To the Graduate Council:

I am submitting herewith a thesis written by Keith Douglas Schofield entitled "Methods for Solving Large-Scale Bounded Linear Programs." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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METHODS FOR SOLVING LARGE-SCALE BOUNDED LINEAR PROGRAMS

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ABSTRACT

The purpose of this thesis is to present two methods for solving large-scale linear programs which impose certain restrictions on the values of the variables. By taking advantage of the restrictions on the variables, the restrictiveness of working with a large basis, which would be required if the simplex procedure was applied directly, is avoided.

Two different types of structure for the restrictions on the variables will be considered. The first type is composed of restrictions which require each variable not to exceed a specified limit. In this case, these upper limits are not explicitly represented in solving the problem. Instead, from a feasible basis an "effective" basis is defined, where the nonbasic variables can equal their upper limit or zero. Conditions are then set to show when an optimal solution is found, to find candidates to enter the effective basis, and what value these candidates should attain. The second type is composed of restrictions that restrict the total value of certain linear sums of the variables. In this case, a "working" basis, much smaller than a normal feasible basis, is derived from a feasible basis. This "working" basis is used to find all the necessary information to optimize the problem.

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INTRODUCTION

Theoretically, it is possible to solve any linear programming problem by the simplex method. However, computer storage capacity or processing time restrictions could prohibit the solving of large-scale problems by directly applying the simplex method. In some large systems, though, the constraints of the problem are structured in such a way that advantage can be taken of the form to facilitate the solving of the problem. The material presented in the following chapters provides methods for solving specially structured larged-scale problems.

In Chapter I, the simple upper-bound procedure is presented as a technique for solving a problem which imposes upper limit restrictions on the value of the variables. This procedure uses the properties these upper limit restrictions have in solving the problem, while not explicitly representing the upper limit restrictions. These properties are used to modify the simplex procedure by defining a new "effective" basis from a feasible basis that is smaller than the feasible basis by the number of upper limit restrictions. Modified simplex rules are then used to let nonbasic ("effective" basis) variables equal to their upper limit, show when an optimal solution is found, find candidates to enter the "effective" basis, and what the new value of these candidates will be. In problems when the number of upper limit restrictions is large in comparison to the normal constraints, this procedure will result in a considerable reduction in the number of constraints and, therefore, a savings in storage and computation.

The generalized-upper bound procedure is presented in Chapter II as a method for solving a problem composed of constraints that require certain linear sums of the variables to equal one (1). From the structure of the constraints, a "working" basis is defined from a feasible basis of order less than the feasible basis, by the number of linear constraints. The "working" basis is then used to calculate the information necessary to apply the revised simplex method to the problem. Some special rules are also used to update the "working" basis. The use of the generalized upper-bound procedure will result in a considerable reduction of constraints and, hence, a saving in storage and computation in problems where the number of linear restrictions is large in comparsion to the number of normal constaints.

CHAPTER I

SOLVING SIMPLE UPPER-BOUNDED PROBLEMS

In this chapter we are going to discuss linear programming problems of the form:

$$\text{maximize} \quad z = \overline{c}'\overline{x} \tag{1-1a}$$

$$A\overline{x} = \overline{b} \tag{1-1b}$$

$$\overline{0} \leq \overline{x} \tag{1-1c}$$

$$x_j \leq u_j \quad \text{for some or all} \quad j \tag{1-1d}$$

where the u_j are positive constants, and the matrix A is $m \times n$.

The constraints of (1-1d), that some or all of the variables x_j are also bounded from above, are called upper-bound constraints. The given upper-bound constraints can represent capacity limitations, production requirements, or resource limitations in linear programming applications. Linear programming problems that have upper-bound constraints are called simple upper-bounded problems. The upper-bound constraints can be expanded to bound all of the variables x_j, by letting each x_j that is not bounded above have as its corresponding u_j, $u_j = \infty$. Thus, we are able to generalize the discussion of simple upper-bounded problems.

If we were to solve problem (1-1) by the standard simplex method, we would have to replace each upper-bound inequality constraint with an equivalent equality constraint using slack variables. The problem with slack variables would have $2n$ variables with $(m + n)$ constraints. In this discussion however, we will develop a modification of the simplex procedure that will solve simple upper-bounded problems without having to represent the upper-bound constraints in an explicit way. This modified procedure, the simple upper-bound procedure as we will now call it, treats only the m constraints $A\bar{x} = \bar{b}$ explicitly. The simple upper-bound procedure therefore, will use a basis of order m instead of a basis of order $(m + n)$ necessary for the standard simplex procedure. Thus, a large savings in storage and computation will occur if n is much larger than m. Our development of the simple upper-bound procedure will be based on three main sources; Linear Programming Methods and Applications by Saul Gass, Methods and Applications of Linear Programming by Cooper and Steinberg, and Optimization Theory for Large Systems by Leon Lasdon.

To begin the development of the simple upper-bound procedure, we will need to investigate some special properties of the constraints in a simple upper-bound problem. We will start by rewriting (1-1), as in the standard simplex procedure, by adding on a slack variable s_j to each of the upper-bound inequality constraints. The new form of (1-1) will be figure (1-2) on page 3 where a_j is the m-element $j^{\underline{th}}$ column of matrix A. A basic feasible solution to a problem in the form of (1-2) would have $(m + n)$ basic variables and $2n - (m + n) = n - m$ nonbasic variables. For each of the n constraints

$$\text{Maximize } z = \bar{c}'\bar{x} \tag{1-2a}$$

Subject to:

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n = \bar{b} \tag{1-2b}$$

$$\left.\begin{aligned} x_1 \qquad\qquad + s_1 \qquad\qquad &= u_1 \\ x_2 \qquad\qquad + s_2 \qquad\qquad &= u_2 \\ \ddots \qquad\qquad \ddots \qquad\qquad &\vdots \\ x_n \qquad\qquad + s_n \qquad\qquad &= u_n \end{aligned}\right\} \tag{1-2c}$$

$$\bar{x} \geq 0 \ , \ s_j \geq 0 \ , \ j = 1, \ldots, n \ . \tag{1-2d}$$

$x_j + s_j = u_j$ we can see that one of the following will be true for a basic feasible solution: $x_j > 0$ and $s_j = 0$, $x_j = 0$ and $s_j > 0$, $x_j > 0$ and $s_j > 0$. The positive x_j's and s_j's will be basic variables for the feasible solution. This will mean that only m of $x_j + s_j = u_j$ constraints can have both variables positive and hence basic. The other $(n - m)$ constraints $x_j + s_j = u_j$ will have only one variable in each constraint positive and basic. If more than m of the $x_j + s_j = u_j$ constraints have both variables basic, there would be more than $(m + n)$ basic variables.

Let us now assume that we have a feasible solution to the problem in (1-2) that has both x_j and s_j basic for the first m constraints $x_j + s_j = u_j$ (i.e., both x_j and s_j positive for $j = 1, \ldots, m$) and one variable basic for each of the last $(n - m)$ constraints $x_j + s_j = u_j$ (i.e., either $x_j = 0$ and $s_j = u_j$ or $x_j = u_j$ and $s_j = 0$ for $j = m + 1, \ldots, n$). This feasible solution will thus have $(m + m)$ columns of its feasible basis of the form $\bar{p}_j = \begin{bmatrix} \bar{a}_j \\ \bar{e}_j \end{bmatrix}$, and $\bar{q}_j = \begin{bmatrix} \bar{0} \\ \bar{e}_j \end{bmatrix}$, $j = 1, \ldots, m$, corresponding to the basic variables x_j and s_j, $j = 1, \ldots, m$. The identity vectory $\bar{e}_j$ is the p element identity vector with unity in the $j\underline{th}$ position.

We next prove the important property that these $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$, $j = 1, \ldots, m$, of the feasible basis are linearly independent if and only if the columns $\bar{a}_j$, $j = 1, \ldots, m$ are linearly independent. To prove the first half of the if and only if statement (if columns $\bar{p}_j$ and $\bar{q}_j$, $j = 1, \ldots, m$ are linearly

independent then the columns $\bar{a}_j$, $j = 1, \ldots, m$, are linearly independent), we assume the columns $\bar{p}_j$ and $\bar{q}_j$, $j = 1, \ldots, m$ are linearly independent. Also assume that the columns $\bar{a}_j$, $j = 1, \ldots, m$, are not linearly independent to get a contradiction.

We know that since the columns $\bar{a}_j$, $j = 1, \ldots, m$, are not linearly independent, that one of the columns is a linear combination of the others. This means that

$$r_1\bar{a}_1 + \ldots + r_{h-1}\bar{a}_{h-1} + r_{h+1}\bar{a}_{h+1} + \ldots + r_m\bar{a}_m = \bar{a}_h$$

for some $\bar{a}_h$, where $1 \leq h \leq m$ and some or all $r_i \neq 0$, $i = 1, \ldots, h-1, h+1, \ldots, m$.

Thus, we can write that

$$r_1 \begin{bmatrix} \bar{a}_1 \\ \bar{e}_1 \end{bmatrix} + \ldots + r_{h-1} \begin{bmatrix} \bar{a}_{h-1} \\ \bar{e}_{h-1} \end{bmatrix} + r_{h+1} \begin{bmatrix} \bar{a}_{h+1} \\ \bar{e}_{h+1} \end{bmatrix} +$$

$$\ldots + r_m \begin{bmatrix} \bar{a}_m \\ \bar{e}_m \end{bmatrix} - r_1 \begin{bmatrix} \bar{0} \\ \bar{e}_1 \end{bmatrix} - \ldots -$$

$$r_{h-1} \begin{bmatrix} \bar{0} \\ \bar{e}_{h-1} \end{bmatrix} + \begin{bmatrix} \bar{0} \\ \bar{e}_h \end{bmatrix} - r_{h+1} \begin{bmatrix} \bar{0} \\ \bar{e}_{h+1} \end{bmatrix} -$$

$$\ldots - r_m \begin{bmatrix} \bar{0} \\ \bar{e}_m \end{bmatrix} = \begin{bmatrix} \bar{a}_h \\ \bar{e}_h \end{bmatrix}$$

or

$$r_1\bar{p}_1 + \ldots + r_{h-1}\bar{p}_{h-1} + r_{h+1}\bar{p}_{h+1} + \ldots +$$

$$r_m\bar{p}_m - r_1\bar{q}_1 - \ldots - r_{h-1}\bar{q}_{h-1} + \bar{q}_h - r_{h+1}\bar{q}_{h+1} -$$

$$\ldots - r_m\bar{q}_m = \bar{p}_h$$

This last line shows us that the columns $\bar{p}_j$ and $\bar{q}_j$ are linearly dependent. Thus, we have a contradiction with the initial assumption that $\bar{p}_j$ and $\bar{q}_j$ are linearly independent.

Therefore, if the $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$ are linearly independent, then the columns $\bar{a}_j$, $j = 1, \ldots, m$, are also linearly independent. We will now use a similar argument to prove the other half of the if and only if statement.

Assume that the columns $\bar{a}_j$, $j = 1, \ldots, m$, are linearly independent. Also assume that the $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$ are not linearly independent to get a contradiction.

We can now write one column of the $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$ as a linear combination of the others. There will be two cases. Either

$$r_1\bar{p}_1 + \ldots + r_m\bar{p}_m + v_1\bar{q}_1 + \ldots + v_{h-1}\bar{q}_{h-1} +$$

$$v_{h+1}\bar{q}_{h+1} + \ldots + v_m\bar{q}_m = \bar{q}_h \qquad (1\text{-}3)$$

for some or all $r_i \neq 0$, $i = 1, \ldots, m$, some or all $v_k \neq 0$, $k = 1, \ldots, h-1, h+1, \ldots, m$, and $1 \leq h \leq m$, or

$$r_1\bar{p}_1 + \ldots + r_{h-1}\bar{p}_{h-1} + r_{h+1}\bar{p}_{h+1} + \ldots + r_m\bar{p}_m + v_1\bar{q}_1 + \ldots + v_m\bar{q}_m = \bar{p}_h \tag{1-4}$$

for some or all $r_i \neq 0$, $i = 1, \ldots, h-1, h+1, \ldots, m$, some or all $v_k \neq 0$, $k = 1, \ldots, m$, and $1 \leq h \leq m$.

For the first case we can rewrite (1-3) as

$$r_1 \begin{bmatrix} \bar{a}_1 \\ \bar{e}_1 \end{bmatrix} + \ldots + r_m \begin{bmatrix} \bar{a}_m \\ \bar{e}_m \end{bmatrix} + v_1 \begin{bmatrix} \bar{0} \\ \bar{e} \end{bmatrix} + \ldots + v_{h-1} \begin{bmatrix} \bar{0} \\ \bar{e}_{h-1} \end{bmatrix} + v_{h+1} \begin{bmatrix} \bar{0} \\ \bar{e}_{h+1} \end{bmatrix} + \ldots + v_m \begin{bmatrix} \bar{0} \\ \bar{e}_m \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{e}_h \end{bmatrix}$$

It is now clear that $r_h \neq 0$, since it is the only column with a nonzero element in its $(m + h)$ position. We also know that

$$r_1\bar{a}_1 + \ldots + r_m\bar{a}_m = 0$$

where $r_h \neq 0$, since the columns $\begin{bmatrix} \bar{a}_j \\ \bar{e}_j \end{bmatrix}$, $j = 1, \ldots, m$, are the only columns with nonzero elements in the first m positions.

This last statement means that the columns $\bar{a}_j$, $j = 1, \ldots, m$, are linearly dependent, since not all the r_j's are zero. Thus we have a contradiction with the assumption that the columns $\bar{a}_j$, $j = 1, \ldots, m$, are linearly independent.

For the second case we can rewrite (1-4) as

$$r_1 \begin{bmatrix} \bar{a}_1 \\ \bar{e}_1 \end{bmatrix} + \ldots + r_{h-1} \begin{bmatrix} \bar{a}_{h-1} \\ \bar{e}_{h-1} \end{bmatrix} + r_{h+1} \begin{bmatrix} \bar{a}_{h+1} \\ \bar{e}_{h+1} \end{bmatrix} +$$

$$r_m \begin{bmatrix} \bar{a}_m \\ \bar{e}_m \end{bmatrix} + v_1 \begin{bmatrix} \bar{0} \\ \bar{e}_1 \end{bmatrix} + \ldots + v_m \begin{bmatrix} \bar{0} \\ \bar{e}_m \end{bmatrix} = \begin{bmatrix} \bar{a}_h \\ \bar{e}_h \end{bmatrix}$$

From this last line we know that some or all of the r_i's are nonzero, $i = 1, \ldots, h-1, h+1, \ldots, m$, since the columns $\begin{bmatrix} \bar{a}_j \\ \bar{e}_j \end{bmatrix}$, $j = 1, \ldots, h-1, h+1, \ldots, m$, are the only columns with nonzero elements in the first m positions. Therefore we can write

$$r_1\bar{a}_1 + \ldots + r_{h-1}\bar{a}_{h-1} + r_{h+1}\bar{a}_{h+1} + \ldots + r_m\bar{a}_m = \bar{a}_h$$

for some or all $r_i \neq 0$, $i = 1, \ldots, h-1, h+1, \ldots, m$, i.e., the columns $\bar{a}_j$, $j = 1, \ldots, m$ are linearly dependent. We now again

have a contradiction with the assumption that the columns $\bar{a}_j$, $j = 1, \ldots, m$ are linearly independent.

This contradiction for the second case along with the contradiction in the first case enables us to see that the second half of the if and only if statement is true; i.e., if the columns $\bar{a}_j$, $j = 1, \ldots, m$ are linearly independent, then the $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$ are also linearly independent.

We now know that the $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$ are linearly independent if and only if the columns $\bar{a}_j$, $j = 1, \ldots, m$ are linearly independent. This completes our proof.

Since we have already assumed that the $(m + m)$ columns $\bar{p}_j$ and $\bar{q}_j$ are columns from the feasible basis, they must be linearly independent. Therefore from the above proof, the columns $\bar{a}_j$, $j = 1, \ldots, m$, must also be linearly independent and thus form a basis for $A\bar{x} = \bar{b}$.

In a generalization of the above discussion, we can see that any feasible solution of problem (1-2) will have the following: m basic variables x_j with the corresponding m variables s_j also basic (i.e., m of the x_j's are nonzero and not at their upper-bound), $(n - m)$ basic variables where either x_j is basic or its corresponding s_j is basic (i.e., $(n - m)$ of the x_j's are either at their upper-bound u_j or equal to zero), and the m columns of matrix A corresponding to the m x_j's that are basic and not at their upper-bound will form a feasible basis for $A\bar{x} = \bar{b}$.

With these results, we can take a given feasible solution to problem (1-2) to form an "effective" basis B from the m columns of matrix A corresponding to the x_j's not at their upper-bound and non-zero. We can also take the m basic variables x_j corresponding to the "effective" basis B and group them into a set called the "effective" basic variables. The remaining $(n - m)$ variables x_j, either at their upper-bound or equal to zero, we will call the noneffective variables. We will show in this development that the simple upper-bound procedure will use the "effective" basis B, the "effective" variables, and the knowledge of what the value of each noneffective variable is equal to (i.e., $x_j = 0$ or $x_j = u_j$ for x_j an noneffective variable) to solve simple upper-bounded problems, like (1-1), while not explicitly representing the slack variables.

Hence, we will be able to solve simple upper-bounded problems using only n instead of $2n$ variables and an "effective" basis B of only dimension m, by dealing only with the original variables x_j. We will now show how to test a current feasible solution to see if it is optimal using only the "effective" basis and the value of the noneffective variables.

To begin with, we will take simplex multipliers $\bar{d} = (d_1, \ldots, d_m)$, associated with the "effective" basis B, multiply them on to (1-1b) and subtract (1-1a) away from this product (note that the simplex multipliers are $\bar{c}_B B^{-1}$, where $\bar{c}_B = \{c_j | x_j$ is an "effective" variable$\}$). Thus we have the following

$$\bar{d}A\bar{x} - \bar{c}'\bar{x} = \bar{d}\bar{b} - z \quad .$$

Or, after rearrangement

$$z = \bar{d}\bar{b} - \bar{d}A\bar{x} + \bar{c}'\bar{x}$$

$$z = \bar{d}\bar{b} - (\bar{d}A - \bar{c}')\bar{x}$$

From the simplex method we know that $\bar{d}\bar{a}_j - c_j = z_j - c_j$, so we have

$$z = \bar{d}\bar{b} - \sum_{j=1}^{n} (z_j - c_j)x_j$$

Since $z_j - c_j = 0$ for all $a_j \in B$, we have

$$z = \bar{d}\bar{b} - \sum_{j \notin B} (z_j - c_j)x_j \qquad (1\text{-}5)$$

where $\sum_{j \notin B}$ is the sum over the set of indices for those vectors not in B . The equation for z in (1-5) is the difference of $\bar{d}\bar{b}$, the current value of the objective function, and the product of the x_j's that are noneffective with their corresponding $z_j - c_j$.

Hence, the current solution will be optimal if two conditions are met. The first condition is that all noneffective $x_j = 0$ have a corresponding $z_j - c_j \geq 0$. This condition would make the current solution optimal, since increasing any of these x_j's would decrease

the value of z . The second condition is that all noneffective $x_j = u_j$ have a corresponding $z_j - c_j \leq 0$. Conversely, this condition would make the current solution optimal, since decreasing any of these x_j would decrease the value of z .

There would therefore be two conditions for which the current solution would not be optimal. One condition would be, if for some noneffective, $x_j = 0$ its corresponding $z_j - c_j$ is less than zero, The other condition would be if for some noneffective $x_j = u_j$ its corresponding $z_j - c_j$ is greater than zero.

Thus, if we wanted increase z , for a nonoptimal solution, we would choose x_k to possibly enter the "effective" basis B . where x_k corresponds to

$$z_k - c_k = \min_j \left\{ \left[-(z_j - c_j | x_j = u \right] \cup \left[z_j - c_j | x_j = 0 \right] \right\}$$

This possible candidate, x_k , to enter the "effective" basis may have a current value of u_k or 0 . We next discuss, in two cases ($x_k = 0$ and $x_k = u_k$) , whether x_k enters the "effectiveness" basis at at what value.

CASE I: $x_k = 0$

Since we want to increase the value of x_k from its current value of 0 , we will also have change some or all of the values of the "effective" variables. In increasing x_k , there are two situations possible. One is that x_k will increase to its upper-bound while none of the "effective" variables increase or decrease beyond

their respective upper-bounds (u_j) or lower-bounds (0). In this case, x_k does not enter the "effective" basis, but has a new value of u_k. The other possible situation is that some "effective" variable does attain one of its bounds before x_k attains its upper-bound. In this case, we bring x_k into the "effective" basis. We next develop a way to determine the value x_k should attain that will keep the constraints satisfied.

The current value of the $i^{\underline{th}}$ "effective" variable, in terms of the simplex tableau for the "effective" basis B, is

$$x_{Bi} = x_{i0} - \sum_{j \in U} x_{ij} u_j$$

where $U = \{j | x_j$ is noneffective and $x_j = u_j\}$, x_{ij} is the $j^{\underline{th}}$ element of $\bar{x}_i = B^{-1}\bar{p}_i$, and $\bar{p}_0 = \bar{b}$. Since $x_k = 0$, we can write

$$x_{Bi} = x_{i0} - \sum_{j \in U} x_{ij} u_j - x_{ik} x_k$$

If we let $\tilde{b}_i = x_{i0} - \sum_{j \in U} x_{ij} u_j$, we now have

$$x_{Bi} = \tilde{b}_i - x_{ik} x_k \qquad (1\text{-}7)$$

for all $i \in B$.

For those $x_{ik} > 0$, we can see that an increase in x_k from zero will decrease the corresponding "effective" x_{Bi}. Since we can not let any x_{Bi} become negative, we have to restrict the in-

crease of x_k so that

$$\tilde{b}_i - x_{ik}x_k \geq 0$$

or

$$x_k \leq \frac{\tilde{b}_i}{x_{ik}}$$

Thus, the increased value of x_k must be less than or equal to the minimum of all the $\frac{\tilde{b}_i}{x_{ik}}$'s, for $x_{ik} > 0$, to keep the x_{Bi}'s from becoming negative; i.e.,

$$x_k \leq \frac{\tilde{b}_r}{x_{rk}} = \min_i \left\{ \frac{\tilde{b}_i}{x_{ik}} \mid x_{ik} > 0 \right\} \qquad (1\text{-}8)$$

However, for those $x_{ik} < 0$, an increase in x_k from zero will increase the corresponding "effective" x_{Bi} in (1-7). Since all of the x_{Bi}'s must be kept less than or equal to their upper-bound u_{Bi}, we have

$$\tilde{b}_i - x_{ik}x_k \leq u_{Bi}$$

or

$$x_k \leq \frac{\tilde{b}_p - u_{Bp}}{x_{pk}} = \min_i \left\{ \frac{\tilde{b}_i - u_{Bi}}{x_{ik}} \mid x_{ik} < 0 \right\} \qquad (1\text{-}9)$$

Finally, we can not increase x_k above its upper-bound u_k ; i.e.,

$$x_k \leq u_k \tag{1-10}$$

Thus, the maximum possible increased value for x_k will be

$$\theta = \min\left\{\frac{\tilde{b}_r}{x_{rk}} , \frac{\tilde{b}_p - u_{Bp}}{x_{pk}} , u_k\right\} \tag{1-11}$$

where equation (1-11) is the combined restrictions on x_k from (1-8), (1-9), and (1-10).

If the new value of x_k is $\theta = u_k$, then x_k will remain non-effective with a new value of u_k . The "effective" variables will be updated accordinly. However, if $\theta = \frac{\tilde{b}_r}{x_{ik}}$, then x_k will replace the variable corresponding to x_{Br} in the "effective" basis. The variable corresponding to x_{Br} will decrease to zero and become non-effective. The new value of $x_{Br} = x_k$ will be $\frac{\tilde{b}_r}{x_{ik}}$ after the transformation of the simplex tableau with the pivot element in the simplex tableau being x_{rk} . Similarly, if $\theta = \frac{\tilde{b}_p - u_{Bp}}{x_{pk}}$, then x_k will replace the variable corresponding to x_{Bp} in the "effective" basis. The variable corresponding to x_{Bp} will increase to its upper bound u_{Bp} and become noneffective. The new value of $x_{Bp} = x_k$ will $\frac{\tilde{b}_p - u_{Bp}}{x_{kp}}$ after the transformation of the simplex tableau with the pivot element in the simplex tableau being x_{pk} .

CASE II: $x_k = u_k$

In this case we want to decrease the value of x_k from its current value of u_k. As in Case I the change in value of x_k will also change the value of some or all of the "effective" variables. Two situations are possible in decreasing the value of x_k. One situation is x_k will decrease in value to zero while none of the "effective" variables increase or decrease beyond their respective upper-bounds (u_j) or lower- bounds (0). In this situation x_k does not enter the "effective" basis but has a new value of zero. The other situation is that some "effective" variable does attain one of its bounds before x_k attains the value of zero. In this case x_k is brought into the "effective" basis. We next develop a way to determine the value x_k should attain that will keep the constraints satisfied.

As in equation (1-6), the current value of the $i^{\underline{th}}$ "effective" variable is terms of the simplex tableau for the "effective" basis is

$$x_{Bi} = x_{i0} - \sum_{\substack{j\in U \\ j\neq k}} x_{ij}u_j - x_{ik}x_k$$

or

$$x_{Bi} + x_{ik}x_k = x_{i0} - \sum_{\substack{j\in U \\ j\neq k}} x_{ij}u_j$$

By letting

$$\hat{b}_i = x_{io} - \sum_{\substack{j \in U \\ j \neq}} x_{ij} u_j \qquad (1\text{-}12)$$

we have that

$$x_{Bi} = \hat{b}_i - x_{ik} u_k \qquad (1\text{-}13)$$

If $x_{ik} > 0$, then we can see that a decrease in x_k from u_k will increase the corresponding "effective" x_{Bi}. Since each u_{Bi} is bounded from above by its corresponding u_{Bi}, we must restrict the decrease in x_k so that

$$x_{Bi} = \hat{b}_i - x_{ik} x_k \leq u_{Bi}$$

or

$$x_k \geq \frac{b_i - u_{Bi}}{x_{ik}}$$

Hence, the decreased value of x_k must be greater than or equal to the maximum of all the $\frac{b_r - u_{Bi}}{x_{ik}}$'s, for $x_{ik} > 0$, so that none of the x_{Bi}'s exceeds its upper-bound, i.e.,

$$x_k \geq \frac{\hat{b}_r - u_{Bi}}{x_{rk}} = \max_i \left\{ \frac{\hat{b}_i - u_{Bi}}{x_{ik}} \middle| x_{ik} > 0 \right\} \qquad (1\text{-}14)$$

Further if $x_{ik} \leq 0$, then a decrease in x_k from u_k will decrease the corresponding "effective" x_{Bi}. Again, none of the "effective" variables can become negative, so we restrict the decrease in x_k so that

$$x_{Bi} = \hat{b}_i - x_{ik}x_k > 0$$

or

$$x_k \geq \hat{b}_i / x_{ik}$$

Thus to keep all of the "effective" variables nonnegative, the decreased value of x_k must be greater than or equal to the maximum of all the $\frac{b_i}{x_{ik}}$'s for $x_{ik} < 0$, i.e.,

$$x_k \geq \frac{\hat{b}_p}{x_{pk}} = \max_i \left\{ \frac{\hat{b}_i}{x_{ik}} \mid x_{ik} < 0 \right\} \qquad (1\text{-}15)$$

Finally, the decrease value of x_k must be nonnegative, i.e.,

$$x_k \geq 0 \qquad (1\text{-}16)$$

Combining the restrictions on x_k from (1-14), (1-15), and (1-16) we have that the minimum value that x_k can attain is given by

$$\theta = \max \left\{ \frac{\hat{b}_r - u_{Br}}{x_{rk}} , \frac{\hat{b}_p}{x_{pk}} , 0 \right\} \qquad (1\text{-}17)$$

If the new value of x_k is $\theta = 0$, then x_k will remain noneffective with a new value of 0. The "effective" variables will then be updated accordingly. However, if $\theta = \frac{\hat{b}_r - u_{Br}}{x_{rk}}$, then x_k will replace the variable corresponding to x_{Br} in the "effective" basis. The variable corresponding to x_{Br} will increase to its upper-bound u_{Br} and become noneffective. The new value of $x_{Br} = x_k$ will be $\frac{\hat{b}_r - u_{Br}}{x_{rk}}$ after the transformation of the simplex tableau with the pivot element in the simplex tableau being x_{rk}. Similarly if $\theta = \hat{b}_{p/x_{pk}}$, then x_k will replace the variable corresponding to x_{Bp} in the the "effective" basis. The variable corresponding to x_{Bp} will decrease to zero and become noneffective. The new value of $x_{Bp} = x_k$ will be $\frac{\hat{b}_p}{x_{pk}}$ after the transformation of the simplex tableau with the pivot element in the simplex tableau being x_{pk}.

After a new value for x_k is found, either from CASE I OR CASE II, we test the new feasible solution to see if it is optimal. If the new solution is not optimal, we repeat the procedure for picking a candidate and finding its new value. This process is repeated until an optimal solution is found.

As an example, we will apply the simple upper-bound procedure to the upper-bounded linear programming problem

$$\text{Max } z = x_1 + 3x_3$$

subject to:

$$x_1 + x_2 + x_3 = 8$$

$$4x_1 \; x_2 \; + x_4 = 26$$

$$-x_1 + x_2 \; + x_5 = 4$$

$$0 \leq x_1 \leq 5$$

$$0 \leq x_2 \leq 7$$

$$0 \leq x_3 \leq \infty$$

$$0 \leq x_4 \leq \infty$$

$$0 \leq x_5 \leq \infty$$

A first feasible solution to this problem will have "effective" variables $x_3 = 8$, $x_4 = 26$, and $x_5 = 4$ and noneffective variables $x_1 = 0$ and $x_2 = 0$. Our initial simplex tableau for the "effective" basis is therefore:

			1	3	0	0	0
Effective vectors	c_B	$\bar{P}_0$	$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$
$\bar{P}_3$	0	8	1	1	1	0	0
$\bar{P}_4$	0	26	4	1	0	1	0
$\bar{P}_5$	0	4	-1	1	0	0	1
$z_j - c_j$		0	-1	-3	0	0	0
u_j			5	7	∞	∞	∞
Current value			0	0	-	-	-

We have added the rows for the value of the u_j's and the current value of the variables to the simplex tableau for record-keeping purposes.

The current solution is not optimal since there $z_j - c_j < 0$ for for $x_j = 0$; namely $z_1 - c_1 = -1$ and $z_2 - c_2 = -3$.

We choose x_2 to be the candidate to enter the "effective" basis since it has the minimum $z_j - c_j$. We now use the CASE I type procedure to determine if x_2 will enter the "effective" basis and what its new value will be.

$$\tilde{b}_1 = 8 - 0 = 8$$

$$\tilde{b}_2 = 26 - 0 = 26$$

$$\tilde{b}_3 = 4 - 0 = 4$$

Since all the x_{i2}'s are positive, we need to calculate $\frac{\tilde{b}_i}{x_{i2}}$ for $i = 1, 2, 3$.

$$\frac{\tilde{b}_1}{x_{12}} = 8/1 = 8$$

$$\frac{\tilde{b}_2}{x_{22}} = 26/1 = 26$$

$$\frac{\tilde{b}_3}{x_{32}} = 4/1 = 4$$

Thus,

$$\text{Min}\{8, 26, 4\} = 4 = \frac{\tilde{b}_3}{x_{22}} = \frac{\tilde{b}_r}{x_{r2}}$$

From the fact that all the x_{i2}'s are positive we have $\frac{\tilde{b}_p - u_{Bp}}{x_{p2}} = \infty$.

Hence,

$$\theta = \min\left\{\frac{\tilde{b}_r}{x_{r2}}, \frac{\tilde{b}_p - u_{Bp}}{x_{p2}}, u_2\right\}$$

$$\theta = \min\{4, \infty, 7\}$$

$$\theta = 4 = \frac{\tilde{b}_3}{x_{32}}$$

Therefore x_2 will replace x_5 in the "effective" basis as the third "effective" variable. The transformed tableau from a pivot on x_{32} is

			1	3	0	0	0
Effective vectors	c_B	$\bar{P}_0$	$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$
$\bar{P}_3$	0	4	2	0	1	0	-1
$\bar{P}_4$	0	22	5	0	0	1	-1
$\bar{P}_2$	3	4	-1	1	0	0	1
$z_j - c_j$		12	-4	0	0	0	3
u_j			5	7	∞	∞	∞
Current value			0	-	-	-	0

We still do not have an optimal solution, since $z_1 - c_1 = -4$ where $x_1 = 0$. Thus x_1 is to be the candidate to enter the "effective" basis. We will again use a CASE I type procedure to determine if x_1 will enter the "effective" basis and what its new value will be.

$$\tilde{b}_1 = 4 - 0 = 4$$

$$\tilde{b}_2 = 22 - 0 = 22$$

$$\tilde{b}_3 = 4 - 0 = 4$$

The positive x_{i1}'s are $x_{11} = 2$ and $x_{21} = 5$. For these x_{i1}'s we calculate $\frac{\tilde{b}_i}{x_{i1}}$.

$$\frac{\tilde{b}_1}{x_{11}} = 4/2 = 2$$

$$\frac{\tilde{b}_2}{x_{21}} = 22/5 = 4.5$$

Thus,

$$\min\{2, 4.5\} = 2 = \tilde{b}_1/x_{11} = \frac{\tilde{b}_r}{x_{r2}}$$

The only negative x_{i1} is $x_{31} = -1$. For this x_{i1} we calculate $\frac{\tilde{b}_i - u_{Bi}}{x_{i1}}$

$$\frac{\tilde{b}_3 - u_{B3}}{x_{31}} = \frac{4 - (7)}{-1} = 3 = \frac{\tilde{b}_p - u_{Bp}}{x_{p1}}$$

Hence,

$$\theta = \min\left\{\tilde{b}_r/x_{p1},\ \frac{\tilde{b}_p - u_{Bp}}{x_{p1}},\ u_1\right\}$$

$$\theta = \min\{2, 3, 5\}$$

$$\theta = 2 = \tilde{b}_1/x_{11}$$

Therefore x_1 will replace x_3 in the "effective" basis as the first "effective" variable. The transformed tableau from a pivot on x_{11} is:

			1	3	0	0	0
Effective vectors	c_B	$\bar{P}_0$	$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$
$\bar{P}_1$	1	2	1	0	1/2	0	-1/2
$\bar{P}_4$	0	12	0	0	-5/2	1	4/2
$\bar{P}_2$	3	6	0	1	1/4	0	1/2
z_j-c_j		20	0	0	4/2	0	2
u_j			5	7	∞	∞	∞
Current value			-	-	0	-	0

Now, all $z_j - c_j > 0$ for all $x_j = 0$ and one of the x_j equals its u_j. Therefore the current solution is optimal, with $x_1 = 2$, $x_2 = 6$, $x_4 = 12$, and $z = 20$.

It should be noted in conclusion of our development of simple upper-bounded procedure that our example problem was not an accurate test of the savings involved. In our example m equaled 3 and n equaled 5. For an accurate test n should be much larger than m.

CHAPTER II

Solving Generalized Upper-Bounded Problems

In this chapter we shall investigate linear programming problems which have the following form:

$$\text{Maximize} \quad z = \bar{c}'\bar{x} \tag{2-1a}$$

subject to:

$$A\bar{x} = \bar{b} \tag{2-1b}$$

$$\left.\begin{aligned} &\sum_{j=n_0+1}^{n_1} x_j = 1 \\ &\sum_{j=n_1+1}^{n_2} x_j = 1 \\ &\quad\vdots \\ &\sum_{j=n_{p-1}+1}^{n_p} x_j = 1 \end{aligned}\right\} \tag{2-1c}$$

$$\bar{x} \geq \bar{0} \tag{2-1d}$$

where A is an $m \times n$ matrix and $n_p = n$

This generalized upper-bound problem, as we shall now call it, has not only the usual m constraints from $A\bar{x} = \bar{b}$ and the non-negativity restrictions, but also p constraints each of the form $\sum x_j = 1$. These p constraints are called the generalized upper-bound constraints. The n variables x_j can appear in at most one of these generalized upper-bound constraints and need not appear in any of them. A generalized upper-bound constraint can be thought of as setting an upper-bound on a subset of the variables.

What we shall do in this development of the generalized upper-bound technique is to show that instead of using a basis of order $m + p$ to solve problem (2.1), that would be necessary according to the simplex-method approach, we shall take advantage of the simple structure of the generalized upper-bound constraints to employ a "working" basis of order m in A modification of the revised simplex algorithm. Since the set of p generalized upper-bound constraints is usually large in comparison to the usual m constraints the problem thus can be solved more effectively in terms of computational requirements. The main sources for this development of the generalized upper-bound problem are Linear Programming Methods and Applications by Saul Gass, Methods and Applications of Linear Programming by Cooper and Steinberg, and Optimization Theory for Large Systems by Leon Lasdon.

The need to efficiently solve generalized upper-bound problems can arise in many situations. Industries find the need to solve these type in a wide range of areas including transportation pro - duction-distribution, and resource allocation. Moreover, generalized

upper-bound problems are formed when solving nonlinear programming problems by separable programming techniques.

To begin our development of the generalized upper-bound procedures, let us rewrite (2-1) in the form of figure (2-2) on page 29 where $\bar{a}_j$ is the $j^{\underline{th}}$ column of matrix A.

We will now denote A_j as the j^{th} column of the coefficient matrix of the entire problem (2-2), where each A_j has $m + p$ elements. The problem as stated in figure (2-2), has $p + 1$ subsets of the A_j's denoted by S_i, for $i = 0, 1, \ldots, p$. The A_j's that have zeros as their $(m + 1)^{\underline{st}}$ to $(m + p)^{\underline{th}}$ coefficients are elements of S_0; i.e., $S_0 = \{ \begin{bmatrix} \bar{a}_j \\ \bar{0} \end{bmatrix} \mid 1 \leq j \leq n_0 \}$. The A_j's with unity as their $(m + 1)^{\underline{th}}$ coefficient are elements of S_i; i.e., $S_i = \{ \begin{bmatrix} \bar{a}_j \\ \bar{e}_i \end{bmatrix} \mid n_{i-1} + 1 \leq j \leq n_i \}$, $i = 1, \ldots, p$, where $\bar{e}_i$ is the p component identity vector with one in the $i^{\underline{th}}$ position.

Assuming full rank for the matrix of the system of equations (2-2) we note that any feasible basis must be of order $(m + p)$. In this feasible basis at least one variable must be positive (hence basic) in each of the generalized upper-bound constraints. This will mean that any feasible basis must contain at least one column from each of the subsets $S_1, S_2, \ldots, S_p$. The remaining m columns of the basis can be in any of the S_i's.

We will now assume that a first initial feasible basis for (2-2) is available, denoted by B. This basis matrix B is of order $\binom{m + p}{}$ and its columns are made up of the A_j's.

$$\text{Maximize} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{2-2a}$$

S_0	S_1	...	S_p		
$\bar{a}_1x_1 + \ldots + \bar{a}_{n_0}x_{n_0}$	$+ \bar{a}_{n_0+1}x_{n_0+1} + \ldots + \bar{a}_{n_1}x_{n_1}$	$+ \ldots +$	$\bar{a}_{n_{p-1}+1}x_{n_{p-1}+1} + \ldots + \bar{a}_{n_p}x_{n_p}$	$= \bar{b}$	(2-2b)
	$x_{n_0+1} + \ldots + x_{n_1}$			$= 1$	(2-2c)
					(2-2c)
			$x_{n_{p-1}+1} + \ldots + x_{n_p}$	$= 1$	(2-2c)

$$x_j \geq 0 \quad \text{for} \quad j = 1, \ldots, n_p \tag{2-2d}$$

Since each subset S_i must have at least one column in B (except S_0), we now choose one column of B for each subset S_i and designate that columns as they key column for that S_i. If the key column in each S_i is denoted by $A_{k_i} = \begin{bmatrix} \bar{a}_{k_i} \\ \bar{e}_i \end{bmatrix}$, $i = 1, \ldots, p$, then the basis B may be ordered so that these p key columns appear first. Now we can partition B as follows:

$$B = \left[\begin{array}{ccc:c} \bar{a}_{k_1} & \bar{a}_{k_2} & \bar{a}_{k_p} & \left\{\begin{matrix} \bar{a}_t \\ \bar{0} \end{matrix}\right\} \vdots \left\{\begin{matrix} \bar{a}_j \\ \bar{e}_i \end{matrix}\right\} \\ \bar{e}_1 & \bar{e}_2 & \bar{e}_p & \end{array}\right] \qquad (2\text{-}3)$$

p key columns; m nonkey columns

where A_t, $A_j \in B$, $A_j \in S_i$ $i \neq 0$, and $A_t \in S_0$. Or,

$$B = \left[\begin{array}{c:c} F_{m\times p} & G_{m\times m} \\ \hline I_{p\times p} & H_{p\times m} \end{array}\right] \qquad (2\text{-}4)$$

key columns; nonkey columns

where $F_{m\times p} = \left[\bar{a}_{k_1}, \bar{a}_{k_2}, \ldots, \bar{a}_{k_p}\right]$, $H_{p\times m}$ is m vectors of p compoments that are either zero vectors for columns in S_0 or unit vectors, $\bar{e}_i$, for columns in S_i, $i \neq 0$, and $G_{m\times m}$ consists of the m columns of $\bar{a}_j$'s for A_j a nonkey column of B.

From (2-2c) we see that each generalized upper bound constraint can be solved for the variable corresponding to the key column

$$x_{k_i} = 1 - \sum_{\substack{j=n_{i-1}+1 \\ j \neq k_i}}^{n_i} x_j \qquad i = 1, \ldots, p .$$

Substituting these expressions for the variable corresponding to the key column into (2-2b) yields the m equations

$$\bar{b} = \bar{a}_1 x_1 + \ldots + \bar{a}_{n_0} x_{n_0} + \bar{a}_{n_0+1} x_{n_0+1} + \ldots +$$

$$\bar{a}_{k_1}\Big(1 - \sum_{\substack{j=n_0+1 \\ j \neq k_1}}^{n_1} x_j\Big) + \ldots + \bar{a}_{n_1} x_{n_1} + \ldots +$$

$$\bar{a}_{k_p}\Big(1 - \sum_{\substack{j=n_{p-1}+1 \\ j \neq k_p}}^{n_p} x_j\Big) + \ldots + \bar{a}_{n_p} x_{n_p}$$

$$\bar{b} = \sum_{j=1}^{n_0} \bar{a}_j x_j + \sum_{j=n_0+1}^{n_1} (\bar{a}_j - \bar{a}_{k_1}) x_j + \ldots +$$

$$\sum_{j=n_{p-1}+1}^{n_p} (\bar{a}_j - \bar{a}_{k_p}) x_j + (\bar{a}_{k_1} + \bar{a}_{k_2} + \ldots + \bar{a}_{k_p})$$

or

$$\bar{b} - \sum_{i=1}^{p} \bar{a}_{k_i} = \sum_{j=1}^{n_0} \bar{a}_j x_j + \sum_{j=n_0+1}^{n_1} (\bar{a}_j - \bar{a}_{k_1}) x_j + \ldots + \sum_{j=n_{p-1}+1}^{n_p} (\bar{a}_j - \bar{a}_{k_p}) x_j \tag{2-5}$$

If we define $\bar{q}_j = \bar{a}_j - \bar{a}_{k_i}$ for $A_j \in S_i$, $i = 1, \ldots, p$ and $\bar{q}_j = \bar{a}_j$ for $A_j \in S_0$, then the m equations of (2-5) can be expressed

$$\bar{b} - \sum_{i=1}^{p} \bar{a}_{k_i} = \sum_{j=1}^{n_0} \bar{q}_j x_j + \sum_{j=n_0+1}^{n_1} \bar{q}_j x_j + \ldots + \sum_{j=n_{p-1}+1}^{n_p} \bar{q}_j x_j$$

since $\bar{q}_j = 0$ for $j = k_i$, $i = 1, \ldots, p$. Or more conveniently

$$\bar{b} - \sum_{i=1}^{p} \bar{a}_{k_i} = \sum_{j=1}^{n} \bar{q}_j x_j \tag{2-6}$$

We will now form our "working" basis and later show that it is a basis for the system of equations in (2-6).

If we recall the form of B in (2-3) and subtract the key columns from the nonkey columns that are in the same S_i, we have a new matrix of the form

$$T = \left[\begin{array}{ccc:c:c} \bar{a}_{k_1} & \cdots & \bar{a}_{k_p} & \left\{\begin{array}{c}\bar{a}_t \\ \bar{0}\end{array}\right\} & \left\{\begin{array}{c}\bar{a}_j - \bar{a}_{k_i} \\ \bar{0}\end{array}\right\} \\ \bar{e}_1 & & \bar{e}_p & & \end{array}\right] \qquad (2\text{-}7)$$

for A_j and A_t nonkey columns, $A_j \in S_i$, $i \neq 0$, and $A_t \in S_0$. By remembering how we defined $\bar{q}_j$ just after the equation (2-5) we have that T is now

$$T = \left[\begin{array}{ccc:ccc} \bar{a}_{k_1} & \cdots & \bar{a}_{k_p} & \bar{q}_{r_1} & \cdots & \bar{q}_{r_m} \\ \bar{e}_1 & & \bar{e}_p & \bar{0} & & \bar{0} \end{array}\right]$$

where $\bar{q}_{r_i}$, $i = 1, \ldots, m$, is the order of the $\bar{q}_j$'s in T

We now define a "working" basis $\bar{B}$ of dimensions $m \times m$ to be the first m rows of the last m column vectors of T. Thus

$$\bar{B} = \{\bar{q}_{r_1}, \ldots, \bar{q}_{r_m}\} \qquad (2\text{-}8)$$

where $\bar{q}_{r_1} = \bar{q}_j$, $i = 1, \ldots, m$, for A_j nonkey column.

Therefore T will now have the form

$$T = \left[\begin{array}{ccc:c} \bar{a}_{k_i} & , \ldots, & \bar{a}_{k_p} & \bar{B}_{m\times m} \\ \bar{e}_i & & \bar{e}_p & 0_{p\times m} \end{array}\right] = \begin{bmatrix} F_{m\times p} & \bar{B}_{m\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix} \qquad (2\text{-}9)$$

We now show $\bar{B}$ is a set of linearly independent vectors, or in other words a basis for m space.

Since we applied elementary column transformations to B to obtain the form T in (2-9) the rank of T is the same as B, i.e., $(m + p)$. Thus the columns vectors of T form a linearly independent set of $\left(m + p\right)$ vectors. Hence, the last m columns $\begin{bmatrix} \bar{B} \\ \bar{0} \end{bmatrix}$, must also linearly independent. From the definition of linear independence, we see that the m-dimensional vectors formed by just the first m components of these last m vectors must also be a linearly independent set; i.e., $\bar{B}$ forms a basis in m space. Now we wish to show that T in (2-9) can be computed by matrix multiplication on B.

First let us return to the form of B in (2-4) where

$$B = \begin{bmatrix} F_{m\times p} & G_{m\times m} \\ I_{p\times p} & H_{p\times m} \end{bmatrix}$$

Since we derived T in (2-7) by elementary column transformations on B, a matrix can be multiplied by B and produce the same form as T in (2-7). The matrix we will show that produces the form is

$$R = \begin{bmatrix} I_{p\times p} & -H_{p\times m} \\ 0_{m\times p} & I_{m\times m} \end{bmatrix} \tag{2-10}$$

where $H_{p\times m}$ is the same $H_{p\times m}$ from B in (2-4).

The product of B with R will be

$$BR = \begin{bmatrix} F_{m\times p} & G_{m\times m} \\ I_{p\times p} & H_{p\times m} \end{bmatrix} \begin{bmatrix} I_{p\times p} & -H_{p\times m} \\ 0_{m\times p} & I_{m\times m} \end{bmatrix}$$

$$BR = \begin{bmatrix} (F_{m\times p}\ I_{p\times p} + G_{m\times m}\ 0_{m\times p}) & (-F_{m\times p}\ H_{p\times m} + G_{m\times m}\ I_{m\times m}) \\ (I_{p\times p}\ I_{p\times p} + H_{p\times m}\ 0_{m\times p}) & (-I_{p\times p}\ H_{p\times m} + H_{p\times m}\ I_{m\times m}) \end{bmatrix}$$

$$BR = \begin{bmatrix} F_{m\times p} & G_{m\times m} - F_{m\times p}\ H_{p\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix}$$

If we recall the definitions of F, G and H in (2-4) that H is made up of column vectors of zeros for $A_j \in S_0$ and unit vectors e_i for $A_j \in S_i$, $i \neq 0$, F is the set of columns $\{a_{k_1}, \ldots, a_{k_p}\}$,

and G is the set of $\bar{a}_j$'s corresponding to the nonkey columns, we see that $G_{m\times m} - F_{m\times p} H_{p\times m}$ is the difference of the columns $\bar{a}_j$ with then columns $\bar{a}_{k_i}$ for $A_j \in S_i$, $i \neq 0$, and that the $\bar{a}_j$'s for $A_j \in S_0$ remain unchanged. We thus have that

$$G_{m\times m} - F_{m\times p} H_{p\times m} = \{\bar{a}_t\} \{\bar{a}_j - \bar{a}_{k_i}\}$$

for $A_j \in S_i$, $i \neq 0$ and $A_t \in S_0$. From the definition of T in equation (2-7) with the above form of $G - FH$ we can see that

$$BR = \left[\begin{array}{ccc:c:c} \bar{a}_{k_1} & \cdots & \bar{a}_{k_p} & \bar{a}_t & \bar{a}_j - \bar{a}_{k_i} \\ \bar{e}_1 & & \bar{e}_p & \bar{0} & \bar{0} \end{array}\right] = T$$

for A_t and A_j nonkey columns, $A_t \in S_0$, and $A_j \in S_i$, $i \neq 0$.

If we recall the form of T in (2-9) we know from the above that

$$G_{m\times m} - F_{m\times m} H_{p\times m} = \bar{B}_{m\times m} \qquad (2\text{-}11)$$

If we replace $G_{m\times m} - F_{m\times m} H_{p\times m}$ with $\bar{B}$, the product of B with R will be

$$BR = \begin{bmatrix} F_{m\times p} & \bar{B}_{m\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix} \qquad (2\text{-}12)$$

Thus we have that T is (2-9) in the same form as the product of B with R in (2-12). We now show that $\overline{B}$ is a basis for the system of equations in (2-6).

We will start with that B is the current feasible basis for (2-2) with $\overline{X}_B$ the current feasible solution, where

$$B\overline{X}_B = \begin{bmatrix} \overline{b} \\ \overline{1} \end{bmatrix}$$

by letting $\overline{1}$ be a p-component column vector of 1's . We will also need to transform the feasible solution by

$$\overline{X}_B = R\overline{Y}_B$$

so that

$$B\overline{X}_B = (BR)\overline{Y}_B = \begin{bmatrix} \overline{b} \\ \overline{1} \end{bmatrix} \qquad (2\text{-}13)$$

We now have BR as the basis matrix for the feasible solution $\overline{Y}_B$. If we partition $\overline{Y}_B$ into a p-component vector and a m-component vector and we write BR in terms of (2-12), we have that (2-13) is now

$$\begin{bmatrix} F_{m\times p} & \bar{B}_{m\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix} \begin{bmatrix} y_{B_1} \\ \vdots \\ y_{B_p} \\ \hline y_{B_{p+1}} \\ \vdots \\ y_{B_{p+m}} \end{bmatrix} = \begin{bmatrix} \bar{b} \\ \bar{1} \end{bmatrix}$$

which written in terms of its components is

$$F_{m\times p} \begin{bmatrix} y_{B_1} \\ \vdots \\ y_{B_p} \end{bmatrix} + \bar{B}_{m\times m} \begin{bmatrix} y_{B_{p+1}} \\ \vdots \\ y_{B_{p+m}} \end{bmatrix} = \bar{b} \qquad (2\text{-}14a)$$

$$I_{p\times p} \begin{bmatrix} y_{B_1} \\ \vdots \\ y_{B_p} \end{bmatrix} = \bar{1} \qquad (1\text{-}4b)$$

We note that from (2-14b) that $y_{B_i} = 1$, $i = 1, \ldots, p$. Using this information in (2-14a) we can show that

$$\sum_{i=1}^{p} \bar{f}_i + \bar{B} \begin{bmatrix} y_{B_{p+1}} \\ \vdots \\ y_{B_{p+m}} \end{bmatrix} = \bar{b} \qquad (2\text{-}15)$$

where $\bar{f}_i$ is the $i^{\underline{th}}$ column of $F_{m\times p}$. Now if we recall from (2-4) that $F_{m\times p} = \left[\bar{a}_{k_1}, \ldots, \bar{a}_{k_p}\right]$ we can rewrite (2-15) in the form

$$\sum_{i=i}^{p} \bar{a}_{k_i} + \bar{B} \begin{bmatrix} y_B \\ \cdot_{p+1} \\ \vdots \\ y_{B_{p+m}} \end{bmatrix} = \bar{b}$$

or

$$\bar{B} \begin{bmatrix} y_{B_{p+1}} \\ \vdots \\ y_{B_{p+m}} \end{bmatrix} = \bar{b} - \sum_{i=1}^{p} \bar{a}_{k_i} \qquad (2\text{-}16)$$

From (2-16) and the definition of B in (2-8) we can see that B is a basis for the system of equations in (2-6).

We now define the system of equations in (2-6) to be the "reduced system" and that the "working" basis $\bar{B}$ is a basis for this system. The use of the "reduced system" is a update procedure of $\bar{B}^{-1}$ which we show later. Next, we shall show that information can be obtained from using $\bar{B}^{-1}$ that will enable us to perform each operation necessary to apply the revised simplex method to the original problem in (2-2).

To apply the revised simplex method to the problem in (2-2), we need the inverse of the current basis to determine: the pricing vector or simplex multipliers D, the vector to be introduced into the basis

and the vector to leave the basis. We will assume that B is the current basis, with the current basic feasible solution $\overline{X}_B$ and that the corresponding "working" basis is $\overline{B}$.

In the revised simplex method we must calculate the row pricing vector $\overline{D}$ for the original system to enable us to determine which vector is to enter the basis. Let $\overline{D} = [d^{(1)}, d^{(2)}]$, where $d^{(1)} = (d_1^{(1)},\ldots, d_m^{(2)})$ and $d^{(2)} = (d_1^{(2)},\ldots, d_p^{(2)})$. The prices corresponding to the m-equations of (2-2b) is $d^{(1)}$ and the prices corresponding to the p-equations of (2-2c) is $d^{(2)}$.

The vector $\overline{D}$ by definition is

$$\overline{D} = \overline{C}_B B^{-1}$$

where $\overline{C}_B$ is the $(m + p)$-component row vector containing the c_j's (see (2-2a)) corresponding to the columns of B. The vector $\overline{C}_B$ is partitioned into a p-component vector for the first p columns called $C_B^{(1)}$ and an m-component vector for the last m-columns of B, called $C_B^{(2)}$. The vector $\overline{C}_B$ can now be written $\overline{C}_B = [C_B^{(1)}, C_B^{(2)}]$.

To determine $\overline{D}$ from B we would need to solve the $(m + p) \times (m + p)$ set of equations

$$\overline{D}B = \overline{C}_B \quad . \tag{2-17}$$

Now, we shall show that we can determine $\overline{D}$ by using the inverse of the "working" basis $\overline{B}$.

By (2-17) we know that

$$\bar{C}_B = \bar{D}B \ .$$

If we post multiply by R, we have

$$\bar{C}_B R = \bar{D}BR \ .$$

With a substitution from (2-4) for BR along with the partitions of $\bar{D}$ we see that

$$\bar{C}_B R = \begin{bmatrix} d^{(1)}, & d^{(2)} \end{bmatrix} \quad \begin{bmatrix} F_{m\times p} & \bar{B}_{m\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix}$$

$$\bar{C}_B R = \begin{bmatrix} d^{(1)}F_{m\times p} + d^{(2)}, & d^{(1)}\bar{B} \end{bmatrix} \qquad (2\text{-}18)$$

If we recall the definition of R in (2-10) and the partitions of $\bar{C}_B$, we can write

$$\bar{C}_B R = \begin{bmatrix} C_B^{(1)} & , & C_B^{(2)} \end{bmatrix} \quad \begin{bmatrix} I_{p\times p} & -H_{p\times m} \\ 0_{m\times p} & I_{m\times m} \end{bmatrix}$$

$$\bar{C}_B R = \begin{bmatrix} C_B^{(1)}, & C_B^{(2)} - C_B^{(1)}H \end{bmatrix}$$

where the last m components are of the form

$$(C_B^{(2)} - C_B^{(1)}H) = (\{c_t\}; \{c_j - c_{k_i}\}) \qquad (2\text{-}19)$$

for $A_t, A_j \in B$, A_t and A_j nonkey, $A_t \in S_0$, and $A_j \in S_i$, $i \neq 0$. Equation (2-19) will be true, since H from (2-4) is either zero vectors for columns $A_t \in S_0$ or unit vectors, $\bar{e}_i$, for columns $A_j \in S_i$, $i \neq 0$.

We now define

$$\bar{w} = (C_B^{(2)} - C_B^{(1)}H) \qquad (2\text{-}20)$$

so that now

$$\bar{C}_B R = \left[C_B^{(1)}, \bar{w}\right]$$

If we take the form of $\bar{D}BR$ above and substitute it in (2-18), we have

$$\left[C_B^{(1)}, \bar{w}\right] = \left[d^{(1)}F + d^{(2)}, d^{(1)}\bar{B}\right]$$

Hence, we have that

$$d^{(1)}F + d^{(2)} = C_B^{(1)} \qquad (2\text{-}21A)$$

$$d^{(1)}\bar{B} = \bar{w} \qquad (2\text{-}21B)$$

since $d^{(1)}_F + d^{(2)}$, like $C_B^{(1)}$, has p-components and $d^{(1)}_{\bar{B}}$, like $\bar{w}$, has m-components.

Thus, we can find $d^{(1)}$ from post multiplication by $\bar{B}^{-1}$ which gives

$$d^{(1)} = \bar{w}\bar{B}^{-1} \tag{2-22}$$

Moreover, since $F_{m\times p} = \left[\bar{a}_{k_1}, \ldots, \bar{a}_{k_p}\right]$ from (2-4), the p components of $d^{(2)}$ can be found from (2-21A) and (2-22)

$$d^{(2)} = C_B^{(1)} - d^{(1)}_F$$

$$d^{(2)} = C_B^{(1)} - d^{(1)}\left[\bar{a}_{k_1}, \ldots, \bar{a}_{k_p}\right]$$

$$d^{(2)} = C_B^{(1)} - \bar{w}\bar{B}^{-1}\left[\bar{a}_{k_1}, \ldots, \bar{a}_{k_p}\right]$$

Thus, we now have that

$$d_i^{(2)} = C_B^{(1)} = \bar{w}\bar{B}^{-1}a_{k_i}, \quad i = 1, \ldots, p \tag{2-23}$$

Therefore, we see from (2-22) and (2-23) that the vector $\bar{D}$, the set of simplex multipliers (prices) in (2-17), can be determined for the current $(m + p) \times (m + p)$ basis B from the inverse of the corresponding "working" basis $\bar{B}$. This vector $\bar{D}$ enables us to determine the $z_j - c_j$'s for those vectors not in the current feasible basis.

By definition $z_j - c_j = \bar{D}A_j - c_j$. Thus, we can price out the nonbasic vectors by calculating $\bar{D}A_j - c_j$ (i.e., $z_j - c_j$) for each A_j that is nonbasic (not an element of B) . Since our problem is to maximize z , if all $\bar{D}A_j - c_j > 0$, then the current solution is optimal (In minimization if all $\bar{D}A_j - c_j < 0$, then current solution is optimal). If some $\bar{D}A_j - c_j < 0$, then we will bring A_u into the current feasible basis, where $DA_u - c_u = \underset{j}{\text{Min}}\{DA_j - c_j \mid A_j \notin B\}$.

In computing $\bar{D}A_j - c_j$ for each A_j not in B , if $A_j \in S_0$ then by definition of S_0 and the partition of D we have

$$z_j - c_j = \bar{D}A_j - c_j = (d^{(1)}, d^{(2)}) \begin{bmatrix} \bar{a}_j \\ \bar{0} \end{bmatrix} - c_j$$

$$z_j - c_j = \bar{D}A_j - c_j = d^{(1)}a_j - c_j$$

From (2-22), where $d^{(1)} = \bar{w}\bar{B}^{-1}$, we have that for $A_j \in S_0$

$$z_j - c_j = \bar{D}A_j - c_j = \bar{w}\bar{B}^{-1}\bar{a}_j - c_j \qquad (2\text{-}24)$$

Otherwise, if $A_j \in S_i$, $i \neq 0$, then from definition of S_i , $i \neq 0$, and the partition of $\bar{D}$ we have

$$z_j - c_j = \bar{D}A_j - c_j = (d^{(1)}, d^{(2)}) \begin{bmatrix} \bar{a}_j \\ \bar{e}_i \end{bmatrix} - c_j$$

$$z_j - c_j = \bar{D}A_j - c_j = d^{(1)}\bar{a}_j + d_i^{(2)} - c_j$$

From (2-22) we have .

$$z_j - c_j = \bar{D}A_j - c_j = \bar{w}\bar{B}^{-1}\bar{a}_j + d_i^{(2)} - c_j$$

From (2-23) we have that $d_i^{(2)} = C_{B_i}^{(1)} - \bar{w}\bar{B}^{-1}\bar{a}_{k_i}$. Thus we have

$$z_j - c_j = \bar{D}A_j - c_j = \bar{w}\bar{B}^{-1}\bar{a}_j + C_{B_i} - \bar{w}\bar{B}^{-1}\bar{a}_{k_i} - c_j$$

$$z_j - c_j = \bar{D}A_j - c_j = \bar{w}\bar{B}^{-1}(\bar{a}_j - \bar{a}_{k_i}) + C_B^{(1)} - c_j$$

$$\text{for } A_j \in S_i \ , \ i \neq 0 \qquad (2\text{-}25)$$

Now that we have found the vector to enter the current basis (A_u: $\bar{D}A_u - c_u = \text{Min}\{\bar{D}A_j - c_j\}$) we must find the vector to be eliminated from the current basis. This vector is selected based on the simplex-method directions for preserving feasibility; i.e., we need to determine the ratio θ , which requires the expression of A_u in terms of the vectors of the current basis B . This expression of A_u is

$$A_u = B\bar{X}_u \qquad (2\text{-}26)$$

where $\bar{X}_u$ is a $(m + p)$-component column vector. If we let $\bar{X}_u = R\bar{Y}_u$ (2-26) becomes

$$A_u = BR\bar{Y}_u \qquad (2\text{-}27)$$

where we partition $\bar{Y}_u$ we will use two cases, One case for $A_u \in S_0$ and the other for $A_u \in S_i$, $i \neq 0$.

CASE I: $A_u \in S_0$

From (2-12) and the definition S_0 , equation (2-27) can be written

$$\begin{bmatrix} \bar{a}_u \\ \bar{0} \end{bmatrix} \begin{bmatrix} F_{m\times p} & \bar{B}_{m\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix} \begin{bmatrix} y_u^{(1)} \\ y_u^{(2)} \end{bmatrix} \tag{2-28}$$

$$\begin{bmatrix} \bar{a}_u \\ \\ \bar{0} \end{bmatrix} \begin{bmatrix} F_{m\times p} y_u^{(1)} + \bar{B}_{m\times m} y_u^{(2)} \\ \\ y_u^{(1)} \end{bmatrix}$$

We can now see that from (2-28)

$$y_u^{(1)} = \bar{0} \tag{2-29}$$

If we use $y_u^{(1)} = \bar{0}$ to solve for the top m elements of (2-28), we have that

$$\begin{aligned} \bar{a}_u &= F_{m\times p}\, \bar{0} + \bar{B} y_u^{(2)} \\ \bar{a}_u &= \bar{B} y_u^{(2)} \end{aligned} \tag{2-30}$$

or

$$\bar{B}^{-1}\bar{a}_u = y_u^{(2)}$$

If we recall that $\bar{X}_u = R\bar{Y}_u$, the definition of R in (2-10), and partition $\bar{X}_u$ into a p-component vector $X_u^{(1)}$ for its first p elements and a m-component vector $X_u^{(2)}$ for the last m elements, we can show that

$$\begin{bmatrix} X_u^{(1)} \\ X_u^{(2)} \end{bmatrix} = \left[\begin{array}{c|c} I_{p\times p} & -H_{p\times m} \\ \hline 0_{m\times p} & I_{m\times m} \end{array}\right] \begin{bmatrix} y_u^{(1)} \\ y_u^{(2)} \end{bmatrix}$$

If we substitute the solutions for $y_u^{(1)}$ from (2-29) and $y_u^{(2)}$ from (2-30) we have

$$\begin{bmatrix} X_u^{(1)} \\ X_u^{(2)} \end{bmatrix} = \begin{bmatrix} I_{p\times p} & -H_{p\times m} \\ 0_{m\times p} & I_{m\times m} \end{bmatrix} \begin{bmatrix} \bar{0} \\ \bar{B}^{-1}\bar{a}_u \end{bmatrix}$$

$$\begin{bmatrix} X_u^{(1)} \\ X_u^{(2)} \end{bmatrix} = \begin{bmatrix} -H\bar{B}^{-1}\bar{a}_u \\ \bar{B}^{-1}\bar{a}_u \end{bmatrix} \qquad (2\text{-}31)$$

Therefore, it is clear that we can compute $\bar{X}_u$ easly from $\bar{B}^{-1}$, since H from (2-4) is made up of columns of zero vectors and unit vectors $\bar{e}_i$.

CASE II: $A_u \in S_\alpha$, $1 \leq \alpha \leq p$

From (2-12) and the definition of S_i , $i \neq 0$, equation (2-27) can be written

$$\begin{bmatrix} \bar{a}_u \\ \bar{e}_\alpha \end{bmatrix} = \begin{bmatrix} F_{m\times p} & \bar{B}_{m\times m} \\ I_{p\times p} & 0_{p\times m} \end{bmatrix} \begin{bmatrix} y_u^{(1)} \\ y_u^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} \bar{a}_u \\ \bar{e}_\alpha \end{bmatrix} = \begin{bmatrix} Fy_u^{(1)} + \bar{B}y_u^{(2)} \\ y_u^{(1)} \end{bmatrix}$$

Thus, the last p-components are

$$\bar{e}_\alpha = y_u^{(1)} \tag{2-32}$$

When we substitute $\bar{e}_\alpha$ in for $y_u^{(1)}$, the first m-components become

$$\bar{a}_u = F\bar{e}_\alpha + \bar{B}y_u^{(2)}$$

If we notice that the α th column of F is the key column for S_α (see (2-4)), we can write

$$\bar{a}_u = \bar{a}_{k_\alpha} + \bar{B}y_u^{(2)}$$

$$(\bar{a}_u - \bar{a}_{k_\alpha}) = \bar{B}y_u^{(2)}$$

or

$$\bar{B}^{-1}(\bar{a}_u - \bar{a}_{k_\alpha}) = y_u^{(2)} \quad .$$

From the definition for the $\bar{q}_j$'s after equation (2-5), we have that

$$\bar{B}^{-1}\bar{q}_u = y_u^{(2)} \tag{2-33}$$

Now if we use the partition of $\bar{X}_u$ from Case I and the definition of R in (2-10) we can write $\bar{X}_u = R\bar{Y}_u$ as

$$\begin{bmatrix} X_u^{(1)} \\ X_u^{(2)} \end{bmatrix} = \begin{bmatrix} I_{p\times p} & -H_{p\times m} \\ 0_{m\times} & I_{m\times m} \end{bmatrix} \begin{bmatrix} y_u^{(1)} \\ y_u^{(1)} \end{bmatrix}$$

If we substitute the solutions for $y_u^{(1)}$ from (2-32) and for $y_u^{(2)}$ from (2-33) we have

$$\begin{bmatrix} X_u^{(1)} \\ X_u^{(2)} \end{bmatrix} = \begin{bmatrix} I_{p\times p} & -H_{p\times m} \\ 0_{m\times p} & I_{m\times m} \end{bmatrix} \begin{bmatrix} \bar{e}_\alpha \\ \bar{B}^{-1}\bar{q}_u \end{bmatrix}$$

$$\begin{bmatrix} X_u^{(1)} \\ X_u^{(2)} \end{bmatrix} = \begin{bmatrix} \bar{e}_\alpha - H\bar{B}^{-1}\bar{q}_u \\ \bar{B}^{-1}\bar{q}_u \end{bmatrix} \tag{2-34}$$

Now that we have found the solution to (2-26), the vector to leave the current basis is chosen according to the usual rules. First, however, if all the components of $\bar{X}_u$ are less than or equal to zero, then the problem has an unbounded solution. Otherwise, if at least one $X_{u_i} > 0$ then

$$\theta = \frac{X_{B_r}}{X_{u_r}} = \min_i \{\frac{X_{B_i}}{X_{u_i}} \mid X_{u_i} > 0\} \tag{2-35}$$

where X_{u_i} is the $i^{\underline{th}}$ component of $\bar{X}_u$. Therefore, the vector to be eliminated is the $r^{\underline{th}}$ column of the current basis B.

From (2-35) we have found that the $r^{\underline{th}}$ element of the current feasible solution will be eliminated. The new feasible solution is computed according to the usual formulas:

$$\begin{aligned} \hat{X}_{B_r} &= \theta \\ \hat{X}_{B_i} &= X_{B_i} - \theta X_{u_i} \qquad i = 1, \ldots, m+p, \quad i \neq r \end{aligned} \tag{2-36}$$

Thus far we have developed formulas based on the revised simplex procedure to determine whether a current basis B for the generalized upper-bound problem is optimal or not, how to select a vector to enter the basis, how to select a vector to leave the basis, and how to compute the new feasible solution. All of these formulas use the

inverse of the current "working" basis. Thus, we need only show how to update the "working" basis and its inverse and the development of generalized upper-bound problem will be complete.

We have already found the vector to be brought into the current basis (A_u) and we have also found that the $r^{\underline{th}}$ column of the current basis B is to be eliminated. We are going to let the $r^{\underline{th}}$ column of B be A_t, where $A_t \in S_\ell$. Also, we let $A_u \in S_\alpha$.

We need to be concerned of the possibility that the subsets S_ℓ and S_α are the same set, and whether A_t is the key column for S_ℓ, for the following reason. Throughout this discussion we have assumed that the current ordered basis B contains p key columns (one from each S_i, $i = 1, \ldots, p$) and m nonkey columns. If one of the key columns has to leave the basis, we must choose another column from the basis, in the same S_i as the key column, and designate it as the key column. In so doing, a reordering of the basis must take place to keep the key columns in the first p columns of B. This will also mean that the "working" basis and its inverse will be changed in the reordering.

There are three possible cases we must consider; A_t is not a key column, A_t is a key column and $\ell \neq \alpha$, A_t is a key column and $\ell = \alpha$.

CASE I: A_t is not a key column.

Since A_t is not a key column, the first p columns of B will remain the same when A_u replaces A_t in the basis B: i.e., the key columns remain unchanged. Hence, a reordering of the columns

will not be necessary when we change the basis. A_t in this case corresponds to a column in the "working" basis $\bar{B}$. Thus, to update $\bar{B}$ we need only replace $\bar{q}_t$ with $\bar{q}_u$ in $\bar{B}$ (see (2-8)). To update $\bar{B}^{-1}$ we return to the "reduced system of (2-6).

The "reduced system of (2-6) we recall is

$$\sum_{j=1}^{n} \bar{q}_j x_j = \bar{b} - \sum_{i=1}^{p} \bar{a}_{k_i}$$

and that the "working" basis $\bar{B}$ is a basis for this system (see (2-16)).

The row vector $(\{c_t\}\,|\,\{c_j - c_{k_i}\})$ where $\bar{q}_t$, $\bar{q}_j \in B$, $A_t \in S_0$ and $A_j \in S_i$, $i \neq 0$, will be the row vector of $\hat{c}_j$'s corresponding to the basis $\bar{B}$ of the "reduced system". The $\hat{c}_j$'s of the "reduced system" are the c_j's of the original system if $A_j \in S_0$, or they are $(c_j - c_k)$ if $A_j \in S_i$, $i \neq 0$.

If we recall equation (2-19)

$$(C_B^{(2)} - C_B^{(1)} H) = (\{c_j\}\,|\,\{c_j - c_{k_i}\}) \quad \text{for } A_t \text{ and}$$

A_j nonkey, $A_t \in S_0$ and $A_j \in S_i$, $i \neq 0$

and equation (2-20)

$$\bar{w} = (C_B^{(2)} - C_B^{(1)} H)$$

we have that

$$\bar{w} = (\{c_t\} \vdots \{c_j - c_{k_i}\}) \quad \text{for} \quad A_t \text{ and } A_j \text{ nonkey,}$$
$$A_t \in S_0 \text{ and } A_j \in S_i \,, \quad i \neq 0 \tag{2-37}$$

i.e., $\bar{w}$, is the row vector of $\hat{c}_j$'s corresponding to the basis B of the "reduced system".

Hence, in the revised simplex method working with the "reduced system" we will have available the $(m + 1)$ order basis inverse

$$E^{-1} = \left[\begin{array}{c|c} 1 & \bar{w}\bar{B}^{-1} \\ \hline 0 & \bar{B}^{-1} \end{array}\right] \tag{2-38}$$

at each iteration of the generalized upper-bound procedure. What we are going to show is that the column vector $\begin{bmatrix} z_u - c_u \\ x_u^{(2)} \end{bmatrix}$ is the pivot column to bring $\bar{q}_u$ into $\bar{B}^{-1}$, where $z_u - c_u = \min\{\bar{D}A_j - c_j\}$ from the original system and $x_u^{(2)}$ is from equation (2-31) or (2-34).

Thus to find the $\hat{z}_j - \hat{c}_j$'s for the "reduced system", we multiply the first row of E^{-1} by the column vectors $\begin{bmatrix} \hat{c}_j \\ \bar{q}_j \end{bmatrix}$, where $\hat{c}_j$ is the "reduced system" c_j and $\bar{q}_j$ is the vector from the "reduced system" being priced out. This $\hat{z}_j - \hat{c}_j$ will be by definition

$$\hat{z}_j - \hat{c}_j = \bar{w}\bar{B}^{-1}\bar{q}_j - \hat{c}_j$$

From the definition of c_j and the definition of $\bar{q}_j$ we have

$$\hat{z}_j - \hat{c}_j = \begin{cases} \bar{w}\bar{B}^{-1}\bar{a}_j - c_j \\ \text{for } A_j \in S_0 \text{ and } c_j \text{ from the} \\ \quad \text{original system (2-2)} \\ \bar{w}\bar{B}^{-1}(\bar{a}_j - \bar{a}_{k_i}) - (c_j - c_{B_i}^{(1)}) \\ \text{for } A_j \in S_i \text{ , } i = 1, \ldots, p \text{ , } c_j \\ \text{and } C_{B_i} \text{ from the original system} \\ (2\text{-}2) \end{cases} \tag{2-39}$$

If we recall equation (2-24) and (2-25) we see that (2-39) is equivalent to them. Therefore, the $\hat{z}_j - \hat{c}_j$'s for the "reduced system" are the same as the $z_j - c_j$'s for the original system.

If we were going to transform $\bar{B}^{-1}$ by bringing $\bar{q}_u$ into $\bar{B}$ in the "reduced system" basis, we would use a pivot column of

$$\begin{bmatrix} z_u - c_u \\ \bar{B}^{-1}\bar{q}_u \end{bmatrix} .$$

The pivot element of $\bar{B}^{-1}\bar{q}_u$ will be the element in the position that corresponds to $\bar{q}_t$'s position in $\bar{B}$; i.e., we will pivot on the $(r-p)^{\underline{th}}$ element of $\bar{B}^{-1}\bar{q}_u$. Note that $\bar{B}^{-1}\bar{q}_u$ has already been calculated in (2-31) or (2-34) and is equal to $X_u^{(2)}$. The update transformation of $\bar{B}^{-1}$ may be shown as follows

$$\left[\begin{array}{c|c|c} 1 & \bar{w}\bar{B}^{-1} & z_u - c_u \\ \hline \bar{0} & \bar{B}^{-1} & \chi_u^{(2)} \end{array}\right] \left[\begin{array}{c|c|c} 1 & \hat{\bar{w}}\hat{\bar{B}}^{-1} & 0 \\ \hline \bar{0} & \bar{B}^{-1} & \bar{e}_{(r-p)} \end{array}\right] \qquad (2\text{-}40)$$

CASE II: A_t is a key column and $\ell \neq \alpha$.

In this case, A_t is the key column for S_ℓ . Since, A_t is leaving the current basis B , a new basic column from S_ℓ will have to be designated as the key column for S_ℓ . Let $A_\beta \in B$ be this new basic column to take the place of A_t as the key column of S_ℓ , where A_β is in the $\beta^{\underline{th}}$ column of the current basis B . This means that we must reorder the columns of the current basis B and then perform a reorder transformation of the "working" basis B and its inverse, so that now A_β will be A_{k_ℓ} : i.e., A_β is moved to the $\ell^{\underline{th}}$ column of B and A_t is moved to the nonkey $\beta^{\underline{th}}$ column of B . After the entire reordering process is completed the reordered $\bar{B}$ and $\bar{B}^{-1}$ will be updated in the same manner as CASE I, since A_t is now a nonkey column. We now show how to perform the reorder transformation of the "working" basis $\bar{B}$ in the reordering process.

Let us now recall that the form of $\bar{B}$ in (2-8) is

$$\bar{B} = \left\{ \bar{q}_{r_1}, \ldots, \bar{q}_{r_m} \right\}$$

for $\bar{q}_{r_i} = \bar{q}_j$, $i = 1, \ldots, m$, where $\bar{q}_j = (\bar{a}_j - \bar{a}_{k_k})$ for $A_j \in S_i$ $i \neq 0$, and $\bar{q}_j = \bar{a}_j$ for $A_j \in S_0$. What we need to do is transform the current "working" basis $\bar{B}$ so that A_β would seem to have

been the vector we chose to be the key column. We also must realize that the only columns of $\bar{B}$ that will change are columns $\bar{q}_j$, corresponding to $A_j \in S_\ell$.

Thus, to accomplish our transformation of $\bar{B}$, we need to change all columns of $\bar{B}$ that are of the form $\bar{q}_j = (\bar{a}_j - \bar{a}_t)$, where A_t is the key column of S_ℓ and $A_j \in S_\ell$, to $\hat{\bar{q}}_j = (\bar{a}_j - \bar{a}_\beta)$, where $A_j \in S_\ell$ and A_β is the new key column of S_ℓ. We can make this transformation as follows:

$$\hat{\bar{q}}_j = (\bar{a}_j - \bar{a}_\beta) = (\bar{a}_j - \bar{a}_t) - (\bar{a}_\beta - \bar{a}_t) = \bar{q}_j - \bar{q}_\beta \quad \text{if} \quad j \neq \beta \tag{2-41}$$

$$\hat{\bar{q}}_t = (\bar{a}_t - \bar{a}_\beta) = -(\bar{a}_\beta - \bar{a}_t) = -\bar{q}_\beta \quad \text{if} \quad j = \beta$$

The transformation in (2-41) can be made in $\bar{B}$ by postmultiplying $\bar{B}$ by a matrix that will subtract $\bar{q}_\beta$ from every column $\bar{q}_j$, where $A_j \in S_\ell$ and A_j nonkey, and also multiply column q_β, in the $(g - p)$ position of $\bar{B}$, by -1 to form $\bar{q}_t$. This transformation matrix is

$$V = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ 0, \ldots, 0, & -1, & \ldots, & -1, & -1, & -1, & \ldots, & -1, & 0, \ldots, 0 \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \end{bmatrix} \text{ - row } (g-p) \quad (2\text{-}42)$$

$(m \times m)$

where V is the unit $(m \times m)$ matrix, except that now row $(g-p)$ (where the $g^{\underline{th}}$ column of B is the current position of A_β) has a diagonal element of -1 and -1's in the positions that corresponds to the column positions of $\bar{q}_j$ in $\bar{B}$, where $A_j \in S_\ell$. The product $\bar{B}V$ will be the new transformed "working" basis $\tilde{\bar{B}}$, i.e., $\bar{B}V = \tilde{\bar{B}}$.

We also need the inverse of this new "working" basis $\tilde{\bar{B}}$ for the reordering process to be complete. The inverse of $\tilde{\bar{B}}$ is simply

$$(\bar{B}V)^{-1} = \tilde{\bar{B}}^{-1}$$

$$V^{-1}\,\bar{B}^{-1} = \tilde{\bar{B}}^{-1} \qquad (2\text{-}43)$$

Thus, since we know the current $\bar{B}^{-1}$ we need to find the form of V^{-1} to be able to find $\tilde{\bar{B}}^{-1}$. We now show that $VV = I_{m\times m}$ i.e., $V = V^{-1}$.

We know that V is the same as $I_{m\times m}$ except for row $(g - p)$. Thus, V can be formed by adding matrix to $I_{m\times m}$ that has the following form

$$L = \begin{bmatrix} 0 & & & 0 & & \\ & \ddots & & \vdots & & \\ & & 0 & & & \\ 0,\ldots,0,-1,\ldots,-1,-2,-1,\ldots,-1,0\ldots 0 & & & & & \text{- row } (g-p) \\ & & & 0 & 0 & \\ & & & \vdots & & \ddots \\ & & & 0 & & 0 \end{bmatrix}_{(m\times m)}$$

Column $(g-p)$

where all elements are zero except in row $(g - p)$. Row $(g - p)$ has -1's in positions corresponding to the column positions of $\bar{q}_j \in \bar{B}$, for $A_j \in S_\ell$, and -2 in the $(g - p)^{\underline{th}}$ position.

If we call the $(g - p)^{\underline{th}}$ row of L, u^T, we can form L from $\bar{e}_{(g-p)}u^T$, i.e.,

$$L = \bar{e}_{(g-p)}u^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{- } (g-p)^{th} \text{ element} \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 0,\ldots,0,-1,\ldots -1,-2,-1,\ldots,-1,-,\ldots,0 \\ \uparrow \\ (g-p)^{\underline{th}} \text{ element} \end{bmatrix}$$

Hence, V is now

$$V = I_{m\times m} + \bar{e}_{(g-p)}u^T$$

and VV will be

$$VV = (I_{m\times m} + \bar{e}_{(g-p)}u^T)\ (I_{m\times m} + \bar{e}_{(g-p)}u^T)$$

$$VV = I_{m\times m} + \bar{e}_{(g-p)}u^T + \bar{e}_{(g-p)}u^T + \bar{e}_{(g-p)}\left(u^T\bar{e}_{(g-p)}\right)u^T$$

$$VV = I_{m\times m} + 2\bar{e}_{(g-p)}u^T + \bar{e}_{(g-p)}(-2)u^T$$

$$VV = I_{m\times m} + 2\bar{e}_{(g-p)}u^T - 2\bar{e}_{(g-p)}u^T$$

$$VV = I_{m\times m}$$

Thus we have that $V = V^{-1}$.

Now that we know the form of V is the same as V^{-1} , equation (2-43) can now be written.

$$V\bar{B}^{-1} = \tilde{\bar{B}}^{-1} \qquad (2\text{-}44)$$

Hence, the reordering process is complete once we have transformed $\bar{B}^{-1}$ into $\tilde{\bar{B}}^{-1}$, since A_t is no longer the key column for S_ℓ . We can now update $\tilde{\bar{B}}^{-1}$ in the same way as CASE I. We will also need to update $\bar{w}$, since $\bar{C}_B$ has also changed order.

CASE III: A_t is a key column and $\ell = \alpha$

In this case the leaving vector A_t is the key column for S_ℓ. Since there is the possibility that there are no other columns from S_ℓ (other than A_t) in the current basis, we have two possibile situations. If there is another column from S_ℓ in the current basis B, we designate it to be the key column and proceed in the same manner as CASE II in reordering B, $\bar{B}$ and $\bar{B}^{-1}$. However, if there is not another column from S_ℓ that is basic, we must let A_β become the new key column for S_ℓ. The fact that there are no other elements of S_ℓ in the current basis will also mean that none of the columns of the "working" basis will be affected by A_β becoming the key column for S_ℓ: i.e., no update in the "working" basis.

We have now finished our development of the generalized upper-bound method. To summarize the steps needed to solve a generalized upper-bound problem we list the following steps as a guide;

Step One: Start with an initial feasible basis and initial feasible solution. Order the basis by designating a basic column for each upper-bound constraint (i.e., S_i, $i \neq 0$) to be the key column Place the key columns in the first p columns of the basis (i.e., form B) (see (2-3)).

Step Two: Form $\bar{B}$ by subtracting the key columns from the non-key columns of $\bar{B}$ that are in the same S_i (i.e., form T) and partition T as in equation (2-12). An alternate way to form $\bar{B}$,

is to solve $\bar{B} = G - FH$, where G , F , and H are the partitioned parts of B (see (2-4) and (2-11)).

Step Three: Calculate $\bar{B}^{-1}$ by any method available.

Step Four: Form $\bar{w}$ and calculate $\bar{w}\bar{B}^{-1}$. (see (2-20)).

Step Five: Begin the iterative process by computing $z_j - c_j$ for each A_j not in the current basis B . (see (2-24) and (2-25)). If all of the computed $z_j - c_j$'s are greater than or equal to zero the current solution is optimal.

Step Six: Find the minimum $z_j - c_j$ and designate the corresponding column to enter the current basis. (i.e., find A_u corresponding to $z_u - c_u = \min_j \{ z_j - c_j | A_j \notin B\} = \min_j \{ \bar{D}A_j - c_j | A_j \notin B\}$)

Step Seven: Compute $\bar{X}_u$ (see (2-31) or (2-34)). If all $X_{u_i} \leq 0$ then the problem has an unbounded solution.

Step Eight: Find θ , the minimum ratio $\frac{X_{B_i}}{X_{u_i}}$ for X_{u_i}'s greater than zero. The X_{B_i} of the minimum ratio will correspond to the column to leave the current basis. (i.e., $\theta = \frac{X_{B_r}}{X_{u_r}}$ implies the r^{th} column of the current basis to leave) (see (2-35)) .

Step Nine: Update $\bar{B}^{-1}$ according to one of the three cases of the update procedure.

Step Ten: Form the new basic variables, (see (2-36)) and return to Step Five.

We will now solve an example of a generalized upper-bound problem. We use for an example the problem

Maximize $z = x_1$

subject to:

$$5x_1 + 10x_2 + 5x_3 + 15x_4 + 10x_5 + 15x_6 + 5x_7 + 10x_8 = 34$$

$$5x_1 + 10x_2 + 10x_3 + 5x_4 + 20x_5 + 15x_5 + 10x_7 + 10x_8 = 38$$

$$x_2 + x_3 = 1$$

$$x_4 + x_5 + x_6 = 1$$

$$x_7 + x_8 = 1$$

where $m = 2$ and $p = 3$.

An initial feasible solution to the problem is $x_1 = 1/5$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $x_5 = 2/5$, $x_6 = 3/5$, $x_7 = 0$, $x_8 = 1$. Our initial unordered feasible basis is therefore $\left[A_1, A_2, A_5, A_6, A_8\right] =$

$$\begin{bmatrix} 5 & 10 & 10 & 15 & 10 \\ 5 & 10 & 20 & 15 & 10 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Our first step is to order our basis by designating key columns. We will let A_2 be the key column for $S_1 = \{A_2, A_3\}$, A_6 be the key column for $S_2 = \{A_4, A_5, A_6\}$, and A_7 be the key column for $S_3 = \{A_7, A_8\}$.

The ordered basis is now

$$B = \left[A_2, A_6, A_8, A_1, A_5\right] = \begin{bmatrix} 10 & 15 & 10 & 5 & 10 \\ 10 & 15 & 10 & 5 & 20 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

If we parition B in the form of (2-4), to form F, G, and H, we have

$$\bar{B} = \left[\begin{array}{ccc:cc} 10 & 15 & 10 & 5 & 10 \\ 10 & 15 & 10 & 5 & 20 \\ \hdashline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c:c} F_{2x3} & G_{2x2} \\ \hdashline I_{3x3} & H_{3x2} \end{array}\right]$$

We now need to find our "working" basis B. Since we know F, G, and H from the parition of B above, we have that

$$\bar{B} = \left[G - FH\right] = \begin{bmatrix} 5 & 10 \\ 5 & 10 \end{bmatrix} - \begin{bmatrix} 10 & 15 & 10 \\ 10 & 15 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix}$$

The inverse of $\bar{B}$ may be calculated in many ways. The one we will use in this case is the elementary row procedure to change $\left[B \,\vdots\, I_{2\times 2}\right]$ to $\left[I_{2\times 2} \,\vdots\, B^{-1}\right]$.

$$\left[\bar{B} \,\vdots\, I_{2x2}\right] = \left[\begin{array}{cc:cc} 5 & -5 & 1 & 0 \\ 5 & 5 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{cc:cc} 1 & -1 & 1/5 & 0 \\ 1 & 1 & 0 & 1/5 \end{array}\right] \approx$$

$$\left[\begin{array}{cc:cc} 1 & -1 & 1/5 & 0 \\ 0 & 2 & -1/5 & 1/5 \end{array}\right] \approx \left[\begin{array}{cc:cc} 1 & 0 & 1/10 & 1/10 \\ 0 & 1 & -1/10 & 1/10 \end{array}\right] \approx$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 1/10 & 1/10 \\ 0 & 1 & -1/10 & 1/10 \end{array}\right] = \left[I_{2x2} \vdots B^{-1}\right]$$

Thus, we have

$$\bar{B}^{-1} = \begin{bmatrix} 1/10 & 1/10 \\ -1/10 & 1/10 \end{bmatrix}$$

To form $\bar{w}$ we use H from the partition of B and $\bar{C}_B = (C^{(1)}, C_B{}^{(2)}) = (0, 0, 0 \quad 1, 0)$. (Note c_1 is the only nonzero c_j) We now have

$$\bar{w} = (C_B^{(2)} - C_B^{(1)}H) = \left(\begin{bmatrix} 1, & 0 \end{bmatrix} - \begin{bmatrix} 0, & 0, & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1, & 0 \end{bmatrix}$$

and

$$\bar{w}\,\bar{B}^{-1}\begin{bmatrix} 1, & 0 \end{bmatrix}\begin{bmatrix} 1/10 & 1/10 \\ -1/10 & 1/10 \end{bmatrix} = \begin{bmatrix} 1/10, & 1/10 \end{bmatrix}$$

We now price out the vectors of the original system not in the basis, i.e. calculate $z_j - c_j$ for A_3, A_4, and A_7 . From (2-25) we now have

$$z_3 - c_3 = \begin{bmatrix} 1/10, & 1/10 \end{bmatrix} \begin{bmatrix} 5 - 10 \\ 10 - 10 \end{bmatrix} + 0 - 0 = -1/2$$

$$z_4 - c_4 = \begin{bmatrix} 1/10, & 1/10 \end{bmatrix} \begin{bmatrix} 15 - 15 \\ 5 - 15 \end{bmatrix} + 0 - 0 = -1$$

$$z_7 - c_7 = \begin{bmatrix} 1/10, & 1/10 \end{bmatrix} \begin{bmatrix} 5 - 10 \\ 10 - 10 \end{bmatrix} + 0 - 0 = -1/2$$

Thus, the minimum $z_j - c_j$ is $z_4 - c_4 = -1$. This means that A_4 is to enter the current basis. Next we calculate $\bar{X}_4$ to determine the mimimum ratio θ . We know that $A_4 \in S_2$ and therefore from equation (2-34) $\bar{X}_4$ is

$$\bar{X}_4 = \begin{bmatrix} \bar{e}_2 - \bar{B}^{-1}\bar{q}_4 \\ \bar{B}^{-1}\bar{q}4 \end{bmatrix}$$

where

$$\bar{q}_4 = \bar{a}_4 - \bar{a}_6 = \begin{bmatrix} 15 \\ 5 \end{bmatrix} - \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

$$\bar{B}^{-1}\bar{q}_4 = \begin{bmatrix} 1/10 & 1/10 \\ -1/10 & 1/10 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$H\bar{B}^{-1}\bar{q}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

So that now

$$\bar{X}_4 = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

Since only the second element of $\bar{X}_4$ is positive, the ratio $\frac{X_{B_2}}{X_{4_2}} = \frac{3/5}{2}$ will be θ; i.e., $\theta = 3/10$. From the fact that the minimum ratio is $\frac{X_{B_2}}{X_{4_2}}$ we know that the second column of the basis, A_6, must leave the basis.

We must use Case III of the update procedure to update $\bar{B}^{-1}$, since A_6 is the key column for S_2 and A_4 is an element of S_2. We now make A_5 the key column for S_2, because it is the only other basic column from S_2. We do this by multipling $\bar{B}^{-1}$ by V.

The form of V is

$$V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where the second row is the 5-3 row (i.e., $(q - p)$ in development),

Thus,

$$V\bar{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/10 & 1/10 \\ -1/10 & 1/10 \end{bmatrix} = \begin{bmatrix} 1/10 & -1/10 \\ 1/10 & -1/10 \end{bmatrix}$$

Note that $\bar{w}$ has not changed since all c_j's = 0 , $j \neq 1$ and hence all $(c_j - c_{k_i}) = 0$ for $A_j \in S_i$, $i \neq 0$.

We now calculate $\bar{w}\bar{B}^{-1}$ with the new $\bar{B}^{-1}$ (i.e., $\bar{V}\bar{B}^{-1}$)

$$\bar{w}\bar{B}^{-1} = \begin{bmatrix} 1, & 10 \end{bmatrix} \begin{bmatrix} 1/10 & -1/10 \\ 1/10 & -1/10 \end{bmatrix} = \begin{bmatrix} 1/10, & 1/10 \end{bmatrix}$$

We can now update $\bar{B}^{-1}$ to $\hat{\bar{B}}^{-1}$ as in equation (2-4), where

$$\left[\begin{array}{c|c} E^{-1} & \begin{matrix} z_4 - c_4 \\ \hline X_4^{(2)} \end{matrix} \end{array}\right] = \left[\begin{array}{c|cc|c} 1 & 1/10 & 1/10 & -1 \\ \hline 0 & 1/10 & -1/10 & -1 \\ 0 & 1/10 & -1/10 & 2 \end{array}\right] \approx$$

$$\left[\begin{array}{c|cc|c} 1 & 3/20 & 1/20 & 0 \\ \hline 0 & 3/20 & 1/20 & 0 \\ 0 & 1/20 & -1/20 & 1 \end{array}\right] = \left[\begin{array}{c|c|c} 1 & \hat{\bar{w}}\,\hat{\bar{B}}^{-1} & 0 \\ \hline 0 & \hat{\bar{B}}^{-1} & \bar{e}_2 \end{array}\right]$$

From the above we see that the new $\bar{B}^{-1}$ and $\bar{w}\bar{B}^{-1}$ are

$$\bar{B}^{-1} = \begin{bmatrix} 3/20 & 1/20 \\ 1/20 & -1/20 \end{bmatrix} \qquad \bar{w}\bar{B}^{-1} = \begin{bmatrix} 3/20 & 1/20 \end{bmatrix}$$

It is important to note, as we form the new basic variables, that then leaving vector A_6 was moved to the $5^{\underline{th}}$ position in the basis and that A_4 has taken its place.

The form of the new basic variables is

$\hat{X}_{B_1} = 1 - (3/10)(0) = 1$ which corresponds to the value of x_2

$\hat{X}_{B_2} = 2/5 - (3/10)(-1) = 7/10$ " " " " " " x_5

$\hat{X}_{B_3} = 1 - (3/10)(0) = 1$ " " " " " " x_8

$\hat{X}_{B_4} = 1/5 - (3/10)(-1) = 1/2$ " " " " " " x_1

$\hat{X}_{B_5} = 0 = 3/10$ " " " " " " x_4

We start the iterative process over by again by calculating the $z_j - c_j$'s for the vectors not in the basis. In this case for A_3, A_6 and A_7.

$$z_3 - c_3 = [3/20, 1/20] \begin{pmatrix} 5 - 10 \\ 10 - 10 \end{pmatrix} + 0 - 0 = -3/4$$

$$z_6 - c_6 = [3/20, 1/20] \begin{pmatrix} 15 - 10 \\ 15 - 10 \end{pmatrix} + 0 - 0 = 1/2$$

$$z_7 - c_7 = [3/20, 1/20] \begin{pmatrix} 5 - 10 \\ 10 - 10 \end{pmatrix} + 0 - 0 = -3/4$$

We arbitrarily choose A_3 to enter the basis, since there is a tie between $z_3 - c_3$ and $z_7 - c_7$ as the minimum $z_j - c_j$.

We now calculate $\bar{X}_3$

$$\bar{X}_3 = \begin{bmatrix} \bar{e}_1 - H\bar{B}^{-1}\bar{q}_3 \\ \bar{B}^{-1}\bar{q}_3 \end{bmatrix}$$

where

$$\bar{q}_3 = \bar{a}_3 - \bar{a}_2 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

$$\bar{B}^{-1}\bar{q}_3 = \begin{bmatrix} 3/20 & 1/20 \\ 1/20 & -1/20 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/4 \\ -1/4 \end{bmatrix}$$

$$H\bar{B}^{-1}\bar{q}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/4 \\ -1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/4 \\ 0 \end{bmatrix}$$

So that now

$$\bar{X}_3 = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1/4 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3/4 \\ -1/4 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \\ 0 \\ -3/4 \\ -1/4 \end{bmatrix}$$

The positive elements of X_3 are $X_{3_1} = 1$ and $X_{3_2} = 1/4$. Thus the possible candidates for θ are $\frac{X_{B_1}}{X_{3_1}} = \frac{1}{1}$ and $\frac{X_{B_2}}{X_{3_2}} = \frac{7/10}{1/4}$.

Since $\frac{X_{B_1}}{X_{3_1}}$ is the minimum of the two, $\theta = \frac{X_{B_1}}{X_{3_1}} = 1$ and the first Column of the basis, A_2, is to leave the basis.

Since A_2 is the key column for S_1 and the only basic column from S_1, the entering vector A_3 must be the new key column for S_1. This is a Case III update, where no change in $\bar{B}^{-1}$ takes place So $\bar{B}^{-1}$ and $\bar{w}\bar{B}^{-1}$ remain unchanged

$$\bar{B}^{-1} = \begin{bmatrix} 3/20 & 1/20 \\ 1/20 & -1/20 \end{bmatrix} \qquad \bar{w}\bar{B}^{-1} = \begin{bmatrix} 3/20 & 1/20 \end{bmatrix}$$

The form of the new basic variables is

$\hat{X}_{B_1} = \theta = 1$ which corresponds to the value of x_3

$\hat{X}_{B_2} = 7/10 - (1)(1/4) = 9/20$ " " " " " " x_5

$\hat{X}_{B_3} = 1 - (1)(0) = 1$ " " " " " " x_8

$\hat{X}_{B_4} = 1/2 - (1)(-3/4) = 5/4$ " " " " " " x_1

$\hat{X}_{B_5} = 3/10 - (1)(-1/4) = 11/20$ " " " " " " x_4

We again start the iterative process over by calculating $z_j - c_j$ for A_2, A_6 and A_7, the vectors not in the basis.

$$z_2 - c_2 = \begin{bmatrix} 3/20, & 1/20 \end{bmatrix} \begin{pmatrix} 10 - 5 \\ 10 - 10 \end{pmatrix} + 0 - 0 = 3/4$$

$$z_6 - c_6 = \begin{bmatrix} 3/20, & 1/20 \end{bmatrix} \begin{pmatrix} 15 - 10 \\ 10 - 20 \end{pmatrix} + 0 - 0 = 1/2$$

$$z_7 - c_7 = \begin{bmatrix} 3/20, & 1/20 \end{bmatrix} \begin{pmatrix} 5 - 10 \\ 10 - 10 \end{pmatrix} + 0 - 0 = -3/4$$

Since $z_7 - c_7$ is the only negative $z_j - c_j$, A_7 is to enter the basis

We now calculate $\bar{X}_7$

$$\bar{X}_7 = \begin{bmatrix} \bar{e}_3 - H\bar{B}^{-1}\bar{q}_7 \\ \bar{B}^{-1}\bar{q}_7 \end{bmatrix}$$

where

$$\bar{q}_7 = \bar{a}_7 - \bar{a}_8 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

$$\bar{B}^{-1}\bar{q}_7 = \begin{bmatrix} 3/20 & 1/20 \\ 1/20 & -1/20 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/4 \\ -1/4 \end{bmatrix}$$

$$H\bar{B}^{-1}\bar{q}_7 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/4 \\ -1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/4 \\ 0 \end{bmatrix}$$

We now have

$$\bar{X}_7 = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1/4 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3/4 \\ -1/4 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \\ 1 \\ -3/4 \\ -1/4 \end{bmatrix}$$

Since the positive elements of X_7 are $X_{7_2} = 1/4$ and $X_{7_3} = 1$, the possible candidates for θ are $\frac{X_{B_2}}{X_{7_2}} = \frac{9/20}{1/4}$ and $\frac{X_{B_3}}{X_{7_3}} = \frac{1}{1}$. Thus, $\theta = \frac{X_{B_3}}{X_{7_3}} = 1$ and the third element of the basis, A_8, is to leave the basis.

This will be a CASE III update, where no change in $\bar{B}^{-1}$ will take place. We have a CASE III update, since the leaving vector A_8 is the key column for S_3 and no other column of S_3 is basic. We will now have that A_7 is the key column of S_3.

The new basic variables are

$\hat{X}_{B_1} = 1 - (1)(0) = 1$ which corresponds to the value of x_3

$\hat{X}_{B_2} = 9/20 - (1)(1/4) = 1/5$ " " " " " " x_5

$\hat{X}_{B_3} = \theta = 1$ " " " " " " x_7

$\hat{X}_{B_4} = 5/4 - (1)(-3/4) = 7/4$ " " " " " " x_1

$\hat{X}_{B_5} = 11/20 - (1)(-1/4) = 4/5$ " " " " " " x_4

We begin the iterative process again by calculating $z_j - c_j$ for A_2, A_3, and A_8, which are the vectors not in the basis.

$$z_2 - c_2 = [3/20,\ 1/20] \begin{pmatrix} 10 - 5 \\ 10 - 10 \end{pmatrix} + 0 - 0 = 3/4$$

$$z_3 - c_3 = [3/20,\ 1/20] \begin{pmatrix} 15 - 10 \\ 15 - 20 \end{pmatrix} + 0 - 0 = 1/2$$

$$z_8 - c_8 = [3/20,\ 1/20] \begin{pmatrix} 10 - 5 \\ 10 - 10 \end{pmatrix} \quad 0 - 0 = 3/4$$

Since there are not any negative $z_j - c_j$'s, the current solution is optimum. The current solution is $\bar{x}_1 = 7/4$, $\bar{x}_2 = 0$, $x_3 = 1$, $\bar{x}_4 = 4/5$, $x_5 = 1/5$, $x_6 = 0$, $x_7 = 1$ and $x_8 = 1$, which gives a maximum value of $7/4$ for x_1, subject to the constraints of the problem.

We have now finished our example of the generalized upper-bound method. We have shown how given an initial feasible solution vector we can form $\bar{B}^{-1}$ and apply the generalized upper-bound method until an optimum solution is found or we determine the solution is unbounded. In conclusion we would like to state that the generalized upper-bound method does entail an extra burden in bookkeeping (i.e., what are the key columns and to which S_i the columns of B belong), but the obvious savings in computational requirements and storage will more than justify the added bookkeeping effort. Problems where the generalized upper-bound method will be of greatest savings are

problems where p is large in comparison to m. It should be noted that the example problem, where $m = 2$ and $p = 3$, is not an accurate test of the savings the generalized upper-bound method can have, since 3 is not large in comparsion to 2. An example of the savings of the generalized upper-bound method has been reported by Hirschfeld in solving an operations planning problem with 50,215 constraints, (627 regular constraints and 49,588 generalized upper-bound constraints) and 282,468 variables. The problem was solved on an IBM System 370/165 in 56.3 central processing unit minutes, with a total elapsed time of 2½ hours.

LIST OF REFERENCES

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