To the Graduate Council:

I am submitting herewith a thesis written by Melissa Ann Henderson entitled "A New Class of Tests for Exponentiality." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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A

A NEW CLASS OF TESTS FOR EXPONENTIALITY

A Thesis

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ABSTRACT

The purpose of this study is to investigate a new class of tests for exponentiality versus broad alternatives. In Chapter 1, a selection of tests for exponentiality against broad alternatives are reviewed. The null cumulative distribution of the test statistic, $T_{r,s}$, as well as its null mean and null variance are developed in Chapter 2. Also, in Chapter 2, a table of the null cumulative distribution function for selected values is provided. In Chapter 3 very easily applied asymptotic distributions for the test statistic are derived. The power of the test proposed in Chapter 2 is investigated in Chapter 4. Two small Monte Carlo studies are used to estimate the power of test versus Weibull alternatives with shape parameters 2 and 1/2. Also, the test $T_{r,s}$ is compared with another test for exponentiality. Finally, the exact value and the asymptotic value of the null cumulative distribution function of $T_{1,5}$ with sample size 5, $F_{5,1,5}$, is compared to give some intuitive feel of the preciseness of the asymptotic distribution.

It is found that the test proposed has considerable power against DFR alternatives and compares well against at least one alternate test for exponentiality. Also, the asymptotic distributions, even with the small sample size of 5 are very accurate.

TABLE OF CONTENTS

CHAPTER PAGE

1. A REVIEW OF THE LITERATURE OF TESTS FOR EXPONENTIALITY. 1

2. A NEW CLASS OF TESTS FOR EXPONENTIALITY 10

3. DEVELOPMENT OF ASYMPTOTIC DISTRIBUTIONS 26

4. INVESTIGATION OF POWER OF THE TEST. 38

LIST OF REFERENCES 44

APPENDIX 47

VITA 55

CHAPTER 1

A REVIEW OF THE LITERATURE OF TESTS FOR EXPONENTIALITY

Let $X_1, \ldots, X_n$ be a collection of independent identically distributed life length observations with common unknown cumulative distribution function $F(x)$. Consider the problem of testing H_0: $F(x) = 1 - e^{-\lambda x}$ for some fixed unknown $\lambda > 0$, versus the broad alternative H_1: $F(x) \neq 1 - e^{-\lambda x}$. Among the class of suitable tests are those of Kolomogorov (1933) and Smirnov (1948) and the χ^2 goodness of fit test. However, if λ is unknown, the above tests are not applicable. The problem of testing for exponentiality versus broad alternatives has been studied by Epstein, Barlow, Proschan and Bartholomew among others. Here a partial survey of results in this area will be presented.

Benjamin Epstein has done much of the early research in testing the null hypothesis of exponentiality versus the broad alternative of nonexponentiality. A graphical procedure (Epstein, 1960) was one of the early approaches to this problem. This procedure of plotting an estimate of $y = \log\{1/[1 - F(t)]\}$ (where $F(t)$ is the cumulative distribution function defined by $F(t) = 0$ if $t < 0$ and $F(t) = 1 - e^{-\lambda t}$ if $t \geq 0$) against t. If the line is approximately straight, exponentiality is suggested. To plot $y = \log\{1/[1 - F(t)]\}$ assume there are n items on a life test (as defined in Barlow and Proschan, 1975) and let $t_1 \leq t_2 \leq \ldots \leq t_n$ be the times when the tested items fail. In plotting the graph the value of $F(t_i)$ associated with t_i will be $F(t_i) = i/(n + 1)$. The obvious advantage to this graphical procedure is that it can be programmed into a computer with a graphics compiler and thus offers a quick result with intuitive appeal to the researcher.

This graphical procedure, however, has two major disadvantages: 1) the sample size must be rather large (≥ 50) (Epstein, 1960) in order to detect substantial departures from an exponential distribution, and 2) even with a large sample size the probability of a Type I or Type II error is unobtainable from this test.

Epstein (1960) has also developed a test for utilizing the χ^2 goodness of fit test. Epstein's test has intuitive appeal since with an extensive body of failure data, one can estimate the parameter λ, use this estimate of λ to find the expected number of failures in each time interval with index k, then use a χ^2 goodness of fit test with $(k - 1)$ degrees of freedom. The disadvantages of this test include a need for a large sample size and a dependence upon the arbitrary choice and number of the intervals into which the time axis is divided, and the fact that only the asymptotic distribution of the test statistic is known.

A test found by Epstein (1960) to be somewhat more sensitive than either the graphical procedure or the χ^2 goodness of fit test is a criterion based on the conditional distribution of total lives. This test makes use of basic properties of Poisson processes. That is, if r events occur in $[0, \tau]$ at times $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_t \leq \tau$ then these times can be considered as independent observations on a random variable uniformly distributed over $[0, \tau]$. In the case where items are not replaced, however, the total lives $T(\tau_i)$ (the total life observed in getting the i^{th} failure) is used instead of the failure times (τ_i). Since the conditional distribution of total lives is uniform over suitable intervals, it is evident this test is a good tool for detecting non-exponential distributions. Thus, if failures become clustered near

some point the condition of uniformity is violated. This test, though, also requires a relatively large sample size to have sufficient power (with small sample sizes this test will only detect large departures from exponentiality).

Since abnormally early failures in exponential data may cause it to be rejected by many tests as non-exponential, Epstein (1960) has designed a test for abnormally early failures. To determine if this type of deviation from the exponential is occuring the following test is used: suppose $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_r$ are the first r failure times, then τ_1 is abnormally small if

$$T(\tau_1) < F^*_\alpha \, T(\tau_r - \tau_1)/(r - 1)$$

where $T(\tau_1)$ is the total life in $[0, \tau_1]$ and $T(\tau_r - \tau_1)$ is the total life in $[\tau_1, \tau_r]$ and F^*_α is a lower α point of the $F(2, 2r - 2)$ distribution (both τ_1 and τ_2 are abnormally small if

$$T(\tau_2) < F^{**}_\alpha \, 2T(\tau_r - \tau_2)/(r - 2)$$

where F^{**}_α is a lower α point of the $F(4, 2r - 4)$ distribution). This test, though, is not an adequate test for exponentiality since essentially all it does is determine if a set of data should be deemed nonexponential immediately or subjected to more tests for exponentiality.

Epstein (1960) has also discussed a test for abnormally long first failure times. If the underlying distribution is really a 2-parameter exponential with minimum life $A > 0$, then the deviation from the exponential can be detected by an abnormally long first failure time. To

test the hypothesis $A = 0$ against the alternative $A > 0$, $A = 0$ will be rejected with a probability of $(1 - \alpha)$ if an abnormally long first failure time τ_1 is obtained or if

$$T(\tau_1) > G^*_\alpha \, T(\tau_r - \tau_1)/(r - 1)$$

where $T(\tau_1)$ and $T(\tau_r - \tau_1)$ are as above and G^*_α is an upper α point of the $F(2, 2r - 2)$ distribution. This test, however, has the same scope and limitations as the test for abnormally early failures outlined above.

Another Epstein (1960) test for exponentiality is used to determine whether the distribution of total lives between successive failures is exponential with parameter λ against the alternative that the distribution of total lives between successive failures is Weibull. This test involves k ordered failure times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_k$. Epstein has shown that the null distribution

$$2k\{\log T(\tau_k)/k - [\log T(\tau_k) + \log T(\tau_2 - \tau_1) + \ldots +$$

$$\log T(\tau_k - \tau_{k-1})]/k\}/[1 + (k + 1)/6k] \sim \chi^2(k - 1)$$

where $T(\tau_k)$ is as defined earlier and $P(\tau_k = \tau_{k-1}$ for any $k) = 0$. This test, under the assumption of life testing (that is, tests involving total lives and not times between failures), is changed to the form

$$-2 \sum_{i=1}^{r-1} \log[T(\tau_i)/T(\tau_r)] \, .$$

Epstein (1960) has also developed another graphical procedure in which the null hypothesis states that since $T(\tau_1)$, $T(\tau_2)$,..., $T(\tau_r)$ are uniformly distributed over $(0, T^*)$ the underlying distribution is exponential with constant θ. A graphical method of testing this hypothesis is to plot a random walk diagram in which the proportion of failures (relative to r) that occur at or before $T(\tau)$ is plotted against $T(r)$. If the null hypothesis is true the diagram should fluctuate around the straight line joining the points $(0, 0)$ and $(T^*, 1)$. A test of significance can be made using the results of Kolmogorov (1933) and Smirnov (1948) on the size of $r^{1/2} D_r$ where

$$D_r = \max_{0<T(\tau)<T^*} |F_r[T(\tau)] - F[T(\tau)]|$$

$$= \max_{0<T(\tau)<T^*} |[\text{number of failures up to } T(\tau)] - rT(\tau)/T^*|$$

where the sample cumulative distribution function $r\,F_r[T(\tau)] =$ number of failures occuring at or before the total life $T(\tau)$, whereas the theoretical cumulative distribution function $F[T(\tau)] = T(\tau)/T^*$ for $0 \leq T(\tau) \leq T^*$. This test has several advantages. Among them is the fact that the null distribution is known. For large sample sizes (≥ 80) (Epstein, 1960) the power of the test is high. For small values of r, Birnbaum (1952) and Massey (1951) have developed some tables which may be used. This graphical procedure has another advantage in that there are several good asymptotic approximations.

Bartholomew (1957) has developed a test for exponentiality given no knowledge of deviations from it. He proposed four alternatives to

the exponential distribution. They are:

1) $p(t) = \lambda[\Gamma(a)]^{-1} (\lambda t)^{a-1} e^{-\lambda t}$ $\quad t \geq 0 , a > 0$

2) $p(t) = \lambda(\lambda t)^{a-1} \exp[-(\lambda t)^{a}/a]$ $\quad t \geq 0 , a > 0$

3) $p(t) = \lambda(1 + a\lambda t)^{-(a+1)/a}$ $\quad t \geq 0 , a > 0$

4) $p(t) = \lambda e^{-\lambda t}[1 + a(\lambda^2 t^2/2 - 2\lambda t + 1)]$ $\quad t \geq 0 .$

He also proposes three test statistics and evaluates the power of the tests using the test statistics. By this comparision process the maximum loss in power which might be incurred by a wrong choice of test is minimized. Bartholomew's proposal, however, is based on asymptotic distributions and is most effective for sample sizes of the order of 100 (Bartholomew, 1957).

Proschan and Pyke (1967) have proposed tests for exponential versus monotone increasing failure rate based on the ranks of normalized spacings. (Let $D_1,\ldots, D_n$ be the normalized sample spacings. Then $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $i = 1,\ldots, n$ where $0 = X_{(0)} < X_{(1)} < \ldots < X_{(n)}$ are the order statistics). These statistics have been shown to be asymptotically normally distributed for fixed alternatives F (where F is a distribution with density f and $q(t) = f(t)/[1 - F(t)]$ is the failure rate of F).

Bickel and Doksum (1967) have shown that the asumptotic normality of the Proschan and Pyke statistics described above holds for sequences of alternatives $\{F_{\theta_n}\}$ that approach the null distribution $(1 - e^{-\lambda t})$, $t \geq 0$ as $n \to \infty$. They also derived statistics which depend on the normalized spacings (as defined above) and are asymptotically most powerful for parametric alternatives $\{F_{\theta_n}\}$ if the scale parameter λ is known. One advantage to this type of test is that unbiasedness has been shown for tests that are monotone in the normalized spacings

(Bickel and Doksum, 1969). A problem associated with this test, though, is that only asymptotic distributions are known and if λ is not known one must use studentization of the linear normalized spacing tests which results in a lower power of the test.

Barlow (1967) has proposed two unbiased tests for testing constant versus increasing (decreasing) failure rates. Both tests are based on statistics which are likelihood ratio statistics for related problems. One disadvantage to these tests is their power has thus far only been determined from small sample sizes. Therefore it is impossible to evaluate their power at this time (Barlow, 1967).

Hollander and Proschan (1972) have developed a distribution-free test for "new better than used" alternatives (where a distribution F is new better than used if $\bar{F}(x + y) \leq \bar{F}(x)\,\bar{F}(y)$ for $x \geq 0$, $y \geq 0$ for $\bar{F}(x) = 1 - F(x)$). The null hypothesis considered is

$$H_0: \; P(X \geq x + y \mid X \geq x) = P(X \geq y) \; \forall x, y \geq 0 .$$

This test, by Feller (1968), is equivalent to testing the null hypothesis that the underlying population is exponential. To test H_0,

1) Reorder the X's so that they are increasing. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote these ordered values.
2) Compute

$$T = \sum_{i>j>k} \psi(X_{(i)}, X_{(j)} + X_{(k)})$$

$$\psi(a, b) = \begin{cases} 1 & \text{if } a > b \\ \\ 0 & \text{if } a < b \end{cases} .$$

3) For a one sided α-level test of H_0 versus "new better than used" alternatives

$$P(X \geq x + y | X = x) \leq P(x \geq y) \quad \forall\, x\,,\ y \geq 0$$

and reject H_0 if $T \leq t_1(\alpha, n)$

and accept H_0 if $T > t_1(\alpha, n)$

where the constant $t_1(\alpha, n)$ satisfies

$$P_0[T \leq t_1(\alpha, n)] = \alpha \ .$$

This test is useful in that tables of $t_1(\alpha, n)$ values are given. These tables, though, are not widely published and are therefore not readily available. Hollander and Proschan (1972), though, have shown that T is asymptotically normal under H_0. This asymptotic distribution makes the test very usable. As in all asymptotic distributions, however, some accuracy is lost from the use of the exact distribution. Also, since the test depends on $\vec{X}$ only through $\psi(X_{(i)}, X_{(j)} + X_{(K)}) = 0$ or 1, it does not use the values of the observations. Hence, information is lost.

Lee, Locke, and Spurrier (1980) have shown that tests based on T_α (where $T_\alpha = (\sum_{i=1}^{n} X_i^\alpha/n)/\bar{X}^\alpha$, $0 < \alpha < 1$ or $\alpha > 1$) are not consistant against certain kinds of non-exponential lifelength distributions. For a class of alternatives that include the distributions with strictly increasing failure rate averages (as defined by Barlow and Proschan, 1975), however, it is possible to construct one-sided tests based on T_α which are consistant. Another set of one-sided tests based on T_α are consistant against a class of alternatives including the distributions with strictly decreasing failure rate averages (as defined by Barlow and Proschan, 1975). Lee, Locke, and Spurrier's (1980) study shows that $T_{1/2}$ and T_2 are quite good for detecting failure rate average and increasing failure rate average alternatives, respectively. This test has two drawbacks: the exact null distribution is not known and the power of the test has only been estimated by Monte Carlo studies. There is also some inconsistency in the test if the alternative distribution is not IFRA or DFRA (as defined by Barlow and Proschan, 1975).

In Chapter 2 a new class of tests for exponentiality will be proposed. The exact null distribution along with its mean and variance will be given. Then in Chapter 3 fairly simple asymptotic distributions will be developed.

CHAPTER 2

A NEW CLASS OF TESTS FOR EXPONENTIALITY

In Chapter 1 a selection of tests for exponentiality against broad alternatives were reviewed. In this chapter a new class of tests for exponentiality against broad alternatives will be proposed. The null cumulative distribution of the test statistic as well as its null mean and null variance will be developed. A table of the null cumulative distribution function for selected values will also be provided.

Consider the random variables $X_1, \ldots, X_n$ which are independent and identically distributed with common unknown life distribution F (where $\lim_{x \to 0^-} F(x) = 0$) . The null hypothesis of the exponentiality of x will be tested against the alternative of non-exponentiality. To develop this test the following theorem is needed:

Theorem 2.1: Let $D_1, D_2, \ldots, D_k$ be successive spacings between order statistics from the exponential (1) distribution (where the k^{th} spacing D_k between order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given by $D_k = X_{(k)} - X_{(k-1)}$ for $k = 1, 2, \ldots, n$ where $X_{(0)} = 0$) . Then

a) $P[D_k \leq t] = -e^{-(n-k+1)t}$ for $k = 1, \ldots, n$, $t \geq 0$

b) $D_1, \ldots, D_n$ are mutually independent

(Barlow and Proschan, 1975, Section 3.2, Theorem 2.5). ■

This theorem is useful for the purposes of the test being developed, since, by Kagan, Linnick, and Rao (1972), the above is a characterization of the exponential distribution.

The test being developed will use weighted spacings which are defined as follows:

Definition 2.1: The weighted spacings $W_1,\ldots, W_n$ are nD_1, $(n-1)D_2,\ldots, D_n$ where $D_1,\ldots, D_n$ are as above (Barlow and Proschan, 1975). ■

Weighted spacings are being used in the test instead of just order statistics, since by Bickel and Doksum (1969) such tests are unbiased. Thus, by Kagan, Linnick, and Rao (1972) and Bickel and Doksum (1969) it seems reasonable to base the test entirely on weighted spacings and test only them for exponentiality.

The test statistic $T_{r,s}$ will be defined as follows:

Definition 2.2: Let $T_{r,s} = W_{(r)}/W_{(s)}$ where $W_{(1)},\ldots, W_{(n)}$ are ordered weighted spacings and r and s are fixed integers such that $1 \leq r < s \leq n$. ■

The following is an example of how $T_{r,s}$ is calculated for a specific data set. (This particular data used is some Proschan (1963) found when investigating the life distribution of the air-conditioning system of a fleet of Boeing 720 jet airliners).

Example 2.1: If the following data are collected

$$X_i = 194, 15, 41, 29, 33, 181$$

$T_{1,5}$ and $T_{2,4}$ are calculated as follows:

1) Order the data: 15, 29, 41, 181, 194.

2) Find the spacings: 15, (29 - 15), (41 - 29), (181 - 41), (194 - 181) = 15, 14, 12, 140, 13

3) Weight the spacings using Definition 2.1:

$$15(5),\ 14(4),\ 12(3),\ 140(2),\ 13(1) = 75,\ 56,\ 36,\ 280,\ 13\ .$$

4) Order the weighted spacings: 13, 36, 56, 75, 280

5) Calculate $T_{1,5} = 13/280 = .046.$

6) Calculate $T_{2,4} = 36/75 = .480.$ ■

By Theorem 4.1 of Langberg, León, and Proschan (1980) the ordered weighted spacing $W_{(i)}$ tends to be small for i near 1 and large for i near n in the decreasing failure rate case (F is a decreasing failure rate distribution if F satisfies the following: $\overline{F}(x|t) = \overline{F}(x + t)/\overline{F}(t)$, where $\overline{F}(x) = 1 - F(x)$, is increasing in $t \geq 0$ for each $x \geq 0$, Barlow and Proschan, 1975); and $W_{(i)}$ tends to be large for i near 1 and large for i near n in the increasing failure rate case (F is an increasing failure rate distribution if F satisfies the following: $\overline{F}(x|t) = \overline{F}(t + x)/\overline{F}(t)$ is decreasing in $-\infty < t < \infty$ for each $x \geq 0$, Barlow and Proschan, 1975). A lower-tailed test is used for both increasing and decreasing failure rate alternatives.

It should be noted that the methods used with statistic $T_{r,s}$ could very easily be applied to the ordered original observations instead of weighted ordered spacings. This alternate test, $T^*_{r,s} = X_{(r)}/X_{(s)}$ would have the same null cumulative distribution function, the same null mean and variance, and the same asymptotic distributions.

Since $T_{r,s}$ and $T^*_{r,s}$ are scale invariant, without loss of generality, the null hypothesis, H_0: $F(x) = 1 - e^{-x}$ can be tested against the alternative, H_1: $F(x) \neq 1 - e^{-\lambda x}$ for any $\lambda > 0$.

The following theorem is a derivation of the null cumulative distribution function of the test statistic:

Theorem 2.2: The null cumulative distribution function of $T_{r,s}$, $F_{n,r,s}(t)$ is

$$\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} n!(-1)^{j+i+2}(t)/$$

$$\{(n-s)!(s-r-j-1)!\, j!(r-i-1)!\, i!$$

$$(n-s+j+1)[n-s+j+1+t(i+s-j-r)]\}.$$

Proof: Let $X_1, \ldots, X_n$ be independent and identically distributed exponential (1). The joint density of the r^{th} and s^{th} order statistics is then

$$[n!/(r-1)!(s-r-1)!(n-s)!]e^{y(s-n-1)}(1-e^{-x})^{r-1}$$

$$(e^{-x}-e^{-y})^{s-r-1}\, e^{-x}\ .$$

Hence, for any t, $0 < t < 1$,

$$F_{n,r,s}(t) = [n!/(r-1)!(s-r-1)!(n-s)!]$$

$$\int_0^{\infty} e^{y(s-n-1)} \int_0^{ty} (1-e^{-x})^{r-1}(e^{-x}-e^{-y})^{s-r-1}\, e^{-x}\, dxdy$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!]$$

$$\int_0^{\infty} -e^{-y}e^{-y(n-s)} \int_0^{ty} (1-e^{-x})^{r-1}(e^{-x}-e^{-y})^{s-r-1}(-e^{-x})dxdy.$$

let $u = e^{-x}$, $v = e^{-y}$, then

$$= [n!/(r-1)!(s-r-1)!(n-s)!]$$

$$\int_1^0 v^{n-s} \int_1^{v^t} (1-u)^{r-1}(u-v)^{s-r-1}\, dudv \ ;$$

through the use of a binomial expansion,

$$= [n!/(r-1)!(s-r-1)!(n-s)!]$$

$$\int_1^0 v^{n-s} \int_1^{v^t} (1-u)^{r-1} \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} u^{(s-r-j-1)}\, v^j (-1)^j\, dudv$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!] \sum_{j=0}^{s-r-1} \binom{s-r-1}{j}(-1)^j$$

$$\int_1^0 v^{n-s+j} \int_1^{v^t} (1-u)^{r-1}\, u^{(s-r-j-1)} dudv \ ;$$

through the use of another binomial expansion,

$$= [n!/(r-1)!(s-r-1)!(n-s)!] \sum_{j=0}^{s-r-1} \binom{s-r-1}{j}(-1)^j$$

$$\int_1^0 v^{n-s+j} \int_1^{v^t} \sum_{i=0}^{r-1} \binom{r-1}{i} u^i (-1)^i\, u^{(s-r-j-1)}\, dudv$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!] \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+1} \int_0^1 v^{n-s+j} \int_1^{v^t} u^{i+s-r-j-1}\, dudv$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!] \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+1} \int_0^1 v^{n-s+j} [u^{i+s-r-j}/(i+s-r-j)]\Big|_1^{v^t} dv$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!]$$

$$[\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \binom{s-r-1}{j}\binom{r-1}{i}(-1)^{j+i+1}/(i+s-r-j)]$$

$$\int_0^1 v^{n-s+j}(v^{t(i+s-r-j)} - 1)\, dv$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!][\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+1}/(i+s-r-j)]\int_0^1 [v^{n-s+j+t(i+s-r-j)} - v^{n-s+j}]\, dv$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!][\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+1}/(i+s-r-j)](\{v^{n-s+j+t(i+s-r-j)+1}/$$

$$[n-s+j+t(i+s-r-j)+1]\} -$$

$$\{v^{n-s+j+1}/(n-s+j+1)\})\Big|_0^1$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!][\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+1}/(i+s-r-j)](\{1/[n-s+j+$$

$$t(i+s-r-j)+1] - 1/[n-s+j+1]\} - 0)$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!]\left[\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}\binom{s-r-1}{j}\binom{r-1}{i}\right.$$

$$\left.(-1)^{j+i+1}/(i+s-r-j)\right]\{[n-s+j+1-$$

$$(n-s+j+ti+ts-tr-tj+1)]/$$

$$(n-s+j+ti+ts-tr-tj+1)(n-s+j+1)\}$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!]\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}\binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+1}(t)(r+j-i-s)/(i+s-r-j)$$

$$(n-s+j+ti+ts-tr-tj+1)(n-s+j+1)$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!]\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}\binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i+2}(t)(i+s-r-j)/(i+s-r-j)$$

$$(n-s+j+ti+ts-tr-tj+1)(n-s+j+1)$$

$$= [n!/(r-1)!(s-r-1)!(n-s)!]\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}\binom{s-r-1}{j}\binom{r-1}{i}$$

$$(-1)^{j+i}(t)/[n-s+j+1+t(i+s-r-j)](n-s+j+1)$$

$$= \sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1} n!(s-r-1)!(r-1)!(-1)^{j+i}(t)/$$

$$(r-1)!(s-r-1)!(n-s)!(s-r-j-1)!j!$$

$$(r - i - 1)!i![n - s + j + 1 + t(i + s - r - j)](n - s + j + 1)$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} n!(-1)^{j+i}(t)/\{(n-s)!(s - r - j - 1)!j!(r - i - 1)!$$

$$i!(n - s + j + 1)[n - s + j + 1 + t(i + s - j - r)]\} \; . \blacksquare$$

Table 2.1 provides values of $F_{n,r,s}(t)$ calculated for selected values of s, r, n, and t. Although it is not an extensive table, it should be noted that it can be expanded through a relatively simple computer program. Also, a very appealing asymptotic distribution is outlined in Chapter 3.

It is of mathematical interest to derive both the mean and the variance of $T_{r,s}$ under H_0. They are as follows:

Theorem 2.3: The mean of $T_{r,s}$ under H_0 is:

$$\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{j+i}/(n - s)!(s - r - j - 1)!j!(r - i - 1)!$$

$$i!(i + s - r - j)^2]\{\log[(n + i - r + 1)/(n + j - s + 1)]$$

$$+ (n - s + j + 1)/(n + i - r + 1) - 1\} \; .$$

Proof: To find the mean, the frequency distribution needs to be determined. It, though, is just the first derivative of the cumulative distribution function. It is found as follows:

$$d(CDF)/dt = \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!j!$$

$$(r - i - 1)!i!(n - s + j + 1)]\{[(n - s + j + ti + ts -$$

$$tr - tj + 1) - t(i + s - r - j)]/[n - s + j + 1 +$$

$$t(i + s - r - j)]^2\}$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!j!(r - i - 1)!$$

$$i!(n - s + j + 1)](n - s + j + ti + ts - tr - tj + 1$$

$$- ti - ts + tr - tj)/[n - s + j + 1 + t(i + s - r - j)]^2$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!j!(r - i - 1)!i!$$

$$[n - s + j + 1 + t(i + s - r - j)]^2 .$$

Then, using the standard method of calculating the mean, it is found as follows:

$$E(t) = \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!$$

$$j!(r - i - 1)!i!]\int_0^1 t/[(n - s + j + 1) +$$

$$t(i + s - r - j)]^2 \, dt$$

through reference to a table of integrals,

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n-s)!(s-r-j-1)!j!(r-i-1)!i!]$$

$$\{\ln[n-s+j+1+t(i+s-r-j)] + (n-s+j+1)/$$

$$[n-s+j+1+t(i+s-r-j)]\}/(i+s-r-j)^2 \Big|_0^1$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n-s)!(s-r-j-1)!j!(r-i-1)!$$

$$i!(i+s-r-j)^2\{[\ln(n-s+j+1+i+s-r-j) +$$

$$(n-s+j+1)/(n-s+j+1+i+s-r-j)] -$$

$$[\ln(n-s+j+1) + (n-s+j+1)/(n-s+j+1)]\}$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n-s)!(s-r-j-1)!j!(r-i-1)!$$

$$i!(i+s-r-j)^2]\{\ln[(n+i-r+1)/(n+j-s+1)] +$$

$$(n-s+j+1)/(n+i-r+1) - 1\} . \blacksquare$$

Theorem 2.4. The variance of $T_{r,s}$ under H_0 is

$$\sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} ([n!(-1)^{i+j}/(n-s)!(s-r-j-1)!j!(r-i-1)!$$

$$i!(i+s-r-j)^3][i-r-n+2s-2j-1 -$$

$$2(n-s+j+1)\ln(n+i-r+1) +$$

$$2(n - s + j + 1) \ln (n + j - s + 1) -$$

$$(n - s + j + 1)^2/(n - r + i + 1)] - [n!(-1)^{i+j}/$$

$$(n - s)!(s - r - j - 1)!j!(r - i - 1)!$$

$$i!(i + s - r - j)^2]^2\{\ln [(n + i - r + 1)/$$

$$(n + j - s + 1)] + (n - s + j + 1)/$$

$$(n + i - r + 1) - 1\}^2) .$$

Proof: To use the standard variance formula, $E(t^2) - [E(t)]^2$, it is first necessary to compute $E(t^2)$. It is as follows:

$$E(t^2) = \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \{[n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!$$

$$j!(r - i - 1)!i!] \int_0^1 t^2/[(n - s + j + 1) +$$

$$t(i + s - r - j)]^2 \, dt\};$$

through reference to a table of integrals,

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} ([n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!j!(r - i - 1)!$$

$$i!](i + s - r - j)^3]\{(n - s + j + 1) + t(i + s - r - j) -$$

$$2(n - s + j + 1)\ \ln[(n - s + j + 1) + t(i + s - r - j)]$$

$$- (n - s + j + 1)^2/[(n - s + j + 1) +$$

$$t(i + s - r - j)]\}\Big|_0^1)$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} ([n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!j!(r - i - 1)!i!]$$

$$(i + s - r - j)^3]\{[n - s + j + 1 + i + s - r - j -$$

$$2(n - s + j + 1)\ \ln(n - s + j + 1 + i + s - r - j) -$$

$$(n - s + j + 1)^2/(n - s + j + 1 + i + s - r - j)] -$$

$$[n - s + j + 1 - 2(n - s + j + 1)\ \ln(n - s + j + 1) -$$

$$(n - s + j + 1)^2/(n - s + j + 1)]\})$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} [n!(-1)^{i+j}/(n - s)!(s - r - j - 1)!j!(r - i - 1)!i!]$$

$$(i + s - r - j)^3][n + 1 + i - r - 2(n - s + j + 1)$$

$$\ln(n + i - r + 1) - (n - s + j + 1)^2/(n + i - r + 1) -$$

$$n + s - j - 1 + 2(n - s + j + 1)\ \ln(n + j - s + 1) -$$

$$n + s - j - 1]$$

$$= \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} \{[n!(-1)^{i+j}/(n-s)!(s-r-j-1)!j!(r-i-1)!$$

$$i!(i+s-r-j)^3][i-r-n+2s-2j-1-2(n-s+j+1)$$

$$\ln(n+i-r+1) + 2(n-s+j+1)\ln(n+j-s+1) -$$

$$(n-s+j+1)^2/(n-r+i+1)]\};$$

then, using the value found above for $E(t^2)$ and the value of $E(t)$ found in Theorem 2.3,

$$E(t^2) - [E(t)]^2 = \sum_{j=0}^{s-r-1} \sum_{i=0}^{r-1} ([n!(-1)^{i+j}/(n-s)!(s-r-j-1)!$$

$$j!(r-i-1)!i!(i+s-r-j)^3][i-r-n+2s-2j-1-$$

$$2(n-s+j+1)\ln(n+i-r+1) + 2(n-s+j+1)$$

$$\ln(n+j-s+1) - (n-s+j+1)^2/(n-r+i+1)] -$$

$$[n!(-1)^{i+j}/(n-s)!(s-r-j-1)!j!(r-i-1)!i!$$

$$(i+s-r-j)^2]^2\{\ln[(n+i-r+1)/(n+j-s+1)] +$$

$$(n-s+j+1)/(n+i-r+1) - 1\}^2) \blacksquare$$

The following is an example of how Table 2.1 can be used:

Example 2.2: From Example 2.1, $T_{1,5} = .046$ and $T_{2,4} = .480$. From Table 2.1 the significance level for $T_{1,5} \approx .37$ and the significance level for $T_{2,4} \approx .69$. The lower significance level of $T_{1,5}$ suggests that $T_{1,5}$ is more powerful than $T_{2,4}$. The data in Example 2.1 was shown to be non-exponential by Proschan (1963). Hence, it seems reasonable to use this data as a basis for a preliminary comparison of significance levels.

Thus, the exact null distribution function has been formulated along with its mean and variance. Obviously, this distribution is not very useful due to the lack of adequate tables and the computational difficulty associated with it. Therefore, in the next chapter, asymtotic distributions are developed to make the test more useful for applied statisticians.

TABLE 2.1

SELECTED VALUES OF $F_{n,r,s}^{*}$

t	$F_{3,1,3}$	$F_{4,1,4}$	$F_{4,1,3}$	$F_{5,1,5}$	$F_{5,2,4}$
.0025	.0112	.0181	.0083	.0257	.0001
.0050	.0222	.0359	.0165	.0506	.0002
.0075	.0331	.0533	.0247	.0748	.0006
.0100	.0439	.0704	.0328	.0983	.0010
.0125	.0545	.0871	.0408	.1211	.0016
.0150	.0650	.1035	.0489	.1433	.0023
.0175	.0754	.1195	.0568	.1648	.0031
.0200	.0857	.1352	.0647	.1858	.0039
.0225	.0958	.1507	.0725	.2061	.0049
.0250	.1058	.1658	.0802	.2259	.0061
.0275	.1157	.1806	.0879	.2452	.0073
.0300	.1255	.1951	.0955	.2639	.0087
.0325	.1351	.2094	.1031	.2822	.0101
.0350	.1447	.2233	.1106	.2999	.0116
.0375	.1541	.2370	.1181	.3171	.0132
.0400	.1634	.2504	.1255	.3338	.0149
.0425	.1726	.2636	.1328	.3501	.0167
.0450	.1817	.2765	.1401	.3660	.0186
.0475	.1907	.2892	.1474	.3815	.0206
.0500	.1996	.3016	.1546	.3965	.0226
.0525	.2083	.3138	.1617	.4111	.0248
.0550	.2170	.3257	.1688	.4253	.0269
.0575	.2256	.3375	.1758	.4393	.0293
.0600	.2340	.3489	.1827	.4528	.0316
.0625	.2424	.3603	.1897	.4659	.0341
.0650	.2507	.3713	.1965	.4787	.0366
.0675	.2589	.3822	.2034	.4912	.0392
.0700	.2669	.3929	.2101	.5034	.0418
.0725	.2749	.4034	.2168	.5152	.0445
.0750	.2829	.4136	.2235	.5268	.0473
.1000	.3571	.5066	.2874	.6273	.0783
.1500	.4830	.6492	.4016	.7666	.1529
.2000	.5844	.7500	.5000	.8529	.2375
.2500	.6667	.8219	.5846	.9072	.3257
.3000	.7337	.8737	.6573	.9417	.4132
.3500	.7885	.9111	.7197	.9636	.4972
.4000	.8333	.9381	.7731	.9776	.5761
.4500	.8700	.9575	.8186	.9865	.6486
.5000	.9000	.9714	.8571	.9921	.7143
.5500	.9244	.9812	.8896	.9955	.7727
.6000	.9440	.9881	.9167	.9975	.8239
.6500	.9598	.9927	.9389	.9987	.8679
.7000	.9722	.9958	.9571	.9994	.9051

TABLE 2.1 (Continued)

t	$F_{3,1,3}$	$F_{4,1,4}$	$F_{4,1,3}$	$F_{5,1,5}$	$F_{5,2,4}$
.7500	.9818	.9978	.9714	.9997	.9356
.8000	.9890	.9989	.9824	.9999	.9597
.8500	.9941	.9996	.9905	.9999	.9779
.9000	.9975	.9998	.9959	.9999	.9904
.9500	.9994	.9999	.9990	.9999	.9977
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

*In the rest of the text $F_{n,r,s}$ will be written as $F_{r,s}$ since the value of n will be clear from the context.

CHAPTER 3

DEVELOPMENT OF ASYMPTOTIC DISTRIBUTIONS

Even though the exact null distribution of $T_{r,s}$ is known, along with its mean and variance, adequate tables are not readily available and there is some difficulty in computing observed signficance levels of a given set of data. Thus, asymptotic distributions have been derived which are as follows (assuming in each case that s and r are fixed):

$$n \ln n \, T_{r,n-s} \sim G(r, 1)$$

and

$$T_{r,s} \sim \beta(r, s - r) \ , \ s > r \ .$$

It is first necessary to consider some standard results from large sample theory. Required definitions are given below.

Definition 3.1: Let $R_1, R_2, \ldots$ be random variables with distribution functions $F_1, F_2, \ldots$ and let R be a random variable with distribution function F. We say the sequence $R_1, R_2, \ldots$ converges in distribution to R (denoted $R_n \xrightarrow{d} R$) if and only if $F_n(x) \to F(x)$ at all points x at which F is continuous.

Definition 3.2: Let $R_1, R_2, \ldots$ be random variables. The sequence $R_1, R_2, \ldots$ is said to converge in probability to R (denoted $R_n \xrightarrow{p} R$) if and only if, for every $\varepsilon > 0$, $P\{|R_n - R| \geq \varepsilon\} \to 0$ as $n \to \infty$.

Two well known theorems from asymptotic theory will be stated for later use. These theorems are as follows:

Theorem 3.1 (Slutsky's Theorem): If $Z_n \xrightarrow{d} Z$ and $U_n \xrightarrow{p} u_0$ (a constant), then

a) $Z_n + U_n \xrightarrow{d} Z + u_0$

b) $Z_n/U_n \xrightarrow{d} Z/u_0$ if $u_0 \neq 0$ (see, for example, Bickel and Doksum, 1977).

Theorem 3.2: Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables. Let $\psi_n(t) = E(e^{tX_n})$ and $\psi(t) = E(e^{tX})$, where $\psi_n(t)$ and $\psi(t)$ exist. Then $X_n \xrightarrow{d} X$ if and only if $\psi_n(t) \to \psi(t)$ in a neighborhood of 0 (see, for example, Hogg and Craig, 1978).

A well known result from advanced calculus will now be stated. This result will be used to aid in the proof of Lemma 3.3.

Lemma 3.1: $\lim_{n\to\infty} \{1 + [b/n] + [\psi(n)/n]\}^{cn} = e^{bc}$ if b and c do not depend on n and $\lim_{n\to\infty} \psi(n) = 0$ (see, for example, Hogg and Craig, 1978, Section 5.3).

Lemma 3.2: A sequence of random variables $R_1, R_2, \ldots, R_n$ converges in probability to a constant c if and only if it converges in distribution to that constant (see, for example, Bickel and Doksum, 1977, Appendix).

It should be noted that for the rest of this chapter $X_{(j)}$ will denote the jth order statistic from a random sample of size n from an exponential (1) distribution.

Limiting distributions of certain order statistics will be investigated:

Lemma 3.3: The statistic $X_{(n)}/\ln n \xrightarrow{p} 1$.

Proof: It is only necessary to show $X_{(n)}/\ln n \xrightarrow{d} 1$ by Lemma 3.2. Now

$$P(X_{(n)} < t \ln n) = [1 - e^{-(t \ln n)}]^n = [1 - (1/n)^t]^n .$$

If $t > 1$, $\lim_{n\to\infty}[1 - (1/n)^t]^n = 1$ by Lemma 3.1. If $0 < t < 1$,

$$\lim_{n\to\infty}[1 - (1/n)^t]^n = \exp \lim_{n\to\infty} n \ln[1 - (1/n)^t]$$

$$= \exp \lim_{m\to 0}[\ln(1 - m^t)]/m$$

$$= \lim_{m\to 0} \exp[\ln(1 - m^t)]/m$$

$$= 0 \quad \text{(by L'Hopital's Rule).}$$

Thus, the lemma is proven. ■

A theorem which will be used later, attributed to Barlow and Proschan (1975, Section 3.2, Theorem 2.5), is as follows:

Lemma 3.4: Let $D_1, D_2, \ldots, D_k$ be successive spacings between order statistics from the exponential (1) distribution (where the k^{th} spacing D_k between order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given by $D_k = X_{(k)} - X_{(k-1)}$ for $k = 1,2,\ldots, n$, where $X_{(0)} = 0$) . Then

a) $P[D_k \leq t] = -e^{-(n-k+1)t}$ for $k = 1,\ldots, n$, $t \geq 0$

b) $D_1,\ldots, D_n$ are mutually independent.

Note: This is a restatement of Theorem 2.1.

The following two lemmas will be the basis for the derivation of the asymptotic approximation to the null distribution:

Lemma 3.5: The statistic $X_{(n+1-k)}/\ln n \xrightarrow{p} 1$ where k is a fixed integer ≥ 1 .

Proof: Through simply adding and subtracting identical terms,

$$X_{(n+1-k)} = X_{(n)} - (X_{(n)} - X_{(n+1-k)})$$

$$= X_{(n)} - [(X_{(n)} - X_{(n-1)}) + (X_{(n-1)} - X_{(n-2)}) +$$

$$\ldots + (X_{(n-k+2)} - X_{(n+1-k)})]$$

$$= X_{(n)} - \sum_{j=1}^{k-1} (X_{(n-j+1)} - X_{(n-j)}) .$$

By Lemma 3.4 $(X_{(n)} - X_{(n-1)})$, $(X_{(n-1)} - X_{(n-2)}),\ldots,$ $(X_{(n-k+2)} - X_{(n+1-k)})$ is a sequence of independent exponential random variables with the respective parameters $1, 2,\ldots, k-1$. Therefore $\sum_{j=1}^{k-1} (X_{(n-j+1)} - X_{(n-j)})$ has a distribution that does not depend on n , and hence

$$\sum_{j=1}^{k-1} (X_{(n-j+1)} - X_{(n-j)})/\ln n \xrightarrow{p} 0 .$$

By Lemma 3.2,

$$\sum_{j=1}^{k-1} (X_{(n-j+1)} - X_{(n-j)})/\ln n \xrightarrow{d} 0 .$$

By Lemma 3.3,

$$X_{(n)}/\ln n \xrightarrow{p} 1 .$$

Hence, by these results and Theorem 3.1,

$$X_{(n+1-k)}/\ln n \xrightarrow{d} 1 .$$

By Lemma 3.2,

$$X_{(n+1-k)}/\ln n \xrightarrow{p} 1$$

which proves the lemma. ■

Lemma 3.6: The statistic $n X_{(r)} \xrightarrow{d} G$ where $G \sim G(r, 1)$ where r is a fixed positive integer.

Proof: The sequence $D_1, D_2, \ldots, D_r$ is distributed as independent exponentials with respective parameters $n , n - 1, \ldots, n - (r - 1)$, by Lemma 3.4. As before, add and subtract identical terms:

$$X_{(r)} = D_r + D_{r-1} + \ldots + D_2 + D_1$$

where $D_j \sim \exp(n - j + 1)$ by Lemma 3.4. Thus,

$$E(e^{tX_{(r)}}) = \{1/(1 - tn^{-1})\}\{1/[1 - t(n - 1)^{-1}]\}$$

$$\ldots \{1/[1 - t(n - r + 1)^{-1}]\}$$

and

$$E(e^{tn\,X_{(r)}}) = \{1/(1 - t)\}\{1/[1 - tn(n - 1)^{-1}]\}$$

$$\ldots \{1/[1 - tn(n - r + 1)^{-1}]\} .$$

Hence,

$$\lim_{n\to\infty} E(e^{tn\,X_{(r)}}) = [1/(1 - t)]^{r}$$

which is by definition the moment generating function of the gamma $G(r, 1)$ distribution. Thus, by Theorem 3.2,

$$n\,X_{(r)} \xrightarrow{d} G$$

where $G \sim G(r, 1)$. ■

The following theorem provides a useful approximation to the null distribution of $T_{r,s}$ when r is small and s is near n:

Theorem 3.3: The statistic $n \ln n\, T_{r,n+1-k}$ is asymptotically distributed as $G(r, 1)$ where r and k are both integers ≥ 1 .

Proof: The statistic $n \ln n\, T_{r,n+1-k} \overset{L}{=} (n \ln n)\, X_{(r)}/X_{(n+1-k)}$

$= n\, X_{(r)}/X_{(n+1-k)}(\ln n)^{-1}$ (where $\overset{L}{=}$ denotes "equals in law").

By Lemma 3.5,

$$X_{(n+1-k)}/\ln n \xrightarrow{p} 1 ,$$

and by Lemma 3.6,

$$n\, X_{(r)} \xrightarrow{d} G \quad \text{where} \quad G \sim G(r, 1) \ .$$

Thus, by Theorem 3.1,

$$(n \ln n) T_{r,n+1-k} = n\, X_{(r)} / [X_{(n+1-k)} (\ln n)^{-1}] \xrightarrow{d} G$$

where $G \sim G(r, 1)$. ■

Note that the proof of this theorem used the fact

$$X_{(n+1-j)}/\ln n = X_{(n)}/\ln n + U_j/\ln n$$

where the random variable U_j has a distribution that does not depend on n . The smaller j is , the nearer to zero U_j is and hence the better the approximation "$U_j/\ln n = 0$" is. Thus, the smaller k is the better the approximation is. It should also be noted that the null distribution does not depend on k since $X_{(n+1-k)}/\ln n \xrightarrow{p} 1$ for any finite positive integer k . Also, the approximation is better for small r . This is because the smaller r is, the closer $E(e^{tn\, X_{(r)}}) = \prod_{i=1}^{r} \{1/[1 - nt/(n + 1 - i)]\}$ is to $1/(1 - t)^r$ for any given n .

One computationally pleasing aspect of this theorem is the fact that $(n \ln n)\, T_{1,n+1-k}$ is asymptotically $\exp(1)$ which requires no tables since natural logarithms can be used. For $r \geq 2$, however,

the cumulative distribution function of a $G(r, 1)$ random variable is not so simple, although it can be found through repeated integration by parts. Often, rather than the entire distribution function, only certain percentiles are needed. These may be obtained from the widely available χ^2 tables, since the asymptotic distribution of $(2n \ln n)\, T_{r,n-s}$ is χ^2_{2r}.

Although Theorem 3.3 provides a useful approximation to the distribution of $T_{r,s}$ when r is small and s is near n, if both r and s are small or if both are near n, Theorem 3.3 is not applicable. In the following theorem, an approximation that is suitable for use when both r and s are small relative to n is provided:

Theorem 3.4: The statistic $T_{r,s}$ is asymptotically distributed as $\beta(r, s - r)$ when r and s are fixed positive integers and $r < s$ under the null hypothesis.

Proof: Since $r < s$,

$$T_{r,s} \overset{L}{=} X_{(r)}/X_{(s)} = n X_{(r)}/[n X_{(r)} + n(X_{(s)} - X_{(r)})] .$$

By Lemma 3.4, $X_{(r)}$ and $(X_{(s)} - X_{(r)})$ are independent. By Lemma 3.6 $n X_{(r)}$ is asymptotically $G(r, 1)$. So by adding and subtracting identical terms:

$$X_{(s)} - X_{(r)} = D_s + D_{s-1} + \ldots + D_{r+1}$$

where D_j, $(j = r + 1, r + 2, \ldots, s - 1, s)$ is distributed as an exponential $(n - j + 1)$ by Lemma 3.4. Thus,

$$E[e^{t(X_{(s)}-X_{(r)})}] = \{1/[1 - t(n - s + 1)^{-1}]\}$$

$$\{1/[1 - t(n - s + 2)^{-1}]\}\ldots\{1/[1 - t(n - r)^{-1}]\}$$

and

$$E[e^{tn(X_{(s)}-X_{(r)})}] = \{1/[1 - tn(n - s + 1)^{-1}]\}\{1/[1 -$$

$$tn(n - s + 2)^{-1}]\} \ldots \{1/[1 - tn(n - r)^{-1}]\} .$$

Therefore,

$$\lim_{n\to\infty} E[e^{nt(X_{(s)}-X_{(r)})}] = [1/(1 - t)]^{s-r}$$

which, by definition, is the moment generating function of the $G(s - r, 1)$ distribution. Thus,

$$n(X_{(s)} - X_{(r)}) \xrightarrow{d} G$$

where $G \sim G(s - r, 1)$ by Theorem 3.2. Let $U \sim G(r, 1)$ and $V \sim G(s - r, 1)$. Since $[n X_{(r)}, n(X_{(s)} - X_{(r)})]$ converges in distribution to (U, V), where

$$E(e^{tU+sV}) = (1 - t)^{-r} (1 - s)^{-(s-r)} ,$$

$$n X_{(r)}/[n X_{(r)} + n(X_{(s)} - X_{(r)})] \xrightarrow{d} U/(U + V) .$$

By Bickel and Doksum (1977, p. 17), $U/(U + V)$ has a $\beta(r, s - r)$ distribution. ■

It should be noted that the $\beta(r, s - r)$ cumulative distribution function is difficult to perform computations with unless $r = 1$ or $s - r = 1$, although tables do exist (see, for example, Pearson's table of the incomplete beta function, 1977). However, since

$$P[U/(U + V) < t] = P[(s - r)U/r(U + V) < (s - r)t/r]$$

$$= P[(s - r)U/rV < (s - r)t/r(1 - t)]$$

and

$$[(s - r)U/rV] \sim F_{2r,\ 2(s-r)},$$

the F tables can be used to give the approximate critical values of the test based on $T_{r,s}$. Hence, if $P(X < m) = \alpha$ where $X \sim F_{2r,2(s-r)}$,

$$\lim_{n\to\infty} P\{T_{r,s} < rm/[rm + (s - r)]\} = \alpha .$$

Similar computations apply to the upper-tail probabilities of $T_{r,s}$.

The asymptotic distributions for $T_{r,s}$ when r is small and s is near n, and for $T_{r,s}$ with both r and s small relative to n have been shown to be fairly simple in that they both have standard tables available (that is, the χ^2 and the F tables). The following theorem describes the more complicated case when r and

s are both large relative to n .

Theorem 3.5: The statistic $(\ln n)[1 - T_{n-j-k+1,n-j+1}]$ (where j and k are fixed positive integers) has a non-degenerate limiting distribution.

Proof: The statistic

$$1 - T_{n-j-k+1,n-j+1} \overset{L}{=} 1 - X_{(n-j-k+1)}/X_{(n-j+1)}$$

$$= (X_{(n-j+1)} - X_{(n-j-k+1)})/X_{(n-j+1)}$$

$$= (D_{j+1} + D_{j+2} + \ldots + D_{j+k})/X_{(n-j+1)}$$

$$= \sum_{i=j+1}^{j+k} D_i/X_{(n-j+1)} .$$

By Lemma 3.4, the D_i for $i = j + 1, j + 2, \ldots, j + k$ are independent $\exp(n - i)$ and hence $\sum_{i=j+1}^{j+k} D_i$ has a non-degenerate (although difficult if $k > 1$) limiting distribution. By Lemma 3.5,

$$X_{(n-j+1)}/(\ln n) \xrightarrow{P} 1 ,$$

so by Theorem 3.1

$$(\ln n)(1 - T_{n-j-k+1,n-j+1})$$

has a non-degenerate limiting distribution. ■

While tables are not available, they could be constructed for this asymptotic distribution quite easily since standard methods would suffice.

Returning to the formulation of Theorem 3.3, let $\{h_n\}$ be any bounded real sequence. By the same argument employed in Theorem 3.3, $\{n + h_n\}(\ln n)\, T_{r,n+1-k}$ is asymptotically $G(r, 1)$. In the derivation of Theorem 3.3 h_n was taken to be identically equal to zero for simplicity. Some other choice for h_n, however, might provide an approximation which is closer to the exact null cumulative distribution function. For example, in Lemma 3.6, the approximation

$$\prod_{j=1}^{r} [1 - nt/(n + 1 - j)] \approx (1 - t)^r$$

is used. It seems reasonable to conjecture that, for a fixed and finite n,

$$\prod_{j=1}^{r} \{1 - [n - (r - 1)/2]t/(n + 1 - j)\} \approx (1 - t)^r$$

is a more accurate approximation, corresponding to $h_n \equiv -(r - 1)/2$. Hence, it is conjectured that the distribution of $[n - (r - 1)/2]$ $(\ln n)\, T_{r,n+1-k}$ is better approximated by a $G(r, 1)$ distribution than that of $(n \ln n)\, T_{r,n+1-k}$.

Since $\{h_n\}$ is bounded, there will be relatively little difference between the distributions of $\{n + h_n\}(\ln n)\, T_{r,n+1-k}$ and $(n \ln n)\, T_{r,n+1-k}$, when n is large. Questions such as an optimal choice of $\{h_n\}$ and the accuracy of approximation for a fixed and finite n will not be dealt with here, but will be left as open questions.

CHAPTER 4

INVESTIGATION OF POWER OF THE TEST

To investigate the power of the test proposed in Chapter 2, two small Monte Carlo studies are used. Also, the test $T_{r,s}$ will be compared with another test for exponentiality, Hollander and Proschan's test (Hollander and Wolfe, 1973), by testing data, thought by some researchers to be non-exponential, with both tests. Finally the exact value and the asymptotic value of $F_{5,1,5}$ will be compared to give some intuitive feel of the preciseness of the asymptotic distribution.

In the Monte Carlo studies 100 Weibull triples were used to estimate the power of the test versus Weibull alternatives with shape parameters 2 and 1/2 . Weibull triples were chosen since the Weibull distribution is very commonly used for modeling failure data, and is often used as a competitor to the exponential distribution.

In the first Monte Carlo study, using standard methods, 100 independent triples of independent identically distributed Weibull (1/2, 1) random variables were generated. Weibull (1/2, 1) random variable triples were used since the Weibull (1/2, 1) distribution is DFR (where DFR means "decreasing failure rate", Barlow and Proschan, 1975). For each triple, the statistic $T_{1,3}$ was calculated, and then used to test the null hypothesis of exponentiality at significance levels of both $\alpha = .05$ and $\alpha = .01$, using the lower-tailed test described in Chapter 2. The proportion of rejections obtained gives us an idea of the power of the test $T_{1,3}$ with a sample of size 3 against Weibull alternatives with shape parameter 1/2 at the specified significance levels. The null hypothesis is rejected if $T_{1,3} < .0125$ for $\alpha = .05$

and if $T_{1/3} < .0025$ for $\alpha = .01$ (see Table 2.1). From the data given in Table A-1 (see Appendix) there are 31 rejections at the .05 significance level and 13 rejections at the .01 significance level. Thus, the power of $T_{1,3}$ against Weibull (1/2, 1) alternatives is estimated by means of this Monte Carlo study to be .31 at significance level .05 and .13 at significance level .01.

Although these estimates of power are not large, they are good when one considers the small sample size. The above suggests that the test is fairly powerful for DFR Wiebull alternatives.

In the second Monte Carlo study, like the first study, standard methods were used to select 100 independent triples of independent identically distributed Weibull (2, 1) random variables. Weibull (2, 1) random variable triples were used since the Weibull (2, 1) distribution is IFR (where IFR means "increasing failure rate", Barlow and Proschan, 1975). As before, for each triple, the statistic $T_{1,3}$ was calculated, and then used to test the null hypothesis of exponentiality at significance levels of both $\alpha = .05$ and $\alpha = .01$ using the lower-tailed test described in Chapter 2. The proportion of rejections obtained gives us an idea of the power of the test $T_{1,3}$ with a sample of size 3 against Weibull alternatives with shape parameter 2 at the specified significance levels. From Table 2.1, if $T_{1,3} < .0125$ for $\alpha = .05$ and if $T_{1,3} < .0025$ for $\alpha = .01$ the null hypothesis is rejected. From the data given in Table A-2 (see Appendix) there are 6 rejections at significance level .05 and 2 rejections at significance level .01. Thus, the power of $T_{1,3}$ against Weibull (2, 1) alternatives is estimated by means of this Monte Carlo study

to be .06 at significance level .05 and .02 at significance level .01 .

Even though the above estimates of power appear low, it does not suggest that the test has low power in general since we are testing samples of only size 3 . Therefore, high power should not be expected. Also, the test results seem to suggest that the test has considerable power against DFR alternatives but lower power against IFR alternatives.

The test developed in Chapter 2 will now be compared against another test for exponentiality. The alternate test is the Distribution-Free Test for "New Better Than Used Alternatives" devised by Hollander and Proschan (Hollander and Wolfe, 1973). The data used in both tests is failure times (in tens of hours) for an alternating current generator (Barker, 1980). Barker (1980) suggests this data to be non-exponential.

By selecting a random sample of size 30 (given in Table A-3 in the Appendix) from the original 60 observations, it was found that $T_{1,30} = .00789$ and $T_{2,29} = .01258$ ($T_{1,30}$ and $T_{2,29}$ were used since it is felt that statistics of the form $T_{r,n+1-r}$ are of most interest to applied statisticians). Through the use of the asymptotic distributions outlined in Chapter 3, the approximate observed significance level of $T_{1,30}$ was .553 and the approximate observed significance level of $T_{2,29}$ was .367.

Hollander and Proschan (Hollander and Wolfe, 1973) define a statistic H , given below, which is equivalent to the Hollander-Proschan statistic described in Chapter 1 and asymptotically normal under H_0 . The statistic H is defined as

$$H = \left[\sum_{i>j>k} \Psi(X_{(i)}, X_{(j)} + X_{(k)}) - \{n(n-1)(n-2)/8\} \right]$$

$$/\{(3/2)\, n(n-1)(n-2)[(5/2592)(n-3)(n-4) +$$

$$(n-3)(7/432) + (1/48)]\}^{1/2}$$

where $\Psi(a, b) = \begin{cases} 1 & \text{if } a > b \\ 0 & \text{if } a < b \,. \end{cases}$

Using the data referred to above, we find $H = -.451$. Although tables for $n = 30$ are not available, Hollander and Proschan showed that the null distribution of H can be approximated by a normal distribution. Using the approximation

$$[H - E_0(H)]/[\text{var}_0(H)]^{1/2} \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$, the approximate observed significance level, using a two tailed test was found to be .652.

A two-tailed test was used since we do not know if the data is new better than used data or new worse than used data prior to the application of the test. If we had known what kind of data we were testing a one-tailed test would have been applied.

The observed significance levels of the test proposed in Chapter 2 and the Hollander-Proschan test indicate that the tests are roughly equivalent when one is not willing to specify a-priori the type of

departure from exponentiality. However, the amount of work (and therefore expense) involved in calculating the observed significance level of $T_{r,s}$ is much less than that involved in calculating the observed significance level of Hollander and Proschan's test.

Finally, to obtain some intuition about the goodness of the asymptotic distribution outlined in Theorem 3.3, exact and asymptotic significance levels of $T_{5,1,5}$ will be compared using the following approximation:

$$P_0(T_{1,5} \leq s) \approx 1 - e^{-5s \ln 5} .$$

Using the above expression we would reject H_0 at the nominal significance level of $\alpha = .01$ if $T \leq .0012$. Using Table 2.1, the exact significance level would be $\alpha = .01$. Again using the above expression H_0 is rejected at the nominal significance level of $\alpha = .05$ if $T \leq .0063$. Using Table 2.1, the exact significance level would be $\alpha = .06$. As before, using the above expression H_0 is rejected at nominal significance level $\alpha = .10$ if $T \leq .0131$. Using Table 2.1, the exact significance level would be .09. These exact and asymptotic significance levels given above are all very close. Thus, it is strongly suggested that the asymptotic distributions are very accurate since samples of only size 5 were used.

In closing, we draw attention to our Monte Carlo studies, which suggested that the class of tests given here have reasonable power against certain alternatives, even with very small sample sizes. Further, the approximation of Theorem 3.3 should be very good, even for small sample sizes, as was suggested in the preceding paragraph.

Hence, tests of the form $T_{r,s}$ are a useful addition to the working tools of the applied statistician.

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APPENDIX

TABLE A-1

TRIPLES OF WIEBULL (1/2, 1) RANDOM VARIABLES

AND $T_{1,3}$ VALUES

X_i	$T_{1,3}$	
3.5965, 5.0882, 17.4716	.2409	
.5798, .8577, 2.2437	.1510	
.5278, 2.0213, 2.2224	.0673	
.0051, .7459, 7.6961	.0022	**
.0402, .8409, .9584	.0698	
.0622, 4.8721, 7.1484	.0194	
.0000, .0999, .3276	.0000	**
.0014, .0070, 8.4548	.0005	**
.0121, .2039, 3.7718	.0102	*
.0247, .6991, .9966	.0548	
.0157, .1316, 1.5400	.0334	
.2055, .5317, .7967	.4062	
.0039, .4120, 5.5806	.0022	**
.0178, .2424, 5.1403	.0109	*
.0007, .1169, 6.9889	.0003	**
.4322, .4513, 4.2389	.0101	*
.4176, 2.3799, 14.0363	.1075	
.3776, 1.2052, 21.1160	.0569	
.3817, .4236, .5909	.0732	
.0006, 1.1779, 7.0339	.0333	**
.0084, .0234, .5188	.0509	
.2876, .3760, 2.2357	.0951	
1.2543, 1.2729, 8.3186	.0053	*
.0174, 1.5044, 1.7143	.0176	
1.2251, 1.5469, 13.6672	.0531	
.0417, .1053, 9.1829	.0138	
.1478, 1.4763, 2.4976	.1669	
.3083, .8937, 24.0715	.0399	
1.8502, 8.5627, 10.0344	.1096	
.0071, .1976, 1.7726	.0135	
.0166, 9.3359, 26.5214	.0027	*
.1284, .2166, 24.3411	.0073	*
.0018, .1376, .1753	.0195	
.2978, 1.4529, 1.8177	.1543	
.0326, .0942, 5.4140	.0184	
.0078, .7285, 1.6249	.0162	
.0089, .3292, 2.9500	.0102	*
.5863, 2.7862, 2.8328	.0106	*
.0028, .0067, .0119	.6309	
.2170, 1.0667, 3.7103	.2463	
.0003, 1.2052, 6.6957	.0001	**
.0143, .1996, 1.1075	.0473	

TABLE A-1 (Continued)

X_i			$T_{i,3}$	
.1266,	.8642,	1.5961	.2575	
.3594,	.3804,	7.0793	.0063	*
.0748,	.7958,	1.4895	.1558	
.0095,	.0305,	.4224	.0727	
.1555,	.1942,	1.2072	.0764	
1.3228	5.8196,	31.9789	.1517	
.0026,	.0045,	.1446	.0271	
2.5129,	3.4375,	5.0839	.2184	
.0399,	3.2684,	10.1255	.0174	
.0396,	2.8255,	8.3914	.0213	
.0981,	.5545,	1.0961	.3224	
.3064,	2.0618,	3.2030	.2618	
.0039,	1.4178,	2.0618	.0041	*
.0231,	1.1021,	3.2182	.0321	
.0011,	.7302,	2.5584	.0018	**
.1964,	.8900,	3.7718	.2045	
.0013,	1.3257,	3.0579	.0015	**
.0561,	2.0812,	2.6775	.0416	
.3152,	1.2091,	10.4751	.1021	
.0873,	.6751,	1.2111	.2228	
2.4671,	5.7662,	22.1891	.4018	
.0193,	.7431,	4.3508	.0160	
1.7636,	3.3242,	6.3793	.5774	
.8063,	1.4641,	2.3684	.3738	
.3005,	2.3513,	15.1113	.0706	
.0339,	.0612,	.3443	.1929	
.0000,	.1605,	.6501	.0002	**
.0082,	.0265,	.5719	.0451	
.0084,	1.3033,	2.4076	.0097	*
.1686,	.4804,	.1632	.2617	
.0995,	.6017,	.0756	.0753	
.0016,	.0306,	1.6666	.0028	*
.0028,	.8813,	.9799	.0048	*
.1249,	.4505,	4.6778	.0887	
.9652,	1.4209,	7.3146	.1547	
.0230,	.1239,	.5859	.1495	
.0433,	.7358,	1.3732	.0937	
.1857,	.2021,	.5837	.0589	
.0001,	.0270,	.0371	.0037	*
.0062,	1.2359,	2.3091	.0076	*
.0002,	.2554,	6.7792	.0001	**
.5736,	.6766,	2.0350	.1198	
.0006,	.0128,	.2109	.0091	*
8.5084,	11.6573,	19.4157	.2467	
.0519,	7.4881,	11.1063	.0105	*
.0241,	.1450,	.5533	.1769	

TABLE A-1 (Continued)

X_i	$T_{1,3}$	
.2390, 2.9348, 3.7664	.1330	
.2505, 3.4399, 6.1588	.1178	
.0523, 1.5270, 10,1408	.0182	
.0059, .0362, 1.7070	.0106	*
.8373, 1.8323, 2.0390	.0823	
.0014, 5.3436, 8.0100	.0004	**
.4552, 1.2091, 1.7447	.3552	
.0613, .1284, .7329	.2221	
.0218, .2135, .5778	.1706	
.3486, .6225, 11.7833	.0491	
.0207, .0419, .4113	.1151	
.0027, .9096, 16.5541	.0005	**

*The single starred (*) entries are those rejected at the .05 significance level but not at the .01 significance level.

**The double starred (**) entries are those rejected at both the .05 and .01 significance levels.

TABLE A-2

TRIPLES OF WIEBULL (2, 1) RANDOM VARIABLES AND $T_{1,3}$ VALUES

X_i	$T_{1,3}$	
1.3771, 1.5019, 2.0444	.0604	
.8742, 1.1675, 1.2239	.0215	
.8523, 1.1924, 1.2209	.0112	*
.2676, .9294, 1.6656	.7362	
.4478, .9576, .9894	.0237	
.4995, 1.4857, 1.6351	.0758	
.0664, .5622, .7565	.1959	
.1942, .2893, 1.7052	.1343	
.3319, .6720, 1.3936	.6831	
.3962, .9144, .9992	.0713	
.3539 .6024, 1.1139	.4681	
.6733, .8539, .9448	.0449	
.2496, .8012, 1.5369	.6670	
.3651, .7017, 1.5057	.6145	
.1645, .5847, 1.6259	.4739	
.8108, .8196, 1.4348	.0072	*
.8039, 1.2420, 1.9356	.2876	
.7839, 1.0478, 2.1436	.2244	
.7860, .8068, .8767	.0176	
.1552, 1.0418, 1.6285	.2626	
.3028, .3910, .8487	.1943	
.7323, .7831, 1.2230	.0462	
1.0583, 1.0622, 1.6983	.0024	**
.3634, 1.1075, 1.1433	.0247	
1.0521, 1.1152, 1.9227	.0400	
.4520, .5696, 1.7408	.1735	
.6201, 1.1023, 1.2571	.0832	
.7452, .9696, 2.2150	.2007	
1.1663, 1.7106, 1.7798	.0198	
.2903, .6667, 1.1539	.5595	
.3589, 1.7479, 2.2693	.1877	
.5986, .6822, 2.2212	.0931	
.2054, .6090, .6471	.0472	
.7388, 1.0991, 1.1611	.0279	
.4251, .5540, 1.5254	.2022	
.2969, .9239, 1.1290	.1636	
.3073, .7573, 1.3106	.5999	
.8751, 1.2919, 1.2973	.0021	**
.2306, .2859, .3309	.0650	
.6825, 1.0163, 1.3879	.1815	
.1262, 1.0478, 1.6086	.2054	
.3459, .6684, 1.0259	.3444	

TABLE A-2 (Continued)

X_i			$T_{1,3}$	
·5965,	·9642,	1·1239	·0893	
·7743,	·7853,	1·6312	·0095	*
·5229,	·9445,	1·1047	·1021	
·3121,	·4179,	·8062	·2262	
·6279,	·6641,	1·0482	·0384	
1·0724,	1·5532,	2·3780	·2564	
·1418,	·2588,	·6166	·5504	
1·2591,	1·3616,	1·5016	·0371	
·4469,	1·3446,	1·7838	·2446	
·4460,	1·2965,	1·7019	·2384	
·5596,	·8629,	1·0232	·0955	
·7440,	1·1983,	1·3378	·0625	
·4184,	1·0912,	1·1983	·0796	
·3899,	1·0246,	1·3394	·2479	
·1812,	·9244,	1·2647	·2289	
·6657,	·9713,	1·3936	·2114	
·5191,	1·0730,	1·3223	·1601	
·4867,	1·2011,	1·2792	·0535	
·7493,	1·0486,	1.7990	·2663	
·5435,	·9065,	1·0490	·0874	
1·2533,	1·5496,	2.1704	·1576	
·3729,	·9285,	1·4443	·4611	
1·1524,	1·3503,	1·5893	·0691	
·9476,	1·0999,	1·2401	·0495	
·7404,	1·2383,	1·9716	·3302	
·0826,	·6329,	·8979	·2251	
·3009,	·4036,	·8697	·2274	
·4290,	·4973,	·7660	·1061	
·3031,	1·0684,	1·2456	·1158	
·6408,	·8326,	·9818	·0776	
·5615,	·6518,	1·7183	·1072	
·1987,	·4184,	1·1362	·6121	
·2299,	·9689,	·9949	·0176	
·5946,	·8193,	1·4707	·2519	
·9912,	1·0918,	1·6445	·0677	
·3896,	·5933,	·8749	·2409	
·4561,	·9262,	1·0825	·1142	
·6565,	·6705,	·8741	·0142	
·0868,	·4055,	·4389	·0524	
·2806,	1·0544,	1·2327	·1152	
·1108,	·7109,	1·6136	·2769	
·8703,	·9069,	1.1944	·0281	
·1542,	·3364,	·6777	·7379	
1·7079,	1·8478,	2·0991	·0491	
·4774,	1·6542,	1·8255	·0728	
·3939,	·6171,	·8624	·2076	
·6992,	1·3089,	1·3931	·0401	

TABLE A-2 (Continued)

X_i	$T_{1,3}$	
.7075, 1.3619, 1.5753	.1006	
.4783, 1.1116, 1.7845	.6729	
.2773, .4362, 1.1431	.3822	
.9566, 1.1635, 1.1949	.0109	*
.1942, 1.5204, 1.6823	.0610	
.8124, 1.0486, 1.1493	.0409	
.4976, .5987, .9253	.1354	
.3843, .6797, .8719	.1667	
.7684, .8883, 1.8527	.1039	
.3794, .4526, .8008	.1285	
.2274, .9766, 2.0171	.4553	

*The single starred (*) entires are those rejected at the .05 significance level but not at the .01 significance level.

**The double starred (**) entries are those rejected at both the .05 and .01 significance levels.

TABLE A-3

AN ORDERED RANDOM SAMPLE OF SIZE 30 FROM
ALTERNATING CURRENT GENERATOR FAILURE TIMES

In Tens of Hours
.30
.38
.52
1.06
3.89
4.09
10.05
14.11
16.08
16.40
20.62
22.90
27.27
27.43
28.22
40.38
46.66
47.26
49.50
50.00
50.25
50.90
51.08
52.07
66.66
74.48
82.71
85.20
86.10
109.73

VITA

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