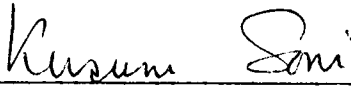


To the Graduate Council:

I am submitting herewith a thesis written by Tohun A. Akinleye entitled "Asymptotic Expansion of Some Laplace Integrals with Explicit Error Terms." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

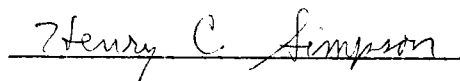


K. Soni, Major Professor

We have read this thesis and
recommend its acceptance:

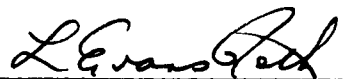


Robert A. Remar



Henry C. Simpson

Accepted for the Council:



Vice Chancellor
Graduate Studies and Research

ASYMPTOTIC EXPANSION OF SOME LAPLACE INTEGRALS
WITH EXPLICIT ERROR TERMS

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Tohun A. Akinleye
December 1981

3055400

To my wife, Juli O. and
children Tokunbo, Fumilayo, Wole and Yomi

ACKNOWLEDGMENTS

The material presented here pertains to the work done between September 1980 and August 1981.

The author wishes to acknowledge with gratitude the guidance of his advisor, Dr. K. Soni, for aiding and seeing to the success of this paper. Her detailed criticism has improved many sections of this paper.

I offer my thanks to my wife Juli and my friend Ikem who have both helped me in the formation of this paper.

ABSTRACT

The asymptotic expansion of the Laplace integral,

$$I(z) = \int_0^{\infty} e^{-zt} f(t) dt, \quad |z| \rightarrow \infty, \quad |\arg z| < \frac{\pi}{2}, \quad (1)$$

where $f(t)$ is a function whose behavior near the origin is known, is generally obtained by Watson's Lemma.

In many applications, $f(t)$ is a function analytic in the right half complex t -plane except that it may have a branch point at the origin. There are many completely different and comparatively more sophisticated techniques being used to obtain the asymptotic expansion of $I(z)$ as $|z| \rightarrow \infty$. Dr. K. Soni's technique for obtaining a uniform asymptotic expansion for finite Laplace transform is comparatively simple and less restrictive. Furthermore, it provides the remainder term explicitly when the expansion is terminated after a finite number of terms. We use this technique to obtain the asymptotic expansion of $I(z)$ together with an explicit representation for the error term. By considering different examples we show how the error, involved in replacing $I(z)$ by a finite number of terms in the asymptotic expansion, can be estimated

TABLE OF CONTENTS

CHAPTER	PAGE
I. INTRODUCTION	1
II. MAIN RESULTS	3
Theorem 1	3
Theorem 2	11
III. APPLICATIONS	16
Example 1: Incomplete Gamma Function	16
Example 2	18
Example 3	30
BIBLIOGRAPHY	34
APPENDIX	36
VITA	40

NOTATIONS

O = Landau order symbol

Γ = gamma function

$t \rightarrow 0+$ = t approaching zero from the right

$R_n(z)$ = the difference between $f(z)$ and n th partial sum of the series

f_{n-1} = n th partial sum of a given series

$f^{(k)}(t)$ = k th derivatives of a function $f(t)$

z = asymptotic variable

$f(z)$ = a function of the real or complex variables z

$\Gamma(.,.)$ = complementary incomplete gamma function

$(.)_p$ = Pochhammer's symbol

$\sim$ = asymptotically equal to

CHAPTER 1

INTRODUCTION

Suppose we are given an integral of the form

$$I(z) = \int_0^a e^{-zt} f(t) dt, \quad (1.1)$$

where $f(t)$ is real and absolutely integrable in the interval $(0, a)$,
 $0 < a < \infty$. If

$$f(t) \sim \sum_{k=0}^{\infty} a_k t^{\Lambda_k}, \quad t \rightarrow 0+, \quad -1 < \Lambda_0 \dots \Lambda_{n-1} < \Lambda_n \dots, \quad (1.2)$$

then, the asymptotic behavior of $I(z)$ is given by Watson Lemma [Olver, 1974, p. 71] and as $z \rightarrow \infty$ in the sector $|\arg z| < \frac{\pi}{2}$,

$$I(z) \sim \sum_{k=0}^{\infty} \Gamma(\Lambda_k + 1) a_k z^{-(\Lambda_k + 1)}. \quad (1.3)$$

The relations (1.2) and (1.3) can also be interpreted in the following sense: If

$$f(t) = \sum_{k=0}^{n-1} a_k t^{\Lambda_k} + R_n(t), \quad t \rightarrow 0+, \quad -1 < \Lambda_0 \dots < \Lambda_{n-1} < \Lambda_n, \quad (1.4)$$

where

$$R_n(t) = o(t^{\Lambda_n}), \quad t \rightarrow 0+, \quad (1.5)$$

then,

$$I(z) = \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{-(\Lambda_k + 1)} + E_n, \quad (1.6)$$

where

$$E_n = O(z^{-(\Lambda_n + 1)}) , \quad z \rightarrow \infty , \quad (1.7)$$

in the sector $|\arg z| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$, and z^{Λ_k} assume the principal values.

Recently, Dr. K. Soni obtained a uniform asymptotic expansion for finite Laplace transform. We use that technique to obtain the asymptotic expansion of the function $I(z)$ as defined by (1.1), together with the remainder term when the expansion is terminated after a finite number of terms.

We apply these results to obtain the estimate of error in the asymptotic expansion of certain integrals. Our results do not follow directly from Watson's Lemma because Watson's Lemma provides only the order of the remainder as $|z| \rightarrow \infty$, [see (2, p. 104)], whereas we obtain an estimate of the error when the asymptotic expansion is terminated after a finite number of terms. Thus, for the type of functions that we consider, we obtain more information than that provided by Watson's Lemma.

CHAPTER 2

MAIN RESULTS

We prove two theorems. In theorem 1, we obtain, under appropriate conditions on f , asymptotic expansion for $I(z)$ as $z \rightarrow \infty$ when $0 < a < \infty$. As stated in the introduction, the technique that we follow here is the same as that given by Dr. Soni [3, 1980]. Then we let $a \rightarrow \infty$ and obtain the corresponding expansion for $I(z)$ when the interval of integration in (1.1) is $(0, \infty)$. This result is given in theorem 2.

A. Theorem 1

Let $f(t)$ be p times continuously differentiable in $0 < t \leq a$ and let

$$f(t) = \sum_{k=0}^{n-1} a_k t^{\Lambda_k} + R_n(t) \quad (2.1)$$

where

$$R_n(t) = O(t^{\Lambda_n}) , \quad t \rightarrow 0+ , \quad -1 < \Lambda_0 < \Lambda_1 \dots < \Lambda_n < \dots . \quad (2.2)$$

If

$$R_n^{(k)}(t) = O(t^{(\Lambda_n - k)}) , \quad t \rightarrow 0+ , \quad (2.3)$$

for $0 < k \leq p$, and $p < \Lambda_n + 1$, then for $|\arg z| < \frac{\pi}{2}$,

$$\begin{aligned}
\int_0^a e^{-zt} f(t) dt &= \sum_{k=0}^{n-1} \Gamma(\Lambda_k + 1) z^{-(\Lambda_k + 1)} a_k \\
&- \sum_{k=0}^{p-1} e^{-za} f^{(k)}(a) z^{-(k+1)} \\
&- z^{-p} \sum_{k=0}^{n-1} \int_a^\infty e^{-zt} (\Lambda_k - p + 1)_p t^{\Lambda_k - p} dt \\
&+ z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt, \tag{2.4}
\end{aligned}$$

where

$$(\Lambda_k - p + 1)_p = (\Lambda_k - p + 1)(\Lambda_k - p + 2) \dots \Lambda_k.$$

PROOF OF THEOREM 1: Let $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$. We see from (2.1) that

$$f(t) = f_{n-1}(t) + R_n(t) \tag{2.5}$$

where

$$f_{n-1}(t) = \sum_{k=0}^{n-1} a_k t^{\Lambda_k}. \tag{2.6}$$

If we multiply both sides of (2.5) by e^{-zt} and integrate from 0 to a , we obtain

$$\int_0^a e^{-zt} f(t) dt = \int_0^a e^{-zt} (f_{n-1}(t) + R_n(t)) dt.$$

Notice that

$$f_{n-1}(t) = o(t^{\Lambda_0}), \quad t \rightarrow 0+,$$

and

$$R_n(t) = O(t^{\Lambda_n}) , \quad t \rightarrow 0+ .$$

Since $\Lambda_0 > -1$ and $\Lambda_n > \Lambda_0 > -1$, each of the integrals in

$$\int_0^a e^{-zt} (f_{n-1}(t) + R_n(t)) dt$$

converges. As a result of this convergence, we can separate the integrals as follows:

$$\begin{aligned} \int_0^a e^{-zt} f(t) dt &= \int_0^a e^{-zt} f_{n-1}(t) dt + \int_0^a e^{-zt} R_n(t) dt \\ &= \int_0^\infty e^{-zt} f_{n-1}(t) dt - \int_a^\infty e^{-zt} f_{n-1}(t) dt \\ &\quad + \int_0^a e^{-zt} R_n(t) dt . \end{aligned} \tag{2.7}$$

Now,

$$\int_0^\infty e^{-zt} f_{n-1}(t) dt = \int_0^\infty e^{-zt} \left(\sum_{k=0}^{n-1} a_k t^{\Lambda_k} \right) dt . \tag{2.8}$$

But, for $0 \leq k \leq n-1$, $t \geq 1$, $z = x + iy$, $x > 0$,

$$|e^{-zt} t^{\Lambda_k}| \leq e^{-xt} t^{\Lambda_k} .$$

Also, for $0 < t \leq 1$, $t^{\Lambda_k} \leq t^{\Lambda_0}$ and $\Lambda_0 > -1$.

Therefore,

$$\begin{aligned}
\left| \int_0^{\infty} e^{-zt} t^{\Lambda_k} dt \right| &\leq \int_0^{\infty} |e^{-zt} t^{\Lambda_k}| dt \\
&\leq \int_0^1 e^{-xt} t^{\Lambda_k} dt + \int_1^{\infty} e^{-xt} t^{\Lambda_k} dt .
\end{aligned}$$

Since both the integrals on the right converge, each of the integrals

$$\int_0^{\infty} e^{-zt} t^{\Lambda_k} dt$$

converges absolutely. Since there are only a finite number of terms involved, we can interchange the order of integration and summation in (2.8). Thus,

$$\int_0^{\infty} e^{-zt} \left(\sum_{k=0}^{n-1} a_k t^{\Lambda_k} \right) dt = \sum_{k=0}^{n-1} a_k \int_0^{\infty} e^{-zt} t^{\Lambda_k} dt , \quad (2.9)$$

[see Appendix item 7]. For $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$,

$$\int_0^{\infty} e^{-zt} t^{\Lambda_k} dt = \frac{\Gamma(\Lambda_k + 1)}{z^{\Lambda_k + 1}} .$$

Therefore, integrating term by term, we get

$$\int_0^{\infty} e^{-zt} f_{n-1}(t) dt = \sum_{k=0}^{n-1} a_k \frac{\Gamma(\Lambda_k + 1)}{z^{\Lambda_k + 1}} .$$

Next, we will show that

$$\int_a^{\infty} e^{-zt} f_{n-1}(t) dt = \sum_{k=0}^{p-1} e^{-za} f_{n-1}(a) z^{-(k+1)} + z^{-p} \int_a^{\infty} e^{-zt} f_{n-1}^{(p)}(t) dt . \quad (2.10)$$

and

$$\begin{aligned} \int_0^a e^{-zt} R_n(t) dt &= - \sum_{k=0}^{p-1} e^{-za} R_n^{(k)}(a) z^{-(k+1)} \\ &\quad + z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt . \end{aligned} \quad (2.11)$$

Using integration by parts in (2.10), we obtain

$$\begin{aligned} \int_a^\infty e^{-zt} f_{n-1}(t) dt &= - \frac{e^{-zt}}{z} f_{n-1}(t) \Big|_a^\infty \\ &\quad + z^{-1} \int_a^\infty e^{-zt} f'_{n-1}(t) dt \\ &= \frac{e^{-za}}{z} f'_{n-1}(a) + \frac{1}{z} \int_a^\infty e^{-zt} f'_{n-1}(t) dt . \end{aligned} \quad (2.12)$$

Similarly,

$$z^{-1} \int_a^\infty e^{-zt} f'_{n-1}(t) dt = z^{-2} e^{-za} f'_{n-1}(a) + z^{-2} \int_a^\infty e^{-zt} f''_{n-1}(t) dt .$$

Now we can prove by mathematical induction that

$$\begin{aligned} \int_0^\infty e^{-zt} f_{n-1}(t) dt &= e^{-za} \sum_{k=0}^{p-1} z^{-(k+1)} f_{n-1}^{(k)}(a) \\ &\quad + z^{-p} \int_a^\infty e^{-zt} f_{n-1}^{(p)}(t) dt , \end{aligned} \quad (2.13)$$

which is Eq. (2.10).

Finally, using integration by parts in (2.12), we obtain

$$\begin{aligned}\int_0^a e^{-zt} R_n(t) dt &= - \frac{e^{-zt}}{z} R_n(t) \Big|_0^a + z^{-1} \int_0^a e^{-zt} R_n'(t) dt \\ &= - \frac{e^{-za}}{z} R_n(a) + z^{-1} \int_0^a e^{-zt} R_n'(t) dt .\end{aligned}$$

Using integration by parts again, and assuming that $\Lambda_n > 1$, we get

$$\begin{aligned}z^{-1} \int_0^a e^{-zt} R_n'(t) dt &= - \frac{e^{-zt}}{z^2} R_n'(t) \Big|_0^a + z^{-2} \int_0^a e^{-zt} R_n''(t) dt \\ &= - \frac{e^{-za}}{z^2} R_n'(a) + z^{-2} \int_0^a e^{-zt} R_n''(t) dt .\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^a e^{-zt} R_n(t) dt &= - \frac{e^{-za}}{z} R_n(0) - \frac{e^{-za}}{z^2} R_n'(a) \\ &\quad + z^{-2} \int_0^a e^{-zt} R_n''(t) dt .\end{aligned}$$

Again, by mathematical induction, we can prove that

$$\int_0^a e^{-zt} R_n(t) dt = - \sum_{k=0}^{p-1} e^{-za} z^{-(k+1)} R_n^{(k)}(a) + z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt$$

which is Eq. (2.11). Combining (2.9), (2.10), and (2.11), we get

$$\begin{aligned}\int_0^a e^{-zt} f(t) dt &= \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{-(\Lambda_k + 1)} \\ &\quad - \sum_{k=0}^{p-1} e^{-za} f_{n-1}^{(k)}(a) z^{-(k+1)}\end{aligned}$$

$$\begin{aligned}
& - z^{-p} \int_a^\infty e^{-zt} f_{n-1}^{(p)}(t) dt - \sum_{k=0}^{p-1} e^{-za} z^{-(k+1)} R_n^{(k)}(a) \\
& + z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt \\
& = \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{-(\Lambda_k + 1)} \\
& - \left(\sum_{k=0}^{p-1} e^{-za} f_{n-1}^{(k)}(a) z^{-(k+1)} + \sum_{k=0}^{p-1} e^{-za} R_n^{(k)}(a) z^{-(k+1)} \right) \\
& - z^{-p} \int_a^\infty e^{-zt} f_{n-1}^{(p)}(t) dt + z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt .
\end{aligned} \tag{2.14}$$

But

$$R_n(t) = f(t) - f_{n-1}(t)$$

so that

$$f_{n-1}^{(k)}(a) + R_n^{(k)}(a) = f^{(k)}(a) . \tag{2.15}$$

Therefore,

$$\sum_{k=0}^{p-1} e^{-za} z^{-(k+1)} (f_{n-1}^{(k)}(a) + R_n^{(k)}(a)) = \sum_{k=0}^{p-1} e^{-za} z^{-(k+1)} f^{(k)}(a) . \tag{2.16}$$

Using (2.16) in (2.14), we get

$$\begin{aligned}
\int_0^a e^{-zt} f(t) dt &= \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{-(\Lambda_k+1)} \\
&\quad - \sum_{k=0}^{p-1} e^{-za} f_{n-1}^{(k)}(a) z^{-(k+1)} \\
&\quad - z^{-p} \int_a^\infty e^{-zt} f_{n-1}^{(p)}(t) dt + z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt .
\end{aligned} \tag{2.17}$$

Since

$$\begin{aligned}
f_{n-1}^{(p)}(t) &= \sum_{k=0}^{n-1} a_k \frac{d^p}{dt^p} t^{\Lambda_k} \\
&= \sum_{k=0}^{n-1} a_k \Lambda_k (\Lambda_k - 1) \dots (\Lambda_k - p + 1) t^{\Lambda_k - p} ,
\end{aligned} \tag{2.18}$$

by using Pochhammer's symbol, we can write

$$f_{n-1}^{(p)}(t) = \sum_{k=0}^{n-1} a_k (\Lambda_k - p + 1)_p t^{\Lambda_k - p} . \tag{2.19}$$

Therefore,

$$\begin{aligned}
\int_a^\infty e^{-zt} f_{n-1}^{(p)}(t) dt &= \int_a^\infty e^{-zt} \sum_{k=0}^{n-1} a_k (\Lambda_k - p + 1)_p t^{\Lambda_k - p} dt \\
&= \sum_{k=0}^{n-1} a_k (\Lambda_k - p + 1)_p \int_a^\infty e^{-zt} t^{\Lambda_k - p} dt .
\end{aligned} \tag{2.20}$$

The interchange of integration and summation is justified because each of the integrals is absolutely convergent. Combining (2.17) and (2.20), we obtain Eq. (2.4).

Notice that the third term in (2.4) can be expressed in terms of complementary incomplete gamma functions. For real z , let $zt = u$, $zdt = du$. Then,

$$\begin{aligned} \int_a^\infty e^{-zt} t^{\Lambda_k - p} dt &= z^{p - \Lambda_k - 1} \int_{az}^\infty e^{-u} u^{\Lambda_k - p} du \\ &= z^{p - \Lambda_k - 1} \Gamma(\Lambda_k - p + 1, az) . \end{aligned} \quad (2.21)$$

Using (2.21) in (2.4), we obtain

$$\begin{aligned} \int_0^a e^{-zt} f(t) dt &= \sum_{k=0}^{n-1} \Gamma(\Lambda_k + 1) z^{-(\Lambda_k + 1)} a_k + \sum_{k=0}^{p-1} e^{-za} f^{(k)}(a) z^{-(k+1)} \\ &\quad + z^{-p} \sum_{k=0}^{n-1} z^{-(\Lambda_k - p + 1)} \Gamma(\Lambda_k - p + 1, az) (\Lambda_k - p + 1)_p a_k \\ &\quad + z^{-p} \int_0^a e^{-zt} R_n^{(p)}(t) dt . \end{aligned} \quad (2.22)$$

By analytic continuation, (2.4) and (2.22) hold for all values of z such that $|\arg z| < \frac{\pi}{2}$.

B. Theorem 2

If $f(t)$ is p times continuously differentiable in $0 < t < \infty$, satisfies (2.1)-(2.3) and if

$$f^{(k)}(t) = o(e^{ct}) , \quad t \rightarrow \infty ,$$

for some $c > 0$ and for $k = 0, 1, 2, \dots, p$, then, for every $|z|$ sufficiently large, and lying in the sector $|\arg z| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$,

$$\int_0^{\infty} e^{-zt} f(t) dt = \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{-(\Lambda_k + 1)} + z^{-p} \int_0^{\infty} e^{-zt} R_n^{(p)}(t) dt . \quad (2.23)$$

PROOF OF THEOREM 2: If we can show that as $a \rightarrow \infty$ in theorem 1,

$$\int_a^{\infty} e^{-zt} f_{n-1}^{(p)}(t) dt \rightarrow 0 ,$$

$$\sum_{k=0}^{p-1} e^{-za} z^{-(k+1)} f^{(k)}(a) \rightarrow 0$$

and,

$$\int_0^a e^{-zt} R_n^{(p)}(t) dt \rightarrow \int_0^{\infty} e^{-zt} R_n^{(p)}(t) dt ,$$

then we would obtain (2.18)

Let z be a fixed number in the sector $-\frac{\pi}{2} + \delta \leq \arg z \leq \frac{\pi}{2} - \delta$. Since $|z|$ is sufficiently large, we can assume that $\operatorname{Re} z = x \geq c + 1$. Then for $k = 0, 1, \dots, n-1$,

$$\begin{aligned} \left| \int_a^{\infty} e^{-zt} t^{\Lambda_k - p} dt \right| &\leq \int_a^{\infty} \left| e^{-zt} \right| t^{\Lambda_k - p} dt = \int_a^{\infty} e^{-xt} t^{\Lambda_k - p} dt \\ &\leq \int_a^{\infty} e^{-t} t^{\Lambda_k - p} dt . \end{aligned}$$

Since the last integral converges when the lower limit is any fixed number greater than zero,

$$\lim_{a \rightarrow \infty} \int_a^\infty e^{-t} t^{\Lambda_k - p} dt = 0 .$$

Therefore,

$$\lim_{a \rightarrow \infty} \int_a^\infty e^{-zt} f_{n-1}^{(p)}(t) dt = 0 .$$

Again, for $0 \leq k \leq p$, and $x \geq c + 1$,

$$\begin{aligned} \left| e^{-zt} f^{(k)}(t) \right| &\leq e^{-xt} \left| f^{(k)}(t) \right| \\ &\leq e^{-ct-t} \left| f^{(k)}(t) \right| . \end{aligned} \quad (1)$$

But $e^{-ct} f^{(k)}(t)$ is bounded as $t \rightarrow \infty$. Therefore

$$\lim_{a \rightarrow \infty} (e^{-za} f^{(k)}(a)) = 0 . \quad (2)$$

Also, for $0 \leq m \leq n$, $0 \leq k \leq p$,

$$\begin{aligned} \left| e^{-zt} t^{\Lambda_m - k} \right| &\leq e^{-xt} t^{\Lambda_m - k} \\ &\leq e^{-ct} e^{-t} t^{\Lambda_m - k} \\ &\leq e^{-t} t^{\Lambda_m - k} . \end{aligned} \quad (3)$$

Since $\Lambda_0 < \Lambda_1, \dots < \Lambda_{n-1} < \Lambda_n$, we can write

$$f_{n-1}(t) = o(t^{\Lambda_n}) , \quad t \rightarrow \infty .$$

Therefore,

$$f_{n-1}^{(p)}(t) = o(t^{\Lambda_n - p}) , \quad t \rightarrow \infty ,$$

$$\lim_{a \rightarrow \infty} e^{-za} R_n^{(k)}(a) = 0 .$$

By (1) and (3), as $t \rightarrow \infty$,

$$\begin{aligned} \left| e^{-zt} R_n^{(p)}(t) \right| &= \left| e^{-zt} (f^{(p)}(t) - f_{n-1}^{(p)}(t)) \right| \\ &\leq \left| e^{-zt} f^{(p)}(t) \right| + \left| e^{-zt} f_{n-1}^{(p)}(t) \right| \\ &\leq M_1 e^{-t} + M_2 e^{-t} t^{\Lambda_n - p} \end{aligned}$$

for some constants M_1 and M_2 . The constant M_1 is obtained as follows:

$$\begin{aligned} \left| e^{-zt} f^{(p)}(t) \right| &\leq \left| e^{-xt} f^{(p)}(t) \right| \leq e^{-ct} e^{-t} \left| f^{(p)}(t) \right| \\ &= e^{-t} e^{-ct} \left| f^{(p)}(t) \right| \\ &\leq M_1 e^{-t} . \end{aligned}$$

Therefore, by (5),

$$\int_0^\infty e^{-zt} R_n^{(p)}(t) dt$$

converges absolutely and so

$$\lim_{a \rightarrow \infty} \int_0^a e^{-zt} R_n^{(p)}(t) dt = \int_0^\infty e^{-zt} R_n^{(p)}(t) dt .$$

Therefore,

$$\begin{aligned}
\int_0^{\infty} e^{-zt} f(t) dt &= \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{\Lambda_k + 1} \\
&+ z^{-p} \int_0^{\infty} e^{-zt} R_n^{(p)}(t) dt, \\
|\arg z| &\leq \frac{\pi}{2} - \delta < \frac{\pi}{2},
\end{aligned}$$

which is Eq. (2.18).

CHAPTER 3

APPLICATIONS

A. Example 1: Incomplete Gamma Function

Consider the integral

$$I(z) = \int_0^{\infty} e^{-zt} (1+t)^{\alpha-1} dt, \quad (|\arg z| < \frac{\pi}{2}) \quad (3.1)$$

where $(1+t)^{\alpha-1}$ is positive when α is real. The asymptotic expansion of $I(z)$ can be obtained directly by using simple integration by parts. However, $f(t) = (1+t)^{\alpha-1}$ can be expanded into a convergent power series when $|t| < 1$.

$$f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)\Gamma(1-\alpha)} t^n, \quad |t| < 1. \quad (3.2)$$

Then,

$$f_{n-1}(t) = \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(1-\alpha)} t^k, \quad (3.3)$$

where $k! = \Gamma(k+1)$. Let $p = n$. Then,

$$I(z) = \sum_{k=0}^{n-1} a_k \Gamma(\Lambda_k + 1) z^{-(\Lambda_k+1)} + E_n(z), \quad (3.4)$$

where, by (2.18),

$$E_n(z) = z^{-n} \int_0^{\infty} e^{-zt} R_n^{(n)}(t) dt. \quad (3.5)$$

$$\begin{aligned}
R_n(t) &= f(t) - f_{n-1}(t) \\
&= f(t) - \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(1-\alpha)} t^k.
\end{aligned} \tag{3.6}$$

From (3.6),

$$\begin{aligned}
R_n^{(n)}(t) &= f^{(n)}(t) \\
&= (-1)^n \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} (1+t)^{\alpha-n-1}.
\end{aligned} \tag{3.7}$$

Therefore,

$$E_n(z) = (-1)^n z^{-n} \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \int_0^\infty e^{-zt} (1+t)^{\alpha-n-1} dt. \tag{3.8}$$

Thus, the expansion and the remainder that we obtain is the same as that obtained by using integration by parts technique.

To obtain an error bound we may assume that α is real and $n+1 > \alpha$. Then, $(1+t)^{\alpha-n-1}$ is a decreasing function of t . By using the Second Mean-Value-Theorem of integral calculus (see Appendix) we obtain,

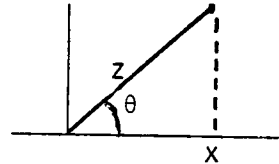
$$\begin{aligned}
\left| z^n E_n(z) \right| &\leq \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \left| \int_0^\infty e^{-zt} (1+t)^{\alpha-n-1} dt \right| \\
&\leq \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \int_0^\infty e^{-xt} (1+t)^{\alpha-n-1} dt \\
&= \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \int_0^N e^{-xt} dt, \quad 0 < N < \infty
\end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} (1 - e^{-Nx}) x^{-1} \\
&\leq \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} x^{-1} \\
&= \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} |z^{-1}| \sec \theta
\end{aligned}$$

where

$$\theta = \arg z$$

$$x = z \cos \theta$$



Therefore,

$$|E_n(z)| \leq \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} |z|^{n+1} \sec \theta.$$

This estimate for the error holds for all z in the half-plane $\operatorname{Re} z > 0$.

However, it is useful only when $|z|$ is reasonably large and

$$|\theta| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}.$$

Notice that the above inequality also implies that

$$E_n(z) = O(|z|^{-n-1}) \quad , \quad |z| \rightarrow \infty.$$

If θ is nearly equal to $\pm 90^\circ$, the bound for error term is not useful because in that case, $\sec \theta$ is very large. In such cases, we have to use other methods to obtain an error bound.

B. Example 2

We consider the integral

$$I(z) = \int_0^\infty e^{-z \cosh u} du. \quad (3.9)$$

In this example, assume $\arg z = 0$ so that $z = x$. The integral above is not a Laplace integral but can be transformed into the desired form by proper substitution.

Let

$$\cosh u - 1 = t .$$

Since $\cosh u - 1$ is an increasing function of u in $(0, \infty)$, the mapping $\cosh u - 1 \rightarrow t$ is one to one. Therefore,

$$I(z) = \int_0^{\infty} e^{-zt} \frac{du}{dt} dt , \quad (3.10)$$

where

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{\sinh u} = \frac{1}{\sqrt{\cosh^2 u - 1}} \\ &= \frac{1}{\sqrt{(t+1)^2 - 1}} \\ &= \frac{1}{\sqrt{t^2 + 2t}} . \end{aligned}$$

Therefore,

$$I(z) = \int_0^{\infty} e^{-zt} \frac{1}{\sqrt{t^2 + 2t}} dt . \quad (3.11)$$

Notice that we cannot apply integration by parts technique here because

$$\begin{aligned}\frac{d}{dt} (t^2 + 2t)^{-1/2} &= -\frac{1}{2} (t^2 + 2t)^{-3/2} (2t + 2) \\ &= -t^{-3/2} (t + 2)^{-3/2} (t + 1) .\end{aligned}$$

This function is not integrable at the origin because

$$t^{-3/2}(t + 2)^{-3/2} (t + 1) = (t^{-3/2}) , t \rightarrow 0+ .$$

Write

$$f(t) = \frac{1}{\sqrt{t^2 + 2t}} = (2t)^{-1/2} \left(1 + \frac{t}{2}\right)^{-1/2}$$

so that

$$\begin{aligned}f(t) &= (2t)^{-1/2} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + 1)\Gamma(1/2)} (t/2)^k , \\ |t| &< 2 .\end{aligned}\tag{3.12}$$

Notice that in this example, $f(t)$ is obtained by substituting $t/2$ for t and $-1/2$ for $\alpha - 1$ in example 1 which has been previously discussed.

As earlier, we define

$$f_{n-1}(t) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + 1)\Gamma(1/2)} (t/2)^{k-1/2} .\tag{3.13}$$

We again take $p = n$. Then,

$$I(z) = \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + 1)\Gamma(1/2)} z^{-(k+1/2)} + E_n(z) ,\tag{3.14}$$

where

$$E_n(z) = z^{-n} \int_0^{\infty} e^{-zt} R_n^{(n)}(t) dt \quad (3.15)$$

and

$$\begin{aligned} R_n(t) &= f(t) - f_{n-1}(t) \\ &= (2t)^{-1/2} (1 + t/2)^{-1/2} - \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k + 1/2)}{\Gamma(k + 1) \Gamma(1/2)} (t/2)^{k-1/2} \\ &= (2t)^{-1/2} \left[(1 + t/2)^{-1/2} - \sum_{k=0}^{n-1} \frac{(-1)^k \Gamma(k + 1/2)}{k! \Gamma(1/2)} (t/2)^k \right]. \end{aligned} \quad (3.16)$$

We shall estimate the error when the expansion is terminated after the first two terms.

For $n = 2$ and $|t| \leq 1$,

$$R_2(t) = (2t)^{-1/2} \left[(1 + t/2)^{-1/2} - \sum_{k=0}^1 \frac{(-1)^k \Gamma(k + 1/2)}{k! \Gamma(1/2)} (t/2)^k \right]. \quad (3.17)$$

But,

$$(1 + t/2)^{-1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + 1/2)}{k! \Gamma(1/2)} (t/2)^k, \quad |t| < 2.$$

Therefore,

$$R_2(t) = \sum_{k=2}^{\infty} \frac{(-1)^k \Gamma(k + 1/2)}{k! \Gamma(1/2)} t^{k-1/2} 2^{-(k + 1/2)}$$

and

$$\begin{aligned} R_2^{(1)}(t) &= \sum_{k=2}^{\infty} \frac{(-1)^k \Gamma(k + 1/2)}{k! \Gamma(1/2)} \cdot \frac{(k-1/2)(k-3/2)}{2^{k-2} 2^{5/2}} t^{k-5/2} \\ &= \sum_{k=2}^{\infty} (-1)^k a_k \end{aligned}$$

where

$$a_k = \frac{\Gamma(k + 1/2)}{k! \Gamma(1/2)} \cdot \frac{(k - 1/2)(k - 3/2)}{2^{k-2} 2^{5/2}} t^{k-5/2}.$$

Note that a_k are positive. We will show that they are decreasing for $k \geq 4$, $t \leq 1$.

For $k \geq 4$, $0 < t \leq 1$,

$$\frac{\Gamma(k + 1/2)(k - 1/2)(k - 3/2)}{k! 2^{k+1/2}} t^k \geq \frac{\Gamma(k + 3/2)(k + 1/2)(k - 1/2)}{(k + 1)! 2^{k+1+1/2}} t^{k+1}$$

$$\text{iff } \frac{\Gamma(k + 1/2)(k - 3/2)(k - 1/2)}{k! 2^k 2^{1/2}} \geq \frac{(k + 1/2)\Gamma(k + 1/2)(k + 1/2)(k - 1/2)}{(k + 1) k! 2^k 2^{1/2}} t$$

$$\text{iff } k - 3/2 \geq \frac{(k + 1/2)^2}{k + 1} (t/2)$$

$$\text{iff } k^2 - 1/2 k - 3/2 \geq (k^2 + k + 1/4) (t/2).$$

For any fixed k , the right-hand side of the last inequality takes its maximum value at $t = 1$ and

$$k^2 - 1/2 k - 3/2 \geq (k^2 + k + 1/4)/2$$

$$\text{iff } k^2 - 2k + 1 \geq 4 + 1/4$$

$$\text{iff } (k - 1)^2 \geq 4 + 1/4.$$

Therefore, for $k \geq 4$, $a_k \geq a_{k+1}$. Also, a_k are positive. Now we will prove that

$$\lim_{k \rightarrow \infty} a_k = 0.$$

To prove this, we write

$$a_k = \frac{\Gamma(k + 1/2)(k - 1/2)(k - 3/2)}{k! 2^k} t^k \left(\frac{t^{-5/2}}{\Gamma(1/2)2^{-2} 2^{5/2}} \right)$$

where the term in the bracket is independent of k . Let

$$b_k = \frac{\Gamma(k + 1/2)}{k!} \frac{(k - 1/2)}{2^k} (k - 3/2) t^k$$

$$\begin{aligned} \left| \frac{b_{k+1}}{b_k} \right| &= \frac{\Gamma(k + 3/2)(k + 1/2)(k - 1/2) |t|^{k+1}}{(k + 1)! 2^{k+1} |t|^k} \cdot \frac{k! 2^k}{\Gamma(k + 1/2)(k - 1/2)(k - 3/2)} \\ &= \frac{(k + 1/2)^2}{2(k + 1)(k - 3/2)} |t|. \end{aligned}$$

$$\lim_{k \rightarrow \infty} \left| \frac{b_{k+1}}{b_k} \right| = \frac{|t|}{2}.$$

Therefore, the series $\sum_{k=2}^{\infty} b_k$ converges absolutely if $|t| < 2$.

Therefore, for $|t| < 2$,

$$\lim_{k \rightarrow \infty} b_k = 0.$$

Therefore,

$$\lim_{k \rightarrow \infty} a_k = 0.$$

Therefore, the alternating series $\sum_{k=4}^{\infty} (-1)^k a_k$ converges and

$$\left| \sum_{k=4}^{\infty} (-1)^k a_k \right| \leq a_4 .$$

We will use this estimate to find a reasonably good estimate for E_2 .

$$\int_0^{\infty} e^{-zt} R_2^{(2)}(t) dt = \int_0^1 e^{-zt} R_2^{(2)}(t) dt + \int_1^{\infty} e^{-zt} R_2^{(2)}(t) dt .$$

Let

$$I_1(x) = \int_0^1 e^{-xt} R_2^{(2)}(t) dt$$

and

$$I_2(x) = \int_1^{\infty} e^{-xt} R_2^{(2)}(t) dt$$

where $z = x$ is real.

$$I_1(x) = \int_0^1 e^{-xt} \left[\sum_{k=2}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} \right] dt .$$

We write $R_2^{(2)}$ as follows:

$$\begin{aligned} R_2^{(2)}(x) &= \left[\frac{\Gamma(5/2)(3/2)(1/2)}{2\Gamma(1/2) 2^{5/2}} t^{-1/2} - \frac{\Gamma(5/2)(5/2)(3/2)}{3(2)\Gamma(1/2)2^{7/2}} t^{1/2} \right. \\ &\quad \left. + \sum_{k=4}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} \right] \\ &= \left[\frac{(3/2)(1/2)\Gamma(1/2)(3/2)(1/2)}{2\Gamma(1/2) 2^{5/2}} t^{-1/2} \right. \\ &\quad \left. - \frac{(5/2)(3/2)(1/2)\Gamma(1/2)(5/2)(3/2)}{6\Gamma(1/2) 2^{7/2}} t^{1/2} \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=4}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} \Bigg\} \\
& = \left[c_1 t^{-1/2} - c_2 t^{1/2} + \sum_{k=4}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} \right]
\end{aligned}$$

where

$$c_1 = \frac{(3/2)(1/2)(3/2)(1/2)}{(2) 2^{5/2}} \approx 0.047$$

and

$$c_2 = \frac{(5/2)(3/2)(1/2)(5/2)(3/2)}{(6) 2^{7/2}} \approx 0.104$$

$$\begin{aligned}
\left| I_1(x) \right| &= \left| \int_0^1 e^{-xt} R_2^{(2)}(t) dt \right| \\
&= \left| \int_0^1 e^{-xt} \left[c_1 t^{-1/2} - c_2 t^{1/2} \right. \right. \\
&\quad \left. \left. + \sum_{k=4}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} dt \right] \right| \\
&\leq \left| \int_0^1 e^{-xt} (c_1 t^{-1/2} - c_2 t^{1/2}) dt \right| \\
&\quad + \left| \int_0^1 \left[\sum_{k=4}^{\infty} \left(\frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} \right) e^{-xt} \right] dt \right|.
\end{aligned}$$

For $0 \leq t \leq 1$,

$$0 \leq \sum_{k=4}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{5-2} 2^{5/2}} t^{k-5/2} \leq a_4$$

$$\begin{aligned} a_4 &= \frac{\Gamma(4+1/2)(4-1/2)(4-3/2)}{4! \Gamma(1/2) 2^{4-2} 2^{5/2}} t^{4-5/2} \\ &= c_3 t^{3/2} \end{aligned}$$

where

$$\begin{aligned} c_3 &= \frac{\Gamma(4+1/2)(4-1/2)(4-3/2)}{4! \Gamma(1/2) 2^2 2^{5/2}}, \text{ a constant} \\ &\approx 0.106. \end{aligned}$$

Thus,

$$\left| \int_0^1 e^{-xt} \sum_{k=4}^{\infty} \frac{(-1)^k \Gamma(k+1/2)(k-1/2)(k-3/2)}{k! \Gamma(1/2) 2^{k-2} 2^{5/2}} t^{k-5/2} dt \right| \leq c_3 \left| \int_0^1 e^{-xt} t^{3/2} dt \right|.$$

Therefore,

$$\left| I_1(x) \right| \leq \left| \int_0^1 e^{-xt} (c_1 t^{-1/2} - c_2 t^{1/2}) dt \right| + \left| \int_0^1 c_3 e^{-xt} t^{3/2} dt \right|.$$

For $x > 0$,

$$\int_0^1 e^{-xt} t^{-1/2} dt \leq \int_0^{\infty} e^{-xt} t^{1/2-1} dt = \frac{\Gamma(1/2)}{x^{1/2}},$$

$$\int_0^1 e^{-xt} t^{1/2} dt \leq \int_0^{\infty} e^{-xt} t^{3/2-1} dt = \frac{\Gamma(3/2)}{x^{3/2}},$$

$$\int_0^1 e^{-xt} t^{3/2} dt \leq \int_0^\infty e^{-xt} t^{5/2-1} dt = \frac{\Gamma(5/2)}{x^{5/2}} .$$

Therefore,

$$\begin{aligned} |I_1(x)| &\leq \frac{c_1 \Gamma(1/2)}{x^{1/2}} + \frac{c_2 \Gamma(3/2)}{x^{3/2}} + \frac{c_3 \Gamma(5/2)}{x^{5/2}} \\ &\leq \frac{\sqrt{\pi}}{\sqrt{x}} \{c_1 + \frac{1}{2} c_2 + \frac{3}{4} c_3\} \\ &\approx \frac{\sqrt{\pi}}{\sqrt{x}} \{0.047 + \frac{1}{2} (0.104) + \frac{3}{4} (0.106)\} \\ &\leq \frac{1}{\sqrt{x}} (0.1785) . \end{aligned} \tag{3.18}$$

Therefore,

$$|I_1(x)| \leq 0.1785 x^{-1/2} \tag{3.19}$$

This bound for $I_1(x)$ holds uniformly in $x \geq 1$. If x is large then we can get a better estimate for $I_1(x)$ by retaining the form as in the first inequality in (3.18). Since

$$R_n(t) = f(t) - f_{n-1}(t) ,$$

$$R_2(t) = (t^2 + 2t)^{-1/2} (2t)^{-1/2} + \left(\frac{1}{2}\right) 2^{-3/2} t^{1/2}$$

$$R_2'(t) = -\frac{1}{2} (t^2 + 2t)^{-3/2} (2t + 2) - 2^{-1/2} \left(-\frac{1}{2}\right) t^{-3/2}$$

$$+ \left(\frac{1}{2}\right) 2^{-3/2} \left(\frac{1}{2}\right) t^{-1/2}$$

$$= -(t+1)(t^2 + 2t)^{-3/2} + 2^{-3/2} t^{-3/2} + 2^{-7/2} t^{-1/2}$$

$$\begin{aligned}
R_2^{(2)}(t) &= -(t^2 + 2t)^{-3/2} + 3(t+1)^2(t^2 + 2t)^{-5/2} - \frac{3}{4\sqrt{2}} t^{-5/2} \\
&\quad - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \frac{1}{\sqrt{2}} t^{-3/2} \\
&= -(t^2 + 2t)^{-3/2} + 3(t+1)^2(t^2 + 2t)^{-5/2} - \frac{3}{4\sqrt{2}} t^{-5/2} - \frac{1}{16\sqrt{2}} t^{-3/2} \\
|R_2^{(2)}(t)| &\leq (t^2 + 2t)^{-3/2} + 3(t+1)^2(t^2 + 2t)^{-5/2} + \frac{3}{4\sqrt{2}} t^{-5/2} + \frac{1}{16\sqrt{2}} t^{-3/2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|I_2(x)| &= \left| \int_1^\infty e^{-xt} R_2^{(2)}(t) dt \right| \\
&\leq \left| \int_1^\infty e^{-xt} \left((t^2 + 2t)^{-3/2} + 3(t+1)^2(t^2 + 2t)^{-5/2} + \frac{3}{4\sqrt{2}} t^{-5/2} + \frac{1}{16\sqrt{2}} t^{-3/2} \right) dt \right| \\
\int_1^\infty e^{-xt} (t^2 + 2t)^{-3/2} dt &\leq \int_1^\infty e^{-xt} t^{-3} dt \\
&\leq \int_1^\infty e^{-xt} dt \\
&= \frac{e^{-xt}}{-x} \Bigg|_1^\infty = \frac{e^{-x}}{x}
\end{aligned}$$

$$(t+1)^2 = t^2 + 2t + 1$$

$$\begin{aligned}
 (t+1)^2(t^2+2t)^{-5/2} &\leq (t^2+2t)^{-3/2} + (t^2+2t)^{-5/2} \\
 &\leq (t^2)^{-3/2} + (t^2)^{-5/2} \\
 &= t^{-3} + t^{-5} .
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_1^\infty e^{-xt} 3(t+1)^2(t^2+2t)^{-5/2} dt &\leq \int_1^\infty e^{-xt} t^{-3} dt + \int_1^\infty e^{-xt} t^{-5} dt \\
 &\leq \int_1^\infty e^{-xt} dt + \int_1^\infty e^{-xt} dt \\
 &= 2 \frac{e^{-x}}{x} .
 \end{aligned}$$

$$\begin{aligned}
 \frac{3}{4\sqrt{2}} \int_1^\infty e^{-xt} t^{-5/2} dt &\leq \frac{3}{4\sqrt{2}} \int_1^\infty e^{-xt} dt \\
 &= \frac{3}{4\sqrt{2}} \frac{e^{-x}}{x}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{16\sqrt{2}} \int_1^\infty e^{-xt} t^{-3/2} dt &\leq \frac{1}{16\sqrt{2}} \int_1^\infty e^{-xt} dt \\
 &= \frac{1}{16\sqrt{2}} \frac{e^{-x}}{x} .
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 |I_2(x)| &\leq |3e^{-x}x^{-1} + 13(16\sqrt{2})^{-1}e^{-x}x^{-1}| \\
 &= |e^{-x}x^{-1} (3 + 13(16\sqrt{2})^{-1})| \\
 &\leq 3.58|e^{-x}x^{-1}| .
 \end{aligned} \tag{3.20}$$

Thus, by (3.19) and (3.20)

$$\begin{aligned}
 |E_2(x)| &= |I_1(x) + I_2(x)|x^{-2} \\
 &\leq |I_1(x)|x^{-2} + |I_2(x)|x^{-2} \\
 &\leq 0.1785 x^{-5/2} + 3.58|e^{-x}x^{-3}| \\
 &\leq 0.1785 x^{-5/2} + 3.58 x^{-3} .
 \end{aligned}$$

Thus, for $n = 2$, $p = 2$

$$\begin{aligned}
 \int_0^\infty e^{-xt}(t^2 + 2t)^{-1/2}dt &= \sum_{k=0}^{n-1} a_k \frac{\Gamma(\Lambda_k + 1)}{x^{\Lambda_k + 1}} + E_n(x) \\
 &= 2^{-1/2}\Gamma(\frac{1}{2}) x^{-1/2} - 2^{-5/2}\Gamma(\frac{3}{2}) x^{-3/2} + E_2(x)
 \end{aligned}$$

where

$$|E_2| \leq 0.1785 x^{-5/2} + 3.58 x^{-3} .$$

C. Example 3

Let

$$f(x) = \log(1 + t^{1/2}) \quad , \quad 0 < t < \infty . \quad (3.21)$$

This function has a branch point at $t = -1$. The expansion of $\log(1 + t)$ about the origin is given by

$$\log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots + \frac{(-1)^{k+1}t^k}{k} + \dots . \quad (3.22)$$

Therefore,

$$\begin{aligned} \log(1 + t^{1/2}) &= t^{1/2} - \frac{(t^{1/2})^2}{2} + \dots + \frac{(-1)^{k+1}(t^{1/2})^k}{k} + \dots, \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} t^{k/2}}{k}, \quad |t| < 1. \end{aligned} \quad (3.23)$$

Therefore,

$$f(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} t^{k/2}}{k}, \quad |t| < 1, \quad (3.24)$$

and if we let

$$f_{n-1}(t) = \sum_{k=0}^{n-1} a_k t^k, \quad a_0 = 0,$$

then

$$f_{n-1}(t) = \sum_{k=1}^{n-1} \frac{(-1)^{k+1} t^{k/2}}{k}. \quad (3.25)$$

$$I(z) = \sum_{k=0}^{n-1} a_k z^{-(\Lambda_k+1)} + E_n(z). \quad (3.26)$$

where

$$E_n(z) = z^{-p} \int_0^{\infty} e^{-zt} R_n^{(p)}(t) dt,$$

$$R_n(t) = f(t) - f_{n-1}(t)$$

$$= \log(1 + t^{1/2}) - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} t^{k/2}. \quad (3.27)$$

Let

$$a_0 = 0$$

$$a_k = \frac{(-1)^{k+1}}{k}, \quad k > 0$$

$$\Lambda_k = \frac{k}{2}.$$

We will estimate the error when the asymptotic expansion is terminated after the first three nonzero terms. Thus, when $n = 4$,

$$f_3(t) = t^{1/2} - \frac{t}{2} + \frac{t^{3/2}}{3}$$

$$R_3(t) = \log(1 + t^{1/2}) - t^{1/2} + \frac{t}{2} - \frac{1}{3} t^{3/2}, \quad 0 < t < \infty.$$

$$R'_3(t) = -\frac{1}{2} \frac{1}{1 + \sqrt{t}} + \frac{1}{2} - \frac{1}{2} t^{1/2}$$

$$R''_3(t) = +\frac{1}{2} \frac{1}{(1 + \sqrt{t})^2} \left(\frac{1}{2} t^{-1/2}\right) - \frac{1}{4} t^{-1/2}$$

$$= \frac{1}{4\sqrt{t}} \left(\frac{1}{1 + 2\sqrt{t} + t} - 1 \right)$$

$$= -\frac{1}{4} \frac{2\sqrt{t} + t}{\sqrt{t} (1 + \sqrt{t})^2}$$

$$= -\frac{1}{4} \frac{2 + \sqrt{t}}{(1 + \sqrt{t})^2}$$

$$= -\frac{1}{4} \left(\frac{1}{1 + \sqrt{t}} + \frac{1}{(1 + \sqrt{t})^2} \right).$$

We take $p = 2$.

$$\begin{aligned}
 \left| \int_0^\infty e^{-xt} R_3'(t) dt \right| &= \frac{1}{4} \int_0^\infty e^{-xt} \left(\frac{1}{1 + \sqrt{t}} + \frac{1}{(1 + \sqrt{t})^2} \right) dt \\
 &\leq \frac{2}{4} \int_0^\infty e^{-xt} \\
 &= \frac{1}{2x} .
 \end{aligned}$$

Therefore, the expansion can be given as

$$\begin{aligned}
 \int_0^\infty e^{-xt} \log(1 + t^{1/2}) dt &= \sum_{k=0}^{n-1} a_k x^{-(\Lambda_k+1)} + E_4(x) \\
 &= x^{-3/2} - \frac{x^2}{2} + E_4(x)
 \end{aligned}$$

where

$$|E_4(x)| \leq \frac{1}{2x^3} .$$

BIBLIOGRAPHY

BIBLIOGRAPHY

- Fulks, Watson. Advanced Calculus. Second edition. New York: John Wiley & Sons, Inc., 1969.
- Olver, F. W. J. Asymptotics and Special Functions. New York: Academic Press, 1974.
- Soni, K. On Uniform Asymptotic Expansions of Finite Laplace and Fourier Integrals. Proceedings of the Royal Society of Edinburgh, 85A, 299-305, 1980.
- Taylor, Angus E. Advanced Calculus. Massachusetts: Blaisdel Publishing Company, 1955.
- Trench, William F. Advanced Calculus. New York: Harper & Row, 1978.

APPENDIX

APPENDIX

1. Gamma Function: The "gamma function" $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad 0 < x.$$

Properties:

(a) $\Gamma(x+1) = x\Gamma(x), \quad x > 0.$

(b) $\Gamma(1/2) = \sqrt{\pi}.$

(c) For any $a > 0,$

$$\int_0^{\infty} e^{-at} t^{x-1} dt = \frac{\Gamma(x)}{a^x}.$$

2. Second Mean-Value-Theorem of Integral Calculus: Suppose f' is nonnegative and integrable and g is continuous on $[a,b]$. Then

$$\int_a^b f(x)g(x)dx = f(a) \int_a^c g(x)dx + f(b) \int_c^b g(x)dx$$

for some c in $[a,b]$. (See Trench, p. 154.)

3. Integration by Parts: If u' and v' are integrable on $[a,b]$, then

$$\int_a^b u(x)v'(x)dx = u(x)v(x) \Big|_a^b - \int_a^b v(x)u'(x)dx.$$

(See Trench, p. 153.)

4. Series:

$$(1 - x)^{-\alpha} = 1 + \alpha x + \frac{(\alpha + 1)}{2!} x^2 + \dots \frac{\alpha(\alpha + 1) \dots (\alpha + n - 1)}{n!} x^n + \dots$$

The series converges to $(1 - \alpha)^{-\alpha}$ when $|x| < 1$.

5. Alternating Series Test:

The series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges if

$$0 \leq a_{n+1} \leq a_n$$

and

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Also, if the above series converges, then

$$0 \leq \sum_{n=0}^{\infty} (-1)^n a_n \leq a_0.$$

6. Sequence of Functions:

Let $\sum a_n x^n$ have a nonzero radius of convergence R . Denote its sum by $f(x)$. Then, f is differentiable for $\{-R < x < R\}$ and,

$$f'(x) = \sum_{n=0}^{\infty} a_n \frac{d}{dx} (x^n), \quad |x| < R.$$

7. Absolute Values of Integrals:

Let $f(x)$ be a function in $a \leq x \leq b$. Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

8. Improper Integral:

We say that the integral $\int_0^{\infty} f(x)dx$ converges (as an improper Riemann integral) if $\int_a^b f(x)dx$ exists for every a and b such that $0 < a < b < \infty$ and the limits

$$\lim_{a \rightarrow \infty} \int_a^b f(x)dx, \quad \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

both exist.

9. The Symbols $\sim$, o , and O :

Suppose we are given a known function $\phi(x)$ where x is assumed to be a real variable, and want to describe the behavior, as $x \rightarrow \infty$, of a wanted function $f(x)$ in terms of the known function $\phi(x)$, then,

(i) If $f(x)/\phi(x) \rightarrow 1$, we write

$$f(x) \sim \phi(x) \quad (x \rightarrow \infty) .$$

In words, $f(x)$ is asymptotic to ϕ or ϕ is an asymptotic approximation to f .

(ii) If $f(x)/\phi(x) \rightarrow 0$, we write

$$f(x) = o\{\phi(x)\} \quad (x \rightarrow \infty) .$$

In words, $f(x)$ is of order less than ϕ .

(iii) If $|f(x)/\phi(x)|$ is bounded, we write

$$f(x) = O\{\phi(x)\} \quad (x \rightarrow \infty) .$$

In words, $f(x)$ is of order not exceeding ϕ .

VITA

Tohun Akinleye was born in September 1945 in Okemesi-Ekiti, Nigeria. He received his primary education at St. Andrew's School in Okemesi-Ekiti. He also received his secondary education in the same town in 1964. In 1970 he obtained a Yaba College of Technology diploma in Mechanical/Electrical Engineering Technology.

During the same year, he obtained the City and Guilds of London diploma in Mechanical/Electrical Engineering Technology.

In 1975, he obtained a Bachelor's Degree in Mechanical Engineering Technology and Mathematics from Metropolitan State College, Denver, Colorado.

Tohun A. Akinleye was selected for Who's Who Among Students in American Universities and Colleges in 1974-1975 while attending Metropolitan State College, Denver, Colorado.

He received his Master of Science in Nuclear Engineering from Tuskegee Institute, Tuskegee, Alabama, in 1980.