

March 16, 1959

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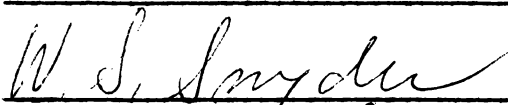
I am submitting herewith a thesis written by Rufus Haynes Ritchie entitled "On the Interaction of Charged Particles with Plasma." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

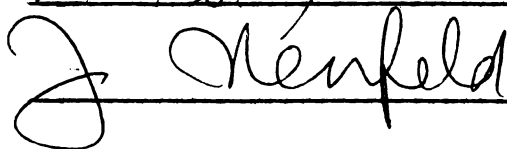

Major Professor

We have read this thesis and
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Accepted for the Council:


Dean of the Graduate School

ON THE INTERACTION OF CHARGED PARTICLES WITH PLASMA

A THESIS

Submitted to
The Graduate Council
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy

by

Rufus Haynes Ritchie

March, 1959

28
33a

ACKNOWLEDGMENTS

The author owes much to the kind encouragement and cooperation of the following people in numerous conversations on the material presented here: Dr. R. D. Present, who supervised this thesis; Dr. J. Neufeld, with whom the author worked closely on much of the material of Chapter II; Dr. David Pines, for criticism and helpful suggestions concerning the material on thin foils; Dr. R. D. Birkhoff and Mr. A. Y. Sakakura for helpful conversations on the material in Chapters III and IV.

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CHAPTER I

INTRODUCTION

The last several years have seen great activity in the study of the properties of the "free electron gas" or plasma. An electronic plasma is understood to mean an assembly of electrons which may be regarded as nearly free in their response to a disturbance. The assembly is assumed to be electrically neutral on the whole, due to a stationary positive charge background which will be assumed uniform in density. These electrons interact with each other via the long-range coulomb potential. Due to the nature of this interaction, the system exhibits a very interesting collective behavior which evinces itself in the existence of plasma oscillations.

Various consequences of this collective behavior are explored below by classical and semi-classical methods. The infinite plasma is investigated by a classical electrodynamic approach, while a finite plasma is treated in the hydrodynamical approximation of Bloch. A quantum mechanical treatment of the infinite plasma is indicated briefly. The emphasis below is on the effects of plasma dynamics on the stopping of charged particles in condensed states of matter, e.g., in the conduction band in metals.

In Chapter II the plasma system will be regarded as having certain electrodynamic properties by virtue of its intrinsic velocity distribution. Without reference to the Boltzmann transport equation, the

dielectric constant of the linearized plasma will be derived. The main features of plasma behavior are found to be codified in the dielectric constant: the potential, polarization charge density and stopping power of the plasma due to uniformly moving point charges are found in a straight forward and physically plausible manner. It is shown that the Debye screening of moving charges in a plasma holds only for slow particles and that the potential ahead of a fast point charge is of the axial quadrupole form

In Chapter III, the spectral distribution of energy and momentum losses experienced by fast electrons interacting with the infinite plasma is derived from a correspondence principle argument. In Chapter IV the finite plasma is treated in the hydrodynamical approximation of Bloch and the effect of the boundaries on the energy loss distribution is investigated. It is suggested that surface oscillations may explain the low-lying characteristic losses observed in electron loss experiments with thin films.

Historical Background

The fundamental collective behavior of the classical plasma has been investigated by Rayleigh,¹ Langmuir², and Langmuir and Tonks³.

¹ Lord Rayleigh, Phil. Mag., 6, 11, 117 (1906).

² I. Langmuir, Proc. Nat. Acad. Sci., 14, 627 (1928).

³ L. Tonks and I. Langmuir, Phys. Rev., 33, 195 (1929).

Vlasov and his associates⁴⁻⁷ have studied the classical plasma extensively, using Boltzmann's transport equation modified to take into account the long-range coulomb forces. Bloch⁸ applied a hydrodynamical theory of the electron gas to the problem of the stopping power of the statistical atom, while Kronig and Korringa⁹ employed another hydrodynamical model to the study of the uniform plasma. Bohm, Pines, Gross and others¹⁰⁻¹⁴ have treated the classical plasma by a "density fluctuation" method in which emphasis is placed upon the distinction between the collective motion of the system, involving plasma oscillations, and the response of individual electrons in the assembly. This separation is accomplished by dividing the coulomb interactions into long and short-range parts. In most of the papers mentioned above, the dynamical behavior of the plasma has been studied after linearization of the exact equations of motion; i.e., one assumes that departures of the plasma from equilibrium are small perturbations around the undisturbed motion of the system. The linear approximation will be used throughout this paper.

⁴ A. Vlasov, Zhur. Eksptl. i. Teort. Fiz., 8, 291 (1938).

⁵ A. Vlasov, J. Phys. (USSR) 9, 25 (1945).

⁶ L. Landau, J. Phys. (USSR) 10, 25 (1946).

⁷ A. Vlasov, Teoria Mnogikh Chastits (Gitl., Moscow-Leningrad, 1950).

⁸ F. Bloch, Ann. Phys., 5, 285 (1933).

⁹ R. Kronig and J. Korringa, Physica, 10, 406 (1943). See also R. Kronig, Physica, 15, 667 (1949).

¹⁰ D. Bohm and D. Pines, Phys. Rev., 82, 625 (1951).

¹¹ D. Bohm and E. D. Gross, Phys. Rev., 75, 1851 and 1864 (1949).

¹² D. Pines and D. Bohm, Phys. Rev., 85, 338 (1952).

¹³ D. Pines, Phys. Rev., 92, 626 (1953).

¹⁴ D. Bohm and D. Pines, Phys. Rev., 92, 609 (1953).

These investigations of Bohm and Pines have culminated in a quantum mechanical treatment^{13,14} of the dynamical properties of plasma. Their approach is an outgrowth of ideas which they developed¹⁰ for the treatment of transverse electromagnetic interactions between plasma electrons. It involves quantization of the longitudinal electromagnetic field and, in direct analogy with their classical treatment¹², requires the introduction of a cut-off momentum which divides the region of collective interaction from the region of individual interactions.

Their treatment has been extremely fruitful in the elucidation of many of the properties of electrons in metals.^{15,16} However, it is attended by mathematical uncertainties such as determination of the cut-off momentum for plasma oscillations and neglect of the subsidiary conditions on the plasma wave function.

¹⁵ D. Pines, Solid State Physics, 1, 386-450 (1955).

¹⁶ D. Pines, Rev. Mod. Phys., 28, 184 (1956).

CHAPTER II

CLASSICAL THEORY OF THE INFINITE PLASMA

Physical Ideas

A simple classical model³ is sufficient to use in order to illustrate the collective nature of the motion of a free-electron gas. Suppose such a system is confined between two plane parallel plates of infinite extent, coinciding with the surfaces $z = 0$ and $z = a$. As mentioned earlier, assume the positive ionic background to be stationary and distributed uniformly over the volume. Neglect any intrinsic motion which the classical point electrons have in the undisturbed state and consider that the motion of the assembly may be described in terms of the displacement $\xi(z)$ of electrons from their original uniform state. If the density of electrons in the original state is n_0 , then the change in the density at z due to the displacement $\xi(z)$ is

$$\delta n = -n_0 \frac{\partial \xi}{\partial z} \quad \left(\frac{\partial \xi}{\partial z} \ll 1 \right) \quad (2.1)$$

From Poisson's equation, one may write

$$\frac{\partial}{\partial z} E(z) = -4\pi e \delta n \quad (2.2)$$

where $E(z)$ is the electric field at z . This may be integrated to give

$$E(z) = 4\pi e n_0 \xi + E_0 \quad (2.3)$$

where E_0 is the constant of integration which may be set equal to zero since externally maintained fields are not considered here.

This shows that a restoring electric field on the electrons is set up when they are displaced from their equilibrium positions and that this restoring field is proportional to the displacement if it is small. The equation of motion of an electron moving under the influence of this field is then

$$m \ddot{\xi} = -eE(x) = -4\pi e^2 n_0 \xi \quad (2.4)$$

or

$$\ddot{\xi} + \omega_p^2 \xi = 0 \quad (2.5)$$

where $\omega_p = \left[4\pi n_0 e^2 / m \right]^{1/2}$. This shows that the electrons undergo simple harmonic oscillation, in this approximation, when disturbed, and that the characteristic "plasma frequency" of oscillation is a function only of the electronic density and the electronic constants. Typical values of n , ω_p and $\hbar\omega_p$ for plasmas are given in Table I.

Table I

	n	ω_p	$\hbar\omega_p$
	cm^{-3}	sec^{-1}	ev
Gaseous Discharge	10^{11}	1.8×10^{10}	1.2×10^{-5}
Ionosphere	10^6	5.6×10^7	3.7×10^{-8}
Conduction Electrons in Metal	10^{23}	1.8×10^{16}	12

The energy $\hbar\omega_p$ is, of course, the minimum quantum mechanical value of the energy required to excite a simple harmonic oscillator of characteristic frequency ω_p .

The consequences of the intrinsic velocity distribution of the free electron gas are very interesting and are investigated below. The velocity distribution of a Fermi-Dirac electron gas in equilibrium is isotropic and is given by¹⁷

$$f_{FD}(\vec{v})d\vec{v} = \frac{2m^3}{n_0 \hbar^3} \left\{ \exp \left[\frac{1}{k_B T} \left(\frac{1}{2} m v^2 - E_F \right) \right] + 1 \right\}^{-1} d\vec{v} \quad (2.6)$$

where n_0 is the electronic density, f_{FD} is normalized so that $\int f_{FD} d\vec{v} = 1$, k_B is the Boltzmann constant, T is the temperature, and E_F is the Fermi energy. At absolute zero

$$E_F^0 = \frac{\hbar^2}{2m} \left(3n_0/8\pi \right)^{2/3} \quad (2.7)$$

In the case of gaseous plasmas in the undisturbed state, it is usually true that $kT \gg E_F^0$, in which case the Fermi-Dirac distribution goes over to the Maxwell-Boltzmann distribution¹⁷, viz.

$$f_{MB}(\vec{v})d\vec{v} = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left\{ - m v^2 / 2 k_B T \right\} d\vec{v} \quad (2.8)$$

¹⁷ G. Joos, THEORETICAL PHYSICS, (Hafner Publishing Co., New York, 1952), p. 623.

In all of the applications to metals considered here, it will be assumed that $k_B T \ll E_F^0$, i.e., that the assembly is completely degenerate and hence that

$$f_{FD}(\vec{v}) d\vec{v} = \begin{cases} (2m^3/n_e h^3) d\vec{v} & (mv^2/2 < E_F) \\ 0 & (mv^2/2 > E_F) \end{cases} \quad (2.9)$$

The derivation of the oscillatory aspect of plasma motion given above assumed that the electrons were stationary until acted upon by external forces. The intrinsic velocity of the plasma distribution must affect its response to perturbations. Consider that there exists in the plasma a harmonic plane wave with field proportional to $\text{Re} \left\{ \exp \left(i \left[\vec{k} \cdot \vec{r} - \omega_p t \right] \right) \right\}$. An electron having the rms velocity $\langle v^2 \rangle^{1/2}$ of the undisturbed plasma may be considered nearly stationary if it moves a small distance compared with the wavelength $\lambda = \frac{2\pi}{k}$ of the disturbance in one period of the plasma oscillation. In that case, the derivation given above for the motion of the electrons should be valid. This condition may be written

$$\langle v^2 \rangle^{1/2} / \omega_p \ll \lambda \quad (2.10)$$

or

$$k \langle v^2 \rangle^{1/2} \ll \omega_p$$

It will be shown below that the resonant frequency, ω_{res} , of the plasma may be expanded in a series in powers of the wave vector k

$$\omega_{\text{res}}^2 = \omega_p^2 + \langle v^2 \rangle k^2 + \dots \quad (2.11)$$

which is in accordance with these qualitative considerations.

The k^2 dependence of the resonance frequency shows itself in that a perturbation of the plasma which is initially localized in a small region of space will propagate in the plasma with a velocity of the order of $[\langle v^2 \rangle]^{1/2}$. The dispersion relation, Equation (2.11) is an important property of plasma, and will be discussed more fully in Chapter II.

Another striking feature of plasma behavior is the "Debye shielding"¹⁸ of charges located within the plasma. If a stationary point charge is placed in the plasma, the electrons move so that the electric potential of the charge is modified from the coulomb law to one which decreases very rapidly with distance. The original derivation of Debye and Hückel was an outgrowth of an investigation of the screening of ions in electrolytes. One may apply their method to treat electrons in a Maxwell-Boltzmann plasma. Suppose that a point charge is introduced into such a plasma and that there then exists in its vicinity the

¹⁸ P. Debye and E. Hückel, Physik Z., 24, 185 (1923).

stationary electric potential $\phi(\vec{r})$ which is the sum of potentials due to the intruder charge and to the plasma electrons. In the volume element $d\vec{r}$ there will then be found

$$n_0 \exp \{ e\phi(\vec{r}) / k_B T \} d\vec{r} \quad (2.12)$$

electrons, if the density of immobile positive charge is n_0 . The net charge density δn will then be

$$\delta n = n_0 \left(1 - \exp \{ e\phi(\vec{r}) / k_B T \} \right)$$

which reduces

$$\delta n = -n_0 e^2 \phi / k_B T$$

if the exponent is small compared with unity.

The potential ϕ must satisfy Poisson's equation:

$$\nabla^2 \phi = -4\pi \delta n$$

A solution of this equation which is bounded as $r \rightarrow \infty$ and is spherically symmetric is

$$\phi(r) = \text{const} \exp(-r/D_{MB}) / r$$

where

$$D_{MB}^2 = \frac{k_B T}{4\pi n_0 e^2} = \langle U_{MB}^2 \rangle / 3\omega_p^2 \quad (2.13)$$

In a treatment of the resistance of metals due to foreign impurity atoms, Mott¹⁹ has derived the shielded coulomb potential by using the Thomas-Fermi equations²⁰. In this derivation, it is assumed that the positive ions are distributed uniformly and are immobile. A charge center introduced into the metal sets up a potential $\phi(r)$ at a radial distance r from the charge. In the Thomas-Fermi method, the electron density at r , $n'(r)$, in a degenerate electron gas is given by

$$n'(r) = \frac{8\pi}{3h^3} \left\{ 2m (\epsilon_F - e\phi(r)) \right\}^{3/2}$$

and at $r = \infty$, $n' = n_0 = \frac{8\pi}{3h^3} (2m\epsilon_F)^{3/2}$

Poisson's equation then gives

$$\nabla^2 \phi = 4\pi e (n_0 - n') = \frac{32\pi^2 e}{3h^3} (2m)^{3/2} \left[\epsilon_F^{3/2} - (\epsilon_F - e\phi)^{3/2} \right]$$

A simple solution is obtained if $e\phi \ll \epsilon_F$: then

$$\begin{aligned} \nabla^2 \phi &= \phi / D_{FD}^2 \\ \phi &= (\text{const.}) \frac{1}{r} e^{-r/D_{FD}} \end{aligned}$$

where

$$D_{FD}^2 = \frac{\hbar^2}{4me^2} \left(\frac{\pi}{3n_0} \right)^{1/3} \quad (2.14)$$

¹⁹ N. F. Mott, Proc. Camb. Phil. Soc., 32, 281 (1936).

²⁰ L. H. Thomas, Proc. Camb. Phil. Soc., 23, 542 (1927).

We shall show that the dielectric method describes the collective behavior of plasma in both the high and low frequency limits, predicting oscillatory behavior when a disturbance possesses Fourier components of frequency greater than the plasma frequency, and predicting the "Debye screening" effect for slowly-moving charges in the plasma.

The most conspicuous properties of metals are their high electrical and thermal conductivities. Drude²¹ first suggested, soon after the discovery of the electron, that the properties might be explained by assuming that metals contain free electrons in thermal equilibrium with positive ions of the solid. The application of Maxwell-Boltzmann statistics led to some well-known difficulties which were largely overcome by the application of quantum statistics by Sommerfeld.²² In Sommerfeld's model, it is assumed that the conduction electrons move in a potential which is constant everywhere inside the metal. Figure 1 illustrates schematically the potential which one assumes in the Sommerfeld picture. Here E_F is the Fermi energy, ϕ_w is the work function of the metal and E_s is the change in potential energy of an electron crossing the metal boundary. In reality, rather than a constant potential in the interior of the metal, one expects a potential periodic with the period of the lattice. Bloch²³ has shown that an electron can move freely through such a periodic field of force, though the connection between its energy

²¹ P. Drude, *Am. Physik*, 1, 566 (1900). Also see H. A. Lorentz, *THE THEORY OF ELECTRONS*, (G. E. Stechert and Co., New York, 1916).

²² A. Sommerfeld, *Z. Physik*, 47, 1 (1928). Also see A. Sommerfeld and N. H. Frank, *Rev. Mod. Phys.*, 3, 1 (1928).

²³ F. Bloch, *Z. Phys.*, 52, 555 (1928).

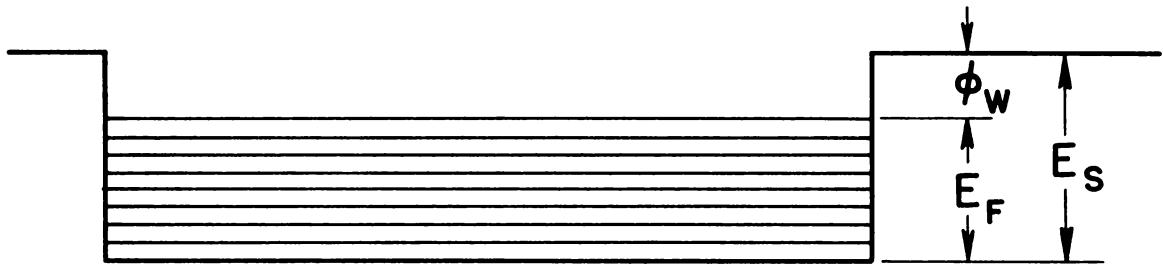


Figure 1. Sommerfeld Model of the Potential Variation in a Metal.

and momentum in general differs from that of a free electron in vacuum. In many cases, the electronic energy E_k is a quadratic function of the momentum $\hbar\vec{k}$ so that

$$E_k = \hbar^2 k^2 / 2m^* \quad (2.15)$$

as in the case of free electrons, though the effective mass m^* in general differs from the electronic mass m . The effect of the lattice upon electronic motion will be ignored below except for the possibility of using the effective electron mass rather than the true mass.

The Boltzmann Transport Equation

Vlasov⁴⁻⁷ and co-workers in Russia have employed a Boltzmann-type transport equation to describe the dynamics of plasma. These equations will be considered briefly but only the linearized electrodynamic equations will be treated in detail below.

In complete analogy with the classical treatments of the theory of gases²⁴ we may write down an equation for the distribution function $f(\vec{r}, \vec{v}, t)$ of plasma electrons such that the number of electrons in the spatial interval $d\vec{r}$ at $\vec{r}$ and in the velocity interval $d\vec{v}$ at $\vec{v}$ at time t is given by $f(\vec{r}, \vec{v}, t) d\vec{r} d\vec{v}$. Suppose that there is a force field of magnitude $\vec{F}(\vec{r})$ existing in the medium. The equation of motion of a

²⁴ J. H. Jeans, THE DYNAMICAL THEORY OF GASES, (Cambridge University Press, 1916).

single electron in this field may be written

$$\dot{\vec{U}} = \frac{1}{m} \vec{F}$$

Let time increase by an infinitesimal amount from t to $t + dt$. During this interval the distribution function will have changed due to the motion of electrons in the force field and due to scattering of electrons upon each other and upon ions in the medium. The change in the number of electrons in the interval $d\vec{r} d\vec{v}$ due to their motion in the force field is

$$\left\{ f(\vec{r} + \vec{U} dt, \vec{v} + \frac{\vec{F}}{m} dt, t + dt) - f(\vec{r}, \vec{v}, t) \right\} d\vec{r} d\vec{v}$$

If there were no collisions, this would be the only change in f . However, collisions between electrons will have removed some of the electrons from the interval $d\vec{v}$ during the time dt and will have deflected electrons which were in other velocity intervals into the interval $d\vec{v}$ at $\vec{v}$. The net gain of electrons in the interval $d\vec{r} d\vec{v} dt$ must be proportional to the interval size and will be denoted by $\left(\frac{\partial f}{\partial t} \right)_{coll} d\vec{r} d\vec{v} dt$. Hence

$$\left\{ f(\vec{r} + \vec{U} dt, \vec{v} + \frac{\vec{F}}{m} dt, t + dt) - f(\vec{r}, \vec{v}, t) \right\} d\vec{r} d\vec{v} = \left(\frac{\partial f}{\partial t} \right)_{coll} d\vec{r} d\vec{v} dt \quad (2.15)$$

One finds the Boltzmann equation by letting $dt \rightarrow 0$ and dividing by $d\vec{r} d\vec{v} dt$;

$$\frac{\partial f}{\partial t} + \vec{U} \cdot \vec{\nabla}_r f + \frac{1}{m} \vec{F} \cdot \vec{\nabla}_v f = \left(\frac{\partial f}{\partial t} \right)_{coll} \quad (2.16)$$

where $\vec{\nabla}_r = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$, $\vec{\nabla}_v = \hat{i} \frac{\partial}{\partial v_x} + \hat{j} \frac{\partial}{\partial v_y} + \hat{k} \frac{\partial}{\partial v_z}$

and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the x , y , z directions, respectively. The force field $\vec{F}$ may be regarded as arising from external electric and magnetic fields $\vec{E}_{\text{ext}}$ and $\vec{H}_{\text{ext}}$ plus the electric and magnetic fields set up by the motion of the plasma electrons themselves: viz,

$$\vec{F} = -e \left[\vec{E} + \frac{1}{c} \vec{v} \times \vec{H} \right] \quad (2.17)$$

$$\vec{E} = \vec{E}_{\text{ext}} - \vec{\nabla} \phi - \frac{1}{c} \frac{\partial}{\partial t} \vec{A} \quad (2.18a)$$

$$\vec{H} = \vec{H}_{\text{ext}} + \vec{\nabla} \times \vec{A} \quad (2.18b)$$

where the scalar and vector potentials ϕ and $\vec{A}$ are the solutions of

$$\nabla^2 \vec{A} - \frac{1}{c^2} \ddot{\vec{A}} = - \frac{4\pi e}{c} \int \vec{v} f(\vec{r}, \vec{v}, t) d\vec{v} \quad (2.19a)$$

$$\nabla^2 \phi - \frac{1}{c^2} \ddot{\phi} = - 4\pi e \int f(\vec{r}, \vec{v}, t) d\vec{v} \quad (2.19b)$$

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \dot{\phi} = 0 \quad (2.19c)$$

The collision term in equation (2.16) will not be considered in detail since it is quite involved for the coulomb interaction law.

The exact solution of equation (2.16) through (2.19) is clearly beyond the reach of analytic methods. One linearizes these equations in order to make any progress in their mathematical analysis. One assumes that

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{v}) + f_1(\vec{r}, \vec{v}, t) + \dots \quad (2.20)$$

and that $f_1 \ll f_0$ so that terms involving squares of f_1 are neglected throughout. One finds

$$\left(\frac{\partial f_0}{\partial t} \right)_{\text{coll}} = 0 \quad (2.21)$$

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f_1 + \frac{1}{m} \vec{F} \cdot \vec{\nabla}_v f_0 = \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} \quad (2.22)$$

where in equations (2.19a) and (2.19b) f is now replaced by f_1 .

The zero-order solution f_0 is usually assumed to be the Maxwell-Boltzmann distribution of particles, equation (2.6) or the Fermi-Dirac distribution equation (2.7). The collision term in equation (2.22) is

neglected or replaced by a schematic damping term. The linearized system forms the basis of Vlasov's approach, and has been investigated in the main for non-relativistic velocities. This system of equations will not be considered further here. Instead, the plasma will be considered as a dispersive medium with certain dielectric properties and a linearized theory will be obtained in an entirely different way below.

The Bohm-Pines Method

As mentioned above, Bohm, Pines and their co-workers¹⁰⁻¹⁶ have devised a method of treating interactions in plasma which does not rely upon the Boltzmann transport equation as modified by Vlasov, but involves the use of collective coordinates in momentum space. Their work has been of great importance in the free electron theory of metals and has shed much light upon problems in this area which had been treated only qualitatively before. However, certain mathematical difficulties attend their treatment. In order to obtain collective coordinates which exhibit oscillatory behavior, they introduce a cutoff wave vector which separates the region of individual interactions from the region of collective interactions.

To illustrate this procedure in a simple manner, following Pines and Bohm¹⁰ one writes for the equation of motion of the i^{th} electron,

considering only the scalar potential,

$$\ddot{\vec{r}}_i = -\vec{\nabla} \left\{ \frac{e}{m} \sum_{j \neq i} \frac{1}{|\vec{r}_i - \vec{r}_j|} \right\} = -\frac{4\pi e^2}{m} \sum_{j \neq i} \sum_{\vec{k}} \frac{\vec{k}}{k^2} e^{i\vec{k} \cdot (\vec{r}_i - \vec{r}_j)} \quad (2.23)$$

where the sum runs over all electrons in the plasma except the i^{th} and the potential has been expanded in a box of unit volume with periodic boundary conditions. They introduce the particle density $\rho(\mathbf{x})$ and its Fourier components $\rho_{\mathbf{k}}$:

$$\begin{aligned} \rho(\vec{r}) &= \sum_{\vec{r}_i} \delta(\vec{r} - \vec{r}_i) \\ \rho_{\vec{k}} &= \int d\vec{r} \rho(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} = \sum_{\vec{r}_i} e^{-i\vec{k} \cdot \vec{r}_i} \end{aligned}$$

so that

$$\rho(\vec{r}) = \sum_{\vec{k}} \sum_{\vec{r}_i} e^{i\vec{k} \cdot (\vec{r} - \vec{r}_i)}$$

The time behavior of the $\rho_{\mathbf{k}}$ is given by

$$\ddot{\rho}_{\vec{k}} = - \sum_{\vec{r}_i} \left[(\vec{k} \cdot \vec{v}_i)^2 + \vec{k} \cdot \ddot{\vec{v}}_i \right] e^{i\vec{k} \cdot \vec{r}_i}$$

and one finds $\ddot{\vec{v}}_i = \ddot{\vec{r}}_i$ from the equations of motion.

Then

$$\ddot{\rho}_{\vec{k}} = - \sum_{\vec{r}_i} (\vec{k} \cdot \vec{v}_i)^2 e^{-i\vec{k} \cdot \vec{r}_i} - \sum_{\vec{k}', \vec{r}_i, \vec{r}_j} \left\{ \frac{4\pi e^2}{m k'^2} \vec{k} \cdot \vec{k}' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}_i} \right\} e^{-i\vec{k}' \cdot \vec{r}_j}$$

Pines and Bohm argue that the sum over k' may be split into two parts, (1) the sum over those values of $\vec{k}' = \vec{k}$ and (2) the sum over $\vec{k}' \neq \vec{k}$ and the second part is negligible because the phase factors $e^{i(\vec{k}-\vec{k}') \cdot \vec{r}_n}$ depend upon the positions of the particles and hence tend to cancel out. They neglect such terms and call this procedure the "random-phase approximation". They then find

$$\ddot{\rho}_{\vec{k}} = - \sum_n \vec{k} \cdot \vec{v}_n e^{-i\vec{k} \cdot \vec{r}_n} - \omega_p^2 \rho_{\vec{k}} \quad (2.24a)$$

and attribute the first term on the right-hand side to random thermal motion of the individual particles and the second term to interparticle actions. For sufficiently small $\vec{k}$, one has

$$\ddot{\rho}_{\vec{k}} + \omega_p^2 \rho_{\vec{k}} = 0 \quad (2.24b)$$

showing that the Fourier components of the density and hence the density itself undergoes harmonic motion.

The equation (2.24a) for $\ddot{\rho}_{\vec{k}}$ is then handled by dividing $\rho_{\vec{k}}$ into a part having oscillatory characteristics and a part possessing individual characteristics. The boundary in $\vec{k}$ space between these two regions is taken to be of the order of k_D , the Debye wave vector.

Using a quantum field approach, employing essentially the same approximations as above, they are able to treat many problems in the free electron theory of metals.

The cutoff wave vector approach has been very fruitful in the hands of Pines and Bohm, but is beset by the uncertainty characteristic of any cutoff procedure. The dielectric approach, on the other hand, is much more straightforward and may be used to treat essentially all the problems considered by Pines and Bohm, in both the classical and quantum mechanical limits.

Electrodynamics of the Plasma

In this section the behavior of a system of classical point electrons is investigated. A phenomenological approach is adopted in which the microscopic behavior of individual electrons in the plasma is ignored. One takes instead a macroscopic point of view and describes the electron assembly as homogeneous, isotropic, dispersive medium. In conventional dispersive media, the dielectric constant is assumed to depend only upon the frequency of the applied field. The dielectric constant of plasma depends not only on the frequency but upon the wave number of the applied field as will be shown below. The dielectric properties of plasma have been considered by Gertsenshtein²⁵ in the study of longitudinal waves passing through ionized media and in an investigation of the scattering of radio waves by the ionosphere. Gertsenshtein used Vlasov's approach and treated electron collisions by means of a generalized form of the Boltzmann transport equation equivalent to the set of equations (2.17) - (2.22) above. However, the phenomenological treatment, which is based

²⁵ M. E. Gertsenshtein, Zhur. Eksptl. i. Teoret. Fiz., 22, 303 (1952); 23, 678 (1952).

on the straightforward application of Maxwell's equations properly modified to take into account the motion of the medium, clarifies the physical aspects of plasma behavior and facilitates the treatment of problems involving the interaction of charged particles with plasma.²⁶

The effect on the plasma of an impressed electric field $\vec{E}(\vec{r}, t)$ will now be considered. One assumes that the field perturbs the equilibrium motion of the electron assembly only slightly and that during the period of time in which the electrons may be considered as responding collectively to the applied field they execute approximately straight-line motion.

Let $nf(\vec{v})d\vec{v}$ be the number of electrons per unit volume possessing velocities in $d\vec{v}$ at $\vec{v}$, and $\int f(\vec{v})d\vec{v} = 1$. Due to this intrinsic motion, the plasma electrons may move through a distance comparable with the wavelength of the applied field in one period of the applied field. This effect may be taken into account by considering that the undisturbed electrons constitute an assembly of directed beams and by treating each direction component separately. Then consider the beam of density $nf(\vec{v})d\vec{v}$. An electron in this beam which is at position $\vec{r}_0$ at $t = 0$ will be at $\vec{r}_0 + \vec{v}t$ at time t and the force on this electron will vary due to its motion in the field. At time t this force will be given by

$$\vec{F} = -e \left\{ \vec{E}(\vec{r}_0 + \vec{v}t, t) + \frac{1}{c} \vec{v} \times \vec{H}(\vec{r}_0 + \vec{v}t, t) \right\}$$

²⁶ J. Neufeld and R. H. Ritchie, Phys. Rev., 98, 1632 (1955).

and if the displacement vector of this electron from its undisturbed path due to the action of the field is denoted by $\vec{\xi}$ we have

$$\left\{ \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} \right\} \vec{\xi} = - \frac{1}{m} \vec{F}$$

from the classical equation of motion. If one defines the polarization vector at the point $\vec{r}$ associated with electrons in the beam having directions in $d\vec{v}$ about $\vec{v}$ to be $\vec{P}_{\vec{v}}(\vec{r}, t) d\vec{v} = -ne f(\vec{v}) \vec{\xi}(\vec{r}, t) d\vec{v}$, and neglects $\dot{\vec{\xi}}$ compared with $\vec{v}$ one has

$$\left\{ \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} \right\} \vec{P}_{\vec{v}}(\vec{r}_0 + \vec{v}t, t) = \frac{ne^2}{m} f(\vec{v}) \left\{ \vec{E}(\vec{r}_0 + \vec{v}t, t) + \frac{1}{c} \vec{v} \times \vec{H}(\vec{r}_0 + \vec{v}t, t) \right\} \quad (2.25)$$

Damping of the motion of the polarization vector due to electron-ion and electron-electron collisions is represented by the damping constant γ . Since the exact nature of this damping is not important to our discussion, it will be neglected at a later point.

$\vec{H}$, $\vec{P}_{\vec{v}}$ and $\vec{E}$ are now expressed as Fourier integrals,

$$\begin{pmatrix} \vec{P}_{\vec{v}}(\vec{r}, t) \\ \vec{E}(\vec{r}, t) \\ \vec{H}(\vec{r}, t) \end{pmatrix} = \frac{1}{(2\pi)^4} \int d\vec{k} d\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \begin{pmatrix} \vec{P}_{\vec{v}}(\vec{k}, \omega) \\ \vec{E}(\vec{k}, \omega) \\ \vec{H}(\vec{k}, \omega) \end{pmatrix} \quad (2.26)$$

and one finds from equation (2.25) the following relation between

$\vec{P}_v(\vec{k}, \omega)$, $\vec{E}(\vec{k}, \omega)$ and $\vec{H}(\vec{k}, \omega)$

$$\vec{P}_v(\vec{k}, \omega) = -\frac{ne^2 f(v)}{m} \frac{\{\vec{E}(\vec{k}, \omega) + \frac{1}{c} \vec{v} \times \vec{H}(\vec{k}, \omega)\}}{(\omega - \vec{k} \cdot \vec{v})^2 + i\gamma(\omega - \vec{k} \cdot \vec{v})} \quad (2.27)$$

Maxwell's equations in a medium moving with velocity $\vec{v}$ are well known²⁷ and may be written, for a nonmagnetic medium

$$\text{Curl } \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{H} \quad (2.28a)$$

$$\text{div } \vec{H} = 0 \quad (2.28b)$$

$$\text{Curl } \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \left[\frac{\partial}{\partial t} \vec{P}_v + \text{Curl } \vec{P}_v \times \vec{v} \right] + \frac{4\pi}{c} \vec{j} \quad (2.28c)$$

$$\text{div } \vec{E} = 4\pi (\rho - \text{div } \vec{P}_v) \quad (2.28d)$$

where $\vec{j}$ and ρ are applied current and charge densities, respectively, relative to the fixed reference system.

²⁷ M. Abraham and R. Becker, Theorie der Elektrizität, II, (Leipzig, B. G. Teubner, 1933) p. 245.

Expanding $\vec{j}(\vec{r}, t)$ and $\rho(\vec{r}, t)$ as in equation (2.26), substituting from equation (2.27), and summing over all velocities, one finds

$$\vec{k} \times \vec{E} = \frac{i\omega}{c} \vec{H}, \quad \vec{k} \cdot \vec{H} = 0 \quad (2.29 \text{ a, b})$$

$$i \vec{k} \times \vec{H} = -\frac{\omega}{c} \vec{E} + \frac{\omega_p^2}{c} \int \frac{f(\vec{v}) d\vec{v}}{(\omega - \vec{k} \cdot \vec{v})^2 + i\gamma(\omega - \vec{k} \cdot \vec{v})} \left\{ (\omega - \vec{k} \cdot \vec{v}) \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{H} \right) + \vec{k} \cdot \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{H} \right) \vec{v} \right\} + 4\pi \vec{I}/c \quad (2.29c)$$

$$\vec{k} \cdot \vec{E} = 4\pi \rho + \omega_p^2 \int \frac{f(\vec{v}) d\vec{v}}{(\omega - \vec{k} \cdot \vec{v})^2 + i\gamma(\omega - \vec{k} \cdot \vec{v})} \left\{ \vec{k} \cdot \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{H} \right) \right\} \quad (2.29d)$$

At this point one discards the Lorentz terms $\frac{1}{c} \vec{v} \times \vec{H}$, arguing that one is interested only in non-relativistic plasmas and that these terms should be smaller than the ones retained by a factor of $\frac{1}{c} \langle v^2 \rangle^{1/2}$ which is less than 1/137 in all cases to be considered. The last term retained in the brackets of equation (2.29c) may be written

$$(\vec{k} \cdot \vec{v}) \vec{v} = \vec{k} \cdot \vec{v} \left\{ \frac{\vec{k} \cdot \vec{v}}{k^2} \vec{k} + \vec{v}^\perp \right\} = \vec{k} \cdot \vec{v} \vec{v}^\parallel + \vec{v}^\perp (\vec{k} \cdot \vec{v})$$

where the superscripts denote the components of the vectors parallel or perpendicular to $\vec{k}$. Since for any isotropic velocity distribution

$$\int \frac{\vec{v}^\perp f(\vec{v}) d\vec{v}}{(\omega - \vec{k} \cdot \vec{v})^2 + i\gamma(\omega - \vec{k} \cdot \vec{v})} = 0$$

one finds

$$\vec{k} \times \vec{E}_{\vec{k},\omega} = \frac{c}{\omega} \vec{k} \omega \quad (2.30a)$$

$$\vec{k} \cdot \vec{H} = 0 \quad (2.30b)$$

$$i \vec{k} \times \vec{H}_{\vec{k},\omega} = - \frac{1}{c} \left[\epsilon_{\vec{k},\omega}^{\perp} \vec{E}_{\vec{k},\omega}^{\perp} + \epsilon_{\vec{k},\omega}^{\parallel} \vec{E}_{\vec{k},\omega}^{\parallel} \right] + \frac{4\pi}{c} \vec{j}_{\vec{k},\omega} \quad (2.30c)$$

$$i \vec{k} \cdot \vec{E}_{\vec{k},\omega} \epsilon_{\vec{k},\omega}^{\parallel} = 4\pi \rho_{\vec{k},\omega} \quad (2.30d)$$

$\vec{E}_{\vec{k},\omega}$ has been divided into components $\vec{E}_{\vec{k},\omega}^{\perp}$ and $\vec{E}_{\vec{k},\omega}^{\parallel}$, perpendicular and parallel, respectively, to the vector $\vec{k}$. The components of $\epsilon_{\vec{k},\omega}$ are

$$\epsilon_{\vec{k},\omega}^{\perp} = 1 - \frac{\omega_p^2}{\omega^2} \int \frac{f(\vec{v}) d\vec{v}}{(\omega - \vec{k} \cdot \vec{v} + i\gamma)} \quad (2.31a)$$

$$\epsilon_{\vec{k},\omega}^{\parallel} = 1 - \omega_p^2 \int \frac{f(\vec{v}) d\vec{v}}{(\omega - \vec{k} \cdot \vec{v})^2 + i\gamma(\omega - \vec{k} \cdot \vec{v})} \quad (2.31b)$$

are the dielectric constants of the plasma associated with the transverse and longitudinal electric fields, respectively. The expressions (2.31a) and (2.31b) agree with those obtained by Gertsenshtein²⁵ who, as stated previously, used a rather cumbersome method based on the Boltzmann

transport equations modified by Vlasov.

It is clear that the foregoing dielectric constants may be easily generalized to include both the effect of bound electrons which may be present in the medium as well as the effect of the motion of ions in the case of an ionized gas. The set of equations (2.30) may be written in terms of the scalar and vector potentials ϕ and $\vec{A}$ respectively. If we let $\vec{H}(\mathbf{r}, t) = \vec{\nabla} \times \vec{A}$, then $\vec{H}_{k\omega} = i\vec{k} \times \vec{A}$ and then from equation (2.30a) $i\vec{k} \times \vec{E} = i\omega/c(i\vec{k} \times \vec{A})$ so that one may write $\vec{E} = i\omega/c \vec{A} - i\vec{k}\phi$. If one supposes that $\vec{A}$ is purely transverse, i.e., that $\vec{k} \cdot \vec{A} = 0$, we find

$$(\kappa^2 - \omega^2 \epsilon^\perp / c^2) \vec{A} = 4\pi \vec{j} / c \quad (2.32a)$$

$$\epsilon^\parallel \kappa^2 \phi = 4\pi \rho \quad (2.32b)$$

In most of the work below only non-relativistic velocities will be considered. In the limit $c \rightarrow \infty$, only the scalar field ϕ and $\epsilon^\parallel$ are needed. Unless otherwise specified, ϵ will be understood to mean $\epsilon^\parallel$.

Let us consider the physical significance of the dielectric constant $\epsilon_{k,\omega}$. Equation (2.30d) may be rewritten in terms of the scalar potential ϕ as

$$\phi_{k\omega} = 4\pi \rho_{k,\omega} / \kappa^2 \epsilon_{k\omega} \quad (2.33)$$

and invoking the convolution property of the Fourier integral,

$$\phi(\vec{r}, t) = \frac{1}{4\pi^3} \int d\vec{r}' \int dt' \left[k^{-2} \epsilon^{-1}(\vec{r}-\vec{r}', t-t') \right] \rho(\vec{r}', t') \quad (2.34)$$

where

$$k^{-2} \epsilon^{-1}(\vec{r}, t) = \frac{1}{(2\pi)^4} \int d\vec{k} \int d\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} / k^2 \epsilon_{\vec{k}\omega} \quad (2.35)$$

is the Green's function for the scalar potential, i.e., the potential at $(\vec{r}, t)$ due to a unit instantaneous charge at $(0, 0)$. Thus $k^{-2} \epsilon^{-1}$ may be regarded as an operator in configuration space, since it describes the potential ϕ at $(\vec{r}, t)$ as the sum of effects due to electrical charge at other space-time points $(\vec{r}', t')$.²⁸ It may be shown by considering the position of the singularities of $1/\epsilon_{\vec{k}\omega}$ in the ω plane that only the retarded solution is to be taken, i.e., that

$$k^{-2} \epsilon^{-1}(\vec{r}, t) = 0 \quad (t < 0)$$

Also, if $\epsilon \rightarrow 1$

$$k^2 \epsilon^{-1} \rightarrow \delta(t) / r$$

or the instantaneous coulomb potential. One may regard $\left\{ k^2 - \omega^2 \epsilon_{\perp} / c^2 \right\}^{-1}$, the Green's function for the vector potential, as an operator in the same sense.

²⁸ U. Fano has given a general discussion of the operator properties of ϵ , Phys. Rev., 103, 1202 (1956).

Scalar Potential due to a Moving Point Charge

It is well known²⁹ that a fast charged particle which interacts with matter may be considered to be a point charge with a well-defined path as long as one is concerned with processes in which the fractional changes in energy and momentum of the particle are small. These conditions are well fulfilled by a proton whose velocity is $\gg c/\hbar$, i.e., for protons of energy much greater than 25 kev. In the case of fast incident electrons, the recoil may not be neglected entirely and the above approximation is valid only for the less violent collisions.

Assume that a point charge of magnitude Ze moves with uniform velocity $\vec{v}_0$ through the plasma. The charge density due to this particle is

$$\rho(\vec{r}, t) = Ze \delta(\vec{r} - \vec{v}_0 t) \quad (2.36)$$

and one finds from equation (2.33)

$$\phi_{\vec{k}\omega} = 8\pi^2 Ze \delta(\omega - \vec{k} \cdot \vec{v}_0) / k^2 \epsilon_{\vec{k}\omega} \quad (2.37)$$

²⁹ E. J. Williams, Kgl. Danske Vidensk. Selskab., Mat-fys. Medd. 13, 4 (1935).

Now assuming $f(\vec{v})$ to be isotropic and integrating over angles equation (2.31b) may be rewritten

$$\begin{aligned} \epsilon_{\omega} &= 1 - \frac{2\pi\omega_p^2}{k^2} \left\{ \int_0^{\infty} \frac{f(u)u du}{(\frac{\omega}{k} - u)} - \int_0^{\infty} \frac{f(u)u du}{(\frac{\omega}{k} + u)} \right\} \\ &= 1 - \frac{2\pi\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{u du f(u)}{(\frac{\omega}{k} - u)} \end{aligned} \quad (2.38)$$

where the damping constant γ has been set equal to zero. The dielectric constant is to be evaluated at $\omega = k \cdot v_0$ from the delta function of equation (2.37) so that the denominator in equation (2.38) has the form $(k \cdot v_0 / k - u)$.

a) Fast Incident Particle

First consider the case where $v_0 \gg [\langle u^2 \rangle]^{1/2}$ in the plasma. The denominator of equation (2.38) may be expanded by the binomial theorem and a series useful for the case of a fast particle will be obtained. One finds

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} \left[1 + \langle u^2 \rangle k^2 / \omega^2 + \langle u^4 \rangle k^4 / \omega^4 + \dots \right] \quad (2.39)$$

where

$$\langle u^{2n} \rangle = \int u^{2n} f(u) d\vec{u} \quad (2.40)$$

One has, e.g., $\langle v^2 \rangle_{FD} = \frac{3}{5} v_F^2$ and $\langle v^2 \rangle_{MB} = \frac{2}{3m} k_B T$. Using the relation between $\phi(\vec{r}, t)$ and its Fourier transform, one may write

$$\phi(\vec{r}, t) = \frac{Ze}{2\pi^2} \int d\vec{k} \int d\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \delta(\omega - \vec{k} \cdot \vec{v}_0) / k^2 \epsilon_{\vec{k}\omega} \quad (2.41)$$

One now chooses a cartesian coordinate system in which the z axis is parallel to $\vec{v}_0$. The components of $\vec{k}$ in this system will be denoted by k_x , k_y and k_z . One may integrate over k_z immediately, obtaining

$$\phi = \frac{Ze}{2\pi^2 v_0} \int dk_x \int dk_y \int d\omega e^{i(xk_x + yk_y + [z/v_0 - t]\omega)} \frac{1}{k^2 \epsilon_{\vec{k}\omega}}$$

where $k^2 = k_x^2 + k_y^2 + \omega^2/v_0^2$. Now one takes $K^2 = k_x^2 + k_y^2$ and writes

$$xk_x + yk_y = \rho K \cos(\theta - \theta')$$

where θ and θ' are the polar angles of the two dimensional vectors $\vec{K}$ and $\vec{\phi}$. Using the identity

$$2\pi J_0(\rho K) = \int_0^{2\pi} d\varphi e^{iK\rho \cos\varphi}$$

one finds

$$\phi(\rho, z, t) = \frac{Ze}{\pi v_0} \int_0^\infty K dK J_0(K\rho) \int_{-\infty}^\infty \frac{\omega^2 d\omega e^{i\omega(z/v_0 - t)}}{k^2 (\omega^2 - \omega_p^2 [1 + \langle v^2 \rangle k^2/\omega^2 + \dots])} \quad (2.42)$$

When $\langle v^2 \rangle^{1/2} \ll v_0$ the term multiplying ω_p^2 in the denominator will be nearly unity. One observes that the roots of the denominator in addition to the pair $\omega = \pm iKv$ are given by $\omega^2 = \omega_p^2(1 + \langle v^2 \rangle k^2/\omega^2 + \dots)$, and correspond to the existence of plasma oscillations. The ω contour should be chosen so as to pass above these singularities. Then for positions ahead of the incident particle ($z > v_0 t$) the ω contour may be closed in the upper half plane and the oscillations will exist only behind the particle. Since there is only the singularity $\omega = ik v_0$ in the upper half plane, the potential ahead of the particle is represented as

$$\phi(\rho, z, t) = Ze \int_0^\infty d\kappa J_0(\kappa \rho) e^{-\kappa(z-v_0 t)} \frac{\kappa^2}{(\kappa^2 + \omega_p^2/v_0^2)} \quad (2.43)$$

if we neglect the velocity distribution of the electrons.

To examine the behavior of this function, one notes that if $r \ll v_0/\omega_p$, one may consider the integrand only for large K . In this case,

$$\phi(r) = Ze \int_0^\infty d\kappa J_0(\kappa \rho) e^{-\kappa(z-v_0 t)} = \frac{Ze}{r} \quad (2.44)$$

where $r = \sqrt{\rho^2 + (z-v_0 t)^2}$.

When $r \gg V/\omega_0$, one looks at the integrand when K is small. We find, approximately,

$$\phi = ze \frac{v_0^2}{\omega_p^2} \frac{1}{r^2} \left(\frac{1}{r} \right) \quad (2.45)$$

which is identical with the potential due to an axial quadrupole.

The physical reason for this form of the potential is that in this approximation the polarization charge lies on a line behind the particle and oscillates with the frequency ω_p . The potential at any point in space is thus the sum of that due to the incident charge and that of a line charge whose linear density varies sinusoidally with distance behind the particle. One sees that the collective effect of the plasma is to cause the potential to fall off approximately as the inverse cube of the distance for large values of r . This result raises some question concerning the procedure of Kwal^{30,31} and Van der Ziel³² of employing an exponentially screened coulomb potential which is valid for a slowly moving ($v_0 \ll \langle v^2 \rangle^{1/2}$) particle when calculating individual energy transfer to electrons in plasma by fast charged particles. The screening parameter for a low velocity incident particle is of the order of the Debye length D_{FD} which in metals may be less than $1 \text{ \AA}$.

³⁰ B. Kwal, Compt. rend. 230, 1669 (1950).

³¹ B. Kwal, J. Phys. Radium, 12, 805 (1951).

³² A. Van der Ziel, Phys. Rev., 92, 35 (1953).

The screening parameter v_o/ω_p for a relativistic incident particle may be much larger, e.g., $\sim 10^2$ A.

For positions behind the incident particle, $z < v_o$, the contour of integration must be closed in the lower half of the ω plane. There will then be the contributions from the pair of poles on the real axis and the pole at $\omega = -iKv_o$. Neglecting the velocities of the plasma electrons, the integral may be written

$$\phi(z) = \frac{zev_o}{\pi} \int_0^\infty \kappa d\kappa J_0(\kappa \rho) \int_{-\infty}^\infty e^{i\frac{\omega}{v_o}(z-v_o t)} \frac{\omega^2 d\omega}{(v_o^2 \kappa^2 + \omega^2)(\omega^2 - \omega_p^2)} \quad (2.46)$$

Let

$$\phi(z) = \phi_I(z) + \phi_{II}(z) \quad (2.47)$$

Corrections to the position of the poles on the real ω axis will be considered in the next section. Evaluation of the residue at the pole at $\omega = -iKv_o$ gives an expression ϕ_I which is the analytical continuation of equation (2.43) into the region $z < Vt$, i.e., ϕ_I is again coulombian for points close to the incident charge and falls off roughly as the inverse cube of the distance for large distances. That part of ϕ arising from the residue at the poles at $\omega = \pm \omega_p$ is purely oscillatory in $(z/v_o - t)$.

$$\phi_{II} = 2(z e \omega_p / v_0) \sin[\omega_p (z/v_0 - t)] K_0 (g \omega_p / v_0) \quad (2.48)$$

This represents the potential due to the polarization charge which is set into oscillation by the passage of the charged particle. In our neglect of damping, the polarization charge continues to oscillate with the plasma eigenfrequency ω_0 . The logarithmic singularity of ϕ_{II} at the track of the particle is due to the neglect of the K dependence of the dielectric constant. In the quantum theory of the dielectric constant described briefly below, this singularity does not occur.

b) Slow Incident Particle

For $v_0^2 \ll \langle v^2 \rangle$ one can approximate equation (2.38) as

$$\epsilon = 1 + k_D^2 / k^2 \left[1 + i B \frac{\omega}{\omega_p} \frac{k_0}{k} + \dots \right] \quad (2.49)$$

where

$$k_D^2 = 4\pi\omega_p^2 \int_0^\infty f(\xi) d\xi \quad (2.50)$$

and the constant B depends upon the kind of statistics assumed for the plasma, i.e., $B_{MB} = (\pi/2)^{1/2}$, $B_{FD} = \sqrt{3} \pi/6$.

Substituting equation (2.49) into equations (2.37) and (2.41), one finds, retaining only the first two terms of equation (2.50),

$$\phi(r) = ze e^{-k_0 r} / r \quad (2.51)$$

where $r = \sqrt{\rho^2 + (z-v_0)^2}$ is the distance from the moving particle to the point at which the potential is measured. For a degenerate Fermi-Dirac gas one finds

$$\frac{1}{D_{FD}^2} \equiv (k_0^2)_{FD} = \frac{3\omega_p^2}{v_F^2} = \frac{4\pi e^2}{k^2} \left(\frac{3n_e}{\pi} \right)^{1/3} \quad (2.52)$$

where $\frac{1}{2} m v_F^2 = E_F$, which agrees exactly with the result of Mott's derivation from the Thomas-Fermi theory, equation (2.14). For a Maxwell-Boltzmann velocity distribution, we find

$$\frac{1}{D_{MB}^2} \equiv (k_0^2)_{MB} = \frac{n \omega_p^2}{k_B T} = \frac{4\pi n_e e^2}{k_B T} \quad (2.53)$$

which agrees with the result obtained by the Debye-Hückel method, equation (2.13).

Validity of the Method

We shall now examine in more detail the restrictions that have to be imposed to justify the present treatment, and we shall consider the following: (A) the relative importance of close binary collisions and long-range interactions, (B) the applicability of a macroscopic picture to a microscopical electron assembly, and (C) the applicability of the classical (orbital) representation. It is understood that the restriction to small disturbances is operative throughout the present treatment.

a) Close Collisions and Long-Range Interactions

The expressions (2.30) and (2.31) are based on the assumption that during the period of the applied fields the electrons execute approximately straight-line motions. In reality, however, each electron undergoes continual collisions with positive charges and other electrons and, as a result of such collisions, deviates progressively from its path and continually loses energy. Any electron in a plasma will follow its rectilinear path at a constant velocity as long as accidental encounters with positive ions and other electrons have no appreciable influence. However, as time passes, the collisions with other electrons will have a cumulative effect as a result of which the direction of the electron motion changes and its velocity becomes appreciably different from its initial velocity. We shall define as the relaxation time τ_D the time interval during which as a result of the cumulative effect of binary

collisions the trajectory of the electron has deviated by an angle of $\pi/2$ from its original direction. Similarly the relaxation time τ_R will designate the time interval during which the energy exchanged in electronic collisions becomes of the same order of magnitude as the initial energy of the electron. The values τ_D and τ_R do not differ appreciably from each other and can be expressed as^{33,34}

$$\tau_D \sim \tau_R \sim \frac{m^2 \langle v^2 \rangle^{3/2}}{4\pi n_0 e^4 \ln(D_{MB} \langle v^2 \rangle / a e^2)} \quad (2.54)$$

The values τ_R and τ_D can be determined approximately by means of the following rough arguments which are given since they provide a better physical insight into the problem than more rigorous calculations.

The energy loss of an incident electron can be expressed as in equation (2.87) below,

$$-\frac{\Delta W}{\Delta z} = \frac{4\pi n_0 e^4}{m v^2} \ln \frac{\rho_{\max}}{\rho_{\min}} \quad (2.55)$$

where ΔW represents the energy loss over a path Δz . Taking as the maximum impact parameter $\rho_{\max} = D_{MB}$ and as the minimum impact parameter the collision diameter $\rho_{\min} = 2e^2/mv^2$, we obtain

$$\Delta z \sim - \frac{\Delta W m^2 \langle v^2 \rangle}{4\pi n_0 e^4 \ln(D_{MB} m \langle v^2 \rangle / a e^2)} \quad (2.56)$$

³³ L. D. Landau, *Zhur. Eksptl. i. Teoret. Fiz.*, 7, 203 (1936).

³⁴ S. Chandrasekhar, *PRINCIPLES OF STELLAR DYNAMICS*, (Chicago, University of Chicago Press, 1942) pp. 48-79.

Taking $\Delta W = -m \langle v^2 \rangle / 2$, we obtain

$$\tau_R = \frac{\Delta E}{\langle v^2 \rangle^{1/2}} \sim \frac{m^2 \langle v^2 \rangle^{3/2}}{8\pi n_0 e^4 \ln(D_0 m \langle v^2 \rangle / 2e^2)} \quad (2.57)$$

The relaxation time for a deviation through an angle of $\pi/2$ from the initial direction may be estimated from the multiple scattering theory of Williams³⁵ which gives for the mean square angle of deviation after traversing a path length X through a medium containing n_0 scattering centers of charge Ze per cm^3 ,

$$\langle \theta^2 \rangle \sim \frac{8\pi Z^2 e^4 n_0 X}{m^2 \langle v^2 \rangle^2} \ln \left(\frac{181}{Z^{1/3}} \right)^2 \quad (2.58)$$

Setting $\langle \theta^2 \rangle \sim \pi^2/4$, $\tau_D = X / \langle v^2 \rangle^{1/2}$, and $Z = 1$, we obtain

$$\tau_D \sim \frac{\pi m^2 \langle v^2 \rangle^{3/2}}{32 e^4 n_0 \ln(181^2)} \quad (2.59)$$

The expressions (2.57) and (2.59) for τ_R and τ_D give values of the same order of magnitude as the expression (2.54) derived by more rigorous considerations.

³⁵ E. J. Williams, Proc. Roy. Soc. (London), 169, 531 (1939).

We shall now consider a term appearing in the denominator of (2.54) which for the specific examples of the MB plasma previously cited is

$$\ln(D_{MB} m \langle v^2 \rangle / 2e^2) \sim 10 \quad (2.60)$$

The individual electron collisions can be neglected if

$$\omega_p^{-1} \ll \tau_R \quad (2.61)$$

Taking into account (2.57) and (2.60), the above expression (2.61) leads to the following inequality:

$$n \ll 10^4 T^3 \quad (2.62)$$

The close interactions are, therefore, negligible for an electron gas of a relatively low density. For a high-density gas such that the close interaction becomes dominant, the assembly loses its "plasma" character and becomes similar to a gaseous assembly of uncharged particles.

b) Macroscopic Treatment

In our description of plasma, we were concerned with the macroscopic behavior of an electron assembly and the Debye length was derived as a macroscopic concept. In order to have an agreement with a microscopic picture in which there are n discrete particles per cm^3 , we must require that

$$D \gg n^{-1/3} \quad (2.63)$$

1. MB Plasma

Substituting (2.53) in (2.63), we obtain

$$n \ll \left(\frac{k_B T}{4\pi e^2} \right)^3 \sim 10^5 T^3 \quad (2.64)$$

2. FD Plasma

Substituting (2.52) in (2.63), we obtain

$$v_F \gg 12.18 v_B \quad (2.65)$$

where $v_B = e^2/\hbar$.

c) Orbital Representation

1. Validity Criterion. It has been shown in a preceding paragraph that an electron having velocity $v_0 \ll \langle v^2 \rangle^{1/2}$ produces in a plasma a potential $\phi = -e \exp(-r/D)r$. To discuss the validity of the orbital representation of electronic motion in the plasma, one argues that each electron is surrounded by a potential approximately equal to ϕ . This argument is strictly applicable to all electrons with $v \ll \langle v^2 \rangle^{1/2}$. For other electrons, the distribution of the potential does not change essentially and qualitatively the potential is still screened at a distance of the order of magnitude of D . It will be shown later that a fundamental change in the screening potential occurs only for electrons have $v \gg \langle v^2 \rangle^{1/2}$, and the number of these electrons is negligible in a plasma in equilibrium.

It will now be determined whether or not the motion of an electron in a potential ϕ can be described by means of the classical (orbital) representation on which the present treatment is based. According to Williams³⁶ the orbital representation is valid if the following two requirements are fulfilled: (a) the uncertainty in the momentum of a perturbed electron is much smaller than the classical value of the momentum transfer caused by the perturbing field ϕ , and (b) the wavelength of the perturbed electron is much smaller than the radius of the perturbing field, which is assumed to be equal to D .

³⁶ E. J. Williams, Rev. Modern Phys., 17, 217 (1945).

The criterion (a) leads to the following inequality:

$$\frac{\phi e r}{\hbar v} = \frac{e^2 \exp(-r/a)}{\hbar v} \gg 1. \quad (2.66)$$

The condition (2.66) can never be satisfied for the whole region of space and consequently the orbital picture cannot be generally applied. We may associate with each value of v a radius

$$R = a \ln(e^2/\hbar v) \quad (2.67)$$

such that only for $r \ll R$ the orbital picture is valid. If, however, the region defined by the radius R is sufficiently large so as to include most of the space occupied by the scattering potential, i.e.,

$$R \gg a \quad \text{or} \quad \ln(e^2/\hbar v) \gg 1 \quad (2.68)$$

then we may assume that the orbital picture applies to the whole space. The inequality (2.68) represents the Williams' requirement (a).

The requirement (b) leads to the following inequality:

$$\hbar/mv a \ll 1 \quad (2.69)$$

2. MB Plasma. We shall apply now the inequalities (2.68) and (2.69). The first of these inequalities corresponds to $v \ll v_B$ or to

$$T \ll 13.5 \times 1.16 \times 10^4 = 1.57 \times 10^5 \text{ OK} \quad (2.70)$$

Substituting (2.53) in the inequality (2.69), we obtain

$$v^2 \gg 4\pi k^2 n_0 e^2 / m^2 R_B T \quad (2.71)$$

We have from (2.62) and (2.64) that $n \ll n_{\max}$ where $n_{\max} = (k_B T / 4\pi e^2)^3$. Substituting in (2.71) $n_{\max}$ in place of n , we obtain

$$v \gg v_{\max} = 2.7 R_B T / m e^2 \quad (2.72)$$

Since v has a range from zero to infinity, not all the electrons satisfy the inequality (2.72). However, if

$$\frac{1}{2} m v_{\max}^2 \ll R_B T \quad (2.73)$$

the number of electrons that do not satisfy the inequality (2.72) may be neglected, and it can be assumed that the inequality (2.69) holds true.

Substituting (2.72) in (2.73), we obtain

$$T \ll 10^4 \text{ } ^\circ\text{K} \quad (2.74)$$

3. FD Plasma. The inequality (2.68) requires that

$$v_F \ll v_B \quad (2.75)$$

and substituting (2.52) in (2.69), we obtain

$$v^2 \gg \frac{4}{\pi} v_B v_F \quad (2.76)$$

Since v has a range of values from zero to v_F , the inequality (2.76) is not satisfied for all the electrons and the necessary requirement is

$$v_F \gg \frac{4}{\pi} v_B \quad (2.77)$$

d. Conclusion

Our treatment of the MB plasma is justified if the following inequalities are satisfied: (2.62), (2.64), (2.70) and (2.74). The inequality (2.62) is more stringent than (2.64) and (2.70) is more stringent than (2.74). Consequently, the criterion for the applicability of our method is expressed by (2.62) and (2.70). This criterion is satisfied for two representative cases of a MB plasma: (1) an ionosphere in which $n_0 \sim 10^6$, $T \sim 300^\circ \text{K}$; (2) a gaseous discharge in which $n_0 \sim 10^{11}$, $T \sim 10^4 \text{ } ^\circ\text{K}$.

The validity of the treatment of the FD plasma is contingent on the inequalities (2.65), (2.75) and (2.77). The inequality (2.75) is not compatible with (2.65) and (2.77) and, therefore, our validity criterion is not satisfied.

The phenomenological treatment of a plasma as a dispersive medium can, therefore, be applied rigorously to such cases as electrical gaseous discharges, or the ionosphere. Our classical treatment is, however, rigorously not applicable to such cases as electrons in the conduction band of a metal. However, the approach sketched in the summary is capable of describing the quantum mechanical properties of plasma.

The Polarization Charge Density due to a Moving Point Charge

a) Fast Particle

The Fourier transform of the polarization charge density, $\rho_{\vec{k}\omega}$, may be expressed as

$$\rho_{\vec{k}\omega} = -i\vec{R} \cdot \vec{P}_{\vec{k}\omega} \quad (2.78)$$

where $\vec{P}_{k\omega}$ is the polarization vector. Further, since

$$\vec{P}_{k\omega} = \frac{\epsilon-1}{4\pi} i \vec{k} \phi \quad (2.79)$$

and $-i \vec{k} \cdot \vec{E} = k^2 \phi$

$$p_{k\omega} = - \frac{\epsilon-1}{4\pi} k^2 \phi = ze \frac{1-\epsilon}{\epsilon} \pi \delta(\vec{k} \cdot \vec{v}_0 - \omega) \quad (2.80)$$

or, in configuration space

$$\rho(\vec{r}, t) = - \frac{ze}{(2\pi)^3} \int_0^\infty k dk J_0(k\rho) \int_{-\infty}^\infty dk_z e^{i k_z z} \int_{-\infty}^\infty d\omega \frac{\epsilon-1}{\epsilon} \delta(v_0 k_z - \omega) e^{-i\omega t}$$

Carrying out the k_z integration and using the expansion (2.39) for $\epsilon_{k\omega}$ one finds

$$\rho = \frac{ze\omega_p^2}{(2\pi)^2 v_0} \int_0^\infty k dk J_0(k\rho) \int_{-\infty}^\infty d\omega e^{i\omega(\frac{z}{v_0} - t)} \frac{d\omega (1 + \langle v^2 \rangle k^2 / \omega^2 + \dots)}{\omega^2 - \omega_p^2 (1 + \langle v^2 \rangle k^2 / \omega^2 + \dots)} \quad (2.81)$$

We have postulated $\langle v^2 \rangle^{1/2} \ll v_0$ so we find the roots of the denominator by successive approximation. The ω contour is chosen to pass above the poles on the real axis, to correspond to zero polarization charge ahead of the particle. Then, the roots of the denominator are approximately

$$\omega = \pm \left[\omega_p^2 + \langle v^2 \rangle (\omega_p^2 / v_0^2 + k^2) \right] \quad (2.82)$$

Evaluating the residues, we find for $(z/v_0 - t) < 0$

$$p = \frac{ze\omega_p}{2\pi v_0} \int_0^\infty k dk J_0(k\rho) \sin [\tilde{\omega} (\frac{z}{v_0} - t)] \quad (2.83)$$

where $\tilde{\omega} = \left[\omega_p^2 (1 + \langle v^2 \rangle / v_0^2) + \langle v^2 \rangle k^2 \right]$

If $\langle v^2 \rangle \rightarrow 0$

$$p \rightarrow \frac{ze\omega_p}{2\pi v_0} \sin(\omega_p [z/v_0 - t]) \frac{\delta(\rho)}{2\pi \rho}$$

To evaluate this expression, we rewrite it as follows:

$$p = - \frac{ze}{2\pi v_0} \frac{d^2}{dt^2} \int_0^\infty k dk J_0(k\rho) \sin [\tilde{\omega} (z/v_0 - t)] / \tilde{\omega}^2 \quad (2.84)$$

Using an identity proven by Lamb³⁷ we find

$$p = - \frac{ze}{2\pi} \frac{d^2}{dt^2} \begin{cases} \frac{\cos [\omega_0 (\{z/v_0 - t\}^2 - \rho^2 / \langle v^2 \rangle)^{1/2}]}{(\{z/v_0 - t\}^2 - \rho^2 / \langle v^2 \rangle)^{1/2}} & \text{if } \alpha > 1 \\ 0 & \text{if } \alpha < 1 \end{cases} \quad (2.85)$$

where $\alpha = (z/v_0 - t) \langle v^2 \rangle^{1/2} / \rho$

The singularity on the surface of the cone

$$\frac{\langle v^2 \rangle^{1/2}}{v_0} (z - v_0 t) = \rho$$

is fictitious; the polarization charge density actually has a finite maximum on this surface.

³⁷ H. Lamb, Proc. London Math. Soc. Sec., 2, No. 7, 122 (1909).

This result illustrates the connection between the ϵ dependence of ϵ and the propagation of disturbances in the plasma. The point charge generates a conical "wake" of polarization behind it, which is due to the spreading out of the disturbance at a velocity given approximately by $[\langle v^2 \rangle]^{1/2}$. This conical wake resembles the Cerenkov radiation emitted by relativistic particles passing through matter.

b) Slow Particle

For the sake of completeness, we quote the result for the polarization charge density due to a slowly moving ($v \ll [\langle v^2 \rangle]^{1/2}$) point charge. One finds for the Fermi-Dirac plasma

$$\rho = \frac{ze}{4\pi D_{FD}^2} \frac{e^{-r/D_{FD}}}{r} + \frac{\sqrt{3}}{8\pi} \frac{ze\omega_p U_0}{U_F^2} \frac{z}{\lambda^3} \left\{ (1-a^2) [e^{-a} E_\lambda(a) - e^a E_\lambda(-a)] \right. \\ \left. + a [e^a E_\lambda(-a) + e^{-a} E_\lambda(a) + 1] \right\} \quad (2.86)$$

where

$$E_\lambda(a) = \int_{-\infty}^a e^x dx / x$$

and $a=r/D_{FD}$. We have taken $r = \sqrt{\rho + (z-v_0 t)^2}$. The origin of coordinates ($r = 0$) is taken at the position of the incident particle and moves with the particle. The z coordinate is measured along the path of the particle and is taken positive in the direction of motion of the particle. For the MB plasma, replace D_{FD} by D_{MB} and the constant $\sqrt{3}/24\pi$ by $(32\pi^3)^{-1/2}$ in equation (2.86).

The effect of the motion of the particle upon the polarization is again apparent. In first approximation the polarization charge is spherically symmetric with the Debye distribution. In second approximation, it lags behind the charge by an amount proportional to v_o/v_F since it has a $\cos \theta$ distribution, where $\cos \theta = z/r$. One may show that the sign of the curly bracket of equation (2.86) is always negative so that the term involving v_o/v_F has the effect of subtracting charge for $z > 0$ and adding when $z < 0$.

Stopping Power of Plasma for a Point Charge

As will be seen below, the energy lost per unit path length to the plasma by a point charge moving with constant velocity cannot be calculated exactly by a classical theory. This is because close collisions involving large energy transfers to plasma electrons are treated incorrectly, leading to a logarithmically divergent stopping power. This is usually circumvented by using the results of binary collision theory for encounters involving small impact parameters and employing classical notions of the collective plasma motion for large impact parameters.³⁸

The first attempt to calculate the stopping power of free electrons in metals was made by Weizacker³⁹ who neglected interelectronic forces and assumed that metal electrons interact only with the lattice. Kronig and Korringa⁴⁰ and Kramers⁴¹ subsequently included the collective effect

³⁸ For an excellent analysis of stopping power from a classical viewpoint, see A. Bohr, Kgl. Danske Vid. Selsk. Mat-fys. Medd., 24, No. 19 (1948).

³⁹ C. F. von Weizacker, Am. Phys., 17, 869 (1933).

⁴⁰ R. Kronig and J. Korringa, Physica, 10, 406, 800 (1943).

⁴¹ H. A. Kramers, Physica, 13, 401 (1947).

of electronic motion in classical treatments of the problem. Fermi⁴² has calculated the stopping power of a system of stationary classical electrons using the dielectric formulation.

The stopping power of the classical plasma will be calculated in this section. In the summary a consistent quantum mechanical treatment of the problem which includes both the effect of plasma oscillations and of individual interactions between the incident charge and the plasma electrons will be sketched briefly.

a) High Velocity Particle

Suppose the space surrounding the particle track is subdivided into two regions: (1) the region of binary collisions within a cylinder having as axis the track of the particle and a radius ρ_1 ; (2) the region of plasma interaction outside this cylinder.

1. Binary Collisions. The contribution of binary collisions to the stopping power as a function of the impact parameter may be found by the following simple argument: Suppose that the incident particle of charge Ze and mass M may be considered to move in a straight line and that a certain electron in the medium located at a distance ρ from the path may be treated as free during the collision. Neglecting the motion of the

⁴² E. Fermi, Phys. Rev., 56, 1242 (1939); 57, 485 (1940).

electron during the collision, we may write for its momentum change $\Delta \vec{p}$,

$$\Delta \vec{p} = \int_{-\infty}^{\infty} \vec{F} dt = Ze^2 \int_{-\infty}^{\infty} \vec{\nabla} \frac{1}{[r^2 + v_0^2 t^2]^{1/2}} dt$$

and in this approximation, only the component of momentum perpendicular to the particle path makes a contribution. Then the energy ϵ acquired by the electron is given by

$$\epsilon = \frac{1}{2m} (\Delta \vec{p})^2 = \frac{Z^2 e^4}{\frac{1}{2} m v_0^2 \rho^2}$$

The number of electrons located between ρ and $\rho + d\rho$ per unit length is $n_0 \times 2\pi\rho d\rho$, so the energy lost per unit path by the incident particle is given by

$$-\left(\frac{dW}{dx}\right)_{\text{Binary}} = 2\pi n_0 \int_{\rho_{\min}}^{\rho_1} \rho d\rho \epsilon(\rho) = \frac{4\pi Z^2 e^4 n_0}{m v_0^2} \ln \frac{\rho_1}{\rho_{\min}} \quad (2.87)$$

The classical value of $\rho_{\min}$ is found by noting that the maximum velocity which can be imparted to the electron in a head-on collision is just $2v_0$. The impact parameter corresponding to this energy transfer is

$$\rho_{\min} \sim Ze^2 / m v_0^2 \quad - \text{classical}$$

This classical treatment is valid only if the coulomb field of the incident particle varies negligibly over the wave packet representing the struck

electron, i.e., over the distance $\lambda = \frac{\hbar}{mv_0}$ in a coordinate system moving with the incident particle. Only values of $\rho_{\min} > \frac{\hbar}{mv_0}$ have a meaning and hence the quantum criterion for $\rho_{\min}$ is

$$\rho_{\min} \sim \hbar / mv_0 \quad - \text{Quantum}$$

One should use the larger of these two values. They are equal when $v_0 \sim Ze^2/\lambda$, hence we may write

$$\rho_{\min} = \begin{cases} Ze^2 / mv_0^2 & Ze^2 / \hbar v_0 \gg 1 \\ \hbar / mv_0 & Ze^2 / \hbar v_0 \ll 1 \end{cases} \quad \begin{matrix} (2.88a) \\ (2.88b) \end{matrix}$$

2. Plasma Region. We wish to determine the energy lost by the incident particle to the portion of the plasma in the region $\rho > \rho_1$ to which the dielectric concept applies. The reaction force of the medium upon the incident particle will be calculated. The result will be in the form of an integral over frequency ω and wave vector k . The resulting expression will be found to diverge for large k and one will be forced to cut the integration off at some intermediate value of k_1 . A reasonable value to assume for k_1 is $1/\rho_1$,

since the omission of the region inside the cylinder of radius ρ_1 is equivalent to the neglect of wavelengths small than ρ_1 .

The retarding force with which the medium acts upon the particle is given by $-Ze \left\{ E_z \Big|_{\vec{r}=\vec{v}_0 t} - E_z^{\text{Self}} \Big|_{\vec{r}=\vec{v}_0 t} \right\}$ where $E_z \Big|_{\vec{r}=\vec{v}_0 t}$ is the component of the total field in the direction of motion of the particle, evaluated at the position of the particle, and $E_z^{\text{Self}} \Big|_{\vec{r}=\vec{v}_0 t}$ is the self-field of the particle. Hence the energy loss per unit path length is given by

$$-\frac{dW}{dz} = -\frac{Ze}{v_0} \left\{ \vec{v} \cdot (\vec{E} - \vec{E}^{\text{self}}) \right\}_{\vec{r}=\vec{v}_0 t} \quad (2.89)$$

We have from equation (2.37) and (2.89)

$$-\frac{dW}{dz} = \frac{i(Ze)^2}{\pi v_0^2} \int_0^{k_\infty} k dk \int_{-\infty}^{\infty} \omega d\omega \left(\frac{1}{\epsilon_{k\omega}} - 1 \right) k^{-2} \quad (2.90)$$

where $k^2 = K^2 + \omega^2/v_0^2$. In the approximation $v_0^2 \gg \langle v^2 \rangle$ one has

$$\left(\frac{1}{\epsilon} - 1 \right) \sim \frac{\omega_p^2 (1 + \langle v^2 \rangle k^2 / \omega^2 + \dots)}{\omega^2 - \omega_p^2 (1 + \langle v^2 \rangle k^2 / \omega^2 + \dots)} \quad (2.91)$$

The zeros of the denominator in the ω plane are located at

$$\omega = \pm (\omega_p^2 + \langle v^2 \rangle k^2)^{1/2} \quad (2.92)$$

Evaluating the ω integral at these poles one finds letting $k^2 = \frac{\langle v^2 \rangle}{v_o^2}$,

$$-\frac{dW}{dz} = \frac{(Ze)^2 \omega_p^2}{v_o^2} \int_0^{K_1} \frac{K dK}{(K^2 + \omega_p^2/v_o^2)} \left\{ 1 + K^2 \langle v^2 \rangle / \omega_p^2 (1 + \beta^2) \right\} \quad (2.93)$$

Carrying out the K integration one obtains

$$-\frac{dW}{dz} = \frac{(Ze)^2 \omega_p^2}{2v_o^2(1+\beta^2)} \left\{ \ln(1 + v_o^2 K_1^2 / \omega_p^2) + \beta^2 K_1^2 v_o^2 / \omega_p^2 \right\} \quad (2.94)$$

In most cases $\frac{v_o^2 K_1^2}{\omega_p^2} \gg 1$ so that we may write

$$-\frac{dW}{dz} = \frac{(Ze)^2 \omega_p^2}{v_o^2(1+\beta^2)} \left\{ \ln \frac{v_o K_1}{\omega_p} + \beta^2 K_1^2 v_o^2 / 2\omega_p^2 \right\} \quad (2.95)$$

This expression is nearly identical with that given by Pines¹⁴, viz, in the present notation

$$-\frac{dW}{dz} = \frac{(Ze)^2 \omega_p^2}{v_o^2} \left\{ \ln \frac{K_1 v_o (1-\beta^2)}{\omega_p} + \frac{\beta^4 (1-\beta^2) K_1^2 v_o^2}{2\omega_p^2} \right\} \quad (2.96)$$

except for the terms in β^2 . If we take $K_1 = k_D$, as Pines and Bohm recommend, we find $(K_1 v_0 / \omega_p)^2 \sim 1/\beta^2$

so that the second term in the brackets is $O(\beta^2)$ according to Pines but of the order of unity in our formula.

In the limit $\beta \rightarrow 0$ we find the usual formula

$$-\frac{dW}{dz} = \left(\frac{Ze\omega_p}{v_0} \right)^2 \ln \left(\frac{v_0}{\omega_p \rho_1} \right) \quad (2.97)$$

3. Total Stopping Power. By adding the contributions (2.87) and (2.97), we obtain for the total stopping power in the case of $\beta \rightarrow 0$

$$-\frac{dW}{dz} = \frac{4\pi Z^2 e^4}{m v_0^2} \ln (v_0 / \omega_p \rho_{\min}) \quad (2.98)$$

where $\rho_{\min}$ is given by (2.77). The exact value of ρ_1 is unimportant in this limit since it subtracts when one adds the stopping power in the two regions.

b) Low Velocity Particle

The stopping power of an incident particle having charge Ze and velocity $v \ll \langle v^2 \rangle^{1/2}$ will now be determined.

The energy loss per unit path length may be thought of as being due to the retarding field of polarization charge located in the "far" region, i.e., to the region outside of a sphere of radius R whose center coincides with the position of the incident particle. Thus, if σ is the polarization charge on the surface of this sphere and p is the polarization density in the far region,

$$-\left(\frac{dN}{dz}\right)_p = Ze \left\{ 2\pi \int_0^\pi \sin\theta \cos\theta \sigma(\theta, R) d\theta + 2\pi \int_0^\pi \sin\theta \cos\theta d\theta \int_R^\infty p(\theta, r) dr \right\} \quad (2.99)$$

where the coordinate system is chosen to move with the incident particle and to have its origin at the particle position. In this expression $\cos\theta = z/r$, $\sigma = -\hat{n} \cdot \vec{P}$ where n is a unit vector normal to the sphere of radius R in the outward direction.

Using the above approximation for $\epsilon_{\vec{k}, \omega}$ in the Fourier integral expressions for $\vec{P}$ and p and expanding the resulting formulas in a power series in $\beta' = v/\langle v^2 \rangle^{1/2}$ we find to the first power in β' and for an F-D plasma,

$$-\frac{dN}{dz} = \frac{Ze^2 \omega_p^2 \beta'}{\sqrt{3} 6 a v_0^2} \left\{ \left(1 + \frac{1}{a} + \frac{1}{a^2}\right) e^{-a} E_a(a) - \left(1 - \frac{1}{a} + \frac{1}{a^2}\right) e^a E_a(-a) - \frac{2}{a} \right\} \quad (2.100)$$

where $a = R/D$. If $a \ll 1$,

$$-\left(\frac{\delta W}{\delta z}\right)_P = \frac{Z^2 e^2 \beta'}{6\sqrt{3} D_{FD}^2} \ln(D_{FD}^2/3.17 R^2) \quad (2.101)$$

In this phenomenological treatment the region inside a sphere of radius R centered at the particle has been excluded from consideration. The classical phenomenological representation is not strictly applicable to an FD plasma. However, the expression (2.101) will be applied to an FD plasma, keeping in mind that in the past a classical treatment has been not infrequently used with satisfactory results in quantum-mechanical problems. The result is confirmed by the quantum mechanical treatment indicated below.

Although the interaction of a fast particle with the surrounding electrons is not subject to an orbital representation, the stopping power has been calculated by using an impact parameter method and the quantum mechanical aspects of the problem have been taken into account by assuming that the minimum impact parameter is equal to $\hbar/mv_0$.

The binary collision of a particle having velocity U_0 with an electron gas having isotropically distributed velocities in the range from zero to v_F is considered. The wavelength of most of the electrons as seen from the moving particle will be somewhat smaller, but of the same order of magnitude as $2\pi\lambda_F$. The space surrounding the incident particle will be divided into two regions: Region I within a sphere having its center at the position of the particle and radius $\pi\lambda_F$; and region II comprising

the space outside the sphere. The contribution to the stopping power from the region I is negligible, since in this region the incident particle traverses a wave packet that exerts on it electrical forces from various direction, thus giving partial cancellation. Therefore, it will be assumed that the total contribution to the stopping power is due to the presence of plasma in the region II.

The expression for the stopping power can, therefore, be obtained by substituting $R = \pi\kappa_F$ in (2.101). Of particular interest is the case of $a_{FD} \ll 1$ which has been treated by Fermi and Teller⁴³ in a different manner. To determine the stopping power for this case, we substitute $R = \pi\kappa_F$ in (2.101), obtaining

$$-\frac{dW}{dz} = \frac{2}{3\pi} \frac{Z^2 e^4 m^2 v_0}{\hbar^3} \ln \left\{ \frac{v_F}{3.17 \times 4\pi v_0} \right\} \quad (2.102)$$

where $v_0 = e^2/\kappa$.

Assuming that $v_F/v_0 \gg 3.17 \times 4\pi$, the formula (2.102) can be expressed as

$$-\frac{dW}{dz} = \frac{2}{3\pi} \frac{Z^2 e^4 m^2 v_0}{\hbar^3} \ln \left\{ \frac{v_F}{v_0} \right\} \quad (2.103)$$

This formula is identical to the corresponding formula by Fermi and Teller.

The Fermi-Teller formula does not apply to the conduction band of metals since it is derived under the assumption that $v_F \gg v_0$.

⁴³ E. Fermi and E. Teller, Phys. Rev., 72, 399 (1947); see also N. Mott, Proc. Phys. Soc., (London) 137, 1462 (1949).

In the more exact quantum-mechanical treatment described in the summary it is not necessary to make the approximations used above.

Stopping Power of Plasma due to the Transverse Field

We now consider briefly the stopping power of a relativistic charged particle in plasma due to the transverse field. Returning to equations (2.3a) and (2.3b) we may write

$$\vec{E}^+ = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\frac{4\pi i \omega}{c} \frac{\vec{F}_\perp}{\{k^2 - \omega^2 \epsilon^+ / c^2\}} \quad (2.104)$$

where

$$\vec{F}_\perp = \pi \delta(\vec{k} \cdot \vec{v}_0 - \omega) \left[\vec{v}_0 - (\vec{k} \cdot \vec{v}_0) \vec{k} / k^2 \right]$$

The energy loss per unit path length of the incident particle denoted by $-\frac{dW}{dx} \Big|_\perp$ is equal to the electric force in the medium acting in the $-\vec{v}_0$ direction evaluated at the position of the particle, i.e., the retarding reaction of the medium on the particle. Hence

$$-\frac{dW}{dx} \Big|_\perp = \frac{4\pi e^2}{(2\pi)^3 v_0 c^2} \int_0^\infty 2\pi k dk \int_{-\infty}^\infty \omega d\omega \frac{v_0^2 - \omega^2 / k^2}{\{k^2 - \omega^2 \epsilon^+ / c^2\}} \quad (2.105)$$

where $k^2 = K^2 + \omega^2/v_o^2$. The transverse dielectric constant may be expanded as follows;

$$\epsilon^+ = 1 - \frac{\omega_p^2}{\omega} \int \frac{f(\vec{v}) d\vec{v}}{(1 - \vec{K} \cdot \vec{v}/\omega)} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2 K^2 \langle v^2 \rangle}{\omega^4} - \dots \quad (2.106)$$

The denominator of equation (2.105) may be written

$$K^2 - \epsilon^+ \omega^2/c^2 = K^2 \left[1 - \frac{\omega_p^2 \langle v^2 \rangle}{\omega^4 c^2} \right] + \frac{\omega_p^2}{c^2} \left[1 + \langle v^2 \rangle/c^2 \right] + (1 - \beta^2) \omega^2/v_o^2 + \dots = D_o$$

where $\beta^2 = v_o^2/c^2$. We neglect terms involving the ratio of $\langle v^2 \rangle/c^2$ and consider the roots of D_o in the ω plane. There are only two in this approximation and they are given by

$$\omega = \pm i \frac{v_o}{\sqrt{1 - \beta^2}} \left[K^2 + \omega_p^2/c^2 \right]^{1/2}$$

The ω integrand of the expression for $\left[-\frac{d\omega}{dx} \right]_{\perp}$ is consequently an odd function of ω and $\left[-\frac{d\omega}{dx} \right]_{\perp} = 0$; there are no radiative losses in a pure plasma in this approximation. Note that if the medium were composed of bound electrons radiative losses could exist. For example, suppose all electrons are bound with a frequency ω_o and that their number density is N ; if $\Omega^2 = \frac{4\pi N e^2}{m}$ then

$$\epsilon^+ = 1 + \frac{\Omega^2}{\omega_o^2 - \omega^2}$$

and

$$D_0 = K^2 + \frac{\omega^2}{v_0^2} (1 - \beta^2 \epsilon^+)$$

Now $\beta^2 \epsilon^+$ may be larger than unity for a range of values of ω ; in this case there may be real roots of D_0 in the ω plane and radiative losses can exist. These losses are identical with the well-known Cerenkov losses⁴⁴ from a charged particle moving faster than the velocity of light in the medium in which it finds itself.

⁴⁴ See, e.g., L. I. Schiff, QUANTUM MECHANICS, (McGraw Hill, New York, 1949), p. 262.

CHAPTER III

DISTRIBUTION OF ENERGY LOSSES EXPERIENCED BY A FAST ELECTRON IN THE INFINITE PLASMA

Consider now the details of the process by which charged particles lose energy to electrons in matter. In case the struck electrons are bound to atoms, they may absorb energy only by making quantum jumps between eigenstates. One would expect that energy analysis of a beam of initially monoenergetic fast electrons after passing through an assembly of such atoms would show losses characteristic of these electronic states.⁴⁵ In direct analogy one also expects discrete energy losses by electrons incident on metals due to the excitation of plasma oscillations in the conduction electrons.

Pines and Bohm¹⁰ have suggested that this mechanism may explain a large body of experimental data on discrete losses in metals.⁴⁶ On this picture, energy spectra such as shown in Figures (2) and (3)⁴⁷ are to be attributed to the successive losses of energy quanta in units of $\hbar\omega_p$ by the fast electrons. Here ω_p is the plasma frequency $\sqrt{4\pi n e^2/m^*}$ where n is the effective number of conduction electrons participating in plasma oscillations and m^* is their effective mass. Blackstock, et. al., have shown that the results shown are not inconsistent with this picture since the value of the 14.7 volt discrete

⁴⁵ Experimental results showing this effect in various gases have been given by L. C. Van Atta, *Phys. Rev.*, 38, 876 (1931); See also Mott and Massey, *THE THEORY OF ATOMIC COLLISIONS*, (Oxford, Second Edition) p. 261.

⁴⁶ For reviews of this subject, see D. Pines, *Rev. Mod. Phys.*, 28, 184 (1956); D. Pines, *Solid State Phys.*, 1, (Academic Press, Inc., New York, 1955); and R. D. Birkhoff, *Handbuch der Physik*, 34, 53 (1957).

⁴⁷ From Blackstock, Ritchie and Birkhoff, *Phys. Rev.*, 100, 1078 (1955).

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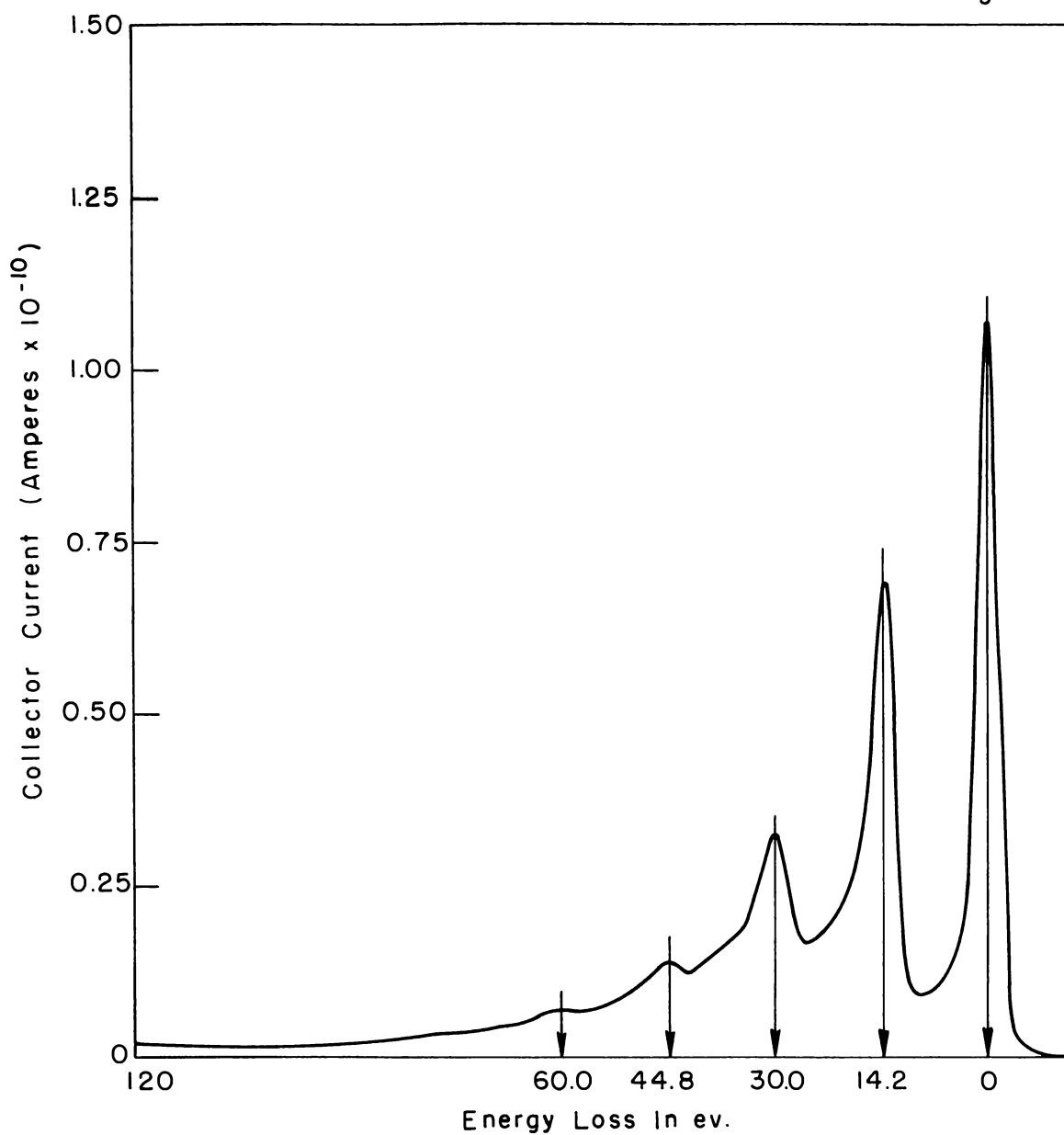


Figure 2. Spectrum of Eigenlosses of 45 keV Electrons in Aluminum.

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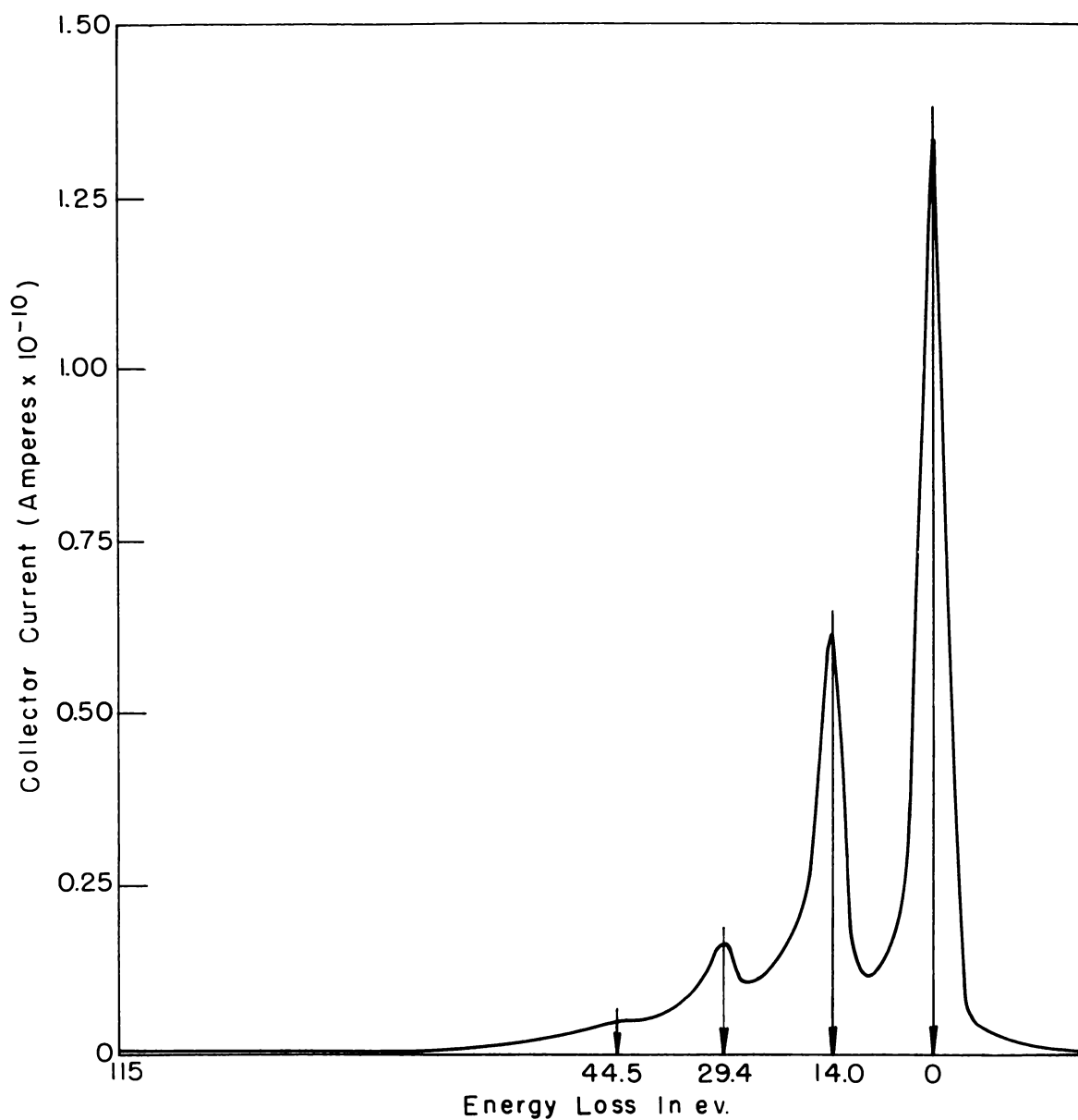


Figure 3. Spectrum of Eigenlosses of 100 keV Electrons in Aluminum.

loss unit is approximately equal to $\hbar\omega_p$ if one assumes three conduction electrons per atom and $m^* =$ the electron mass. They have compared the energy loss to plasma with a formula similar to equation (2.72), using $\rho_1 \sim 1/k_0$ and find good agreement.

An alternate explanation for these losses, which has been advanced by various workers, is that they are due to interband transitions of individual conduction electrons. Evidence in support of this has been presented⁴⁸ showing correlation between the fine structure of X-ray absorption edges and the characteristic loss lines. The plasma interpretation has been strengthened by Watanabe's⁴⁹ experimental verification of the Pines-Bohm dispersion relation (equation 3.10 below).

However, it may be shown⁵⁰ that such a dispersion relation is not uniquely a property of plasma but is associated with collective interactions in solids and appears whenever these interactions are important, e.g., in the case of the energy shift in insulators due to coulomb interactions between different atoms in the solid. Since interelectronic coulomb forces are less important in the case of bound electrons than in the case of free electrons, one expects the dependence of energy loss on momentum transfer to be correspondingly weaker in the first case.

⁴⁸ Leder, Mendlowitz and Marton, Phys. Rev., 101, 1460 (1956).

⁴⁹ H. Watanabe, J. Phys. Soc., Japan, 11, 112 (1956).

⁵⁰ R. H. Ritchie, ORNL Health Physics Annual Progress Report, (1957), ORNL-2384, p. 93.

It is probably true that no electronic transition in a solid can be distinguished unambiguously as a one-electron phenomenon, or as a collective occurrence involving many electrons. However, it is certainly important that one be able to say which description best fits a given case.

Probably the most convincing evidence for the existence of plasma oscillations has been given by Pines^{13,14} who has compared $\hbar\omega_p$ with the observed values of characteristic energy loss in several metals. (ω_p was calculated assuming that all electrons outside the appropriate closed shell are free.) He finds good agreement in just those substances for which one would expect the free electron approximation to be most nearly valid. See Table II¹⁴,

Table II.

Element	Be	B	C	Mg	Al	Si	Ge
Number of Valence Electrons Assumed	2	3	4	2	3	4	4
$\hbar\omega_p$ (ev)	19	24	25	11	16	17	16
ΔE_{obs} (ev)	19	19	22	10	15	17	17

It may be noted that optical data on the reflection and absorption of light by metal surfaces is useful in interpreting discrete loss data. In particular, optical determination of the transition frequency, i.e., that frequency at which a metal ceases to be a reflector and becomes transparent,

furnishes an independent determination of the plasma frequency for that metal. Such measurements are available for the alkali metals. The optical values agree well with characteristic loss values where the latter are available.⁵¹ Hubbard¹⁶ and Ferrell⁵² have investigated the angular dependence of the energy losses of fast electrons to plasma, the first author by a correspondence principle argument and the second by the Pines-Bohm¹¹ method. Neither of these results include the dispersive properties of the plasma.

In this section a more general semi-classical formula for the transition probability of a fast electron in plasma will be derived, while the quantum mechanical method described briefly in the summary yields a result valid for all energy and momentum losses.⁵³

In order to determine the interaction probability of a fast particle, one may employ Hubbard's¹⁶ modification of the Weizsacker-Williams⁵⁴ treatment of scattering. One assumes as before that the incident particle may be replaced by a uniformly moving point charge insofar as its effect on the medium is concerned. One then calculates the rate of energy absorption by the medium from the moving point charge. This energy absorption per unit path length may be expressed as an integral over ω and $\vec{k}$, i.e.,

$$-\frac{dW}{dz} = \int d\vec{k} \int d\omega W(\vec{k}, \omega) \quad (3.1)$$

⁵¹ Apparently the first comparison of $h\nu_{\text{optical}}$ with characteristic loss values in the alkali metals was made by Blackstock, Ritchie and Birkhoff, Ref. 47.

⁵² R. A. Ferrell, Phys. Rev., 101, 554 (1956).

⁵³ See also R. H. Ritchie, Phys. Rev., 106, 874 (1957).

⁵⁴ E. J. Williams, Kgl. Danske. Vid. Selsk. Mat. Fys. Medd., 13, No. 4 (1935).

In all the work above, this energy absorption was assumed to take place in a continuous manner. When considering elementary processes in the medium, one must take into account that the quantum of energy corresponding to a frequency ω is $\hbar\omega$ and that associated with the wave vector $\vec{k}$ there is a momentum $\hbar\vec{k}$. Then one may consider that the probability $P(\vec{k}, \omega)$ that the electron shall lose energy $\hbar\omega$ and momentum $\hbar\vec{k}$ per unit path length is given by

$$P(\vec{k}, \omega) = \frac{1}{\hbar\omega} W(\vec{k}, \omega) \quad (3.2)$$

as long as $\hbar\omega$ and $\hbar\vec{k}$ are small compared with the energy and momentum possessed by the particle. This result is proved more generally in appendix A. Referring to equation (2.27) one has

$$\phi_{\vec{k}\omega} = - \frac{4\pi^2 e}{\epsilon_{\vec{k}\omega}} \frac{\delta(\vec{k} \cdot \vec{v}_0 - \omega)}{k^2} \quad (3.3)$$

when the incident particle is an electron. The energy loss per unit path length in the medium may be found from

$$-\frac{dW}{dz} = e E_z \Big|_{\vec{k} \cdot \vec{v}_0 = \omega} = \frac{e}{v_0} \vec{v}_0 \cdot \vec{E} \Big|_{\vec{k} \cdot \vec{v}_0 = \omega} \quad (3.4)$$

where E_z is the component of electric field in the z direction, if the

incident electron proceeds in the positive z direction. Expressing φ in terms of its Fourier representation

$$-\frac{\partial \varphi}{\partial z} = \frac{e^2}{\pi^2 v_0} \int d\vec{k} \int_0^\infty \omega d\omega \operatorname{Im} \left(\frac{1}{\epsilon_{\vec{k}, \omega}} \right) \frac{\delta(\vec{k} \cdot \vec{v}_0 - \omega)}{k^2} \quad (3.5)$$

we may write immediately for the transition probability per unit path length

$$P(\vec{k}, \omega) = \frac{e^2}{\pi^2 k v_0} \operatorname{Im} \left(\frac{1}{\epsilon_{\vec{k}, \omega}} \right) \delta(\omega - \vec{k} \cdot \vec{v}_0) / k^2 \quad (3.6)$$

In the appendix it is shown that one should replace $\delta(\omega - \vec{k} \cdot \vec{v}_0)$ by $\delta(\omega - \vec{k} \cdot \vec{v}_0 + \hbar k^2 / 2m)$ to account for the recoil of the incident electron. Now divide $\vec{k}$ into $\vec{k}_\perp$ and $\vec{k}_{\parallel}$, which are perpendicular and parallel, respectively to $\vec{v}_0$. We may integrate over $\vec{k}_{\parallel}$ immediately, obtaining

$$P(\vec{k}_\perp, \omega) = \frac{e^2}{\pi^2 k v_0^2} \frac{1}{k_\perp^2 + \omega^2 / v_0^2} \operatorname{Im} \left(\frac{1}{\epsilon_{\vec{k}, \omega}} \right) \quad (3.7)$$

since one considers only small momentum transfers, i.e., $\vec{k} \cdot \vec{v}_0 \gg \hbar k^2 / 2m$.

Retaining the damping constant γ in the expression for $\epsilon_{k\omega}$, equation (2.31b) and assuming $\omega \gg k$ one finds

$$\text{Im} \left(\frac{1}{\epsilon_{k\omega}} \right) = \frac{2\gamma\omega\omega_p^2}{\{\omega^2 - \omega_p^2[1 + \langle v^2 \rangle k^2/\omega^2 + 1]\}^2 + 4\gamma^2\omega^2} \quad (3.8)$$

If one then substitutes equation (3.8) into (3.7) and employs the approximation $k_{\perp} \approx \Theta m v_0 / \hbar$ ($\Theta \ll 1$), where Θ is the angular deviation of the electron and $m v_0$ is its momentum (See Figure 4), one finds

$$P(\theta, \omega) d\Omega d\omega = \frac{\omega_p^2 e^2}{\pi^2 \hbar v_0^2} \left\{ \frac{2\gamma\omega d\omega}{[\omega^2 - \omega_p^2(1 + \delta)]^2 + 4\gamma^2\omega^2} \right\} \left[\frac{d\Omega}{\Theta^2 + (\hbar\omega/mv_0^2)^2} \right] \quad (3.9)$$

In this equation $P(\theta, \omega) d\Omega d\omega$ is the probability per unit path length that the incident electron shall lose energy $\hbar\omega$ in the interval $\hbar d\omega$ and scatter through the angle θ into the small solid angle $d\Omega$ at θ .

The quantity $\delta = \frac{3}{5} \hbar^2 k_F^2 k^2 / \omega^2$, where the explicit relation $\langle v^2 \rangle_{FD} = \frac{3}{5} \frac{\hbar^2}{m} k_F^2$ has been used. The term in curly brackets exhibits a strong resonance if $\gamma \ll \omega_p$. The resonance occurs when

$$\omega^2 = \omega_p^2 \left(1 + \frac{3}{5} \frac{\hbar^2 k_F^2}{m^2} \frac{k^2}{\omega^2} + \dots \right)$$

Solving this equation by iteration,

$$\omega^2 \approx \omega_p^2 + \frac{3}{5} \frac{\hbar^2 k_F^2}{m^2} k^2$$

and from Figure (4), the energy loss $\Delta E = \hbar\omega$ is given by

$$\Delta E \approx \hbar\omega_p + \frac{6}{5} \frac{E_0 E_F}{\hbar\omega_p} \theta^2 \quad (3.10)$$

where $E_0 = \frac{1}{2} m v_0^2$. This is just the plasma dispersion relation obtained by Pines¹³, except for terms $O(k^4)$.⁵⁵

Equation (3.9) contains the major features of the characteristic energy-angle distributions of fast electrons to plasma. The resonance term yields explicitly the parabolic connection between energy loss and angular deflection found experimentally by Watanabe⁴⁹. Figure (5) shows a schematic representation of Watanabe's result for the energy-angle distribution of losses. Curves B and B' represent single and double eigenlosses, respectively, while curve D illustrates diffuse losses which Watanabe attributes to interband transitions in the foils. He finds that an equation equivalent to equation (3.10) above represents the eigenlosses with good accuracy for several metals (Al, Be, Mg, Ge and C) if he assumes the valence electrons are free to participate in plasma oscillations.

The factor in the square brackets of equation (3.9) has the coulomb variation given by Ferrell⁵². To obtain the total angular distribution of the fast electron, one integrates over all energy losses, obtaining

$$P(\theta) d\Omega = \frac{e^2 \omega_p^2}{4\pi \hbar v_0^2} \left[\omega_p^2 + \frac{3\hbar^2 k_p^2 k^2}{8m^2} \right]^{-1/2} \frac{d\Omega}{\left\{ \theta^2 + \frac{1}{2E_0} \left(\omega_p^2 + \frac{3\hbar^2 k_p^2 k^2}{8m^2} \right) \right\}} \quad (3.11)$$

⁵⁵ A term proportional to $\hbar^2 k^4 / 4m^2$ appearing in the dispersion relation of Pines is due to the recoil of the plasma electrons and is included in the quantum mechanical treatment described below.

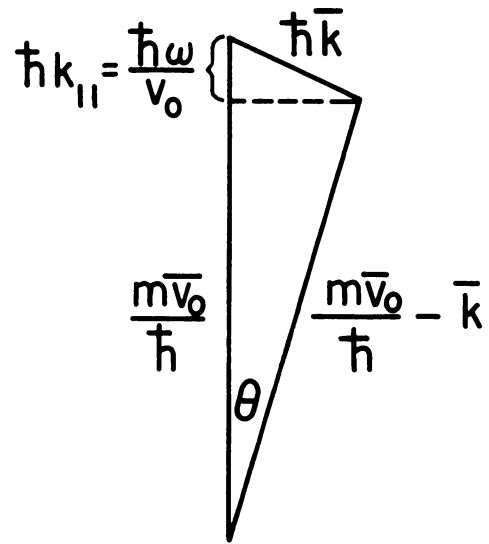
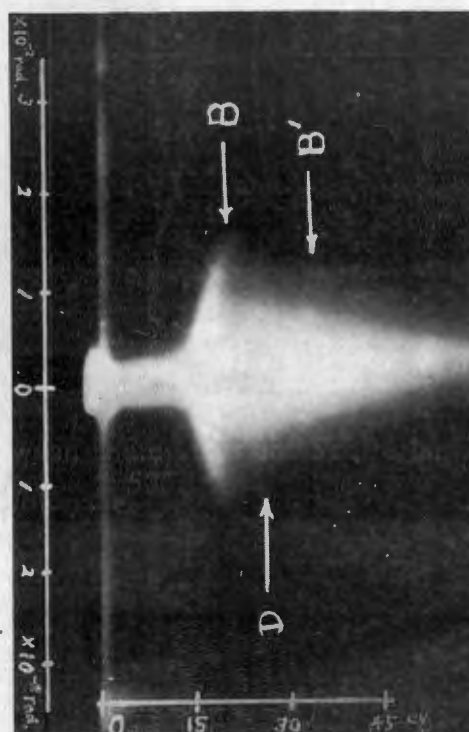
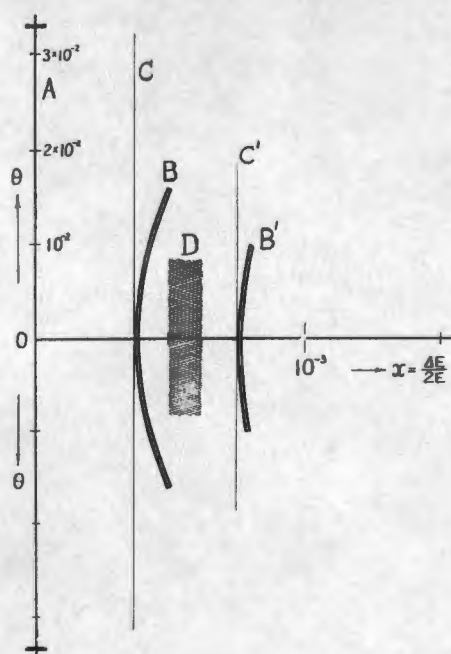


Figure 4. Energy-Momentum Diagram for a Fast Electron Losing Energy to a Plasma.



5a



5b

Figure 5. Relation Between Energy Loss and Scattering Angle for Electrons Losing Energy in Metals (After Watanabe⁵³).

where $k^2 = \omega_p^2/v_o^2 + (mv\theta/\hbar)^2$. Now, collecting terms and neglecting higher powers of $\hbar\omega_p$ and $\hbar^2k^2/2m$ compared with E_o , the energy of the incident electron, we find

$$P(\theta)d\Omega = \frac{e^2\omega_p^2}{2\pi\hbar v_o^2} \left[\omega_p^2 + \frac{e}{5} \omega_F \omega_E \theta^2 \right]^{-1/2} \frac{2\Omega}{\left\{ \theta^2 + (\omega_p/2\omega_F)^2 \right\}} \quad (3.12)$$

where $\omega_F = \hbar k_F^2/2m$ and $\omega_E = E/\hbar$.

Now, if one neglects the θ^2 term in this expression, one finds Ferrell's equation (8)⁵² and Hubbard's equation (29)¹⁶ for the angular distribution. Further, one sees that instead of $\epsilon(\vec{k}, \omega)$ for the free electron gas, one could have used the classical formula for the dielectric constant,

$$\epsilon(\omega) = 1 + \frac{4\pi N e^2}{m} \sum_i \frac{f_i}{\omega_i - i g_i \omega - \omega^2} \quad (3.13)$$

where N is the atomic density of the medium, and f_i , ω_i and g_i are the oscillator strength, transition frequency and damping constant of the i^{th} interband transition, respectively. Assuming that in the neighborhood of the j^{th} transition only the j^{th} term in this series is important, substituting this in equation (3.7) and integrating over $\vec{k}$ and ω , one

finds, for the probability of scattering through the angle θ , after losing energy to the j^{th} transition,

$$P_j(\theta) d\Omega = \frac{e^2}{2\pi k v_0^2} \frac{\alpha_j^2}{[\alpha_j^2 + \omega_j^2]^{1/2}} \frac{d\Omega}{\{\theta^2 + (\omega_j^2 + \alpha_j^2)/4\omega_E^2\}} \quad (3.14)$$

where $\alpha_j^2 = \frac{4\pi N e^2 f_j}{m}$. This again has the same form as Ferrell's equation (8)⁵² except that in place of the plasma frequency, there appears the one-electron transition frequency ω_j , shifted due to mutual interaction between electrons in the medium. In the frequency ω_j corresponding to the case of free electrons, then $\alpha_j = \omega_p^2$. Thus, if the k dependence of the dielectric constant of the conduction electrons is neglected, one obtains the same general form for the angular distribution, whether the fast electron has excited interband transitions in the solid or whether it has lost energy to plasma oscillations. It may be shown⁵⁰, using a special model for the dielectric constant of a solid, that this is still true even if the k dependence is retained, although in general one expects a much weaker k dependence in the case of interband transitions.

CHAPTER IV

THE INTERACTION OF ELECTRONS WITH A FINITE PLASMA

The rapid growth of that field of physics devoted to the study of "characteristic" energy losses in solids⁴⁶⁻⁴⁹ and the promise shown by the theoretical interpretation of Pines and others⁴⁶ of these losses make it likely that fast electron beams will become even more important in future studies of electronic states of solids. One of the central problems in this field is whether such losses occur with the excitation of collective oscillations in the conduction electron sea or with inter-band transitions in the solid.

In reality these two descriptions are complementary and represent opposite views of the dynamical behavior of electrons in solids. A given solid partakes of both of these extreme characteristics but may be best described by one or the other of the models. It is probably true that no electronic transition can be distinguished unambiguously as a one-electron phenomenon or as a collective phenomenon: e.g., in Chapter III it was shown that the angular distribution of charged particles losing energy to solids is insensitive to the kind of electronic interaction occurring. However, it is certainly important to be able to say which of the descriptions best fits a given case.

One expects that the presence of boundaries in a solid will have more effect on the response of a plasma than on the response of the one-electron model of the solid and hence that surface effects may be useful in distinguishing between the two types of transitions; e.g., in the extreme tight binding approximation for electronic states in a solid, coulomb interactions between electrons bound to different ion cores are very weak and in this case the surface effects treated below are unimportant. It is felt that the bounded plasma model is of considerable interest, since it represents a case where quantum effects should appear on a scale of distance much larger than is usual in physics.

In this chapter, electron energy losses to an idealized model of a finite plasma will be considered and the effect of the finiteness of the foil on characteristic energy loss experiments will be investigated.⁵⁶

Gabor⁵⁷ has treated the finite foil problem by making the assumption that the electric field intensity is zero at the boundaries of the foil. This is not in accord with experimental work on the optics of thin films, since it results in the prediction that a metal grain does not interact with transverse electromagnetic radiation originating outside of the grain. His conclusion that the probability for plasma loss should decrease strongly with decreasing foil thickness is a direct consequence of this assumption.

⁵⁶ R. H. Ritchie, Phys. Rev., 106, 874 (1957).

⁵⁷ D. Gabor, Phil. Mag., 1, 1 (1956).

The difference which he finds between "coherent" and "incoherent" illumination is due to the same assumption.

To compare with Gabor's treatment, one may expand the field given by equation (3.3) in terms of a Fourier integral in x , y and t as a Fourier series involving only cosine functions of the z coordinate, thereby assuming that the electric field intensity is zero both at the boundaries and everywhere outside the foil. Then, using the method described above to calculate the interaction probability $P(\vec{k}, \omega)$ in the foil, one finds a factor which shows the same limitation obtained by Gabor on x -momentum transfer to the foil.⁵⁸ More realistic boundary conditions will be used in the treatment given below and it will be seen that similar z momentum transfer restrictions occur, but that the behavior of the interaction cross section predicted below is different from that given by Gabor.

⁵⁸ Gabor's unconventional application of first-order time dependent perturbation theory leads to a result which disagrees with the Pines-Bohm formula for the mean free path for plasma loss in thick films. He assumes that the wave function of the incident electron is a plane wave with duration $\tau \gg 1/\omega_p$, where $\hbar\omega_p$ is the energy loss of the incident electron. He finds that the total probability of plasma excitation is proportional to the duration τ which leads to an unrealistic result for an infinitely long wave-train. The more conventional perturbation theory (L. I. Schiff, QUANTUM MECHANICS, McGraw-Hill Book Co., Inc., New York, 1949, second edition, p. 200) shows that instead of τ one should use a/v , the electron "transit time" in the foil. Experimental evidence on the energy variation of mean free path for plasma loss (Blackstock, Ritchie and Birkhoff, Phys. Rev., 100, 1078 (1955) shows good agreement with the Pines-Bohm formula and does not agree as well with Gabor's Equation (34).

Since the methods of Chapter II are not readily adaptable to the study of the bounded plasma, the hydrodynamical treatment of Bloch^{59,60} will be employed to describe the response of the finite plasma to an incident charged particle. Taking account of the pressure-density relation for a degenerate electron gas, Bloch derived the equations of motion of the statistical atom. He showed that these equations admit not only the steady state solutions familiar from the work of Thomas and Fermi, but also oscillations about this state. He used the approximate correspondence between the frequencies of these hydrodynamical disturbances and the actual excitation frequencies of the atomic electrons to derive a general formula for the stopping power of atoms of atomic number Z for fast charged particles. Wheeler and Fireman⁶¹ have extended the work of Bloch to obtain a universal photoelectric absorption curve for the statistical atom.

In the following it will be assumed that the plasma has uniform density inside the bounded region and that the density decreases abruptly to zero at the boundary.

Only irrotational motion of the electron gas will be considered and all velocities will be taken small compared with c . The usual equation

⁵⁹ F. Bloch, Z. Physik, 81, 363 (1933); Helv. Phys. Acta. 1, 358 (1934).

⁶⁰ H. Jensen, Z. Physik, 106, 620 (1937).

⁶¹ J. A. Wheeler and Fireman, "On a Universal Photoelectric Absorption Curve", unpublished manuscript.

of motion in hydrodynamics⁶² may be written

$$\frac{\partial}{\partial t} \vec{v} + \frac{1}{2} \nabla (v^2) = \vec{F} - (\nabla p) / \rho \quad (4.1)$$

where $\vec{v}(r,t)$ is the linear velocity of the electron gas at point with position vector r and at time t , $\vec{F}$ is the body force per unit mass of the gas, p is the hydrodynamical pressure and ρ is the mass density of the gas. The equation of continuity is

$$\nabla \cdot (\rho \vec{v}) = - \partial \rho / \partial t \quad (4.2)$$

while the body force F is taken to be due to the coulomb field of charges located in the medium;

$$\vec{F} = \frac{e}{m} \nabla \phi$$

where

$$\nabla^2 \phi = 4\pi \left[\frac{e}{m} \rho(\vec{r}, t) - q_{ext} \right] \quad (4.3)$$

⁶² See, e.g., L. Page, INTRODUCTION TO THEORETICAL PHYSICS, (New York, D. Van Nostrand, 1935, second edition), page 220.

and q_{ext} is the charge density of applied charge in the plasma. In order to derive a connection between the pressure and density of the electron gas. We shall suppose that the gas is a Fermi-Dirac degenerate system at absolute zero. Then n , the density of particles is given by

$$n = N/V = \frac{8\pi}{3} \frac{p_F^3}{h^3}$$

where N is the total number of electrons in the volume V . The total kinetic energy E_1 is

$$E_1 = \frac{3}{10} \frac{h^2}{m} \left(\frac{3}{8\pi} \right)^{2/3} N^{5/3} V^{-2/3} \quad (4.4)$$

and the pressure p is given by

$$p = - \frac{dE_1}{dV} = \alpha (p/m)^{5/3} \quad (4.5)$$

where

$$\alpha = \frac{h^2}{5m} \left(\frac{3}{8\pi} \right)^{2/3}$$

One obtains

$$\frac{1}{2} \bar{U} + \frac{1}{2} \bar{\nabla}(\bar{U}^2) = \frac{e}{m} \bar{\nabla} \phi - \frac{\alpha}{\rho} \bar{\nabla}(\rho/m)^{5/3} \quad (4.6)$$

which, together with equations (4.2) and (4.3), defines the Bloch approximation to the plasma system.

Although somewhat simpler in appearance than the plasma equations dealt with before, these equations must be linearized before progress can be made in finding their solutions. Expanding ρ and $\vec{v}$ about their equilibrium values

$$\begin{aligned}\rho &= \rho_0 + \rho_1 + \dots \\ \vec{v} &= \vec{v}_1 + \dots\end{aligned}$$

setting $\rho = mn$, where n is the number density of electrons and neglecting terms involving squares of n_1 , v , or their products, equations (4.2), (4.3) and (4.6) become

$$\frac{\partial}{\partial t} \bar{v}_1 = \frac{e}{m} \nabla \phi - \frac{5kT}{3n_0} \nabla n_1 \quad (4.7a)$$

$$n_0 \nabla \cdot \bar{v}_1 = - \partial n_1 / \partial t \quad (4.7b)$$

$$\nabla^2 \phi = 4\pi e \{ n_1 - q_{ext} \} \quad (4.7c)$$

Assuming that the velocity $\bar{v}_1$ is derivable from a potential function ψ , so that $\bar{v}_1 = - \nabla \psi$, the Bloch equations become

$$\gamma \phi + \frac{\partial}{\partial t} \psi = - \frac{e}{m} \phi + \beta^2 n / n_0 \quad (4.8a)$$

$$\frac{\partial}{\partial t} n = n_0 \nabla^2 \psi \quad (4.8b)$$

$$\nabla^2 \phi = 4\pi e n + 4\pi e \delta(x) \delta(y) \delta(z - v_0 t) \quad (4.8c)$$

where $\beta^2 = \frac{1}{3} (v_F^2)$ is the velocity of propagation in the Bloch electron gas. The term $\gamma\psi$ in equation (4.8a) represents a schematic damping of the electronic motion in analogy with the damping introduced in Chapter I in the derivation of the dielectric constant of the free electron gas. The subscript on the number density perturbation n_1 will be dropped henceforth. The applied density of charge has been taken to be that of a classical point charge moving with uniform velocity.

To establish the connection between this hydrodynamical approach and the dielectric constant approach, consider the unbounded plasma. Operating on equation (4.8a) with the Laplacian, on equation (4.8b) with $\frac{\partial}{\partial t}$ and combining the three equations (4.8) we find

$$\nabla^2 \psi + \gamma \dot{\psi} + \omega_p^2 \psi - \beta^2 \nabla^2 \dot{\psi} = -\omega_p^2 \delta(x) \delta(y) \delta(z - v_0 t) \quad (4.9)$$

Going to Fourier space,

$$n_{\vec{k}, \omega} = \frac{-2\pi \omega_p^2 \delta(\omega - \vec{k} \cdot \vec{v}_0)}{\omega_p^2 + \beta^2 k^2 - i\gamma\omega - \omega^2} \quad (4.10)$$

and using the definition of the dielectric constant $\epsilon = n_{\text{real}} \left(\frac{1}{\epsilon} - 1 \right)$, where n_{real} is the density of externally introduced electrons in the plasma, we find

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega - \beta^2 k^2} \quad (4.11)$$

In case $v_0 \ll v_F$, this reduces to the low velocity form of the dielectric constant, equation (2.49)

$$\epsilon \rightarrow 1 + 3\omega_p^2/v_F^2 k^2$$

If $v_0 \gg v_F$, we may expand equation (4.11) in powers of k^2 ; the result is

$$\epsilon \rightarrow 1 - \frac{\omega_p^2}{\omega^2} \left(1 + v_F^2 k^2 / 3\omega^2 + \dots \right) \quad (4.12)$$

which has the same form as equation (2.39). The numerical coefficient of the k^2 term in equation (4.12) is $\frac{1}{3}$ instead of $\frac{3}{5}$ as given by equation (2.39). One sees that the main features of the extreme high and low velocity behavior of plasma are reproduced by the hydrodynamical model. In the work below, when the plasma response to high velocity particles is considered, it is understood that the relation $\beta^2 = \frac{3}{5} v_F^2$ is to be employed.

Note that the equation (4.9) for n has the same form as the Klein-Gordon equation and thus may be thought of as describing a meson-like "particle" field around a charged particle in plasma.⁶³

One may employ this similarity to investigate qualitatively quantum effects arising because the plasma is confined to a finite range in the z direction. The homogeneous form of the equation for the variation of

⁶³ See J. Lindhard, Kgl. Danske. Vid. Sels., 28, No. 8 (1954).

plasma density around equilibrium may be written

$$\nabla^2 n - \frac{1}{\beta^2} \frac{\partial^2 n}{\partial t^2} - \frac{\omega_p^2}{\beta^2} n = 0 \quad (4.13)$$

One could carry out a quantization of this classical field equation in a manner similar to that employed by Landau⁶⁴ in his quantum treatment of the classical hydrodynamical equations. The field n and its conjugate n^* then become operators which in the case of an infinite medium could be expanded in plane waves, i.e.,

$$n \sim e^{i(k \cdot r - \omega t)}$$

If one substitutes this into equation (4.13) one finds the following relation between the energy and momentum of the "plasmon"

$$\hbar \omega_p = \hbar \omega = \sqrt{(\hbar \omega_p)^2 + (\beta \hbar k)^2} \quad (4.14)$$

showing that the rest energy of the plasmon is $\hbar \omega_p$ and that it has a rest mass of

$$m_p = \hbar \omega_p / \beta^2$$

which is of the order of the electron mass. In this model, a charged particle is capable of exciting "plasmons", the quanta of the hydrodynamical field. An amount of energy $\sqrt{(\hbar \omega_p)^2 + (\beta \hbar k)^2}$ is required to create a plasmon and simultaneously to impart to it momentum $\hbar k$. In the case

⁶⁴ L. D. Landau, J. Phys. USSR, 5, 71 (1941). For reference to later work in this field, see Van Kampen, Kgl. Danske. Vid. Sels., 26, No. 15 (1953).

of the finite plasma confined between parallel planes at distance a apart, one would expand the field n in terms of eigenfunctions of the bounded region. One knows generally that the minimum value which can be assumed by the z component of the wave vector $\vec{k}$ in this case is

$$k_z \sim 1/a$$

and hence that the minimum amount of z momentum which can be absorbed by the finite medium plasmon is $\sim \hbar/a$. This effect will appear in the thin foil case below.

A Slowly Moving Particle in the Bounded Plasma

Consider now the case of a stationary or slowly moving particle in the semi-infinite plasma which is confined to the region $z > 0$. Assume that there is a point charge located on the z axis at $-z_0$, outside of the plasma. In the time-independent case, the equation (4.9) becomes,

$$\nabla^2 n - \frac{1}{d^2} n = 0 \quad (d^2 = \beta^2/\omega_p^2) \quad (4.15)$$

while Poisson's equation reads

$$\nabla^2 \phi = 4\pi e \left[n + \delta(x) \delta(y) \delta(z + z_0) \right] \quad (4.16)$$

subject to the boundary condition

$$\frac{\partial \phi}{\partial z} = 4\pi e d^2 \frac{\partial n}{\partial z} \quad (4.17)$$

This follows from the requirement that the component of $\vec{v} = -\nabla\psi$ normal to the surface shall vanish.

If one expands both n and ψ in Fourier integrals in the x and y coordinates, e.g.,

$$n(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y e^{i(k_x x + k_y y)} n(k_x, k_y | z)$$

The equations (4.15) through (4.17) become

$$\left\{ \frac{d^2}{dz^2} - (k^2 + \frac{1}{d^2}) \right\} n = 0 \quad (4.15a)$$

$$\left\{ \frac{d^2}{dz^2} - k^2 \right\} \psi = 4\pi e \left\{ n + \delta(z+z_0) \right\} \quad (4.16a)$$

$$\frac{d}{dz} \psi = 4\pi e d^2 \frac{\partial}{\partial z} n \quad (z \rightarrow 0+) \quad (4.17a)$$

where $K^2 = k_x^2 + k_y^2$. The source free solution of equation (4.15a) which is bounded as $z \rightarrow \infty$ may be written

$$n(k|z) = A e^{-\sqrt{k^2 + d^{-2}} z}$$

and employing the Green's function for equation (4.16a) one may write

$$\psi(k|z) = -\frac{2\pi e}{k} \left\{ e^{-k|z+z_0|} + \int_0^\infty e^{-k|z-z'|} n(k|z') dz' \right\}$$

If one imposes the boundary condition (4.17a) the constant A may be evaluated. One finds for the potential outside the plasma

$$\phi(\rho, z < 0) = -\frac{e}{\sqrt{\rho^2 + (z+z_0)^2}} + \frac{e}{d} \int_0^\infty \frac{J_0(\xi \rho/d) e^{-\xi(z-z_0)/d}}{(\xi + \sqrt{\xi^2 + 1})^2} d\xi \quad (4.18)$$

$$= \phi_0 + \phi_1$$

where $\rho^2 = x^2 + y^2$. The first term on the right hand side of equation (4.18) is just the potential due to the point charge at $(0,0,-z_0)$.

The second term represents the potential in $z < 0$ due to electrons in the plasma in the region $z > 0$. To look at its behavior for $(z-z_0), \rho \gg d$, examine the integrand of ϕ_1 for $\xi \ll 1$. One finds

$$\phi_1 \rightarrow e / \sqrt{\rho^2 + (z-z_0)^2}$$

This is just the image potential which one finds in the classical case of a point charge located approximately distance Z_0 from an infinite metal surface.

The expression for ϕ_1 is of interest in the theory of the potential barrier at a metal surface. The reflection and transmission of electrons through metal surfaces is of importance in the theory of the Shottky effect and in predicting thermionic emission from metals as well as in the prediction of experiments on the reflection of monoenergetic electrons

impinging on metals. Reflection calculations⁶⁵ are usually made by assuming that the potential acting on an electron outside a metal is of the image form for large distances but breaks down at a few angstroms from the surface. Bardeen⁶⁶ has made a quantum mechanical calculation of this image potential by assuming free electron wave functions inside the metal and requiring that the potential be self-consistent. His results show that the classical image potential is valid a few atomic radii from the surface but that the potential approaches a nearly constant value inside at smaller distances. His results are mainly numerical. More recently, Sachs and Dexter⁶⁷ have calculated an approximate quantum mechanical correction to the classical image force theory using the Thomas-Fermi equations. They find a correction term to the classical image formula which gives approximate agreement with the result of Bardeen. Although they formulate an equation which could be solved in theory to give the potential function continuously through the surface and into the metal, the final expression is only approximately applicable outside the metal.

The Bloch equations employed here, on the other hand, yield a prediction of the potential at all points in and outside of the metal and have the same limits of validity, since they are based in part on the Thomas-Fermi equations.

⁶⁵ See, e.g., L. A. MacColl, Phys. Rev., 56, 699 (1939); L. W. Nordheim, Proc. Roy. Soc., A121, 626 (1928); E. Guth and C. J. Mullin, Phys. Rev., 59, 575 (1941); C. Herring and M. H. Nichols, Rev. Mod. Phys. 21, 185 (1949).

⁶⁶ J. Bardeen, Phys. Rev., 49, 653 (1936).

⁶⁷ R. G. Sachs and D. L. Dexter, J. Appl. Phys., 21, 1304 (1950).

The image potential evaluated at the position of the point charge is, from equation (4.18)

$$\phi(0, z_0) \equiv \phi_{\text{image}} = \frac{e}{d} \int_0^\infty \frac{e^{-2\zeta z_0/d}}{(\zeta + \sqrt{\zeta^2 + 1})^2} d\zeta = \frac{e}{2d} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\frac{z_0}{d}\right)^n \left\{ E_{n+2}\left(\frac{z_0}{d}\right) + E_{n+4}\left(\frac{z_0}{d}\right) \right\}$$

where

$$E_n(x) = \int_0^\infty e^{-xu} du / u^n$$

To obtain ϕ image inside of the metal, one proceeds as before except that $z_0 \rightarrow -z_0$ and there is a source term $\frac{1}{2} \delta(Z - z_0)$ on the right hand side of equation (4.15a). One finds

$$\phi(\xi, z > 0) = -e \frac{e^{-\sqrt{\xi^2 + (z-z_0)^2}/d}}{\sqrt{\xi^2 + (z-z_0)^2}} - \frac{e}{d} \int_0^\infty \frac{\zeta d\zeta J_0(\zeta \rho/d) e^{-(z+z_0)\sqrt{1+\zeta^2}/d}}{\sqrt{1+\zeta^2} (\zeta + \sqrt{1+\zeta^2})^2}$$

As in the case of the infinite plasma, the potential in the neighborhood of a point charge at greater than a few angstroms from the surface is of the Debye screened form. The image potential is given by $\phi(0, Z_0)$

$$\begin{aligned} \phi_{\text{image}} &= -\frac{e}{d} \int_0^\infty \frac{\zeta d\zeta e^{-2z_0\sqrt{1+\zeta^2}/d}}{\sqrt{1+\zeta^2} (\zeta + \sqrt{1+\zeta^2})^2} \\ &= -\frac{e}{2d} \sum_{n=0}^\infty \frac{1}{n!} \left(-\frac{2z_0}{d}\right)^n \left\{ E_{n+2}\left(\frac{z_0}{d}\right) - E_{n+4}\left(\frac{z_0}{d}\right) \right\} \end{aligned}$$

The form of the image potential is sketched in Figure (6). For comparison, the dotted line shows the simple image potential $\phi = -e/2z_0$.

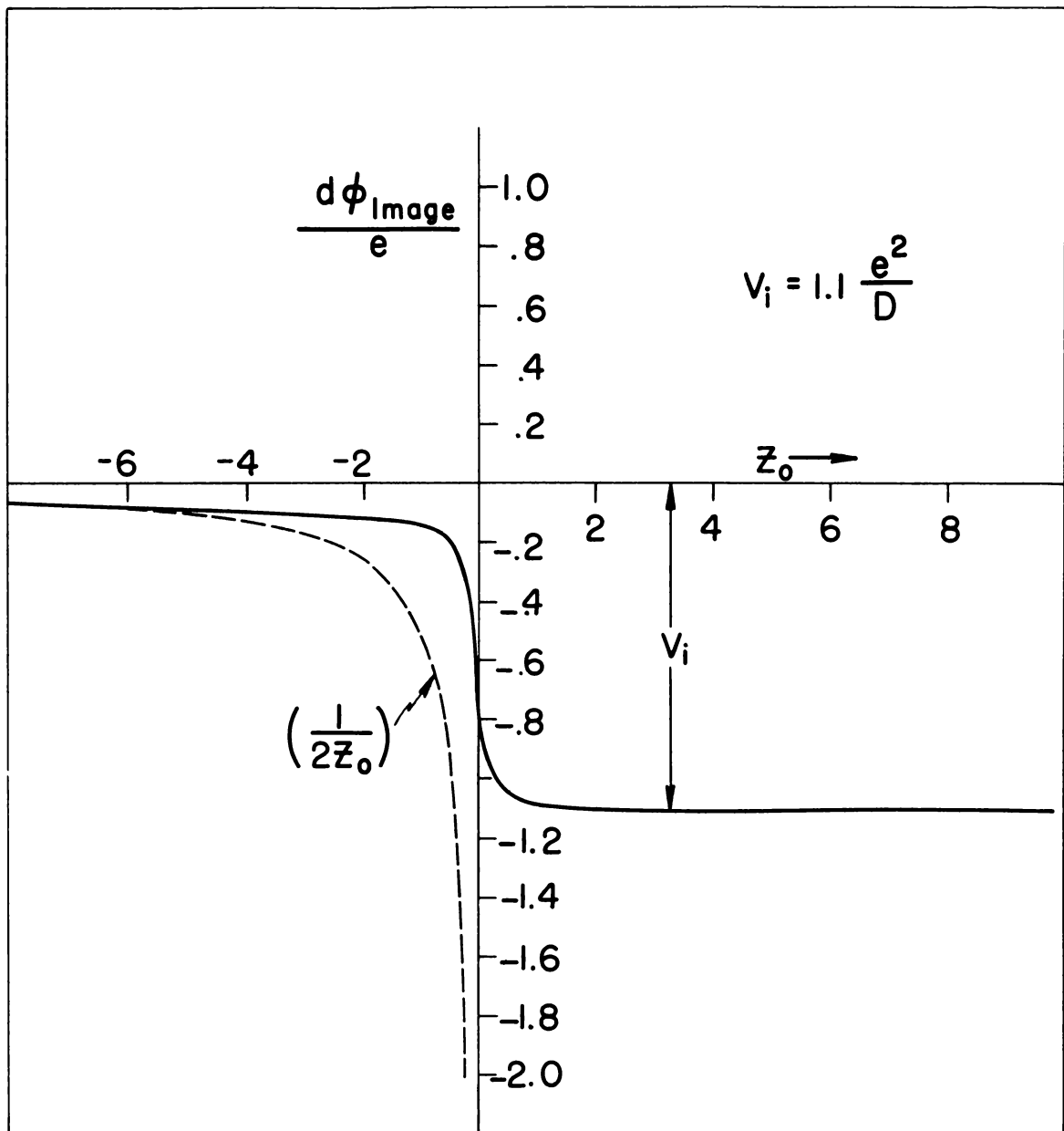


Figure 6. Image Potential in the Neighborhood of a Metal Surface.

The discontinuity at $z_0 = 0$ arises from the Sommerfeld assumption that the potential in the neighborhood of a metal surface changes abruptly at the surface. The inner potential V_1 has been taken to be $1.1 e^2/d$ in the case illustrated in Figure (6). Although the result is of qualitative significance only, due to the schematic step function representation assumed for the static variation in potential at the surface, it does indicate that the simple image potential is valid only at relatively large distances from the surface. The approximation, employed by MacColl⁶⁵, that $\phi_{\text{image}} = -e/2z_0$ for z_0 larger than the value at which $-e/2z_0$ is equal to the inner potential V_1 , seems to be very unrealistic in view of the present results.

A Fast Particle in the Bounded Plasma

The case of a fast moving charge normally incident on the plane boundary of a hydrodynamical electron gas will now be considered. As before, one eliminates the velocity potential ψ from equation (4.8) and finds a set of coupled equations in the electronic density n and the scalar potential ϕ . After expanding both of these quantities in Fourier integrals the equations become

$$\left(\frac{1}{d^2} - k^2\right) n(k, \omega | z) = \frac{1}{v_0 d^2} e^{i\omega z/v_0} \quad (4.19a)$$

$$\left(\frac{1}{d^2} - k^2\right) \phi(k, \omega | z) = 4\pi e \left(n + \frac{1}{v_0} e^{i\omega z/v_0}\right) \quad (4.19b)$$

with the condition that the z component of velocity at the surface located at $z = 0$ be nil, i.e.,

$$\frac{\partial \phi}{\partial z} = 4\pi e d^2 \frac{\partial n}{\partial z} \quad @ \quad z = 0 + \quad (4.19c)$$

where $\Gamma^2 = \kappa^2 + \frac{1}{d^2} = \frac{1}{\rho^2} (\omega^2 + i\gamma\omega)$

The solution of equation (4.19a) which is bounded as $z \rightarrow \infty$ is

$$n(\kappa, \omega | z) = A e^{-\Gamma z} - \frac{e^{i\omega z/v_0}}{v_0 d^2 (\Gamma^2 + \omega^2/v_0^2)} \quad (4.20)$$

where A is a constant to be determined from equation (4.19c). The solution of (4.19b) may be written in terms of the Green's function for the differential equation, i.e.,

$$\phi(\kappa, \omega | z) = -\frac{2\pi e}{\kappa} \int_0^\infty e^{-\kappa|z-z'|} n(z') dz' - \frac{+\pi e}{v_0} \frac{e^{i\omega z/v_0}}{(\kappa^2 + \omega^2/v_0^2)}$$

Carrying out the indicated integration after inserting the expression for $n(z')$ from (4.20), one then evaluates the constant A . The potential ϕ may be written

$$\phi^+(\kappa, \omega | z) = A_1 e^{-\Gamma z} + A_2 e^{-\kappa z} + A_3 e^{i\omega z/v_0} \quad (z > 0) \quad (4.21a)$$

$$\phi^-(\kappa, \omega | z) = B_1 e^{\kappa z} + B_2 e^{i\omega z/v_0} \quad (z < 0) \quad (4.21b)$$

where the constants A_1, B_1 are very lengthy and will not be written in full.

Consider the energy absorbed by the medium from the incident electron. Let the electric potential in the medium minus the vacuum field of the particle be $\phi_m(\vec{r}, t)$ and let $\phi_m(\vec{k}, \omega)$ be its Fourier transform. The total energy loss of the electrons is given by

$$W = -e \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \phi_m(\vec{r}, t) \Big|_{\substack{r=y=z=0 \\ z=v_0 t}} dt = -\frac{ie}{(2\pi)^3 v_0} \int d\vec{k} \int d\omega \omega \delta(k_z - \omega/v_0) \phi_m(\vec{k}, \omega) \quad (4.22)$$

where k_z is the z component of $\vec{k}$. This result is an expression of the fact that if the electron undergoes momentum change $\hbar k_z$ in the z direction and is deflected through an angle θ (where $\theta \ll 1$) then from conservation of energy and momentum $\hbar k_z = \Delta E / v_0$ where ΔE is its energy loss. As shown in appendix A, the wave properties of the incident electron are taken into account, within the domain of validity of the Born approximation, by merely replacing $\delta(k_z - \omega/v_0)$ by $v_0 \delta(\omega - v_0 k_z + \hbar k^2/2m)$ in equation (4.22). The additive term $\hbar k^2/2m$ accounts for the recoil of the electron.

Substituting from equation (4.21) into (4.22) and carrying out the indicated integrations, one finds in the limit $v_0 \gg \beta$,

$$W = \lim_{a \rightarrow \infty} \frac{ae^2 v_0}{\pi v_0^2} \int_0^\infty k dk \int_{-\infty}^\infty \frac{\omega d\omega}{(k^2 + \omega^2/v_0^2)} \left(\frac{1}{\epsilon} - 1 \right) \quad (4.23)$$

$$= \frac{2Le^2}{\pi v_0^2} \int_0^\infty k^2 dk \int_{-\infty}^\infty \frac{\omega d\omega}{(k^2 + \omega^2/v_0^2)^2} \frac{(1-\epsilon)^2}{\epsilon [1 + \epsilon + 2a^2 (k^2 - \omega^2/v_0^2)]}$$

where a is the thickness of the foil and where $\epsilon = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega - \beta^2 k^2}$.

The first term clearly represents the conventional energy loss in the absence of the boundary, while the second contains the effect of the surface⁶⁸. As in Chapter III, set $W = \int_0^\infty 2\pi K dK \int_0^\infty d\omega \hbar \omega P(K, \omega)$ where $P(K, \omega)$

is the probability that the electron shall undergo energy loss $\hbar \omega$ and momentum change $\hbar K$ perpendicular to the z direction. Then writing $K = k_\perp$ to emphasize the similarity with equation (3.7), one has

$$P(k_\perp, \omega) = \frac{e^2}{\pi^2 \epsilon_0^2} \left\{ \frac{a}{k_\perp^2 + \omega^2 / v_0^2} \operatorname{Im} \left(\frac{1}{\epsilon} \right) - \frac{a k_\perp}{(k_\perp^2 + \omega^2 / v_0^2)^2} \operatorname{Im} \left(\frac{(1-\epsilon)^2}{\epsilon (1+\epsilon + 2a^2 [k_\perp^2 - \omega^2 / v_0^2])} \right) \right\} \quad (4.24)$$

$$\equiv a P_\omega' + P_b$$

where

$$P_\omega' = \frac{e^2}{\pi^2 \epsilon_0^2} \left\{ \frac{1}{(k_\perp^2 + \omega^2 / v_0^2)} \cdot \frac{\gamma \omega \omega_p^2}{[\omega^2 - (\omega_p^2 + \beta^2 k_\perp^2)]^2 + \gamma^2 \omega^2} \right\}$$

The P_ω' distribution agrees with that found in Chapter III (equation 3.8) for the infinite medium. Energy resonance occurs at the value given in equation (3.10) showing the same connection between energy loss and angular deviation as is found for the infinite medium, providing one takes

$\beta^2 = \frac{3}{5} v_F^2$. The P_b distribution may be rewritten as follows, again using the fact that $\beta/v \ll 1$;

$$P_b = \frac{2e^2 \omega_p^2}{\pi^2 \epsilon_0^2} \frac{k_\perp}{(k_\perp^2 + \omega^2 / v_0^2)^2} \cdot \left\{ \frac{\gamma \omega}{[\omega^2 - \frac{\omega_p^2}{2} - \beta k_\perp \omega / v_0]^2 + \gamma^2 \omega^2} - \frac{\gamma \omega}{[\omega^2 - (\omega_p^2 + \beta^2 k_\perp^2)]^2 + \gamma^2 \omega^2} \right\} \quad (4.25)$$

$$\equiv P_{b1} + P_{b2}$$

⁶⁸

The second term found in this derivation has been multiplied by 2 to take into account both surfaces of a thick but finite foil. As is shown below, a foil is considered thick here if $a \gg v_0 / \omega_p$.

The P_{b2} distribution is closely related to P_{ω}' . Resonance occurs at the same energy, and the connection between energy loss and angular deflection is the same as that occurring in the infinite medium, but the angular distribution of eigenlosses is very different. The net effect of the P_{b2} term is to decrease the total energy loss at $\hbar\omega_p$.

The P_{b1} distribution shows that energy loss may occur at a lowered value, viz, $\hbar\omega_g$, where

$$\omega_g^2 = \omega_p^2/2 + \beta k_{\perp} \omega_p/\sqrt{2} \quad (4.26)$$

or at an energy $\Delta E_g = \hbar\omega_p/\sqrt{2} + \frac{1}{2}\sqrt{3/5}\sqrt{E_0 E_F}\theta$ and that the connection between energy loss and angular deflection is markedly different than for the infinite medium. Figure (7) shows the distribution of eigenlosses predicted by the hydrodynamical theory of the finite plasma. The heavy parabola, curve A, indicates the losses of high intensity at the full plasma energy. The lighter straight line indicates the losses of lower intensity at the lowered plasma frequency, $\hbar\omega_p/\sqrt{2}$.

If one now integrates over ω , neglecting the k dependence of the resonance, one finds

$$P_{b1} = \frac{\theta^2 \omega_p^2}{\pi k v_0^2} \frac{\sqrt{2}}{\omega_p} \frac{k_{\perp}}{(k_{\perp}^2 + \omega_p^2/2v_0^2)^2} \quad (4.27)$$

and

$$P_{b2} = - \frac{e^2 \omega_p^2}{\pi k v^2} \frac{1}{\omega_p} \frac{k_{\perp}}{(k_{\perp}^2 + \omega_p^2/v_0^2)^2} \quad (4.28)$$

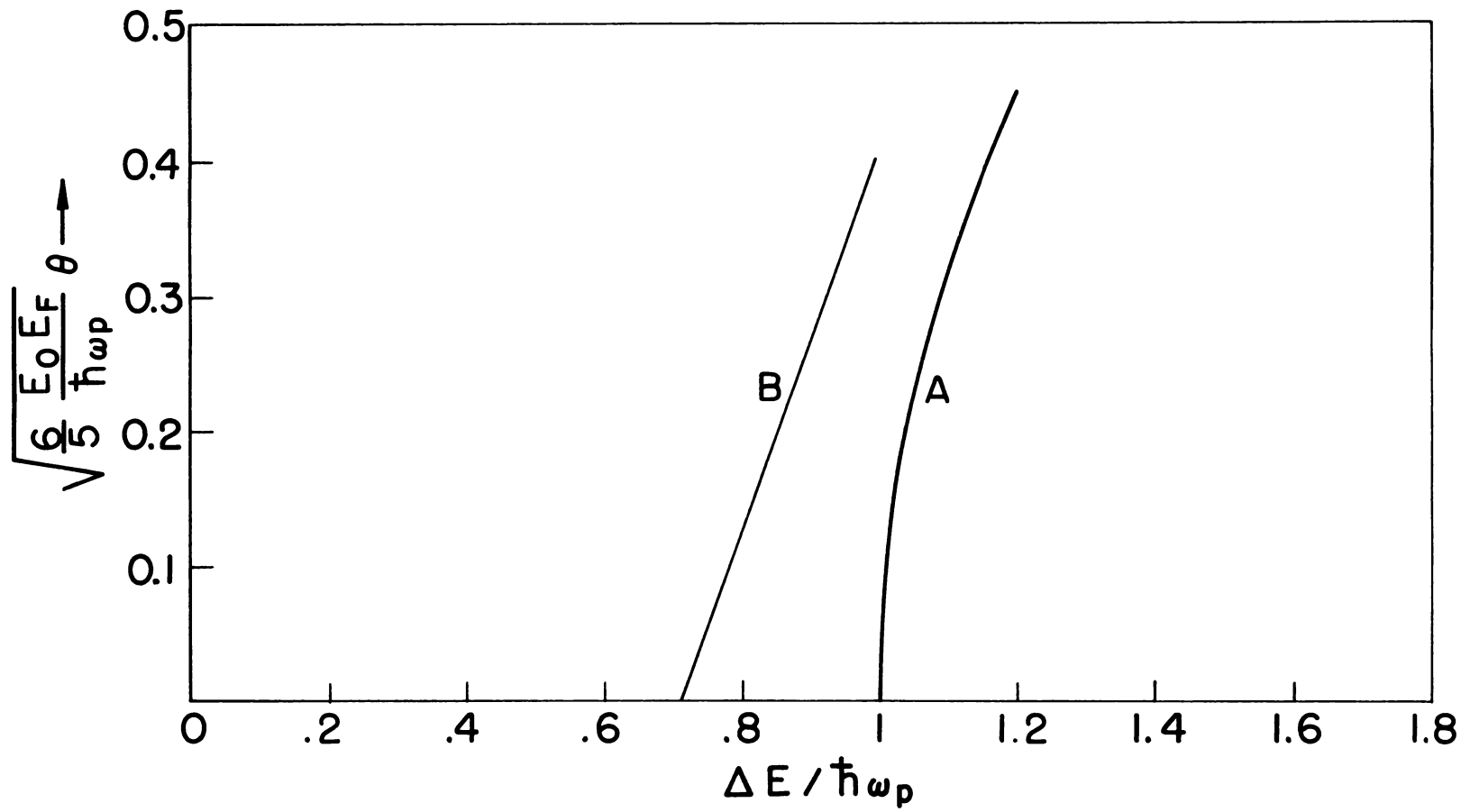


Figure 7. The Relation Between Energy Loss and Angular Deflection in the Semi-Infinite Plasma.

If one replaces $k_{\perp}$ by $mv_0 \theta / \hbar$ as in Chapter III one sees that both distributions are strongly peaked at angles greater than zero and decrease like $\frac{1}{\theta^3}$ for $\theta \gg \frac{\hbar \omega_p}{2E}$, in contrast to the $\frac{1}{\theta^2}$ dependence found for the infinite medium. An integration over all angles does not require the cutoff necessary to obtain a finite result in the case of P_{∞}' .

Carrying out the integrations over angle, one finds

$$P_{b1} = \pi e^2 / \kappa \sigma_0 \quad (4.29)$$

$$P_{b2} = -\pi e^2 / 2 \kappa \sigma_0 \quad (4.30)$$

The P_{b2} term subtracts from $a \cdot P_{\infty}'$, showing that the loss at $\hbar \omega_p$ should decrease rapidly as the foil thickness decreases, while the loss at $\hbar \omega_p / \sqrt{2}$ is negligible compared with $a P_{\infty}'$ for thick foils but increases in importance as a decreases.

The resonance at the lowered frequency is due to the depolarizing effect of the surfaces of the foil.⁶⁹ Jensen⁶⁰ has shown that the resonance frequency of plasma contained in a small sphere is less than the value appropriate to an infinite plasma by a factor of $1/\sqrt{3}$ and that this shift is due to the depolarizing effect of surface charge on the sphere. That the analogous effect occurs for the plane foil may be verified by a simple classical analysis.

⁶⁹ The writer is indebted to Dr. David Pines for pointing out this fact to him.

Suppose there exists a uniform surface charge of superficial density σ on the bounding plane $z = 0$. From Gauss' law the electric field $E_z(0)$ in the plasma at $z = 0$ is given by

$$E_z(0) = 2\pi\sigma$$

The charge density σ is equal to the normal component of the polarization vector $\vec{P}$ at the boundary. Since $\vec{P}(z) = +n_0 e \vec{\xi}(z)$, where $\vec{\xi}(z)$ is the displacement vector of electrons in the plasma, the equation of motion of electrons at the boundary is

$$m \ddot{\xi}_z(0) = -e E_z(0) = -2\pi n_0 e^2 \xi_z(0)$$

or

$$\ddot{\xi}_z(0) + \frac{2\pi n_0 e^2}{m} \xi_z(0) = 0$$

Hence electrons at the boundary may undergo harmonic oscillation at the reduced plasma frequency $\omega = \omega_p/\sqrt{2}$.

The case of a finite plasma confined between the parallel planes $z = 0$ and $z = a$ will now be considered. The equations to be solved are

$$\left(\frac{d^2}{dz^2} - \Gamma^2\right) n(k, \omega | z) = \frac{1}{\sigma_0 d^2} e^{\omega z/\sigma_0} \quad (0 < z < a) \quad (4.31a)$$

$$\left(\frac{d^2}{dz^2} - \kappa^2\right) \phi(k, \omega | z) = 4\pi e \left(n + \frac{1}{\sigma_0} e^{\omega z/\sigma_0}\right) \quad (4.31b)$$

with the conditions that

$$\frac{d}{dz} \phi = 4\pi e d^2 \frac{d}{dz} n \quad z = 0, a \quad (4.31c)$$

where, as before $\Gamma^2 = k^2 + d^{-2} - \frac{1}{\beta^2} (\omega^2 + i\gamma\omega)$.

The solution of (4.31a) has the form

$$\eta(k, \omega | z) = A e^{\Gamma z} + B e^{-\Gamma z} - \frac{e^{i\omega z/v_0}}{v_0 d^2 (\Gamma^2 + \omega^2/v_0^2)} \quad (4.32)$$

and

$$\phi = - \frac{4\pi e e^{i\omega z/v_0}}{v_0 (k^2 + \omega^2/v_0^2)} - \frac{4\pi e}{2k} \int_{-\infty}^{\infty} e^{-k|z-z'|} \eta(k, \omega | z') dz' \quad (4.33)$$

The constants A and B are evaluated from equation (4.31c), but are very lengthy and will not be given in full. To simplify the result, one neglects terms involving β^2 entirely. The solution found above for the semi-infinite plasma showed that the consequence of retaining terms $O(\beta^2)$ was to introduce a weak coupling between energy loss and angular deflection. It will be seen that for a foil of thickness $a \sim v_0/\omega_p$ a much stronger electrodynamic coupling becomes important.

After integrating over k_z and neglecting the recoil of the incident electron, one finds for the interaction probability in the foil in the limit $\beta \rightarrow 0$,

$$P(k_1, \omega) = \frac{e^2}{\pi^2 k v_0^2} \left\{ \frac{a}{k_1^2 + \omega^2/v_0^2} \operatorname{Im} \left(\frac{1}{\epsilon_{k_1, \omega}} \right) + \frac{2k_1}{(k_1^2 + \omega^2/v_0^2)^2} \operatorname{Im} \left[\frac{\epsilon - 1}{\epsilon} \cdot \frac{2(\epsilon - 1) \cos \frac{\omega a}{v_0} + (\epsilon - 1)^2 e^{-k_1 a} + (1 - \epsilon) e^{k_1 a}}{(\epsilon - 1)^2 e^{-k_1 a} - (\epsilon + 1) e^{k_1 a}} \right] \right\} \quad (4.34)$$

where $\epsilon = 1 - \omega_p^2/\omega^2$. As in the case $a \gg v_0/\omega_p$ one may identify terms P_{b1} and P_{b2} , corresponding to surface effects at the lowered plasma energy and at the full plasma energy $\hbar\omega_p$, respectively.

The angular distribution of losses at the energy $\hbar\omega_p$ is of some interest. Integrating over the resonance at $\omega = \omega_p$, one finds

$$P_{k_{\perp}} \equiv \int \{ a P_{k_{\perp}}' + P_{k_{\perp}} \} d\omega \quad (4.35)$$

$$= \frac{e^2}{\pi^2 \kappa v_0^2} \left\{ \frac{a}{k_{\perp}^2 + \omega_p^2/v_0^2} - \frac{2k_{\perp}}{(k_{\perp}^2 + \omega_p^2/v_0^2)^2} \left(\frac{\cosh k_{\perp} a - \cos \omega_p a/v_0}{\sinh k_{\perp} a} \right) \right\}$$

As in the case of large a , the effect of the surfaces is to decrease the probability for scattering through small angles, but is less important for large angles. For $k_{\perp} \gg \omega_p/v_0$ the probability distribution reduces to that appropriate to the infinite foil. For $k_{\perp}$ comparable with ω_p/v_0 and $a \ll v_0/\omega_p$, $P_{k_{\perp}} \rightarrow 0$. The effect of the finiteness of the foil on $P_{k_{\perp}}$ is illustrated in Figure (8).

This figure shows the probability of energy loss $\hbar\omega_p$ and transverse momentum loss $\hbar\vec{k}_{\perp}$ in a foil of thickness a , where $\vec{k}_{\perp}$ is a two-dimensional vector perpendicular to the z direction. Dimensionless variables have been employed: the variable $t = a\omega_p/v_0$ has been plotted on the abscissa and $p = \frac{2\pi\hbar v_0^2}{e^2 a \omega_p} P_{k_{\perp}}$, the probability per unit foil thickness, is shown on the ordinate. p is given as a function of t for two different angles of scatter: $\theta = 0$ and $\theta = \theta_E = \frac{\hbar\omega_p}{2E_0}$. One sees that there is a strong reduction in scattering probability for $t < 2\pi$, and that in general p is an oscillating function of t , tending to have larger values when $t = 2\pi n$, where n is an integer > 0 , and smaller values when $t = (n+1/2)\pi$. This effect becomes less pronounced as θ increases.

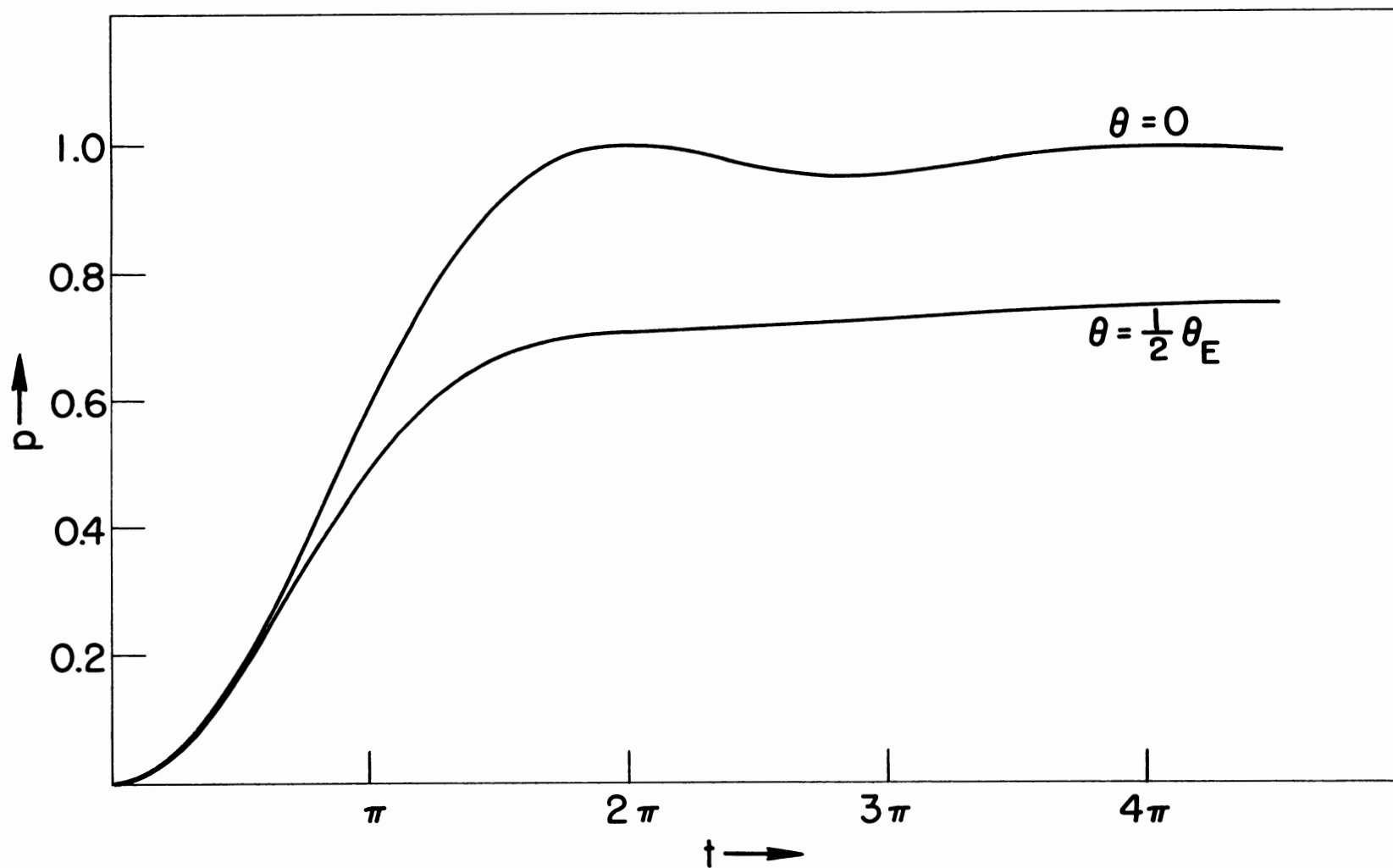


Figure 8. Probability of Electron Scattering in Plasma Through the Angle θ with Loss of Energy $\hbar\omega_p$ in a Foil of Thickness $a = \frac{v_0}{\omega_p} t$.

This reduction in scattering probability as a function of angle should be observable experimentally in thin films and if observed would show that the losses associated with this distribution are collective in nature.

This effect is understandable on the basis of the uncertainty principle, i.e., when $\Delta p_x \Delta x = \hbar k_x a \sim \hbar$ one expects fewer plasma eigenlosses to occur in the foil. This condition is just the range $t = \frac{a \omega_p}{v} \approx 2\pi$.

This eigenloss distribution does not agree with that derived by Ferrell⁷⁰ who used essentially the same assumptions regarding the physical response of the foil as are employed here. The reason for the discrepancy seems to lie in the fact that Ferrell employed the procedure of calculating the scattering of the incident electron on a certain normal mode of oscillation in the foil, while the present result was found by computing scattering on the particular combination of modes excited by the electron itself.

One finds for the total probability P_{b_2} integrated over all angles

$$P_{b_2} = \frac{-2e^2}{\hbar v_0} + \int_0^\infty \frac{x^2 dx}{(x^2 + t^2)^2} \frac{\cosh x - \cosh t}{\sinh x} \quad (4.36)$$

⁷⁰ R. A. Ferrell, Phys. Rev., 111, 1214 (1958).

Now substituting the rational fraction expansions of $\coth x$ and $\operatorname{csch} x$ into this equation and taking $t \ll 1$, one finds

$$P_{b2} = -\frac{e^2}{\pi v_0} t \left[\ln \frac{1}{t} + 1.57 \dots + i \right]$$

One may also write

$$aP_{b2}' = \frac{e^2 t}{\pi v_0} \left[\ln \frac{1}{t} + \ln a k_c \right]$$

where k_c is a cutoff wave vector corresponding to that K_1 in equation (2.95). Then the total transition probability for loss at the plasma energy in very thin films should be

$$aP_{b2}' + P_{b2} = \frac{e^2}{\pi v_0} t \ln \left(\frac{a k_c}{4.82 \dots} \right)$$

This formula is clearly wrong when $ak_c < 4.82 \dots$. However, the present approach will surely break down before the foil thickness becomes this small, since k_c is of the order of the reciprocal of the mean inter-electronic spacing. P_{b2} is plotted versus t in Figure (9).

The P_{b1} term shows a coupling between θ and $\hbar\omega$ for small a . It is found that resonance occurs at

$$\omega = \frac{\omega_p}{\sqrt{2}} \left(1 \pm e^{-ak_{\perp}} \right)^{1/2} = \frac{\omega_p}{\sqrt{2}} \alpha_{\pm}^{1/2} \quad (4.37)$$

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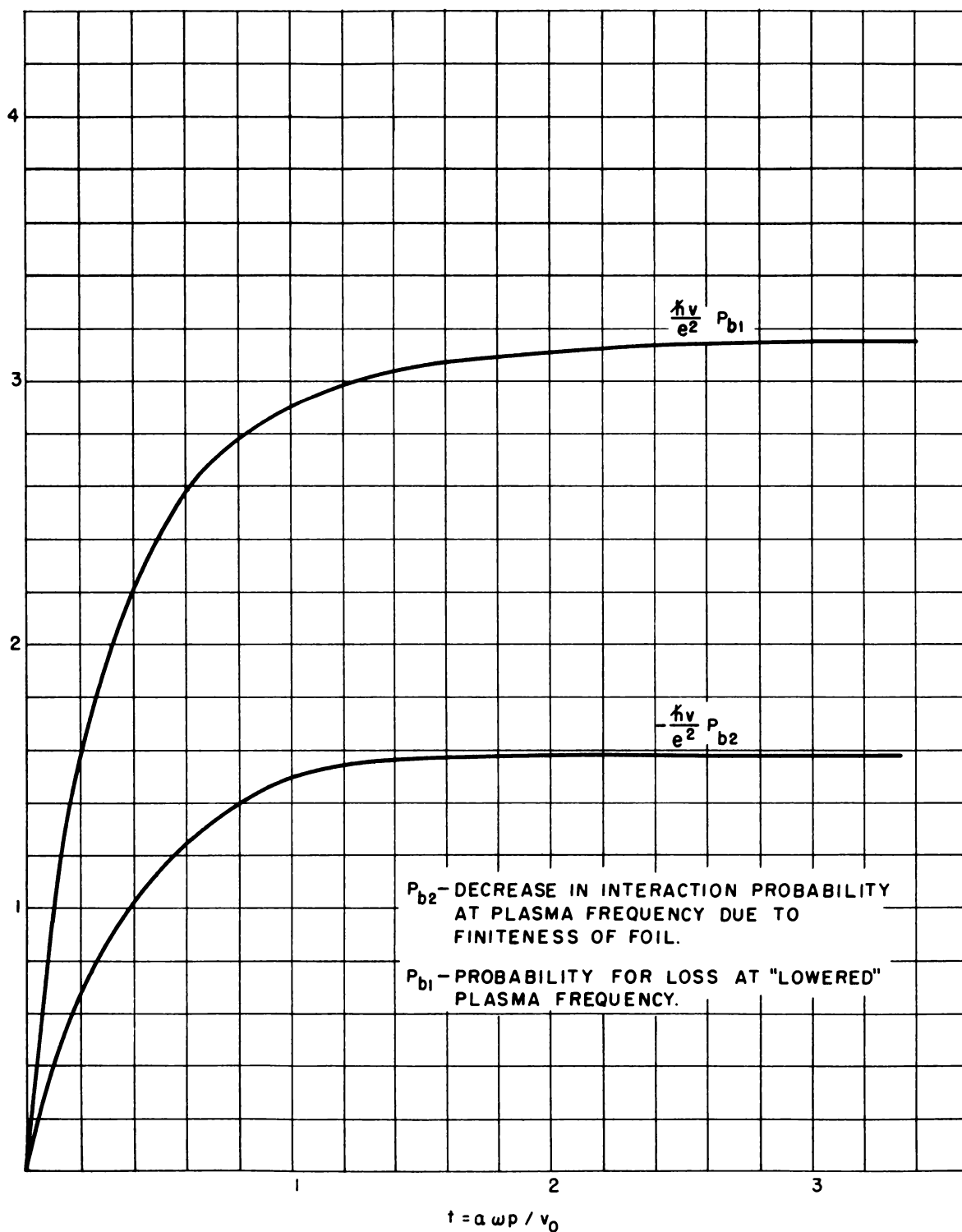


Figure 9. Interaction Probability in Thin Foil vs Foil Thickness.

showing a splitting in the lowered loss for thin foils. The angular spread of the lowered loss was found to be $k_{\perp} = \frac{mv}{\hbar} \theta \sim \frac{\omega_p}{\sqrt{2}v_e}$ for the case of $a \rightarrow \infty$ above. The probability of loss at larger angles decreases rapidly as is seen by inspecting equation (4.27). One expects that the splitting will become important when the exponent in equation (4.37) is of the order of unity, i.e., for $a \sim \sqrt{2}v_e/\omega_p$. This length is of the order of 50 Å for fast electrons in materials such as Al in which the characteristic loss value, assumed to be equal to $\hbar\omega_p$ is 15 ev. The intensity distribution as a function of $k_{\perp}$ is found by integrating over ω and is given by

$$P_{b_1}(k_{\perp}) 2\pi k_{\perp} dk_{\perp} = \frac{\sqrt{2} \omega_p}{\hbar \sigma_0^2} k_{\perp}^2 dk_{\perp} \left\{ \frac{\alpha_+^{1/2}}{\alpha_-} \frac{1 - \cos(\frac{1}{2} \alpha_+^{1/2}/\sqrt{2})}{(k_{\perp}^2 + \omega_p^2 \alpha_+ / 2 \sigma_0^2)^2} + \frac{\alpha_-^{1/2}}{\alpha_+} \frac{1 + \cos(\frac{1}{2} \alpha_-^{1/2}/\sqrt{2})}{(k_{\perp}^2 + \omega_p^2 \alpha_- / 2 \sigma_0^2)^2} \right\} \quad (4.38)$$

The first term in this equation comes from the resonance at $\omega = \omega_p \alpha_+^{1/2}/\sqrt{2}$ and the second from $\omega = \omega_p \alpha_-^{1/2}/\sqrt{2}$. To find the total interaction probability one may now integrate over $k_{\perp}$. The result is shown in Figure (9).

The net boundary effect is an increase in total energy loss to the conduction electrons above the value which would exist in its absence. The total energy loss to the conduction electrons per unit thickness increases logarithmically as the foil thickness decreases.

The possibility occurs to one that these sub-plasma energy losses may be identified with the low-lying losses observed by some experimenters using thin foils. It does not seem that the observed values of the losses

are $1/\sqrt{2}$ times the "characteristic" losses observed in the same metals. However, it should be noted that thin metallic films may have a strongly granular structure. The strong variation of the grain structure with substrate composition, rate and amount of condensation, etc., of thin evaporated metallic films has been discussed by Heavens.⁷¹ In this reference are given electron micrographs which clearly show the transition from small grain size to a state in which the grains merge to form a nearly uniform film as the amount of material deposited is increased in a series of films. The surface depolarization effect will certainly be larger for a small grain of average dimension a than for the semi-infinite plane foil of thickness a which was treated above. Thus one would expect the "lowered" losses in an actual foil to lie closer to the value $h\nu_p/\sqrt{3}$ appropriate to a spherical grain, than to the value $h\nu_p/\sqrt{2}$. This seems to be approximately true for the low-lying losses which have been observed.

To compare the probability of loss at the lowered eigenfrequency with the loss probability at the plasma eigenfrequency, we note that

$$\frac{P_{b1}}{2P_{b1}'} \sim \frac{2\pi\sigma_0}{2\omega_p \ln(\nu_0 k_c / \omega_p)}$$

⁷¹ O. S. Heavens, Optical Properties of Thin Solid Films, (Academic Press, Inc., New York, 1955).

now $\frac{v_o}{w_p} \sim 30 \text{ \AA}$ and $1/k_c \sim 1 \text{ \AA}$ in a typical situation where one has ~ 14 kev electrons incident on Al. In most foil experiments done to date a has been of the order of several thousand angstroms. Hence, typically, if $a = 3000 \text{ \AA}$

$$P_{+1}/a P_{\infty}' \sim 0.02$$

This ratio is not inconsistent with ratios of low-lying loss probabilities to loss probabilities at the characteristic values which have been observed experimentally.

The two most puzzling features about the experimental results on low-lying losses are (a) that such loss lines seem to be narrower and to occur with higher probability than one would expect if they were due to inter-band transitions of individual electrons,¹⁶ and (b) that some experimenters observe these losses while others do not. The depolarization effect seems to offer a possible explanation of these features.

Experimental evidence has recently been obtained by Marton and co-workers⁷² on the low-lying loss in Al. They find that it has a value of 7 ev compared with the characteristic loss value of 15 ev and conclude that the low-lying loss is not due to the excitation of surface oscillations. However, such a shift downward from the value of $15/\sqrt{2}$ ev might be explained by irregularities in the foil, or, as suggested by Ferrell⁷³, it might be due to the presence of a thick layer of aluminum oxide on the surface of

⁷² L. Marton, et. al., Phys. Rev., 110, 1057 (1958).

⁷³ R. A. Ferrell, "Surface Effects", talk at NRC Subcommittee Conference on the Penetration of Charged Particles in Matter, Sept. 15-18, 1958, at Gatlinburg, Tenn.

the foil. Another effect, which has not been considered in the treatment given above, is that the electronic density does not go to zero in a discontinuous manner at the surface of a metal but varies smoothly with distance from the nominal surface. This density variation has been computed by the Thomas-Fermi method⁷⁴.

A simple classical model which takes into account in a schematic way the variation of electronic density at the surface is the following: Suppose that the density decreases to zero in two steps at distance δ apart, instead of varying from the full value to zero in a discontinuous manner at the surface. Assume that at the first step the density goes from the value n_0 to $n_0/2$ and that at the second step it decreases from $n_0/2$ to zero. Then if one assumes that a dielectric constant can be associated with each value of the density, one may look for source-free solutions of Poisson's equation which satisfy the boundary condition that the normal component of the displacement vector shall be zero at each density step. One finds that for disturbances of long wavelength, surface oscillations occur at frequencies equal to $\omega_p / \sqrt{4 \pm 2\sqrt{2}}$ where ω_p is the plasma frequency appropriate to the bulk density n_0 . One finds

$$\omega_{ms} \approx \begin{cases} 0.34 \omega_p \\ 0.93 \omega_p \end{cases}$$

⁷⁴ B. Mrowka and A. Recknagel, Phys. Zeits., 38, 758 (1937).

showing a marked deviation from the value $\omega_p/\sqrt{2}$ found above. One expects in general that surface oscillations will be rather strongly influenced by the condition of the surface and that experimental studies of electron losses should reveal much about electronic surface states.

It is clear that the classical argument for the existence of surface oscillation losses presented here is based on firm enough ground that one would expect such a phenomenon to occur in fact. Even though it may turn out that the experimentally observed low-lying losses referred to above are not attributable to surface oscillations, such oscillations should exist in nature. A study of these phenomena should be very fruitful in the elucidation of surface states of solids.

CHAPTER V

SUMMARY

In the preceding chapters the interaction of non-relativistic charged particles with the free electron plasma has been investigated in the classical approximation. The wave vector dependent dielectric constant of an infinite plasma has been derived by considering the electrodynamic behavior of an isotropic mixture of directed electron beams. The polarization charge density in the neighborhood of an intruder charge has been found to decrease exponentially with distance from the charge if the charge moves slowly and to exhibit a "wake" in the case of a rapidly moving charge. The scalar potential at large distances from a fast charge was found to be of the quadrupole form rather than the Debye shielded form as has often been assumed in the literature. The stopping power of the free electron plasma for fast particles was shown to depend on the mean square velocity distribution in the undisturbed medium and to have a different form than that given in the literature. This dependence was shown to evince itself in a dependence of energy loss on angular deflection in differential stopping power experiments with fast electrons interacting with metals.

The case of a bounded plasma has been treated by assuming that the plasma electrons obey the Bloch hydrodynamical equations. The image potential problem for a slowly moving electron approaching a metal surface has been solved in this approximation and an explicit representation

for the potential has been found which differs markedly from the result obtained from simple electrostatic theory. The effect of the finiteness of a foil on the losses observed in differential stopping power experiments with thin metallic films has been calculated in the Bloch approximation. It was shown that the angular distribution of electrons losing energy at the bulk "plasma" resonance value in thin films is strongly modified from the distribution appropriate to a thick foil. The strong decrease in intensity at small angles of scatter is shown to be a consequence of the indeterminacy principle and is an effect which should appear in characteristic loss experiments with thin films. This model predicts that the excitation of surface oscillations in foils will give rise to losses in these experiments which may account for experimentally observed low-lying characteristic losses. In Appendix A the equivalence of the semi-classical impact parameter treatment and the Born approximation for the interaction of fast particles with plasma is proved more generally than has been done before. A simple generalization of the impact parameter treatment which takes into account the recoil of the incident electron is described.

A quantum mechanical generalization of the dielectric treatment of plasma is possible.⁷⁵ In this approach, the interaction probability of an incident charged particle with a degenerate Fermi gas is calculated in

⁷⁵ R. H. Ritchie, "The Interaction of Charged Particles with a Fermi-Dirac Electron Gas", submitted to Physical Review.

the Born approximation, employing a propagator for longitudinal interactions in the electron gas in analogy with field theory methods and using Feynman's⁷⁶ method for calculating self-energy. This treatment takes into account the wave properties of both the incident and struck electrons within the framework of the Born approximation.

The relativistic extension of the work presented above is of considerable interest. In particular, a study of the emission of photons by solids bombarded with fast charged particles should be of value in distinguishing between collective and one-electron phenomena in metals.⁷⁷ Such a generalization of the approach used above is straightforward.

⁷⁶ R. P. Feynman, Phys. Rev., 76, 749 (1949); 76, 769 (1949).

⁷⁷ R. A. Ferrell (Phys. Rev., 111, 1214, 1958) has investigated this problem in a very illuminating manner but his results do not include retardation effects in a consistent way.

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APPENDIX

APPENDIX

Equivalence of Classical and Quantum Mechanical Treatments of the Scattering of Fast Electrons in Plasma

In this paper, the scattering probability of a fast electron interacting with plasma is computed from a correspondence principle argument. The energy W transferred from the particle to the medium is written as an integral over the particle path of the reaction force of the medium on the particle, viz

$$W = -e \int_{-\infty}^{\infty} dz \left[\frac{\partial}{\partial z} \phi_m(\bar{r}, t) \right]_{\substack{x=y=0 \\ t=z/v_0}} \quad (A-1)$$

where $-\frac{\partial}{\partial z} \phi_m$ is the electric field in the medium due to the electron minus the self-field of the electron and the electron path is taken to be along the z axis. ϕ_m may be written quite generally as

$$\phi_m(\bar{r}, t) = -e \int d\bar{r}' \int dt' V_G(\bar{r}, t; \bar{r}', t') \delta(x') \delta(y') \delta(z' - v_0 t') \quad (A-2)$$

where $V_G(\vec{r}, t; \vec{r}', t')$ is the Green's function for the electric potential in the medium, i.e., the electric potential at $\vec{r}, t$ due to a unit instantaneous electrical charge appearing at $\vec{r}', t'$. $V_G(\vec{r}, t; \vec{r}', t')$ is a solution of the quantum mechanical equation of motion of the plasma. In this paper V_G is essentially obtained by semi-classical methods but is applied to problems in which quantum effects are important. It will now be

shown that all quantum effects due to the wave nature of the incident electron (in the Born approximation) are contained in the result obtained by the correspondence principle argument of Chapter III.

In general, $V_G(r, t; r', t')$ may be expanded in a Fourier integral in the variable $t-t'$. Defining, $V_G(\vec{r}, t; \vec{r}', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} V_G(\vec{r}; \vec{r}' | \omega) d\omega$ one has from equation (A-2)

$$\phi_m(\vec{r}, t) = -\frac{e}{2\pi v_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int d\tau V_G(\vec{r}; \vec{r}' | \omega) \delta(x) \delta(y') e^{i\omega z/v_0} \quad (A-3)$$

In Chapter IV we assume that the plasma is infinite in the x and y directions; it follows that $V_G(\vec{r}; \vec{r}' | \omega)$ may be expanded as follows:

$$V_G(\vec{r}; \vec{r}' | \omega) = \frac{1}{(2\pi)^2} \int d\vec{k} d\vec{k}_z e^{i[\vec{k} \cdot (\vec{r} - \vec{r}') + z k_z]} V_G(\vec{r}' | \vec{k}, k_z, \omega) \quad (A-4)$$

where $\vec{\rho}$ is a two dimensional vector which is the projection of $\vec{r}$ in the x, y plane and $\vec{k}$ is in the same plane. It is convenient but

not necessary to assume that the medium is infinite in the x and y directions. Then from equations (A-1), (A-3) and (A-4) it follows that

$$W = \frac{-2e^2}{(2\pi)^2 v_0^2} \int d\vec{k}_z \int d\vec{k} \int_0^{\infty} d\omega \cdot \omega \delta(k_z - \frac{\omega}{v_0}) \partial_m \left\{ \int_{-\infty}^{\infty} d\tau e^{i\omega \tau/v_0} V_G(\vec{r}' | \vec{k}, k_z, \omega) \right\} \quad (A-5)$$

Arguing as in Chapter III that W must also be given by

$$W = \int d\vec{k} \int_0^{\infty} d\omega k \omega P(\vec{k}, \omega) \quad (A-6)$$

and setting the integrands of A-5 and A-6 equal, one obtains for the probability $P(\vec{k}, \omega)$ of energy loss $\hbar\omega$ and momentum transfer $\hbar\vec{k}$, where $k^2 = K^2 + k_z^2$,

$$P(\vec{k}, \omega) = \frac{-2e^2}{(2\pi)^3 \hbar v_0} \int_{-\infty}^{\infty} dz' e^{iz' \hbar k_z} V_G(z' | \vec{K}, \hbar k_z, \omega, k_z) \quad (A-7)$$

Now one may obtain precisely the same equation by a quantum mechanical perturbation-theoretic argument in which the incident electron is treated not as a classical point charge but as a plane wave. Although the arguments presented below are not entirely rigorous, they may be made so using methods in modern quantum field theory⁷⁸.

Suppose that the incident free electron is expanded in terms of momentum eigenfunctions, so that its wave function $\Psi(\vec{r}, t)$ may be written

$$\Psi(\vec{r}, t) = \frac{1}{L^{3/2}} \sum_f C_f(t) e^{i(\vec{k}_f \cdot \vec{r} - \omega_f t)}$$

where $\omega_f = \frac{\hbar}{2m} k_f^2$ and normalization is in a large cube of side L . One takes $C_f(t) \rightarrow \delta_{of}$ as $t \rightarrow -\infty$, so that the electron is initially in the state characterized by momentum $\hbar\vec{k}_0$. Due to its interaction with the plasma, it may undergo transitions to states of lower energy. The charge

⁷⁸ See, e.g., S. S. Schweber, H. A. Bethe and F. deHoffman, "Mesons and Fields", Volume I (Row, Peterson and Co., Evanston, Ill., 1955).

density⁷⁹ ρ_{of} corresponding to the transition $\vec{k}_o \rightarrow \vec{k}_f$ is given by

$$\rho_{of}(\vec{r}, t) = \frac{-e}{\omega^3} \left\{ \exp \left[i(\vec{k}_o - \vec{k}_f) \cdot \vec{r} - i(\omega_o - \omega_f)t \right] + \text{complex conjugate} \right\} \quad (\text{A-8})$$

The amplitude of the electron in the initial state is altered due to such transitions and the change in amplitude may be written

$$\left(e^{-\lambda \Delta E T / \hbar} - 1 \right) \simeq -\lambda \Delta E T / \hbar \quad (\text{A-9})$$

at the end of a time interval T. The retention of only first-order terms in this expansion is equivalent to the use of first-order perturbation theory. ΔE is the change in energy of the incident electron due to its interaction with the plasma. ΔE is a complex quantity and may be written more explicitly as

$$\Delta E = (\Delta E)_{real} - i \hbar \Gamma / 2 \quad (\text{A-10})$$

where Γ is the total transition rate of the incident electron to all accessible lower energy states.

⁷⁹ See, e.g., N. F. Mott and H. S. W. Massey, "The Theory of Atomic Collisions", (Oxford, Second Edition, 1949), p. 362. The method of transition charge densities was used by Møller in his original derivation of the relativistic electron-electron scattering formula (Z. f. Physik, 70, 786, 1931).

If the potential in the plasma due to charge density ρ_{of} is $\phi_m^{of}(\vec{r}, t)$, the interaction energy ΔE will be given by

$$\Delta E = \frac{1}{T} \int_{-T/2}^{T/2} dt \int d\vec{r} \rho_{of}(\vec{r}, t) \phi_m^{of}(\vec{r}, t) \quad (A-11)$$

where an average over the time interval T is indicated. Hence we put for the rate for the transition $\vec{k}_0 \rightarrow \vec{k}_f$

$$\Gamma_{of} = \frac{-2}{\hbar T} \Im_m \left\{ \int d\vec{r} \int_{-T/2}^{T/2} dt \rho_{of}(\vec{r}, t) \phi_m^{of}(\vec{r}, t) \right\} \quad (A-12)$$

The scalar potential ϕ_m may be written in terms of the Green's function $V_G(\vec{r}, t; \vec{r}', t')$ as before;

$$\phi_m(\vec{r}, t) = - \frac{e}{L^3} \int d\vec{r}' dt' V_G(\vec{r}, t; \vec{r}', t') \left(e^{i(\vec{q} \cdot \vec{r} - \Omega t)} + c.c. \right) \quad (A-13)$$

where $\vec{q} = \vec{k}_0 - \vec{k}_f$ and $\Omega = \omega_0 - \omega_f$ and one must consider only the term corresponding to energy absorption by the medium. Expanding V_G in the cube of side L and in the time interval T , one has

$$V_G(\vec{r}, t; \vec{r}', t') = \frac{1}{L^3 T} \sum_{\omega} \sum_{\vec{k}} e^{i\{(\vec{r}-\vec{r}') \cdot \vec{k} + (\vec{q} \cdot \vec{r}') - \omega t + \omega t'\}} \times V_G(\vec{r}' | \vec{k}, \omega) \quad (A-14)$$

Combining equations (A-8), (A-12), (A-13) and (A-14) one finds

$$\Gamma_{of} = -\frac{2e^2}{\hbar L^4} \Im_m \int_{-\infty}^{\infty} dz' e^{i\omega z'} V_G(z'|\vec{q}, \Omega)$$

which shows explicitly that the momentum and energy lost by the electron in the transition $o \rightarrow f$ are absorbed by the field. Now $P_{of}(\vec{q}, \Omega)$ the total transition probability in the foil per incident electron is equal to the transition rate per cm^2 of foil divided by the incident electron flux, v/L^3 . Thus

$$P_{of}(\vec{q}, \Omega) = \frac{1}{v_o} \Gamma_{of} = -\frac{2e^2}{\hbar v_o L^3} \Im_m \int_{-\infty}^{\infty} dz' e^{i\omega z'} V_G(z'|\vec{q}, \Omega)$$

Now to compare with equation (B-7) in which the vector $\vec{k}$ was assumed to be distributed continuously in a three-dimensional space of infinite extent, one converts from a sum over final states $\vec{k}_f$ to an integral, i.e.,

$$\sum_{\vec{k}_f} \rightarrow \left(\frac{L}{2\pi}\right)^3 \int d\vec{k}_f$$

Then

$$P_{of}(\vec{q}, \Omega) = -\frac{2e^2}{(2\pi)^3 \hbar v_o} \delta(\omega - \Omega) \int_{-\infty}^{\infty} dz' e^{i\omega z'} V_G(z'|\vec{q}, \omega)$$

The only difference in this result and the parallel equation (A-7) is that instead of setting $\omega = vk_z$, as in the classical case, one uses

$$\Omega = \frac{\hbar}{2m} (k_o^2 - k_f^2)$$

which together with the relation $\vec{q} = \vec{k} = \vec{k}_0 - \vec{k}_f$ yields the more general result

$$\omega = \Omega = \frac{\hbar}{2m} (\vec{k}_0 \cdot \vec{k} - k^2) = v_0 k_z - \hbar k^2 / 2m$$

which accounts for the recoil of the electron in its interaction with the plasma.

Thus it has been shown that the interaction probability obtained by assuming that the incident particle is a classical point charge and employing a correspondence principle argument gives essentially the same result which one finds from the Born approximation to the corresponding quantum mechanical solution.