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To the Graduate Council:

I am submitting herewith a thesis written by William Douglas Foland entitled "Hydrodynamical Theory of Spontaneous Fission with Applications to Mendelevium." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

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HYDRODYNAMICAL THEORY OF SPONTANEOUS FISSION
WITH APPLICATIONS TO MENDELEVIUM

A THESIS

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The Graduate Council
of
The University of Tennessee
in
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by
William Douglas Foland
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It is a pleasure for the author to be able to express his gratitude to Professor I. H. Tipton for her continued interest in him and his work. Professor Tipton's encouragement and advice have helped the author throughout his period of residence at the university.

The author wishes to thank the members of his committee for their care in reading the thesis and for their many suggestions for improving the phraseology.

CORRECTIONS

The nuclide considered in this thesis should be $^{256}_{100}\text{Fm}$ instead of $^{256}_{101}\text{Mv}$. The reference cited on page 94 states that the spontaneous-fission half-life of $^{256}_{100}\text{Fm}$ is definitely $3\frac{1}{2}$ hr and that $^{256}_{101}\text{Mv}$ is thought to decay by β^- emission (to $^{256}_{100}\text{Fm}$). The value of x for $^{256}_{100}\text{Fm}$ (when calculated with the constants used on page 96) is 0.814.

In the calculation of the Gamow integral (page 100 ff) a factor of $\sqrt{2}$ was omitted. This omission causes an error in the calculated half-life. If the correctly calculated half-life is 10^u sec and the incorrectly calculated one is 10^w sec, the relationship $u = \sqrt{2}(w + 20.5) - 20.5$ relates the two half-lives.

The two errors change the comparison of calculated and experimental half-lives. The latter error, for example, changes the $10^{1.8}$ -sec half-life in Table V to 10^{11} sec. The change of nuclides will increase the activation energy E_f slightly; for $^{256}_{100}\text{Fm}$ E_f is probably between 4.8 and 5.3 Mev. This increase in E_f will increase the half-life calculated by the method adopted in Chapter VI. When both of these errors are removed, the best value obtained for the spontaneous-fission half-life of $^{256}_{100}\text{Fm}$ is 10^8 sec. (The table on the following page replaces Table V when the two corrections are made.)

William Douglas Foland
Knoxville, June 25, 1958

TABLE V (CORRECTED)

CONSTANTS OF THE BARRIER AND SPONTANEOUS-FISSION HALF-LIFE
WHEN E_f DETERMINES x

E_f (Mev)	last term used	x	$(a_2)_{\max}$	a	$\tau_{1/2} [m(a_2^+)]$	$\tau_{1/2} [m(a_2^3)]$
4.8	a_2^7	0.840	0.68	0.90	$10^{7.9} \text{ sec}$	$10^{8.7} \text{ sec}$
4.8	a_2^8	0.845	0.72	0.93	$10^{9.2} \text{ sec}$	$10^{10.1} \text{ sec}$
5.3	a_2^7	0.836	0.69	0.91	$10^{9.5} \text{ sec}$	$10^{10.2} \text{ sec}$
5.3	a_2^8	0.840	0.73	0.94	$10^{11.2} \text{ sec}$	$10^{12.1} \text{ sec}$

TABLE OF CONTENTS

CHAPTER	PAGE
I. DEVELOPMENT OF THE MODEL	1
Developments prior to the discovery of the	
neutron	1
The Weizsäcker semi-empirical formula	4
The completion of the model following the	
discovery of nuclear fission	7
The liquid-drop model as an extension of	
Weizsäcker's formula	22
II. THE PROBLEM CONSIDERED IN THIS THESIS	24
Justification for the thesis	24
Assumptions characterizing the model used . .	26
The liquid drop	27
WKB approximation	27
Shape of the drop	28
The kinetic energy	28
Adjustment of the potential-energy	
barrier	29
Symmetry of fission	31
Outline of the remainder of the thesis . . .	32
III. THE GAMOW PENETRATION FACTOR	33
The simplest example of the Gamow factor . .	33
The WKB solution of the many-body wave	
equation	34
Penetration of the barrier	38

CHAPTER	PAGE
III. (continued)	
The Gamow factor for a continuum	44
IV. ELECTROSTATIC ENERGY OF THE DEFORMED DROP . . .	47
Formulation of the two-part problem	47
A sample calculation	60
The general calculations	64
V. A HYDRODYNAMICAL CALCULATION	78
The drop and its distortion	78
The boundary condition	79
A simplifying demand and its justification .	80
Evaluation of the kinetic energy	81
VI. SPONTANEOUS FISSION OF ${}_{101}\text{Mv}^{256}$	94
VII. EVALUATION OF RESULTS AND CONCLUSIONS	108
BIBLIOGRAPHY	115
APPENDIX I	118
APPENDIX II	126
APPENDIX III	128

LIST OF TABLES

TABLE	PAGE
I. Saddle-Point Shapes for Five Nuclei	19
II. Constants of the Barrier and Spontaneous-Fission Half-Life When 0.82	103
III. Constants of the Barrier and Spontaneous-Fission Half-Life When 0.83	104
IV. Constants of the Barrier and Spontaneous-Fission Half-Life When 0.845	105
V. Constants of the Barrier and Spontaneous-Fission Half-Life When Determines	106
VI. Constants of the Parameterized Barrier and Spontaneous-Fission Half-Life When 5.0 Mev and 0.83	129
VII. Constants of the Parameterized Barrier and Spontaneous-Fission Half-Life When 4.8 Mev and 0.83	130
VIII. Constants of the Parameterized Barrier and Spontaneous-Fission Half-Life When 5.0 Mev and 0.82	131

LIST OF FIGURES

FIGURE		PAGE
1.	Potential for Uncharged and Charged Liquid	
	Drops	4
2.	The Barrier to Fission	9
3.	The Barrier to be Penetrated	39
4.	Barrier Penetration from Left to Right	41
5.	A Contour Map Showing a Simple Saddle-Point	
	Path	46
6.	A Profile of the Drop	49
7.	Quadrant I of Figure 6	51
8.	Intersection of a Sphere of Radius κ ($\kappa > b$)	
	With the Drop at $\vartheta' = \vartheta'_1$ and $\vartheta' = \vartheta'_2$	54
9.	The $\kappa-\mu'$ Area of Integration	57
10.	Points for Evaluation of Derivatives	68

CHAPTER I

DEVELOPMENT OF THE MODEL

Many theoretical studies of charged liquid drops have been made. There are two reasons for such studies: (1) Under certain conditions a charged liquid drop can be in a dynamical state which is unstable (either classically, quantum-mechanically, or both). (2) The liquid drop is an example of a system of particles held together by saturated¹ forces. These two characteristics of the charged liquid drop have led--both separately and jointly--to the liquid-drop model of the atomic nucleus.

Developments Prior to the Discovery of the Neutron

Lord Rayleigh² made the first study of the small deformations of a charged liquid drop in 1882. He was considering a charged conducting drop (which might be of mercury)

¹A saturated force can arise in two ways, both of which can be most concisely expressed in mathematical terms: (1) The potential energy of interaction between any two particles approaches $+\infty$ when their separation approaches either some small finite value or zero. (These are "hard-core" particles.) (2) The potential energy of interaction contains an exchange operator which changes the sign of the potential energy of interaction for anti-symmetric and not for symmetric states. When two particles have small average separations, their symmetry property determines whether they attract or repel one another. Saturation is manifested by the total binding energy and the volume of the system being both directly proportional to the number of constituent particles.

²J. W. S. Rayleigh, Phil. Mag. 14, 184 (1882).

which he found to be dynamically unstable against small deformations when the ratio of electrostatic to surface energy of the undistorted drop became as great as two. For a charged, conducting, spherical drop with charge Q and radius R_0 , the electrostatic energy is $E_s^0 = \frac{1}{2} Q^2/R_0$; its surface energy can be written as $E_s^0 = 4\pi R_0^2 \cdot \sigma$ where σ is the energy per unit area. The Rayleigh condition for instability is thus $E_s^0/E_s^0 \geq 2$.

Rayleigh used the Lagrangian formulation of mechanics for considering the small deformations of the drop. With spherical coordinates for the drop he assumed distortions such that the distance from the center of the drop to a point on the surface is

$$r(\cos \vartheta) = \sum_{n=0}^{\infty} a_n P_n(\cos \vartheta)$$

where the P_n 's are the Legendre polynomials. The drop thus has axial symmetry: there is no dependence on the azimuthal angle φ . For the kinetic energy of the drop he obtained

$$T = 2\pi \rho R_0^3 \sum (2n^2 + n)^{-1} \left(\frac{da_n}{dt} \right)^2$$

where ρ is the mass density (which is uniform and constant).

The potential energy can be divided into two parts: a surface part (V_s) and an electrostatic part (V_E) with $V = V_s + V_E$.

For V_s Rayleigh obtained $V_s = 2\pi \sigma \sum \frac{(n-1)(n+2)}{2n+1} a_n^2$

and for V_E

$$V_E = -\frac{Q^2}{2R_0^3} \sum \frac{(n-1)}{2n+1} a_n^2.$$

The Lagrangian $L = T - V$ is separable $L = \sum L_n$,
 and the equation $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{a}_n} \right) - \frac{\partial L}{\partial a_n} = 0$ gives the equation
 for uncoupled coordinates

$$\ddot{a}_n + \frac{n(n-1)}{\rho R_0^3} \left[(n+2) \cdot 0 - \frac{Q^2}{4\pi R_0^3} \right] a_n = 0.$$

The system behaves like a collection of uncoupled oscillators
 whenever the coefficient of a_n in the differential equation
 is positive.³

Rayleigh's study of the charged, liquid drop was
 motivated principally by the first of the two reasons noted
 above. The second important study of charged liquid drops
 was motivated by the second of the two reasons. This study
 was begun by Gamow⁴⁻⁶ in 1929. He was considering the similar-
 ity of the liquid drop to the atomic nucleus. From the avail-
 able nuclear data Gamow had concluded that α particles retain
 their identity inside the nucleus; this implies saturated
 forces between particles with resultant surface tension. The
 macrophysical analogue is the liquid drop. Gamow extended
 his considerations to include charge. The important change
 introduced by the charge occurs for the potential: for an

³Part of this formulation is found in the work already
 cited. The remainder is in

J. W. S. Rayleigh, Theory of Sound (New York: Dover
 Publications, 1945), Vol. II, pp. 371-4.

⁴G. Gamow, Proc. Roy. Soc. (London) A, 123, 386 (1929).

⁵Ibid. 126, 632 (1930).

⁶G. Gamow, Physik. Zeitschr. 30, 717 (1929).

uncharged drop one has a potential well on a desert; for a charged one, he has a well on a hill (see Figure 1).

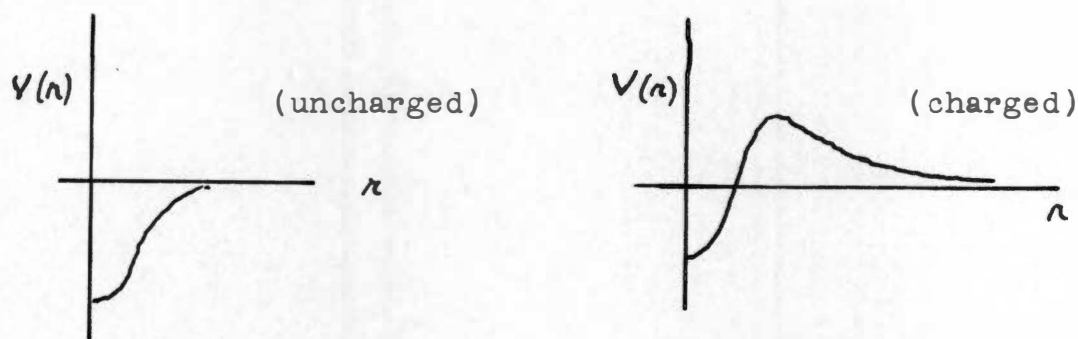


Figure 1. Potential for uncharged and charged liquid drops. ($V(r)$ is potential, a function of position. It is the potential for a molecule being added to the drop.)

Using his quantum-mechanical solution for barrier penetration,⁷ he was able to give a qualitative explanation of α -decay based on this analogy of the charged liquid drop.

The Weizsäcker Semi-Empirical Formula

Studies of charged liquid drops have become numerous due to the liquid-drop model of the atomic nucleus. These studies are usually based on Weizsäcker's semi-empirical formula for nuclear binding energies published in 1935.⁸ This formula expresses the binding energy (B) of a nucleus with

⁷G. Gamow, Zeits. f. Phys. 51, 204 (1928).

⁸C. F. von Weizsäcker, Zeits. f. Phys. 96, 431 (1935).

A nucleons and Z protons as

$$B = \alpha A \left[1 - \beta \left(\frac{N-Z}{A} \right)^2 \right] - \gamma A^{2/3} - \frac{3}{5} \frac{Z^2 e^2}{r_0 A^{1/3}}$$

in which α , β , γ and r_0 are parameters and e is the magnitude of the charge on the electron. ($N = A - Z$ is the number of neutrons.) The parameters are adjusted to fit empirical data. It was known that the density of nuclear matter was nearly constant from one nucleus to another. This leads to interpretation of the first term as a volume energy, of the $A^{2/3}$ term as a surface energy, and of the $A^{-1/3}$ term as electrostatic energy of a spherical body with radius $r_0 A^{1/3}$ and with a uniformly distributed volume charge of magnitude Ze .

The Weizsäcker formula gives the binding energy for the ground state of the nucleus. If the nucleus is distorted from the spherical shape assumed for the ground state, the binding energy is changed. The new binding energy cannot be learned through use of the formula alone. The formula can, however, suggest how one might obtain the new binding energy. To gain maximum information from the formula--with the aim of extending this information--one must consider the various quantities appearing in the formula. The αA part of the volume energy is the contribution of the charge-independent, short-range, nuclear force for A nucleons--all saturated. There are now two corrections to this: (1) The $-\gamma A^{2/3}$ corrects for those nucleons lying in or quite near the surface of the nucleus; they cannot have completely saturated forces.

(2) The symmetry energy term--the second part of the volume energy--is a qualitative expression of the Pauli exclusion principle and the exchange character of the forces. A correction of this form is necessary because both the kinetic energy and the potential energy of the nucleus increase with the $\left(\frac{N-Z}{A}\right)^2$ term.⁹ This is understandable only in terms of quantum mechanics. The nucleus can have at most four nucleons in a given energy level--spin difference allows two like nucleons in a single level. When a level has its quota of one kind of nucleons, other nucleons of the same kind must be promoted to higher energy levels. The effect is associated with both the Pauli principle and the exchange operator used to represent the potential. The increased energy reduces the binding of the nucleus. The electrostatic term in the binding energy is purely classical--the mutual repulsion of the Z protons. Weizsäcker recognized some of the similarity between a liquid drop and the nuclear model expressed by his formula; the extension of the information contained in the formula, however was achieved only after the discovery of neutron-induced fission. This extension came through the study of the liquid-drop model.

⁹J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (New York: John Wiley and Sons, 1952), Chapter VI.

The Completion of the Model Following the
Discovery of Nuclear Fission

The discovery by Fermi and his co-workers of the creation of new isotopes by nuclear reactions in which a neutron combined with a stable nucleus led him to suggest that it might be possible to create nuclei beyond uranium--i.e., with atomic numbers greater than ninety-two.¹⁰ Experiments in which uranium was bombarded with thermal neutrons did lead to the discovery of transuranic nuclei; they also led to the discovery of neutron-induced fission.

Shortly after the discovery of fission Meitner and Frisch¹¹ suggested that the fission phenomenon might be qualitatively explained through use of the analogy between the nucleus and the charged liquid drop. They emphasized the competition between the surface tension and the electrostatic repulsion--the former tending to hold the drop together, the latter tending to cause it to fly apart. They therefore suggested that for uranium the electrostatic repulsion should be almost large enough to cancel the surface tension--the nucleus should have only slight stability in this sense. If one could cause the nucleus to vibrate in a mode corresponding to the collective vibrations of a charged drop, he could cause the nucleus to divide provided the amplitude of vibration became

¹⁰E. Fermi, Nature 133, 898 (1934).

¹¹L. Meitner and O. R. Frisch, Nature 143, 239 (1939).

large enough to enable the electrostatic repulsion to overcome the surface tension. In order to cause the vibration, one would have to supply energy to the nucleus. The energy could be supplied by the neutron: when it enters the nucleus, a new nucleus--a "compound nucleus"--is formed; and the binding energy of the neutron becomes kinetic energy of the compound nucleus. One has, therefore, a qualitative picture of fission.

Meitner and Frisch further suggested that the neutron-induced fission could be looked upon as a purely classical phenomenon. They gave two arguments supporting their suggestion: (1) It would seem that the distortions necessary to produce fission must be large-amplitude distortions. This would mean that the zero-point distortions demanded by quantum mechanics must be negligible. (2) Practically all of the fissions are caused by large-amplitude distortions. The probability-per-unit-time that a large-amplitude distortion produce fission via quantum-mechanical "tunnel effects" must be quite small because of the large mass and barrier involved.

The barrier to be penetrated can be represented by a simple picture. In Figure 2 $V(a)$ is the potential energy of distortion--the work one must do on the nucleus to produce a given distortion--and a is a parameter which defines a given distortion. E_f is the minimum work one must do on the nucleus to cause it to divide.

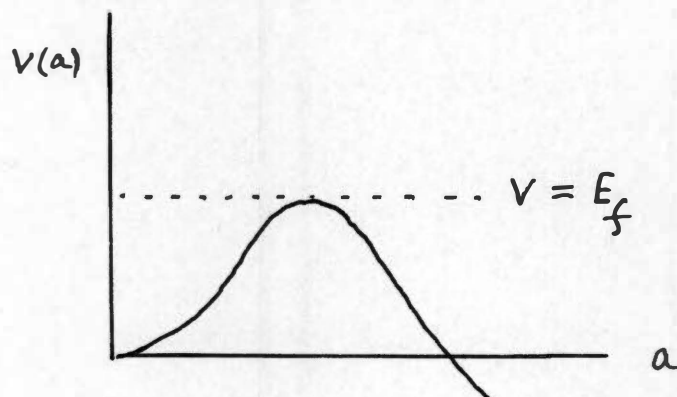


Figure 2. The barrier to fission.

Bohr immediately recognized that the proposal of Meitner and Frisch was promising.¹² He emphasized the competition among modes of decay possible for the compound nucleus formed when the neutron enters a stable nucleus. The compound nucleus has excess energy which it must lose. It can lose this energy through radiation, emission of particles, or fission.¹³

The analogies proposed by Meitner and Frisch and by Bohr completed the outline for the liquid-drop model of the nucleus. The details of the model and its applicability were first studied by Bohr and Wheeler¹⁴ who determined some

¹²N. Bohr, Nature 143, 330 (1939).

¹³The liquid drop contains, in addition to the analogue of fission, an analogue of the emission of particles from the compound nucleus in a nuclear reaction: thermal energy becomes concentrated in some molecule of the drop which, as a result, evaporates. This analogue was not considered by Meitner and Frisch.

¹⁴N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).

of the quantitative predictions of the model and checked these predictions against known experimental data.

Bohr had suggested that the model contains an explanation of the stability of heavy nuclei with respect to spontaneous fission. To examine this aspect of the model, he and Wheeler considered the energy released by fission processes for a number of heavier nuclei. These fission processes were conceptual binary fissions in which the original nucleus divided into two identical nuclei. (Whenever $2k+1$ particles were to be divided, one daughter nucleus got k , the other $k+1$ particles.) They estimated that such fissions release energy whenever the original nucleus contains around one hundred or more nucleons. The amount of energy released grows rapidly with the number of nucleons: they estimated that such a fission for ${}_{28}\text{Ni}^{61}$ requires the addition of 11 Mev, for ${}_{50}\text{Sn}^{117}$ releases 10 Mev, and for ${}_{92}\text{U}^{239}$ releases 200 Mev. They then investigated the liquid-drop model and found that it provided an explanation for the stability of the nuclei.

The model says that the heavy nuclei are stable because in a small deformation the surface energy increases more rapidly than the electrostatic energy decreases; before fission can occur, distortions of the drop must become large enough to reverse this behavior. To examine the explanation, Bohr and Wheeler described the model quantitatively. They assumed:

- (1) The liquid is incompressible since nuclear matter has constant density.
- (2) The initial state corresponds to a

spherical shape. (3) The charge distribution is uniform; the total charge is Ze . (4) The surface tension is the same for all nuclei; it is a constant. The center of the drop was chosen as the origin for spherical coordinates. Distortions of the drop were then assumed to be such that the distance from the origin to any point on the surface becomes

$$r(\vartheta) = R [1 + a_0 + a_2 P_2(\cos \vartheta) + a_3 P_3(\cos \vartheta) + \dots]$$

where ϑ is the colatitude, R is the original radius, the a_n 's characterize the distortion, and $P_n(\cos \vartheta)$ is the Legendre polynomial of degree n .

Upon calculating the change in the energy (surface plus electrostatic) produced by small distortions, they learned that the drop, or the nucleus, first becomes unstable for the P_2 distortions. (These produce "dumbbell" shapes.) The spherical drop is unstable with respect to any small distortion of the P_2 type when the ratio of electrostatic energy to surface energy is greater than two.¹⁵ For a nucleus this ratio is $\frac{3}{5} \frac{Z^2 e^2}{R} / 4\pi R^2 \sigma$ where σ is the surface tension. With $R = r_0 A^{1/3}$ (from the assumption of incompressibility) and with r_0 and σ from experimental mass-defect data, they obtained the ratio for ^{238}U

$$\frac{(\text{electrostatic energy})}{(\text{surface energy})} = \frac{E_c^0}{E_s^0} = 1.71.$$

¹⁵This is the same as the Rayleigh criterion (p. 2). Note, however, that this electrostatic energy is due to a volume charge whereas the Rayleigh electrostatic energy is due to a surface charge.

The ratio E_c^0/E_s^0 is an important quantity. Since it occurs so frequently, it is used to define a quantity x as $x = \frac{1}{2} E_c^0/E_s^0$. (The zero superscript is the conventional ground-state notation. The subscript "c" on the electrostatic energy is frequently used; it comes from the words "Coulomb energy.") From this value one sees that uranium is below the point where instability sets in. One can furthermore conclude that all nuclei below uranium will be below the point where instability sets in: the ratio of the two energies grows as Z^2/R^3 -- or Z^2/A .

The fact that the model does not predict instability for nuclei known to be stable is, of course, no severe test of the model. Bohr and Wheeler extended their calculations of the change in energy produced by distortions in order to be able to consider large-amplitude distortions. They wanted to calculate the minimum energy one must give the nucleus in order to cause distortions so large that the nucleus takes on a critical shape--a shape for which the space-rate of decrease of electrostatic energy becomes larger than the space-rate of increase of surface energy. For this shape the nucleus is unstable; it has enough energy for fission. (The critical shape for the conditions under which Figure 2 is applicable occurs for a' such that $V(a') = E_f$ --this is the "top of the hill.")

The energy necessary for the critical distortion is denoted by E_f --the minimum energy necessary for fission.

They wrote for this energy

$$E_f = 4\pi \kappa_0^2 A^{2/3} \cdot f(x) \quad (= E_f^0 \cdot f(x))$$

where $f(x)$ is some unknown function of the x defined on the previous page. They found $f(x)$ for values of x near unity-- the instability limit for P_2 deformations

$$f(x) = \frac{98}{135} (1-x)^3 - \frac{11368}{34425} (1-x)^4 + \dots$$

and near zero (very small charges)

$$f(x) = 0.260 - 0.215 x,$$

They then

connected the two regions with a plausible curve. As uranium is near $x \approx 1$, they could achieve a rough prediction of E_f for uranium from their curve. Instead of doing this, they took E_f from estimates based on experimental data in order better to determine x for uranium. They conclude that $E_f = 6$ Mev and $x = 0.74$ for U^{239} .¹⁶

The classical treatment assumed in the liquid-drop model was justified by Bohr and Wheeler for the P_2 distortions. They found the time average of the square of the amplitudes for the zero-point oscillations: $\langle a_n^2 \rangle_{z.p.}$

For a_2 they have a frequency of oscillation, ω_2 , for

which they calculate $\frac{1}{2} \hbar \omega_2 = \sim 0.4$ Mev. Then they have¹⁷

¹⁶This value of x is smaller than the one given in the previous case. This value allows the 0 used in the other case to be retained and increases κ_0 by five per cent. Bohr and Wheeler considered this agreement a satisfying check on 0 and κ_0 .

¹⁷As Rayleigh showed (see page 3), the different modes of motion are independent for small amplitudes where they behave like simple harmonic oscillators. The time average of

$$\frac{\langle a_z^2 \rangle_f}{\langle a_z^2 \rangle_{z.p.}} \approx \frac{E_f}{\frac{1}{2} \hbar \omega_z} = \frac{6 \text{ Mev}}{0.4 \text{ Mev}} = 15$$

where $\langle a_z^2 \rangle_f$ refers to the large-amplitudes which lead to fission. The large ratio justifies the use of classical methods for this mode.¹⁸ The magnitude of the contribution of quantum-mechanical "tunnel effect" was also estimated--the effect had been assumed negligible. Their calculation was for a very crude barrier, so their estimate of the spontaneous-fission half-life has little value. It did indicate the valid validity of this classical treatment of fission.

A number of other aspects of fission were considered by Bohr and Wheeler. Among them are the cross-sections for various events: capture of the neutron by the nucleus; decay of the compound nucleus by radiation, neutron emission, and fission; and capture of other particles and of photons by the nucleus and the subsequent decay of the resulting compound nucleus. Their results are generally good; but the treatment does not further the development of the liquid-drop model, and it will not be considered in this thesis.

Experimental data showed that uranium fission was pre-

the square of the amplitude of a simple harmonic oscillator is proportional to its energy.

¹⁸ There are two steps that require justification for the liquid-drop model. Replacing quantum mechanics by classical mechanics is the step considered here. The other step is replacing the many-body problem by the problem of the continuous medium.

ponderantly asymmetric. For the large-amplitude distortions which can lead to fission Bohr and Wheeler had considered only P_2 and P_4 deformations and their coupling¹⁹ which combination must lead to symmetric fission; i.e., division into two nuclides of equal size. They suggested that asymmetry might arise after the nucleus passed through the critical shape. The state of the nucleus is represented by a point in the hyperspace of the a_n 's; the critical shape corresponds to a saddle point on the potential-energy surface in this hyperspace. Their suggestion was that the nucleus point might take some path to fission other than the path of steepest ascent to and descent from the saddle point.

The large-amplitude distortions were studied in more detail by Present and Knipp.²⁰ Their estimates of the quantum excitation energies--for the distortions represented by the a_n 's--indicated that one need consider distortions through a_5 and none higher. Their calculation of the potential energy of distortion--the potential-energy surface--showed strong coupling between even and odd harmonics. If ΔE is the potential energy of distortion,

$$\Delta E = E_s^\circ F(x; a_n)$$

where $F(x; a_n)$ is a multiple-power series in the a_n 's

¹⁹Small-amplitude distortions can be treated as uncoupled (see page 3), but large-amplitude distortions cannot.

²⁰R. D. Present and J. K. Knipp, Phys. Rev. 57, 751 and 1188 (1940).

with coefficients linear in X --which has its previous definition. The $F(X; a_n)$ of Present and Knipp extends beyond the comparable one of Bohr and Wheeler,²¹ and should provide a stronger test of the model.

The odd-even coupling of a_2 and a_3 was considered for two values of X : $X = 0.85$, corresponding to the first value obtained by Bohr and Wheeler for uranium, and $X = 0.75$, corresponding to the value they thought more nearly correct. In order to be consistent in retaining small quantities, Present and Knipp assumed that $a_3 \lesssim a_2^{3/2}$. (This assumption was justified in the result obtained.) For $X = 0.85$ they found a saddle point at $a_2 = 0.65$, $a_3 = 0.25$ with $E_f = 6$ Mev; for $X = 0.75$: $a_2 = 0.66$, $a_3 = 0.30$ and $E_f = 13$ Mev. (For $X = 0.85$ the value of E_f is acceptable; for $X = 0.75$ it is not. The application is to uranium.) Present and Knipp tentatively concluded that the a_2 - a_3 coupling would lead to asymmetric fission with a mass ratio of 2:3 for the resulting nuclei.

The 0.65 is a large value for a_2 ; the power series converges slowly for such large values. Present, Reines,

²¹With the exception of $a_2^3 a_4$ and $a_2 a_4^2$, the multiple-power series of Present and Knipp includes all $a_1^{k_1} a_2^{k_2} a_3^{k_3}$ such that $10 \geq \sum_{i=1}^3 l_i \cdot k_i$ with $2 \leq l_i \leq 5$ and $0 \leq k_i \leq 5$. The comparable series of Bohr and Wheeler contains a_2 and a_4 only, and the inequality has ≤ 8 instead of ≤ 10 .

and Knipp²² extended the $F(x; a_n)$ to include powers to a_2^6 . The convergence remained unsatisfactory, however, for the saddle point of uranium. The difficulty was due to the large values of a_2 at the saddle points. (The P_2 distortion produces the dumbbell shape which represents pretty well the critical shape.) To avoid the expansion in terms of the large a_2 , Present, Reines, and Knipp chose specific values of a_2 and performed expansions in powers of a_4 . In this way they obtained an expansion which is valid for $x \geq 0.80$. An extrapolation to $x = 0.75$ gave a satisfactory activation energy, E_f , for uranium. By including a_1 , and a_3 in the calculations, they determined the odd-even coupling for the large values of x . They concluded that the critical shape was symmetrical for $x \geq 0.80$; the odd-even coupling did not account for asymmetric fission as it had seemed to do in the earlier paper of Present and Knipp.

Present, Reines, and Knipp also attempted to calculate the spontaneous-fission lifetime for U^{235} . (The method used was the one which is used in this thesis. For details see Chapter III.) The predicted lifetime was shorter than the experimental one.

The difficulty of obtaining precise predictions for uranium from the liquid-drop model is illustrated by the problem arising in the paper of Present, Reines, and Knipp:

²²R. D. Present, F. Reines, and J. K. Knipp, Phys. Rev. 70, 557 (1946).

the calculations necessary for the potential energy of distortion are quite easy when one is interested in small distortions; when one considers the large distortions which produce fission, however, these calculations become quite laborious. When the convergence is as slow as Present, Reines, and Knipp indicate, the calculations become prohibitive. With the help of an electronic computer Frankel and Metropolis²³ attacked, and were able to solve, the problem of these calculations--not only for uranium, but for other examples also. They indicate that their numerical calculations should be accurate to within a per cent or so for a number of saddle points; a half-life for spontaneous fission was given for uranium only.

The model used by Frankel and Metropolis is the same as the model already considered. The distortions they considered for most of their work include only even harmonics through P_{10} . They did consider P_1 and P_3 for the investigation of asymmetric fission; for all other work no odd harmonics were considered.

In Table I are given coefficients for saddle-point shapes for five nuclei as determined by Frankel and Metropolis.²⁴ From the table it is immediately apparent that power-series methods are impossible for small values of X and become good

²³S. Frankel and N. Metropolis, Phys. Rev. 72, 914 (1947).

²⁴They also give drawings of the saddle-point profiles for these five nuclei.

TABLE I

SADDLE POINT SHAPES FOR FIVE NUCLEI^a

x	a ₂	a ₄	a ₆	a ₈	a ₁₀
0.9	0.23	0.019	-0.0016	----	----
.81	.47	.083	- .006	-0.006	----
.77	.5 ₅	.1 ₁	- .01	- .01	----
.74	.70	.193	- .008	- .020	-0.0025
.65	2.4	1.15	- .11	- .22	+ .03

^aSource: S. Frankel and N. Metropolis,
loc. cit. Table I.

for large values of χ . One can also see that the higher harmonics contribute decreasingly less and less to the deformations for the critical shapes; P_2 and P_4 describe the critical shape pretty well for the larger values of χ .

Frankel and Metropolis conclude that $\chi = 0.65$ is near the (lower) limit of the χ -region for which the saddle-point shapes of the nucleus can be described through the Legendre polynomials: the saddle-point shapes for $\chi < \sim 0.65$ demand a multiple-valued function $R(\cos\theta)$ rather than the single-valued ones possible through the Legendre polynomials. (For $\chi \rightarrow 0$, for example, the critical shape approaches two spheres in contact.)

The activation energy, E_f , was calculated by Frankel and Metropolis for nuclei near uranium. (Their paper presents these values graphically.) For U^{238} they obtained 6.97 Mev. ($\chi = 0.74$ and $E_s^0 = 538$ Mev were assumed.)

The paper of Frankel and Metropolis is particularly interesting in that they calculate the half-life for spontaneous fission for two values of χ . The calculation they performed is based on a "Gamow penetration factor" to be considered in some detail in Chapter III. This factor is intimately connected with the kinetic energy of the nucleus, and their method of solution can be easily pictured through use of the kinetic energy.

The nucleus was assumed to be distorted so that

$$R(\cos\theta) = R_0 [1 + a_2 P_2(\cos\theta) + O(a_2^2)]$$

(This equation has the same meaning as it has on page 11.) It was also assumed that a velocity potential exists. They obtained for the velocity field

$$\vec{T} = \frac{da_z}{dt} \vec{V} (h^2 p_z (\cos \theta)) .$$

The kinetic energy of the nucleus was then written as

$$T = \frac{1}{2} \mathcal{M} \left(\frac{da_z}{dt} \right)^2$$

in which $\mathcal{M}$ was named the "effective mass." The "mass" $\mathcal{M}$ was found to be

$$\mathcal{M} = 0.3 M (1 + a_z) + O(a_z^2)$$

with M the total mass of the nucleus. One can picture the "mass" $\mathcal{M}$ penetrating the potential barrier with both $\mathcal{M}$ and the barrier height depending on a_z . (See page 39.)

Frankel and Metropolis conclude that the Gamow penetration factor is well represented by $G = 10^{-7.85 E_f}$ with E_f in Mev. For $X = 0.74$ and 0.75 they obtained the respective lifetimes 10^{26} and 10^{17} years. (In addition to X and E_f^0 from above, they assumed $\kappa_0 = 1.47 \times 10^{-13}$ cm.)

Both the activation energy and the lifetime for spontaneous fission were good values for uranium (experimental half-life for U^{238} is $\sim 10^{16}$ years). Both must be regarded as successes of the liquid-drop model. The model failed, however, to predict preferentially asymmetric fission. Frankel and Metropolis did not study this aspect of the model as thoroughly as they did the other two aspects. They added small amounts of P_1 and P_3 deformations to symmetric expansions to try to determine whether the deformation energy increases or decreases with increasing asymmetry. Their results seem to them to suggest strongly that the liquid-drop model does

not lead to preferentially asymmetric fission.

There have been a number of studies of the liquid-drop model other than the ones considered in the preceding pages. Of these studies many work with the model already considered and follow the procedures already described.²⁵ An unpublished thesis by Beynam²⁶ considers small asymmetric deformations of a charged liquid drop for which the distance from the center of the drop to the surface is a function of both the azimuthal and colatitude angles. Considered from the viewpoint of the distortion energies, these more general distortions seem less likely to lead to fission than do the Bohr and Wheeler distortions: the increase in surface energy would be greater for the more distorted surfaces, and the reduction in electrostatic energy would not be great enough to compensate for the increase. The asymmetric distortions might be more likely to occur, but they would seem less likely to lead to fission.

The Liquid-Drop Model as an Extension of Weizsäcker's Formula

Upon reflection one can see how the liquid-drop model is interpreted by the semi-empirical formula of Weizsäcker

²⁵For a review of the fission calculations see: R. D. Present, *Nucleonics* 3, 25 (1948); for simple but detailed treatment of small deformations see: M. S. Plesset, *Am. J. Phys.* 9, 1 (1941).

²⁶E. M. Beynam, "Small, Asymmetric Deformations of a Charged Drop" (Unpublished Master's thesis, Department of Physics, The University of Tennessee, Knoxville, June 1949).

and how it is an extension of this formula to states other than the ground states of nuclei. The mass and charge densities of the drop are assumed to be uniform and constant--the corresponding densities for the nucleus are, respectively, the number densities of nucleons and protons. Constant and uniform density of nucleons for any distortion or division demands conservation of the volume energy denoted by $\propto A$ --the main contribution to the binding energy, arising from the purely nuclear force. The added assumption of constant and uniform density of protons requires constant and uniform density of un-paired particles, which, in turn, demands conservation of the energy denoted by $\propto A \beta \left(\frac{N-Z}{A} \right)^2$ --the pairing, or symmetry energy. ~~The only energy changes possible occur for the surface and electrostatic energies--precisely those considered by the~~ liquid-drop model.

CHAPTER II

THE PROBLEM CONSIDERED IN THIS THESIS

This thesis will consider in some detail the calculation of some of the quantities considered by the papers reviewed in Chapter I. The objective of the thesis is the prediction of spontaneous-fission half-lives for nuclei with high atomic numbers.

Justification for the Thesis

The Rayleigh criterion for instability of a charged conducting drop (page 2) is also the criterion for instability of the drop assumed in the liquid-drop model (page 11). As the charge of the drop increases, it approaches instability. Nuclei, considered in the light of this criterion, are expected to become less stable as their charge increases. This condition is expressed by the formula for the activation energy as given by Bohr and Wheeler

$$E_f = E_s^0 f(x) = E_s^0 \left[\frac{98}{135} (1-x)^3 - \frac{11,368}{34,425} (1-x)^4 + \dots \right]$$

for large values of x (see page 13).

The high-energy accelerators built during the past decade have made possible the creation of a number of nuclei with high atomic numbers--the possible atomic numbers extend now through $Z = 102$. Many of these nuclei decay via spontaneous fission. It is these high- Z nuclei which should provide a good test of

the liquid-drop model; their values of X are nearer unity than the X -values of previously known nuclei, and the power series employed in liquid-drop calculations are expected to converge rapidly enough to give good results with a few terms from these series. The predictions made by the liquid-drop model for one of these high- Z nuclides will be considered in this thesis.

The effective mass, $\mathcal{M}$, used by Frankel and Metropolis in their calculations of the spontaneous-fission half-life of uranium is incorrect.¹ This effective mass will be calculated in this thesis, and the series for it will be extended beyond the linear term obtained by Frankel and Metropolis. The error in the predicted spontaneous-fission half-life introduced by this error in the effective mass will be evaluated for the nuclide considered in this thesis.

A problem which is interesting, both physically and mathematically, arises in the calculation of the change in electrostatic energy produced by the distortion of the charged liquid drop. This problem has not been mentioned in any of the published papers. A contribution to the electrostatic energy of distortion can be easily overlooked in the formulation of the problem. The evaluation of this particular contribution shows that it is actually null for a number of the lower powers in the multiple-power series expansion. In none

¹The author was told of the existence of this error by R. D. Present.

of the published papers which calculate the electrostatic energy of distortion would an error be introduced by the omission of this particular contribution. The assumption, made by the author, that this contribution has been overlooked is based on the fact that it has not been mentioned, and the calculation itself is not trivial.² The calculation of this quantity will be made in this thesis, and a number of terms in its power series expansion will be shown to be zero. (The entire series has not been shown to be zero, so the quantity is not known to be identically zero--or to be non-zero.)

Assumptions Characterizing the Model Used

A number of assumptions as to the nature of the entire phenomenon of spontaneous fission have been made in this thesis. For some aspects of the problem assumptions are absolutely necessary; for others, assumptions are made because they simplify the problem. The assumptions will not be divided into the two categories for consideration--some will obviously belong in the one or the other category; others are borderline cases.

²There is one exception to this statement. The contribution was overlooked by Present, Reines, and Knipp in one form of an unpublished calculation. The calculation was performed by them in another, independent way for which the contribution does not explicitly occur. Since the result of the second method agreed with that of the first, the omission of the particular contribution was not discovered. Only the result of the calculations was published; the result is correct. (The author learned of this contribution to the energy and of this instance of its omission in talks with R. D. Present.)

The Liquid Drop

The change in potential energy produced by a deformation of the drop is due to the changes of surface and electrostatic energies. The surface energy of the drop is proportional to the surface area for any condition of the drop--or, the surface tension, σ , is a constant. The densities of mass and charge are uniform (no spatial variation) and constant (no temporal variation). The values of σ and κ_0 are taken from the Weizsäcker formula. No explicit assumption about nuclear forces is involved.

WKB Approximation

The Schrödinger wave equation has to be used for one aspect only of this thesis: for the calculation of the probability-per-unit-time of a spontaneous fission. The first-order WKB approximation to the wave function is assumed to be a satisfactory solution to the wave equation for barrier-penetration problems.³ (An intrinsic part of this WKB approximation is the use of the classical action function.)

It is expected that this assumption is quite satisfactory for spontaneous fission; the assumption has been made in theoretical studies of α -decay and has led to good results.

³For the WKB method applied to a many-body system see W. Pauli, "Kapitel 2, Die allgemeinen Prinzipien der Wellenmechanik," Handbuch der Physik Vol. 24/1 (2nd. ed.; Berlin: Springer, 1933), p. 166 ff.

Shape of the Drop

The shape of the drop, on its way to fission, is assumed to be satisfactorily represented by the P_2 deformation⁴

$$r(\cos \vartheta) = R_0 [1 + a_0 + a_2 P_2(\cos \vartheta)].$$

This means that the potential energy of deformation is a function of a_2 only.

This assumption would seem to be qualitatively justified on the basis of the importance of the P_2 deformation: using a P_2 deformation only, one can approximate the critical shape fairly well for the larger values of χ (see Table I, page 19); instability of the drop occurs first for the P_2 deformation (see page 11).

The Kinetic Energy

The velocity field, $\vec{v}$, which describes the flow of the fluid inside the drop, is assumed to be irrotational:

$\vec{\nabla} \times \vec{v} = 0$. This means that a velocity potential, ϕ , exists such that ϕ is a function of position, of a_2 , and $\dot{a}_2$:

$$\phi = \phi(r, \mu; a_2, \dot{a}_2).$$

The irrotational $\vec{v}$ minimizes the classical kinetic energy and, therefore, the effective mass, m , which enters the penetration factor, and, consequently, decreases the half-life for spontaneous fission. The assumption that ϕ

⁴See page 11.

depends on a_2 and not on the other a_n agrees with the shape assumed above.

Adjustment of the Potential Energy Barrier

For the calculation of the half-life for spontaneous fission the potential energy of distortion is assumed to be fairly well represented for large χ by a function of the single distortion-amplitude, a_2 . If ΔE is the potential energy of distortion, then ΔE becomes a function of a_2 , denoted by $\Delta E(a_2)$.

As ΔE is the barrier to fission, the error introduced through use of the $\Delta E(a_2)$ must be evaluated. There are three reasons for making an adjustment of this $\Delta E(a_2)$ barrier.

- (1) The maximum of the $\Delta E(a_2)$ barrier is larger than the maximum of the saddle-path barrier for which ΔE is a function of all the distortion-amplitudes; i.e., $\Delta E(a_n)$. (See Table I.) The saddle-path barrier represented by $\Delta E(a_n)$ should be the correct barrier to fission; the discrepancy therefore suggests that one should lower the $\Delta E(a_2)$ barrier to make it agree with the $\Delta E(a_n)$ barrier.
- (2) The maximum of the $\Delta E(a_n)$ barrier should be equal to the activation energy E_f of the nucleus; the theoretical value is found to be lower than the empirical estimates

for $Z \sim 100.5$ The $\Delta E(a_n)$ barrier should therefore be raised. (This is a circumstance which should improve predictions based on the $\Delta E(a_z)$ barrier, as this barrier is higher than the $\Delta E(a_n)$ barrier.)

- (3) The height and width of both the $\Delta E(a_n)$ and $\Delta E(a_z)$ barriers vary with X , and the spontaneous-fission half-life is therefore very sensitive to the value of X . As X is somewhat uncertain for a given nuclide because of uncertainties in γ and κ_0 , one has some difficulty in making predictions of half-lives.

Two methods of adjusting the potential energy barrier to compensate for the three difficulties considered have been investigated: (1) The value of X is chosen to give agreement between the maximum of $\Delta E(a_z)$ and the experimental value⁶ of E_f . (2) The potential energy of distortion $\Delta E(a_z)$ is taken through the a_z^{κ} term to give a power series denoted by $\Delta E_{\kappa}(a_z)$; then an additive term is introduced to make

$$\Delta E(a_z) = \Delta E_{\kappa}(a_z) + \kappa a_z^{\kappa+1}.$$

The value of κ is fixed by making the maximum of this $\Delta E(a_z)$ equal to the experimental value of E_f .

⁵Estimates to be considered in Chapter VI place the activation energy of $^{101}\text{Mv}^{256}$ (the nucleus treated in that chapter) between 4.7 and 5.0 Mev. The prediction of Frankel and Metropolis (Phys. Rev. 72, 914 (1947).) places this energy between 1.5 and 2.5 Mev.

⁶For the nuclide considered in this thesis this value is actually predicted by an extrapolation of experimental values for neighboring nuclides.

The first of these two methods of adjustment should be satisfactory if the adjusted value of χ differs only slightly from the value one would otherwise predict for the nuclide. The evaluation of the error caused by using the second method of adjustment is difficult. One would expect a small error when κa_2^{k+1} is small compared with $\Delta E_k(a_2)$ throughout the range of a_2 for the spontaneous-fission process. The criterion for the validity of the first method of adjustment is met for the nuclide considered in Chapter VI; the criterion for the second is not. Thus the method of variation of χ was used for adjusting the potential energy barrier.

Symmetry of Fission

We assume that the fission process is symmetrical or nearly symmetrical in the initial stages, or that asymmetry sets in only after the barrier is penetrated. The assumption is made in order to permit considering only those distortions represented by combinations of even Legendre polynomials.⁷ The asymmetry appears to be unimportant for calculating E_f and the lifetimes.⁸

⁷The potential energy of distortion is treated as a function of a_2 only for the spontaneous-fission calculation; it is not so restricted throughout other parts of the thesis.

⁸R. D. Present, *Nucleonics* 3, 25 (1948).

Outline of the Remainder of the Thesis

The remainder of the thesis will consider the aspects of fission discussed in the first section of this chapter. Chapter III will consider the Gamow penetration factor for the many-body problem of the nucleus. The problem will be considered as a particle problem, starting with the Schrödinger equation for the system. The form of the Gamow penetration factor which applies to many particles will be changed into the form one must have for a continuum--for a liquid drop. Chapter IV will formulate the electrostatic energy calculation for the deformed drop and solve for one contribution to this energy--the contribution which was discussed in the first section of this chapter and which can easily be overlooked in formulating the problem. Chapter V will consider the hydrodynamics of the liquid drop: the kinetic energy will be calculated, and the effective mass, m , will be found. Chapter VI will apply the accumulated results to a particular nucleus-- $^{101}\text{Mo}^{256}$. The spontaneous-fission half-life for this nucleus will be evaluated. Chapter VII will evaluate the entire process as considered in the light of the results achieved in Chapter VI.

CHAPTER III

THE GAMOW PENETRATION FACTOR

The kinetic energy of the nucleus does not occur in the expression for the half-life for spontaneous-fission if the nucleus is treated as a collection of particles. If the nucleus is treated as a charged liquid drop, however, the kinetic energy does occur. It occurs in the Gamow penetration factor. In this chapter the Gamow penetration factor will be obtained. The kinetic energy will arise in a natural way; it will lead to the definition of the effective mass. Both of these quantities will be obtained in Chapter V.

The Simplest Example of the Gamow Factor

Consider the Gamow factor which arises for the simplest problem possible: a single particle of mass m moves in a space of a single dimension with a potential energy $V(x)$. Let the energy eigenvalue be E . Furthermore, let $V(x)$ be greater than E when $x_1 < x < x_2$ -- this is a classically forbidden region of "negative kinetic energy." Quantum-mechanically there is a non-zero probability that the particle will pass through the barrier. The probability that the particles passes through the barrier upon reaching it is called the transparency of the barrier and is given by the Gamow factor
$$e^{-\frac{2}{\hbar} \int_{x_1}^{x_2} dx \sqrt{2m(V(x) - E)}}.$$

This expression for the transparency can be obtained via the WKB approximation method. One can assume a wave function

$$\psi(x) = \varphi(x) e^{\pm \frac{i}{\hbar} A(x)} \quad \text{and use the Schrödinger wave equation to get approximate functions } \varphi(x) \text{ and } A(x).$$

When one leaves this simplest of problems and considers the many-body problem of the nucleus, the Gamow factor is hard to obtain.

The WKB Solution of the Many-Body Wave Equation

The time-independent wave equation for the nucleus is

$$\sum_{l=1}^A -\frac{\hbar^2}{2m_l} \nabla_l^2 \Psi + U \Psi = E \Psi$$

where U is the potential energy--a function of all the position coordinates of the A particles-- E is the total energy eigenvalue, and m_l is the mass of the l th particle. One can now assume a wave function analogous to the one assumed for the one-dimensional case above. It is, however, easier to use the more straightforward assumption of

$$\Psi = e^{\frac{i}{\hbar} S} \quad \text{with} \quad S = S_0 + \left(\frac{\hbar}{i}\right) S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \dots$$

Now $\vec{\nabla}_l \Psi = e^{\frac{i}{\hbar} S} \left(\frac{i}{\hbar} \vec{\nabla}_l S \right)$

$$\nabla_l^2 \Psi = e^{\frac{i}{\hbar} S} \left[\frac{i}{\hbar} \vec{\nabla}_l^2 S - \frac{i}{\hbar} \vec{\nabla}_l S \cdot \vec{\nabla}_l S + \frac{i}{\hbar} \nabla_l^2 S \right]$$

and hence

$$\sum_{l=1}^A -\frac{\hbar^2}{2m_l} \left[-\frac{1}{\hbar^2} (\vec{\nabla}_l S)^2 + \frac{i}{\hbar} \nabla_l^2 S \right] = E - U$$

is the resulting form of the wave equation. Using now the approximation

$$S = S_0 + \frac{\hbar}{i} S_1, \quad \text{one has}$$

$$(\vec{\nabla} S)^2 = (\vec{\nabla} S_0 + \frac{\hbar}{i} \vec{\nabla} S_1)^2 = (\vec{\nabla} S_0)^2 + 2 \frac{\hbar}{i} \vec{\nabla} S_0 \cdot \vec{\nabla} S_1 + \left(\frac{\hbar}{i}\right)^2 (\vec{\nabla} S_1)^2,$$

and putting this in the wave equation, he gets

$$\sum_{l=1}^A \frac{-\hbar^2}{2m_l} \left[-\frac{1}{\hbar^2} \left\{ (\vec{\nabla} S_0)^2 + 2 \frac{\hbar}{i} \vec{\nabla} S_0 \cdot \vec{\nabla} S_1 + \left(\frac{\hbar}{i}\right)^2 (\vec{\nabla} S_1)^2 \right\} \right. \\ \left. + \frac{i}{\hbar} \left\{ \nabla_l^2 S_0 + \frac{\hbar}{i} \nabla_l^2 S_1 \right\} \right] = E - U,$$

or

$$\sum_{l=1}^A \frac{1}{2m_l} \left[(\vec{\nabla} S_0)^2 + 2 \frac{\hbar}{i} \vec{\nabla} S_0 \cdot \vec{\nabla} S_1 - i \hbar \nabla_l^2 S_0 \right] = E - U$$

to linear terms in $\hbar$. This is the first WKB approximation. (By including no terms in $\hbar$, one gets the zero approximation.)

The WKB method assumes that $\hbar$ can be treated as an expansion parameter--as a small one, in fact, so that increasing powers of $\hbar$ give higher and higher orders of approximations. By equating "coefficients of $\hbar$ " on the left and right of the last equation, one obtains

$$\sum_{l=1}^A \frac{1}{2m_l} (\vec{\nabla} S_0)^2 = E - U \quad \text{for } (\hbar)^0$$

$$\text{and} \quad \sum_{l=1}^A \frac{1}{2m_l} [2 \vec{\nabla} S_0 \cdot \vec{\nabla} S_1 + \nabla_l^2 S_0] = 0 \quad \text{for } (\hbar)^1.$$

The first of these two equations is the Hamilton-Jacobi differential equation for the characteristic action function

of classical mechanics. The second of the two can be rewritten as

$$\sum_{l=1}^A \frac{1}{2m_l} [2 \vec{\nabla}_l S_0 \cdot \vec{\nabla}_l S_1 + \nabla_l^2 S_0] e^{2S_1} = 0$$

by multiplying by the factor e^{2S_1} .

When all the m_l are equal, this last equation can be written for this hyperspace as¹

$$\vec{\nabla}_h \cdot (e^{2S_1} \vec{\nabla}_h S_0) = 0$$

where $\vec{\nabla}_h$ is a hypergradient defined by

$$\vec{\nabla}_h \equiv \hat{e}_{11} \frac{\partial}{\partial x_1} + \hat{e}_{12} \frac{\partial}{\partial y_1} + \hat{e}_{13} \frac{\partial}{\partial z_1} + \hat{e}_{21} \frac{\partial}{\partial x_2} + \dots + \hat{e}_{A3} \frac{\partial}{\partial z_A}$$

and $\hat{e}_{ij} \cdot \hat{e}_{kl} = \delta_{ik} \cdot \delta_{jl}$. The $\hat{e}_{ij}$ are unit vectors, and the δ_{ij} are the Kronecker deltas. The subscripts on the x , y , and z refer to the particular particles.

A solution to the last differential equation above is

$$e^{2S_1} \vec{\nabla}_h S_0 = \vec{K}_h \quad \text{where } \vec{K}_h \text{ is a constant}$$

vector--in the hyperspace. The $\vec{\nabla}_h S_0$ and $\vec{K}_h$ are parallel,

so

$$e^{S_1} = \left(\frac{|\vec{K}_h|}{|\vec{\nabla}_h S_0|} \right)^{\frac{1}{2}}$$

A WKB-type solution to the differential equation² is thus

¹The assumption of equality of mass for all nucleons is generally made for the liquid-drop model.

²The boundary conditions of wave mechanics are continuity of ψ and of all of its first partial derivatives. These conditions are never met by WKB wave functions at the classical turning points--where e^{S_1} is infinite.

$$e^{S_1} \propto \frac{1}{\sqrt{|\vec{\nabla}_h S_0|}} \quad .^3$$

The Hamilton-Jacobi differential equation for the characteristic function can be written

$$(\vec{\nabla}_h S_0)^2 = 2m(E - U) \quad (\text{All } m_i = m.)$$

So $e^{S_1} \propto \frac{1}{\sqrt{E - U}}$, and S_0 , alone remains, to be found.

The solution for S_0 is conveniently written in a special form. This form assumes a parameterization of all the Cartesian coordinates for the particles: $X_i = X_i(\alpha)$; where α is a common physical parameter, defining in some way the path of the system in the hyperspace, and the X_i can be any x , y , or z for any of the particles by fixing the subscript (x_1 , x_2 , and x_3 refer to x , y , and z for particle 1, etc.). With this parameterization S_0 becomes⁴

$$S_0 \propto \int \sqrt{2(E - U)} \sqrt{\sum_{i=1}^{3A} m_i \left(\frac{dx_i}{d\alpha} \right)^2} d\alpha$$

³This solution for S_1 is not a general solution of the differential equation for S_1 . This expression for S_1 corresponds to the usual, one-dimensional, WKB-approximation function. The explicit expression for S_1 is not used in any way in either the penetration factor or the lifetime calculations, both of which depend only on the formula for S_0 . If a more general expression for S_1 were used throughout the remainder of this heuristic derivation, the results for the penetration factor would still be in no way modified.

⁴The m_i are not necessarily equal for the validity of this function.

which is called Jacobi's function.⁵

The WKB first approximation to the wave function,

$$\Psi = e^{\frac{i}{\hbar} S} = e^{\frac{i}{\hbar} [S_0 + \frac{\hbar}{i} S_1,]}$$

then becomes

$$\Psi \propto \frac{1}{\sqrt{E-U}} e^{\pm \frac{i}{\hbar} \int \sqrt{2(E-U)} \sqrt{\sum_{i=1}^{3A} m_i \left(\frac{dx_i}{d\alpha}\right)^2} d\alpha}$$

The $\pm$ signs in the exponent allow for either sign for the solution for meeting the demands of a particular problem.

The function $E-U$ is positive in classically allowed regions and negative in forbidden regions. The exponentials in the wave function are therefore imaginary in allowed regions and real in forbidden regions. This means that the wave function is oscillatory in allowed regions only; in forbidden regions it is a function of exponential growth or decay.

For the case of the barrier-penetration problem the indefinite integral in the exponent of the wave function is replaced by a definite integral: either the upper or lower limit is fixed as one turning point of the barrier; the other limit is made the variable limit α .

Penetration of the Barrier

The barrier which is to be penetrated in the spontaneous fission process can be expressed as a function of α : a particular value of this parameter determines a particular point

⁵Herbert Goldstein, Classical Mechanics (Cambridge: Addison-Wesley Press, Inc., 1951), pp.228-35.

in the hyperspace; U is assumed to be a point function and $U - E$ is thus a function of α .

In the neighborhood of the range of α for which $U - E$ is a barrier, there are three regions to consider. These regions are indicated in Figure 3. Region I is determined by $\alpha < \alpha_1$, region II by $\alpha_1 \leq \alpha \leq \alpha_2$, and region III by $\alpha_2 < \alpha$. Region II is the classically forbidden region. The other two are classically allowed regions.

For the three different regions the wave functions take different forms. The form for any one region is a linear com-

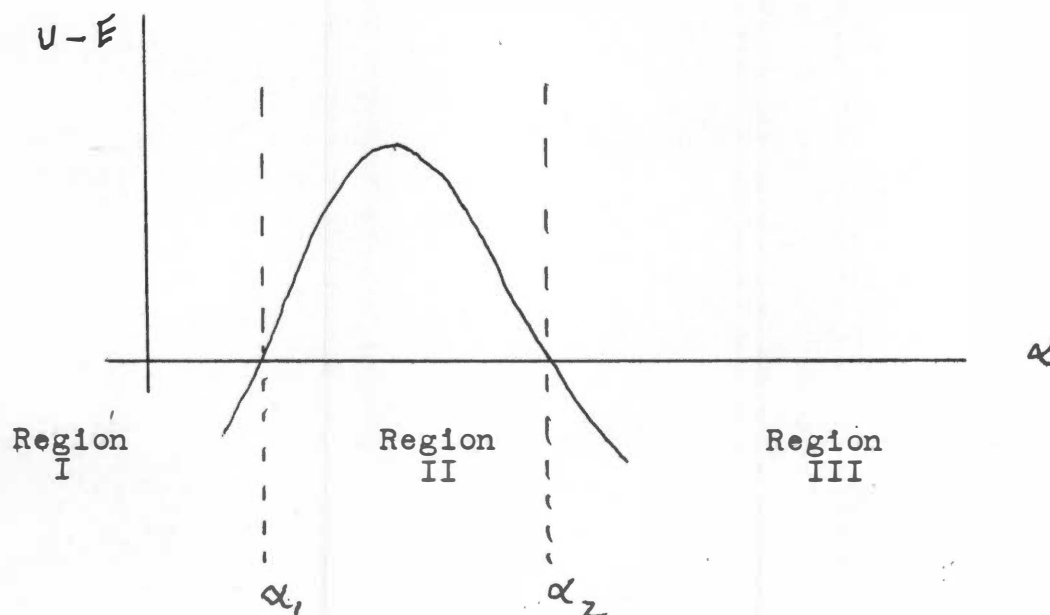


Figure 3. The barrier to be penetrated.

ination of two terms which differ only in the sign of the integral in the exponent. (See the general form above.)

With $f(\alpha)$ defined as

$$f(\alpha) \equiv \sqrt{2|E-U|} \sqrt{\sum_{i=1}^{3A} m_i \left(\frac{dX_i}{d\alpha}\right)^2}$$

the wave functions for the three regions can be written as

$$\begin{aligned}\Psi_I &= \frac{1}{\sqrt{E-U}} \left[A_1 e^{+\frac{i}{\hbar} \int_{\alpha}^{\alpha_1} f(\alpha) d\alpha} + A_2 e^{-\frac{i}{\hbar} \int_{\alpha}^{\alpha_1} f(\alpha) d\alpha} \right] \\ \Psi_{II} &= \frac{1}{\sqrt{U-E}} \left[B_1 e^{+\frac{1}{\hbar} \int_{\alpha_1}^{\alpha} f(\alpha) d\alpha} + B_2 e^{-\frac{1}{\hbar} \int_{\alpha_1}^{\alpha} f(\alpha) d\alpha} \right] \\ \Psi_{III} &= \frac{1}{\sqrt{U-E}} \left[C_1 e^{+\frac{i}{\hbar} \int_{\alpha_2}^{\alpha} f(\alpha) d\alpha} + C_2 e^{-\frac{i}{\hbar} \int_{\alpha_2}^{\alpha} f(\alpha) d\alpha} \right]\end{aligned}$$

where the subscript on a given Ψ indicates the region in which it is valid.

The wave functions are to be applied to the barrier-penetration problem. For this purpose it is essential that one have the interpretation of the wave functions. The interpretations of the functions for regions I and III will be given first. These are easily seen and are unambiguous. The interpretation of the solution in region II will be achieved indirectly in considering the physics of the penetration of the barrier.

In region I A_1 is the amplitude of a wave traveling in the direction of decreasing α , and A_2 is the amplitude of a wave traveling in the direction of increasing α . In region III C_1 is the amplitude of a wave traveling in the direction of increasing α , and C_2 is the amplitude of a

wave traveling in the direction of decreasing α . The meaning of the wave function in region II is most easily seen by considering an actual penetration of the barrier.

Figure 4a shows the barrier of Figure 3 with its three regions and the turning points α_1 , and α_2 . In Figure 4b the wave functions are drawn for the three regions for the case of penetration of the barrier from left to right (increasing α). These functions are qualitative only and are based on the physical nature of penetration.

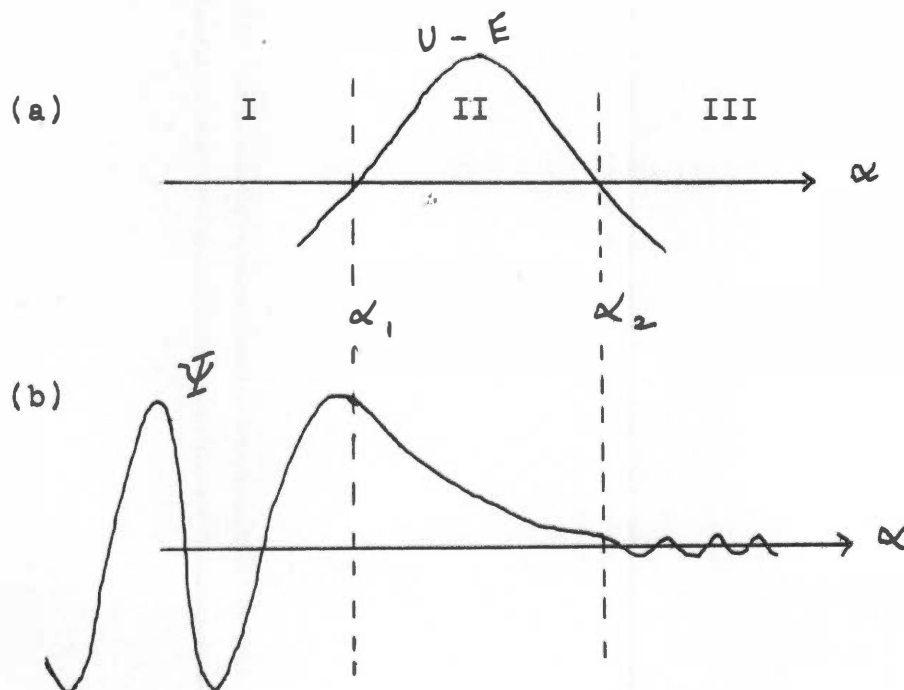


Figure 4. Barrier penetration from left to right.

The large-amplitude wave in region I means that the wave is incident from the left; the small-amplitude wave in region III means that there is a small but non-zero probability that the wave penetrates the barrier.

For regions I and III the interpretations of the approximate wave-functions show that these wave functions can be chosen so that they correspond to this penetration phenomenon: in region III there is a transmitted wave only; thus $C_2 = 0$, and $C_1 \neq 0$. In region I there is an incident wave--so $A_2 \neq 0$ --and a reflected wave-- $A_1 \neq 0$. The proper choice, then, of A_1 , A_2 , and C_1 , will make the approximate wave functions correspond to this phenomenon--except, of course, for values of α near the turning points.

Figure 4b shows that Ψ_{II} can be the wave function inside region II only if $|B_1| \ll |B_2|$: the exponential decrease of Ψ_{II} with increasing α occurs only for

$$B_2 e^{-\frac{i}{\hbar} \int_{\alpha}^{\alpha_2} f(\alpha) d\alpha}$$

If one changes the limit α_1 on the integrals in Ψ_{II} , replacing it with α_2 , he can write

$$\Psi_{II} = \frac{1}{\sqrt{V-E}} \left[B'_1 e^{-\frac{i}{\hbar} \int_{\alpha_2}^{\alpha} f(\alpha) d\alpha} + B'_2 e^{\frac{i}{\hbar} \int_{\alpha_2}^{\alpha} f(\alpha) d\alpha} \right]$$

In the case of the single-particle, one-dimensional, barrier-penetration problem, Kramers' connection formula can be used to connect the proper solution Ψ_{II} to the Ψ_{III} , transmitted wave. For this case the magnitude of B'_1 is twice that of B'_2 -- Ψ_{II} being written in the form indicated above,⁶

Assuming that the magnitudes of B'_1 and B'_2 are roughly the

⁶W. Pauli, "Kapitel 2, Die allgemeinen Prinzipien der Wellenmechanik," Handbuch der Physik Vol. 24/1 (2nd. ed.; Berlin: Springer, 1933), p. 171.

same for the case being considered here, one can use B_1' and B_2' to solve this penetration problem. Rewriting the last form for Ψ_{II} by changing limits, one obtains

$$\Psi_{II} = \frac{1}{\sqrt{U-E}} \left[\left(B_1' e^{\frac{1}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha} \right) e^{-\frac{1}{\hbar} \int_{\alpha_1}^{\alpha} f(\alpha) d\alpha} + \left(B_2' e^{-\frac{1}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha} \right) e^{\frac{1}{\hbar} \int_{\alpha_1}^{\alpha} f(\alpha) d\alpha} \right].$$

This identifies B_1 and B_2 as

$$B_1 = B_2' e^{-\frac{1}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha}$$

$$B_2 = B_1' e^{\frac{1}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha}$$

which shows that $|B_2| \gg |B_1|$.

Since $|B_2| \gg |B_1|$,

$$\frac{[\sqrt{U-E} \Psi_{II}]_{\alpha=\alpha_2}}{[\sqrt{U-E} \Psi_{II}]_{\alpha=\alpha_1}} = \frac{B_1' + B_2'}{B_2}$$

where both forms of Ψ_{II} have been used, each use being obvious, and where B_1 has been neglected. The relationships between B_1' and B_2' and B_1 and B_2 yield

$$\frac{[\sqrt{U-E} \Psi_{II}]_{\alpha=\alpha_2}}{[\sqrt{U-E} \Psi_{II}]_{\alpha=\alpha_1}} \cong e^{-\frac{1}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha}.$$

If one assumes that

$$\frac{\lim_{\alpha \rightarrow \alpha_2} \sqrt[4]{U-E}}{\lim_{\alpha \rightarrow \alpha_1} \sqrt[4]{U-E}} \sim 1,$$

then $\left| \frac{\Psi_{II}(\alpha \rightarrow \alpha_2)}{\Psi_{II}(\alpha \rightarrow \alpha_1)} \right|^2 \sim e^{-\frac{2}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha}$

is the probability of penetration of the barrier. For this event, then, the probability of penetration of the barrier is

$$e^{-\frac{2}{\hbar} \int_{\alpha_1}^{\alpha_2} f(\alpha) d\alpha}$$

or written explicitly

$$e^{-\frac{2}{\hbar} \int_{\alpha_1}^{\alpha_2} \sqrt{2(U-E)} \sqrt{\sum_{i=1}^{3A} m_i \left(\frac{dx_i}{d\alpha}\right)^2} d\alpha}$$

This is the desired Gamow penetration factor for the spontaneous fission process.

The Gamow Factor for a Continuum

To change the Gamow factor obtained for the case of particles to the one needed for the case of a continuous medium, one must change the factor $\sum_{i=1}^{3A} m_i \left(\frac{dx_i}{d\alpha}\right)^2$ to the equivalent continuum expression. Before changing to the

continuum expression, change the $\frac{dx_i}{d\alpha}$ to $\frac{dx_i}{dt} \cdot \frac{1}{\dot{\alpha}} \cdot \left(\dot{\alpha} \equiv \frac{d\alpha}{dt}\right)$

Define now a function

$$\rho(\vec{n}) \equiv \sum_{i=1}^A m_i \delta_i(\vec{n} - \vec{n}_i)$$

where $\delta(\vec{n} - \vec{n}_i)$ is the Dirac delta function. Then

$$\iiint \frac{\rho(\vec{n}) \cdot v^2(\vec{n})}{\dot{\alpha}^2} d\tau = \sum_{i=1}^A \frac{m_i v_i^2}{\dot{\alpha}^2}$$

which is precisely the expression to be changed. Now the change to the continuum is immediate: exchange $\rho(\vec{r})$ for the mass density of the continuum $\rho_m(\vec{r})$. The summation for particles is thus replaced by an integral for the continuous medium:

$$\sum_{i=1}^{3A} m_i \left(\frac{dx_i}{d\alpha} \right)^2 \text{ becomes } \frac{2}{\dot{\alpha}^2} \left[\frac{1}{2} \iiint \rho_m(\vec{r}) v^2(\vec{r}) d\tau \right].$$

The bracket is recognizable as T -- the kinetic energy of the continuous medium.

The importance of the kinetic energy is now made clear. Full analogy with the single-particle Gamow factor is obtained by defining $\mathcal{M} = \frac{2T}{\dot{\alpha}^2}$ and noting that $T = \frac{1}{2} \mathcal{M} \dot{\alpha}^2$. The quantity $\mathcal{M}$ is a mass; it is the effective mass defined by Frankel and Metropolis. The quantity $\dot{\alpha}$ is a speed. The resulting form of the Gamow factor

$$e^{-\frac{2}{\hbar} \int_{\alpha_1}^{\alpha_2} d\alpha \sqrt{2\mathcal{M}(U-E)}}$$

for the continuous medium is completely analogous to the single-particle, one-dimensional case considered at the beginning of this chapter. The sequence of values through which α ranges defines the path (in the hyperspace) taken by the point representing the system as the system undergoes spontaneous fission. The sequence of shapes leading to spontaneous fission is described by the values of the parameters along the saddle-point path over the energy surface. (see Figure 5). The penetration factor is greatest for this saddle-point path, since $U-E$ has its least value.

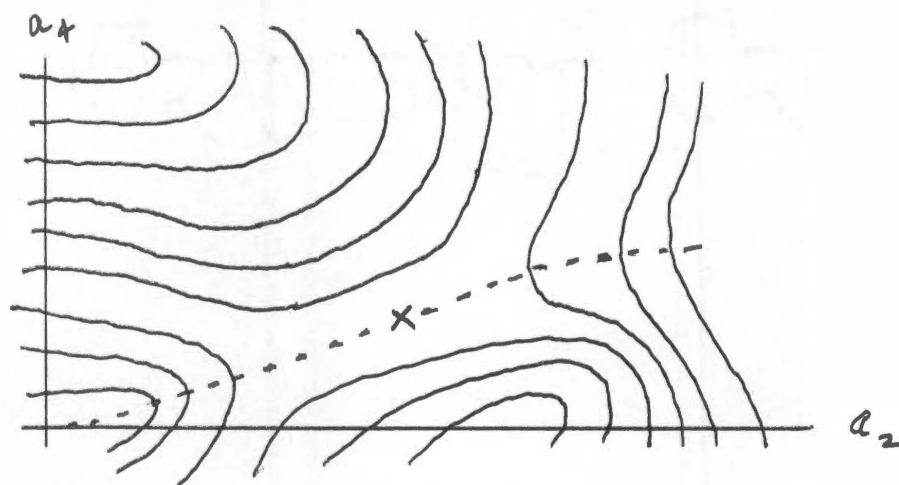


Figure 5. A contour map showing a simple saddle-point path (a schematic figure).

The penetration factor is also greatest when the effective mass m has its least value, i.e., for an irrotational motion of the fluid (see page 80).

CHAPTER IV

ELECTROSTATIC ENERGY OF THE DEFORMED DROP

The calculation of the electrostatic energy of the deformed drop is a laborious problem when there are large deformations. In calculating this contribution to the potential energy of deformation, one finds it convenient to express it as the sum of two contributions.

Formulation of the Two-Part Problem

The electrostatic energy of the deformed drop can be written as

$$E_c = E_c^0 + E_c'' + \delta E_c$$

where E_c^0 is the electrostatic energy of the spherical drop and E_c'' and δE_c are changes in the electrostatic energy produced by the distortion. It was stated on page 25 that a particular contribution to the electrostatic energy of distortion can easily be overlooked; this contribution is δE_c . Since only the δE_c term will be of particular interest in this thesis, write $E_c = E_c' + \delta E_c$ incorporating both E_c^0 and E_c'' in E_c' .

The shape of the drop is determined by the equation of its surface. For this calculation, the equation of the surface can be fairly general. Spherical coordinates are used; the origin of coordinates is fixed at the center of the undistorted drop. It is assumed that any ray radiating from the

center of the drop cuts the surface only once. The equation for the surface is

$$R(\mu) - R_0 \left[1 + \sum_{n=0}^{\infty} a_n(t) P_n(\mu) \right] = 0$$

where

- (1) $R(\mu)$ is the distance from the center of the drop to a point on the surface--the point is fixed by fixing the values of φ and μ . This distance is thus independent of φ . ($\mu = \cos \vartheta$)
- (2) R_0 is the radius of the drop when it is spherical in shape.
- (3) The $a_n(t)$ are distortion parameters; the size of any given one may vary with time, t , only.
- (4) $P_n(\mu)$ is the Legendre polynomial of degree n :

$$P_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n.$$
- (5) The upper limit on the index n is finite.

The absence of dependence on φ means that the drop is a figure of revolution. The absence of odd n means that the drop is symmetric with respect to the equatorial plane:

$$R(-\mu) = R(\mu).$$

The charge density of the drop is assumed to be uniform. (See Chapter II.) Denote charge density by $\rho_z(\vec{r})$.

The total electrostatic self-energy of the volume charge is

$$E_c = \frac{1}{2} \int_{\Omega} \rho_z(\vec{r}) d\tau \int_{\Omega'} \frac{\rho_z(\vec{r}') d\tau'}{r_{12}}$$

where r_{12} is the separation of the points $\vec{r}$ and $\vec{r}'$ and the volumes Ω and Ω' coincide. Denoting the electrostatic potential by V , one has

$$V(\vec{r}) = \int_{\Omega'} \frac{\rho_z(\vec{r}') d\tau'}{r_{12}}, \text{ and he can write}$$

$E_c = \frac{1}{2} \int_{\Omega} \rho_z(\vec{r}) V(\vec{r}) d\tau$. The evaluation of $V(\vec{r})$ will now be considered.

The restrictions on the surface of the drop allow it to have the profile exhibited in Figure 6. (The restrictions demand that it be a figure of revolution.) There are two regions denoted in the figure: the shaded area is region I; it represents the largest sphere one could locate with its center at the origin and which lies wholly within the drop.

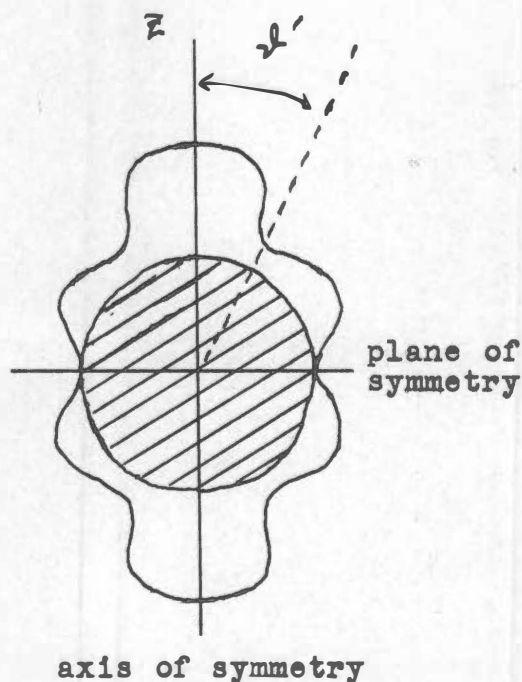


Figure 6. A profile of the drop (a figure of revolution).

Let the radius of this sphere be b . The unshaded area represents region II; it is the remainder of the drop.

The two regions of the drop suggest two regions for considering $V(\vec{r})$. In region I one has a valid expression for $V(\vec{r})$ in the expression

$$V_I(\vec{r}) = \int_0^{\pi} d\varphi' \int_0^{\pi} \sin \vartheta' d\vartheta' \int_0^{\kappa} \frac{r'^2 dr'}{\kappa_{12}} \\ + \int_0^{\pi} d\varphi' \int_0^{\pi} \sin \vartheta' d\vartheta' \int_{\kappa}^{R(\cos \vartheta')} \frac{r'^2 dr'}{\kappa_{12}^*}$$

where κ_{12} demands evaluation with $\kappa > \kappa'$ and κ_{12}^* demands evaluation with $\kappa < \kappa'$. In region II the limits of integration become a problem. When $\kappa > b$, there are some values of κ' for which the elemental volume ring $2\pi \kappa' \sin \vartheta' d\vartheta' \cdot \kappa' d\kappa'$ may be either inside of the drop or outside of it, depending on the value of ϑ' . This means that one must form a number of integrals over the ϑ' range so that the volume integration covers only the volume of the drop. (It is not profitable to change the order of integration: the limits on the two κ' integrations must remain as they are in $V_I(\vec{r})$ above in order that the $|\vec{r} - \vec{r}'|^{-1}$ expansions can be used.) When the distortions are small--i.e., when there are small ripples on a spherical surface--the error caused by ignoring these limit problems should be small. So for region II one could use for $V(\vec{r})$ the expression which is valid in region I plus a correction

$$V_{II}(\vec{r}) = V_I(\vec{r}) + \Delta V(\vec{r})$$

where $\Delta V(\vec{r})$ is expected to be small. The $\Delta V(\vec{r})$ must be determined.

There is in $\Delta V(\vec{r})$ a correction for each of the two integrals appearing in V_I . Figure 7 will help to show how each of these arises and how the corresponding correction is to be formulated. The area bounded by the curve representing

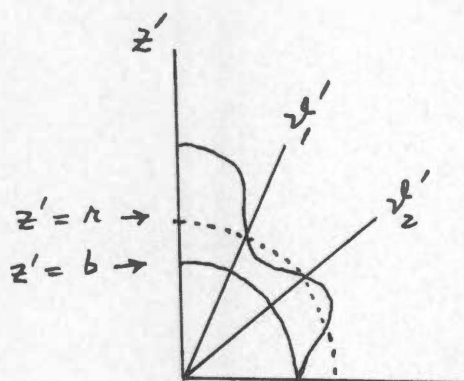


Figure 7. Quadrant I of Figure 6 (modified).

the profile and the arc of radius a and lying between ϑ'_1 and ϑ'_2 becomes on revolution an annular region lying outside the drop. When V_I is used in region II, each of the two integrals composing V_I produces an error for this volume. For the integral in which a_{12} occurs the a' integration is from 0 to a . This covers the volume being discussed; it is not a part of the drop. To correct for this one must subtract the quantity

$$p_z \int_0^{2\pi} d\varphi' \int_{\vartheta'_1(a)}^{\vartheta'_2(a)} \sin \vartheta' d\vartheta' \int_0^a \frac{a'^2 da'}{R(\cos \vartheta')} .$$

For the second integral, the one in which a_{12}^* occurs, the a' integration is from a to $R(\cos \vartheta')$. This also covers

(with a negative $d\kappa'$) the volume being discussed. The error this integral introduces must be subtracted. It is

$$\rho_z \int_0^{2\pi} d\varphi' \int_{\vartheta_1'(n)}^{\vartheta_2'(n)} \sin \vartheta' d\vartheta' \int_n^{\kappa(\cos \vartheta')} \frac{\kappa'^2 d\kappa'}{\kappa_{12}^*} .$$

The contribution to $\Delta V(\vec{\kappa})$ due to the volume being considered is

$$\begin{aligned} & - \rho_z \int_0^{2\pi} d\varphi' \int_{\vartheta_1'(n)}^{\vartheta_2'(n)} \sin \vartheta' d\vartheta' \int_{\kappa(\cos \vartheta')}^n \frac{\kappa'^2 d\kappa'}{\kappa_{12}^*} \\ & - \rho_z \int_0^{2\pi} d\varphi' \int_{\vartheta_1'(n)}^{\vartheta_2'(n)} \sin \vartheta' d\vartheta' \int_n^{\kappa(\cos \vartheta')} \frac{\kappa'^2 d\kappa'}{\kappa_{12}^*} . \end{aligned}$$

This will be denoted by $[\Delta V(\vec{\kappa})]_1$. The quantity may differ from zero as $\kappa_{12}^* \neq \kappa_{12}$. Both ϑ_1' and ϑ_2' depend on κ ; this dependence is denoted by writing $\vartheta_1'(\kappa)$ and $\vartheta_2'(\kappa)$. The $\kappa(\cos \vartheta')$ is the same function as the $\kappa(\mu)$ which defines the surface (page).

There may be a number of these regions requiring correction. Thus

$$\Delta V(\vec{\kappa}) = \sum_{\text{regions}} [\Delta V(\vec{\kappa})]_j .$$

Formulating $\Delta V(\vec{\kappa})$ for one particular value of κ is not, in general, sufficient. The angles ϑ' for which these regions occur are angles ϑ_j' such that $\kappa(\cos \vartheta_j') = \kappa$. As κ takes on different values, the number of solutions ϑ_j' may differ; the number of terms in the summation $\Delta V(\vec{\kappa})$ may thus depend on κ . (If κ in Figure 5 were slightly larger, only one ϑ_j' would exist for the range $0 \leq \vartheta' \leq \pi/2$; three

exist for the value of κ chosen.) The problem of obtaining $\Delta V(\vec{\kappa})$ is thus quite difficult for the general surface thus far accepted. For actual treatment of $\Delta V(\vec{\kappa})$ two further restrictions on the surface are made: (1) Only those surfaces are permitted for which $R(\cos \vartheta') = \kappa \leq \kappa_{\max}$ has one solution in region II when $0 \leq \vartheta' \leq \pi/2$. (2) $R(\cos 0) > R(\cos \frac{\pi}{2})$.¹ For the surface thus restricted only one term occurs for the sum $\Delta V(\vec{\kappa})$ above, and $\Delta V(\vec{\kappa}) = [\Delta v(\vec{\kappa})]_{\text{only one}}$. This problem can be solved as the greatest difficulties have been removed.

Using the expressions obtained for the potential, we can now write the electrostatic energy in a new form. Let Ω_1 be the volume of region I and Ω_2 that of region II.

$$E_c = \frac{1}{2} \rho_z \left[\int_{\Omega_1} v_I(\vec{\kappa}) d\tau + \int_{\Omega_2} \{v_I(\vec{\kappa}) + \Delta V(\vec{\kappa})\} d\tau \right]$$

$$= \frac{1}{2} \rho_z \left[\int_{\Omega_1 + \Omega_2} v_I(\vec{\kappa}) d\tau + \int_{\Omega_2} \Delta V(\vec{\kappa}) d\tau \right]$$

The definitions of E'_c and δE_c have now evolved:

$$E'_c \equiv \frac{1}{2} \rho_z \int_{\Omega} v_I(\vec{\kappa}) d\tau \quad (\Omega = \Omega_1 + \Omega_2)$$

$$\delta E_c \equiv \frac{1}{2} \rho_z \int_{\Omega_2} \Delta V(\vec{\kappa}) d\tau.$$

(These definitions are general and will give the correct E_c

¹These restrictions are consistent with the case where only the a_2 distortion occurs provided $a_2 > 0$. The shape of the drop is, however, not restricted to the shapes possible with a_2 alone; the other a_n can be non-zero.

when $\Delta V(\vec{r})$ is treated for the more general surface. The summation will not give, however, a single expression for $\Delta V(\vec{r})$; and one will have to subdivide further the drop into regions of validity of the different $\Delta V(\vec{r})$ arising.)

The quantity δE_C is now to be calculated. Some changes in $\Delta V(\vec{r})$ simplify this calculation.

$$\Delta V(\vec{r}) = \rho \int_{\vec{r}} \int_0^{2\pi} d\phi' \int_{\vartheta_1'}^{\vartheta_2'} \sin \vartheta' d\vartheta' \int_{R(\cos \vartheta')}^{\kappa} \lambda'^2 d\lambda' \left(\frac{1}{\lambda_{12}^*} - \frac{1}{\lambda_{12}} \right).$$

The demands on the surface allow only one such integral for $\Delta V(\vec{r})$. The symmetry condition gives ϑ_2' as $\pi - \vartheta_1'$. Figure 8 illustrates this relationship between ϑ_1' and ϑ_2' for the simplest of shapes of the drop. The relationship

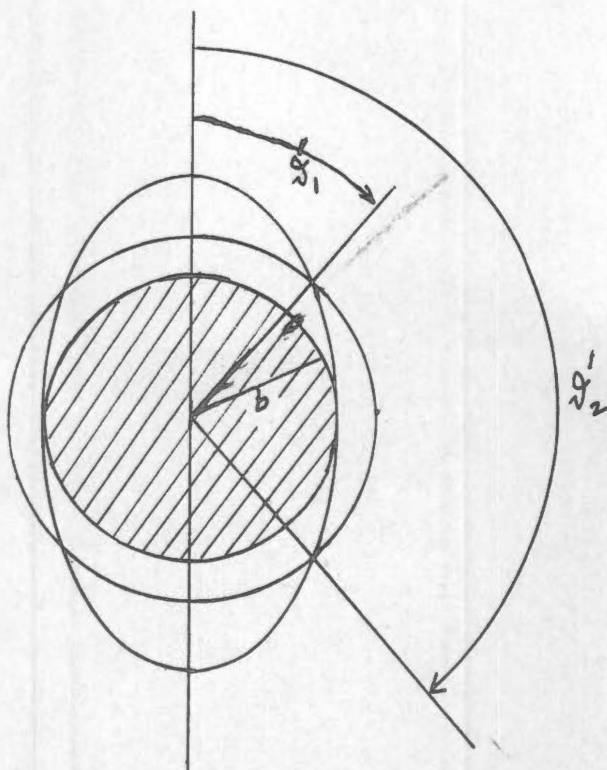


Figure 8. Intersection of a sphere of radius κ ($\kappa > b$) with the drop at $\vartheta_1' = \vartheta_1'$ and $\vartheta_2' = \vartheta_2'$.

between these two angles simplifies the integration. The expansions for $(\kappa_{12})^{-1}$ and $(\kappa_{12}^*)^{-1}$ are

$$(\kappa_{12})^{-1} = \sum_{l=0}^{\infty} \frac{\kappa'^l}{\kappa^{l+1}} P_l(\cos \gamma)$$

$$(\kappa_{12}^*)^{-1} = \sum_{l=0}^{\infty} \frac{\kappa^l}{\kappa'^{l+1}} P_l(\cos \gamma)$$

where γ is the angle between vectors $\vec{\kappa}$ and $\vec{\kappa}'$. Explicitly, then,

$$\begin{aligned} \Delta V(\vec{\kappa}) &= \rho_z \int_0^{2\pi} d\varphi' \int_{\vartheta'}^{\pi-\vartheta'} \sin \vartheta' d\vartheta' \int_{R(\cos \vartheta')}^{\kappa} \kappa'^2 d\kappa' \sum_{l=0}^{\infty} \left[\frac{\kappa^l}{\kappa'^{l+1}} - \frac{\kappa'^l}{\kappa^{l+1}} \right] P_l(\cos \gamma) \\ &= \rho_z \sum_{l=0}^{\infty} \int_{\vartheta'}^{\pi-\vartheta'} \sin \vartheta' d\vartheta' \int_{R(\cos \vartheta')}^{\kappa} \kappa'^2 d\kappa' \left[\frac{\kappa^l}{\kappa'^{l+1}} - \frac{\kappa'^l}{\kappa^{l+1}} \right] \int_0^{2\pi} d\varphi' P_l(\cos \gamma) \end{aligned}$$

and as $\int_0^{2\pi} d\varphi' P_l(\cos \gamma) = 2\pi P_l(\cos \vartheta) P_l(\cos \vartheta')$, the φ' integration is immediate. The appearance of the cosines makes it convenient to use $\mu = \cos \vartheta$ and $\mu' = \cos \vartheta'$. With these changes

$$\Delta V(\vec{\kappa}) = 2\pi \rho_z \sum_{l=0}^{\infty} \int_{-\mu'_0}^{\mu'_0} d\mu' \int_{R(\mu')}^{\kappa} \kappa'^2 d\kappa' \left[\frac{\kappa^l}{\kappa'^{l+1}} - \frac{\kappa'^l}{\kappa^{l+1}} \right] P_l(\mu) P_l(\mu')$$

where $\mu'_0 = \cos \vartheta'$. After minor rearrangement the equation becomes

$$\Delta V(\vec{\kappa}) = 2\pi \rho_z \sum_{l=0}^{\infty} P_l(\mu) \int_{-\mu'_0}^{\mu'_0} d\mu' P_l(\mu') \int_{R(\mu')}^{\kappa} \kappa'^2 d\kappa' \left[\frac{\kappa^l}{\kappa'^{l+1}} - \frac{\kappa'^l}{\kappa^{l+1}} \right].$$

Note now two things about the expression for $\Delta V(\vec{\kappa})$:

(1) $\mu'_0 (= \mu'_0(r)) \geq 0$. (μ'_0 is the cosine of an angle lying between 0 and $\pi/2$.) (2) The integral

$$\int_0^R r'^2 dr' \left[\frac{r'^l}{r'^{l+1}} - \frac{r'^l}{r^{l+1}} \right] \text{ is a function of } r \text{ and } \mu' \text{ and is } R(\mu')$$

even with respect to exchanging μ' for $-\mu'$. (See the function $R(\mu)$, page 48.) These two facts allow the elimination of all odd values of l above:

$$\int_{-\mu'_0}^{\mu'_0} d\mu' P_{\text{odd}}(\mu') Q_{\text{even}}(r, \mu') = 0.$$

One can thus write

$$\Delta V(r) = 4\pi \rho_z \sum_{\substack{l=0 \\ \text{even}}}^{\infty} P_l(\mu) \int_0^{\mu'_0} d\mu' P_l(\mu') \int_0^R r'^2 dr' \left[\frac{r'^l}{r'^{l+1}} - \frac{r'^l}{r^{l+1}} \right].$$

For δE_c one has

$$\delta E_c = \frac{1}{2} \rho_z \int_{\Omega_z} \Delta V d\tau = \frac{1}{2} \rho_z \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \int_b^{R(\mu)} r^2 dr \Delta V(r, \mu)$$

$$\text{or } \delta E_c = \pi \rho_z \int_{-1}^1 d\mu \int_b^{R(\mu)} r^2 dr \Delta V(r, \mu).$$

The $\Delta V(r, \mu)$ is an even function of μ . The explicit dependence is obviously even. The implicit dependence occurs in $R(\mu)$ which is an even function of μ . One can, therefore write

$$\begin{aligned} \delta E_c &= 2\pi \rho_z \int_0^1 d\mu \int_b^{R(\mu)} r^2 dr \Delta V(r, \mu) \\ &= 8\pi^2 \rho_z^2 \sum_{\substack{l=0 \\ \text{even}}}^{\infty} \int_0^1 d\mu \int_b^{R(\mu)} r^2 dr \int_0^{\mu'_0} d\mu' \int_0^R r'^2 dr' P_l(\mu) P_l(\mu') \left[\frac{r'^l}{r'^{l+1}} - \frac{r'^l}{r^{l+1}} \right]. \end{aligned}$$

Now exchange the order of integration for κ and μ' . Note:
 (1) μ'_0 is the value of μ' for which $R(\mu') = \kappa$. (This was previously expressed in $R(\mu'_j) = \kappa$.) (2) $R(\mu')$ is a monotonically non-decreasing function of μ' for $0 \leq \mu' \leq 1$. (Only one value of μ'_0 was allowed to occur on this range. See the two restrictions on page 53.)

By considering Figure 9, one sees that through the ex-

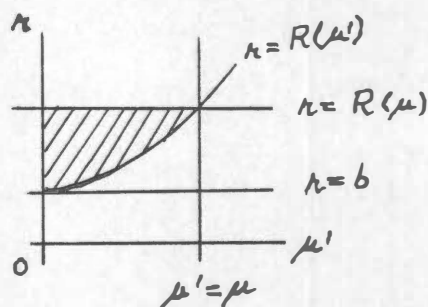


Figure 9. The κ - μ' area of integration (shaded).

change desired the integral $\int_{\kappa=b}^{R(\mu)} d\kappa \int_{\mu'=0}^{\mu'_0} d\mu'$ becomes $\int_0^{\mu} d\mu' \int_{\kappa=R(\mu')}^{R(\mu)} d\kappa$.

(μ'_0 is, remember, a function of κ .) The change of order of integration has now changed the expression for δE_c to

$$\delta E_c = \beta \pi^2 \sum_{\substack{l=0 \\ \text{even}}}^{\infty} \int_0^1 d\mu \int_0^{\mu} d\mu' \int_{R(\mu')}^{R(\mu)} d\kappa \int_{R(\mu')}^{\kappa} d\kappa' \left[\frac{\kappa^{l+2}}{\kappa'^{l-1}} - \frac{\kappa'^{l+2}}{\kappa^{l-1}} \right] P_l(\mu) P_l(\mu').$$

By defining a function $G(\kappa, \mu, \kappa', \mu')$ as

$$G(\kappa, \mu, \kappa', \mu') \equiv \sum_{\substack{l=0 \\ \text{even}}}^{\infty} P_l(\mu) P_l(\mu') \left[\frac{\kappa^{l+2}}{\kappa'^{l-1}} - \frac{\kappa'^{l+2}}{\kappa^{l-1}} \right],$$

one can write

$$\delta E_c = 8\pi^2 p^2 \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} d\kappa \int_{R(\mu')}^\kappa d\kappa' G(\kappa, \mu, \kappa', \mu').$$

The function $G(\kappa, \mu, \kappa', \mu')$ is convenient for the solution of the problem.

The quantity δE_c will be a multiple-power series in the a_n 's. The method of evaluation of δE_c will be the following: the expression obtained above will be differentiated a number of times with respect to the a_n . Then all a_n will be made zero. The coefficient of the chosen term--determined by the differentiation--is then obtainable from the remaining expression. It will be seen in Chapter V that a_0 is a function of the other a_n 's. (The a_0 is chosen to keep the volume of the drop constant.) This dependence will not be used initially in δE_c . The a_0 powers will be allowed to appear; they can be then removed at will with $a_0 = a_0(a_n's)$. From the last explicit form for δE_c , on the previous page, one sees immediately that δE_c contains third or higher powers from the power series, coefficients of all a_j , a_k and a_j^2 terms vanish: the ranges of integration for κ and κ' are each of first order in the a_n 's, and the integrand is a difference of terms such that it must be of first order also.

It is convenient to use R_0 as the unit of length; this unit will be adopted. The expression for $R(\mu)$ becomes

$$R(\mu) = 1 + \sum_{n=0, \text{ even}}^N a_n P_n(\mu)$$

It is important that N be a finite number. (This restriction

is not a strong restriction; the other restrictions (page 48) limit the surface of the drop tremendously, but this restriction alone allows any continuous surface meeting the symmetry restrictions to be approximated to any desired degree.) The many restrictions on the surface of the drop will leave a fairly general surface. This drop is the one for which δE_c will be obtained.

The differentiations to be performed can lead to confusion with symbols. The confusion can be reduced by two devices:

(1) To denote an arbitrary a_n , use the notation a_{i_j} . This becomes, then, a_{i_1} , a_{i_2} , etc.; and for four or more arbitrary a_n 's, the second subscript is easier than different single subscripts for each one. (The i_1 , i_2 , ... subscripts are still as arbitrary as a_k , a_l ,)

(2) To denote any k -fold differentiation, use the symbol $\frac{\partial^k}{(\partial a_i)^k}$. This shall include any arbitrary choice of possible a_{i_j} 's. (For example: $\frac{\partial^3}{\partial a_{i_1} \partial a_{i_2} \partial a_{i_3}}$; $\frac{\partial^3}{\partial a_{i_1}^3}$;

$\frac{\partial^3}{\partial a_{i_1} \partial a_{i_2}^2}$, etc.; are included in the single notation $\frac{\partial^3}{(\partial a_i)^3}$.)

(If one or more of the subscripts i_j do not appear in the answer, the answer is the same for all of the possible choices of these i_j .)

A Sample Calculation

The process used for finding the general term in the δE_c series is easier to follow if one has considered a specific example. Consider the simplest, non-trivial case: the a_2^3 term in δE_c .

Differentiate the $\delta E_c / 8\pi^2 \rho_z^2$ one time with respect to a_2 :²

$$\begin{aligned} \frac{\partial}{\partial a_2} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) &= \frac{\partial}{\partial a_2} \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} dn \int_{R(\mu')}^n dn' G(n, \mu, n', \mu') \\ &= \int_0^1 d\mu \int_0^\mu d\mu' \left\{ \int_{R(\mu')}^n dn' G(n, \mu, n', \mu') \right\} \frac{\partial R(\mu)}{\partial a_2} - \int_{R(\mu')}^{R(\mu)} dn G(n, \mu, n', \mu') \left\{ \frac{\partial R(\mu')}{\partial a_2} \right\}. \end{aligned}$$

By interchanging n and $R(\mu')$ in the second part of this integrand, and then interchanging the names of n and n' , one changes the integral to

$$\frac{\partial}{\partial a_2} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} dn' \left\{ G(R(\mu'), \mu, n', \mu') \cdot \frac{\partial R(\mu)}{\partial a_2} + G(n', R(\mu'), \mu, \mu') \cdot \frac{\partial R(\mu')}{\partial a_2} \right\}.$$

(Note that $G(n, \mu, n', \mu')$ is an odd function with respect to interchange of n and n' .) Now $\frac{\partial R(\mu)}{\partial a_2} = P_2(\mu)$; so

$$\frac{\partial}{\partial a_2} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} dn' \left\{ G(R(\mu'), \mu, n', \mu') \cdot P_2(\mu) + G(n', R(\mu'), \mu, \mu') \cdot P_2(\mu') \right\}.$$

Define, now the indefinite integral of

$$\int dn' G(n, \mu, n', \mu') \quad \text{as } g(n, \mu, n', \mu'). \quad \text{Then}$$

²Notation: $\int dn' G(n, \mu, n', \mu') \Big|_{n=R(\mu)}$ means that the variable n in the integral is replaced by $R(\mu)$. This notation is adopted in order to show the differentiation that has been performed.

$$\frac{\partial}{\partial a_2} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \left\{ \left[g(R(\mu), \mu, R(\mu), \mu') - g(R(\mu), \mu, R(\mu'), \mu') \right] \cdot P_2(\mu) \right. \\ \left. + \left[g(R(\mu'), \mu, R(\mu), \mu') - g(R(\mu'), \mu, R(\mu'), \mu') \right] \cdot P_2(\mu') \right\}.$$

By differentiating two more times, one gets

$$\frac{\partial^3}{\partial a_2^3} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \left\{ \left[\frac{\partial^2}{\partial a_2^2} g(R(\mu), \mu, R(\mu), \mu') \right. \right. \\ \left. \left. - \frac{\partial^2}{\partial a_2^2} g(R(\mu), \mu, R(\mu'), \mu') \right] \cdot P_2(\mu) \right. \\ \left. + \left[\frac{\partial^2}{\partial a_2^2} g(R(\mu'), \mu, R(\mu), \mu') \right. \right. \\ \left. \left. - \frac{\partial^2}{\partial a_2^2} g(R(\mu'), \mu, R(\mu'), \mu') \right] \cdot P_2(\mu') \right\}.$$

As $R(\mu)$ --and $R(\mu')$ --depend linearly on a_2 ,

$$\frac{\partial^2 g}{\partial a_2^2} (R(\mu), \mu, R(\mu), \mu') = \left[\frac{\partial^2 g}{\partial \kappa^2} (\kappa, \mu, R(\mu), \mu') \right]_{\kappa=R(\mu)} \cdot P_2^2(\mu) \\ + 2 \left[\frac{\partial^2 g}{\partial \kappa \partial \kappa'} (\kappa, \mu, \kappa', \mu') \right]_{\substack{\kappa=R(\mu) \\ \kappa'=R(\mu)}} \cdot P_2^2(\mu) \\ + \left[\frac{\partial^2 g}{\partial \kappa'^2} (R(\mu), \mu, \kappa', \mu') \right]_{\kappa'=R(\mu)} \cdot P_2^2(\mu)$$

and

$$\frac{\partial^2 g}{\partial a_2^2} (R(\mu), \mu, R(\mu'), \mu') = \left[\frac{\partial^2 g}{\partial \kappa^2} (\kappa, \mu, R(\mu'), \mu') \right]_{\kappa=R(\mu)} \cdot P_2^2(\mu) \\ + 2 \left[\frac{\partial^2 g}{\partial \kappa \partial \kappa'} (\kappa, \mu, \kappa', \mu') \right]_{\substack{\kappa=R(\mu) \\ \kappa'=R(\mu')}} \cdot P_2(\mu) P_2(\mu') \\ + \left[\frac{\partial^2 g}{\partial \kappa'^2} (R(\mu), \mu, \kappa', \mu') \right]_{\kappa'=R(\mu')} \cdot P_2^2(\mu')$$

where $P_2^2(x)$ is $[P_2(x)]^2$ --not an associated Legendre polynomial.

The a_2^3 term in $\delta E_c / 8\pi^2 \rho_z^2$ is obtained upon setting all a_n equal to zero. This means that the results of the above differentiations are to be used with $R(\mu) = 1 = R(\mu')$. Those differentiations for which only $\frac{\partial^2 g}{\partial n^2}$ occurs arise from differentiating the integrand of

$$\int_{R(\mu')}^{R(\mu)} dn' G(R(\mu), \mu, n', \mu') \cdot P_2(\mu)$$

or of $\int_{R(\mu')}^{R(\mu)} dn' G(R(\mu'), \mu, n', \mu') \cdot P_2(\mu')$.

When $R(\mu)$ and $R(\mu')$ both become one, these terms become zero; the range of integration becomes zero. One need not consider these terms. For all other terms one can change from $\frac{\partial}{\partial n} \left(\frac{\partial g}{\partial n'} \right)$ and $\frac{\partial}{\partial n'} \left(\frac{\partial g}{\partial n} \right)$ to $\frac{\partial G}{\partial n}$ and $\frac{\partial G}{\partial n'}$, respectively.

Let N_3 be the coefficient of a_2^3 in the expansion of $\delta E_c / 8\pi^2 \rho_z^2$. Taking account of the points developed in the previous two paragraphs, one can write

$$\begin{aligned} 3! N_3 = & \int_0^1 d\mu \int_0^1 d\mu' \left\{ \left[2 \frac{\partial G}{\partial n} (1, \mu, 1, \mu') \cdot P_2^2(\mu) \right. \right. \\ & + \frac{\partial G}{\partial n'} (1, \mu, 1, \mu') \cdot P_2^2(\mu) \\ & - 2 \frac{\partial G}{\partial n} (1, \mu, 1, \mu') \cdot P_2(\mu) P_2(\mu') \\ & \left. \left. - \frac{\partial G}{\partial n'} (1, \mu, 1, \mu') \cdot P_2^2(\mu') \right] \cdot P_2(\mu) \right. \end{aligned}$$

(continued equation)

(continued equation)

$$\begin{aligned}
 & + \left[2 \frac{\partial G}{\partial \kappa} (1, \mu, 1, \mu') \cdot P_2(\mu) P_2(\mu') \right. \\
 & \quad + \frac{\partial G}{\partial \kappa'} (1, \mu, 1, \mu') \cdot P_2^2(\mu) \\
 & \quad - 2 \frac{\partial G}{\partial \kappa} (1, \mu, 1, \mu') \cdot P_2^2(\mu') \\
 & \quad \left. - \frac{\partial G}{\partial \kappa'} (1, \mu, 1, \mu') \cdot P_2^2(\mu') \right] \cdot P_2(\mu') \}.
 \end{aligned}$$

Now
$$\left[\frac{\partial G}{\partial \kappa} (\kappa, \mu, \kappa', \mu') \right]_{\substack{\kappa=1 \\ \kappa'=1}} = \sum_{\substack{l=0 \\ l \text{ even}}}^{\infty} (2l+1) P_l(\mu) P_l(\mu') = - \left[\frac{\partial G}{\partial \kappa'} (\kappa, \mu, \kappa', \mu') \right]_{\substack{\kappa=1 \\ \kappa'=1}},$$

so
$$\begin{aligned}
 3! N_3 &= \sum_{\substack{l=0 \\ l \text{ even}}}^{\infty} (2l+1) \int_0^1 d\mu \int_0^{\mu} d\mu' P_l(\mu) P_l(\mu') \left\{ \left[2 P_2^3(\mu) - P_2^2(\mu) \right. \right. \\
 & \quad \left. \left. - 2 P_2(\mu) P_2(\mu') + P_2^2(\mu') \right] \cdot P_2(\mu) \right. \\
 & \quad \left. + \left[2 P_2(\mu) P_2(\mu') - P_2^2(\mu) \right. \right. \\
 & \quad \left. \left. - 2 P_2^2(\mu') + P_2^2(\mu') \right] P_2(\mu') \right\} \\
 &= \sum_{\substack{l=0 \\ l \text{ even}}}^{\infty} (2l+1) \int_0^1 d\mu \int_0^{\mu} d\mu' P_l(\mu) P_l(\mu') \left\{ P_2^3(\mu) - 3 P_2^2(\mu) P_2(\mu') + 3 P_2(\mu) P_2^2(\mu') - P_2^3(\mu') \right\}.
 \end{aligned}$$

Then
$$3! N_3 = \sum_{\substack{l=0 \\ l \text{ even}}}^{\infty} (2l+1) \int_0^1 d\mu \int_0^{\mu} d\mu' P_l(\mu) P_l(\mu') \cdot [P_2(\mu) - P_2(\mu')]^3.$$

The evaluation of N_3 is not difficult. A more general cubic term will be considered below, so N_3 need not be evaluated.

The steps of this sample calculation may help clarify the general calculations now to be considered.

The General Calculations

Return now to the general calculations which were interrupted for the sample calculation. Perform now the first general differentiation of $\delta E_c / 8\pi^2 \rho_z^2$:

$$\begin{aligned} \frac{\partial}{\partial a_i} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) &= \frac{\partial}{\partial a_i} \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} dn \int_{R(\mu')}^n dn' G(n, \mu, n', \mu') \\ &= \int_0^1 d\mu \int_0^\mu d\mu' \left\{ \int_{R(\mu')}^n dn' G(n, \mu, n', \mu') \right\} \cdot \frac{\partial R(\mu)}{\partial a_i} - \int_{R(\mu')}^{R(\mu)} dn G(n, \mu, n', \mu') \cdot \frac{\partial R(\mu')}{\partial a_i} \\ &= \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} \left\{ dn' G(R(\mu), \mu, n', \mu') \cdot \frac{\partial R(\mu)}{\partial a_i} - dn G(n, \mu, R(\mu'), \mu') \cdot \frac{\partial R(\mu')}{\partial a_i} \right\} \end{aligned}$$

Upon interchanging n and $R(\mu')$ in $G(n, \mu, R(\mu'), \mu')$ in the second term on the integrand and then interchanging the names of n and n' in the resulting term, one finds that the above expression becomes

$$\frac{\partial}{\partial a_i} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} dn' \left[G(R(\mu), \mu, n', \mu') \cdot \frac{\partial R(\mu)}{\partial a_i} + G(R(\mu'), \mu, n', \mu') \cdot \frac{\partial R(\mu')}{\partial a_i} \right].$$

The functional form of $R(\mu)$ is given on page 58. Using this form, one can change the above expression to

$$\frac{\partial}{\partial a_i} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \int_{R(\mu')}^{R(\mu)} dn' \left[G(R(\mu), \mu, n', \mu') \cdot P_i(\mu) + G(R(\mu'), \mu, n', \mu') \cdot P_i(\mu') \right].$$

One sees that the coefficients of all linear terms in the expansion for δE_c are zero: to obtain these coefficients, one makes both $R(\mu)$ and $R(\mu')$ equal to one; this makes the limits on the n' integration equal. (See also page 58.)

Let $g(n, \mu, n', \mu')$ be an indefinite integral of

$\int d\mu' G(\kappa, \mu, \mu', \mu)$. (This was also done in the sample calculation.) One can rewrite $\frac{\partial}{\partial a_i} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right)$ as

$$\begin{aligned} \frac{\partial}{\partial a_i} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \int_0^1 d\mu \int_0^\mu d\mu' \{ [g(R(\mu), \mu, R(\mu), \mu') - g(R(\mu), \mu, R(\mu'), \mu'))] P_i(\mu) \\ + [g(R(\mu'), \mu, R(\mu), \mu') - g(R(\mu'), \mu, R(\mu'), \mu'))] P_i(\mu') \}. \end{aligned}$$

Differentiate this expression κ additional times (allowing different i_j in the manner discussed on page 59):

$$\begin{aligned} \frac{\partial^{n+1}}{(\partial a_i)^{n+1}} \left(\frac{\delta E_c}{8\pi^2 \rho_z^2} \right) = \\ \int_0^1 d\mu \int_0^\mu d\mu' \{ \left[\frac{\partial^n}{(\partial a_i)^n} g(R(\mu), \mu, R(\mu), \mu') - \frac{\partial^n}{(\partial a_i)^n} g(R(\mu), \mu, R(\mu'), \mu') \right] P_i(\mu) \\ + \left[\frac{\partial^n}{(\partial a_i)^n} g(R(\mu'), \mu, R(\mu), \mu') - \frac{\partial^n}{(\partial a_i)^n} g(R(\mu'), \mu, R(\mu'), \mu') \right] P_i(\mu') \}. \end{aligned}$$

When this differentiation has been performed and the a_{i_j} are all made zero, the resulting expression is proportional to the coefficient of the general $n+1$ -power term. The particular term evaluated depends on the choice of a_{i_j} in the differentiation. Let the coefficient of the term $a_{i_1} a_{i_2} a_{i_3} \dots a_{i_{n+1}}$ in $\delta E_c / 8\pi^2 \rho_z^2$ be $N_{n+1}(i_1, i_2, \dots, i_{n+1})$, and omit the explicit dependence noted in the parenthesis.

Define κ by³

³This is not a trivial quantity. The general $n+1$ -power term includes a_2^{n+1} , $a_2^n a_4$, $a_2 a_4^n$, etc. κ thus depends on the choice of the i_j . The range of values on κ is from 1 to $(n+1)!$, and κ must be determined for each particular choice.

$$\kappa \equiv \frac{\partial^{n+1}}{(\partial a_i)^{n+1}} a_{i_1} a_{i_2} \dots a_{i_{n+1}}$$

and M_{n+1} as κN_{n+1} . Then

$$M_{n+1} = \int_0^1 d\mu \int_0^1 d\mu' \left\{ \left[\frac{\partial^n}{(\partial a_i)^n} g(R(\mu), \mu, R(\mu'), \mu') - \frac{\partial^n}{(\partial a_i)^n} g(R(\mu'), \mu, R(\mu), \mu') \right] P_{i_1}(\mu) \right. \\ \left. + \left[\frac{\partial^n}{(\partial a_i)^n} g(R(\mu'), \mu, R(\mu), \mu') - \frac{\partial^n}{(\partial a_i)^n} g(R(\mu), \mu, R(\mu'), \mu') \right] P_{i_1}(\mu') \right\} \\ \begin{matrix} R(\mu) = 1 \\ R(\mu') = 1 \end{matrix}$$

For the differentiations, now

$$\frac{\partial^n}{(\partial a_i)^n} g(R(\mu), \mu, R(\mu'), \mu') = \sum_{s=0}^n \binom{n}{s} \frac{\partial^{n-s}}{\partial \mu^{n-s}} \frac{\partial^s}{\partial \mu'^s} g \cdot \prod_{j=2}^{n+1} P_{i_j}(\mu)$$

$$\frac{\partial^n}{(\partial a_i)^n} g(R(\mu), \mu, R(\mu'), \mu') = \sum_{s=0}^n \frac{\partial^{n-s}}{\partial \mu^{n-s}} \frac{\partial^s}{\partial \mu'^s} g \cdot \sum_{\alpha_n} \prod_{j=2}^{n-s+1} P_{i_j}(\mu) \prod_{k=n-s+2}^{n+1} P_{i_k}(\mu')$$

where

$\binom{n}{s}$ is the binomial coefficient; $\prod$ is the standard product notation; and $\sum_{\alpha_n}$ sums all possible, non-repeating permutations of indices i_j and i_k . Some care must be exercised in forming the α_n summation: (1) The summation must be performed before the actual choice of the a_{i_j} is made; the permutations are made with the $a_{i_1}, a_{i_2}, a_{i_3}$, etc. and not with a_2, a_4, a_6 , etc. (2) A permutation must give a term that has not occurred. Interchanging indices among either the μ or μ' polynomials does not produce new terms for the summation: $P_{i_2}(\mu) P_{i_3}(\mu) P_{i_4}(\mu')$ is the same as $P_{i_3}(\mu) P_{i_2}(\mu) P_{i_4}(\mu')$, and of the two only one can occur in

the summation.

Note now that no term with $\frac{\partial^0}{\partial \kappa^0} g$ can contribute; these terms are due to differentiations $\int d\kappa' \frac{\partial^n G}{\partial \kappa^n}$, and when $R(\kappa) = 1 = R(\kappa')$, the term becomes $\int_{+1}^{+1} d\kappa' \frac{\partial^n G}{\partial \kappa^n}$.

All $\sum_{s=0}^n$ summations can be replaced, then, by $\sum_{s=1}^n$ summations. This allows $\frac{\partial^s}{\partial \kappa^s} g$ to be replaced by $\frac{\partial^{s-1}}{\partial \kappa^{s-1}} G(\kappa, \mu, \kappa', \mu')$ in which the appropriate functional dependence of κ and κ' are entered. After performing the differentiations, one sets both κ and κ' equal to 1.

Form now the integrands for M_{n+1} after removing the terms with $\frac{\partial^0}{\partial \kappa^0} g$ -- the $\Lambda=0$ terms:

$$\begin{aligned} & \sum_{s=1}^n \binom{n}{s} \frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^s}{\partial \kappa'^s} g(R(\kappa), \mu, R(\kappa'), \mu') \cdot P_i(\mu) \prod_{j=2}^{n+1} P_{ij}(\mu) \\ & - \sum_{s=1}^n \binom{n}{s} \frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^s}{\partial \kappa'^s} g(R(\kappa'), \mu, R(\kappa), \mu') \cdot P_i(\mu') \prod_{j=2}^{n+1} P_{ij}(\mu') \\ & - \sum_{s=1}^n \frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^s}{\partial \kappa'^s} g(R(\kappa), \mu, R(\kappa'), \mu') \cdot P_i(\mu) \sum_{\alpha_n} \prod_{j=2}^{n-s+1} P_{ij}(\mu) \prod_{k=n-s+2}^{n+1} P_{ik}(\mu') \\ & + \sum_{s=1}^n \frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^s}{\partial \kappa'^s} g(R(\kappa'), \mu, R(\kappa), \mu') \cdot P_i(\mu') \sum_{\alpha_n} \prod_{j=2}^{n-s+1} P_{ij}(\mu') \prod_{k=n-s+2}^{n+1} P_{ik}(\mu). \end{aligned}$$

For all of these cases $\frac{\partial^s g}{\partial \kappa^s \partial \kappa'^s} = \frac{\partial^{s-1}}{\partial \kappa^{s-1} \partial \kappa'^{s-1}} G$. Note now what each differentiation asks: $\frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^{s-1}}{\partial \kappa'^{s-1}} G$ asks about the variations of G in the κ and κ' directions; when $R(\kappa)$ or $R(\mu')$ replaces κ and/or κ' , the position of evaluation of these derivatives is given. The evaluations of these derivatives are made at the four corners of the square indicated in

the κ, κ' -plane in Figure 10. After the evaluations are made, then, the $R(\mu)$ and $R(\mu')$ are set equal to one--a move to the point x , say, in the figure. Thus to evaluate any of

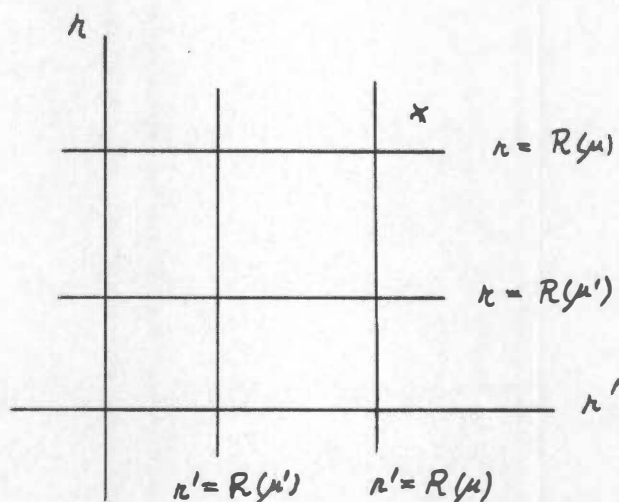


Figure 10. Points for the evaluation of derivatives.

the derivatives, one uses

$$\left[\frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^{s-1}}{\partial \kappa'^{s-1}} G(\kappa, \mu, \kappa', \mu') \right] \text{ set } \kappa = 1 = \kappa'$$

Only one derivative (this general one) need be calculated for the four sums.

Perform now the differentiation

$$\frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^{s-1}}{\partial \kappa'^{s-1}} G(\kappa, \mu, \kappa', \mu') .$$

$$G(\kappa, \mu, \kappa', \mu') = \kappa^2 \kappa' - \kappa'^2 \kappa + \sum_{\substack{l=2 \\ \text{even}}} P_l(\mu) P_l(\mu') \left[\frac{\kappa^{l+2}}{\kappa'^{l-1}} - \frac{\kappa'^{l+2}}{\kappa^{l-1}} \right]$$

$$\frac{d^m}{d\kappa^m} \kappa^{-(l-1)} = (-1)^m \frac{(l+m-2)!}{(l-2)!} \kappa^{-(l+m-1)}$$

$$\frac{d^t}{dn^t} n^{l+2} = \begin{cases} 0 & \text{when } t > l+2 \\ (l+2)! & \text{when } t = l+2 \\ \frac{(l+2)!}{(l+2-t)!} n^{l+2-t} & \text{when } t < l+2. \end{cases}$$

When G is differentiated four or more times--or when $n \geq 5$ --the two terms preceding the summation contribute nothing.

Consider first, then, the case of $n \geq 5$:

$$\begin{aligned} \frac{\partial^{n-5}}{\partial n^{n-5}} \frac{\partial^{s-1}}{\partial n^{s-1}} G &= \sum_{\substack{2 \leq l \leq n-s-2 \\ \text{even}}} P_l(\mu) P_l(\mu') \left[\frac{(l+2)! n^{l+2-(n-5)}}{(l+2-n+5)!} \cdot (-1)^{s-1} \frac{(l+s-3)!}{(l-2)!} n^{-(l+s-2)} \right] \\ &+ \sum_{\substack{2 \leq l \leq s-3 \\ \text{even}}} P_l(\mu) P_l(\mu') \left[\frac{-(l+2)! n^{l+2-(s-1)}}{(l+2-s+1)!} \cdot (-1)^{n-5} \frac{(l+n-s-2)!}{(l-2)!} n^{-(l+n-s-1)} \right] \end{aligned}$$

Make now two changes: (1) Set the n and n' equal to one.

(2) Let the factorials of negative numbers that may occur in the denominators (none occur in numerators) be infinite.

(This makes $[(-t)!]^{-1} = 0$.) Then

$$\frac{\partial^{n-5}}{\partial n^{n-5}} \frac{\partial^{s-1}}{\partial n^{s-1}} G(l, \mu, \mu') = \sum_{\substack{l=2 \\ \text{even}}} P_l(\mu) P_l(\mu') \cdot \frac{(l+2)!}{(l-2)!} \left[\frac{(-1)^{s-1} (l+s-3)!}{(l+2-n+5)!} - (-1)^{n-5} \frac{(l+n-s-2)!}{(l+2-s+1)!} \right]$$

Factoring $(-1)^{s-1}$:

$$= (-1)^{s-1} \sum_{\substack{l=2 \\ \text{even}}} P_l(\mu) P_l(\mu') \cdot \frac{(l+2)!}{(l-2)!} \left\{ \frac{[l-2+(s-1)]!}{[l+2-(n-5)]!} + (-1)^n \frac{[l-2+(n-5)]!}{[l+2-(s-1)]!} \right\}$$

Note that the summation can be extended to include

$l=0$ without adding anything: the $\frac{(l+2)!}{(l-2)!}$ factor makes $[(l-2)!]^{-1} = 0$.

Thus

$$\frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^{s-1}}{\partial \kappa'^{s-1}} G(1, \mu, 1, \mu') = (-1)^{s-1} \sum_{\substack{\ell=0 \\ \ell \in \mathbb{N}}} P_{\ell}(\mu) P_{\ell}(\mu') \cdot \frac{(\ell+2)!}{(\ell-2)!} \left\{ \frac{[\ell-2+(s-1)]!}{[\ell+2-(n-s)]!} + (-1)^n \frac{[\ell-2+(n-s)]!}{[\ell+2-(s-1)]!} \right\}.$$

A check shows that the extended sum is valid for all $n \geq 1$; extending the sum does nothing for those $n \geq 5$; but it makes the differentiation valid for all n .

The actual computation of the differentiation for specific n and s is made easier by the chosen method of writing the above expression: if one exchanges $n-s$ and $s-1$ in the differentiation, he exchanges the first and last terms in the brace--the $(-1)^n$ causes sign exchanges for odd n . The computation of factorials is made easy for negative factorials in the denominator by cancelling as many factors as possible. Suppose some quantity arises $\frac{7!}{(-3)!}$: this is zero by virtue of the $(-3)!$. One can, however, get the same result by taking for this fraction the product of all consecutive integers between -3 and 8 : $(-2)(-1)(0)(1)(2)(3)(4)(5)(6)(7)$. Any time a negative factorial occurs in the denominator, a zero will occur in this product.

Go now to explicit formation of some of the M_{n+1} .

$$M_{n+1} = \int_0^1 d\mu \int_0^{\mu} d\mu' \sum_{s=1}^n \frac{\partial^{n-s}}{\partial \kappa^{n-s}} \frac{\partial^{s-1}}{\partial \kappa'^{s-1}} G(1, \mu, 1, \mu') \cdot \left\{ \checkmark \right\}$$

$$\left\{ \binom{n}{s} \left[\prod_{j=1}^{n+1} P_{i_j}(\mu) - \prod_{j=1}^{n+1} P_{i_j}(\mu') \right] \right. \\ \left. - P_{i_1}(\mu) \sum_{j=2}^{n-s+1} \prod_{i_j} P_{i_j}(\mu) \prod_{k=n-s+2}^{n+1} P_{i_k}(\mu') \right\}$$

(continued equation)

(continued equation)

$$+ P_{i_1}(\mu') \sum_{\alpha_n} \prod_{j=2}^{n-s+1} P_{i_j}(\mu') \prod_{k=n-s+2}^{n+1} P_{i_k}(\mu) \}$$

$$\underline{n=1} \quad \frac{\partial^0}{\partial n^0} \frac{\partial^0}{\partial n'^0} G(\mu, \mu') = G(\mu, \mu') = 0.$$

$M_2 = 0$. All quadratic terms are zero. (This was previously shown in a far easier way--page 58.)

$$\underline{n=2} \quad \frac{\partial G}{\partial n} = \sum_{\substack{l=0 \\ \text{even}}} (2l+1) P_l(\mu) P_l(\mu') = - \frac{\partial G}{\partial n'}$$

$$M_3 = \int_0^1 d\mu \int_0^1 d\mu' \sum_{\substack{l=0 \\ \text{even}}} (2l+1) P_l(\mu) P_l(\mu') \cdot \{ \checkmark \}$$

$$\left\{ 2 \left[\frac{3}{\pi} P_{i_1}(\mu) - \frac{3}{\pi} P_{i_1}(\mu') \right] - P_{i_1}(\mu) \sum_{\alpha_2} P_{i_2}(\mu) P_{i_3}(\mu') + P_{i_1}(\mu') \sum_{\alpha_2} P_{i_2}(\mu') P_{i_3}(\mu) \right. \\ \left. - \left[\frac{3}{\pi} P_{i_1}(\mu) - \frac{3}{\pi} P_{i_1}(\mu') \right] + P_{i_1}(\mu) \sum_{\alpha_2} P_{i_2}(\mu') P_{i_3}(\mu') - P_{i_1}(\mu') \sum_{\alpha_2} P_{i_2}(\mu) P_{i_3}(\mu) \right\}.$$

$$M_3 = \int_0^1 d\mu \int_0^1 d\mu' \sum_{\substack{l=0 \\ \text{even}}} (2l+1) P_l(\mu) P_l(\mu') \frac{3}{\pi} \left[P_{i_1}(\mu) - P_{i_1}(\mu') \right]$$

The evaluation of this quantity will be considered below.

$$\underline{n=3} \quad \frac{\partial^2 G}{\partial n^2} = \sum_{\substack{l=0 \\ \text{even}}} 2(2l+1) P_l(\mu) P_l(\mu') = - \frac{\partial^2 G}{\partial n'^2} ; \quad \frac{\partial^2 G}{\partial n' \partial n} = 0.$$

$$M_4 = \int_0^1 d\mu \int_0^1 d\mu' \sum_{\substack{l=0 \\ \text{even}}}^{\infty} z(2l+1) P_l(\mu) P_l(\mu') \cdot \{ \checkmark \}$$

$$\begin{aligned} & \left\{ 3 \left[\frac{4}{\pi} P_{i_1}(\mu) - \frac{4}{\pi} P_{i_2}(\mu') \right] - P_{i_1}(\mu) \sum_{\alpha_3} P_{i_2}(\mu) P_{i_3}(\mu) P_{i_4}(\mu') \right. \\ & \quad + P_{i_1}(\mu') \sum_{\alpha_3} P_{i_2}(\mu') P_{i_3}(\mu') P_{i_4}(\mu) \\ & \quad - \left[\frac{4}{\pi} P_{i_1}(\mu) - \frac{4}{\pi} P_{i_2}(\mu') \right] + P_{i_1}(\mu) \sum_{\alpha_3} \frac{4}{\pi} P_{i_2}(\mu') \\ & \quad \left. - P_{i_1}(\mu') \sum_{\alpha_3} \frac{4}{\pi} P_{i_2}(\mu) \right\} \end{aligned}$$

$$M_4 = \int_0^1 d\mu \int_0^1 d\mu' \sum_{\substack{l=0 \\ \text{even}}}^{\infty} z(2l+1) P_l(\mu) P_l(\mu') \cdot \{ \checkmark \}$$

$$\left\{ z \left[\frac{4}{\pi} P_{i_1}(\mu') - \frac{4}{\pi} P_{i_2}(\mu) \right] + \sum_{\alpha'_4} [P_{i_1}(\mu) - P_{i_1}(\mu')] \times \left[\frac{4}{\pi} P_{i_2}(\mu') + \frac{4}{\pi} P_{i_2}(\mu) \right] \right\}$$

(The second form of this brace is chosen because it emphasizes the equivalence of the i_j .) The α' is a permutation different from the α : for α' one permutes the indices on the functions in the first factor; the i_1 , i_2 , i_3 , and i_4 appear in this factor. This permutation allows four negative $P_{i_1}(\mu') P_{i_2}(\mu') P_{i_3}(\mu') P_{i_4}(\mu')$ and four positive $P_{i_1}(\mu) P_{i_2}(\mu) P_{i_3}(\mu) P_{i_4}(\mu)$ to occur. The α permutation would have allowed only one of each.

The evaluation of M_4 will be considered below.

$$\underline{n=4} \quad \frac{\partial^3 G}{\partial n^3} = \sum_{\substack{l=0 \\ \text{even}}} l(l+1)(2l+1) P_l(\mu) P_l(\mu') = -\frac{\partial^3 G}{\partial n^3}$$

$$\begin{aligned} \frac{\partial^3 G}{\partial n^2 \partial n'} &= \sum_{\substack{l=0 \\ \text{even}}} -(l+2)(l-1)(2l+1) P_l(\mu) P_l(\mu') \\ &= \sum_{\substack{l=0 \\ \text{even}}} -[l(l+1)-2](2l+1) P_l(\mu) P_l(\mu') = -\frac{\partial^3 G}{\partial n \partial n'} \end{aligned}$$

$$M_5 = \int_0^1 d\mu \int_0^1 d\mu' \sum_{\substack{l=0 \\ \text{even}}} l(l+1)(2l+1) P_l(\mu) P_l(\mu') \cdot \{ \checkmark \}$$

$$\left\{ (4-6+4-1) \left[\frac{5}{\pi} P_{\frac{5}{2}}(\mu) - \frac{5}{\pi} P_{\frac{5}{2}}(\mu') \right] \right.$$

$$- P_{\frac{5}{2}}(\mu) \sum_{\alpha_4} \frac{4}{\pi} P_{\frac{4}{2}}(\mu) P_{\frac{4}{2}}(\mu') + P_{\frac{5}{2}}(\mu') \sum_{\alpha_4} \frac{4}{\pi} P_{\frac{4}{2}}(\mu') P_{\frac{4}{2}}(\mu)$$

$$+ P_{\frac{5}{2}}(\mu) \sum_{\alpha_4} \frac{3}{\pi} P_{\frac{3}{2}}(\mu) \frac{5}{\pi} P_{\frac{5}{2}}(\mu') - P_{\frac{5}{2}}(\mu') \sum_{\alpha_4} \frac{3}{\pi} P_{\frac{3}{2}}(\mu') \frac{5}{\pi} P_{\frac{5}{2}}(\mu)$$

$$- P_{\frac{5}{2}}(\mu) \sum_{\alpha_4} P_{\frac{3}{2}}(\mu) \frac{5}{\pi} P_{\frac{5}{2}}(\mu') + P_{\frac{5}{2}}(\mu') \sum_{\alpha_4} P_{\frac{3}{2}}(\mu') \frac{5}{\pi} P_{\frac{5}{2}}(\mu)$$

$$+ P_{\frac{5}{2}}(\mu) \sum_{\alpha_4} \frac{5}{\pi} P_{\frac{5}{2}}(\mu') - P_{\frac{5}{2}}(\mu') \sum_{\alpha_4} \frac{5}{\pi} P_{\frac{5}{2}}(\mu) \}$$

$$+ \int_0^1 d\mu \int_0^1 d\mu' \sum_{\substack{l=0 \\ \text{even}}} 2(2l+1) P_l(\mu) P_l(\mu') \cdot \{ \checkmark \}$$

$$\left\{ - P_{\frac{5}{2}}(\mu) \sum_{\alpha_4} \frac{3}{\pi} P_{\frac{3}{2}}(\mu) \frac{5}{\pi} P_{\frac{5}{2}}(\mu') + P_{\frac{5}{2}}(\mu') \sum_{\alpha_4} \frac{3}{\pi} P_{\frac{3}{2}}(\mu') \frac{5}{\pi} P_{\frac{5}{2}}(\mu) \right.$$

$$+ P_{\frac{5}{2}}(\mu) \sum_{\alpha_4} P_{\frac{3}{2}}(\mu) \frac{5}{\pi} P_{\frac{5}{2}}(\mu') - P_{\frac{5}{2}}(\mu') \sum_{\alpha_4} P_{\frac{3}{2}}(\mu') \frac{5}{\pi} P_{\frac{5}{2}}(\mu) \}$$

(The incorporation of terms in the first integral is in accord with the form of $\frac{\partial^3 G}{\partial n^2 \partial n'}$ adopted above.)

$$\begin{aligned}
 M_5 = & \int_0^1 d\mu \int_0^\mu d\mu' \sum_{\substack{l=0 \\ \text{even}}} l(l+1)(2l+1) P_l(\mu) P_l(\mu') \cdot \frac{5}{\pi} \left[P_{\frac{5}{2}}(\mu) - P_{\frac{5}{2}}(\mu') \right] \\
 & + \int_0^1 d\mu \int_0^\mu d\mu' \sum_{\substack{l=0 \\ \text{even}}} 2(2l+1) P_l(\mu) P_l(\mu') \cdot \left\{ \checkmark \right\} \\
 & \left\{ -P_{\frac{1}{2}}(\mu) \sum_{\alpha_4} \frac{3}{\pi} P_{\frac{3}{2}}(\mu) \frac{5}{\pi} P_{\frac{5}{2}}(\mu') + P_{\frac{1}{2}}(\mu') \sum_{\alpha_4} \frac{3}{\pi} P_{\frac{3}{2}}(\mu') \frac{5}{\pi} P_{\frac{5}{2}}(\mu) \right. \\
 & \left. + P_{\frac{1}{2}}(\mu) \sum_{\alpha_4} P_{\frac{3}{2}}(\mu) \frac{5}{\pi} P_{\frac{5}{2}}(\mu') - P_{\frac{1}{2}}(\mu') \sum_{\alpha_4} P_{\frac{3}{2}}(\mu') \frac{5}{\pi} P_{\frac{5}{2}}(\mu) \right\}
 \end{aligned}$$

The evaluation of M_5 will be considered below.

$$\underline{n=5} \quad \frac{\partial^4 G}{\partial n^4} ; \frac{\partial^4 G}{\partial n^3 \partial n'} ; \frac{\partial^4 G}{\partial n^2 \partial n'^2} ; \frac{\partial^4 G}{\partial n \partial n'^3} ; \frac{\partial^4 G}{\partial n' ^4}$$

All four are zero.

$$M_6 = 0$$

$$\underline{n=6} \quad \frac{\partial^5 G}{\partial n^{5-p} \partial n'^p} = (-1)^p \sum_{\substack{l=0 \\ \text{even}}} (2l+1) \cdot l(l+1)(l+2)(l-1) P_l(\mu) P_l(\mu')$$

$$\begin{aligned}
 M_7 = & \int_0^1 d\mu \int_0^\mu d\mu' \sum_{\substack{l=0 \\ \text{even}}} (2l+1) \cdot l(l+1)(l+2)(l-1) P_l(\mu) P_l(\mu') \cdot \left\{ \checkmark \right\} \\
 & \left\{ (6-15+20-15+6-1) \left[\frac{7}{\pi} P_{\frac{7}{2}}(\mu) - \frac{7}{\pi} P_{\frac{7}{2}}(\mu') \right] \right. \\
 & \left. - P_{\frac{1}{2}}(\mu) \sum_{\alpha_6} \frac{6}{\pi} P_{\frac{3}{2}}(\mu) P_{\frac{5}{2}}(\mu') + P_{\frac{1}{2}}(\mu') \sum_{\alpha_6} \frac{6}{\pi} P_{\frac{3}{2}}(\mu') P_{\frac{5}{2}}(\mu) \right\}
 \end{aligned}$$

(continued equation)

(continued equation)

$$\begin{aligned}
& + P_{i_1}(\mu) \sum_{\alpha_6} \frac{5}{\pi} P_{i_j}(\mu) \frac{7}{\pi} P_{i_k}(\mu') - P_{i_1}(\mu') \sum_{\alpha_6} \frac{5}{\pi} P_{i_j}(\mu') \frac{7}{\pi} P_{i_k}(\mu) \\
& - P_{i_1}(\mu) \sum_{\alpha_6} \frac{4}{\pi} P_{i_j}(\mu) \frac{7}{\pi} P_{i_k}(\mu') + P_{i_1}(\mu') \sum_{\alpha_6} \frac{4}{\pi} P_{i_j}(\mu') \frac{7}{\pi} P_{i_k}(\mu) \\
& + P_{i_1}(\mu) \sum_{\alpha_6} \frac{3}{\pi} P_{i_j}(\mu) \frac{7}{\pi} P_{i_k}(\mu') - P_{i_1}(\mu') \sum_{\alpha_6} \frac{3}{\pi} P_{i_j}(\mu') \frac{7}{\pi} P_{i_k}(\mu) \\
& - P_{i_1}(\mu) \sum_{\alpha_6} P_{i_2}(\mu) \frac{7}{\pi} P_{i_k}(\mu') + P_{i_1}(\mu') \sum_{\alpha_6} P_{i_2}(\mu') \frac{7}{\pi} P_{i_k}(\mu) \\
& + P_{i_1}(\mu) \sum_{\alpha_6} \frac{7}{\pi} P_{i_k}(\mu') - P_{i_1}(\mu') \sum_{\alpha_6} \frac{7}{\pi} P_{i_j}(\mu) \}
\end{aligned}$$

$$M_7 = \int_0^1 d\mu \int_0^\mu d\mu' \sum_{\substack{l=0 \\ \text{even}}} (2l+1) P_l(\mu) P_l(\mu') \cdot l(l+1)(l+2)(l-1) \prod_{j=1}^7 [P_{i_j}(\mu) - P_{i_j}(\mu')]]$$

The evaluation of this quantity will be considered shortly.

Consider briefly larger values of n .

$$\underline{n=7} \quad \frac{\partial^6 G}{\partial n^{6-p} \partial n^p} = (-1)^p \sum_{\substack{l=0 \\ \text{even}}} (2l+1) P_l(\mu) P_l(\mu') (l+2)(l+1)(l)(l-1) \cdot f(p)$$

$$f(0) = -6 ; \quad f(1) = -4 ; \quad f(2) = -2 ; \quad f(3) = 0$$

$$f(4) = 2 ; \quad f(5) = 4 ; \quad f(6) = 6$$

$$\underline{n=8} \quad \frac{\partial^7 G}{\partial n^{7-p} \partial n^p} = (-1)^p \sum_{\substack{l=0 \\ \text{even}}} (2l+1) P_l(\mu) P_l(\mu') \cdot l(l+1)(l+2)(l-1) \cdot [l(l+1) + g(p)]$$

$$g(0) = g(7) = 36 ; \quad g(1) = g(6) = 18 ; \quad g(2) = g(5) = 6$$

$$g(3) = g(4) = 0$$

The term $\prod_{j=1}^{n+1} [P_{i,j}(\mu) - P_{i,j}(\mu')]$ can be found, perhaps, for $n = 2m$ as it was for $n = 6, 4$, and 2 . The modifications such as the $g(\rho)$ cause other terms to appear. For $n = 2m + 1$ the terms in the m th powers do not appear: $\frac{\partial^m}{\partial \mu^m} \frac{\partial^m}{\partial \mu'^m} G = 0$. Thus the $\prod_{j=1}^{n+1} [P_{i,j}(\mu) - P_{i,j}(\mu')]$ does not appear--or only by adding and subtracting the missing term will it appear.⁴

It would be satisfying to find a closed, simple expression for the general M_{n+1} . The usefulness of the higher orders is, however, questionable: in the power series, which is the object of these calculations, one does not take a large number of terms. (This point will be again briefly considered below.) For this study we shall be content with the terms obtained as explicit functions.

The evaluation of the M_{n+1} obtained is straightforward forward. In Appendix I there are established three

⁴It might be noted that in those cases where no complications of the $g(\rho)$ form occur the $\prod_{j=1}^{n+1} [P_{i,j}(\mu) - P_{i,j}(\mu')]$ form is assured. The change of signs occurring and the permutations assure all terms except the $\prod_{j=1}^{n+1} P_{i,j}(\mu)$ and $-\prod_{j=1}^{n+1} P_{i,j}(\mu')$. These are also assured: Take the $(-1)^{s-1}$ multiplier from the $\frac{\partial^{n-s}}{\partial \mu^{n-s}} \frac{\partial^{s-1}}{\partial \mu'^{s-1}} G$ (page 69), and put it in the summation $\sum_{s=1}^n \binom{n}{s} \left[\prod_{j=1}^{n+1} P_{i,j}(\mu) - \prod_{j=1}^{n+1} P_{i,j}(\mu') \right]$. The resulting $-\sum_{s=1}^n (-1)^s \binom{n}{s}$ is just a multiplier for the brace. Now $-\sum_{s=0}^n (-1)^s \binom{n}{s} = -(1-1)^n = 0$, and hence $-\sum_{s=1}^n (-1)^s \binom{n}{s} = 1$. The complete form is thus assured.

lemmas⁵ which show the M_{n+1} to be zero for $n=2$ to 6. Lemma 1 shows M_3 , M_4 , and the second part of M_5 to be zero; Lemma 2 shows the first part of M_5 to be zero; and Lemma 3 shows M_7 to be zero. Thus

$$M_{n+1} = 0 \quad \text{for } n=2 \text{ to } 6.$$

Now $\delta E_c / 8\pi^2 \rho_z^2$ was to be a power series in the a_n coefficients. $M_{n+1} = 0$ for $n \leq 6$ says that the leading term--the one of lowest power--of the power series for $\delta E_c / 8\pi^2 \rho_z^2$ is of eighth or higher power:

$$\frac{\delta E_c}{8\pi^2 \rho_z^2} = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_N} A(p_{i_1}, p_{i_2}, \dots, p_{i_N}) a_{i_1}^{p_{i_1}} a_{i_2}^{p_{i_2}} \cdots a_{i_N}^{p_{i_N}}$$

has now a subsidiary condition

$$\sum_{j=1}^N p_{i_j} \geq 8. \quad 6$$

From above $E_c = E'_c + \delta E_c$. The result of the calculations for δE_c shows that E'_c is correct for all terms below the eighth power in the a_n 's. (This statement is true if the surfaces are such that $\Delta V(\kappa^2)$ consists of a single term only as discussed on page 53.)

As a non-zero term in δE_c has not been found, the value of M_{n+1} for $n \geq 7$ would be interesting. It might be that δE_c is actually zero for the drop which has been considered. Evaluation of these quantities seems to be very hard.

⁵Each lemma has a restriction on its validity. The applicability of the lemmas to these M_{n+1} is assured by the restriction on $R(\kappa)$ expressed on page 58; this is the only reason for the particular restriction.

⁶The definition of this N is contained in the restriction on page 58.

CHAPTER V

A HYDRODYNAMICAL CALCULATION

In this chapter the kinetic energy will be calculated. The steps necessary for the calculation will be kept as general as possible until calculational details demand the treatment of a particular case. The drop and its distortion will be characterized. Then the boundary conditions for a free liquid surface will be derived. This boundary condition is the constraint relating the shape of the drop to the velocity field in the interior. This equation of constraint will then be used to determine the velocity field. Once the velocity field is obtained, the determination of the kinetic energy is a straightforward calculation. This calculation will then be performed.

The Drop and Its Distortion

The drop to be treated is incompressible. The mathematical statement of incompressibility is

$$\frac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla} \rho + \frac{\partial \rho}{\partial t} = 0.$$

Combining the equation of continuity

$$\vec{\nabla} \cdot (\rho \vec{v}) + \frac{\partial \rho}{\partial t} = 0$$

and the equation of incompressibility, one finds

$$\vec{\nabla} \cdot \vec{v} = 0.$$

The drop is assumed to have rotational symmetry about

the z -axis and reflection symmetry in the x, y -plane. If the surface of the drop is given in spherical coordinates by: $S(r, \vartheta, \varphi; t) = 0$, then $S = S(r, \vartheta; t) = S(r, \pi - \vartheta; t)$; where r , ϑ , and φ are spherical coordinates. (The surface used in Chapter IV had this symmetry, but the expansion in Legendre polynomials contained a finite number of terms.)

This surface will shortly be restricted to a very limited surface; some preliminary calculations can be done with this general surface.

The Boundary Condition

There is a perfectly general boundary condition for a free surface of a fluid: the surface is defined by the moving fluid. This statement must be expressed mathematically. In terms of Cartesian coordinates we define the moving surface with the equation $F(x, y, z; t) = 0$. Consider a point P in the surface at time t ; let the surface move for an interval of time Δt during which interval P moves to a new position Q . (Mathematically time produces a mapping which carries P into Q ; physically a volume element at P moves in time Δt to the point Q .) Evaluate F at Q by a Taylor expansion about the point P :

$$F(Q, t + \Delta t) = F(P, t) + \frac{\partial F}{\partial x}(x_Q - x_P) + \frac{\partial F}{\partial y}(y_Q - y_P) + \frac{\partial F}{\partial z}(z_Q - z_P) + \frac{\partial F}{\partial t} \Delta t + \text{higher-order terms.}$$

Now $F(Q) = F(P) = 0$. Divide the remaining equation

by Δt : $\frac{\partial F}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial F}{\partial y} \frac{\Delta y}{\Delta t} + \frac{\partial F}{\partial z} \frac{\Delta z}{\Delta t} + \frac{\partial F}{\partial t} + \text{higher-order terms} = 0$.

In the limit as $\Delta t \rightarrow 0$

$$\vec{v} \cdot \vec{\nabla} F = -\frac{\partial F}{\partial t}$$

where $\vec{v}$ is defined by

the limit process and is recognizable as the velocity evaluated for a fluid element located at the point P at time t . So the boundary condition is

$$\vec{v} \cdot \vec{\nabla} F = -\frac{\partial F}{\partial t} \quad (\text{on the surface}).$$

The equation is perfectly general.

A Simplifying Demand and Its Justification

It is assumed that the velocity field within the drop is irrotational. If $\vec{v}$ is the velocity field,

$$\vec{v} = \vec{v}(r, \vartheta; t) = \vec{\nabla} \phi(r, \vartheta; t)$$

where the spatial dependence of both $\vec{v}$ and ϕ agrees with the surface assumed above--i.e., no φ dependence.

A theorem of Kelvin states that a fluid system with an irrotational velocity field has less kinetic energy than it would have with any velocity field that is not irrotational.¹ When one assumes an irrotational velocity field, then, he has minimum kinetic energy of the drop. It was seen at the end

¹Sir Horace Lamb, Hydrodynamics (6th ed.; New York: Dover Publications, 1932), pp. 47-8.

of Chapter III that the kinetic energy is proportional to the effective mass defined there. This effective mass occurs in the Gamow penetration factor. A minimum effective mass gives a maximum penetration probability and a minimum half-life for spontaneous fission. Those motions of the nucleus which correspond to greater effective masses will have longer half-lives and will make negligible contributions to the experimentally determined half-life. It is for this reason that an irrotational velocity field is assumed.

Incompressibility demanded that the divergence of the velocity field be zero. This means that ϕ is a harmonic function: $\nabla^2 \phi = 0$.

Evaluation of the Kinetic Energy

The solution demanded above for ϕ was $\phi(r, \vartheta; t)$, and as ϕ is a harmonic function

$$\phi = \sum_{n=0}^{\infty} b_n(t) r^n P_n(\mu)$$

where $\mu = \cos \vartheta$ and P_n is the Legendre polynomial of order n . The b_n are parameters. (The other solution for ϕ behaves like $r^{-n-1} P_n(\mu)$ and diverges at the origin.)

The equation for the surface $S(r, \vartheta; t) = 0$ will be

$$S(r, \vartheta; t) = r - R_0 \left[1 + \sum_{\substack{n=0 \\ \text{even}}}^{\infty} a_n(t) P_n(\mu) \right] = 0.$$

Note that only even Legendre polynomials arise; this meets the demand that $S(r, \vartheta; t) = S(r, \pi - \vartheta; t)$. (Equating to zero merely preserves the form of $F(\vec{r}, t)$ of the general

case.)

The general boundary condition now relates the a_n 's to the b_n 's by

$$\vec{\nabla} \phi \cdot \vec{\nabla} S = - \frac{\partial S}{\partial t} \quad (\text{surface})$$

Before developing this equation any further, allow the unit of length to be R_0 --this is, note, the radius of the undistorted drop. Form now the boundary-condition equation.

$$\vec{\nabla} \phi = \hat{n} \sum_{n=1} n b_n R^{n-1} P_n - \hat{\theta} (1-\mu^2)^{\frac{1}{2}} \sum_{n=1} b_n R^{n-1} \frac{d}{d\mu} P_n$$

$$\vec{\nabla} S = \hat{n} + \hat{\theta} (1-\mu^2)^{\frac{1}{2}} R^{-1} \sum_{\substack{n=0 \\ \text{even}}} a_n \frac{dP_n}{d\mu}$$

$$\frac{\partial S}{\partial t} = \sum_{\substack{n=0 \\ \text{even}}} \dot{a}_n P_n. \quad (\dot{a}_n \equiv \frac{d}{dt} a_n(t).)$$

Here $\hat{n}$ and $\hat{\theta}$ are unit vectors in the spherical-coordinate space. Denote by $P_{n,m}(\mu)$ the function $(1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_n(\mu)$. (The standard notation is P_n^m ; as powers of functions arise, it is less ambiguous to save the superscript for powers and to use the m as a second subscript.) In forming the equation, one must replace all R by the surface value. Denote the value of R at the surface by $R(\mu)$. By virtue of the symmetry of $S(R, \vartheta; t)$, $R(\mu) = 1 + \sum_{\substack{n=0 \\ \text{even}}} a_n P_n(\mu)$. Then one has

$$\sum_{n=1} n b_n R^{n-1}(\mu) P_n - \sum_{\substack{l=0 \\ \text{even}}} \sum_{n=1} a_l b_n R^{n-2}(\mu) P_n P_{l,1} = \sum_{\substack{l=0 \\ \text{even}}} \dot{a}_l P_l.$$

It is possible to eliminate all odd powers of n in the above equation. $\vec{\nabla}$ and ϕ are mutually orthogonal functions. The θ -component of $\vec{\nabla}$, v_θ , must be an odd function with respect to replacing μ by $-\mu$ since $\vec{\nabla}$ must

be consistent with the surface of the drop. In order that ψ_θ be an odd function, ϕ must be an even function; hence ϕ cannot have the odd Legendre polynomials in its expansion. Thus for the drop and the distortions considered

$$\sum_{\substack{n=2 \\ \text{even}}} n b_n R^{n-1}(\mu) P_n - \sum_{\ell=0} \sum_{\substack{n=2 \\ \text{even}}} b_n a_\ell R^{n-2}(\mu) P_n P_\ell = \sum_{\substack{\ell=0 \\ \text{even}}} a_\ell P_\ell$$

relates the b_n 's and the a_n 's.

The general surface used thus far is very hard to treat; a particular surface will now be used: allow $S(\mu, \theta; t)$ to be $1 - (1 + a_0 + a_2 P_2)$. This is the simplest, non-trivial surface satisfying the demands on the surface. This surface is not unrealistic: an indication of the relative importance of the P_2 , P_4 , and P_6 can be obtained from the values of a_2 , a_4 , and a_6 for the saddle point of the barrier to fission. Interpolation of the data in Table I gives for $X = 0.83$ (the X of interest in the thesis) the saddle point values: $a_2 = 0.38$, $a_4 = 0.045$, and $a_6 = -0.0046$. One would thus expect P_4 and P_6 to make small contributions to the velocity field for the nucleus with $X = 0.83$.

Treat now the particular case. With the relationships

$$P_2 P_n = \frac{3}{2} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)} P_n + \frac{3}{2} \frac{n(n-1)}{(2n-1)(2n+1)} P_{n-2}$$

$$P_2, P_n = \frac{3n(n+1)}{2n+1} \left[-\frac{(n+2)}{2n+3} P_{n+2} + \frac{2n+1}{(2n-1)(2n+3)} P_n + \frac{n-1}{2n-1} P_{n-2} \right]$$

the boundary condition can be written as

$$\sum_{\substack{n=2 \\ \text{even}}} n b_n R^{n-2}(\mu) \left[a_2 \cdot \frac{3}{2} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \left\{ (1+a_0) + \frac{(n+1)(n-3)a_2}{(2n-1)(2n+3)} \right\} P_n - \frac{3}{2} \frac{(n-1)(n+2)}{(2n-1)(2n+1)} a_2 P_{n-2} \right] = \dot{a}_0 + \dot{a}_2 P_2.$$

The problem of relating the b_n 's to the a_2 is to be solved by treating it as a perturbation problem in which the perturbation is a_2 . The orders of perturbation will be defined by the powers of a_2 retained: when a_2^k is the highest power of a_2 retained, the order of the perturbation is the k^{th} . The condition of incompressibility gives a constraint

$$\frac{4\pi}{3} = \iiint_{\Omega} d\tau = 2\pi \int_{-1}^1 d\mu \int_0^{R(\mu)} r^2 dr = \frac{4\pi}{3} \int_0^1 R^3(\mu) d\mu;$$

and $\int_0^1 R^3(\mu) d\mu = 1$ determines a_0 as a function of a_2 . (This is also treated as a power series--assuming a_2 to be smaller than one.) From the equation one gets

$$a_0 = -\frac{a_2^2}{5} \left(1 + \frac{2}{3 \cdot 7} a_2 + 0 \cdot a_2^2 + \frac{2}{3 \cdot 5 \cdot 7} a_2^3 + \frac{4}{3^2 \cdot 5 \cdot 7^2} a_2^4 + \dots \right).$$

It is convenient, before performing the algebra required in the solution, to change two expressions which occur: (1) The various orders of perturbation give various expressions for the b_n . Change this to denote the value of b_n attained in the k^{th} approximation by $^k b_n$. (2) The quantity $R^{n-2}(\mu)$ is clumsy. It is

$$\begin{aligned} R^{n-2}(\mu) &= (1 + a_0 + a_2 P_2)^{n-2} \\ &= (1 + a_0)^{n-2} + (n-2)(1 + a_0)^{n-3} a_2 P_2 + \frac{1}{2}(n-2)(n-3)(1 + a_0)^{n-4} a_2^2 P_2^2 + \dots \end{aligned}$$

This is made a simpler expression by defining quantities

$\beta_\ell^{(n)}$ such that

$$R^{n-2}(\mu) = \beta_0^{(n)} + (n-2)\beta_1^{(n)} P_2 + \frac{1}{2}(n-2)(n-3)\beta_2^{(n)} P_2^2 + \dots$$

where the subscript is the lowest power of a_2 occurring in the particular β . (The $1+Q_0$ is a function of a_2 .) Any desired approximation of the β 's is attainable.

Go now to the various orders of approximation. The linear independence of the P_n allows the equating of coefficients of P_n on the left-hand side of the boundary-condition equation to those on the right-hand side.

Zero Order: ($a_2' = 0$)

From P_2 coefficients: $2 {}^0b_2 \beta_0^{(2)} = \dot{a}_2$, ${}^0b_2 = \frac{1}{2} \dot{a}_2$

From P_n coefficients ($n \geq 4$): $n {}^0b_n \beta_0^{(n)} = 0$, ${}^0b_n = 0$
($n \geq 4$)

First Order: ($a_2^2 = 0$)

From P_0 coefficients: $2 {}^1b_2 \beta_0^{(2)} \left\{ -\frac{3}{2}, \frac{1.4}{3.5} \right\} a_2 = \dot{a}_0 = -\frac{2}{5} a_2 \dot{a}_2$

(This merely reproduces the 0b_2 coefficients.)

From P_2 coefficients:

$$2 {}^1b_2 \beta_0^{(2)} \left\{ 1 - \frac{3}{3.7} a_2 \right\} + 4 {}^1b_4 \{ \text{powers of } a_2 \} = \dot{a}_2$$

Since ${}^0b_4 = 0$, ${}^1b_4 \sim a_2$ and ${}^1b_4 a_2 \sim a_2^2$ or zero in this order. (The tilde $\sim$ will mean "of order.") So

$$2 {}^1b_2 \left\{ 1 - \frac{1}{7} a_2 \right\} = \dot{a}_2$$

$${}^1b_2 = \frac{\dot{a}_2}{2} \left(1 + \frac{1}{7} a_2 \right)$$

From P_4 coefficients:

$$2 {}^1b_2 \left[\beta_0^{(2)} \left\{ a_2 \frac{9}{2} \frac{3.4}{5.7} \right\} \right] + 4 {}^1b_4 \beta_0^{(4)} \cdot 1 = 0$$

$$a_2 \frac{9.3}{5.7} 'b_2 + 'b_4 = 0$$

$$'b_4 = -\frac{27}{70} a_2 \dot{a}_2$$

Now $'b_2$ does not contribute to P_n coefficients for $n \geq 6$. The $'b_4$ has a multiplier of a_2 in the P_n coefficients except for $n=4$. So for $n \geq 6$ the equation for the coefficients is $n 'b_n \cdot 1 = 0$. The result for the first order is

$$'b_2 = \dot{a}_2 \left[\frac{1}{2} + \frac{1}{2 \cdot 7} a_2 \right]$$

$$'b_4 = \dot{a}_2 \left[\frac{-27}{2 \cdot 5 \cdot 7} a_2 \right]$$

$$'b_n = 0 \quad (n \geq 6)$$

Second Order: ($a_2^3 = 0$)

It is convenient to rewrite the equation.

$$\sum_{\text{even}} \left\{ n b_n \beta_0^{(n)} \left[a_2 \cdot \frac{9}{2} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \left\{ (1+a_2) + \frac{(n+1)(n-3)a_2}{(2n-1)(2n+3)} \right\} P_n - \frac{3}{2} \frac{(n-1)(n+2)a_2}{(2n-1)(2n+1)} P_{n-2} \right] \right.$$

$$+ n b_n \cdot \frac{(n-2)}{2} \beta_1^{(n)} P_2 \left[a_2 \cdot \frac{9}{2} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \left\{ (1+a_2) + \frac{(n+1)(n-3)a_2}{(2n-1)(2n+3)} \right\} P_n - \frac{3}{2} \frac{(n-1)(n+2)a_2}{(2n-1)(2n+1)} P_{n-2} \right]$$

$$\left. + n b_n \cdot \frac{(n-2)(n-3)}{2} \beta_2^{(n)} P_2^2 P_n \right\} = \dot{a}_0 + \dot{a}_2 P_2$$

(One needs $P_2(P_2 P_n)$ which is available above.) From the P_0 coefficients:

$$2^2 b_2 \beta_0^{(2)} \left[\frac{-3}{2} \frac{1.4}{3.5} a_2 \right] + 4^2 b_4 \frac{2.1}{2} \beta_2^{(4)} \cdot \frac{3}{2} \cdot \frac{4.3}{7.9} \cdot \frac{3}{2} \cdot \frac{2.1}{3.5} = \dot{a}_0$$

Now $2^2 b_4 \beta_2 \sim a_2^3$. So

$$2^2 b_2 \cdot 1 \left(-\frac{2}{5} a_2 \right) = -\frac{2}{5} a_2 \dot{a}_2 \left(1 + \frac{1}{7} a_2 \right) \quad \text{This constitutes,}$$

then, merely a check on $'b_2$.

From P_2 coefficients:

$$2^2 b_2 \beta_0^{(2)} \left[1 - \frac{1}{5} a_2^2 + \frac{3(-1)}{3.7} a_2 \right] + 4^2 b_4 \left[\beta_0^{(4)} \left(-\frac{3}{2} \cdot \frac{3.6}{7.9} a_2 \right) + 2 \beta_1^{(4)} \left(\frac{3}{2} \cdot \frac{4.3}{7.9} \right) \right] = \dot{a}_2$$

$$2^2 b_2 \left(1 - \frac{1}{7} a_2 - \frac{1}{5} a_2^2 \right) + 4^2 b_4 \left(\frac{1}{7} a_2 \right) = \dot{a}_2$$

Let $2^2 b_2 = 'b_2 + \frac{\gamma_2}{2} a_2^2 \dot{a}_2$. The γ_2 is to be determined.

$$2 \left[\frac{\dot{a}_2}{2} \left(1 + \frac{a_2}{7} + \gamma_2 a_2^2 \right) \right] \left(1 - \frac{1}{7} a_2 - \frac{1}{5} a_2^2 \right) + 4 \left[\dot{a}_2 \left(\frac{-2.7}{2.5.7} a_2 \right) \right] \cdot \frac{1}{7} a_2 = \dot{a}_2$$

This yields $\gamma_2 = \frac{108}{5.7^2}$.

From P_4 coefficients:

$$2^2 b_2 \beta_0^{(2)} \left[\frac{9}{2} \cdot \frac{3.4}{5.7} a_2 \right] + 4^2 b_4 \left\{ \beta_0^{(4)} \left[1 + \frac{5.1}{7.11} a_2 \right] + 2 \beta_1^{(4)} \left[\frac{4.5}{7.11} \right] \right\} = 0$$

$$2^2 b_2 \frac{5.4}{5.7} a_2 + 4^2 b_4 \left(1 + \frac{4.5}{7.11} a_2 \right) = 0$$

Let $2^2 b_4 = 'b_4 + a_2 \gamma_4 a_2^2$ with γ_4 to be determined. The equation yields

$$\gamma_4 = \frac{459}{5.7^2.11} .$$

From P_6 coefficients:

$$4 {}^2b_4 \left\{ \beta_0^{(4)} \left[a_2 \cdot \frac{9}{2} \cdot \frac{5.6}{9.11} \right] \right\} + 6 {}^2b_6 \beta_0^{(6)} \cdot 1 + 4 {}^2b_4 \cdot 2 \beta_1^{(4)} \cdot \frac{3}{2} \cdot \frac{5.6}{9.11} = 0$$

$${}^2b_6 = -4 {}^2b_4 a_2 \left(\frac{5.5}{2 \cdot 3.11} \right) = -4 \left(\frac{5.5}{2 \cdot 3.11} \right) a_2 \left[a_2 \left(\frac{-27}{2 \cdot 5.7} \right) \right]$$

$${}^2b_6 = \frac{5.9}{7.11} a_2 a_2^2$$

From the coefficients for P_n with $n \geq 8$:

The $n=2$ does not contribute, so no a_2^0 occurs in the equation for the coefficients. The $n=4$ occurs with $b_4 a_2^2$ (for $n=8$ only). This is of order a_2^3 . Then $b_6 a_2$ occurs for $n=8$; again this is of order a_2^3 . The resulting equation is $n {}^2b_n = 0$ for $n \geq 8$.

The end result for the second order is

$${}^2b_2 = a_2 \left[\frac{1}{2} + \frac{1}{2.7} a_2 + \frac{54}{5.72} a_2^2 \right]$$

$${}^2b_4 = a_2 \left[\frac{-27}{2 \cdot 5.7} a_2 + \frac{459}{5.72 \cdot 11} a_2^2 \right]$$

$${}^2b_6 = a_2 \left[\frac{5.9}{7.11} a_2^2 \right]$$

$${}^2b_n = 0 \quad \text{for } (n \geq 8).$$

The third- and fourth-order approximations can be determined; there is a tremendous amount of simple algebra in the work, so only the result will be given. Each order adds a new term--in the pattern suggested by the grouping

of the second-order result. The fourth-order result is

$${}^4b_2 = \dot{a}_2 \left[\frac{1}{2} + \frac{1}{2 \cdot 7} a_2 + \frac{54}{5 \cdot 7^2} a_2^2 - \frac{757}{2 \cdot 3 \cdot 7^3 \cdot 11} a_2^3 + \frac{25,116,867}{2 \cdot 5^2 \cdot 7^4 \cdot 11^2 \cdot 13} a_2^4 \right]$$

$${}^4b_4 = \dot{a}_2 \left[\frac{-27}{2 \cdot 5 \cdot 7} a_2 + \frac{459}{5 \cdot 7^2 \cdot 11} a_2^2 - \frac{9 \cdot 9 \cdot 153,991}{5^2 \cdot 7^3 \cdot 11^2 \cdot 13} a_2^3 + \frac{18 \cdot 8,131,801}{5 \cdot 7^4 \cdot 11^3 \cdot 13} a_2^4 \right]$$

$${}^4b_6 = \dot{a}_2 \left[\frac{5 \cdot 9}{7 \cdot 11} a_2^2 - \frac{9 \cdot 317}{5 \cdot 7 \cdot 11^2} a_2^3 + \frac{7,230,700,953}{5^2 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 17} a_2^4 \right]$$

$${}^4b_8 = \dot{a}_2 \left[\frac{-4 \cdot 7 \cdot 27}{5 \cdot 11 \cdot 13} a_2^3 + \frac{801,954}{5^2 \cdot 7 \cdot 11^2 \cdot 19} a_2^4 \right]$$

$${}^4b_{10} = \dot{a}_2 \left[\frac{4 \cdot 9 \cdot 9 \cdot 27}{13 \cdot 17 \cdot 19} a_2^4 \right]$$

$${}^4b_n = 0 \quad \text{for } n \geq 12.$$

The purpose of the determination of the b_n 's has been determining $\vec{v}$ in order to calculate T --the kinetic energy. As $\vec{v} = \vec{\nabla} \phi = \vec{\nabla} \sum_{n=2}^{\infty} b_n n^n P_n(\mu)$, the approximation to $\vec{v}$, in the power series of a_2 , has been obtained through the a_2^4 term. The evaluation of T is now possible through the a_2^4 term (T is obviously going to be a power series in a_2): quite apart from possible cancellation of terms due to the orthogonality of the P_n , no power of a_2 less than the fifth can have been dropped by the fourth-order obtained.

In order to evaluate T , it is assumed that the mass

density ρ is uniform. (Incompressibility has previously been assumed.) Uniformity demands $\vec{\nabla}\rho = 0$. This assumption allows the change

$$T = \frac{1}{2} \iiint \rho v^2(\vec{r}) d\tau \Rightarrow \frac{1}{2} \rho \iiint v^2(\vec{r}) d\tau$$

It is convenient to keep the b_n 's (in the velocity) until the integrations are performed. With them

$$\begin{aligned} T &= \frac{1}{2} \rho \iiint \left\{ \left[\sum_{\substack{n=2 \\ \text{even}}}^{\infty} b_n r^{n-1} P_n(\mu) \right]^2 + (1-\mu^2) \left[\sum_{\substack{n=2 \\ \text{even}}}^{\infty} b_n r^{n-1} \frac{d}{d\mu} P_n(\mu) \right]^2 \right\} dz \\ &= \frac{1}{2} \rho \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \int_0^{R(\mu)} r^2 dr \left\{ \text{the brace above} \right\} \\ &= 2\pi \rho \int_0^1 d\mu \int_0^{R(\mu)} r^2 dr \left\{ \sum_{\substack{l=2 \\ \text{even}}}^{\infty} \sum_{\substack{n=2 \\ \text{even}}}^{\infty} n l b_n b_l r^{n+l-2} P_n(\mu) P_l(\mu) \right. \\ &\quad \left. + \sum_{\substack{l=2 \\ \text{even}}}^{\infty} \sum_{\substack{n=2 \\ \text{even}}}^{\infty} b_n b_l r^{n+l-2} P_{n,1}(\mu) P_{l,1}(\mu) \right\} \\ T &= 2\pi \rho \sum_{\substack{l=2 \\ \text{even}}}^{\infty} \sum_{\substack{n=2 \\ \text{even}}}^{\infty} \left\{ n l b_n b_l \int_0^1 d\mu \int_0^{R(\mu)} dr r^{n+l} P_n(\mu) P_l(\mu) \right. \\ &\quad \left. + b_n b_l \int_0^1 d\mu \int_0^{R(\mu)} dr r^{n+l} P_{n,1}(\mu) P_{l,1}(\mu) \right\} \end{aligned}$$

Now with the fourth-order b_n 's one gets a fourth-order T by stopping both summations at ten. (Keep the notation T ; one could denote the approximation by 4T , conforming with the above notation for the b_n 's.)

$$\begin{aligned} T &= 2\pi \rho \sum_{\substack{l=2 \\ \text{even}}}^{10} \sum_{\substack{n=2 \\ \text{even}}}^{10} \left\{ \frac{n l b_n b_l}{n+l+1} \int_0^1 d\mu R^{n+l+1}(\mu) P_n(\mu) P_l(\mu) \right. \\ &\quad \left. + \frac{b_n b_l}{n+l+1} \int_0^1 d\mu R^{n+l+1}(\mu) P_{n,1}(\mu) P_{l,1}(\mu) \right\}. \end{aligned}$$

Some of the larger values of n and l do not contribute. The $R^{n+l+1}(\mu)$ introduces terms like $a_2^8 P_2^8$. The value of q necessary to make $\int_0^1 d\mu P_2^8 P_n P_l$ non-zero causes the whole contribution to be of fifth or higher order. (A similar result occurs for the other integral.) All terms through fourth order are contained in the following expression:

$$\begin{aligned}
 T = 2\pi p \Big\{ & \frac{4}{5} b_2^2 \int_0^1 \left[(1+a_2)^5 + 5(1+a_2)^4 a_2 P_2 + 10(1+a_2)^3 a_2^2 P_2^2 + 10 a_2^3 P_2^3 + 5 a_2^4 P_2^4 \right] P_2 P_2 d\mu \\
 & + \frac{8}{7} b_2 b_4 \cdot 2 \int_0^1 \left[7(1+a_2)^6 a_2 P_2 + 21(1+a_2)^5 a_2^2 P_2^2 + 35 a_2^3 P_2^3 \right] P_2 P_4 d\mu \\
 & + \frac{12}{9} b_2 b_6 \cdot 2 \int_0^1 \left[36(1+a_2)^7 a_2^2 P_2^2 \right] P_2 P_6 d\mu \\
 & + \frac{16}{9} b_4 b_4 \int_0^1 \left[(1+a_2)^9 + 9(1+a_2)^8 a_2 P_2 + 36 a_2^2 P_2^2 \right] P_4 P_4 d\mu \\
 & + \frac{24}{11} b_4 b_6 \cdot 2 \int_0^1 (11 a_2 P_2) P_4 P_6 d\mu + \frac{36}{13} b_6 b_6 \int_0^1 P_6 P_6 d\mu \\
 & + \frac{b_2^2}{5} \int_0^1 \left[(1+a_2)^5 + 5(1+a_2)^4 a_2 P_2 + 10(1+a_2)^3 a_2^2 P_2^2 + 10 a_2^3 P_2^3 + 5 a_2^4 P_2^4 \right] P_{2,1} P_{2,1} d\mu \\
 & + \frac{b_2 b_4}{7} \cdot 2 \int_0^1 \left[7(1+a_2)^6 a_2 P_2 + 21(1+a_2)^5 a_2^2 P_2^2 + 35 a_2^3 P_2^3 \right] P_{2,1} P_{4,1} d\mu \\
 & + \frac{b_2 b_6}{9} \cdot 2 \int_0^1 \left[36(1+a_2)^7 a_2^2 P_2^2 \right] P_{2,1} P_{6,1} d\mu \\
 & + \frac{b_4 b_4}{9} \int_0^1 \left[(1+a_2)^9 + 9(1+a_2)^8 a_2 P_2 + 36 a_2^2 P_2^2 \right] P_{4,1} P_{4,1} d\mu \\
 & + \frac{b_4 b_6}{11} \cdot 2 \int_0^1 (11 a_2 P_2) P_{4,1} P_{6,1} d\mu + \frac{b_6 b_6}{13} \int_0^1 P_{6,1} P_{6,1} d\mu
 \end{aligned}$$

The result of performing the integrations (see Appendix II) and replacing a_0 by $a_0(a_2)$ is

$$\begin{aligned}
T = \pi \rho \left\{ \frac{4}{5} b_2 b_2 \left[1 + a_2 + \frac{11}{7} a_2^2 + \frac{8}{15} a_2^3 - \frac{613}{3 \cdot 5 \cdot 7 \cdot 11} a_2^4 \right] \right. \\
+ \frac{16}{7} b_2 b_4 \left[\frac{9}{5} a_2 + \frac{12}{5} a_2^2 + \frac{681}{5^2 \cdot 11} a_2^3 \right] \\
+ b_2 b_6 \frac{4 \cdot 36^2}{7 \cdot 9 \cdot 11} a_2^2 + b_4 b_6 \frac{80}{11} a_2 \\
\left. + \frac{8}{9} b_4 b_4 \left[1 + \frac{15}{7} a_2 + \frac{771}{7 \cdot 11} a_2^2 \right] + \frac{12}{13} b_6 b_6 \right\}.
\end{aligned}$$

The introduction of the values obtained for b_n gives the following result:

$$\begin{aligned}
2T = 2\pi \rho \dot{a}_2^2 \left\{ 0.20 + 0.257143 a_2 - 0.109388 a_2^2 \right. \\
\left. - 0.279045 a_2^3 - 0.034399 a_2^4 \right\}
\end{aligned}$$

The effective mass m was defined in Chapter III through $2T = m \dot{a}_2^2$. The a_2 is in the present case the α used in Chapter III. Thus

$$\begin{aligned}
m = \frac{3MA}{10} \left\{ 1 + \frac{9}{7} a_2 - 0.546940 a_2^2 \right. \\
\left. - 1.395225 a_2^3 - 0.171995 a_2^4 \right\}
\end{aligned}$$

where ρ has been eliminated through its definition: if M is the mass and Ω the volume of the drop, $\rho = M/\Omega$. For a nucleus with A nucleons, each of mass m , $\rho = \frac{MA}{\Omega} = \frac{MA}{\frac{4\pi}{3}}$. (The first two coefficients in m are exact as written.)

As was stated in Chapter II, the value of T obtained by Frankel and Metropolis is incorrect. They use the same

definition of m as is used here, so their error in T is exhibited by their value of m . They give²

$$m = \frac{3MA}{10} \{1 + a_2 + O(a_2^2)\}$$

This value of m can be obtained by using the zero-order b_n 's above. Neglecting the higher powers of a_2 will introduce small errors if a_2 remains quite small. The error in the linear term should not be negligible. The question of the effect of using the incorrect m will be considered in Chapter VI.

²S. Frankel and N. Metropolis, Phys. Rev. 72, 914 (1947).

CHAPTER VI

SPONTANEOUS FISSION OF ${}_{101}\text{Mv}^{256}$

The predictions of fission activation energies and lifetimes based on series approximations using the liquid-drop model are expected to become increasingly better--and easier to obtain--as the atomic number of the nucleus, against which the predictions are checked, becomes greater. (See page 24.) This chapter will consider the half-life for spontaneous fission of ${}_{101}\text{Mv}^{256}$.

The creation of the nucleus with atomic number 101 was reported by Ghiorso et al.¹ in 1955. They suggested for it the name Mendeleevium and the symbol Mv. The mass number was not definitely determined, but it was thought to be 256. The spontaneous-fission half-life was measured; it was about three and one-half hours.

The liquid-drop model will predict the activation energy E_f (see page 29) as well as the half-life for spontaneous fission. To find these predictions one needs the potential energy of distortion for the liquid drop. There are two contributions to this potential energy--one a surface, the other an electrostatic contribution. Neither contribution is determined in this thesis; both have been determined in other studies, and the results of one of these studies will be used

¹A. Ghiorso, B. G. Harvey, G. R. Choppin, S. G. Thompson, and G. T. Seaborg, Phys. Rev. 98, 1518 (1955).

here. The calculations are straightforward for both contributions. The electrostatic contribution was formulated in Chapter IV; the surface contribution will now be formulated, and then the results of the calculations will be quoted for both.

The surface energy E_s is proportional to the surface area:

$$E_s = O \cdot \iint dS$$

where O is the surface-energy density. The part of the surface energy due to distortion is

$$E_s - E_s^0 = O \cdot \iint dS - O \cdot 4\pi R_0^2 A^{2/3} \quad \text{where } E_s^0 \text{ is identifiable on the right. (The } 4\pi R_0^2 \cdot O \text{ is the } \gamma \text{ of the Weizsäcker formula.) The } dS \text{ is easily found, and}$$

$$\iint dS = 2\pi R_0^2 A^{2/3} \int_0^1 d\mu R^2(\mu) \cdot \left[1 + (1-\mu^2) \left\{ \frac{1}{R} \frac{dR}{d\mu} \right\}^2 \right]^{1/2}$$

where $R(\mu) = 1 + \sum_{n \neq 0} a_n P_n(\mu)$ is defined in Chapter IV.

The two contributions to the potential energy of distortions for the case of $a_n = 0$ for $n \neq 0, 2$ are²

$$E_s - E_s^0 = E_s^0 \left[0.4 a_2^2 - 0.03810 a_2^3 - 0.2171 a_2^4 + 0.0935 a_2^5 + 0.05316 a_2^6 - 0.08883 a_2^7 + 0.004357 a_2^8 + \dots \right]$$

$$E_c - E_c^0 = E_c^0 \left[-0.2 a_2^2 - 0.03810 a_2^3 + 0.1282 a_2^4 + 0.01603 a_2^5 - 0.04241 a_2^6 + 0.005794 a_2^7 + 0.002780 a_2^8 + \dots \right]$$

²R. D. Present and J. K. Knipp, Unpublished calculations.

With $2x = E_c^0/E_s^0$, this approximation to the potential energy of distortion $\Delta E'$ becomes

$$\begin{aligned} \Delta E' = E_s^0 a_z^2 \{ & 0.4(1-x) - 0.03810(1+2x)a_z - (0.2171 - 0.2564x)a_z^2 \\ & + (0.0935 + 0.03206x)a_z^3 + (0.05316 - 0.08482x)a_z^4 \\ & + (-0.08883 + 0.011588x)a_z^5 + (0.004357 + 0.005560x)a_z^6 \}. \end{aligned}$$

This $\Delta E'$ is identifiable with the V in the Gamow factor of Chapter III. The maximum of $\Delta E'$ is the activation energy E_f .³

The $\Delta E'(a_z)$ above should be the barrier to fission. It is going to be a higher barrier than the $\Delta E(a_n)$ saddle-path barrier (see Chapter II). To check this statement, one needs complete specification of $\Delta E'(a_z)$ and values for the maxima of both $\Delta E'(a_z)$ and $\Delta E(a_n)$. The $\Delta E'(a_z)$ is determined by fixing the values of E_s^0 and x . When γ (of the Weizsäcker formula) is 16 Mev and x for U^{239} is 0.74 ($x \propto Z^2/A$),⁴ E_s^0 and x for Mv^{256} are, respectively, 650 Mev and 0.83.

With the E_s^0 and x fixed, as above indicated, for

³Actually zero-point energies should be taken into account. The difference between the zero-point energy for very small distortions and that for the large-amplitude distortions is assumed to be small enough to be negligible. (See page 12.)

⁴These values of γ and x are from R. D. Present, *Nucleonics* 3, 25 (1948).

Mv^{256} , the maximum of $\Delta E'(a_z)$ is 6.4 Mev--this should be the activation energy, E_f , for this nuclide. This value is much larger than the maximum of $\Delta E(a_n)$ predicted by Reines⁵ whose results would give 2.2 Mev (for $X = 0.83$ and $E_s^0 = 650$ Mev.) Another comparison of the two maxima is possible: extrapolation of the results of Frankel and Metropolis⁶ places E_f for Mv^{256} between 1.5 and 2.5 Mev. The discrepancy between the maximum of $\Delta E'(a_z)$ and these maxima for $\Delta E(a_n)$ becomes only slightly smaller by omitting the a_z^8 th term from $\Delta E'(a_z)$: the resulting maximum is 5.9 Mev.

An evaluation of these activation-energy predictions must depend on experimental verification. The activation energy of Mv^{256} is not known; determinations of the activation energies of other nuclides have, however, been made, and extrapolation of these results can be used to estimate the activation energy of Mendelevium. According to the data assembled by Hill and Wheeler,⁷ the activation energy of Mendelevium, $E_f(Mv)$, should be 4.7 Mev; according to the data of Seaborg,⁸ $E_f(Mv)$ 4.8 Mev; and according

⁵Frederick Reines, "Nuclear Fission and the Liquid-Drop Model of the Nucleus" (Unpublished Ph. D. thesis, Department of Physics, New York University, December 1943), Figure 9.

⁶S. Frankel and N. Metropolis, Phys. Rev. 72, 914 (1947).

⁷D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).

⁸G. T. Seaborg, Phys. Rev. 88, 1429 (1952).

to Wheeler's formulae,⁹ $E_f(\text{MeV})$ might be 4.7, 4.9, or 5.0 Mev. These results might fix E_f , then, between 4.7 and 5.0 Mev. As the predictions of these experimental data lie midway between the predictions of $\Delta E(a_1)$ and $\Delta E'(a_2)$, neither of the two functions gives satisfactory values of E_f .

Because of the extreme sensitivity of the half-life to the height and width of the barrier, the Mv^{256} barrier to fission must have almost the correct height and shape if the half-life for spontaneous fission is to be even nearly correct. The barrier represented by $\Delta E'(a_2)$ cannot hope to meet this requirement unless its height is adjusted. Two methods of adjusting the barrier have been considered:

- (1) Adjust $\Delta E'(a_2)$ by taking from this power series in a_2 the terms in powers of a_2 less than or equal to a_2^k , and add to this a term κa_2^{k+1} :

$$\Delta E(a_2) = \Delta E'_k(a_2) + \kappa a_2^{k+1}$$

Now fix κ by demanding that the maximum of this $\Delta E(a_2)$ be equal to the value of E_f as predicted from experimental results.

- (2) Take κ to be an adjustable parameter, and choose it to make the maximum of $\Delta E'(a_2)$ equal to the predicted value of E_f .

The first method was unsatisfactory. For $5 \leq k \leq 8$ the

⁹J. A. Wheeler, "Nuclear Fission and Nuclear Stability," in W. Pauli (ed.), Niels Bohr and the Development of Physics (New York: McGraw-Hill, Inc., 1955), p. 175.

required values of K are too large, i.e., the correction is comparable to the $\Delta E'_n(a_2)$ for a part of the range of values taken on by a_2 in the barrier-penetration process. The barrier would be distorted by such corrections; its shape could not correspond to the physical phenomenon. (The calculations made with this method are summarized in Appendix III.) The second method of adjusting the barrier proved satisfactory. Calculations were made for $X = 0.82, 0.83, \text{ and } 0.845$. These three values covered the range of X -values needed, and interpolations between calculations were used to get the desired predictions.

The prediction of the spontaneous-fission half-life requires the use of the Gamow factor of Chapter III. This requires the identification of, and the determination of values for, quantities occurring in this factor. The $U-E$ in the Gamow factor is equal to the $\Delta E'(a_2)$ just discussed--with the adjustment of X . The $\mathcal{M}$ of the Gamow factor is taken from Chapter V. The α_1 and α_2 --the limits on the integral in the Gamow factor--remain to be fixed. The transformation from nucleon coordinates to deformation parameters of a continuous medium, made at the end of Chapter III, shows that for the continuous medium α has the dimension of length. In the present form α becomes $R_0 a_2$, where R_0 is the radius of the undistorted drop representing the Mendelevium nucleus--it is the unit of length used in the previous chapters. The α_1 becomes $R_0 a_2 = R_0 \cdot 0 = 0$; the α_2 becomes

$R_0 a_2 = R_0 a$ where a is the second zero of $\Delta E'(a_2)$. The value of R_0 is fixed by letting $R_0 = n_0 A^{1/3}$ and choosing n_0 as¹⁰ 1.30×10^{-13} cm. The Gamow factor is now completely determined.

The Gamow factor does not completely determine the half-life for spontaneous fission. It is the probability of penetration of the barrier each time the system moves in the direction of the barrier. For small a_2 the system behaves like an harmonic oscillator with amplitude a_2 and with a characteristic frequency $\omega_2/2\pi$. If G is the Gamow penetration factor, then the probability-per-unit-time of spontaneous fission is $(\omega_2/2\pi) G$, and the half-life for spontaneous fission is $(\ln 2)/(2\pi/\omega_2) G^{-1}$. The expression for ω_2 given by Bohr and Wheeler¹¹ is

$$\omega_2 = 2 \left\{ \frac{2\gamma(1-x)}{3MA n_0^2} \right\}^{1/2}.$$

(All the symbols have already been defined.) For Mv^{256} this is equal to $1.27 \times 10^{21} \text{ sec}^{-1}$. Thus the expression for the half-life is

$$\tau_{1/2} = \frac{2\pi \ln 2}{1.27 \times 10^{21} \text{ sec}^{-1}} G^{-1}.$$

The evaluation of the half-life requires the evaluation of the Gamow integral for which the integrand is $\sqrt{m \Delta E}$ --the square root of a polynomial. The ΔE is a_2^2 times a poly-

¹⁰The value comes from R. D. Present, loc. cit.

¹¹N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).

nomial in a_2 ; $\mathcal{M}$ begins with a constant term. So

$$\sqrt{\mathcal{M} \Delta E} = a_2 \sqrt{\mathcal{M} (\Delta E/a_2^2)}.$$

As $\mathcal{M}$ is known only through the a_2^4 term,¹² the function $\Delta E/a_2^2$ is used through the a_2^4 term only--or, ΔE is used in the integral through the a_2^6 term; the higher terms are dropped.¹³ The resulting problem of integration is solved by "extracting the square root"--in this sense: the polynomial

$f(a_2) = \sum_{l=0}^4 c_l a_2^l$ is found for which $[f(a_2)]^2$ is equal to $\mathcal{M}(\Delta E/a_2^2)$ through the a_2^4 term. Then $\sqrt{\mathcal{M} \Delta E/a_2^2}$ is replaced by $f(a_2)$ which is integrable.

The process of calculation of the half-life for spontaneous fission, the $\tau_{1/2}$, has now been fixed. Two variations have been made in the process described:

- (1) Calculations made with the effective mass $\mathcal{M}$ which includes the a_2^4 term (see Chapter V) have been repeated after dropping the a_2^4 term. These two sets of calculations have been distinguished by using labels $\mathcal{M}(a_2^4)$ to indicate the former $\mathcal{M}$ and $\mathcal{M}(a_2^3)$ to indicate the latter.
- (2) Calculations made with the $\Delta E'(a_2)$ from above have been repeated after dropping the a_2^6 th term, and, in some

¹²The calculation of additional terms is algebraically very laborious.

¹³The higher terms have been used in evaluating the upper limit of the Gamow integral.

cases, again repeated after dropping the a_2^7 th term.

Tables II, III, and IV give the results of the calculations for $x = 0.82, 0.83, \text{ and } 0.845$. In these tables the column headed "last term used" gives a quantity which when multiplied by E_f^0 is the last term used in the power series for $\Delta E'(a_2)$. The $(\Delta E)_{\max}$ is the maximum value of $\Delta E'(a_2)$, and $(a_2)_{\max}$ is the value of a_2 for which this occurs. The a is the non-zero root of $\Delta E'(a_2) = 0$. $\tau_{1/2}$ is the half-life, and the $\mathcal{M}(a_2^4)$ and $\mathcal{M}(a_2^3)$ have the meaning given them above.

By comparing corresponding entries from the three tables, one can see the tremendous influence produced in the predicted half-life by a small change in x .

Table V contains the results of interpolating the results of Tables II, III, and IV--in the process of adjusting the x -value to make E_f take on a chosen value (see the discussion above). With the exception of the first and third columns in Table V, all quantities in Table V appear in the preceding three tables. The first column gives the value of E_f accepted for the adjustment process. (The range of 4.7 to 5.0 Mev embraces the predicted values.) The third column gives the value of x necessary for making the maximum of $\Delta E'(a_2)$ equal to the corresponding E_f of the first column.

Some of the calculations presented in Tables II, III, and IV have been repeated after replacing $\mathcal{M}$ by the incorrect Frankel-Metropolis expression for $\mathcal{M}$. (The results of these calculations are presented in tabular form in Appendix III.)

TABLE II

CONSTANTS OF THE BARRIER AND SPONTANEOUS-FISSION HALF-LIFE
WHEN $x = 0.82$

Last Term Used	$(AE)_{\max}$	$(a_2)_{\max}$	a	$\tau_{1/2} [m(a_2^+)]$	$\tau_{1/2} [m(a_2^3)]$
-0.016392 a_2^6 - (No zero occurs short of $a_2 = 1.00$.)					
-0.079328 a_2^7	7.1 Mev	0.72	0.94	$10^{4.9}$ sec (1 day)	$10^{5.5}$ sec (4 days)
+0.008916 a_2^8	7.6 Mev	0.76	0.98	$10^{7.8}$ sec (2 yrs)	$10^{8.7}$ sec (16 yrs)

TABLE III

CONSTANTS OF THE BARRIER AND SPONTANEOUS-FISSION HALF-LIFE
WHEN $x = 0.83$

Last Term Used	$(\Delta E)_{\max}$	$(a_2)_{\max}$	a	$\tau_{1/2} [m(a_2^4)]$	$\tau_{1/2} [m(a_2^3)]$
-0.01724 a_2^6	- (No zero occurs short of $a_2 = 1.00..$)				
-0.07921 a_2^7	5.9 Mev	0.71	0.92	$10^{2.3}$ sec (3½ min)	$10^{2.9}$ sec (13 min)
+0.008972 a_2^8	6.4 Mev	0.75	0.97	$10^{5.7}$ sec (6 days)	$10^{6.5}$ sec (40 days)

TABLE IV

CONSTANTS OF THE BARRIER AND SPONTANEOUS-FISSION HALF-LIFE
WHEN $x = 0.845$

Last Term Used	$(\Delta E)_{\max}$	$(a_z)_{\max}$	a	$\tau_{1/2} [m(a_z^4)]$	$\tau_{1/2} [m(a_z^3)]$
-0.01851 a_2^6 - (No zero occurs short of $a_z = 1.00$.)					
-0.07904 a_2^7	4.4 Mev	0.68	0.89	$10^{-1.5}$ sec	$10^{-0.8}$ sec
+0.009055 a_2^8	4.7 Mev	0.72	0.93	$10^{0.7}$ sec (5 sec)	$10^{1.7}$ sec (1 min)

TABLE V See Errata at front of Dissertation
preceding Table of Contents

CONSTANTS OF THE BARRIER AND SPONTANEOUS-FISSION HALF-LIFE
WHEN E_f DETERMINES x

E_f (Mev)	Last Term Used	x	$(a_z)_{max}$	a	$\tau_{1/2} [m(a_z^+)]$	$\tau_{1/2} [m(a_z^-)]$
4.7	a_2^7	0.842	0.685	0.895	$10^{-0.8}$ sec (1/10 sec)	$10^{-0.1}$ sec (1 sec)
4.7	a_2^8	0.845	0.72	0.93	$10^{0.7}$ sec (5 sec)	$10^{1.7}$ sec (50 sec)
5.0	a_2^7	0.839	0.695	0.901	$10^{0.0}$ sec (1 sec)	$10^{0.6}$ sec (4 sec)
5.0	a_2^8	0.842	0.725	0.94	$10^{1.8}$ sec (1 min)	$10^{2.6}$ sec ($6\frac{1}{8}$ min)

The change produced in the predicted half-life by the change from $\mathcal{M}(a_2^+)$ to $\mathcal{M}(\text{FM})$ is roughly equal to the change produced by changing from $\mathcal{M}(a_2^+)$ to $\mathcal{M}(a_2^3)$. There is always an increase in half-life; the Frankel-Metropolis $\mathcal{M}$ gives a half-life five to eight times as great as that given by the $\mathcal{M}(a_2^+)$.

The tables show a number of significant points:

- (1) Small changes in χ can produce remarkably large changes in the predicted activation energy and in the predicted half-life.
- (2) The values of $(a_2)_{\text{max}}$ are very large. (Compare these $(a_2)_{\text{max}}$ with the saddle-point a_2 in Table I.)
- (3) The values of a are quite large for a power-series development.¹⁴ (For example: $(0.9)^8$ is 0.43.)
- (4) Replacing $\mathcal{M}(a_2^+)$ by $\mathcal{M}(a_2^3)$ increases the predicted half-life by as much as a factor of ten.
- (5) Neglecting the a_2^8 th term in $\Delta E'(a_2)$ shortens the predicted half-life considerably; the changed prediction is about one-fiftieth of the original prediction for the χ -values of interest.

The evaluation of the results presented in this chapter will be attempted in Chapter VII.

¹⁴The coefficients in the power series have not been shown to decrease in order to assure convergence. Convergence of the series depends on the decreasing size of higher powers of the variable a_2 .

CHAPTER VII

EVALUATION OF RESULTS AND CONCLUSIONS

The problem of comparing the predictions of the liquid-drop model with empirical results for Mv^{256} --or any nucleus, for that matter--is subject to some uncertainty. A rigorous calculation would require the evaluation of the contributions of the many types of distortions (represented by the a_n 's) and justification of the omission of all but a small finite number of these distortions. Solution of the general problem would seem to require some sort of electronic computer. When the χ -value for the nuclide is near one, however, few of the a_n distortions should be required, and one would think that the power-series expansions would converge rapidly. The results of the calculations which were given in Chapter VI allow a comparison of experimental evidence with predictions based on such simplified calculations.

Consider first the convergence of the power-series for the potential energy of deformation. When one has few terms in this series, it will converge rapidly if the values of a_2 required remain small. This does not occur. The values of a (see Chapter VI) are quite large for series methods. These large values of a_2 in the penetration process make one skeptical of the $\Delta E(a_2)$ convergence. (The slow convergence with large a_2 is somewhat ameliorated in the evaluation of the half-life: in the integration a_2^8 integrates as $\frac{1}{9} a^9$, and

the coefficient caused by integration helps the convergence.)

The adjustment of the barrier through the choice of κ seems to be essential. The κ , itself, for any given nuclide, is somewhat uncertain. It is inversely proportional to the γr_0 , both of which are somewhat uncertain. Values of r_0 may be found to vary from ~ 1.2 to ~ 1.5 (in units of 10^{-13}cm), and values of γ used by various authors range from ~ 14 to ~ 18 Mev. The product of the two, γr_0 , is not so variable; values range from 20.7 to 21.7 (in units corresponding to the separate units). The change produced in κ when it was modified to adjust the barrier was small in comparison with uncertainty in κ resulting from uncertainty in γr_0 .¹ From this viewpoint, then, the adjustment is satisfactory. The adjustment is satisfactory from a second viewpoint: the change of κ leaves the barrier a true liquid-drop potential-energy barrier--the change of the potential energy of distortion with a_2 represents an actual physical phenomenon. (This was not the case with the other method of adjusting the barrier as it distorted the barrier considerably for

¹If one considers values of γr_0 that correspond to various κ values for Mv^{256} , he finds for $\kappa = 0.83$: $\gamma r_0 = 20.7$, for $\kappa = 0.845$: $\gamma r_0 = 20.4$, and for $\kappa = 0.77$: $\gamma r_0 = 22.3$. (Units for γr_0 are Mev $\times (10^{-13}\text{cm})$.) The value of κ , $\kappa = 0.77$, is roughly the value needed to cause the saddle-path maximum of Reines' $\Delta E(a_n)$ to become 4.7 to 5.0 Mev.

Frederick Reines, "Nuclear Fission and the Liquid-Drop Model of the Nucleus" (Unpublished Ph. D. thesis, Department of Physics, New York University, December 1943), Fig. 9.

larger values of a_2 .)

It should be noted that a single parameter is used in adjusting the barrier; the adjustment changes the height to meet a fixed value, and other changes in the barrier are a consequence of this change--not changes introduced by further parameterization.

If the liquid-drop model is correct, the true barrier to fission is the $\Delta E(a_n)$ for the saddle-point path. There is uncertainty in χ , so $\Delta E(a_n)_{\chi=0.83}$ may not be the true barrier for Mv^{256} . The value of χ might vary as much as the variation of γR_0 would demand ($\chi \propto 1/\gamma R_0$). The predicted value of the maximum of the $\Delta E(a_n)$ barrier is proportional to γ , which is more uncertain than χ , so the values of the maximum of the $\Delta E(a_n)_{\chi=0.83}$ considered in Chapter VI may deviate from the activation energy for Mv^{256} due to errors in both χ and γ . The position of the maximum of the $\Delta E(a_n)$ is uncertain because of the uncertainty of χ alone. Using the adjusted $\Delta E(a_2)$ -barrier of this thesis changes both the position and the value of the maximum of the liquid-drop barrier. The position change should cause minor deviations in the shape of the barrier: the saddle points of Table I indicate that a_4 is small and a_6 negligible for the saddle point when $\chi \sim 0.83$. The change in the value of the maximum of the barrier is demanded by experimental evidence. The author feels that the adjusted $\Delta E(a_2)$ -barrier should be a close approximation to the true, liquid-

drop barrier.

The values obtained for the half-life are considered satisfactory; an inaccurate representation of the barrier can lead to very large errors in this prediction (see below)--Reines,² for example, obtained 10^{-1} sec for the spontaneous-fission half-life of U^{238} (experiment gives 10^{23} sec). The major variations possible in the predicted half-life occur in the Gamow factor. The calculations of half-lives can hardly do better than give the right power of ten when the half-life is written as 10^l sec.

There are only two uncertainties possible in the Gamow factor-- $\mathcal{M}$ and ΔE . The $\mathcal{M}$ used in this thesis may be wrong by enough to cause an error of a factor of ten in the half-life. This is suggested in Table V by the change in half-life in going from $\mathcal{M}(a_z^3)$ to $\mathcal{M}(a_z^4)$. It was noted in Chapter VI that the omission of the eighth-power term in $\Delta E(a_z)$ could change the predicted half-life by roughly a factor of fifty. It is thus possible that the slow convergence of $\mathcal{M}$ and ΔE might cause an error in the half-life of a factor of 10^3 .

In considering the results of Chapter VI, one should keep in mind some of the limitations of the liquid-drop model itself. The model is understandable when interpreted by the Weizsäcker formula, as seen in Chapter I; it has also the

²Frederick Reines, op. cit., p. 20.

same sort of limitations. No account of shell effects or of odd-even effects is taken by the model. It can thus hope to agree with the general trend of experimental results, but not to agree with nuclides that deviate considerably from the trend. Odd-even effects can cause changes in activation energies and bring about changes in half-lives as great as a factor of $10^3 - 10^{5.3}$. It is also possible that the ground state of the nucleus corresponds to a non-spherical shape; this could cause deviations from predictions of the model. Neither of these effects should matter for an adjusted barrier such as the one used in this thesis.

Table V gives the results of applying the simplified, liquid-drop calculations to Mv^{256} . The half-life predictions vary from $10^{-0.8}$ sec to $10^{2.6}$ sec (experimental value: $10^{4.4}$ sec). By including the a_2^8 th term in $\Delta E(a_2)$, one multiplies the predicted half-life by ~ 50 ; by including the a_2^4 th term in M , he multiplies the predicted half-life by $\sim 1/10$. Increasing the assumed activation energy from 4.7 to 5.0 Mev increases the predicted half-life by a factor of ~ 10 . The varied κ is nearly 0.84 for all barriers considered; $(a_2)_{max}$ and a are also changed little from one barrier to another, being ~ 0.7 and ~ 0.9 , respectively, for all.

It seems to the author that Mv^{256} is quite close to

³W. J. Whitehouse and W. Galbraith, Nature 169, 494 (1952).

those nuclei which can be treated with the much simplified liquid-drop approximations and that experimental results for nuclei with slightly larger χ -values might agree quite well with predictions based on these simplified calculations.



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APPENDIX

APPENDIX I

THREE LEMMAS

In Chapter IV there occur three integrals which are easy to evaluate. The three lemmas to follow make these evaluations immediate. For the three lemmas two equations are needed. One is for a summation; the other is for an integration. The summation will first be performed; then the integration will be done. Finally the three lemmas will be stated and proved.

The recurrence formula for the Legendre polynomials is¹

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

Using this relationship twice, one can write

$$(2n+1)x^2P_n(x) = \frac{(n+1)(n+2)}{2n+3}P_{n+2}(x) + \left[\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1} \right]P_n(x) + \frac{n(n-1)}{2n-1}P_{n-2}(x)$$

$$(2n+1)y^2P_n(y) = \frac{(n+1)(n+2)}{2n+3}P_{n+2}(y) + \left[\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1} \right]P_n(y) + \frac{n(n-1)}{2n-1}P_{n-2}(y)$$

Upon multiplying the first of these by $P_n(y)$ and the second by $P_n(x)$ and subtracting the latter result from the former, one obtains

$$\begin{aligned} (2n+1)(x^2-y^2)P_n(x)P_n(y) = & \frac{(n+1)(n+2)}{2n+3} [P_{n+2}(x)P_n(y) - P_{n+2}(y)P_n(x)] \\ & - \frac{(n-1)n}{2n-1} [P_n(x)P_{n-2}(y) - P_n(y)P_{n-2}(x)]. \end{aligned}$$

¹A good reference book for those aspects of Legendre polynomials used in this thesis is

T. M. MacRobert, Spherical Harmonics (2nd revised ed.; New York: Dover Publications, Inc., 1948).

The desired summation

$$\sum_{\substack{n=0 \\ \text{even}}}^N (2n+1)(x^2-y^2) P_n(x) P_n(y) = \frac{(N+1)(N+2)}{2N+3} [P_{N+2}(x) P_N(y) - P_{N+2}(y) P_N(x)]$$

follows immediately.²

The integral needed is $\int_0^1 dx P_n(x) P_m(x)$ where m is odd and n is even. The $P_m(x)$ is replaced by its equivalent (from Legendre's equation) $\frac{-1}{m(m+1)} \frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_m(x) \right]$.

Then

$$\begin{aligned} -m(m+1) \int_0^1 dx P_n(x) P_m(x) &= \int_0^1 dx P_n(x) \frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_m(x) \right] \\ &= P_n(x) \cdot (1-x^2) \frac{d}{dx} P_m(x) \Big|_0^1 - \int_0^1 dx (1-x^2) \frac{d}{dx} P_m(x) \cdot \frac{d}{dx} P_n(x) \end{aligned}$$

$$\begin{aligned} \text{and } m(m+1) \int_0^1 dx P_n(x) P_m(x) &= P_n(0) P_m'(0) + \int_0^1 dx \frac{d}{dx} P_m(x) \cdot (1-x^2) \frac{d}{dx} P_n(x) \\ &= P_n(0) P_m'(0) + P_m(x) \cdot (1-x^2) \frac{d}{dx} P_n(x) \Big|_0^1 \\ &\quad - \int_0^1 dx P_m(x) \frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_n(x) \right] \end{aligned}$$

So

$$m(m+1) \int_0^1 dx P_n(x) P_m(x) = P_n(0) P_m'(0) + n(n+1) \int_0^1 dx P_m(x) P_n(x)$$

$$\text{and } \int_0^1 dx P_n(x) P_m(x) = \frac{P_n(0) P_m'(0)}{[m(m+1) - n(n+1)]}.$$

²It is also immediate that one can form the summation for $n=1$, odd, to N , odd; the answer is the same, except for N 's being odd. These summations are similar to Christoffel's first summation formula (MacRobert, p. 100):

$$\sum_{\substack{n=0 \\ \text{integral}}}^N (2n+1)(x-y) P_n(x) P_n(y) = (N+1) \{ P_{N+1}(x) P_N(y) - P_N(x) P_{N+1}(y) \}$$

Finally

$$\int_0^1 dx P_n(x) P_m(x) = (-1)^{\frac{n+m+1}{2}} \frac{m}{(n-m)(n+m+1)} \left\{ \frac{n!}{[2^{\frac{n}{2}} (\frac{n}{2})!]^2} \right\} \left\{ \frac{(m-1)!}{[2^{\frac{m-1}{2}} (\frac{m-1}{2})!]^2} \right\}.$$

This is the integral needed for the lemmas.

Lemma 1: When $F(x, y)$ is an even-multiple power series of x and y with only finite powers of either occurring (i.e.,

$$F(x, y) = \sum_{\mu}^M \sum_{\lambda}^{\Lambda} a_{\lambda\mu} x^{2\lambda} y^{2\mu} \text{ with } M \text{ and } \Lambda \text{ finite.})$$

$\mu, \lambda \text{ integral } \geq 0$

then

$$\int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1)(x^2-y^2) P_n(x) P_n(y) \cdot F(x, y) = 0$$

Proof: Replace the infinite sum by a limit process

$$\lim_{N \rightarrow \infty} \int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^N (2n+1)(x^2-y^2) P_n(x) P_n(y) \cdot F(x, y),$$

Perform the summation above, and look at one term only.

$$\frac{(N+1)(N+2)}{2N+3} \int_0^1 dx P_{N+2}(x) \int_0^x dy P_N(y) \cdot F(x, y).$$

Now let $F(x, y)$ be written as a multiple-power series in Legendre polynomials (only even ones occur)

$$F(x, y) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} A(l, m) P_{2l}(x) P_{2m}(y)$$

Look at one term from this series

$$\begin{aligned} & \frac{(N+1)(N+2)}{2N+3} \int_0^1 dx P_{N+2}(x) \int_0^x dy P_N(y) \cdot A(j, k) P_{2j}(x) P_{2k}(y) \\ &= \frac{(N+1)(N+2)}{2N+3} A(j, k) \int_0^1 dx P_{N+2}(x) P_{2j}(x) \int_0^x dy P_N(y) P_{2k}(y) \end{aligned}$$

Of the many terms arising when one removes the products of

polynomials (via $P_N(x) P_m(x) = \sum_{l=N-m}^{N+m} a_l P_l(x)$,

consider only one

$$\frac{(N+1)(N+2)}{2N+3} A \cdot B \int_0^1 dx P_N(x) \int_0^x dy P_{N'}(y).$$

Both N and N' are even; each is nearly the size of N :

$$N+2-2j \leq N \leq N+2+2j ; \quad N-2k \leq N' \leq N+2k.$$

B is a function of zero degree in N for large N (the recurrence formula is $x P_N(x) = \frac{N+1}{2N+1} P_{N+1} + \frac{N}{2N+1} P_{N-1}$).

Now perform the y -integration:

$$\frac{(N+1)(N+2)}{2N+3} A \cdot B \cdot \frac{1}{2N'+1} \int_0^1 dx P_N(x) \cdot [P_{N'+1}(x) - P_{N'-1}(x)].$$

Consider now one term here:

$$\frac{(N+1)(N+2)}{2N+3} A \cdot B \frac{1}{2N'+1} \int_0^1 dx P_N(x) P_{N'+1}(x)$$

How does this behave as $N \rightarrow \infty$?

The integration of $\int_0^1 dx P_{\text{even}}(x) P_{\text{odd}}(x)$ gives the behavior of the integral. The coefficient of the integral converges to a constant value as N^2 occurs for both the denominator and the numerator. ($N' = N + \ell$ with ℓ finite.) The behavior of the term for large N is the same as the behavior of $\int_0^1 dx P_N(x) P_m(x)$ with m odd,

N even and with $m = N + \alpha$. This behavior is

$$\lim_{N \rightarrow \infty} \frac{N + \alpha}{\alpha (2N)} \left\{ \frac{N!}{[2^{\frac{N}{2}} (\frac{N}{2})!]^2} \right\}^2.$$

or $\sum_{n \rightarrow \infty} n^{-1}$ which is zero in the limit.. (Stirling's formula for large N is $N! = \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$.)

The term which has been considered converges to zero. There are many such terms; their number is, however, finite. Thus each term approaches zero as N^{-1} , and the number of terms is finite; the original integral is thus zero, and the lemma has been proved.

Lemma 2: When $Q(x, y)$ meets the same demands as $F(x, y)$ of lemma 1

$$\int_0^1 dx \int_0^x dy \sum_{n=0}^{\infty} (2n+1) n(n+1) P_n(x) P_n(y) \cdot (x^2 - y^2)^3 Q(x, y) = 0.$$

Proof: Perform some partial integrations:

$$\begin{aligned} & \int_0^x dy n(n+1) P_n(y) \cdot (x^2 - y^2)^3 \cdot Q(x, y) \\ &= - \int_0^x dy \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} P_n(y) \right] \cdot (x^2 - y^2)^3 Q(x, y) \\ &= - (1-y^2) \frac{d}{dy} P_n(y) \cdot (x^2 - y^2)^3 Q(x, y) \Big|_0^x + \int_0^x dy \frac{d}{dy} P_n(y) \cdot (1-y^2) \frac{d}{dy} \left[(x^2 - y^2)^3 Q(x, y) \right] \end{aligned}$$

(The integrated part is obviously zero.)

$$\begin{aligned} &= \int_0^x dy \frac{d}{dy} P_n(y) \cdot (1-y^2) \frac{d}{dy} \left[(x^2 - y^2)^3 Q(x, y) \right] \\ &= P_n(y) \cdot (1-y^2) \frac{d}{dy} \left[(x^2 - y^2)^3 Q(x, y) \right] \Big|_0^x - \int_0^x dy P_n(y) \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2 - y^2)^3 Q(x, y) \right\} \right] \end{aligned}$$

For the integrated part the upper limit is obviously zero; the lower limit is also zero: $(x^2 - y^2)^3 Q(x, y)$ is even with respect to the operation of replacing y by $-y$. The

differentiation creates a function which is odd. At zero the function vanishes. Thus

$$\int_0^x dy \, n(n+1) P_n(y) \cdot (x^2 - y^2)^3 Q(x, y) = \\ - \int_0^x dy \, P_n(y) \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \{ (x^2 - y^2)^3 Q(x, y) \} \right].$$

Now this integrand has a factor $(x^2 - y^2)$: there occur a maximum of two y -differentiations for $(x^2 - y^2)^3$.

$\frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \{ (x^2 - y^2)^3 Q(x, y) \} \right]$ satisfies the demands on $F(x, y)$ of Lemma 1. So

$$\int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1)n(n+1) P_n(x) P_n(y) \cdot (x^2 - y^2)^3 Q(x, y) \\ = - \int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1) P_n(x) P_n(y) \cdot (x^2 - y^2) \cdot F^*(x, y)$$

which by Lemma 1 is zero. The lemma is thus proved.

Lemma 3: When $Q(x, y)$ meets the same demands as $F(x, y)$ of lemma 1

$$\int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1)n(n+1)(n-1)(n+2) P_n(x) P_n(y) \cdot (x^2 - y^2)^5 Q(x, y) = 0$$

Proof: Replace $(n-1)(n+2)$ by $[n(n+1) - 2]$.

$$\int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1)n(n+1)[n(n+1) - 2] P_n(x) P_n(y) \cdot (x^2 - y^2)^5 Q(x, y) \\ = \int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1)n(n+1)(n)(n+1) P_n(x) P_n(y) \cdot (x^2 - y^2)^5 Q(x, y).$$

(The -2 contributes nothing: all requirements of Lemma 2 are met for its contribution.)

Using the partial integrations performed in Lemma 2, one can write instead of this last form

$$= \int_0^x dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1) P_n(x) \left\{ -n(n+1) P_n(y) \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \right\}.$$

(The conditions from Lemma 2 for the disappearance of the parts integrated also obtain here.) Now perform the partial integration of

$$\begin{aligned} & \int_0^x dy \ (-1)(n)(n+1) P_n(y) \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right]. \\ &= \int_0^x dy \ \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} P_n(y) \right] \cdot \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \\ &= (1-y^2) \frac{d}{dy} P_n(y) \cdot \frac{d}{dy} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \Big|_0^x \\ &\quad - \int_0^x dy \ \frac{d}{dy} P_n(y) \cdot (1-y^2) \frac{d^2}{dy^2} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \end{aligned}$$

The integrated part is zero: lower limit, $P_n'(0) = 0$; upper, (x^2-y^2) is a factor of the term. One integrates the remaining part to get

$$\begin{aligned} & -P_n(y) \cdot (1-y^2) \frac{d^2}{dy^2} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \Big|_0^x \\ & + \int_0^x dy \ P_n(y) \frac{d}{dy} \left\{ (1-y^2) \frac{d^2}{dy^2} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \right\}. \end{aligned}$$

The integrated part is an odd function of y ; the lower limit is zero. Also (x^2-y^2) is a factor of this part, so the upper

limit is zero.

The integrations have shown that

$$\begin{aligned} & \int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1) n(n+1)(n-1)(n+2) P_n(x) P_n(y) \cdot (x^2-y^2)^5 Q(x,y) \\ &= \int_0^1 dx \int_0^x dy \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (2n+1) P_n(x) P_n(y) \frac{d}{dy} \left\{ (1-y^2) \frac{d^2}{dy^2} \left[(1-y^2) \frac{d}{dy} \left\{ (x^2-y^2)^5 Q(x,y) \right\} \right] \right\}. \end{aligned}$$

As (x^2-y^2) is a root of the $\frac{d}{dy} \left\{ \frac{d}{dy} \right\}$, the factor satisfies the demands made on $(x^2-y^2) F(x,y)$ in Lemma 1. Lemma 1 thus establishes the validity of Lemma 3.

APPENDIX II

TABULATION OF INTEGRALS OCCURRING IN CHAPTER V

(The general form for the definite integral is

$$\int_1^x F(\mu) d\mu .)$$

F	value
P_2^2	$\frac{2}{5}$
P_2^3	$\frac{4}{35}$
P_2^4	$\frac{6}{35}$
P_2^5	$\frac{8}{7 \cdot 11}$
P_2^6	$\frac{106}{7 \cdot 11 \cdot 13}$
$P_2^2 P_4$	$\frac{4}{5 \cdot 7}$
$P_2^3 P_4$	$\frac{24}{5 \cdot 7 \cdot 11}$
$P_2^4 P_4$	$\frac{408}{5 \cdot 7 \cdot 11 \cdot 13}$
$P_2^3 P_6$	$\frac{36}{7 \cdot 11 \cdot 13}$

F	value
P_4^2	$\frac{2}{9}$
$P_2 P_4^2$	$\frac{40}{7 \cdot 9 \cdot 11}$
$P_2^2 P_4^2$	$\frac{3578}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}$
$P_2 P_4 P_6$	$\frac{10}{11 \cdot 13}$
P_6^2	$\frac{2}{13}$

F	value
$P_{2,1} P_{2,1}$	$\frac{12}{5}$
$P_2 P_{2,1} P_{2,1}$	$\frac{12}{5.7}$
$P_2^2 P_{2,1} P_{2,1}$	$\frac{12}{5.7}$
$P_2^3 P_{2,1} P_{2,1}$	$\frac{12}{7.11}$
$P_2^4 P_{2,1} P_{2,1}$	$\frac{636}{5.7 \cdot 11 \cdot 13}$
$P_{2,1} P_{4,1}$	0
$P_2 P_{2,1} P_{4,1}$	$\frac{8}{7}$
$P_2^2 P_{2,1} P_{4,1}$	$\frac{32}{7.11}$
$P_2^3 P_{2,1} P_{4,1}$	$\frac{408}{7.11 \cdot 13}$
$P_{2,1} P_{6,1}$	0
$P_2 P_{2,1} P_{6,1}$	0
$P_2^2 P_{2,1} P_{6,1}$	$\frac{72}{11 \cdot 13}$

F	value
$P_{4,1} P_{4,1}$	$\frac{40}{9}$
$P_2 P_{4,1} P_{4,1}$	$\frac{680}{7.9 \cdot 11}$
$P_2^2 P_{4,1} P_{4,1}$	$\frac{12,200}{7.9 \cdot 11 \cdot 13}$
$P_{4,1} P_{6,1}$	0
$P_2 P_{4,1} P_{6,1}$	$\frac{280}{11 \cdot 13}$
$P_{6,1} P_{6,1}$	$\frac{84}{13}$

APPENDIX III

On pages 30 and 98 was discussed a method of adjusting the potential energy of distortion different from the one used in Chapter VI. Some calculations were made with this method of adjusting the barrier; the results were unsatisfactory. Tables VI, VII, and VIII show the results of these calculations. The first and second entries in these tables are readily identified by the $\Delta E(a_z)$ which is applicable to this method (see page 98)

$$\Delta E(a_z) = \Delta E'_{\kappa}(a_z) + \kappa a_z^{\kappa+1}.$$

The third, fourth, and fifth entries in these tables have the meanings given them in Tables II through V. The last entry is the half-life calculated by using the incorrect, Frankel-Metropolis M .

Note that a in these tables is very large for power-series methods. The undesirability of this method of correcting the potential-energy barrier is apparent when one compares E_f/E_s° with κ/E_s° from the tables: E_f/E_s° is 0.0077 (when E_s° is 650 Mev and $E_f = 5.0$ Mev), and κ/E_s° is larger than this value. As a_z can be large, the correction is quite large. The method is entirely unsatisfactory.

TABLE VI

CONSTANTS OF THE PARAMETERIZED BARRIER AND SPONTANEOUS-FISSION
 HALF-LIFE WHEN $E_f = 5.0$ Mev AND $x = 0.83$

k	κ/E_s^0	$(a_2)_{max}$	a	$\tau_{1/2} [m(a_z^+)]$	$\tau_{1/2} [m(fm)]$
5	-0.084298	0.69	0.98	$10^{2.2}$ sec ($2\frac{1}{2}$ min)	$10^{2.4}$ sec (4 min)
6	- .102132	.63	.84	$10^{-1.9}$ sec	$10^{-1.3}$ sec
7	- .036565	.62	.82	$10^{-2.8}$ sec	$10^{-2.3}$ sec
8	- .073464	.62	.80	$10^{-3.7}$ sec	$10^{-3.2}$ sec

TABLE VII

CONSTANTS OF THE PARAMETERIZED BARRIER AND SPONTANEOUS-FISSION
 HALF-LIFE WHEN $E_f = 4.8$ Mev AND $\alpha = 0.83$

k	κ/E_s°	$(a_2)_{\max}$	a	$\tau_{1/2} [\mathcal{M}(a_2^+)]$	$\tau_{1/2} [\mathcal{M}(FM)]$
5	-0.088465	0.66	0.935	$10^{-0.2}$ sec ($\frac{1}{2}$ sec)	$10^{0.2}$ sec ($1\frac{1}{2}$ sec)

TABLE VIII

CONSTANTS OF THE PARAMETERIZED BARRIER AND SPONTANEOUS-FISSION
 HALF-LIFE WHEN $E_f = 5.0 \text{ Mev}$ AND $\chi = 0.82$

k	κ/E_s^0	$(a_2)_{\max}$	a	$\tau_{1/2} [m(a_2^+)]$	$\tau_{1/2} [m(FM)]$
7	-0.121791	0.56	0.74	$10^{-5.4} \text{ sec}$	$10^{-4.9} \text{ sec}$
8	-0.234070	0.55	0.72	$10^{-6.2} \text{ sec}$	$10^{-5.7} \text{ sec}$