$ of C.

This traditional covering space theory works well for spaces that behave nicely with respect to paths. The lifting property only works for maps from path connected and locally path connected spaces. Also, the existence of a universal covering space is only guaranteed for path connected, locally path connected, and semilocally simply connected spaces. For such spaces, the endpoint map $p : \widetilde{X} \to X$ where $\widetilde{X}$ is the set of fixed endpoint homotopy classes of paths in X starting at some basepoint is the universal covering map.

One approach to extend covering space theory to spaces that are not so nice is to drop the requirement of existence of evenly covered neighborhoods. Most if not all of the applications of covering space theory hold provided the map has unique path lifting. In particular the lifting property and the correspondence theorem mentioned above still hold. Fischer and Zastrow [5] take this approach by asking when $p : \widetilde{X} \to X$ has unique path lifting. In this case $\widetilde{X}$ is guaranteed to be simply connected. It is shown that, provided X is path connected, p has unique path lifting if the natural homomorphism $\pi_1(X) \to \check{\pi}_1(X)$ from the fundamental group of X to the first shape group of X is a monomorphism.

Another approach is to drop the consideration of paths altogether. One can shift the focus from paths to chains, the discrete version of paths. Chains cannot be effectively defined for arbitrary topological spaces but can be defined for metric spaces. In a metric space, an ϵ-chain can be defined as a finite sequence $x_0, \ldots, x_n$ where $d(x_i, x_{i+1}) \leq \epsilon$ for each $0 \leq i < n$. However, chains can be effectively defined for a wider class of spaces called uniform spaces. This approach is taken by Berestovskii and Plaut [1]. A (generalized) covering map is defined as a projection $X \to X/G$ where G is a particular type of group action on X. A (generalized) universal covering space is constructed for a class of spaces called coverable spaces. An analog to the fundamental group is defined as well as an analog to the lifting property.

In [4] we extend the ideas of Berestovskii and Plaut by providing a definition for generalized uniform covering maps that does not rely on group actions. We provide an alternate approach to defining the generalized universal covering space introduced by Berestovskii and Plaut and prove that this space is indeed universal in a particular category. In addition, we relate elements of this generalized universal covering space to generalized paths of Krasinkiewicz and Minc [7] and prove that the generalized fundamental group introduced by Berestovskii and Plaut is isomorphic to the first shape group for metric continua.

In the same paper we also define uniform covering maps which were introduced by James [6].

This report classifies uniform covering maps and generalized uniform covering maps in terms of subgroups of the uniform fundamental group. It also explores characterizations of generalized uniform covering maps as inverse limits of uniform covering maps.

Chapter 2

Previous and preliminary results

2.1 Uniform spaces

We deal with uniform spaces because they are the widest class of spaces for which chains can be effectively defined. Most of the following definitions and facts can be found in [2].

A uniform structure on X is a family $\mathcal{E}$ of subsets of $X \times X$ called entourages such that

- $\mathcal{E}$ forms a filter under set inclusion,
- each $E \in \mathcal{E}$ contains the diagonal of $X \times X$,
- for each $E \in \mathcal{E}$, its transpose $E^T = \{(x, y) : (y, x) \in E\} \in \mathcal{E}$,
  and
- for each $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ so that $F^2 \subset E$ (F^2 consists of pairs $(x, z) \in X \times X$ such that there is $y \in X$ with $(x, y) \in F$ and $(y, z) \in F$).

A **basis** for a uniform structure is a family $\mathcal{B}$ of subsets of $X \times X$ such that

- each $E \in \mathcal{B}$ contains the diagonal of $X \times X$,
- each $E \in \mathcal{B}$ is symmetric (that means $(x, y) \in E$ implies $(y, x) \in E$),
- for each $E, F \in \mathcal{B}$ there is an $H \in \mathcal{B}$ so that $H \subset E \cap F$,
  and
- for each $E \in \mathcal{B}$ there is an $F \in \mathcal{B}$ with $F^2 \subset E$.

A uniform structure is generated from a basis by including all supersets.

Every uniform space is a topological space. Given a uniform space X, the topology induced by the uniform structure is defined by taking U to be open if for each $x \in U$ there is an entourage E of X so that $B(x, E) \subset U$, where $B(x, E) = \{y : (x, y) \in E\}$. Given a topological space X, a uniform structure on X is said to be compatible with the topology if it induces that topology. A topological space is uniformizable if it has a uniform structure that is compatible with the topology.

Every metric space is uniformizable. Given a metric space X, for each $\epsilon > 0$ define $E_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. Then $\{E_\epsilon : \epsilon > 0\}$ forms a basis for a uniform structure on X that is compatible with the topology. Every topological group is uniformizable. Given

a topological group G, for each neighborhood U of the identity $e \in G$ define $U_d = \{(x, y) \in G \times G : yx^{-1} \in U\}$. Then $\{U_d : U \text{ is a neighborhood of } e\}$ forms a basis for a uniform structure on G that is compatible with the topology. Any compact space X has exactly one compatible uniform structure, the one consisting of all neighborhoods of the diagonal in $X \times X$. Finally, a topological space is uniformizable if and only if it is completely regular.

Given a function $f : X \to Y$ between uniform spaces and an entourage E of X, $f(E)$ is notation for $(f \times f)(E)$ [10]. Similarly, for an entourage E of Y, $f^{-1}(E)$ is notation for $(f \times f)^{-1}(E)$. The function f is uniformly continuous if $f^{-1}(E)$ is an entourage of X for each entourage E of Y.

Given a function $f : X \to Y$ with X a uniform space, the function generates a uniform structure on Y [4] if the family $\{f(E) : E \text{ is an entourage of } X\}$ forms a basis for a uniform structure on Y. Notice the function must be surjective. If Y is given the uniform structure generated by f then f is necessarily uniformly continuous.

Normally we are given a function $f : X \to Y$ between uniform spaces. In this case we say f generates the uniform structure on Y if the same family $\{f(E) : E \text{ is an entourage of } X\}$ forms a basis for the uniform structure on Y. Notice f generates the uniform structure on Y if and only if f is uniformly continuous and $f(E)$ is an entourage of Y for each entourage E of X. Such a function is called bi-uniformly continuous in [1].

A uniform space is Hausdorff if and only if the intersection of all entourages is the diagonal. A uniform space is complete if and only if every Cauchy filter has a limit point. A filter is Cauchy if for each entourage E there is an E-bounded member of the filter. Note a subset A of a uniform space X is called E-bounded if for each $x, y \in A$, $(x, y) \in E$. Equivalently, $A \times A \subset E$. A point y is a limit point of a filter if for each entourage E of X there is a member A of the filter with $A \subset B(x, E)$.

Given an inverse system of uniform spaces $\{X_\alpha, \phi_{\beta\alpha}\}$, a basis for the inverse limit uniform structure on $\varprojlim X_\alpha$ is $\{\pi_\alpha^{-1}(E_\alpha)\}$ where α runs through the inverse system and E_α runs through a basis of entourages of X_α.

2.2 Uniform covering maps

Uniform covering maps were introduced by James [6]. The definition below is from [4] and is equivalent to James' definition. [1]

Suppose X is a uniform space. Given an entourage E of X, an ***E*-chain** in X is a finite sequence $x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in E$ for each $i \leq n$. Inverses and concatenations of E-chains are defined in the obvious way. X is called **chain connected** if for each entourage E of X and any $x, y \in X$ there is an E-chain starting at x and ending at y.

A surjective function $f : X \to Y$ from a uniform space X has **chain lifting** if for every entourage E of X there is an entourage F of X so that for any $x \in X$, any $f(F)$-chain in Y starting at $f(x)$ can be lifted to an E-chain in X starting at x.

A function $f : X \to Y$ from a uniform space X has **uniqueness of chain lifts** if for every entourage E of X there is an entourage $F \subset E$ so that any two F-chains in X starting at the same point with identical images must be equal. The function has **unique chain lifting** if it has both chain lifting and uniqueness of chain lifts.

[1] James fails to explicitly assume that a uniform covering map should generate the uniform structure on its image. It appears that he may have meant to do so implicitly, and in this case the definitions are equivalent.

Definition 2.2.1. *A function $f : X \to Y$ between uniform spaces is a uniform covering map if it generates the uniform structure on Y and has unique chain lifting.*

James uses the concept of uniform openness.

Definition 2.2.2. *[6] A function $f : X \to Y$ is uniformly open if for each entourage E of X there is an entourage F of Y such that for if $(f(x), y) \in F$, there is a $y' \in X$ with $f(y') = y$ and $(x, y) \in E$.*

Lemma 2.2.3. *A function $f : X \to Y$ between uniform spaces is uniformly continuous and uniformly open if and only if it generates the uniform structure on Y and has chain lifting.*

Proof. This lemma is essentially from [4]. It is easy to see that uniform continuity plus uniform openness imply that the function generates the uniform structure on Y. □

We used $f(F)$ instead of an entourage F of Y in the definition of chain lifting so that we could talk about chain lifting without needing f to generate the uniform structure on Y or needing Y to have a uniform structure at all.

James also uses the concept of a transverse entourage.

Definition 2.2.4. *[6] An entourage E_0 of X is transverse to f if $(x, y) \in E_0$ with $f(x) = f(y)$ implies $x = y$.*

Lemma 2.2.5. *[4] A map $f : X \to Y$ between uniform spaces has uniqueness of chain lifts if and only if there is an entourage of X that is transverse to f.*

Then we have the James' characterization of uniform covering maps.

Proposition 2.2.6. *A map $f : X \to Y$ is a uniform covering map if and only if*

1. *f is surjective,*
2. *f is uniformly continuous,*
3. *f is uniformly open,*

   and

4. *there exists an entourage transverse to f.*

The following characterization relates uniform covering maps to topological covering maps.

Proposition 2.2.7. *A map $f : X \to Y$ is a uniform covering map if and only if it generates the uniform structure on Y and there is a basis for the uniform structure consisting of entourages E that evenly cover $f(E)$.*

We say an entourage E **evenly covers** $f(E)$ if $B(x, E)$ is mapped bijectively onto $B(f(x), f(E))$.

An analog to the classical theorem of existence of a universal covering map is proved in [4]. Therefore analogs to local path connectedness and semi-local simply connectedness are defined.

Definition 2.2.8. *A uniform space X is **uniformly locally path connected** if for each entourage E of X there is an entourage F of X such that if $(x, y) \in F$, x and y can be joined by an E-bounded path.*

Definition 2.2.9. *A uniform space is* ***uniformly semilocally simply connected*** *if there exists an entourage E_0 of X such that for each $x \in X$, any E_0-bounded loop in X at x is null homotopic.*

As in the classical case, the universal covering space of a path connected, uniformly locally path connected, and uniformly semilocally simply connected space X is $\widetilde{X}$, the space of homotopy equivalence classes of paths. Give $\tilde{X}$ a uniform structure as follows. Given an entourage E of X, define an entourage E^* to be pairs of equivalence classes of paths $([\alpha], [\beta])$ such that $\alpha^{-1}\beta$ is homotopic to an E-bounded path.

Theorem 2.2.10. *[4] The endpoint map $\widetilde{X} \to X$ is a uniform covering map if and only if X is path connected, uniformly locally path connected, and uniformly semilocally simply connected.*

2.3 Generalized uniform covering maps

The idea behind the definition of generalized uniform covering maps is to find a covering space theory for spaces that are not locally path connected and semilocally simply connected. Since the use of paths is somewhat unsatisfactory for such spaces, a natural strategy is to pass one's attention from paths to chains which are the discrete version of paths. We thank Conrad Plaut for introducing us to the concept of chains in a uniform space. More on Berestovskii and Plaut's work and the relationship to our theory is discussed in Chapter 5.

Like in the setting of paths, we wish to have homotopies of chains. Homotopies between chains were successfully defined in [1]. The following is an equivalent definition from [4] that relies on homotopies already defined for paths. It utilizes Rips complexes which are a fundamental tool for studying chains in a uniform space.

Definition 2.3.1. *Given an entourage E of X the Rips complex $R(X, E)$ is the subcomplex of the full complex over X whose simplices are finite E-bounded subsets of X.*

Rips complexes are also used as the definition for uniform covering maps in [4]. We list it as a proposition.

Proposition 2.3.2. *A map $f : X \to Y$ is a uniform covering map if and only if it generates the uniform structure on Y and the family $\mathcal{E}$ of entourages of X such that the induced map $f_E : R(X, E) \to R(Y, f(E))$ is a simplicial covering map forms a basis of the uniform structure on X.*

The reason Rips complexes are so useful is because E-chains in X can be related to paths between vertices of $R(X, E)$. Any E-chain $x_1, \ldots, x_n$ determines a path in $R(X, E)$ by simply joining successive terms x_i, x_{i+1} by an edge path, i.e., a path along the edge joining x_i and x_{i+1}. Denote the edge path between points x and y by $\boldsymbol{e(x, y)}$ (since only homotopy classes of paths will be considered any two such paths will be equivalent). Conversely, any path in $R(X, E)$ joining two vertices x and y can be realized, up to homotopy, as a concatenation of edge paths (see [11, Section 3.4]). Now homotopies between E-chains can be defined in terms of homotopies between paths in $R(X, E)$. Since the identity function $K_w \to K_m$, K a simplicial complex, from K equipped with the CW (weak) topology to K equipped with the metric topology is a homotopy equivalence (see [8, page 302]), it does not matter which topology is chosen for $R(X, E)$. Two E-chains starting at the same point

x and ending at the same point y are said to be ***E*-homotopic relative endpoints** if the corresponding paths in $R(X, E)$ are homotopic relative endpoints. Therefore E-chains and paths in $R(X, E)$ between vertices can be used interchangeably.

In [1], an E-homotopy between E-chains c and d is defined as a finite sequence of E-chains $c_0, \ldots, c_n$ where $c_1 = c$ and $c_n = d$, each c_i has the same endpoints as c and d, and c_{i+1} is obtained from c_i by either adding or deleting a link from c_i. It is easy to see that this definition is equivalent to the above formulation. The above formulation has the advantage that it relies on the well understood concept of path homotopies. However, to show in practice that two chains are homotopic one often ends up using the definition from [1].

We wish to consider finer and finer chains in a space and therefore come to the concept of generalized paths. Equivalent objects were studied in [1]. They are a generalization of generalized paths introduced by Krasinkiewicz and Minc [7].

A **generalized path** is a collection of homotopy classes of chains $\alpha = \{[\alpha_E]\}_E$ where E runs over all entourages of X and for any $F \subset E$, α_F is E-homotopic relative endpoints to α_E. Inverses and concatenations of generalized paths are defined in the obvious way. The set of generalized paths in X starting at x_0 is denoted as $GP(X, x_0)$ and given a uniform structure generated by basic entourages defined as follows. For each entourage E of X let E^* be the set of all pairs (α, β), $\alpha, \beta \in GP(X, x_0)$, such that $\alpha^{-1}\beta$ is E-homotopic to the chain x, y where x is the endpoint of α and y is the endpoint of β. Call such a generalized path **E-short**.

Analogously to the way the fundamental group is defined, define the group $\check{\pi}_1(X, x_0)$ to be the group of all generalized loops based at x_0. It is isomorphic to $\varprojlim(\pi_1(R(X, E), x_0))$. Also, as the notation suggests, $\check{\pi}_1(X, x_0)$ is isomorphic to the first shape group for metric compacta.

Given a uniformly continuous function $f : X \to Y$, there is an induced function $f_* : GP(X, x_0) \to GP(Y, y_0)$. Suppose $\alpha = \{[c_E]_E\}_E \in GP(X, x_0)$. Given an entourage E of Y, let $F = f^{-1}(E)$ and define $f_*(\alpha)_E = [f(c_F)]_E$.

More definitions are needed in order to define a generalized uniform covering map. Suppose $f : X \to Y$ is a function between uniform spaces. This function has **approximate uniqueness of chain lifts** if for each entourage E of X there is an entourage $F \subset E$ such that any two F-chains that start at the same point and have identical images under f are E-close. Two chains $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are ***E*-close** if $(x_i, y_i) \in E$ for each $i \leq n$. The function f has **generalized path lifting** if for any $x \in X$, any generalized path starting at $f(x)$ lifts to a generalized path starting at x.

Definition 2.3.3. *A map $f : X \to Y$ is a generalized uniform covering map if:*

- *f generates the uniform structure on Y,*
- *f has chain lifting,*
- *f has approximate uniqueness of chain lifts,*
  and
- *f has generalized path lifting.*

The definition from [4] includes a useful property that is later proved to be redundant.

Lemma 2.3.4. *Suppose $f : X \to Y$ is a generalized uniform covering map. Then for each entourage E of X there is an entourage F of Y so that any two generalized paths in X starting at the same point are E^*-close if their images under f_* are F^*-close. In particular, F-short generalized paths in Y lift to E-short generalized paths in X.*

This property can be used to show that a generalized uniform covering map has uniqueness of generalized path lifts provided X is Hausdorff. This result is proved for a special case in [4].

Proposition 2.3.5. *Suppose $f : X \to Y$ is a generalized uniform covering map with X Hausdorff. Then f has uniqueness of generalized path lifts.*

Proof. Suppose two generalized paths α and β in X start at the same point and have $f_*(\alpha) = f_*(\beta)$. Given an entourage E of X, by 2.3.4, α and β are E^*-close. Then the endpoints of α and β are E-close. Since X is Hausdorff this implies that the endpoints are equal. Then $\alpha = \beta$ since α_E and β_E are E-homotopic rel. endpoints for each entourage E of X. □

In fact if Y is Hausdorff, then a generalized uniform covering map $f : X \to Y$ having unique generalized path lifting is equivalent to X being Hausdorff.

Proposition 2.3.6. *Suppose $f : X \to Y$ is a generalized uniform covering map with Y Hausdorff. If f has uniqueness of generalized path lifts then X is Hausdorff.*

Proof. Suppose X is not Hausdorff. Then there is $x, y \in X$ with $(x, y) \in E$ for each entourage E of X but $x \neq y$. Consider the constant generalized path at x and the generalized path $\{[x, y]_E\}_E$. They are not equal since they have different endpoints but they are both lifts of the constant generalized path at $f(x)$. Indeed, $f(x) = f(y)$ since $(f(x), f(y)) \in f(E)$ for each entourage E of X and f generates the uniform structure on Y. □

The requirement that a generalized uniform covering map have generalized path lifting is a general condition that we wish to be satisfied. It is automatically satisfied in the following case.

Proposition 2.3.7. *[4] A map $f : X \to Y$ between uniform spaces with complete fibers is a generalized uniform covering map if it generates the uniform structure on Y and has chain lifting and approximate uniqueness of chain lifting.*

Conditions for the endpoint map $GP(X, x_0) \to X$ to be a generalized uniform covering map are introduced in [4].

Definition 2.3.8. *A uniform space X is uniform joinable if any two points in X can be joined by a generalized path.*

Definition 2.3.9. *A uniform space X is locally uniform joinable if for each entourage E of X there is an entourage $F \subset E$ such that if $(x, y) \in F$, x and y can be joined by an E-short generalized path.*

It is easy to see that if X is locally uniform joinable and chain connected then X is uniform joinable.

Proposition 2.3.10. *[4] The endpoint map $GP(X, x_0) \to X$ is a generalized uniform covering map if and only if X is locally uniform joinable chain connected.*

In particular there is the following equivalence.

Proposition 2.3.11. *Suppose X is chain connected. Then the endpoint map $GP(X, x_0) \to X$ generates the uniform structure on X if and only if X is locally uniform joinable.*

2.4 Additional basic results

Proposition 2.4.1. *[4] If X is locally uniform joinable, then $GP(X)$ is locally uniform joinable chain connected.*

Proposition 2.4.2. *Suppose X is Hausdorff. Then $GP(X)$ is Hausdorff.*

Proof. Suppose $\alpha, \beta \in GP(X)$ with $(\alpha, \beta) \in E^*$ for each entourage E of X. Then $(x, y) \in E$ for each entourage E of X where x and y are the endpoints of α and β. Since X is Hausdorff $x = y$. Then α_E is E-homotopic to β_E for each entourage E of X so $\alpha = \beta$. □

Lemma 2.4.3. *[4] If $f : X \to Y$ generates the uniform structure on Y and X is locally uniform joinable then Y is locally uniform joinable.*

Lemma 2.4.4. *[4] If $f : X \to Y$ is uniformly continuous and surjective and X is chain connected then Y is chain connected.*

Proposition 2.4.5. *[4] Suppose X is path connected, uniformly locally path connected, and uniformly semilocally simply connected. Then the natural map $\widetilde{X} \to GP(X)$ is a uniform equivalence.*

Proposition 2.4.6. *[4] If X is locally uniform joinable then $\check{\pi}_1(GP(X)) = 1$.*

In Chapter 7, pointed 1-movable continua are considered. The following equivalence was proved in [4].

Proposition 2.4.7. *[4] The following are equivalent for a metric continuum X.*

1. *X is uniform joinable.*
2. *X is locally uniform joinable.*
3. *X is pointed 1-movable.*

2.5 Group actions

"Group Actions and Covering Maps in the Uniform Category" [3] considered group actions that induce uniform covering maps and generalized uniform covering maps. Unless otherwise noted, the following definitions and results are from that paper. Suppose a group G acts on a uniform space X. The action is **neutral** [6] if for each entourage E of X there is an entourage F of X such that if $(x, gy) \in F$ there is an $h \in G$ with $(hx, y) \in E$. Note that if G acts neutrally on X, then the projection $p : X \to X/G$ has chain lifting and generates a uniform structure on X/G. This is the structure that will always be considered. The action is **uniformly properly discontinuous** if there is an entourage E_0 of X such that if $(x, gx) \in E_0$ for some $x \in X$ then $g = 1$. Given an entourage E of X define $\boldsymbol{G_E}$ to be the subgroup generated by $\{g \in G : (x, gx) \in E \text{ for some } x \in X\}$ [1]. The action is **small scale uniformly continuous** if for each entourage E of X there is an entourage F of X so

that for each $g \in G_F$, $g^{-1}(E)$ is an entourage of X. The action is **small scale uniformly equicontinuous** if for each entourage E of X there is an entourage F of X such that for each $g \in G_F$, $F \subset g^{-1}(E)$. The action is **uniformly equicontinuous** [6] if for each entourage E of X there is an entourage F of X such that for each $g \in G$, $F \subset g^{-1}(E)$. Equivalently, X has a basis of G-invariant entourages. An entourage E is **G-invariant** if for each $g \in G$, $gE = E$. Finally, the action has **small scale bounded orbits** if for each entourage E of X there is an entourage F of X such that the action of G_F on X has E-bounded orbits.

Proposition 2.5.1. *Suppose G is a group acting on a uniform space X. If the action is neutral and uniformly properly discontinuous then the projection $p : X \to X/G$ is a uniform covering map.*

Proposition 2.5.2. *Suppose G is a group acting faithfully on a chain connected uniform space X. The action is small scale uniformly continuous and the projection $p : X \to X/G$ is a uniform covering map if and only if the action is neutral and uniformly properly discontinuous.*

Proposition 2.5.3. *Suppose G acts on a uniform space X. Suppose the action is neutral and has small scale bounded orbits. Suppose also that the projection $p : X \to X/G$ has complete fibers. Then p is a generalized uniform covering map*

Proposition 2.5.4. *Suppose G is a group acting on a chain connected uniform space X. Suppose that the projection $p : X \to X/G$ has complete fibers. Then the action is small scale uniformly equicontinuous and p is a generalized uniform covering map if and only if the action is neutral and has small scale bounded orbits.*

Proposition 2.5.5. *Suppose a group G acts faithfully on a uniform Hausdorff space X. If the action has small scale bounded orbits, then the action is free.*

2.6 The lim one functor

Given an inverse sequence of groups $\{G_i, \psi_i\}$, consider the function $f : \Pi G_i \to \Pi G_i$ that sends (g_i) to $(g_i \psi_{i+1}(g_{i+1})^{-1})$. Then $\varprojlim G_i$ and $\varprojlim^1 G_i$ can be defined as sets for which $1 \to \varprojlim G_i \hookrightarrow \Pi G_i \xrightarrow{f} \Pi G_i \hookrightarrow \varprojlim^1 G_i \to 1$ is an exact sequence. Of importance to us is the case when $\varprojlim^1 G_i = 1$. Notice $\varprojlim^1 G_i = 1$ if and only if f is surjective. Therefore we have the following.

Lemma 2.6.1. *Given an inverse sequence of groups $\{G_i, \psi_i\}$, $\varprojlim^1 G_i = 1$ if and only if for each sequence $(g_i) \in \Pi G_i$, there is a sequence $(h_i) \in \Pi G_i$ with $g_i = h_i \psi_{i+1}(h_{i+1})^{-1}$ for all $i \geq 1$.*

Notice if each ψ_i is surjective then $\varprojlim^1 G_i = 1$. Indeed, given $(g_i) \in \Pi G_i$, set $h_1 = 1$. Then for each $i \geq 1$ choose h_{i+1} so that $\psi_{i+1}(h_{i+1}) = g_i^{-1} h_i$.

It is helpful to also have the following.

Lemma 2.6.2. *If $\varprojlim^1 G_i = 1$ then for each sequence $(g_i) \in \Pi G_i$, there is a sequence $(h_i) \in \Pi G_i$ with $g_i = \psi_{i+1}(h_{i+1})^{-1} h_i$ for all $i \geq 1$.*

Proof. Given the sequence (g_i), there is a sequence (k_i) with $g_i^{-1} = k_i \psi_{i+1}^{-1}(k_{i+1})$ for each i. Then $g_i = \psi_{i+1}(k_{i+1})k_i^{-1}$. Set $h_i = k_i^{-1}$. Then $g_i = \psi_{i+1}(h_{i+1}^{-1})h_i = \psi_{i+1}(h_{i+1})^{-1}h_i$. □

Chapter 3

Generalized uniform covering maps relative to subgroups of the uniform fundamental group

The existence of a universal generalized uniform covering map for a locally uniform joinable chain connected space was treated in [4]. The next natural step is to extend this result to match the corresponding result in the classical setting. In this setting, one not only has a theorem about the existence of a universal covering map but a theorem giving the existence of a covering map corresponding to any subgroup of the fundamental group of the base space. This chapter investigates an analogous result for generalized uniform covering maps. In addition, a classification theorem for generalized uniform covering maps over a space X in terms of subgroups of $\check{\pi}_1(X)$ is given.

3.1 Existence of generalized uniform covering maps

Given a subgroup H of $\check{\pi}_1(X)$, we wish to see what conditions imply the existence of a generalized uniform covering map $p_H : Z \to X$ with $p_*(\check{\pi}_1(Z)) = H$. First we will consider when a generalized uniform covering map exists relative to a subgroup H of $\check{\pi}_1(X)$. The covering space will be constructed as the quotient of the action of H on $GP(X)$. Given $h \in H$ and $\alpha \in GP(X)$, let $h \cdot \alpha = h\alpha$. Let $q_H : GP(X) \to GP(X)/H$ be the induced projection. Given $\alpha \in GP(X)$, let $[\alpha]_H$ denote the orbit of α under the action.

Lemma 3.1.1. *Given a subgroup H of $\check{\pi}_1(X)$, the action of H on $GP(X)$ is uniformly equicontinuous. In particular, given an entourage E of X, the entourage E^* of $GP(X)$ is invariant under the action of H.*

Proof. Let E be an entourage of X. Suppose $(\alpha, \beta) \in E^*$ and $h \in H$. Then $(h\alpha)^{-1}(h\beta) = \alpha^{-1}h^{-1}h\beta = \alpha^{-1}\beta$ is E-short so $(h \cdot \alpha, h \cdot \alpha) \in E^*$. □

Since equicontinuous actions are neutral, q_H has chain lifting and generates a uniform structure on $GP(X)/H$. The following lemma gives that the endpoint map $p_H : GP(X)/H \to X$ has approximate uniqueness of chain lifts provided that X is locally uniform joinable chain connected.

Lemma 3.1.2. *Suppose $f : X \to Y$ generates the uniform structure on Y and $g : Y \to Z$ is any function between uniform spaces.*

1. *If $g \circ f$ generates the uniform structure on Z then g generates the uniform structure on Z.*
2. *If f has chain lifting and $g \circ f$ has approximate uniqueness of chain lifts then g has approximate uniqueness of chain lifts.*
3. *If $g \circ f$ has uniqueness of chain lifts then g has uniqueness of chain lifts.*
4. *If $g \circ f$ has chain lifting then g has chain lifting.*

Proof.

1. Given an entourage E of Y, $g \circ f(f^{-1}(E)) \subset g(E)$ and $g \circ f(f^{-1}(E))$ is an entourage of Z.
2. Given an entourage E of Y, choose an entourage $F \subset f^{-1}(E)$ so that two F-chains starting at the same point are $f^{-1}(E)$-close if their images under $g \circ f$ are identical. Choose an entourage $K \subset E$ so that K-chains lift to F-chains. Suppose two K-chains start at a point $y \in Y$ have identical images under g. Choose $x \in f^{-1}(y)$ and lift the two chains to F-chains in X starting at x. These chains have identical images under $g \circ f$ so they are $f^{-1}(E)$-close. Therefore the K-chains are E-close.
3. Suppose E_0 is an entourage of X that is transverse to $g \circ f$. Then $f(E_0)$ is an entourage of Y that is transverse to g.
4. Given an entourage E of Y, choose an entourage $F \subset f^{-1}(E)$ so that $g \circ f(F)$-chains lift to $f^{-1}(E)$-chains. Notice $f(F) \subset E$. Let us see that $g(f(F))$-chains in Z lift to E-chains in Y. Suppose $x \in Y$ and $(g(x), y) \in g(f(F))$. Choose $x' \in f^{-1}(x)$. Then there is a $y' \in X$ with $g \circ f(y') = y$ and $(x', y') \in f^{-1}(E)$. Notice $(f(x'), f(y')) = (x, f(y')) \in E$.

□

Proposition 3.1.3. *Suppose X is a uniform space and H is a subgroup of $\breve{\pi}_1(X)$. Then X is locally uniform joinable chain connected if and only if the endpoint map $p_H : GP(X)/H \to X$ is a generalized uniform covering map.*

Proof. Suppose X is uniform joinable chain connected. Since $p_H \circ q_H = p$ where $p : GP(X) \to X$ is the endpoint map, according to 3.1.2, p_H has chain lifting. Also according to 3.1.2, p_H has approximate uniqueness of chain lifts. Finally, since p has generalized path lifting, p_H has generalized path lifting.

Now suppose p_H is a generalized uniform covering map. In particular it generates the uniform structure on X so $p = p_H \circ q_H$ generates the uniform structure on X. Recall p generates the uniform structure on X if and only if X is uniform joinable chain connected (2.3.11). □

We wish to have that $p_{H*}(\check{\pi}_1(GP(X)/H, [\alpha_0]_H)) = H$ where α_0 is the constant generalized path at the basepoint of X. For this proof we need p_H to have unique generalized path lifting. Recall that a generalized uniform covering map $f : X \to Y$ has unique generalized path lifting if X is Hausdorff (2.3.5).

Lemma 3.1.4. *Suppose X is Hausdorff. Then $GP(X)/H$ is Hausdorff if and only if H is a closed subgroup of $\check{\pi}_1(X)$.*

Proof. It is easy to see that for a uniform space X and a subset $A \subset X$, $x \in X$ is in the closure of A if and only if $B(x, E) \cap A$ is nonempty for each entourage E of X.

Given that X is Hausdorff, $GP(X)/H$ being Hausdorff means that if $\alpha, \beta \in GP(X)$ end at the same point and for each entourage E of X there is a $\gamma(E) \in H$ with $(\alpha, \gamma(E)\beta) \in E^*$, then $\alpha\beta^{-1} \in H$. Now H being a closed subgroup of $\check{\pi}_1(X)$ means that if $\lambda \in \check{\pi}_1(X)$ and for each entourage E of X there is a $\gamma(E) \in H$ with $(\lambda, \gamma(E)) \in E^*$, then $\lambda \in H$. Since E^* is invariant under the action of H, taking $\lambda = \alpha\beta^{-1}$ shows the equivalence. □

The following indicates that if X is Hausdorff then H being closed in $\check{\pi}_1(X)$ is equivalent to H being closed in $GP(X)$.

Lemma 3.1.5. *If X is Hausdorff then $\check{\pi}_1(X)$ is closed in $GP(X)$.*

Proof. Suppose $\alpha \in GP(X)$ and for each entourage E of X there is a $\gamma(E) \in \check{\pi}_1(X)$ with $(\alpha, \gamma(E)) \in E^*$. Let x be the endpoint of α and x_0 be the basepoint of X. Then $(x, x_0) \in E$ for each entourage E of X. Since X is Hausdorff $x = x_0$ and $\alpha \in \check{\pi}_1(X)$. □

Proposition 3.1.6. *Suppose X is Hausdorff and H is a closed subgroup of $\check{\pi}_1(X)$. Suppose p_H is a generalized uniform covering map. Then $p_{H*}(\check{\pi}_1(GP(X)/H, [\alpha_0]_H)) = H$.*

Proof. By the previous lemma $GP(X)/H$ is Hausdorff. Therefore p_H has unique generalized path lifting. Suppose $\alpha \in \check{\pi}_1(X)$. Now $\alpha \in p_{H*}(\check{\pi}_1(GP(X)/H, [\alpha_0]_H))$ if and only if α lifts to a generalized loop $\widetilde{\alpha}_H$ in $GP(X)/H$. In that case $\widetilde{\alpha}_H(1) = [\alpha_0]_H$. But α lifts to a generalized path $\widetilde{\alpha}$ in $GP(X)$ and $\widetilde{\alpha}(1) = \alpha$. Then $q_{H*}(\widetilde{\alpha})$ is another lift of α in $GP(X)/H$ so $q_{H*}(\widetilde{\alpha}) = \widetilde{\alpha}_H$ and $\widetilde{\alpha}_H(1) = [\alpha]_H$. Therefore $[\alpha]_H = [\alpha_0]_H$ so $\alpha \in H$. □

Theorem 3.1.7. *Suppose a uniform space X is locally uniform joinable chain connected. Then for each closed subgroup H of $\check{\pi}_1(X)$ there is a Hausdorff, locally uniform joinable, and chain connected space Z and a generalized uniform covering map $p_H : Z \to X$ such that $p_{H*}(\check{\pi}_1(Z)) = H$.*

Proof. Accoring to 3.1.3, the endpoint map $p_H : GP(X)/H \to X$ is a generalized uniform covering map. Then by 3.1.6, $p_{H*}(\check{\pi}_1(GP(X)/H, [\alpha_0]_H)) = H$. Notice $GP(X)/H$ is Hausdorff by 3.1.4, locally uniform joinable by 2.4.3, and chain connected by 2.4.4. □

3.2 Equivalence of generalized uniform covering maps

Recall the following theorem from the classical setting of topological covering maps.

Theorem 3.2.1. *[9] Suppose $f : X \to Y$ and $g : Z \to Y$ are two covering maps. Suppose X, Y, and Z are path connected and locally path connected. Then f and g are equivalent if and only if the groups $f_*(\pi_1(X))$ and $g_*(\pi_1(Z))$ are conjugate in $\pi_1(Y)$.*

Note saying that f and g are equivalent means that there is a homeomorphism $h : X \to Z$ with $f = g \circ h$.

We wish to have an analog of this result for generalized uniform covering maps. The most important property of covering maps is the lifting lemma, and generalized uniform covering maps have this property.

Lemma 3.2.2. *[4] Suppose $f : X \to Y$ is a generalized uniform covering map and $g : Z \to Y$ is uniformly continuous. Suppose X is Hausdorff and Z is locally uniform joinable chain connected. Let $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ with $f(x_0) = g(z_0) = y_0$. Then there is a unique uniformly continuous lift $h : Z \to X$ of g with $h(z_0) = x_0$ if and only if $g_*(\check{\pi}_1(Z, z_0)) \subset f_*(\check{\pi}_1(X, x_0))$.*

Lemma 3.2.3. *Suppose $f : X \to Y$ is a map between uniform spaces with X uniform joinable. Let $y_0 \in Y$ and $x_0, x_1 \in f^{-1}(y_0)$. Then $f_*(\check{\pi}_1(X, x_0))$ and $f_*(\check{\pi}_1(X, x_1))$ are conjugate.*

Proof. This lemma is proved in the same manner that it is proved in the classical setting. Let α be a generalized path in X from x_0 to x_1. Consider the induced function $f_* : GP(X) \to GP(Y)$. First suppose γ_1 is a generalized loop in X at x_1. Then $f_*(\alpha)f_*(\gamma_1)f_*(\alpha)^{-1} = f_*(\alpha\gamma_1\alpha^{-1})$ so $f_*(\alpha)f_*(\check{\pi}_1(X, x_1))f_*(\alpha)^{-1} \subset f_*(\check{\pi}_1(X, x_0))$. Now suppose γ_0 is a generalized loop in X at x_0. Then $f_*(\gamma_0) = f_*(\alpha\alpha^{-1}\gamma_0\alpha\alpha^{-1}) = f_*(\alpha)f_*(\alpha^{-1}\gamma_1\alpha)f_*(\alpha)^{-1}$ so $f_*(\check{\pi}_1(X, x_0)) \subset f_*(\alpha)f_*(\check{\pi}_1(X, x_1))f_*(\alpha)^{-1}$. □

Lemma 3.2.4. *Suppose a map $f : X \to Y$ between uniform spaces has unique generalized path lifting and X is uniform joinable. Let $y_0 \in Y$ and $x_0 \in f^{-1}(y_0)$. Given a subgroup H of $\check{\pi}_1(Y, y_0)$ that is conjugate to $f_*(\check{\pi}_1(X, x_0))$, there is an $x_1 \in X$ so that $f_*(\check{\pi}_1(X, x_1)) = H$.*

Proof. Suppose $\alpha H\alpha^{-1} = f_*(\check{\pi}_1(X, x_0))$ for some $\alpha \in \check{\pi}_1(Y, y_0)$. Let $\widetilde{\alpha}$ be the lift of α in X and x_1 be the endpoint of $\widetilde{\alpha}$. By 3.2.3, $\alpha f_*(\check{\pi}_1(X, x_1))\alpha^{-1} = f_*(\check{\pi}_1(X, x_0))$. Therefore $f_*(\check{\pi}_1(X, x_1)) = H$. □

We say two maps $f : X \to Y$ and $g : Z \to Y$ are uniform equivalent if there is a uniform equivalence $h : X \to Z$ with $f = g \circ h$.

Theorem 3.2.5. *Suppose $f : X \to Y$ and $g : Z \to Y$ are two generalized uniform covering maps. Suppose X and Z are Hausdorff, locally uniform joinable, and chain connected. Let $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ with $f(x_0) = g(z_0) = y_0$. Then f and g are uniform equivalent if and only if the groups $f_*(\check{\pi}_1(X, x_0))$ and $g_*(\check{\pi}_1(Z, z_0))$ are conjugate in $\check{\pi}_1(Y, y_0)$.*

Proof. Suppose f and g are uniform equivalent with equivalence $h : X \to Z$ with $f = g \circ h$. Therefore $f_*(\check{\pi}_1(X, x_0)) = g_* \circ h_*(\check{\pi}_1(X, x_0)) = g_*(\check{\pi}_1(Z, h(x_0)))$. By 3.2.3 $g_*(\check{\pi}_1(Z, h(x_0)))$ is conjugate to $g_*(\check{\pi}_1(Z, z_0))$.

Now suppose the groups $f_*(\check{\pi}_1(X, x_0))$ and $g_*(\check{\pi}_1(Z, z_0))$ are conjugate. By 3.2.4 there is an $x_1 \in X$ with $f_*(\check{\pi}_1(X, x_1)) = g_*(\check{\pi}_1(Z, z_0))$. Then by 3.2.2 there is a lift $h : X \to Z$ of f with respect to the uniform covering map g and there is a lift $k : Z \to X$ of g with respect to the uniform covering map f. Now $h \circ k$ is the identity on X since it and the identity are both lifts of f with respect to itself. Similarly $k \circ h$ is the identity on Y. Therefore h is a uniform equivalence. □

3.3 Classifying generalized uniform covering maps

We wish to combine Sections 3.1 and 3.2 to classify generalized uniform covering maps of a locally uniform joinable chain connected space in terms of subgroups of its uniform fundamental group. We will only consider generalized uniform covering maps where the covering space is Hausdorff, locally uniform joinable, and chain connected.

Proposition 3.3.1. *Suppose $f : X \to Y$ is a generalized uniform covering map. Suppose X is Hausdorff. Let $x_0 \in X$ and set $y_0 = f(x_0)$. Then $f_*(\check{\pi}_1(X, x_0))$ is closed in $\check{\pi}_1(Y, y_0)$.*

Proof. Set $H = f(\check{\pi}_1(X, x_0))$. Suppose $\gamma \in \mathrm{Cl}(H)$. Lift γ to a generalized path $\tilde{\gamma}$ in X starting at x_0. It suffices to show that the endpoint of $\tilde{\gamma}$ is x_0. Given an entourage E of X, choose an entourage F of X so that if $\alpha, \beta \in GP(X)$ with $(f_*(\alpha), f_*(\beta)) \in F^*$, then $(\alpha, \beta) \in E^*$ (see 2.3.5). Now there is an $h \in H$ with $(\gamma, h) \in F^*$. Let $\tilde{h} \in \check{\pi}_1(X)$ with $f_*(\tilde{h}) = h$. Then $(\tilde{\gamma}, \tilde{h}) \in E^*$. In particular, $(x, x_0) \in E$ where x is the endpoint of $\tilde{\gamma}$. Since X is Hausdorff, $x = x_0$. □

Theorem 3.3.2. *Suppose X is Hausdorff, locally uniform joinable, and chain connected. Then there is a bijective correspondence between conjugacy classes of closed subgroups of $\check{\pi}_1(X)$ and generalized uniform covering maps over X with the covering space being Hausdorff, locally uniform joinable, and chain connected.*

Proof. Let $x_0 \in X$ and $f : Z \to X$ be a generalized uniform covering map where Z is Hausdorff, locally uniform joinable and chain connected. We identify f with the conjugacy class of the closed subgroup $H = f_*(\check{\pi}_1(Z, z_0)$ where $z_0 \in f^{-1}(x_0)$. This identification is well defined and bijective by 3.2.5. The identification is surjective by 3.1.7. □

Chapter 4

Uniform covering maps relative to subgroups of the uniform fundamental group

Now we wish to translate the results of Chapter 3 to the setting of uniform covering maps. The existence of a universal uniform covering map for a path connected, uniformly locally path connected, and uniformly semilocally simply connected space was treated in [4]. First we will define an analog to such spaces in the setting of uniform generalized paths and prove that $GP(X) \to X$ is a uniform covering map for such spaces. Then we will examine uniform covering maps relative subgroups of the uniform fundamental group and will classify uniform covering maps of a class of spaces in terms of those subgroups. Finally, we will classify uniform covering maps of path connected, uniformly locally path connected, and uniformly semilocally simply connected spaces.

4.1 Existence of uniform covering maps

We already know that the endpoint map $\widetilde{X} \to X$ is a uniform covering map if and only if X is path connected, uniformly locally path connected, and uniformly semilocally simply connected. Let us see when the endpoint map $GP(X) \to X$ is a uniform covering map.

Definition 4.1.1. *A uniform space is simply uniform joinable if every generalized loop is trivial.*

Equivalently, a space X is simply uniform joinable if $\check{\pi}_1(X, x_0) = 1$ for each $x_0 \in X$.

Definition 4.1.2. *A uniform space is semilocally simply uniform joinable if there is an entourage E so that a generalized loop is trivial if its E-term is trivial.*

Equivalently, a space X is semilocally simply uniform joinable if there is an entourage E so that the projection $\check{\pi}_1(X, x_0) \to \pi_1(R(X, E), x_0)$ is a monomorphism for all $x_0 \in X$.

Proposition 4.1.3. *Suppose $f : X \to Y$ is a generalized uniform covering map with X Hausdorff. If Y is semilocally simply uniform joinable then X is semilocally simply uniform joinable.*

Proof. Suppose Y is semilocally simply uniform joinable with entourage E so that any generalized loop is trivial if its E-term is trivial. Suppose c is a generalized loop at x_0 with $c_{f^{-1}(E)}$ trivial. Then $f(c)_E$ is trivial so $f(c)$ is trivial. But then c and the constant generalized path at x_0 are two lifts of the same generalized path so they must be equal since X is Hausdorff. □

Proposition 4.1.4. *The following are equivalent.*

1. *The projection $\pi_X : GP(X, x_0) \to X$ is a uniform covering map for some $x_0 \in X$.*
2. *The projection $\pi_X : GP(X, x_0) \to X$ is a uniform covering map for all $x_0 \in X$.*
3. *X is locally uniform joinable, chain connected, and semilocally simply uniform joinable.*

Proof. 1. $\implies$ 3. Since π_X is surjective, X is uniform joinable. Since π_X generates the structure on X, by 2.3.11, X is locally uniform joinable. To see that X is semilocally simply uniform joinable, let E_0^* be a basic transverse entourage of X. Suppose γ is a generalized loop in X at a point $x \in X$ whose E_0-term is trivial. Let α be a generalized path from x_0 to x. Then $(\alpha, \alpha\gamma) \in E_0^*$ and $\pi_X(\alpha) = \pi_X(\alpha\gamma)$ so $\alpha = \alpha\gamma$ and γ is trivial.

3. $\implies$ 2. Let $x_0 \in X$. By 2.3.11 $\pi_X : GP(X, x_0) \to X$ generates the structure on X (uniform joinability and local uniform joinability are used here). Since X is semilocally simply uniform joinable there is an entourage E_0 of X such that any generalized loop in X is trivial if its E_0-term is trivial. To see that E_0^* is transverse to π_X, suppose $(\alpha, \beta) \in E_0^*$ with $\pi_X(\alpha) = \pi_X(\beta)$. Then $\alpha^{-1}\beta$ is a generalized loop in X whose E-term is trivial so it is trivial. Therefore $\alpha = \beta$. Finally, to see that π_X has chain lifting, let E^* be a basic entourage of $GP(X, x_0)$ and let $F \subset E$ be an entourage of X so that any two $x, y \in X$ with $(x, y) \in F$ can be joined by an E-short generalized path. Suppose $\alpha \in GP(X, x_0)$ and $(\pi_X(\alpha), y) \in F$. Join $\pi_X(\alpha)$ and y by an E-short generalized path β. Then $(\alpha, \alpha\beta) \in E^*$. □

Now we wish to see that for a locally uniform joinable, chain connected, and semilocally simply uniform joinable space X, for each subgroup H of $\check{\pi}_1(X, x_0)$ there is a uniform covering map $p_H : Z \to X$ with $p_*(\check{\pi}_1(Z, z_0)) = H$.

Proposition 4.1.5. *If X is semilocally simply uniform joinable then every subgroup H of $\check{\pi}_1(X)$ is closed in $\check{\pi}_1(X)$.*

Proof. Suppose α is a generalized loop in X at the basepoint of X and for each entourage E of X there is an $h(E) \in H$ with $(\alpha, h(E)) \in E^*$. There is an entourage E_0 of X so that any generalized loop is trivial if its E_0-term is trivial. Therefore $\alpha h(E_0)^{-1}$ is trivial, i.e., $\alpha = h(E_0) \in H$. □

Theorem 4.1.6. *Suppose a uniform space X is Hausdorff, locally uniform joinable, chain connected, and semilocally simply uniform joinable. Suppose $x_0 \in X$. Then for each subgroup H of $\check{\pi}_1(X, x_0)$ there is a Hausdorff, locally uniform joinable, and chain connected space Z, a uniform covering map $p_H : Z \to X$ and a point $z_0 \in p_H^{-1}(x_0)$ such that $p_{H*}(\check{\pi}_1(Z, z_0)) = H$.*

Proof. Consider the endpoint map $p_H : GP(X, x_0)/H \to X$. According to 4.1.4, $\pi_X : GP(X, x_0) \to X$ is a uniform covering map. According to 3.1.1, $q_H : GP(X, x_0) \to GP(X, x_0)/H$ generates the uniform structure on $GP(X, x_0)/H$. Then, since $\pi_X = p_H \circ q_H$, p_H is a uniform covering map by 3.1.2. Notice $GP(X)/H$ is locally uniform joinable by 2.4.3, and chain connected by 2.4.4. Now H is closed by 4.1.5. Then $GP(X)/H$ is Hausdorff by 3.1.4. Also, $p_{H*}(\check{\pi}_1(GP(X, x_0), \alpha_0)) = H$ by 3.1.6. □

Now we consider the case of X being path connected, uniformly locally path connected, and uniformly semilocally simply connected. Existence of uniform covering maps follows immediately since in this case, $GP(X, x_0)$ is uniformly equivalent to $\widetilde{X}$.

Theorem 4.1.7. *Suppose X is Hausdorff, path connected, uniformly locally path connected, and uniformly semilocally simply connected. Suppose $x_0 \in X$. Then for each subgroup H of $\pi_1(X, x_0)$ there is a uniform covering map $p_H : Z \to X$ and a point $z_0 \in p_H^{-1}(x_0)$ such that $p_{H*}(\pi_1(Z, z_0)) = H$.*

Proof. By 2.4.5, the natural map from $\widetilde{X}$ to $GP(X, x_0)$ is a uniform equivalence. Then $GP(X, x_0) \to X$ is a uniform covering map which implies that X is locally uniform joinable, chain connected, and semilocally simply uniform joinable. Therefore the theorem follows from 4.1.6 since $\check{\pi}_1(X)$ is isomorphic to $\pi_1(X)$. □

The fact that if X is Hausdorff, path connected, uniformly locally path connected, and uniformly semilocally simply connected then X is semilocally simply uniform joinable is nontrivial since it is not true that if X is uniformly semilocally simply connected then X is semilocally simply uniform joinable, even for a path connected space (see 4.1.9 below). Since it is nontrivial, let us include it as a proposition.

Proposition 4.1.8. *If X is Hausdorff, path connected, uniformly locally path connected, and uniformly semilocally simply connected then X is semilocally simply uniform joinable.*

Of course the converse is not true. The Warsaw circle is locally uniform joinable, chain connected, and semilocally simply uniform joinable but not locally path connected.

Example 4.1.9. We will consider a subspace of the Hawaiian Earing to show that a path connected uniformly semilocally simply connected space need not be semilocally simply uniform joinable. Let the Hawaiian earring be the union of circles C_n of diameter $1/n$ in $\mathbb{R}^2$ with center $(0, 1/2n)$ for $n \in \mathbb{N}$. Obtain a space X by removing, for each n, the point of $(0, 1/n)$ from circle C_n. Notice X is path connected and uniformly semilocally simply connected (in fact X is simply connected). However, X is not semilocally simply uniform joinable. Given an entourage E of X, there is an ϵ so that the basic entourage $E_\epsilon = \{(x, y) : d(x, y) \leq \epsilon\} \subset E$. Then, if $n > 1/\epsilon$, the generalized path that runs around the circle C_n has trivial E_ϵ term (and therefore trivial E term) but is not trivial.

Similarly, it is not true that a simply connected space is necessarily simply joinable (consider the circle with one point removed). However, we do have the following.

Proposition 4.1.10. *If X is Hausdorff, path connected, uniformly locally path connected, and simply connected then X is simply uniform joinable.*

Proof. If X is Hausdorff, path connected, uniformly path connected, and simply connected then X is uniformly equivalent to $\widetilde{X}$. But X is path connected, uniformly locally path connected, and uniformly semilocally simply connected so $\widetilde{X}$ is uniformly equivalent to $GP(X)$. Since $GP(X)$ is simply uniform joinable 2.4.6, so is X. □

4.2 Classifying uniform covering maps

We wish to classify uniform covering maps with of locally uniform joinable, chain connected, and semilocally simply uniform joinable spaces. Section 3.2 was treated in terms of generalized uniform covering maps. These results can be used since uniform covering maps are generalized uniform covering maps.

We will only consider uniform covering maps where the covering space is chain connected. Note that we do not need to explicitly assume that the covering space is Hausdorff or locally uniform joinable since these properties are inherited via uniform covering maps.

Lemma 4.2.1. *Suppose $f : X \to Y$ is a uniform covering map.*

1. *If Y is Hausdorff then X is Hausdorff.*
2. *If Y is locally uniform joinable then X is locally uniform joinable.*

Proof.

1. Suppose $x, y \in X$ and $(x, y) \in E$ for each entourage E of X. Then $(f(x), f(y)) \in f(E)$ for each entourage E and f generates the uniform structure on Y so $f(x) = f(y)$. But there is an entourage of X that is transverse to f so $x = y$.
2. Given an entourage E of X, there is an entourage $F \subset E$ so that any two F-chains in X starting at the same point who have identical images must be equal. By 2.3.4 there is an entourage K of Y so that K-short uniform generalized paths lift to F-short uniform generalized paths. Since Y is locally uniform joinable there is an entourage D of Y so that if $(x, y) \in D$, x and y can be joined by an K-short uniform generalized path. Suppose $(x, y) \in f^{-1}(D) \cap F$. Then $(f(x), f(y)) \in D$ so $f(x)$ and $f(y)$ can be joined by an K-short uniform generalized path. This uniform generalized path lifts to an F-short uniform generalized path starting at x. But then the F-term of this uniform generalized path must be identical to x, y.

□

Theorem 4.2.2. *Suppose X is Hausdorff, locally uniform joinable, chain connected, and semilocally simply uniform joinable. Then there is a bijective correspondence between the the conjugacy classes of subgroups of $\check{\pi}_1(X)$ and uniform covering maps over X with the covering space being chain connected.*

Proof. Let $x_0 \in X$ and $f : Z \to X$ be a generalized uniform covering map where Z is chain connected. Notice Z is Hausdorff and locally uniform joinable by the previous lemma. We identify f with the conjugacy class of the subgroup $H = f_*(\check{\pi}_1(Z, z_0)$ where $z_0 \in f^{-1}(x_0)$. Notice this subgroup is closed by 4.1.5. This identification is well defined and bijective by 3.2.5. The identification is surjective by 4.1.6. □

Again, we have the corresponding result for path connected, uniformly locally path connected, and uniformly semilocally simply connected spaces.

Corollary 4.2.3. *Suppose X is Hausdorff, path connected, uniformly locally path connected, and uniformly semilocally simply connected. Then there is a bijective correspondence between the conjugacy classes of subgroups of $\pi_1(X)$ and uniform covering maps over X with the covering space being chain connected.*

Proof. As above, it follows from 4.1.7 and 3.2.5 since $\pi_1(X)$ is isomorphic to $\check{\pi}_1(X)$. □

Chapter 5

Characterizing generalized uniform covering maps as inverse limits of uniform covering maps

The space $GP(X)$ was considered by Berestovskii and Plaut as the inverse limit of spaces X_E for which $X_E \to X$ is realized as the projection associated with a group acting on X_E that is uniformly equicontinuous and uniformly properly discontinuous. Thus, in that situation, the map $X_E \to X$ is a uniform covering map. Since the endpoint map $GP(X) \to X$ is the model generalized uniform covering map, it makes sense to see if generalized uniform covering maps can be characterized as inverse limits of uniform covering maps.

Let us fix a basepoint of a uniform space X. Following Berestovskii and Plaut, given an entourage E of X, denote the set of E-homotopy classes of E-chains in X starting at the basepoint as X_E. Berestovskii and Plaut considered the following uniform structure on X_E. For each entourage $F \subset E$, let $\widehat{F}$ be the set of all pairs $([c],[d]) \in X_E \times X_E$, such that $c^{-1}d$ is E-homotopic relative endpoints to the edge path between the endpoints of c and d and those endpoints are F-close. [1] Then $\{\widehat{F} : F \subset E\}$ is a basis for a uniform structure on X_E.

For entourages $F \subset E$, an F-chain is also an E-chain and if two F-chains are F-homotopic relative endpoints then they are also E-homotopic relative endpoints. Therefore there is a map $\phi_{FE} : X_F \to X_E$ that sends an equivalence class $[c_F]_F$ in X_F to the equivalence class $[c_F]_E$ in X_E. Then the corresponding inverse limit $\varprojlim X_E$ is uniformly equivalent to $GP(X)$.

5.1 Generalized uniform covering maps approximated by uniform covering maps

Suppose X is a chain connected uniform space. Then, given any entourage E of X, the endpoint map $p_E : X_E \to X$ is a uniform covering map. It generates the uniform structure on X since if $F \subset E$, $F = p_E(\widehat{F})$ (chain connectivity is used here). To see that p_E has chain lifting, suppose $F \subset E$, $[c] \in X_E$, and $y \in X$ with $(p_E([c]), y) \in F$. Then c concatenated with y is an E-chain whose equivalence class is $\widehat{F}$-close to $[c]$. Finally, $\widehat{E}$ is transverse to p_E

[1] Berestovskii and Plaut denote this basic entourage as F^*.

so that p_E has unique chain lifting.

Since $GP(X)$ is uniformly equivalent to $\varprojlim X_E$, the endpoint map $\pi_X : GP(X) \to X$ is approximated by uniform covering maps in the sense that for any basic entourage E^* of $GP(X)$, $GP(X) \to X$ factors as $GP(X) \overset{\pi_E}{\to} X_E \overset{p_E}{\to} X$ with g_E a uniform covering map and π_E having E^*-bounded fibers, where π_E is the projection $([c_E]) \mapsto [c_E]$.

Definition 5.1.1. *A map $f : X \to Y$ between uniform spaces is approximated by uniform covering maps if for every entourage E of X, there is an entourage $F \subset E$ so that f factors as $X \overset{h_F}{\to} Z_F \overset{g_F}{\to} Y$ where g_F is a uniform covering map and h_F has F-bounded fibers.*

Investigating when a generalized uniform covering map is approximated by uniform covering maps will be a step toward characterizing generalized uniform covering maps that are inverse limits of uniform covering maps. There is a stronger condition than approximate uniqueness of chain lifts that is necessary for a generalized uniform covering map to be approximated by uniform covering maps.

Definition 5.1.2. *A map $f : X \to Y$ between uniform spaces has strong approximate uniqueness of chain lifts if for every entourage E of X there is an entourage $F \subset E$ so that any two F-chains starting at the same point that have identical images are F-close.*

This condition is stronger than approximate uniqueness of chain lifts because the chains are required to be F-close. Whether the condition is strictly stronger is unknown. In particular, the following question should be posed.

Question. *Does every generalized uniform covering map have strong approximate uniqueness of chain lifts?*

The following shows that the condition is necessary for a map to be approximated by uniform covering maps.

Proposition 5.1.3. *Suppose $f : X \to Y$ is a map between uniform spaces that is approximated by uniform covering maps. Then f has strong approximate uniqueness of chain lifts.*

Proof. Given an entourage E of X, take an entourage $F \subset E$ so that f factors as $X \overset{h_F}{\to} Z_F \overset{g_F}{\to} Y$ where g_F is a uniform covering map and h_F has F-bounded fibers. Let E_0 be an entourage of Z_F that is transverse to g_F. Suppose there are two $h_F^{-1}(E_0) \cap F$-chains c and d starting at the same point with identical images under f. Then $h_F(c)$ and $h_F(d)$ are E_0-chains starting at the same point with identical images so they are identical. Since h_F has F-bounded fibers, c and d are F-close. Of course they are also $h_F^{-1}(E_0)$-close since their images under h_F are identical. □

Now we wish to mimic the factoring of $GP(X) \to X$ as $GP(X) \to X_E \to X$ for arbitrary generalized uniform covering maps with strong approximate uniqueness. Given a map $f : X \to Y$ between uniform spaces, an entourage E of X, a set $A \subset X$, and an $x \in A$, the ***E*-component** of x in A is the set of all $y \in A$ that can be joined to x by an E-chain in A. Given a map $f : X \to Y$ and an entourage E of X, let X_f/E be the set obtained by identifying the E-components of the fibers of f to a point. Let $q_E : X \to X_f/E$ denote the quotient function associated with the identification. Note that if F is an entourage such that two F-chains with the same images under f are F-close then $q_F(x) = q_F(y)$ if and

only if $f(x) = f(y)$ and $(x, y) \in F$. Define a map $g_F : X_f/F \to Y$ to send $q_F(x)$ to $f(x)$. Note it is well defined since $q_F(x) = q_F(x')$ implies $f(x) = f(x')$.

To give X_f/E a uniform structure, of course we could just pull back the uniform structure of Y, but then g_E will not be a uniform covering map. In fact we will only consider a uniform structure on X_f/F for basic entourages F of X.

Lemma 5.1.4. *Suppose a map $f : X \to Y$ has chain lifting and F is an entourage of X such that two F-chains that start at the same point and have the same images under f are F-close. Then q_F generates a uniform structure on X_f/F.*

Proof. It suffices to show that q_F has chain lifting. Given an entourage D of X choose $H \subset D \cap F$ so that $f(H)$-chains in Y lift to $D \cap F$-chains in X. Suppose $x, y \in X$ with $(q_F(x), q_F(y)) \in q_F(H)$. Then $q_F(x) = q_F(x')$ and $q_F(y) = q_F(y')$ for some $(x', y') \in H$. Then $(f(x), f(y)) \in f(H)$ so there is a $y'' \in X$ with $f(y'') = f(y)$ and $(x, y'') \in D \cap F$. But then x, x, y'' and x, x', y' are two F-chains with identical images under f so $(y', y'') \in F$. Then $q_F(y'') = q_F(y') = q_F(y)$. □

Proposition 5.1.5. *Suppose $f : X \to Y$ generates the uniform structure on Y, has chain lifting, and has strong approximate uniqueness of chain lifts. Then f is approximated by uniform covering maps.*

Proof. By 5.1.4, for each entourage E of X there is an entourage $F \subset E$ so that f factors as $X \overset{q_F}{\to} X_f/F \overset{g_F}{\to} Y$ where q_F has F-bounded fibers. By 3.1.2 g_F has chain lifting. Note that g_F generates the uniform structure on Y since $g_F(q_F(G)) = f(G)$. It remains to show that there is an entourage of X_f/F that is transverse to g_F. Suppose $(q_F(x), q_F(y)) \in q_F(F)$ with $f(x) = f(y)$. Then $q_F(x) = q_F(x')$ and $q_F(y) = q_F(y')$ for some $(x', y') \in F$. Then x, x', y', y is an F-chain with $f(x) = f(x') = f(y') = f(y)$ so $q_F(x) = q_F(y)$. □

As justification for the definition of X_f/E, we will show that if X is locally uniform joinable chain connected then $GP(X)_{\pi_X}/E^*$ is uniformly equivalent to $\pi_E(GP(X))$. First notice that there is a uniform structure on $GP(X)_{\pi_X}/E^*$ for any basic entourage E^* of $GP(X)$.

Lemma 5.1.6. *For any basic entourage E^* of $GP(X)$, any two E^*-chains that start at the same point and have identical images are E^*-close.*

Proof. Suppose $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are E^*-chains with identical images under π and $\alpha_1 = \beta_1$. It suffices to show that if $(\alpha_i, \beta_i) \in E^*$ then $(\alpha_{i+1}, \beta_{i+1}) \in E^*$. If $(\alpha_i, \beta_i) \in E^*$ then $[\alpha_{i+1}^{-1}\beta_{i+1}]_E = [\alpha_{i+1}^{-1}\alpha_i\beta_i^{-1}\beta_{i+1}]_E$ which is E-homotopic to the constant chain at the endpoint of α_{i+1}. □

Proposition 5.1.7. *If X is locally uniform joinable chain connected then $GP(X)_{\pi_X}/E^*$ is uniformly equivalent to $\pi_E(GP(X))$.*

Proof. First note there is a bijective correspondence between the two sets. Let $i : GP(X)_{\pi_X}/E^* \to X_E$ send $q_{E^*}(\alpha)$ to α_E. The function is well defined and injective. Of course it need not be surjective since X_E contains all E-chains starting at x_0 while $i(GP(X)_{\pi_X}/E^*)$ only contains E-chains that are terms of generalized paths. But it therefore does map onto $\pi_E(GP(X))$. To see that i generates the uniform structure on $\pi_E(GP(X))$, first note that i is uniformly continuous since for an entourage $F \subset E$, $q_{F^*} \subset i^{-1}(\widehat{F})$. Now

consider a basic entourage $q_{E^*}(F^*)$ of $GP(X)_{\pi_X}/E^*$. Let $H \subset E$ be an entourage of X so that if $(\pi_X(\alpha), y) \in H$ there is a $\beta \in GP(X)$ with $(\alpha, \beta) \in E^* \cap F^*$ and $\pi_X(\beta) = y$. Suppose $\alpha, \beta \in GP(X)$ with $(\pi_E(\alpha), \pi_E(\beta)) \in \widehat{H}$. Set $\pi_X(\alpha) = x$ and $\pi_X(\beta) = y$. Then $(x, y) \in H$ so there is a $\beta' \in GP(X)$ with $\pi_X(\beta') = y$ and $(\alpha, \beta') \in E^* \cap F^*$. Then α, β and α, β' are two E^* chains with identical images so $(\beta, \beta') \in E^*$. Therefore $q_{E^*}(\beta) = q_{E^*}(\beta')$ so $(q_{E^*}(\alpha), q_{E^*}(\beta)) \in q_{E^*}(F^*)$. Notice $i(q_{E^*}(\alpha), q_{E^*}(\beta)) = (\pi_E(\alpha), \pi_E(\beta))$. □

5.2 Inverse limits of generalized uniform covering maps

Now inverse limits are investigated. We will look at inverse limits of generalized uniform covering maps with complete fibers. In particular we are interested in characterizing inverse limits of strong Mittag-Leffler inverse systems of uniform covering maps.

The strong Mittag-Leffler condition is assumed for many of the results of this section but it is not needed for strong approximate uniqueness of chain lifts to be preserved by inverse limits.

Lemma 5.2.1. *Suppose there are inverse systems of uniform spaces $\{X_\alpha, \phi_{\beta\alpha}\}$ and $\{Y_\alpha, \psi_{\beta\alpha}\}$ with compatible maps $f_\alpha : X_\alpha \to Y_\alpha$. Set $f = \varprojlim f_\alpha$.*

1. *If each f_α has approximate uniqueness of chain lifts then f has approximate uniqueness of chain lifts.*

2. *If each f_α has strong approximate uniqueness of chain lifts then f has strong approximate uniqueness of chain lifts.*

Proof. 1. Consider a basic entourage $\pi_\alpha^{-1}(E_\alpha)$ of $\varprojlim(X_\alpha)$. Now there is an entourage $F_\alpha \subset E_\alpha$ so that any two F_α-chains in X_α starting at the same point that have identical images under f_α are E_α-close. Suppose c and d are two $\pi_\alpha^{-1}(F_\alpha)$-chains in $\varprojlim(X_\alpha)$ starting at the same point that have identical images under f. Then $\pi_\alpha(c)$ and $\pi_\alpha(d)$ are two F_α-chains in X_α starting at the same point with identical images under f_α so they are E_α-close. Therefore c and d are $\pi_\alpha^{-1}(E_\alpha)$-close.

2. The proof is similar to the proof of 1. □

The following shows that having complete fibers is a necessary condition for a generalized uniform covering map to be the inverse limit of uniform covering maps.

Lemma 5.2.2. *Suppose there are inverse systems $\{X_\alpha, \phi_{\beta\alpha}\}$ and $\{Y_\alpha, \psi_{\beta\alpha}\}$ of uniform spaces and compatible maps $f_\alpha : X_\alpha \to Y_\alpha$ that have complete fibers. Suppose each X_α is Hausdorff. Then $f = \varprojlim f_\alpha$ has complete fibers.*

Proof. Suppose $(y_\alpha) \in \varprojlim Y_\alpha$. Notice $f^{-1}((y_\alpha))$ is identical to $\varprojlim f_\alpha^{-1}(y_\alpha)$ as sets and uniform spaces. Since the inverse limit of complete Hausdorff spaces is complete [2], $f^{-1}((y_\alpha))$ is complete. □

Now consider a strong Mittag-Leffler inverse system.

Lemma 5.2.3. *Suppose there is a strong Mittag-Leffler inverse system of uniform spaces $\{X_\alpha, \phi_{\beta\alpha}\}$ with compatible maps $f_\alpha : X_\alpha \to Y$. Set $f = \varprojlim f_\alpha$.*

1. *If each f_α generates the uniform structure on Y then f generates the uniform structure on Y.*

2. *If each f_α has chain lifting then f has chain lifting.*

Proof. Consider a basic entourage $\pi_\alpha^{-1}(E_\alpha)$ of $\varprojlim X_\alpha$. Let $\beta < \alpha$ so that $\phi_{\beta\alpha}(X_\beta) = \pi_\alpha(\varprojlim X_\alpha)$.

1. First note that f is uniformly continuous since each f_α is. To see that $f_\beta\phi_{\beta\alpha}^{-1}(E_\alpha) \subset f(\pi_\alpha^{-1}(E_\alpha))$, suppose $(x,y) \in f_\beta\phi_{\beta\alpha}^{-1}(E_\alpha)$. Then $(x,y) = f_\beta(x'_\beta, y'_\beta)$ for some $(x'_\beta, y'_\beta) \in \phi_{\beta\alpha}^{-1}(E_\alpha)$. Now $\phi_{\beta\alpha}(x'_\beta, y'_\beta) = \pi_\alpha((x_\alpha),(y_\alpha))$ for some $(x_\alpha),(y_\alpha) \in \varprojlim X_\alpha$. Note $f((x_\alpha),(y_\alpha)) = (x,y)$. Also $\pi_\alpha((x_\alpha),(y_\alpha)) = \phi_{\beta\alpha}(x'_\beta, y'_\beta) \in E_\alpha$ so $(x,y) \in f(\pi_\alpha^{-1}(E_\alpha))$.

2. There is an entourage F_β of X_β so that an if $(f_\beta(x), y) \in f_\beta(F_\beta)$, there is a $y' \in X_\beta$ with $f_\beta(y') = y$ and $(x,y') \in \phi_{\beta\alpha}^{-1}(E_\alpha)$. Suppose $(f((x_\alpha)), y) \in f(\pi_\beta^{-1}(F_\beta))$. Then $(f_\beta(x_\beta), y) \in f_\beta(F_\beta)$ so there is a $y'_\beta \in X_\beta$ with $f_\beta(y'_\beta) = y$ and $(x_\beta, y'_\beta) \in \phi_{\beta\alpha}^{-1}(E_\alpha)$. Now $\phi_{\beta\alpha}(y'_\beta) = \pi_\alpha((y_\alpha))$ for some $(y_\alpha) \in \varprojlim X_\alpha$. Note $f((y_\alpha)) = y$ and $((x_\alpha),(y_\alpha)) \in \pi_\alpha^{-1}(E_\alpha)$. □

Notice it is unclear how to generalize this result to an inverse system $\{Y_\alpha, \psi_{\beta\alpha}\}$ and maps $f_\alpha : X_\alpha \to Y_\alpha$. There is even a problem showing that the resulting map f is surjective.

Proposition 5.2.4. *Suppose there is a strong Mittag-Leffler inverse system $\{X_\alpha, \phi_{\beta\alpha}\}$ of Hausdorff uniform spaces and compatible generalized uniform covering maps $f_\alpha : X_\alpha \to Y$ with complete fibers. Then the inverse limit $f = \varprojlim f_\alpha$ is a generalized uniform covering map.*

Proof. By 5.2.2 the inverse limit has complete fibers. By 5.2.3 it generates the uniform structure on Y and has chain lifting. By 5.2.1 it has approximate uniqueness of chain lifts. □

In particular, the inverse limit of a strong Mittag-Leffler inverse system of uniform covering maps over a Hausdorff space is a generalized uniform covering map.

5.3 Realizing a generalized uniform covering map as the inverse limit of uniform covering maps

Now we wish the express a generalized uniform covering map as the inverse limit of uniform covering maps. According to 5.1.3, strong approximate uniqueness is a necessary condition. Then, according to 5.1.5, given a generalized uniform covering map $f : X \to Y$ that has strong approximate uniqeness of chain lifts, X has a basis of entourages such that for each basic entourage E, $q_E : X \to X_f/E$ has E-bounded fibers and $g_E : X_f/E \to Y$ is a uniform covering map. Consider the inverse system $\{X_f/E, \phi_{FE}\}$ given by this basis where for basic entourages $F \subset E$ of X, $\phi_{FE} : X_f/F \to X_f/E$ is defined to send $q_F(x)$ to $q_E(x)$. These functions are well-defined since each equivalence class of X_f/F is contained in a single equivalence class of X_f/E. The functions are also uniformly continuous and compatible with the maps $\{q_E\}$ and $\{g_E\}$.

Proposition 5.3.1. *Suppose $f : X \to Y$ is a generalized uniform covering map that has strong approximate uniqueness. Consider the inverse system $\{X_f/E, \phi_{FE}\}$ given by the*

basis as above and set $q = \varprojlim q_E$. Suppose X is Hausdorff. Then q is a uniform embedding. If f has complete fibers then q is a uniform equivalence.

Proof. Since each q_E is uniformly continuous q is as well. To see that q is injective, suppose $x, y \in X$ with $q(x) = q(y)$. Since $q_E(x) = q_E(y)$ for all basic of entourages E of X, $(x, y) \in E$ for each basic entourage E. Then since X is Hausdorff, $x = y$. To see that q is a uniform embedding, suppose E is a basic entourage of X. Choose a basic entourage F of X so that $F^3 \subset E$. Then to show that q is a uniform embedding it suffices to show that for any $x, y \in X$ with $(q(x), q(y)) \in \pi_F^{-1}(q_F(F))$, $x, y \in E$ where π_F is the projection from $\varprojlim X_f/E$ to X_f/F. Suppose $(q(x), q(y)) \in \pi_F^{-1}(q_F(F))$. Then there are $x', y' \in X$ with $(x, x'), (x', y'), (y', y) \in F$. Therefore $(x, y) \in E$.

Finally, to see that q is surjective and therefore a uniform equivalence if f has complete fibers, suppose $(q_E(x_E)) \in \varprojlim X_f/E$. Set $y = f(x_E)$. Note y is independent of the entourage E and the choice of representative x_E. For each basic entourage E let $A_E \subset X$ be the equivalence class of x_E. Then $\{A_E\}$ is a Cauchy filter base in the fiber $f^{-1}(y)$ so it has a limit point x. Now, given an entourage E of X there is an entourage F so that $A_F \subset B(x, E)$. Then $A_{F \cap E} \subset B(x, E)$ so $q_E(x_{F \cap E}) = q_E(x)$. But $q_E(x_{F \cap E}) = q_E(x_E)$ so $q(x) = (q_E(x_E))$. □

Theorem 5.3.2. *Suppose $f : X \to Y$ is a map between uniform spaces with Y Hausdorff. Then the following are equivalent.*

1. *There is a strong Mittag-Leffler inverse system $\{X_\alpha, \phi_{\beta\alpha}\}$ and uniform covering maps $f_\alpha : X_\alpha \to Y$ with $f = \varprojlim f_\alpha$.*
2. *X is Hausdorff and f generates the uniform structure on Y, has chain lifting, has strong approximate uniqueness of chain lifts, and has complete fibers.*

Proof. 1. $\implies$ 2. By 5.2.3, f generates the uniform structure on Y and has chain lifting. By 5.2.1 or 5.1.3 f has strong approximate uniqueness. Notice that each X_α is Hausdorff as Y is Hausdorff and f_α is a uniform covering map. Then f has complete fibers by 5.2.2. Notice that the inverse limit of Hausdorff spaces is Hausdorff.

2. $\implies$ 1. By 5.3.1, f is uniformly equivalent to the map $\varprojlim X_f/E \to Y$. Notice the inverse system is strong Mittag-Leffler since $X \to X_f/E$ is surjective for each basic entourage E. □

Chapter 6

Characterizing regular generalized uniform covering maps as inverse limits of regular uniform covering maps

Since generalized uniform covering maps can be characterized as the inverse limit of uniform covering maps (provided that the map has strong approximate uniqueness of chain lifts), we now attempt to characterize generalized uniform covering maps induced by group actions as inverse limits of uniform covering maps induced by group actions.

6.1 Inverse limits of regular generalized uniform covering maps

Since we are interested in inverse limits of uniform covering maps, we will consider the inverse limit of neutral and uniformly properly discontinuous group actions (see 2.5.1).

Lemma 6.1.1. *Suppose there is an inverse system of groups $\{G_\alpha, \psi_{\beta\alpha}\}$, an inverse system of uniform spaces $\{X_\alpha, \phi_{\beta\alpha}\}$, and compatible actions of G_α on X_α that are neutral. Suppose the inverse system $\{G_\alpha, \psi_{\beta\alpha}\}$ is strong Mittag-Leffler. Set $G = \varprojlim G_\alpha$ and $X = \varprojlim X_\alpha$. Then the induced action of G on X is neutral.*

Proof. Let $\pi_\alpha^{-1}(E_\alpha)$ be a basic entourage of X. There is a $\beta < \alpha$ with $\psi_{\beta\alpha}(G_\beta) = \pi_\alpha(G)$. Now there is an entourage $F_\beta \subset \phi_{\beta\alpha}^{-1}(E_\alpha)$ so that if $(x, hy) \in F_\beta$ then there is a $g \in G_\beta$ with $(gx, y) \in \phi_{\beta\alpha}^{-1}(E_\alpha)$. Suppose $((x_\alpha), (h_\alpha)(y_\alpha)) \in \pi_\beta^{-1}(F_\beta)$. Then $(x_\beta, h_\beta y_\beta) \in F_\beta$ so there is a $g'_\beta \in G_\beta$ with $(g'_\beta x_\beta, y_\beta) \in \phi_{\beta\alpha}^{-1}(E_\alpha)$. Now there is a $(g_\alpha) \in G$ with $g_\alpha = \psi_{\beta\alpha}(g'_\beta)$. Then $((g_\alpha)(x_\alpha), (y_\alpha)) \in \pi_\alpha^{-1}(E_\alpha)$. □

Lemma 6.1.2. *Suppose there is an inverse system of groups $\{G_\alpha, \psi_{\beta\alpha}\}$, an inverse system of uniform spaces $\{X_\alpha, \phi_{\beta\alpha}\}$, and compatible actions of G_α on X_α that have small scale bounded orbits. Set $G = \varprojlim G_\alpha$ and $X = \varprojlim X_\alpha$. Then the induced action of G on X has small scale bounded orbits.*

Proof. Let $\pi_\alpha^{-1}(E_\alpha)$ be a basic entourage of X. Let F_α be an entourage of X_α so that orbits of the action of $G_{\alpha\, F_\alpha}$ on X_α are E_α-bounded. Suppose $(g_\alpha) \in G_{\pi_\alpha^{-1}(F_\alpha)}$ and $(x_\alpha) \in X$. Then $g_\alpha \in G_{\alpha\, F_\alpha}$ so $(x_\alpha, g_\alpha x_\alpha) \in E_\alpha$. Therefore $((x_\alpha), (g_\alpha)(x_\alpha)) \in \pi_\alpha^{-1}(E_\alpha)$. □

In particular the inverse limit of uniformly properly discontinuous actions has small scale bounded orbits.

Given a strong Mittag Leffler inverse system of groups $\{G_\alpha, \psi_{\beta\alpha}\}$, an inverse system of Hausdorff uniform spaces $\{X_\alpha, \phi_{\beta\alpha}\}$, and compatible actions of G_α on X_α that are neutral and properly discontinuous, we wish to have the projection $p : X \to X/G$ be a generalized uniform covering map where $G = \varprojlim G_\alpha$ and $X = \varprojlim X_\alpha$. By 6.1.1 the action is neutral and by 6.1.2 the action has small scale bounded orbits. It remains to have that the projection $X \to X/G$ has complete fibers. For this proof we need a strong assumption on the inverse system $\{G_\alpha, \psi\beta\alpha\}$. In particular we need it to be a countable sequence with $\varprojlim{}^1 G_i = 1$.

Proposition 6.1.3. *Suppose there is an inverse sequence of uniform spaces $\{X_i, \phi_i\}$ and a Mittag-Leffler inverse sequence of groups $\{G_i, \psi_i\}$ with compatible neutral and free actions of G_i on X_i. Let $X = \varprojlim X_i$ and $G = \varprojlim G_i$. Suppose that $\varprojlim{}^1 G_i = 1$. Then $\varprojlim(X_i/G_i)$ is uniformly equivalent to X/G.*

Proof. We assume the actions are neutral so that there are uniform structures on X_i/G_i. Notice the action of G on X is neutral by 6.1.1 so that there is a uniform structure on X/G. Given $[(x_i)] \in X/G$, define $f : X/G \to \varprojlim(X_i/G_i)$ to send $[(x_i)]$ to $([x_i])$. Notice it is well defined. To see that it is injective, suppose $[(x_i)], [(y_i)] \in X/G$ with $([x_i]) = ([y_i])$. Then for each n there is a $g_n \in G_n$ so that $x_n = g_n y_n$. It suffices to show that (g_i) is a thread. But $\phi_n(x_n) = \phi_n(g_n y_n) = \psi_n(g_n)\phi_n(y_n) = \psi_n(g_n)y_{n-1}$ and $\phi_n(x_n) = x_{n-1} = g_{n-1}y_{n-1}$. Then since the action is free we have $\psi_n(g_n) = g_{n-1}$.

Now we will see that f is uniformly continuous. For each n, let $q_n : X_n \to X_n/G_n$ be the quotient map associated with the action of G_n on X_n, $\pi_n : \varprojlim X_i \to X_n$ be the projection to X_n, and $\Pi_n : \varprojlim(X_i/G_i) \to X_n/G_n$ be the projection to X_n/G_n. Let $Q : X \to X/G$ be the quotient map associated with the action of G on X. Given an entourage E_n of X_n, let us see that $f(Q(\pi_n^{-1}(E_n))) \subset \Pi_n^{-1}(q_n(E_n))$. Suppose $([(x_i)], [(y_i)]) \in Q(\pi_n^{-1}(E_n))$. Then $(x_i) = (g_i)(x_i')$ and $(y_i) = (h_i)(y_i')$ with $(x_n', y_n') \in E_n$ for some $(g_i), (h_i) \in \varprojlim G_i$. Then $(([x_i]), ([y_i])) \in \Pi_n^{-1}(q_n(E_n))$.

Finally, let us see that set $f(Q(\pi_n^{-1}(E_n)))$ is an entourage of $\varprojlim(X_i/G_i)$ so that f is a uniform equivalence. Choose $m > n$ so that if $g_m \in G_m$, there is an $(h_i) \in \varprojlim G_i$ with $h_n = \psi_{mn}(g_m)$ where $\psi_{mn} = \psi_{n+1} \circ \psi_{n+2} \circ \cdots \circ \psi_{m-1} \circ \psi_m$. Suppose $(([x_i]), ([y_i])) \in \Pi_n^{-1}(\phi_{mn}^{-1}(E_n))$. Without loss of generality we can assume $(x_i, y_i) \in \phi_{mn}^{-1}(E_n)$. We will define $(x_i'), (y_i') \in \varprojlim X_i$ with $([(x_i')], [(y_i')]) \in Q(\pi_n^{-1}(E_n))$ and $([x_i']) = ([x_i])$ and $([y_i']) = ([y_i])$.

Since $([x_i])$ is a thread, for each i there is a $g_i \in G_i$ with $\phi_{i+1}(x_{i+1}) = g_i x_i$. Because $\varprojlim{}^1 G_i = 1$ there is a sequence $\{k_i\}$ with $g_i = \psi_{i+1}(k_{i+1})^{-1}k_i$ for each i (see 2.6.2). Define $x_i' = k_i x_i$. Then $\phi_{i+1}(x_{i+1}') = \psi_{i+1}(k_{i+1})\phi_{i+1}(x_i) = k_i g_i^{-1} g_i x_i = x_i'$ so $(x_i') \in \varprojlim X_i$. Notice $([x_i']) = ([x_i])$. Define $(y_i') \in \varprojlim X_i$ analogously.

We need $([(x_i')], [(y_i')]) \in Q(\pi_n^{-1}(E_n))$. Now $(\phi_{mn}(x_m), \phi_{mn}(y_m)) \in E_n$. Let us calculate $\phi_{mn}(x_m)$. $\phi_{mn}(x_m) = \psi_{m-1\ n}(g_{m-1})\psi_{m-2\ n}(g_{m-2}) \cdots \psi_{n+2\ n}(g_{n+2})\psi_{n+1} g_{n+1} g_n x_n$. Now $g_i = \psi_{i+1}(k_{i+1})^{-1}k_i$ so $\psi_{in}(g_i) = \psi_{in}(\psi_{i+1}(k_{i+1})^{-1}k_i) = \psi_{i+1\ n}(k_{i+1})^{-1}\psi_{in}(k_i)$. Therefore $\psi_{m-1\ n}(g_{m-1})\psi_{m-2\ n}(g_{m-2}) \cdots \psi_{n+1\ n}(g_{n+1})$ is a telescoping product and is equal to $\psi_{mn}(k_m)^{-1}k_n x_n$. Now there is $(h_i) \in \varprojlim G_i$ with $h_n = \psi_{mn}(k_m)^{-1}$ so $\pi_n((h_i)(x_i')) = \phi_{mn}(x_m)$. Similarly there is $(l_i) \in \varprojlim G_i$ with $\pi_n((l_i)(y_i')) = \phi_{mn}(y_m)$. □

Proposition 6.1.4. *Suppose there is a Mittag-Leffler inverse sequence of groups $\{G_i, \psi_i\}$, an inverse sequence of Hausdorff uniform spaces $\{X_i, \phi_i\}$, and compatible actions of G_i on X_i that are neutral and have small scale bounded orbits. Suppose the projections associated with the actions of G_i on X_i have complete fibers. Suppose $\varprojlim^1 G_i = 1$. Set $G = \varprojlim G_i$ and $X = \varprojlim X_i$. Then the projection $p : X \to X/G$ associated with the induced action of G on X is a generalized uniform covering map.*

Proof. By 6.1.1 the induced action is neutral. By 6.1.2 the induced action has small scale bounded orbits. Therefore by 2.5.3 it is enough to show that p has complete fibers. Notice the action is free by 2.5.5. Therefore 6.1.3 applies so the inverse limit uniform structure on X/G coincides with the structure generated by p. But then p has complete fibers by 5.2.2. □

In particular the inverse limit of neutral and uniformly properly discontinuous actions induces a generalized uniform covering map provided the inverse sequence of groups is Mittag-Leffler and has trivial $\varprojlim^1$.

6.2 Realizing a regular generalized uniform covering map as the inverse limit of regular uniform covering maps

Now, given an action of a group G on a uniform space X that induces a generalized uniform covering map $p : X \to X/G$ we wish to express that action as the inverse limit of neutral and properly discontinuous actions.

Lemma 6.2.1. *Suppose a group G acts faithfully and uniformly equicontinuously on a uniform space X. Then for each entourage E of X there is an entourage $F \subset E$ such that the induced action of G/G_F on X/G_F is uniformly properly discontinuous.*

Proof. First note that an action is uniformly equicontinuous if and only if there is a basis of invariant entourages [3]. An entourage F of X is G-invariant if $(gx, gy) \in F$ for each $(x, y) \in F$ and $g \in G$. Given an entourage E of X, choose $F \subset E$ to be G-invariant. Notice since F is invariant, G_F is normal in G [10]. Then G/G_F is a group and its action on X/G_F is well defined. Let $p_F : X \to X/G_F$ be the projection associated with the action of G_F on X. Since the action of G on X is uniformly equicontinuous, so is the action of G_F on X and that action is therefore neutral. Therefore p_F generates a uniform structure on X/G_F. Suppose $([x], [g][x]) \in p_F(F)$ for some $[x] \in X/G_F$ and $[g] \in G/G_F$. Then $(g_F x, h_F g x) \in H$ for some $g_F, h_F \in G_F$. Then $(x, g_F^{-1} h_F g x) \in F$ so $g_F^{-1} h_F g \in G_F$ and $g \in G_F$. Therefore $[x] = [g][x]$. Notice the action of G/G_F on X/G_F is faithful since G_F is precicely the stabilizer of the action of G on X/G_F. Therefore $[g] = e_{G/G_F}$. □

Notice the action of G/G_F on X/G_F in the above lemma is also uniformly equicontinuous. Therefore if a group G acts faithfully and uniformly equicontinuously on a uniform space X, X has a basis of entourages so that for each basic entourage E, the action of G/G_E on X/G_E induces a uniform covering map. Consider the inverse systems $\{G/G_E, \psi_{FE}\}$ and $\{X/G_E, \phi_{FE}\}$ where for $F \subset E$, ψ_{FE} sends the equivalence class of g in G/G_F to the equivalence class of g in G/G_E and ϕ_{FE} sends the equivalence class of x in X/G_F to the

equivalence class of x in X/G_E. The functions are well defined, ψ_{FE} is a group homomorphism, ϕ_{FE} is uniformly continuous, and these maps are compatible with the actions of G/G_E on X/G_E.

Proposition 6.2.2. *Suppose a group G acts faithfully and uniformly equicontinuously on a uniform space X. Suppose X is Hausdorff and the action has small scale bounded orbits. Consider the inverse system $\{G/G_E, \psi_{FE}\}$ given by the basis as above. Then G is isomorphic to a subgroup of $\varprojlim G/G_E$. If the projection $X \to X/G$ induced from the action of G on X has complete fibers then G is isomorphic to $\varprojlim G/G_F$.*

Proof. Let $f : G \to \varprojlim G/G_E$ send g to $([g]_E)$. Note f is a homomorphism. Suppose $f(g) = f(h)$ for some $g, h \in G$. Then $gh^{-1} \in G/G_E$ for each basic E. Now for each entourage D of X there is a basic entourage E of X so that the orbits of the action of G_E on X are D-bounded. Therefore for each $x \in X$, $(x, gh^{-1}x) \in D$ for each entourage D of X. Since X is Hausdorff $x = g^{-1}hx$ and since the action is faithful $g = h$. Therefore f is injective and G is isomorphic to a subgroup of $\varprojlim G/G_E$.

Now suppose the projection induced from the action of G on X has complete fibers. To see that f is surjective and therefore X is isomorphic to $\varprojlim G/G_E$, suppose $([g_E]_E) \in \varprojlim G/G_E$. Let $x \in X$ and set $A_E = \{g_F x : F \subset E\}$. Note that each A_E is contained in the fiber of $[x] \in X/G$. To see the $\{A_E\}$ is Cauchy, given a basic entourage E of X choose a basic entourage F so that the orbits of the action of G_F on X are E-bounded. Suppose $g_H x, g_K x \in A_F$. Now $g_K^{-1} g_H \in G_F$ so $(g_K^{-1} g_H x, x) \in E$. Since E is invariant $(g_H x, g_K x) \in E$. Therefore $\{A_E\}$ is Cauchy so there is a limit point gx. Now, given an entourage E of X there is an entourage F of X so that $A_F \subset B(gx, E)$. Then $A_{F\cap E} \subset B(gx, E)$ so $(g_{F\cap E}x, gx) \in E$. Since E is invariant $(g^{-1}g_{F\cap E}x, x) \in E$ so $g^{-1}g_{F\cap E} \in G_E$. Then $[g_{F\cap E}]_E = [g]_E$. But $[g_{F\cap E}]_E = [g_E]_E$ so $f(g) = ([g_E]_E)$. □

Proposition 6.2.3. *Suppose a group G acts uniformly equicontinuously on a uniform space X. Suppose X is Hausdorff and the action has small scale bounded orbits. Consider the inverse system $\{X/G_E, \phi_{FE}\}$ given by the basis as above. Then X embeds in $\varprojlim X/G_E$. If the projection $X \to X/G$ induced from the action of G on X has complete fibers then X is uniformly equivalent to $\varprojlim X/G_E$.*

Proof. Let $f : X \to \varprojlim X/G_E$ send x to $([x]_E)$. First, to see that f is injective, suppose $f(x) = f(y)$ for some $x, y \in X$. Now for each entourage D of X there is a basic entourage E so that the orbits of the action of G_E on X are D bounded. Since $f(x) = f(y)$, there is a $g_E \in G_E$ with $x = g_E y$. Then $(x, y) = (g_E y, y) \in D$. Since $(x, y) \in D$ for each entourage D of X and X is Hausdorff, $x = y$. To see that f is a uniform embedding, first note it is uniformly continuous since the projections $X \to X/G_E$ are uniformly continuous. Suppose D is an entourage of X. Choose E so that $E^2 \subset D$ and choose a basic entourage F so that the orbits of the action of G_F on X are E-bounded. Suppose $x, y \in X$ with $(f(x), f(y)) \in \pi_F^{-1}(p_F(E))$ where π_F is the projection from $\varprojlim X/G_E$ to X/G_F and p_F is the projection associated with the action of G_F on X. Now $(x, g_F y) \in E$ for some $g_F \in G_F$. Also $(g_F y, y) \in E$ so $(x, y) \in E^2 \subset D$ and $(f(x), f(y)) \in f(D)$.

Finally, to see that if f is surjective and therefore a uniform equivalence if the projection induced from the action of G on X has complete fibers, suppose $([x_E]_E) \in \varprojlim X/G_E$. For each basic entourage E of X let A_E be the orbit of x_E under the action of G_E. Now each x_E gets sent to the same equivalence class in X/G, say $[x]$, and each A_E is contained in

the fiber $p^{-1}([x])$ where p is the projection associated with the action of G on X. Note $\{A_E\}$ is a Cauchy filter base since given an entourage D of X there is a basic entourage E of X so that the orbits of the action of G_E on X are D-bounded. Therefore there is a limit y of $\{A_E\}$. Let us see that $f(y) = ([x_E]_E)$. Given E, there is an $A_F \subset B(x, E)$. Then $A_{F\cap E} \subset B(y, E)$. Now $y = gx_{F\cap E}$ for some $g \in G$. Then $(x_{F\cap E}, gx_{F\cap E}) \in E$ so $g \in G_E$. Therefore $[y]_E = [x_{F\cap E}]_E$. But $[x_E]_E = [x_{F\cap E}]$. □

Theorem 6.2.4. *Suppose a group G acts faithfully and uniformly equicontinuously on a metrizable uniform space X. Then the following are equivalent.*

1. *The action has small scale bounded orbits and the projection $p : X \to X/G$ has complete fibers.*

2. *There is a Mittag-Leffler inverse sequence of groups $\{G_i, \psi_i\}$ with trivial $\varprojlim^1$, an inverse sequence of uniform spaces $\{X_i, \phi_i\}$, and compatible actions of G_i on X_i that are uniformly properly discontinuous and neutral with G isomorphic to $\varprojlim G_i$, X uniformly equivalent to $\varprojlim X_i$, and the action of G on X equivalent to the induced action of $\varprojlim G_i$ on $\varprojlim X_i$.*

Proof. 1. $\implies$ 2. Since X is metrizable it is Hausdorff. Then by 6.2.1 and the paragraph succeeding it there are inverse systems $\{G/G_E, \psi_{FE}\}$ and $\{X/G_E, \phi_{FE}\}$ and compatible actions of G/G_E on X/G_E that are uniformly properly discontinuous. Since X is metrizable it has a countable basis of entourages so these systems can be realized as sequences. Notice the actions are neutral since the action of G on X is uniformly equicontinuous. By 6.2.2 G is isomorphic to $\varprojlim G_\alpha$. By 6.2.3 X is uniformly equivalent to $\varprojlim G/G_E$. The inverse sequence $\{G/G_E, \psi_{FE}\}$ is Mittag-Leffler and has trivial $\varprojlim^1$ since in fact each ψ_{FE} is surjective. Indeed, given $F \subset E$ and $[g]_E \in G/G_E$, $[g]_F \in G/G_F$ gets mapped to $[g]_E$. Notice the action of G on X is equivalent to the induced action of $\varprojlim G/G_E$ on $\varprojlim X/G_E$ since given $x \in X$ and $g \in G$, $([gx]_E) = ([g]_E)([x]_E)$.

2. $\implies$ 1. By 6.1.2 the action of G on X has small scale bounded orbits. According to the proof of 6.1.4 p has complete fibers. □

Chapter 7

Characterizing generalized uniform covering maps of locally uniform joinable spaces as inverse limits of uniform covering maps

Sergey Melikhov visited the University of Tennesse during the 2007-2008 academic year. During this time he became interested in generalized uniform covering maps. He later asked the following question. If a pointed 1-movable continuum X is represented by the inverse limit of its nerves $\{P_i\}$, is it true that generalized uniform covering spaces over X are the same thing as the inverse limits of covering spaces over the P_i's? What follows grew out of examining this question and I would like to thank Sergey Melikhov for posing it.

A pointed 1-movable continuum is locally uniform joinable chain connected ([4]). We will only consider the case where the generalized uniform covering space is also locally uniform joinable chain connected.

Let $f : (X, x_0) \to (Y, y_0)$ be a pointed map between uniform spaces. Given an entourage E of X, there is an induced map $f_E : R(X, E) \to R(Y, f(E))$ which was defined in [4]. Therefore there is a well-defined induced map $f_{E*} : (X_E, x_0) \to (Y_{f(E)}, y_0)$. Given an entourage E of Y we can view the induced map $f_{f^{-1}(E)*} : (X_{f^{-1}(E)}, x_0) \to (Y_{f(f^{-1}(E))}, y_0)$ as having (Y_E, y_0) as its codomain. Let us denote this function as f_{E*}. Then we can define the induced map $f_* : GP(X, x_0) \to GP(Y, y_0)$, first mentioned in [4], as follows. Given $c \in GP(X, x_0)$, let $f_*(c) = (f_{E*}(c_{f^{-1}(E)}))_E$. Notice for maps $f : X \to Y$ and $g : Y \to Z$, $(g \circ f)_* = g_* \circ f_*$.

Proposition 7.0.5. *Suppose $\{X_s, \phi_{ts}\}$ is a Strong Mittag-Leffler inverse system of uniform spaces. Let $X = \varprojlim X_s$ and $x_0 = (x_{0s}) \in X$. Consider the inverse system $\{GP(X_s, x_{0s}), \phi_{ts*}\}$. Then the map $f = \varprojlim \pi_{s*}$ from $GP(X, x_0)$ to $\varprojlim GP(X_s, x_{0s})$ is a uniform equivalence.*

Proof. Let us see that f is injective. Suppose $\alpha, \beta \in GP(X, x_0)$ with $f(\alpha) = f(\beta)$. Given a basic entourage $\pi_s^{-1}(E_s)$ of X, we wish to show that $\alpha_{\pi_s^{-1}(E_s)} = \beta_{\pi_s^{-1}(E_s)}$. Choose $t > s$ so that $\phi_{ts}(X_t) = \pi_s(X)$. Now $\pi_{t*}(\alpha) = \pi_{t*}(\beta)$ so $\pi_t(\alpha_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)})$ is $\phi_{ts}^{-1}(E_s)$-homotopic to $\pi_t(\beta_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)})$. Therefore $\phi_{ts}\pi_t(\alpha_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)}) = \pi_s(\alpha_{\pi_s^{-1}(E_s)})$ is E_s-homotopic in $\phi_{ts}(X_t)$ to

$\phi_{ts}\pi_t(\beta_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)}) = \pi_s(\beta_{\pi_s^{-1}(E_s)})$. Therefore this homotopy lifts to a $\pi_s^{-1}(E_s)$-homotopy in X from $\alpha_{\pi_s^{-1}(E_s)}$ to $\beta_{\pi_s^{-1}(E_s)}$.

Let us see that f is surjective. Suppose $(\alpha_s) \in \varprojlim GP(X_s, x_{s0})$. Given a basic entourage $\pi_s^{-1}(E_s)$ of X we wish to construct a $\pi_s^{-1}(E_s)$-chain. Choose $m(s) > s$ so that $\phi_{m(s)s}(X_{m(s)}) = \pi_s(X)$. Say that the $\phi_{m(s)s}^{-1}(E_s)$-term of $\alpha_{m(s)}$ is represented by the chain $y_0, \ldots, y_{n(E_s)}$. Now for each $i \leq n(E_s)$ there is an $x_i^{E_s} \in X$ with $\phi_{m(s)s}(y_i) = \pi_s(x_i^{E_s})$. Notice $x_0^{E_s}, \ldots, x_{n(E_s)}^{E_s}$ is a $\pi_s^{-1}(E_s)$-chain and we can assume that $x_0^{E_s} = x_0$. Define $\beta_{\pi_s^{-1}(E_s)} = \left[x_0^{E_s}, \ldots, x_{n(E_s)}^{E_s}\right]_{\pi_s^{-1}(E_s)}$ and let $\beta = \left(\beta_{\pi_s^{-1}(E_s)}\right)$. Notice $f(\beta) = (\alpha_s)$ since $\pi_s(\beta_{\pi_s^{-1}(E_s)})$ is the E_s-term of α_s.

To see that β is a generalized path, suppose $\pi_t^{-1}(E_t) \subset \pi_s^{-1}(E_s)$. Choose r so that $r > m(s)$ and $r > m(t)$. Set $E_r = \phi_{rs}^{-1}(E_s) \cap \phi_{rt}^{-1}(E_t)$. Since $E_r \subset \phi_{rt}^{-1}(E_t)$, $\phi_{m(r)r}^{-1}(E_r) \subset \phi_{m(r)r}^{-1}\phi_{rt}^{-1}(E_t) = \phi_{m(r)t}^{-1}(E_t)$. Therefore the $\phi_{m(r)r}^{-1}(E_r)$-term of $\alpha_{m(r)}$ is $\phi_{m(r)t}^{-1}(E_t)$-homotopic to the $\phi_{m(r)t}^{-1}(E_t)$-term of $\alpha_{m(r)}$. This homotopy maps via $\phi_{m(r)t}$ to an E_t-homotopy between the images of these two chains. First notice the image of the $\phi_{m(r)r}^{-1}(E_r)$-term of $\alpha_{m(r)}$ under $\phi_{m(r)t}$ is $\pi_t(x_0^{E_r}), \ldots, \pi_t(x_{n(E_r)}^{E_r})$. Now, since $\phi_{m(r)m(t)*}(\alpha_{m(r)}) = \alpha_{m(t)}$, the image of the $\phi_{m(r)t}^{-1}(E_t)$-term of $\alpha_{m(r)}$ under $\phi_{m(r)m(t)}$ is $\phi_{m(t)t}^{-1}(E_t)$-homotopic to the $\phi_{m(t)t}^{-1}(E_t)$-term of $\alpha_{m(t)}$. Therefore the image of the $\phi_{m(r)t}^{-1}(E_t)$-term of $\alpha_{m(r)}$ under $\phi_{m(r)t}$ is E_t-homotopic to $\pi_t(x_0^{E_t}), \ldots, \pi_t(x_{n_{E_t}}^{E_t})$. Then we have that $x_0^{E_r}, \ldots, x_{n_{E_r}}^{E_r}$ is $\pi_t^{-1}(E_t)$-homotopic to $x_0^{E_t}, \ldots, x_{n(E_t)}^{E_t}$. Similarly, $x_0^{E_r}, \ldots, x_{n(E_r)}^{E_r}$ is $\pi_s^{-1}(E_s)$-homotopic to $x_0^{E_s}, \ldots, x_{n(E_s)}^{E_s}$. Since $\pi_t^{-1}(E_t) \subset \pi_s^{-1}(E_s)$, $x_0^{E_t}, \ldots, x_{n(E_t)}^{E_t}$ is $\pi_s^{-1}(E_s)$-homotopic to $x_0^{E_s}, \ldots, x_{n(E_s)}^{E_s}$.

Let us see that f generates the uniform structure on $\varprojlim GP(X_\alpha, x_{0s})$. Notice f is uniformly continuous since each π_{s*} is. Given a basic entourage $\pi_s^{-1}(E_s)^*$ of $GP(X, x_0)$, choose $t > s$ so that $\phi_{ts}(X_t) = \pi_s(X)$. Suppose $\alpha, \beta \in GP(X, x_0)$ with $(f(\alpha), f(\beta)) \in \Pi_t^{-1}(\phi_{ts}^{-1}(E_s)^*)$ where $\Pi_t : \varprojlim GP(X_s, x_{0s}) \to GP(X_t, x_{0t})$ is the projection. Then $(\pi_{t*}(\alpha), \pi_{t*}(\beta)) \in \phi_{ts}^{-1}(E_s)^*$, i.e., $\pi_t(\alpha_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)})^{-1}\pi_t(\beta_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)})$ is $\phi_{ts}^{-1}(E_s)$-homotopic to the edgepath. Then $\phi_{ts}\pi_t(\alpha_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)})^{-1}\phi_{ts}\pi_t(\beta_{\pi_t^{-1}\phi_{ts}^{-1}(E_s)}) = \pi_s(\alpha_{\pi_s^{-1}(E_s)})^{-1}\pi_s(\beta_{\pi_s^{-1}(E_s)})$ is E_s-homotopic in $\phi_{ts}(X_t)$ to the edge. This homotopy lifts to a $\pi_s^{-1}(E_s)$-homotopy from $\alpha_{\pi_s^{-1}(E_s)}^{-1}\beta_{\pi_s^{-1}(E_s)}$ to the edge. Therefore $(\alpha, \beta) \in \pi_s^{-1}(E_s)^*$. □

Corollary 7.0.6. *Suppose $\{X_i, \phi_i\}$ is a Mittag-Leffler inverse sequence of uniform spaces. Let $X = \varprojlim X_i$ and $x_0 = (x_{0i}) \in X$. Suppose H_i are subgroups of $\check{\pi}_1(X_i, x_{0i})$ such that $\phi_{i+1*}(H_{i+1}) \subset H_i$. Notice there are well defined bonding maps $GP(X_{i+1}, x_{0i+1})/H_{i+1} \to GP(X_i, x_{0i})/H_i$ that send $[\alpha]$ to $[\phi_{i*}(\alpha)]$. Suppose $\{H_i, \phi_{i+1*}\}$ is Mittag-Leffler and $\varprojlim{}^1 H_i = 1$. Then $\varprojlim GP(X_i, x_{0i})/H_i$ is uniformly equivalent to $GP(X, x_0)/H$ where $H = \varprojlim H_i$.*

Proof. This result follows from 6.1.3 and 7.0.5. □

Recall that if X_i is compact, then local path connectedness implies uniform local path connectedness and semilocal simple connectedness implies uniform semilocal simple connectedness (.0.12). Also, if X_i is path connected, uniformly locally path connected, and uniformly semilocally simply connected then $GP(X_i)$ is uniformly equivalent to $\widetilde{X}_i$ and $\check{\pi}_1(X_i)$ is isomorphic to $\pi_1(X_i)$. Therefore we have the following.

Corollary 7.0.7. *Suppose a locally uniform joinable chain connected space X is the inverse limit of a Mittag-Leffler inverse sequence of compact spaces that are path connected, locally path connected, and semilocally simply connected. Suppose there are subgroups H_i of $\pi_1(X_i)$ such $\phi_{i+1*}(H_{i+1}) \subset H_i$, $\{H_i, \phi_{i*}\}$ is Mittag-Leffler, and $\varprojlim{}^1 H_i = 1$. Then the inverse limit of the covering spaces C_i corresponding to the subgroups H_i is uniformly equivalent to the generalized uniform covering space $GP(X)/H$ where $H = \varprojlim H_i$. Note that $GP(X)/H$ is Hausdorff since H is closed in $GP(X)$ as the inverse limit of closed subsets.*

Now, given a generalized uniform covering map of $X = \varprojlim X_i$, we wish to express it as the inverse limit of uniform covering maps of X_i. We will assume that the generalized uniform covering space is Hausdorff and locally uniformly chain connected so that it is uniformly equivalent to $GP(X)/H$ for some closed subgroup H of $\check{\pi}_1(X)$ (see 3.3.2).

Lemma 7.0.8. *Suppose $\{X_s, \phi_{ts}\}$ is a Strong Mittag-Leffler inverse system of uniform spaces. Let $X = \varprojlim X_s$ and $x_0 = (x_{0s}) \in X$. Consider the inverse system $\{GP(X_s, x_{0s}), \phi_{ts*}\}$ and the map $f = \varprojlim \pi_{s*}$. If $A \subset GP(X, x_0)$ is closed then $f(A)$ is uniformly equivalent to $\varprojlim \pi_{s*}(A)$.*

Proof. Since f is a uniform equivalence it suffices to show that if $\alpha \in GP(X, x_0)$ with $f(\alpha) \in \varprojlim \pi_{s*}(A)$ then $\alpha \in A$. Given an entourage E of X, $f(E)$ is an entourage of $\varprojlim GP(X_s, x_{s0})$ so $\pi_t^{-1}(E_t) \subset f(E)$ for some t and some entourage E_t of X_t. Now $\pi_{t*}(\alpha) = \pi_{t*}(\beta)$ for some $\beta \in A$. Then $(f(\alpha), f(\beta)) \in \pi_t^{-1}(E_t) \subset f(E)$. Since f is injective that implies $(\alpha, \beta) \in E$. Then since A is closed $\alpha \in A$. □

Proposition 7.0.9. *Suppose $\{X_i, \phi_i\}$ is a Mittag-Leffler inverse sequence of uniform spaces. Let $X = \varprojlim X_i$ and $x_0 = (x_{0i}) \in X$. Given a closed subgroup H of $\check{\pi}_1(X, x_0)$, let $H_i = \pi_{i*}(H)$. Then $GP(X, x_0)/H$ is uniformly equivalent to $\varprojlim GP(X_i, x_{0i})/H_i$.*

Proof. The action of H_i on $GP(X_i, x_{0i})$ is neutral (3.1.1) and free. Then $\varprojlim GP(X_i, x_{0i})/H_i = \varprojlim GP(X_i, x_0)/\varprojlim H_i$ by 6.1.3. But $\varprojlim H_i = H$ by 7.0.8 and $\varprojlim GP(X_i, x_{0i}) = GP(X, x_0)$ by 7.0.5. □

Notice the bonding maps ϕ_{i*} are surjective since $\pi_{i-1*} = \phi_{i*} \circ \pi_{i*}$ and each restriction $\pi_{i*} : H \to H_i$ is surjective by definition. Therefore $\{H_i, \phi_{i*}\}$ is Mittag-Leffler and $\varprojlim{}^1 H_i = 1$. Then we have the converse to 7.0.6.

Again, in the case that the X_i are compact, path connected, locally path connected, and semilocally simply connected we obtain covering maps of X_i relative to subgroups of the fundamental groups. Therefore we have the following.

Theorem 7.0.10. *Suppose a locally uniform joinable chain connected space X is represented as an inverse limit of a Mittag-Leffler inverse sequence $\{X_i, \phi_i\}$ of compact spaces X_i that are path connected, locally path connected, and semilocally simply connected. Then generalized uniform covering spaces of X that are locally uniform joinable chain connected coincide with inverse limits of covering spaces of X_i relative to subgroups H_i of $\pi_1(X_i)$ such that $\phi_{i+1*}(H_{i+1}) \subset H_i$ and the inverse sequence $\{H_i, \phi_{i*}\}$ is Mittag-Leffler with trivial $\varprojlim{}^1$.*

Bibliography

Bibliography

[1] V. Berestovskii and C. Plaut. Uniform universal covers of uniform spaces. *Topology App.*, 154:1748–1777, 2007.

[2] N. Bourbaki. *General topology.* Springer, Berlin, 1989.

[3] N. Brodskiy, J. Dydak, B. LaBuz, and A. Mitra. Group actions and covering maps in the uniform category. 2008. preprint.

[4] N. Brodskiy, J. Dydak, B. LaBuz, and A. Mitra. Rips complexes and covers in the uniform category. 2008. preprint.

[5] A. Zastrow H. Fischer. Generalized universal coverings and the shape group. *Fund. Math.*, 197:167–196, 2007.

[6] I. M. James. *Topological and uniform spaces.* Springer, Berlin, 1987.

[7] J. Krasinkiewicz and P. Minc. Generalized paths and pointed 1-movability. *Fund. Math.*, 104:141–153, 1979.

[8] S. Mardešić and J. Segal. *Shape theory.* North-Holland Publ.Co., Amsterdam, 1982.

[9] J.R. Munkres. *Topology.* Prentice Hall, Upper Saddle River, NJ, 2000.

[10] C. Plaut. Quotients of uniform spaces. *Topology App.*, 153:2430–2444, 2006.

[11] E. Spanier. *Algebraic topology.* McGraw-Hill, New York, 1966.

Appendix

Uniform structures and topology

Lemma .0.11. *For any uniform space X, any entourage E of X, and any $x \in X$, $x \in \mathrm{Int}B(x, E)$. Further, if $y \in X$ and there is an entourage F of X such that $B(y, F) \subset B(x, E)$, then $y \in \mathrm{Int}B(x, E)$.*

Proof. Given an entourage E of X and an $x \in X$, let U be the set of all $y \in X$ for which there is an entourage F_y of X such that $B(y, F_y) \subset B(x, E)\}$. Note $x \in U \subset B(x, E)$; we will see that U is open. Suppose $y \in U$. Let F_y be an entourage so that $B(y, F_y) \subset B(x, E)$. Choose an entourage H of X so that $H^2 \subset F_y$. To see that $B(y, H) \subset U$, suppose $(y, z) \in H$. Now $B(z, H) \subset B(x, E)$ since if $(z, w) \in H$, $(y, w) \in H^2 \subset F_y$ so $(w, x) \in E$. Therefore $z \in U$ and U is open. □

Proposition .0.12. *Suppose X is a compact uniform space.*

1. *If X is locally path connected then X is uniformly locally path connected.*

2. *If X is semilocally simply connected then X is uniformly semilocally simply connected.*

Proof. 1. Given an entourage E if X, for each $x \in X$ there is a neighborhood U_x of x so that if $y, z \in U_x$, y and z can be joined by an E-bounded path. Choose an entourage F_x so that $B(x, (F_x)^2) \subset U_x$. Since $x \in \mathrm{Int}B(x, F_x)$ (.0.11), $\{\mathrm{Int}B(x, F_x)\}_{x \in X}$ is an open cover of X. Since X is compact it has a finite subcover, say $\{\mathrm{Int}B(x_i, F_{x_i})\}_{i=1}^n$. Set $F = \cap_{i=1}^n F_i$. We wish to see that if $(x, y) \in F$, x and y can be joined by an E-bounded path. Now $x \in \mathrm{Int}B(x_i, F_{x_i})$ for some i. Notice since $(x, y) \in F_{x_i}$, $(y, x_i) \in F_{x_i}^2$ so $y \in U_{x_i}$. Now $x \in U_{x_i}$ as well so x and y can be joined by an E-bounded path.

2. For each $x \in X$ there is a neighborhood U_x of x so that any loop contained in U_x is homotopically trivial. Choose an entourage E_x so that $B(x, E_x) \subset U_x$. Then $\{\mathrm{Int}B(x, E_x)\}_{x \in X}$ is an open cover of X so it has a finite subcover, say
$\{\mathrm{Int}B(x_i, E_{x_i})\}_{i=1}^n$. Set $E = \cap_{i=1}^n E_i$. Now given any E-bounded loop, choose a point x on that loop. Now $x \in \mathrm{Int}B(x_i, E_{x_i})$ for some i and since the loop is E-bounded, the loop is contained in U_{x_i}. Therefore the loop is homotopically trivial. □

Vita

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