

August 1, 1949

To the Committee on Graduate Study:

I am submitting to you a thesis written by Kenneth Scott Carman entitled "Semigroup Ideals." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Arts with a major in Mathematics.

D. D. Miller

Major Professor

We have read this thesis
and recommend its acceptance:

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SEMIGROUP IDEALS

5

A THESIS

Submitted to
The Committee on Graduate Study
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Master of Arts

by

Kenneth Scott Carman

August 1949

ACKNOWLEDGMENT

The author wishes to express his appreciation for the valuable assistance rendered him by Dr. D. D. Miller, under whose direction this paper was written.

FOREWORD

In the literature of ideals a left ideal L of a system S has usually been defined by the inclusion $SL \subseteq L$ and a right ideal R by the inclusion $RS \subseteq R$. On the other hand a left zero element z is usually defined by the equation $za = z$ and a right zero element by the equation $az = z$; similarly a left or right identity element e is defined by the equation $ea = a$ or $ae = a$ respectively. The author has in this paper made the terms left and right when applied to ideals consistent with the use of the terms left and right when applied to zeros or identities; thus L is a left ideal of a multiplicative system S if $LS \subseteq L$; and R is a right ideal of S if $SR \subseteq R$. The reader is therefore cautioned to interchange the words left and right throughout this paper when referring to other literature on the subject. If such an interchange of words is effected, no change in the symbolism is required.

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SEMIGROUP IDEALS

0. Introduction

In this paper we undertake an analysis of the ideal structure of semigroups, with particular reference to finite semigroups of orders two, three, and four. In Section 1 we define the terms to be used throughout the paper, and review some elementary and well-known properties of sets, semigroups, partially ordered systems, and ideals. Section 2 is devoted to a discussion of the semigroups of ideals of a semigroup, and Section 3 to the semilattices of ideals. In Section 4 are exhibited the multiplication tables of the semigroups of ideals and the Hasse diagrams of the semilattices of ideals for the 143 semigroups of orders two, three, and four.

Theorems on one-sided ideals will generally be stated and proved for left ideals; the corresponding theorems for right ideals will then be obvious by left-right duality, and resulting theorems on two-sided ideals will follow immediately. However, a number of theorems on two-sided ideals will be given separately.

The Appendix contains a list, compiled by J. C. Harden, E. E. Posey, and the author, of all semigroups of orders two, three, and four, and their subsemigroups. Throughout the thesis we shall refer to these semigroups by the reference numbers assigned to them in the Appendix.

1. Fundamental concepts

We assume that the reader is familiar with the fundamental concepts, terminology, and notation of set theory, remarking only that we shall use the symbol $\subseteq$ for inclusion and $\subset$ for proper inclusion,

capital Latin letters for sets and systems, lower case Latin letters for elements, $\cup$ for union and $\cap$ for intersection.

A system is a set which is "organized" by the introduction of one or more relations or operations among its elements or among its subsets. An important organizing relation of this sort is that of partial order,¹ defined between pairs of elements (but not necessarily all pairs of elements) of a set. A set P is said to be partially ordered if there is defined in it a binary relation $\leq$ such that

- (1) $a \leq a$ for all $a \in P$ (reflexivity),
- (2) if $a \leq b$ and $b \leq a$ then $a = b$ (asymmetry),
- (3) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

We note that the relation $a \leq b$ need not be defined for all $a, b \in P$. If it is so defined, i.e., if we adjoin to the above three properties the requirement

(4) if $a \in P$ and $b \in P$ then either $a \leq b$ or $b \leq a$ (dichotomy), then P is said to be simply ordered, or totally ordered, or merely ordered.² A simply ordered subset of a partially ordered set is often called a chain. It is sometimes convenient to write $b \geq a$ as a synonym for $a \leq b$, to write $a < b$ as an abbreviation for the simultaneous pair $a \leq b$ and $a \neq b$, and to write $a > b$ as an

¹A comprehensive treatise on partially ordered sets is Garrett Birkhoff, Lattice Theory, 2nd ed. (New York: American Mathematical Society, 1949).

²Some writers are addicted to using "ordered" as a synonym for "partially ordered."

abbreviation for the simultaneous pair $a \geq b$ and $a \neq b$.

An element a of a partially ordered set P is called an upper bound for a subset X of P if $x \in X$ implies $x \leq a$; note that a need not be an element of X . An upper bound a for a subset X is called a least upper bound (l.u.b.) or supremum (sup) if, whenever p is an upper bound for X , $a \leq p$; in other words, a is a least upper bound for X provided that (a) $x \leq a$ for all $x \in X$, and (b) if $x \leq p$ for all $x \in X$ then $a \leq p$. Clearly, if a subset X has a least upper bound, the latter is unique; for if a and a' are both least upper bounds for X then $a \leq a'$ and $a' \leq a$, whence, by the symmetry of the order relation, $a = a'$. By the same argument, if $a \in X$ and a is an upper bound for X then $a = \sup X$; in this case, a may be called the greatest element of X . We shall also consider maximal elements of a subset X , an element $p \in X$ being maximal in X if there is no element $x \in X$ such that $x > p$; note that the greatest element of X (if any) is maximal, but that a maximal element need not be greatest, and, in the absence of a greatest element, maximal elements need not be unique.

Analogous definitions will now be given, dual to those in the preceding paragraph, for the "lower" concepts. Throughout this paragraph, P will be a partially ordered set, and X an arbitrary subset of P . If $a \in P$, a is a lower bound for X if $a \leq x$ for all $x \in X$; if, in addition, $p \in P$ and $p \leq x$ for all $x \in X$ imply $p \leq a$, then a is the (unique) greatest lower bound (g.l.b.) or infimum (inf) of X . If $a \in X$ and a is a lower bound for X , then $a = \inf X$.

and a may be called the least element of X . An element $p \in X$ is said to be minimal in X if there is no element $x \in X$ such that $x < p$; X may contain more than one minimal element, but if there is a least element of X it is minimal and no other element of X is minimal.

It is easy to prove³ that if X is a chain in P then any maximal element of X is the greatest element of X , and any minimal element is the least element.

If a partially ordered set P has the properties

(5) each subset consisting of just two elements has a least upper bound,

(6) each subset consisting of just two elements has a greatest lower bound, then P is called a lattice. It follows readily by mathematical induction that if a partially ordered set P has property (5) then every finite subset of P has a least upper bound, and similarly that if P has property (6) then every finite subset has a greatest lower bound. If a partially ordered set P be assumed merely to have property (5), or merely to have property (6), then P is called a semilattice, and whenever it is necessary to distinguish between these two assumptions we may speak of an upper semilattice in the former case and a lower semilattice in the latter. A complete upper semilattice is one in which every subset (finite or infinite) has a least upper bound, and it is obvious that any finite upper semilattice is complete; similarly for complete lower semilattices and complete lattices.

³Birkhoff, op. cit., p. 9, Th. 1.5.

The most important partial ordering relation is set inclusion, and an important example of a lattice is the set of all subsets of a given set, partially ordered by inclusion; this lattice is complete, any set of subsets having for its least upper bound the union of those subsets, and for its greatest lower bound their intersection. The given set itself is the greatest element of this lattice, and the empty set is the least element. If we modify the example by requiring the given set A to contain at least two elements and by letting P be the class of all non-empty subsets of A , we have an upper semilattice in which A is the greatest element, in which there is no least element, and in which each element of A is a minimal element of P .

A convenient pictorial representation of a finite partially ordered set is a Hasse diagram, in which elements of the set are represented as the vertices of a (not necessarily connected) graph in the plane, with a line segment joining the vertices representing elements a and b whenever a covers b (i.e., when $a > b$, and $a \geq x \geq b$ implies either $a = x$ or $x = b$), the covering element being represented by a vertex higher than the vertex representing the element covered. For example, the Hasse diagram of the upper semilattice formed by the set of all non-empty subsets of a set of three elements $\{a, b, c\}$, ordered by inclusion, is illustrated by Figure 1.

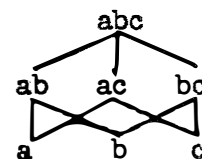


Figure 1

A second method of organizing a set is by introducing an operation which associates with each ordered pair of its elements a unique element of the set. Such an operation is often called multiplication, and if c

is the unique element associated with the ordered pair (a, b) this relationship is written $a \circ b = c$ or $ab = c$. If such an operation is defined in a set S and if the operation is associative, i.e., if $a(bc) = (ab)c$ for all elements a, b , and c of S , then the system consisting of the set and the operation is called a semigroup; no confusion need arise from using the same letter S to denote both the set and the system. In general, ab and ba need not be the same element of S ; but if $ab = ba$ for all elements a and b of S , then S is called a commutative semigroup. Familiar examples of semigroups are the system of all real numbers under ordinary multiplication, the system of all rational numbers under arithmetical addition, and the system of 2×2 matrices over a field under matrix multiplication. The first two of these examples are commutative, the third non-commutative; all three, having infinitely many elements, are examples of infinite semigroups. As an example of a finite semigroup (i.e., one having only a finite number of elements) we may cite the system consisting of the numbers $+1$ and -1 , under ordinary multiplication. The number of elements in a finite semigroup S is called the order of S . In the Appendix are listed all finite semigroups of orders less than 5. The notion of product of elements of a semigroup is easily extended to product of subsets of a semigroup; if A and B are non-empty subsets of a semigroup S , the product AB is defined to be the set of all elements ab of S , where $a \in A$, $b \in B$. It is easily seen that the associativity of multiplication of elements implies the associativity of multiplication of subsets: $A(BC) = (AB)C$.

A finite semigroup may be described completely by giving its "multiplication table." Thus, if s_1 and s_2 are the elements of a semigroup S , and if $s_1s_1 = s_1$, $s_1s_2 = s_1$, $s_2s_1 = s_2$, and $s_2s_2 = s_2$, we have the table

	s_1	s_2
s_1	s_1	s_1
s_2	s_2	s_2

in which the first factor in a product of two elements is a row-label and the second factor is a column-label. We may abbreviate by omitting the "s" and writing only the subscripts:

	1	2
1	1	1
2	2	2

and we may (and, in the sequel, shall) abbreviate further by omitting the row- and column-labels, understanding that in a semigroup of order k the row-labels and column-labels are each the sequence $1, \dots, k$. Thus, in the present example, the multiplication table may be written simply

	1	2
1	1	1
2	2	2

Two mathematical systems, each consisting of a set of elements and an operation or a relation defined among the elements, are said to be isomorphic if there is a one-to-one correspondence between the elements of the two sets under which the operation or relation is preserved; the correspondence is then called an isomorphism. In particular, two semigroups S and T are isomorphic if there is a 1 - 1 correspondence between the elements of S and the elements of T such that

if s_1 corresponds to t_1 and s_2 corresponds to t_2 then s_1s_2 corresponds to t_1t_2 . And two partially ordered sets P and Q are isomorphic if there is a 1 - 1 correspondence between their elements such that if p_1 corresponds to q_1 and p_2 corresponds to q_2 then $p_1 \leq p_2$ if and only if $q_1 \leq q_2$. An isomorphism may indeed be looked upon as a re-naming of the elements of a system, carried out in such a way that if a is renamed a' and b is renamed b' then ab is renamed $a'b'$. An anti-isomorphism is a 1 - 1 correspondence between two systems which reverses a binary operation or relation; for example, in an anti-isomorphism between the semigroups above, s_1s_2 corresponds to t_2t_1 ; and in an anti-isomorphism between the partially ordered sets above, $p_1 \leq p_2$ if and only if $q_2 \leq q_1$. We shall usually not distinguish between two systems that are either isomorphic or anti-isomorphic. For example, semigroup No. 2.3 in the Appendix may be written

$$\begin{array}{c} 1 \ 2 \\ 1 \left| \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \right. \end{array} \quad \text{or} \quad \begin{array}{c} 0 \ 1 \\ 0 \left| \begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right. \end{array} \quad \text{or} \quad \begin{array}{c} 1 \ 2 \\ 1 \left| \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right. \end{array},$$

but we list only the first of these in the Appendix.

A non-empty subset R of a semigroup S is a subsemigroup of S if the set R and the multiplication defined in S constitute a semigroup. Since the multiplication is assumed to be associative throughout S , in order that a non-empty subset R be a subsemigroup it is necessary and sufficient that R be closed under multiplication; symbolically: $RR \subseteq R$. Clearly, S itself is a subsemigroup of S ; all other subsemigroups of S are said to be proper subsemigroups. If L is a non-empty subset of S such that not only is $LL \subseteq L$ but $LS \subseteq L$, then

L is called a left ideal of S ; similarly, if R is non-empty and $SR \subseteq R$ then R is a right ideal of S ; and a subset of S that is both a right ideal and a left ideal is said to be a two-sided ideal, or simply an ideal, of S . For example, in the semigroup S displayed on the left above, the left ideals are $\{1\}$, $\{2\}$, and S itself, while S is the only right ideal and hence the only two-sided ideal. It is obvious that a semigroup S is always a two-sided ideal of S , and that in a commutative semigroup all ideals are two-sided.

Similarly, if T is a subset of a semilattice P , T is a subsemilattice of P if, with respect to the partial order defined in P , T is itself a semilattice; and T is a sublattice of P if T is itself a lattice with respect to the same partial order.

2. Semigroups of Ideals

If L_1 and L_2 are left ideals of a semigroup S , then, since multiplication of subsets of S is associative,

$$(L_1 L_2)S = L_1(L_2 S) \subseteq L_1 L_2.$$

Hence the product of two left ideals of S is a left ideal of S .

Therefore we have

Theorem 2.1. The set of left ideals of a semigroup forms a semigroup under multiplication.

We shall usually refer to this semigroup formed by the left ideals of a semigroup S as the left ideal semigroup of S ; similarly, we speak of the right ideal semigroup of S , and, the two-sided ideals of S obviously forming a semigroup, we shall refer to it as the two-sided ideal semigroup of S or simply as the ideal semigroup of S . A commutative semigroup clearly has a commutative ideal semigroup.

The accompanying Tables I - III list the semigroups formed by the left, right, and two-sided ideals of the semigroups of orders two, three and four. For each (left, right, or two-sided) ideal semigroup, either its multiplication table is displayed, or the reference number already assigned to that particular semigroup (or to an isomorph or anti-isomorph of it) in the Appendix is given. When the multiplication table of a (left, right, or two-sided) ideal semigroup is given, a numerical label is assigned to each ideal and used in the Table. When the tabular form is omitted, the ideals are simply listed, followed by the reference number of the semigroup which they form. The omission of a list of left (or right) ideals of a semigroup indicates that the left (or right) ideals are all two-sided.

Having defined product of non-empty subsets of a semigroup S (we could, as will be remarked later, extend the definition to include the empty set, but this is irrelevant to our present purposes), and having observed this multiplication of subsets to be associative, we see that under this operation the class of all non-empty subsets of S forms a semigroup which we shall denote by Σ . Now let us enquire into the position occupied in Σ by the semigroup of left ideals of S .

Theorem 2.2. The class Γ of left ideals of a semigroup S is a right ideal of Σ .

Proof. Let $M \in \Sigma$ and let $L \in \Gamma$. Then $(ML)S = M(LS) \subseteq ML$, whence $ML \in \Gamma$ and therefore $\Sigma\Gamma \subseteq \Gamma$.

If A is a left ideal of a semigroup S such that $AS = A$ (instead of merely $AS \subseteq A$), we shall call A a high left ideal of S .

Similarly, we speak of high right and two-sided ideals. In the tables we shall distinguish a high ideal A by marking it with an asterisk, thus: $*A$.

Theorem 2.3. The class Δ of high left ideals of a semigroup S is a right ideal of Σ , and hence a right subideal of the right Σ -ideal of left S -ideals.

Proof. Let $M \in \Sigma$ and let $H \in \Delta$. Then $(MH)S = M(HS) = MH$, whence $MH \in \Delta$ and therefore $\Sigma\Delta \subseteq \Delta$.

Two "distributive laws" that we shall use constantly without explicit mention (but shall not tarry to prove⁴) are

$$A \circ \bigcup B_i = \bigcup AB_i, \quad A \circ \bigcap B_i \subseteq \bigcap AB_i,$$

where A and B_i are arbitrary subsets of a semigroup and i ranges over an index set of arbitrary cardinality. If we define the product of the empty set by any set to be the empty set, we see that these laws are valid even when some of the sets involved are empty. Note that the inclusion in the second of these is not in general reversible; a finite example is found in semigroup No. 4.49, in which, if we let S_1 be the subsemigroup $\{1, 3, 4\}$, S_2 the subsemigroup $\{1, 2\}$, and S_3 the subsemigroup $\{3, 4\}$, we find that $S_1(S_2 \cap S_3)$ is empty since $S_2 \cap S_3$ is empty, while $S_1S_2 \cap S_1S_3 = \{1, 2\} \cap \{1, 3, 4\} = \{1\}$. An example in which $S_1S_2 \cap S_1S_3 \not\subseteq S_1(S_2 \cap S_3)$ even though both sides are non-empty is given by Dubreil.⁵

⁴For a proof, see Paul Dubreil, Algèbre, v. 1 (Paris: Gauthier-Villars, 1946), p. 31.

⁵Ibid., p. 32.

3. Semilattices of ideals

Our first concern will be to establish the fact that the left (right) ideals of a semigroup, partially ordered by inclusion, form a semilattice, of which the two-sided ideals are a subsemilattice.

Theorem 3.1. The union of any collection of left ideals of a semigroup S is a left ideal of S .

Proof. Let $\{L_i\}$ be any collection of left ideals of S , and let $\bigcup L_i$ be their union. Then $(\bigcup L_i)S = \bigcup (L_i S) \subseteq \bigcup L_i$, whence $\bigcup L_i$ is a left ideal of S . A similar proof holds for right ideals, whence the theorem follows for two-sided ideals.

Theorem 3.2. The intersection of any collection of left ideals of a semigroup S is either empty or a left ideal of S .

Proof. Let $\{L_i\}$ be any collection of left ideals of S , let $\bigcap L_i$ be their intersection, and suppose $\bigcap L_i$ is non-empty. Then $(\bigcap L_i)S \subseteq \bigcap (L_i S) \subseteq \bigcap L_i$, whence $\bigcap L_i$ is a left ideal of S . Similarly for right and two-sided ideals.

Theorem 3.3. The left ideals of a semigroup S form a complete upper semilattice, with respect to inclusion, of which the two-sided ideals of S form a complete upper subsemilattice.

Proof. By Theorem 3.1, the union of any set $\{L_i\}$ of left ideals of a semigroup is a left ideal, and since the union of any collection of sets is the smallest set containing all of them, $\bigcup L_i = \sup \{L_i\}$. Hence the left ideals of S form a complete upper semilattice. Since the union of two-sided ideals is a two-sided ideal, the two-sided ideals of S also form a complete upper semilattice, which is a subsemilattice of

the semilattice of left ideals since every two-sided ideal is a left ideal. The theorem and its proof hold equally, of course, for right ideals.

We note that the semilattices of right ideals and of left ideals need not be lattices, for two right (or left) ideals may have empty intersection and hence no lower bound; for example, No. 2.3 in the Appendix. The example just cited is non-commutative and we shall see presently that any such example must be.

Theorem 3.4. The intersection of any finite set of two-sided ideals of a semigroup is non-empty.

Proof. If A and B are two-sided ideals of a semigroup S , then $AB \subseteq AS \subseteq A$ and $AB \subseteq SB \subseteq B$, whence $AB \subseteq A \cap B$. Now denote by $\prod_{i=1}^n A_i$ the product $A_1 A_2 \cdots A_n$ of any set of n two-sided ideals of S . We shall show by induction that $\prod_{i=1}^n A_i \subseteq \bigcap_{i=1}^n A_i$. First, as we have just shown, $A_1 A_2 \subseteq A_1 \cap A_2$. Now suppose that $\prod_{i=1}^k A_i \subseteq \bigcap_{i=1}^k A_i$ for some arbitrary $k \leq n$. Then $\prod_{i=1}^{k+1} A_i = (\prod_{i=1}^k A_i) A_{k+1} \subseteq (\prod_{i=1}^k A_i) \cap A_{k+1}$, since $\prod_{i=1}^k A_i$ is an ideal (being an element of the semigroup of ideals of S) and the product of two ideals is contained in their intersection. But, by our supposition, $\prod_{i=1}^k A_i \subseteq \bigcap_{i=1}^k A_i$, whence $(\prod_{i=1}^k A_i) \cap A_{k+1} \subseteq (\bigcap_{i=1}^k A_i) \cap A_{k+1} = \bigcap_{i=1}^{k+1} A_i$. Hence $\prod_{i=1}^{k+1} A_i \subseteq \bigcap_{i=1}^{k+1} A_i$, and the induction is complete. Since $\prod_{i=1}^n A_i$ is non-empty, $\bigcap_{i=1}^n A_i$ is non-empty, as was to be proved.

We now have immediately the

Corollary 1. The two-sided ideals of a semigroup form a lower semilattice.

Corollary 2. The two-sided ideals of a semigroup form a lattice.

The lattice of two-sided ideals is not necessarily complete,

however, even though the upper semilattice is complete, for not every set of two-sided ideals need have non-empty intersection. For example, consider the multiplicative semigroup of positive integers, and the infinite set $\{I_k\}$ of two-sided ideals, where I_k is defined to be the set of integers greater than k ; the intersection of this set of ideals is clearly vacuous.

Corollary 3. If a semigroup S has only a finite number of two-sided ideals (and à fortiori if S itself is finite), the lattice of two-sided ideals is complete.

Although, as we have seen, two left ideals of a semigroup S may have empty intersection, so that the semilattice of left ideals of S may not be a lattice, nonetheless we can show that some elements of the semilattice may (and, under circumstances to be mentioned presently, must) have a certain "minimal" character.

We recall that in any partially ordered system P an element a of a subset X is said to be a minimal element of X (or to be minimal in X) if $x \in X$ and $x \leq a$ jointly imply $x = a$. Since any subset X of a partially ordered system P is obviously itself a partially ordered system with respect to the partial order defined in P , it was not necessary to define the notion of minimal element relative to an arbitrary subset of P ; we might as well have given the definition in the "absolute" form: an element a of P is minimal in P if $x \in P$, $x \leq a$ jointly imply $x = a$. We gave the "relativized" form of the definition because we wished to emphasize the fact that a minimal element of a subset X of P need not be minimal in P , although of course an

element minimal in P is necessarily minimal in every subset X containing that element. This is particularly important because our main concern will be with semilattices (including lattices), and a subset of a semilattice T need not be a subsemilattice of T , nor need a subset of a lattice be a sublattice. For example, in the lattice of all subsets of a given set of two or more elements the non-empty sets form an upper subsemilattice but not a sublattice, and the subsets each consisting of a single element do not even constitute a subsemilattice.

Returning to minimal elements, let us consider the complete upper semilattice $\mathcal{L}$ of left ideals of a semigroup S , partially ordered by inclusion. A left ideal L of S is then minimal in a subsemilattice $\mathcal{L}_1$ of $\mathcal{L}$ if L properly contains no left ideal K of S such that $K \in \mathcal{L}_1$; such an ideal L minimal in $\mathcal{L}_1$ need not, of course, be minimal in $\mathcal{L}$. A left ideal L of S is minimal in $\mathcal{L}$ if it properly contains no left ideal of S . Accordingly we may define a minimal left ideal of a semigroup S as a left ideal of S which properly contains no left ideal of S ; in other words, a subset L of a semigroup S is a minimal left ideal of S if (1) $IS \subseteq L$ and (2) $N \subset L$ implies $NS \not\subseteq N$. It is important to note that L may be a minimal left ideal of S even though there is a proper subset K of L such that $KL \subseteq K$; in this case K is a left ideal of the semigroup L , but need not be a left ideal of S and hence need not be an element of $\mathcal{L}$. Of course, in the semilattice of left ideals of the semigroup L , K is an element; and if K properly contains no left ideal of the semigroup L then K is a minimal element of the semilattice of left

ideals of L . Hence it is necessary to distinguish carefully between the statement " L properly contains no left ideal of S ," which by definition is equivalent to " L is a minimal left ideal of S ," and the statement " L has no proper ideals," in which L is thought of as a semigroup quite apart from the other left ideals of S . The statement " L properly contains no left ideals" is ambiguous, and some writers have sought to avoid confusion by making the following definitions: if L is a left ideal of a semigroup S and A is a subset of L , then (a) if $AL \subseteq A$, A is a left ideal of L , and (b) if $AS \subseteq A$, A is a left subideal of L (i.e., a left ideal of S contained in L).

With these tedious terminological preliminaries out of the way, we observe that a finite semilattice must have at least one minimal element, whence we have at once

Theorem 3.5. If a semigroup S has only a finite number of left ideals, then S has at least one minimal left ideal.

Corollary 1. Any finite semigroup has at least one minimal left ideal.

Theorem 3.6. Every minimal left ideal of a semigroup S is a high left ideal of S .

Proof. Let A be a left ideal of S . Then $AS \subseteq A$ and $(AS)S = A(SS) \subseteq AS$, whence AS is a left ideal of S contained in the left ideal A . Hence, if A is minimal, $AS = A$ and A is therefore a high left ideal of S .

The converse of Theorem 3.6 is false: Semigroup No. 4.59 in the Appendix, for example has a high left ideal $\{1, 2, 4\}$ of which the high left ideals $\{1\}$, $\{2\}$, and $\{4\}$ are proper subsets.

Theorem 3.7. If L_1 is a left ideal of a semigroup S and if L_2 is a high left ideal of the semigroup L_1 , then L_2 is a left ideal of S .

Proof. By hypothesis, $L_1 S \subseteq L_1$ and $L_2 L_1 = L_2$. Hence $L_2 S = (L_2 L_1) S = L_2 (L_1 S) \subseteq L_2 L_1 = L_2$, whence L_2 is a left ideal of S .

By Theorem 3.6 we have at once the

Corollary. If L_1 is a left ideal of a semigroup S and if L_2 is a minimal left ideal of the semigroup L_1 , then L_2 is a left ideal of S .

The Hasse diagrams of the semilattices of one-sided and two-sided ideals of semigroups of orders two, three, and four may be found in Tables I - III. The one-sided ideals are indicated by subscripts L and R for left ideals and right ideals, respectively; two-sided ideals bear no subscripts. For each semigroup, all its subsemigroups are also listed, and with each subsemigroup is given the reference number of its isomorph or anti-isomorph in the Appendix.

4. Ideal structure of semigroups of orders two, three, and four

We have observed that the following semigroups are isomorphic to their semigroups of two-sided ideals: 2.2, 2.4, 3.2, 3.7, 3.8, 3.10, 3.16, 4.1, 4.8, 4.38, 4.41, 4.42, 4.39, 4.45, 4.47, 4.76, 4.105, 4.115, 4.116, 4.118. No case was found in which a semigroup was isomorphic to a semigroup of one-sided ideals of that semigroup.

Among the semigroups of order four, there are 28 which are not isomorphic to their semigroups of left, right, or two-sided ideals but which have left, right, or two-sided ideal semigroups of order four. Of these, 4.22, 4.27, 4.75, 4.79, 4.80, and 4.82 have two-sided ideal semigroups isomorphic to 4.9; 4.16, 4.19, 4.73, 4.91, 4.92, and 4.93 have

two-sided ideal semigroups isomorphic to 4.49; 4.30, 4.31, 4.32, 4.77, and 4.78 have two-sided ideal semigroups isomorphic to 4.55. Also, the right ideal semigroups of 4.74, 4.87, 4.106, 4.120 and the left ideal semigroup of 4.107 are isomorphic to 4.68; the right ideal semigroups of 4.60 and 4.98, and the left ideal semigroup of 4.63, are isomorphic to 4.70; the left ideal semigroups of 4.72, 4.98, and 4.103 are isomorphic to 4.02.

Many distinct semigroups have isomorphic lattices of ideals. In fact, the 55 commutative semigroups of order four fall into ten classes of ideal-lattice-isomorphic semigroups, and the entire set of 121 semigroups of order four is divided into just 31 such classes. Table IV gives the Hasse diagrams of these ideal lattices and lists the semigroups whose ideal lattices are isomorphic to each of them.

TABLE I

IDEAL ANALYSIS 2x2 SEMIGROUPS

Subsemigroups	Semilattice	Ideals	Ideal Table
2.1 $\begin{array}{ c } \hline 12 \\ \hline 21 \\ \hline \end{array}$	S	*S	$\begin{array}{c} 1 \\ 1 \mid \overline{=1.1} \end{array}$
2.2 $\begin{array}{ c } \hline 11 \\ \hline 12 \\ \hline \end{array}$	$\begin{array}{c} S \\ \\ (1) \end{array}$	$\begin{array}{l} *(1) \\ *S \end{array}$	$\begin{array}{c} 1 \ 2 \\ 1 \ 2 \mid \overline{=2.2} \end{array}$
2.3 $\begin{array}{ c } \hline 11 \\ \hline 22 \\ \hline \end{array}$	$\begin{array}{c} S \\ \swarrow \quad \searrow \\ (1)_L \quad (2)_L \end{array}$	$\begin{array}{l} \text{Left} \\ *(1) \\ *(2) \\ *S \end{array}$	$\begin{array}{c} 1 \ 2 \ 3 \\ 1 \ 2 \ 3 \mid \overline{=3.14} \end{array}$
2.4 $\begin{array}{ c } \hline 11 \\ \hline 11 \\ \hline \end{array}$	$\begin{array}{c} S \\ \\ (1) \end{array}$	$\begin{array}{l} *(1) \\ S \end{array}$	$\begin{array}{c} 1 \ 2 \\ 1 \ 2 \mid \overline{=2.4} \end{array}$

TABLE II

IDEAL ANALYSIS 3x3 SEMIGROUPS

Subsemigroups		Semilattice	Ideals	Ideal Table
3.1	(12)--2.1	$\begin{array}{c} S \\ \\ (12) \end{array}$	$\begin{array}{c} *(12) \\ S \end{array}$	$\begin{array}{c cc} & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 2 & 4 \end{array}$
3.2	(12)--2.4	$\begin{array}{c} S \\ \\ (12) \\ \\ (1) \end{array}$	$\begin{array}{c} *(1) \\ (12) \\ S \end{array}$	$\begin{array}{c ccc} & 1 & 2 & 3 \\ 1 & 3 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{array}$
3.3		S	*S	$\begin{array}{c c} 1 \\ 1 & 1.1 \end{array}$
3.4	(12)--2.4 (13)--2.1	$\begin{array}{c} S \\ \\ (13) \end{array}$	$\begin{array}{c} *(13) \\ S \end{array}$	$\begin{array}{c cc} & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 2 & 4 \end{array}$
3.5	(12)--2.2 (13)--2.1	$\begin{array}{c} S \\ \\ (13) \end{array}$	$\begin{array}{c} *(13) \\ *S \end{array}$	$\begin{array}{c cc} & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 2 & 4 \end{array}$
3.6	(12)--2.4 (13)--2.2		$\begin{array}{c} *(1) \\ *(13) \\ (12) \\ S \end{array}$	$\begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 9 & 9 \\ 2 & 4 & 9 & 9 & 9 \\ 3 & 4 & 9 & 9 & 9 \\ 4 & 4 & 9 & 9 & 9 \end{array}$

TABLE II

IDEAL ANALYSIS 3x3 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
3.7	(12) → 2.2 (13) → 2.4		Two Sided *(1) *(13) * S Left *(1) *(13) *(12) * S	$ \begin{array}{c ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \approx 3.7 $ $ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \approx 4.68 $
3.8	(12) → 2.4 (13) → 2.2		*(1) *(12) * S	$ \begin{array}{c ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \approx 3.8 $
3.9	(12) → 2.3 (13) → 2.4		*(12), S = 2.4 Left *(1) *(2) *(12) (13) S	$ \begin{array}{c ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 3 & 3 & 3 & 3 \end{array} $
3.10	(13) → 2.2 (12) → 2.4		*(3) *(13) S	$ \begin{array}{c ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \approx 3.10 $
3.11	(12) → 2.1 (13) → 2.2		*(3) * S	$ \begin{array}{c cc} & 1 & 2 \\ \hline 1 & & \\ 2 & & \end{array} \approx 2.2 $

TABLE II

IDEAL ANALYSIS 3x3 SEMIGROUPS (Continued)

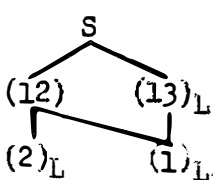
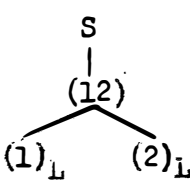
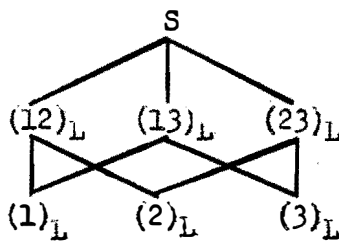
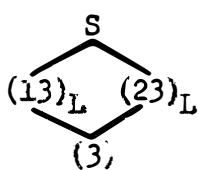
Subsemigroups		Semilattice	Ideals	Ideal Table																																																								
3.12	(12)—2.3 (13)—2.2		*(12), *S = 2.2 Left *(1) *(2) *(12) *(13) *S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>4</td><td>1</td><td>1</td><td>1</td><td>4</td></tr><tr><td>5</td><td>3</td><td>3</td><td>3</td><td>5</td></tr></table>	1	2	3	4	5	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	1	1	1	4	5	3	3	3	5																										
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3.13	(12)—2.3 (13)—2.2 (23)—2.2		*(12), *S = 2.2 Left *(1) *(2) *(12) *S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>1</td><td></td><td></td><td></td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> =4.70	1	2	3	4	1				2				3				4																																							
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3.14	(12)—2.3 (13)—2.3 (23)—2.3		Left *(1) *(2) *(3) *(12) *(13) *(23) *S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td></tr><tr><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr><tr><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td></tr><tr><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td></tr></table>	1	2	3	4	5	6	7	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	5	6	6	6	6	6	6	6	7	7	7	7	7	7	7
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3.15	(12)—2.3 (13)—2.2 (23)—2.2		*(3), *S = 2.2 Left *(3) *(23) *(13) *S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>1</td><td></td><td></td><td></td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> =4.102	1	2	3	4	1				2				3				4																																							
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TABLE II

IDEAL ANALYSIS 3×3 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
3.16	(12) $\rightarrow$ 2.2 (13) $\rightarrow$ 2.2 (23) $\rightarrow$ 2.2	$ \begin{array}{c} S \\ \\ (12) \\ \\ (1) \end{array} $	$ \begin{array}{l} *(1) \\ *(12) \\ *S \end{array} $	$ \begin{array}{c ccc} & 1 & 2 & 3 \\ 1 & & & \\ 2 & & \equiv 3.16 & \\ 3 & & & \end{array} $
3.17	(12) $\rightarrow$ 2.2 (13) $\rightarrow$ 2.2	$ \begin{array}{ccc} & S & \\ (12) & \diagdown & \diagup (13) \\ & (1) & \end{array} $	$ \begin{array}{l} *(1) \\ *(12) \\ *(13) \\ *S \end{array} $	$ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ 1 & & & & \\ 2 & & \equiv 4.49 & & \\ 3 & & & & \\ 4 & & & & \end{array} $
3.18	(12) $\rightarrow$ 2.4 (13) $\rightarrow$ 2.4	$ \begin{array}{ccc} & S & \\ (12) & \diagdown & \diagup (13) \\ & (1) & \end{array} $	$ \begin{array}{l} *(1) \\ (12) \\ (13) \\ S \end{array} $	$ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ 1 & & & & \\ 2 & & \equiv 4.55 & & \\ 3 & & & & \\ 4 & & & & \end{array} $

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS

Subsemigroups	Semilattice	Ideals	Ideal Table
4.1 1111 1222 1233 1233 (12)→2.2 (13)→2.2 (23)→2.2 (33)→2.4 (123)→3.16 (134)→3.10 (234)→3.10	S (123) (12) (1)	*(1) *(12) *(123) S	1 2 3 4 1 2 3 4 ■ 4.1
4.2 1111 1211 1111 1111 (12)→2.2 (13)→2.4 (14)→2.4 (123)→3.6 (124)→3.6 (134)→3.18	S (123) (124) (134) (12) (13) (14) (1)	*(1) *(12) (13) (14) (123) (124) (134) S	1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 2 1 2 1 1 2 2 1 2 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 2
4.3 1111 1111 1122 1122 (12)→2.4 (123)→3.2 (124)→3.2	S (123) (124) (12) (1)	*(1) (12) (123) (124) S	1 2 3 4 5 1 2 3 4 5 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 1 1 2 2 2 1 1 2 2 2
4.4 1111 1214 1111 1411 (12)→2.2 (13)→2.4 (14)→2.4 (123)→3.6 (124)→3.8 (134)→3.18	S (124) (134) (14) (13) (1)	*(1) *(14) *(124) (13) (134) S	1 2 3 4 5 6 1 2 3 4 5 6 1 1 1 1 1 1 1 1 2 1 1 2 1 2 3 1 2 3 1 1 1 1 1 1 1 1 2 1 1 2 1 2 3 1 2 3

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.5 1111 1212 1111 1214 (12)→2.2 (14)→2.2 (24)→2.2 (13)→2.4 (123)→3.6 (134)→3.6 (124)→3.16	S (123) (124) (13) (12) (1)	*(1) *(12) *(124) (13) (123) S	1 2 3 4 5 6 1 1 1 1 1 1 2 1 2 2 1 2 2 3 1 2 3 1 2 3 4 1 1 1 1 1 1 5 1 2 2 1 2 2 6 1 2 3 1 2 3
4.6 1111 1211 1111 1113 (12)→2.2 (13)→2.4 (123)→3.6 (134)→3.2	S (123) (134) (12) (13) (1)	*(1) *(12) (13) (123) (134) S	1 2 3 4 5 6 1 1 1 1 1 1 2 1 2 1 2 1 2 3 1 1 1 1 1 1 4 1 2 1 2 1 2 5 1 1 1 1 3 3 6 1 2 1 2 3 4
4.7 1111 1111 1112 1122 (12)→2.4 (13)→2.4 (123)→3.18 (124)→3.2	S (123) (124) (12) (1)	*(1) (12) (123) (124) S	1 2 3 4 5 1 1 1 1 1 2 1 1 1 1 1 3 1 1 1 2 2 4 1 1 2 2 2 5 1 1 2 2 2
4.8 1111 1113 1111 1312 (12)→2.4 (13)→2.4 (123)→3.18	S (123) (13) (1)	*(1) (13) (123) S	1 2 3 4 1 2 4.8 3 4

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

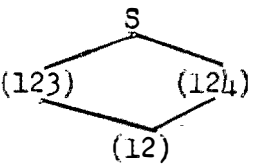
Subsemigroups		Semilattice	Ideals	Ideal Table
4.14	(12) → 2.1 (34) → 2.1	S (12)	*(12) * S	$\begin{array}{c cc} & 1 & 2 \\ 1 & & \\ 2 & =2.2 & \end{array}$
$\overline{1211}$	(123) → 3.5			
2122	(134) → 3.11			
1234				
1243				
4.15	(12) → 2.1 (34) → 2.1	S (12)	*(12) * S	$\begin{array}{c cc} & 1 & 2 \\ 1 & & \\ 2 & =2.2 & \end{array}$
$\overline{1212}$	(123) → 3.5			
2121				
1234				
2143				
4.16	(12) → 2.1 (13) → 2.2 (14) → 2.2		*(12) *(123) *(124) * S	$\begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & =4.49 & & & \end{array}$
2122	(124) → 3.5			
1231	(123) → 3.5			
1214	(134) → 3.17			
4.17	(12) → 2.1 (14) → 2.2	S (124) (12)	*(12) *(124) * S	$\begin{array}{c ccc} & 1 & 2 & 3 \\ 1 & & & \\ 2 & =3.16 & & \\ 3 & & & \end{array}$
$\overline{1211}$	(13) → 2.2			
2122	(34) → 2.2			
1234	(123) → 3.5			
1244	(124) → 3.5 (134) → 3.16			
4.18	(12) → 2.1 (13) → 2.2	S (123) (3)	*(3) (123) S	$\begin{array}{c ccc} & 1 & 2 & 3 \\ 1 & & & \\ 2 & =3.16 & & \\ 3 & & & \end{array}$
$\overline{1231}$	(34) → 2.2			
2132	(14) → 2.2			
3333	(123) → 3.11			
1234	(134) → 3.16 (124) → 3.5			

TABLE III

IDEAL ANALYSIS $4x4$ SEMIGROUPS (Continued)

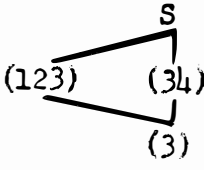
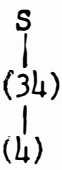
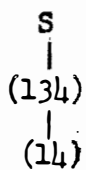
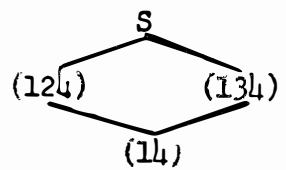
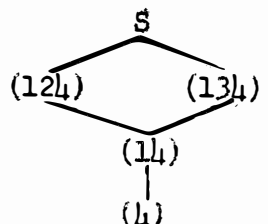
Subsemigroups		Semilattice	Ideals	Ideal Table																														
4.19	(12) — 2.1 (13) — 2.2 (14) — 2.2 (34) — 2.2 (123) — 3.11 (134) — 3.17		*(3) *(34) *(123) * S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>1</td><td></td><td></td><td></td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> = 4.49	1	2	3	4	1				2				3				4													
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4.20	(12) — 2.1 (13) — 2.2 (14) — 2.2 (34) — 2.2 (123) — 3.11 (124) — 3.11 (134) — 3.16		*(4) *(34) * S	<table><tr><th>1</th><th>2</th><th>3</th></tr><tr><td>1</td><td></td><td></td></tr><tr><td>2</td><td></td><td></td></tr><tr><td>3</td><td></td><td></td></tr></table> = 3.16	1	2	3	1			2			3																				
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4.21	(12) — 2.2 (13) — 2.4 (14) — 2.1 (123) — 3.8 (124) — 3.5 (134) — 3.4		*(14) *(134) * S	<table><tr><th>1</th><th>2</th><th>3</th></tr><tr><td>1</td><td></td><td></td></tr><tr><td>2</td><td></td><td></td></tr><tr><td>3</td><td></td><td></td></tr></table> = 3.16	1	2	3	1			2			3																				
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4.22	(12) — 2.2 (13) — 2.4 (14) — 2.1 (123) — 3.6 (124) — 3.5 (134) — 3.4		*(14) *(134) *(124) * S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>1</td><td></td><td></td><td></td></tr><tr><td>2</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> = 4.9	1	2	3	4	1				2				3				4													
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4.23	(12) — 2.2 (13) — 2.2 (14) — 2.2 (24) — 2.2 (34) — 2.2 (123) — 3.17 (124) — 3.16 (134) — 3.16		*(4) *(14) *(124) *(134) S	<table><tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>1</td><td>2</td><td>2</td><td>2</td></tr><tr><td>3</td><td>1</td><td>2</td><td>3</td><td>2</td></tr><tr><td>4</td><td>1</td><td>2</td><td>2</td><td>4</td></tr><tr><td>5</td><td>1</td><td>2</td><td>3</td><td>4</td></tr></table>	1	2	3	4	5	1	1	1	1	1	2	1	2	2	2	3	1	2	3	2	4	1	2	2	4	5	1	2	3	4
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2	1	2	2	2																														
3	1	2	3	2																														
4	1	2	2	4																														
5	1	2	3	4																														

TABLE III

IDEAL ANALYSIS L_4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
4.24	(12) — 2.2	S		1 2 3
$\overline{1133}$	(13) — 2.1		*(13)	1
1234	(123) — 3.5	(134)	*(134)	2 = 3.8
3311	(134) — 3.1		* S	3
$\overline{3411}$		(13)		
4.25	(12) — 2.2	S		1 2 3
	(13) — 2.4		*(2)	1
$\overline{1214}$	(14) — 2.1	(124)	*(124)	2 = 3.10
$\overline{2222}$	(123) — 3.10		S	3
$\overline{1214}$	(124) — 3.11	(2)		
$\overline{4241}$	(134) — 3.4			
4.26	(12) — 2.2	S		1 2 3
	(14) — 2.1		*(2)	1
$\overline{1244}$	(124) — 3.11	(124)	*(124)	2 = 3.10
$\overline{2222}$	(134) — 3.1		S	3
$\overline{4211}$		(2)		
$\overline{4211}$				
4.27	(12) — 2.2	S		1 2 3 4
	(13) — 2.4		*(1)	1
$\overline{1111}$	(123) — 3.6	(124)	(13)	2 = 4.9
$\overline{1214}$	(124) — 3.11		*(124)	3
$\overline{1111}$		(13)	S	4
$\overline{1412}$		(1)		
4.28	(12) — 2.2	S		1 2 3
	(13) — 2.4		*(1)	1
$\overline{1111}$	(123) — 3.8	(13)	*(13)	2 = 3.8
$\overline{1234}$	(124) — 3.11		* S	3
$\overline{1313}$		(1)		
$\overline{1432}$				

TABLE III

IDEAL ANALYSIS $4x4$ SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.34 (12)—2.4 (13)—2.1 (123)—3.4 <u>1133</u> 1133 3311 3312	$ \begin{array}{c} S \\ \\ (123) \\ \\ (13) \end{array} $	$ \begin{array}{c} *(13) \\ (123) \\ S \end{array} $	$ \begin{array}{c ccc} & 1 & 2 & 3 \\ 1 & & & \\ 2 & & & \\ 3 & & & \end{array} = 3.2 $
4.35 (123)—3.3 <u>1233</u> 2311 3122 3122	$ \begin{array}{c} S \\ \\ (123) \end{array} $	$ \begin{array}{c} *(123) \\ S \end{array} $	$ \begin{array}{c cc} & 1 & 2 \\ 1 & & \\ 2 & & \end{array} = 2.4 $
4.36 (12)—2.2 (13)—2.4 (14)—2.4 (123)—3.8 (134)—3.18 <u>1111</u> 1233 1311 1311	$ \begin{array}{c} S \\ \swarrow \quad \searrow \\ (123) \quad (134) \\ \swarrow \quad \searrow \\ (13) \\ \\ (1) \end{array} $	$ \begin{array}{c} *(1) \\ *(13) \\ *(123) \\ (134) \\ S \end{array} $	$ \begin{array}{c ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 2 \\ 3 & 1 & 2 & 3 & 2 & 3 \\ 4 & 1 & 1 & 2 & 1 & 2 \\ 5 & 1 & 2 & 3 & 2 & 3 \end{array} $
4.37 (12)—2.2 (13)—2.4 (14)—2.4 (123)—3.8 (134)—3.18 <u>1111</u> 1234 1311 1411	$ \begin{array}{c} S \\ \\ (134) \\ \swarrow \quad \searrow \\ (13) \quad (14) \\ \swarrow \quad \searrow \\ (1) \end{array} $	$ \begin{array}{c} *(1) \\ *(13) \\ *(14) \\ *(134) \\ *S \end{array} $	$ \begin{array}{c ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 2 \\ 3 & 1 & 1 & 1 & 1 & 3 \\ 4 & 1 & 1 & 1 & 1 & 4 \\ 5 & 1 & 2 & 3 & 4 & 5 \end{array} $
4.38 (12)—2.2 (13)—2.4 (24)—2.4 (123)—3.8 (124)—3.10 <u>1111</u> 1232 1313 1212	$ \begin{array}{c} S \\ \\ (123) \\ \\ (13) \\ \\ (1) \end{array} $	$ \begin{array}{c} *(1) \\ *(13) \\ *(123) \\ S \end{array} $	$ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} = 4.38 $

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
4.39	(12) — 2.2 (13) — 2.4 (123) — 3.8 (134) — 3.2	$ \begin{array}{c} S \\ \\ (134) \\ \\ (13) \\ \\ (1) \end{array} $	$ \begin{array}{l} *(1) \\ *(13) \\ *(134) \\ *S \end{array} $	$ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \equiv 4.39 $
4.40	(12) — 2.2 (23) — 2.4 (24) — 2.4 (123) — 3.10 (124) — 3.10 (234) — 3.18	$ \begin{array}{ccc} & S & \\ (123) & \swarrow \quad \searrow & (124) \\ & (12) & \\ & & \\ & (1) & \end{array} $	$ \begin{array}{l} *(1) \\ *(12) \\ (123) \\ (124) \\ S \end{array} $	$ \begin{array}{c ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 2 \\ 3 & 1 & 2 & 2 & 2 & 2 \\ 4 & 1 & 2 & 2 & 2 & 2 \\ 5 & 1 & 2 & 2 & 2 & 2 \end{array} $
4.41	(12) — 2.2 (23) — 2.4 (123) — 3.10 (234) — 3.2	$ \begin{array}{c} S \\ \\ (123) \\ \\ (12) \\ \\ (1) \end{array} $	$ \begin{array}{l} *(1) \\ *(12) \\ (123) \\ S \end{array} $	$ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \equiv 4.41 $
4.42	(12) — 2.2 (13) — 2.2 (14) — 2.2 (23) — 2.2 (24) — 2.2 (34) — 2.2 (123) — 3.16 (124) — 3.16 (134) — 3.16 (234) — 3.16	$ \begin{array}{c} S \\ \\ (123) \\ \\ (12) \\ \\ (1) \end{array} $	$ \begin{array}{l} *(1) \\ *(12) \\ *(123) \\ *S \end{array} $	$ \begin{array}{c cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \equiv 4.42 $

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table																																																	
4.43 <u>1111</u> 1221 1231 1114 (12)—2.2 (13)—2.2 (14)—2.2 (23)—2.2 (123)—3.16 (124)—3.17 (134)—3.17		$*(1)$ $*(12)$ $*(14)$ $*(123)$ $*(124)$ $*S$	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr><tr><th>1</th><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><th>2</th><td>1</td><td>2</td><td>1</td><td>2</td><td>2</td><td>2</td></tr><tr><th>3</th><td>1</td><td>1</td><td>3</td><td>1</td><td>3</td><td>3</td></tr><tr><th>4</th><td>1</td><td>2</td><td>1</td><td>4</td><td>2</td><td>4</td></tr><tr><th>5</th><td>1</td><td>2</td><td>3</td><td>2</td><td>5</td><td>5</td></tr><tr><th>6</th><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr></table>		1	2	3	4	5	6	1	1	1	1	1	1	1	2	1	2	1	2	2	2	3	1	1	3	1	3	3	4	1	2	1	4	2	4	5	1	2	3	2	5	5	6	1	2	3	4	5	6
	1	2	3	4	5	6																																														
1	1	1	1	1	1	1																																														
2	1	2	1	2	2	2																																														
3	1	1	3	1	3	3																																														
4	1	2	1	4	2	4																																														
5	1	2	3	2	5	5																																														
6	1	2	3	4	5	6																																														
4.44 <u>1111</u> 1221 1234 1141 (12)—2.2 (13)—2.2 (23)—2.2 (123)—3.16 (124)—3.6 (134)—3.8		$*(1)$ $*(12)$ $*(14)$ $*(124)$ $*S$	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><th>1</th><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><th>2</th><td>1</td><td>2</td><td>1</td><td>2</td><td>2</td></tr><tr><th>3</th><td>1</td><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><th>4</th><td>1</td><td>2</td><td>1</td><td>2</td><td>4</td></tr><tr><th>5</th><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr></table>		1	2	3	4	5	1	1	1	1	1	1	2	1	2	1	2	2	3	1	1	1	1	3	4	1	2	1	2	4	5	1	2	3	4	5													
	1	2	3	4	5																																															
1	1	1	1	1	1																																															
2	1	2	1	2	2																																															
3	1	1	1	1	3																																															
4	1	2	1	2	4																																															
5	1	2	3	4	5																																															
4.45 <u>1111</u> 1224 1234 1141 (12)—2.2 (13)—2.2 (23)—2.2 (123)—3.16 (124)—3.8 (134)—3.8		$*(1)$ $*(14)$ $*(124)$ $*S$	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><th>1</th><td>1</td><td></td><td></td><td></td></tr><tr><th>2</th><td></td><td>$\cong 4.45$</td><td></td><td></td></tr><tr><th>3</th><td></td><td></td><td></td><td></td></tr><tr><th>4</th><td></td><td></td><td></td><td></td></tr></table>		1	2	3	4	1	1				2		$\cong 4.45$			3					4																												
	1	2	3	4																																																
1	1																																																			
2		$\cong 4.45$																																																		
3																																																				
4																																																				
4.46 <u>1111</u> 1222 1232 1222 (12)—2.2 (13)—2.2 (23)—2.2 (123)—3.16 (124)—3.10 (234)—3.6		$*(1)$ $*(12)$ $*(123)$ (124) S	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><th>1</th><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><th>2</th><td>1</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><th>3</th><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td></tr><tr><th>4</th><td>1</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><th>5</th><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td></tr></table>		1	2	3	4	5	1	1	1	1	1	1	2	1	2	2	2	2	3	1	2	3	2	3	4	1	2	2	2	2	5	1	2	3	2	3													
	1	2	3	4	5																																															
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2	1	2	2	2	2																																															
3	1	2	3	2	3																																															
4	1	2	2	2	2																																															
5	1	2	3	2	3																																															

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table	
4.47	(12) $\rightarrow$ 2.2 (13) $\rightarrow$ 2.2		$*(1)$ $*(12)$ $*(124)$ $*S$	1 2 3 4 1 2 3 4	1 2 3 4 =4.47
<u>1111</u>	(23) $\rightarrow$ 2.2				
1222	(24) $\rightarrow$ 2.4				
1234	(123) $\rightarrow$ 3.16				
1242	(124) $\rightarrow$ 3.10 (234) $\rightarrow$ 3.8				
4.48	(12) $\rightarrow$ 2.2 (13) $\rightarrow$ 2.2 (14) $\rightarrow$ 2.2		$*(1)$ $*(12)$ $*(13)$ $*(14)$ $*(123)$ $*(124)$ $*(134)$ $*S$	1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 2 1 1 3 1 3 1 3 3 1 1 1 4 1 4 4 4 1 2 3 1 5 2 3 5 1 2 1 4 2 6 4 6 1 1 3 4 3 4 7 7 1 2 3 4 5 6 7 8
<u>1111</u>	(14) $\rightarrow$ 2.2				
1211	(123) $\rightarrow$ 3.17				
1131	(124) $\rightarrow$ 3.17				
1114	(134) $\rightarrow$ 3.17				
4.49	(12) $\rightarrow$ 2.2 (13) $\rightarrow$ 2.2 (14) $\rightarrow$ 2.2		$*(1)$ $*(12)$ $*(13)$ $*(123)$ $*S$	1 2 3 4 5 1 2 3 4 5	1 2 3 4 5 1 1 1 1 1 1 2 1 2 2 1 1 3 3 3 1 2 3 4 4 1 2 3 4 5
<u>1111</u>	(14) $\rightarrow$ 2.2				
1212	(34) $\rightarrow$ 2.2				
1133	(123) $\rightarrow$ 3.17				
1234	(124) $\rightarrow$ 3.16 (134) $\rightarrow$ 3.16				
4.50	(12) $\rightarrow$ 2.2 (13) $\rightarrow$ 2.2 (14) $\rightarrow$ 2.4		$*(1)$ $*(12)$ $*(13)$ $*(14)$ $*(123)$ (124) (134) S	1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 2 1 1 3 1 3 1 3 3 1 1 1 1 1 1 1 1 1 2 3 1 5 2 3 5 1 2 1 1 2 2 1 2 1 1 3 1 3 1 3 3 1 2 3 1 5 2 3 5
<u>1111</u>	(14) $\rightarrow$ 2.4				
1211	(123) $\rightarrow$ 3.17				
1131	(124) $\rightarrow$ 3.6				
1111	(134) $\rightarrow$ 3.6				

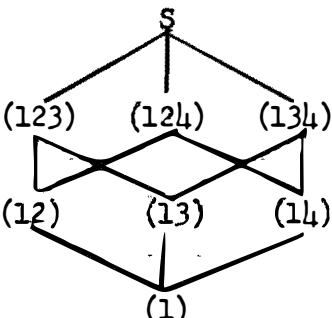
TABLE III

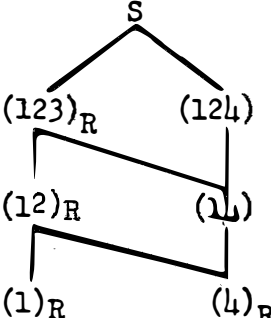
IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table	
4.51	(12)—2.2 (13)—2.2 <u>1111</u> (14)—2.4 1211 (123)—3.17 1134 (124)—3.6 1141 (134)—3.8		*(1) *(12) *(14) *(124) *(134) *S	1	2 3 4 5 6
				1	1 1 1 1 1 1
				2	1 2 1 2 1 2
				3	1 1 1 1 3 3
				4	1 2 1 2 3 4
				5	1 1 3 3 5 5
				6	1 2 3 4 5 6
4.52	(12)—2.2 (13)—2.2 <u>1111</u> (123)—3.17 1212 (124)—3.10 1131 1212		*(1) *(12) *(13) *(123) (124) S	1	2 3 4 5 6
				1	1 1 1 1 1 1
				2	1 2 1 2 2 2
				3	1 1 3 3 1 3
				4	1 2 3 4 2 4
				5	1 2 1 2 2 2
				6	1 2 3 4 2 4
4.53	(12)—2.4 (13)—2.4 <u>1111</u> (14)—2.4 1111 (123)—3.18 1112 (124)—3.18 1121		*(1) (12) (123) (124) S	1	2 3 4 5
				1	1 1 1 1 1
				2	1 1 1 1 1
				3	1 1 1 2 2
				4	1 1 2 1 2
				5	1 1 2 2 2
4.54	(12)—2.4 (13)—2.4 <u>1111</u> (123)—3.18 1111 (124)—3.2 1111 1112		*(1) (12) (13) (123) (124) S	1	2 3 4 5 6
				1	1 1 1 1 1 1
				2	1 1 1 1 1 1
				3	1 1 1 1 1 1
				4	1 1 1 1 1 1
				5	1 1 1 1 2 2
				6	1 1 1 1 2 2

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table									
4.55	(12)--2.4 (13)--2.4 (14)--2.4 (123)--3.18 (124)--3.18 (134)--3.18		*{1} {12} {13} {14} {123} {124} {134} S	1 2 3 4 5 6 7 8	1	2	3	4	5	6	7	8	
					1	1	1	1	1	1	1	1	1
					2	1	1	1	1	1	1	1	1
					3	1	1	1	1	1	1	1	1
					4	1	1	1	1	1	1	1	1
					5	1	1	1	1	1	1	1	1
					6	1	1	1	1	1	1	1	1
					7	1	1	1	1	1	1	1	1
					8	1	1	1	1	1	1	1	1

4.56	(12)--2.4 (14)--2.3 (123)--3.2 (124)--3.9		*{14}, {124}, S = 3.2 Right *{1} *{4} *{14} {12} {123} {124} S	1 2 3 4 5 6 7	1	2	3	4	5	6	7	
					1	1	2	3	1	1	3	3
					2	1	2	3	1	1	3	3
					3	1	2	3	1	1	3	3
					4	1	2	3	1	1	3	3
					5	1	2	3	1	4	3	3
					6	1	2	3	1	1	3	3
					7	1	2	3	1	4	3	6

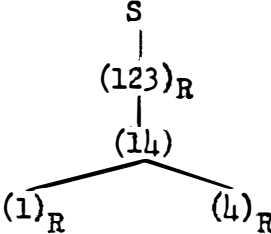
4.57	(12)--2.2 (14)--2.3 (23)--2.1 (123)--3.11 (124)--3.12		*{14}, S = 2.2 Right *{1} *{4} *{14} *{123} *S	1 2 3 4 5	1	2	3	4	5	
					1	1	2	3	1	3
					2	1	2	3	1	3
					3	1	2	3	1	3
					4	1	2	3	4	5
					5	1	2	3	4	5

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table																																																																																																				
4.58 1134 1134 3334 3334		$*(34), *(134), S = 3.10$ Right	<table> <tr> <th></th> <th>1</th> <th>2</th> <th>3</th> <th>4</th> <th>5</th> <th>6</th> <th>7</th> </tr> <tr> <td>$*(3)$</td> <td>1</td> <td>1</td> <td>2</td> <td>3</td> <td>1</td> <td>3</td> <td>1</td> </tr> <tr> <td>$*(4)$</td> <td>2</td> <td>1</td> <td>2</td> <td>3</td> <td>1</td> <td>3</td> <td>1</td> </tr> <tr> <td>$*(34)$</td> <td>3</td> <td>1</td> <td>2</td> <td>3</td> <td>1</td> <td>3</td> <td>1</td> </tr> <tr> <td>$*(13)$</td> <td>4</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>4</td> </tr> <tr> <td>$*(134)$</td> <td>5</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>4</td> </tr> <tr> <td>(123)</td> <td>6</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>4</td> </tr> <tr> <td>S</td> <td>7</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>4</td> </tr> </table>		1	2	3	4	5	6	7	$*(3)$	1	1	2	3	1	3	1	$*(4)$	2	1	2	3	1	3	1	$*(34)$	3	1	2	3	1	3	1	$*(13)$	4	1	2	3	4	5	4	$*(134)$	5	1	2	3	4	5	4	(123)	6	1	2	3	4	5	4	S	7	1	2	3	4	5	4																																				
	1	2	3	4	5	6	7																																																																																																
$*(3)$	1	1	2	3	1	3	1																																																																																																
$*(4)$	2	1	2	3	1	3	1																																																																																																
$*(34)$	3	1	2	3	1	3	1																																																																																																
$*(13)$	4	1	2	3	4	5	4																																																																																																
$*(134)$	5	1	2	3	4	5	4																																																																																																
(123)	6	1	2	3	4	5	4																																																																																																
S	7	1	2	3	4	5	4																																																																																																
4.59 1111 2222 1112 4444		$*(124), S = 2.4$ Left	<table> <tr> <th></th> <th>1</th> <th>2</th> <th>3</th> <th>4</th> <th>5</th> <th>6</th> <th>7</th> <th>8</th> <th>9</th> </tr> <tr> <td>$*(1)$</td> <td>1</td> <td>1</td> <td>1</td> <td>1</td> <td>1</td> <td>1</td> <td>1</td> <td>1</td> <td>1</td> </tr> <tr> <td>$*(2)$</td> <td>2</td> <td>2</td> <td>2</td> <td>2</td> <td>2</td> <td>2</td> <td>2</td> <td>2</td> <td>2</td> </tr> <tr> <td>$*(4)$</td> <td>3</td> <td>3</td> <td>3</td> <td>3</td> <td>3</td> <td>3</td> <td>3</td> <td>3</td> <td>3</td> </tr> <tr> <td>$*(12)$</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> </tr> <tr> <td>$*(14)$</td> <td>5</td> <td>5</td> <td>5</td> <td>5</td> <td>5</td> <td>5</td> <td>5</td> <td>5</td> <td>5</td> </tr> <tr> <td>$*(24)$</td> <td>6</td> <td>6</td> <td>6</td> <td>6</td> <td>6</td> <td>6</td> <td>6</td> <td>6</td> <td>6</td> </tr> <tr> <td>$*(124)$</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> </tr> <tr> <td>(123)</td> <td>8</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> <td>4</td> </tr> <tr> <td>S</td> <td>9</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> <td>7</td> </tr> </table>		1	2	3	4	5	6	7	8	9	$*(1)$	1	1	1	1	1	1	1	1	1	$*(2)$	2	2	2	2	2	2	2	2	2	$*(4)$	3	3	3	3	3	3	3	3	3	$*(12)$	4	4	4	4	4	4	4	4	4	$*(14)$	5	5	5	5	5	5	5	5	5	$*(24)$	6	6	6	6	6	6	6	6	6	$*(124)$	7	7	7	7	7	7	7	7	7	(123)	8	4	4	4	4	4	4	4	4	S	9	7	7	7	7	7	7	7	7
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$*(2)$	2	2	2	2	2	2	2	2	2																																																																																														
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$*(12)$	4	4	4	4	4	4	4	4	4																																																																																														
$*(14)$	5	5	5	5	5	5	5	5	5																																																																																														
$*(24)$	6	6	6	6	6	6	6	6	6																																																																																														
$*(124)$	7	7	7	7	7	7	7	7	7																																																																																														
(123)	8	4	4	4	4	4	4	4	4																																																																																														
S	9	7	7	7	7	7	7	7	7																																																																																														
4.60 1234 2134 3334 4434		$*(34), *S = 2.3$ Right	<table> <tr> <th></th> <th>1</th> <th>2</th> <th>3</th> <th>4</th> </tr> <tr> <td>$*(3)$</td> <td>1</td> <td></td> <td></td> <td></td> </tr> <tr> <td>$*(4)$</td> <td>2</td> <td></td> <td></td> <td></td> </tr> <tr> <td>$*(34)$</td> <td>3</td> <td></td> <td></td> <td></td> </tr> <tr> <td>*S</td> <td>4</td> <td></td> <td></td> <td></td> </tr> </table> $= 4.70$		1	2	3	4	$*(3)$	1				$*(4)$	2				$*(34)$	3				*S	4																																																																														
	1	2	3	4																																																																																																			
$*(3)$	1																																																																																																						
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$*(34)$	3																																																																																																						
*S	4																																																																																																						

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
4.61	(12)→2.4 (13)→2.2 <u>1134</u> 1134 (14)→2.2 3334 (34)→2.3 1434 (134)→3.10 (123)→3.10 (124)→3.13		*(34), *(134), S = 3.9 Right *(3) *(4) *(34) *(134) S	1 2 3 4 5 1 1 2 3 3 3 2 1 2 3 3 3 3 1 2 3 3 3 4 1 2 3 4 4 5 1 2 3 4 4
4.62	(12)→2.3 (13)→2.3 <u>1111</u> 2222 (14)→2.2 3333 (24)→2.2 1224 (123)→3.14 (124)→3.13 (234)→3.12		*(123), *S = 2.2 Left *(1) *(2) *(3) *(12) *(13) *(23) *(123) *(124) *S	1 2 3 4 5 6 7 8 9 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 6 6 6 6 6 6 6 6 6 7 7 7 7 7 7 7 7 7 8 4 4 4 4 4 4 8 8 9 7 7 7 7 7 7 9 9
4.63	(12)→2.3 (13)→2.2 <u>1111</u> 2222 (23)→2.2 1234 (34)→2.1 2143 (123)→3.13		*(12), *S = 2.2 Left *(1) *(2) *(12) *S	1 2 3 4 1 2 =4.70 3 4
4.64	(12)→2.3 (34)→2.3 <u>1133</u> 2244 (13)→2.3 (24)→2.3 1133 2244		Left *(13) *(24) *S Right *(12) *(34) S	1 2 3 1 2 =3.14 3 1 2 3 1 2 =3.14 3

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.65 1234 2143 1234 2143	S (12)_R (34)_R	Right *(12) *(34) * S	1 2 3 1 2 3 =3.14
4.66 1111 2222 1133 2244	S (123)_L (124)_L (13)_L (24)_L (12)	*(12), *S = 2.2 Left *(1) *(2) *(12) *(13) *(24) *(123) *(124) * S	1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 4 1 1 1 4 4 4 4 5 2 2 2 5 5 5 5 6 3 3 3 6 6 6 6 7 3 3 3 7 7 7 7 8 3 3 3 8 8 8 8
4.67 1133 2233 3333 4444	S (123)_L (134)_L (234)_L (13)_L (23)_L (34)_L (3)_L (4)_L	*(34), *S = 2.3 Left *(3) *(4) *(34) *(23) *(13) *(234) *(134) *(123) * S	1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 3 4 1 1 1 4 4 4 4 4 5 1 1 1 5 5 5 5 5 6 3 3 3 6 6 6 6 6 7 3 3 3 7 7 7 7 7 8 1 1 1 8 8 8 8 8 9 3 3 3 9 9 9 9 9

TABLE III

IDEAL ANALYSIS $4x4$ SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.68 (12)→2.3 (13)→2.2 (23)→2.2 (34)→2.4 (123)→3.15 (134)→3.7 (234)→3.7	S (123)_L (134)_L (234)_L (13)_L (23)_L (34) (3)	*(3), *(34), *S = 3.6 Left *(3) *(34) *(13) *(23) *(134) *(234) *(123) *S	1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 3 3 3 3 3 3 1 1 4 4 4 4 4 4 1 1 5 5 5 5 5 5 1 1 6 6 6 6 6 6 1 1 7 7 7 7 7 7 1 1 8 8 8 8 8 8
4.69 (12)→2.4 (13)→2.2 (14)→2.2 (34)→2.3 (123)→3.8 (124)→3.8 (134)→3.15	S (123)_R (124)_R (12) (1)	*(1), *(12), *S = 3.8 Right *(1) *(12) *(123) *(124) *S	1 2 3 4 5 1 2 3 4 5 1 1 1 1 1 1 1 2 2 2 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5
4.70 (12)→2.3 (13)→2.3 (23)→2.3 (34)→2.2 (123)→3.14 (134)→3.12 (234)→3.12	S (123) (134)_L (234)_L (12)_L (13)_L (23)_L (34)_L (1)_L (2)_L (3)_L	*(123), *S = 2.2 Left *(1) *(2) *(3) *(12) *(13) *(23) *(34) *(123) *(134) *(234) *S	1 2 3 4 5 6 7 8 9 10 11 1 2 3 4 5 6 7 8 9 10 11 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 3 3 3 4 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5 5 6 6 6 6 6 6 6 6 6 6 6 7 3 3 3 3 3 3 7 3 7 7 8 8 8 8 8 8 8 8 8 8 8 9 5 5 5 5 5 5 9 5 9 9 10 6 6 6 6 6 6 10 6 10 10 11 8 8 8 8 8 8 11 8 11 11

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table																																																																																																																																				
4.71 (13)--2.4 (12)--2.3 <u>1111</u> (14)--2.3 2222 (24)--2.3 1111 (123)--3.9 4444 (134)--3.9 (124)--3.14		$*(124), S = 2.4$ Left *(1) 1 *(2) 2 *(4) 3 *(12) 4 *(14) 5 *(24) 6 *(124) 7 (13) 8 (134) 9 (123) 10 S 11	<table> <tr> <th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th></tr> <tr> <td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr> <tr> <td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr> <tr> <td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr> <tr> <td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td></tr> <tr> <td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr> <tr> <td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td><td>6</td></tr> <tr> <td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td></tr> <tr> <td>8</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr> <tr> <td>9</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr> <tr> <td>10</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td></tr> <tr> <td>11</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td><td>7</td></tr> </table>	1	2	3	4	5	6	7	8	9	10	11	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	7	8	1	1	1	1	1	1	1	1	1	1	9	5	5	5	5	5	5	5	5	5	5	10	4	4	4	4	4	4	4	4	4	4	11	7	7	7	7	7	7	7	7	7	7
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2	2	2	2	2	2	2	2	2	2	2																																																																																																																													
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10	4	4	4	4	4	4	4	4	4	4																																																																																																																													
11	7	7	7	7	7	7	7	7	7	7																																																																																																																													

4.72 (12)--2.3 (34)--2.3 <u>1134</u> (23)--2.2 2234 (123)--3.15 3334 (134)--3.12 3334 (234)--3.12		$*(34), *S = 2.2$ Right *(3) 1 *(4) 2 *(34) 3 *(123) 4 *S 5	<table> <tr> <th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr> <tr> <td>1</td><td>1</td><td>2</td><td>3</td><td>1</td></tr> <tr> <td>2</td><td>1</td><td>2</td><td>3</td><td>1</td></tr> <tr> <td>3</td><td>1</td><td>2</td><td>3</td><td>1</td></tr> <tr> <td>4</td><td>1</td><td>2</td><td>3</td><td>4</td></tr> <tr> <td>5</td><td>1</td><td>2</td><td>3</td><td>4</td></tr> </table>	1	2	3	4	5	1	1	2	3	1	2	1	2	3	1	3	1	2	3	1	4	1	2	3	4	5	1	2	3	4
1	2	3	4	5																													
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2	1	2	3	1																													
3	1	2	3	1																													
4	1	2	3	4																													
5	1	2	3	4																													
		Left *(34) 1 *(234) 2 *(134) 3 *S 4	<table> <tr> <th>1</th><th>2</th><th>3</th><th>4</th></tr> <tr> <td>1</td><td colspan="3" rowspan="4">=4.102</td></tr> <tr> <td>2</td></tr> <tr> <td>3</td></tr> <tr> <td>4</td></tr> </table>	1	2	3	4	1	=4.102			2	3	4																			
1	2	3	4																														
1	=4.102																																
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TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

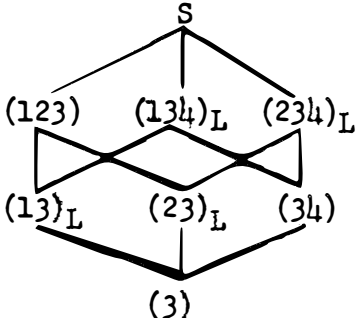
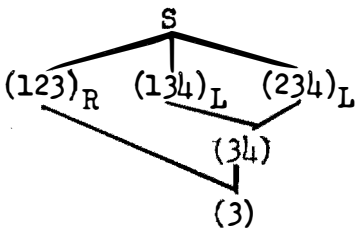
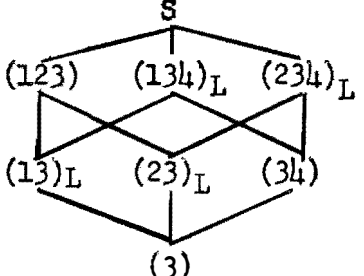
Subsemigroups		Semilattice	Ideals	Ideal Table
4.73	(12) -- 2.3 (13) -- 2.2 (23) -- 2.2 (34) -- 2.2 (123) -- 3.15 (134) -- 3.17 (234) -- 3.17		* (3), * (34), * (123), * S = 4.49	
<u>1133</u> 2233 3333 3334			Left	1 2 3 4 5 6 7 8
			* (3)	1 1 1 1 1 1 1 1
			* (34)	2 1 2 1 1 1 2 2
			* (23)	3 1 1 3 3 3 3 3
			* (13)	4 1 1 4 4 4 4 4
			* (123)	5 1 1 5 5 5 5 5
			* (134)	6 1 2 4 4 4 6 6
			* (234)	7 1 2 3 3 3 7 7
			* S	8 1 2 5 5 5 8 8
4.74	(12) -- 2.3 (13) -- 2.2 (23) -- 2.2 (34) -- 2.4 (123) -- 3.15 (134) -- 3.7 (234) -- 3.7		* (3), * (34), * S = 3.7	
<u>1134</u> 2234 3333 3333			Right	1 2 3 4
			* (3)	1
			* (34)	2
			* (123)	3
			* S	4
			Left	1 2 3 4 5
			* (3)	1 1 1 1 1
			* (34)	2 1 1 1 1
			* (134)	3 1 2 3 3
			* (234)	4 1 2 4 4
			* S	5 1 2 5 5
4.75	(12) -- 2.3 (13) -- 2.2 (23) -- 2.2 (34) -- 2.4 (123) -- 3.15 (134) -- 3.6 (234) -- 3.6		* (3), * (123), (34), S = 4.9	
<u>1133</u> 2233 3333 3333			Left	1 2 3 4 5 6 7 8
			* (3)	1 1 1 1 1 1 1 1
			* (23)	2 1 2 2 2 1 2 2
			* (13)	3 1 3 3 3 1 3 3
			* (123)	4 1 4 4 4 1 4 4
			(34)	5 1 1 1 1 1 1 1
			(134)	6 1 3 3 3 1 3 3
			(234)	7 1 2 2 2 1 2 2
			S	8 1 4 4 4 1 4 4

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table	
4.76	(12)—2.4 (13)—2.4 (14)—2.2 (123)—3.18 (134)—3.8		$*(1), *(13), *(123), *S=4.76$		
$\overline{1111}$			Right	1 2 3 4 5 6	
$\overline{1113}$			$*(1)$	1	1 1 1 1 1 1
$\overline{1113}$			$*(12)$	2	1 1 1 1 3 3
$\overline{1113}$			$*(13)$	3	1 1 1 1 3 3
$\overline{1234}$			$*(123)$	4	1 1 1 1 3 3
			$*(134)$	5	1 2 3 4 5 6
			$*S$	6	1 2 3 4 5 6
4.77	(12)—2.3 (13)—2.4 (14)—2.4 (123)—3.9 (124)—3.9 (134)—3.18		$*(12), (123), (124), S=4.55$		
$\overline{1111}$			Left	1 2 3 4 5 6 7 8 9	
$\overline{2222}$			$*(1)$	1	1 1 1 1 1 1 1 1 1
$\overline{1111}$			$*(2)$	2	2 2 2 2 2 2 2 2 2
$\overline{1111}$			$*(12)$	3	3 3 3 3 3 3 3 3 3
$\overline{1111}$			(13)	4	1 1 1 1 1 1 1 1 1
			(14)	5	1 1 1 1 1 1 1 1 1
			(123)	6	3 3 3 3 3 3 3 3 3
			(124)	7	3 3 3 3 3 3 3 3 3
			(134)	8	1 1 1 1 1 1 1 1 1
			S	9	3 3 3 3 3 3 3 3 3
4.78	(12)—2.3 (13)—2.4 (24)—2.4 (123)—3.9 (124)—3.9		$*(12), (123), (124), S=4.55$		
$\overline{1111}$			Left	1 2 3 4 5 6 7 8	
$\overline{2222}$			$*(1)$	1	1 1 1 1 1 1 1 1
$\overline{1111}$			$*(2)$	2	2 2 2 2 2 2 2 2
$\overline{2222}$			$*(12)$	3	3 3 3 3 3 3 3 3
			(13)	4	1 1 1 1 1 1 1 1
			(24)	5	2 2 2 2 2 2 2 2
			(123)	6	3 3 3 3 3 3 3 3
			(124)	7	3 3 3 3 3 3 3 3
			S	8	3 3 3 3 3 3 3 3

TABLE III

IDEAL ANALYSIS $4x4$ SEMIGROUPS (Continued)

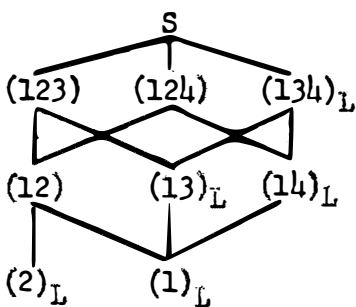
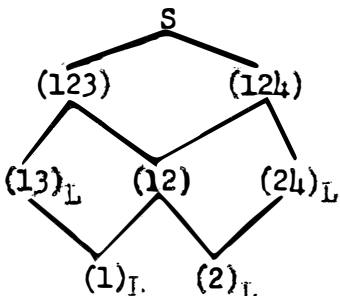
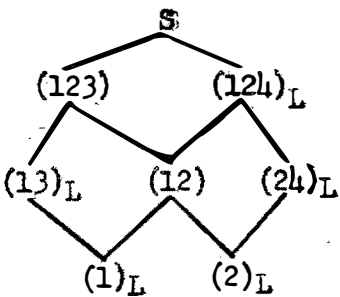
Subsemigroups	Semilattice	Ideals	Ideal Table																																																																																																				
4.79 $\overline{1111}$ $\overline{2222}$ $\overline{1111}$ $\overline{1114}$ (12)—2.3 (13)—2.4 (14)—2.2 (123)—3.9 (124)—3.12 (134)—3.6		*(12), *(124), (123), S=4.9 Left *(1) 1 *(2) 2 *(12) 3 *(14) 4 *(124) 5 (13) 6 (134) 7 (123) 8 S 9	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th></tr><tr><td>*(1)</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>*(2)</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>*(12)</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>*(14)</td><td>4</td><td>1</td><td>1</td><td>4</td><td>4</td><td>1</td><td>4</td><td>1</td><td>4</td></tr><tr><td>*(124)</td><td>5</td><td>3</td><td>3</td><td>5</td><td>5</td><td>3</td><td>5</td><td>3</td><td>5</td></tr><tr><td>(13)</td><td>6</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>(134)</td><td>7</td><td>1</td><td>1</td><td>4</td><td>4</td><td>1</td><td>4</td><td>1</td><td>4</td></tr><tr><td>(123)</td><td>8</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>S</td><td>9</td><td>3</td><td>3</td><td>3</td><td>5</td><td>5</td><td>3</td><td>5</td><td>3</td></tr></table>		1	2	3	4	5	6	7	8	9	*(1)	1	1	1	1	1	1	1	1	1	*(2)	2	2	2	2	2	2	2	2	2	*(12)	3	3	3	3	3	3	3	3	3	*(14)	4	1	1	4	4	1	4	1	4	*(124)	5	3	3	5	5	3	5	3	5	(13)	6	1	1	1	1	1	1	1	1	(134)	7	1	1	4	4	1	4	1	4	(123)	8	3	3	3	3	3	3	3	3	S	9	3	3	3	5	5	3	5	3
	1	2	3	4	5	6	7	8	9																																																																																														
*(1)	1	1	1	1	1	1	1	1	1																																																																																														
*(2)	2	2	2	2	2	2	2	2	2																																																																																														
*(12)	3	3	3	3	3	3	3	3	3																																																																																														
*(14)	4	1	1	4	4	1	4	1	4																																																																																														
*(124)	5	3	3	5	5	3	5	3	5																																																																																														
(13)	6	1	1	1	1	1	1	1	1																																																																																														
(134)	7	1	1	4	4	1	4	1	4																																																																																														
(123)	8	3	3	3	3	3	3	3	3																																																																																														
S	9	3	3	3	5	5	3	5	3																																																																																														
4.80 $\overline{1111}$ $\overline{2222}$ $\overline{1111}$ $\overline{2224}$ (12)—2.3 (13)—2.4 (24)—2.2 (123)—3.9 (124)—3.12		*(12), *(124), (123), S=4.9 Left *(1) 1 *(2) 2 *(12) 3 *(24) 4 *(124) 5 (13) 6 (123) 7 S 8	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th></tr><tr><td>*(1)</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>*(2)</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>*(12)</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>*(24)</td><td>4</td><td>2</td><td>2</td><td>4</td><td>4</td><td>2</td><td>2</td><td>4</td></tr><tr><td>*(124)</td><td>5</td><td>3</td><td>3</td><td>3</td><td>5</td><td>5</td><td>3</td><td>5</td></tr><tr><td>(13)</td><td>6</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>(123)</td><td>7</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>S</td><td>8</td><td>3</td><td>3</td><td>3</td><td>5</td><td>5</td><td>3</td><td>5</td></tr></table>		1	2	3	4	5	6	7	8	*(1)	1	1	1	1	1	1	1	1	*(2)	2	2	2	2	2	2	2	2	*(12)	3	3	3	3	3	3	3	3	*(24)	4	2	2	4	4	2	2	4	*(124)	5	3	3	3	5	5	3	5	(13)	6	1	1	1	1	1	1	1	(123)	7	3	3	3	3	3	3	3	S	8	3	3	3	5	5	3	5																			
	1	2	3	4	5	6	7	8																																																																																															
*(1)	1	1	1	1	1	1	1	1																																																																																															
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*(12)	3	3	3	3	3	3	3	3																																																																																															
*(24)	4	2	2	4	4	2	2	4																																																																																															
*(124)	5	3	3	3	5	5	3	5																																																																																															
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S	8	3	3	3	5	5	3	5																																																																																															
4.81 $\overline{1111}$ $\overline{2222}$ $\overline{1113}$ $\overline{2224}$ (12)—2.3 (13)—2.4 (24)—2.2 (123)—3.9 (124)—3.12		*(12), *(123), *S = 3.7 Left *(1) 1 *(2) 2 *(12) 3 *(24) 4 *(13) 5 *(123) 6 *(124) 7 *S 8	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th></tr><tr><td>*(1)</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>*(2)</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>*(12)</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>*(24)</td><td>4</td><td>2</td><td>2</td><td>4</td><td>2</td><td>2</td><td>4</td><td>4</td></tr><tr><td>*(13)</td><td>5</td><td>1</td><td>1</td><td>1</td><td>5</td><td>1</td><td>1</td><td>5</td></tr><tr><td>*(123)</td><td>6</td><td>3</td><td>3</td><td>3</td><td>6</td><td>3</td><td>3</td><td>6</td></tr><tr><td>*(124)</td><td>7</td><td>3</td><td>3</td><td>3</td><td>7</td><td>3</td><td>3</td><td>7</td></tr><tr><td>*S</td><td>8</td><td>3</td><td>3</td><td>3</td><td>8</td><td>3</td><td>3</td><td>8</td></tr></table>		1	2	3	4	5	6	7	8	*(1)	1	1	1	1	1	1	1	1	*(2)	2	2	2	2	2	2	2	2	*(12)	3	3	3	3	3	3	3	3	*(24)	4	2	2	4	2	2	4	4	*(13)	5	1	1	1	5	1	1	5	*(123)	6	3	3	3	6	3	3	6	*(124)	7	3	3	3	7	3	3	7	*S	8	3	3	3	8	3	3	8																			
	1	2	3	4	5	6	7	8																																																																																															
*(1)	1	1	1	1	1	1	1	1																																																																																															
*(2)	2	2	2	2	2	2	2	2																																																																																															
*(12)	3	3	3	3	3	3	3	3																																																																																															
*(24)	4	2	2	4	2	2	4	4																																																																																															
*(13)	5	1	1	1	5	1	1	5																																																																																															
*(123)	6	3	3	3	6	3	3	6																																																																																															
*(124)	7	3	3	3	7	3	3	7																																																																																															
*S	8	3	3	3	8	3	3	8																																																																																															

TABLE III.

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
4.82	(12) -- 2.3 (13) -- 2.2 (23) -- 2.2 (14) -- 2.4 (123) -- 3.13 (124) -- 3.9 (134) -- 3.2		$*(12), *(123), (124), S=4.9$	
<u>1111</u>			Left	1 2 3 4 5 6 7
2222			$*(1)$	1 1 1 1 1 1 1
1231			$*(2)$	2 2 2 2 2 2 2
<u>1111</u>			$*(12)$	3 3 3 3 3 3 3
			$*(123)$	4 3 3 3 4 3 3 4
			(14)	5 1 1 1 1 1 1 1
			(124)	6 3 3 3 3 3 3 3
			S	7 3 3 3 4 3 3 4
4.83	(12) -- 2.3 (13) -- 2.4 (14) -- 2.2 (24) -- 2.2 (123) -- 3.9 (124) -- 3.13 (134) -- 3.7		$*(12), *(123), *S = 3.7$	
<u>1111</u>			Left	1 2 3 4 5 6 7
2222			$*(1)$	1 1 1 1 1 1 1
1113			$*(2)$	2 2 2 2 2 2 2
<u>1214</u>			$*(12)$	3 3 3 3 3 3 3
			$*(13)$	4 1 1 1 1 1 4 4
			$*(123)$	5 3 3 3 3 3 5 5
			$*(124)$	6 3 3 3 3 3 6 6
			$*S$	7 3 3 3 3 3 7 7
4.84	(12) -- 2.2 (14) -- 2.3 (123) -- 3.7 (124) -- 3.12 (134) -- 3.9		$*(14), *(134), *S = 3.7$	
<u>1114</u>			Right	1 2 3 4 5 6 7
1214			$*(1)$	1 2 3 1 3 1 3
1314			$*(4)$	2 1 2 3 2 3 1 3
<u>1114</u>			$*(14)$	3 1 2 3 1 3 1 3
			$*(123)$	4 1 2 3 4 5 1 3
			$*S$	5 1 2 3 4 5 1 3
			(13)	6 1 2 3 6 7 1 3
			(134)	7 1 2 3 6 7 1 3

TABLE III

IDEAL ANALYSIS $L \times L$ SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table																																																																										
4.85	(12) -- 2.3 (13) -- 2.4 (14) -- 2.2 (123) -- 3.9 (124) -- 3.12 (134) -- 3.8		*(12), *(123), *S = 3.8	<table><tr><th>Left</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th></tr><tr><td>*(1)</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>*(2)</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>*(12)</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>*(13)</td><td>4</td><td>1</td><td>1</td><td>1</td><td>1</td><td>4</td><td>4</td></tr><tr><td>*(123)</td><td>5</td><td>3</td><td>3</td><td>3</td><td>3</td><td>5</td><td>5</td></tr><tr><td>*(134)</td><td>6</td><td>1</td><td>1</td><td>1</td><td>4</td><td>4</td><td>6</td></tr><tr><td>*S</td><td>7</td><td>3</td><td>3</td><td>3</td><td>5</td><td>5</td><td>7</td></tr></table>	Left	1	2	3	4	5	6	7	*(1)	1	1	1	1	1	1	1	*(2)	2	2	2	2	2	2	2	*(12)	3	3	3	3	3	3	3	*(13)	4	1	1	1	1	4	4	*(123)	5	3	3	3	3	5	5	*(134)	6	1	1	1	4	4	6	*S	7	3	3	3	5	5	7										
Left	1	2	3	4	5	6	7																																																																							
*(1)	1	1	1	1	1	1	1																																																																							
*(2)	2	2	2	2	2	2	2																																																																							
*(12)	3	3	3	3	3	3	3																																																																							
*(13)	4	1	1	1	1	4	4																																																																							
*(123)	5	3	3	3	3	5	5																																																																							
*(134)	6	1	1	1	4	4	6																																																																							
*S	7	3	3	3	5	5	7																																																																							
4.86	(12) -- 2.3 (13) -- 2.4 (14) -- 2.2 (24) -- 2.2 (123) -- 3.9 (124) -- 3.10 (134) -- 3.10		*(4), *(124), S = 3.10	<table><tr><th>Left</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><td>*(4)</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>*(24)</td><td>2</td><td>1</td><td>2</td><td>2</td><td>2</td></tr><tr><td>*(124)</td><td>3</td><td>1</td><td>3</td><td>3</td><td>3</td></tr><tr><td>(134)</td><td>4</td><td>1</td><td>3</td><td>3</td><td>3</td></tr><tr><td>S</td><td>5</td><td>1</td><td>3</td><td>3</td><td>3</td></tr></table>	Left	1	2	3	4	5	*(4)	1	1	1	1	1	*(24)	2	1	2	2	2	*(124)	3	1	3	3	3	(134)	4	1	3	3	3	S	5	1	3	3	3																																						
Left	1	2	3	4	5																																																																									
*(4)	1	1	1	1	1																																																																									
*(24)	2	1	2	2	2																																																																									
*(124)	3	1	3	3	3																																																																									
(134)	4	1	3	3	3																																																																									
S	5	1	3	3	3																																																																									
4.87	(12) -- 2.3 (13) -- 2.4 (14) -- 2.2 (24) -- 2.2 (123) -- 3.9 (124) -- 3.13 (134) -- 3.7		*(12), *(123), *S = 3.7	<table><tr><th>Left</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr><tr><td>*(1)</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>*(2)</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>*(12)</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>*S</td><td>4</td><td>3</td><td>3</td><td>3</td><td>4</td><td>6</td></tr><tr><td>(13)</td><td>5</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>(123)</td><td>6</td><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr></table> <table><tr><th>Right</th><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><td>*(12)</td><td>1</td><td></td><td></td><td></td></tr><tr><td>*(123)</td><td>2</td><td>= 4.68</td><td></td><td></td></tr><tr><td>*(124)</td><td>3</td><td></td><td></td><td></td></tr><tr><td>*S</td><td>4</td><td></td><td></td><td></td></tr></table>	Left	1	2	3	4	5	6	*(1)	1	1	1	1	1	1	*(2)	2	2	2	2	2	2	*(12)	3	3	3	3	3	3	*S	4	3	3	3	4	6	(13)	5	1	1	1	1	1	(123)	6	3	3	3	3	3	Right	1	2	3	4	*(12)	1				*(123)	2	= 4.68			*(124)	3				*S	4			
Left	1	2	3	4	5	6																																																																								
*(1)	1	1	1	1	1	1																																																																								
*(2)	2	2	2	2	2	2																																																																								
*(12)	3	3	3	3	3	3																																																																								
*S	4	3	3	3	4	6																																																																								
(13)	5	1	1	1	1	1																																																																								
(123)	6	3	3	3	3	3																																																																								
Right	1	2	3	4																																																																										
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*(123)	2	= 4.68																																																																												
*(124)	3																																																																													
*S	4																																																																													

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table	
4.88	(12)—2.3 (13)—2.4 (14)—2.2 (24)—2.2 (123)—3.9 (124)—3.13 (134)—3.8		*(12), *(123), *S = 3.8	Left	1 2 3 4 5 6
1111			*(1)	1	1 1 1 1 1 1
2222			*(2)	2	2 2 2 2 2 2
1113			*(12)	3	3 3 3 3 3 3
1234			*(13)	4	1 1 1 1 1 4
			*(123)	5	3 3 3 3 3 5
			*S	6	3 3 3 5 5 6
4.89	(12)—2.3 (13)—2.4 (14)—2.2 (123)—3.9 (124)—3.12 (134)—3.7		*(12), *(123), *S = 3.7	Left	1 2 3 4 5 6 7 8 9
1111			*(1)	1	1 1 1 1 1 1 1 1 1
2222			*(2)	2	2 2 2 2 2 2 2 2 2
1113			*(12)	3	3 3 3 3 3 3 3 3 3
1114			*(13)	4	1 1 1 1 4 1 4 4 4
			*(14)	5	1 1 1 1 5 1 5 5 5
			*(123)	6	3 3 3 3 6 3 6 6 6
			*(124)	7	3 3 3 3 7 3 7 7 7
			*(134)	8	1 1 1 1 8 1 8 8 8
			*S	9	3 3 3 3 9 3 9 9 9
4.90	(12)—2.3 (13)—2.2 (14)—2.2 (24)—2.2 (34)—2.2 (123)—3.12 (124)—3.15 (134)—3.16		*(4), *(124), *S = 3.16	Left	1 2 3 4 5 6
1111			*(4)	1	1 1 1 1 1 1
2224			*(14)	2	1 2 2 2 2 2
1134			*(24)	3	1 3 3 3 3 3
1444			*(134)	4	1 2 2 4 2 4
			*(124)	5	1 5 5 5 5 5
			*S	6	1 5 5 6 5 6

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
4.91	(12)—2.3 (13)—2.2 (14)—2.2		*(12), *(123), *(124), *S = 4.49	
1111	(123)—3.12		Left	1 2 3 4 5 6 7 8 9
2222	(124)—3.12		*(1)	1
1131	(134)—3.17		*(2)	2
1114			*(12)	3
			*(13)	4
			*(14)	5
			*(123)	6
			*(124)	7
			*(134)	8
			*S	9
4.92	(12)—2.3 (13)—2.2 (24)—2.2		*(12), *(123), *(124), *S = 4.49	
1111	(123)—3.12		Left	1 2 3 4 5 6 7 8
2222	(124)—3.12		*(1)	1
1131			*(2)	2
2224			*(12)	3
			*(13)	4
			*(24)	5
			*(124)	6
			*(123)	7
			*S	8
4.93	(12)—2.3 (13)—2.2 (14)—2.2		*(12), *(123), *(124), *S = 4.49	
1111	(23)—2.2		Left	1 2 3 4 5 6 7
2222	(123)—3.16		*(1)	1
1231	(124)—3.12		*(2)	2
1114	(134)—3.17		*(12)	3
			*(14)	4
			*(123)	5
			*(124)	6
			*S	7

TABLE III

IDEAL ANALYSIS $L \times L$ SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.94		$*(12), *(123), *S = 3.16$	
<u>1111</u>		Left	1 2 3 4 5 6 7
2222		$*(1)$	1
1133		$*(2)$	2
1134		$*(12)$	3
		$*(13)$	4
		$*(123)$	5
		$*(134)$	6
		$*S$	7
			1 1 1 1 1 1 1
			2 2 2 2 2 2 2
			3 3 3 3 3 3 3
			4 1 1 1 4 4 4
			5 3 3 3 5 5 5
			6 1 1 1 4 4 6
			7 3 3 3 5 5 7
4.95		$*(12), *(123), *S = 3.16$	
<u>1111</u>		Left	1 2 3 4 5 6
2222		$*(1)$	1
1133		$*(2)$	2
1234		$*(12)$	3
		$*(13)$	4
		$*(123)$	5
		$*S$	6
			1 1 1 1 1 1
			2 2 2 2 2 2
			3 3 3 3 3 3
			4 1 1 1 4 4
			5 3 3 3 5 5
			6 3 3 3 5 5
4.96		$*(123), *S = 2.2$	
<u>1111</u>		Left	1 2 3 4 5 6 7 8
2222		$*(1)$	1
3333		$*(2)$	2
1234		$*(3)$	3
		$*(12)$	4
		$*(13)$	5
		$*(23)$	6
		$*(123)$	7
		$*S$	8
			1 1 1 1 1 1 1 1
			2 2 2 2 2 2 2 2
			3 3 3 3 3 3 3 3
			4 4 4 4 4 4 4 4
			5 5 5 5 5 5 5 5
			6 6 6 6 6 6 6 6
			7 7 7 7 7 7 7 7
			8 7 7 7 7 7 7 8

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table																																																		
4.97 <u>1111</u> 2222 1233 1234 (12)---2.3 (13)---2.2 (14)---2.2 (23)---2.2 (24)---2.2 (34)---2.2 (123)---3.13 (124)---3.13 (134)---3.16 (234)---3.16	<pre> S (123) (12) / \ (1)L (2)L </pre>	$*(12), *(123), *S = 3.16$ Left $*(1)$ $*(2)$ $*(12)$ $*(123)$ $*S$	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><th>1</th><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><th>2</th><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><th>3</th><td>3</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><th>4</th><td>3</td><td>3</td><td>3</td><td>4</td><td>4</td></tr><tr><th>5</th><td>3</td><td>3</td><td>3</td><td>4</td><td>5</td></tr></table>		1	2	3	4	5	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	3	3	3	4	4	5	3	3	3	4	5														
	1	2	3	4	5																																																
1	1	1	1	1	1																																																
2	2	2	2	2	2																																																
3	3	3	3	3	3																																																
4	3	3	3	4	4																																																
5	3	3	3	4	5																																																
4.98 <u>1134</u> 2234 3334 4434 (12)---2.3 (34)---2.3 (13)---2.2 (14)---2.2 (23)---2.2 (24)---2.2 (123)---3.15 (124)---3.15 (134)---3.13 (234)---3.13	<pre> S / \ (134)L (234)L \ / (34) / \ (3)R (4)R </pre>	$*(34), *S = 2.2$ Right $*(3)$ $*(4)$ $*(34)$ $*S$ Left $*(34)$ $*(234)$ $*(134)$ $*S$	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><th>1</th><td>1</td><td>2</td><td>2</td><td>2</td></tr><tr><th>2</th><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><th>3</th><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><th>4</th><td>4</td><td>4</td><td>4</td><td>4</td></tr></table> = 4.70 <table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr><tr><th>1</th><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><th>2</th><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><th>3</th><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><th>4</th><td>4</td><td>4</td><td>4</td><td>4</td></tr></table> = 4.102		1	2	3	4	1	1	2	2	2	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4		1	2	3	4	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
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3	3	3	3	3																																																	
4	4	4	4	4																																																	
4.99 <u>1114</u> 2224 1234 4444 (12)---2.3 (13)---2.2 (14)---2.2 (23)---2.2 (24)---2.2 (34)---2.2 (123)---3.13 (124)---3.15 (134)---3.16 (234)---3.16	<pre> S (124) / \ (14)L (24)L \ / (4) </pre>	$*(4), *(124), *S = 3.16$ Left $*(4)$ $*(14)$ $*(24)$ $*(124)$ $*S$	<table><tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><th>1</th><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><th>2</th><td>1</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><th>3</th><td>1</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><th>4</th><td>1</td><td>4</td><td>4</td><td>4</td><td>4</td></tr><tr><th>5</th><td>1</td><td>4</td><td>4</td><td>4</td><td>5</td></tr></table>		1	2	3	4	5	1	1	1	1	1	1	2	1	2	2	2	2	3	1	3	3	3	3	4	1	4	4	4	4	5	1	4	4	4	5														
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2	1	2	2	2	2																																																
3	1	3	3	3	3																																																
4	1	4	4	4	4																																																
5	1	4	4	4	5																																																

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.100 (12) — 2.3 (34) — 2.3 (13) — 2.2 (14) — 2.2 (23) — 2.2 (24) — 2.2 (123) — 3.15 (124) — 3.15 (134) — 3.15 (234) — 3.15		*(34), *S = 2.2 Left *(3) *(4) *(34) *(134) *(234) *S	1 2 3 4 5 6 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 3 4 3 3 4 4 4 5 3 3 5 5 5 6 3 3 6 6 6
4.101 (12) — 2.3 (13) — 2.3 (14) — 2.3 (23) — 2.3 (24) — 2.3 (123) — 3.15 (124) — 3.15 (134) — 3.15 (234) — 3.15		Left *(1) *(2) *(3) *(4) *(12) *(13) *(14) *(23) *(24) *(34) *(123) *(124) *(134) *(234) *S	1 15 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 Table of all left Zeros

TABLE III

IDEAL ANALYSIS $4x4$ SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.102 $\overline{1114}$ $\overline{2224}$ $\overline{3334}$ $\overline{4444}$ (12)--2.3 (13)--2.3 (23)--2.3 (14)--2.2 (24)--2.2 (34)--2.2 (123)--3.14 (124)--3.15 (134)--3.15 (234)--3.15		$*(4), *S = 2.2$ Left $*(4)$ $*(14)$ $*(24)$ $*(34)$ $*(124)$ $*(134)$ $*(234)$ $*S$	1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 2 1 2 2 2 2 2 2 3 1 3 3 3 3 3 3 4 1 4 4 4 4 4 4 5 1 5 5 5 5 5 5 6 1 6 6 6 6 6 6 7 1 7 7 7 7 7 7 8 1 8 8 8 8 8 8
4.103 $\overline{1134}$ $\overline{2234}$ $\overline{3334}$ $\overline{4443}$ (12)--2.3 (13)--2.2 (23)--2.2 (34)--2.1 (123)--3.15 (134)--3.5 (234)--3.5		$*(34), *S = 2.2$ Left $*(34)$ $*(134)$ $*(234)$ $*S$	1 2 3 4 1 2 $\equiv 4.102$ 3 4
4.104 $\overline{1134}$ $\overline{2234}$ $\overline{3334}$ $\overline{4444}$ (12)--2.3 (13)--2.2 (14)--2.2 (23)--2.2 (24)--2.2 (34)--2.2 (123)--3.15 (124)--3.15 (134)--3.16 (234)--3.16		$*(4), *(34), *S = 3.16$ Left $*(4)$ $*(34)$ $*(134)$ $*(234)$ $*S$	1 2 3 4 5 1 1 1 1 1 2 1 2 2 2 3 1 2 3 3 4 1 2 4 4 5 1 2 5 5

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups		Semilattice	Ideals	Ideal Table
4.105	(12)--2.4 (34)--2.4		*(3), *(34), *(134), S=4.106	
<u>1134</u>	(13)--2.2		Right	1 2 3 4 5 6
<u>1134</u>	(123)--3.10		*(3)	1 1 1 1 1 1
<u>3333</u>	(134)--3.7		*(13)	2 1 2 3 4 2 4
<u>3333</u>			*(34)	3 1 1 1 1 1 1
			*(134)	4 1 2 3 4 2 4
			(123)	5 1 2 3 4 2 4
			S	6 1 2 3 4 2 4
4.106	(12)--2.1 (13)--2.2		*(3), *(34), *S = 3.7	
<u>1234</u>	(34)--2.4		Right	1 2 3 4
<u>2134</u>	(123)--3.11		*(3)	1
<u>3333</u>	(134)--3.7		*(34)	2 = 4.68
<u>3333</u>			*(123)	3
			*S	4
4.107	(12)--2.2 (13)--2.4		*(14), *(134), *S = 3.7	
<u>1114</u>	(14)--2.1		Left	1 2 3 4
<u>1214</u>	(123)--3.7		*(14)	1
<u>1314</u>	(124)--3.5		*(124)	2 = 4.68
<u>1414</u>	(134)--3.4		*(134)	3
			*S	4

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.108 (12)--2.2 (13)--2.4 (14)--2.4 (123)--3.7 (134)--3.18 <u>1111</u> 1211 1311 1311		Two Sided *(1) *(13) *(123) (134) S Left *(1) *(12) *(13) *(123) (134) S Right *(1) *(123) (13) (14) (134) S	1 2 3 4 5 1 1 1 1 1 2 1 1 2 1 2 3 1 1 3 1 3 4 1 1 2 1 2 5 1 1 3 1 3 1 2 3 4 5 6 1 1 1 1 1 1 2 1 2 1 2 1 2 3 1 3 1 3 1 3 4 1 4 1 4 1 4 5 1 3 1 3 1 3 6 1 4 1 4 1 4 1 2 3 4 5 6 1 1 1 1 1 1 2 1 2 1 1 1 2 3 1 3 1 1 1 3 4 1 3 1 1 1 3 5 1 3 1 1 1 3 6 1 2 1 1 1 2
4.109 (12)--2.2 (14)--2.2 (13)--2.4 (123)--3.7 (124)--3.17 (134)--3.6 <u>1111</u> 1211 1311 1111		Two Sided *(1) *(13) *(123) (14) (134) S Left *(1) *(12) *(13) *(123) (14) (124) (134) S	1 2 3 4 5 6 1 1 1 1 1 1 2 1 1 2 1 1 2 3 1 1 3 1 1 3 4 1 1 1 1 1 1 5 1 1 2 1 1 2 6 1 1 3 1 1 3 1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 2 1 2 1 2 1 2 1 2 3 1 3 1 3 1 3 1 3 4 1 4 1 4 1 4 1 4 5 1 1 1 1 1 1 1 1 6 1 2 1 2 1 2 1 2 7 1 3 1 3 1 3 1 3 8 1 4 1 4 1 4 1 4

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.110 (12) -- 2.4 (13) -- 2.4 (14) -- 2.2 (123) -- 3.18 (124) -- 3.7 (134) -- 3.7		Two Sided *(1) *(12) *(13) *(123) *S Right *(1) *(12) *(13) *(14) *(123) *(124) *(134) *S	1 2 3 4 5 1 1 1 1 1 2 1 1 1 1 3 1 1 1 1 4 1 1 1 1 5 1 2 3 4 5 1 2 3 4 5 6 7 8 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 3 1 1 1 1 1 1 1 4 1 2 3 4 5 6 7 8 5 1 1 1 1 1 1 1 6 1 2 3 4 5 6 7 8 7 1 2 3 4 5 6 7 8 8 1 2 3 4 5 6 7 8
4.111 (12) -- 2.2 (13) -- 2.4 (14) -- 2.4 (123) -- 3.7 (124) -- 3.7 (134) -- 3.18		Two Sided *(1) *(13) *(14) *(134) *S Right *(1) *(14) *(123) (13) (134) *S Left *(1) *(13) *(14) *(124) *(134) *S	1 2 3 4 5 1 1 1 1 1 2 1 1 1 2 3 1 1 1 1 4 1 1 1 2 5 1 1 3 3 5 1 2 3 4 5 6 1 1 1 1 1 1 2 1 1 1 1 1 3 1 2 3 1 2 6 4 1 1 4 1 1 4 5 1 1 4 1 1 4 6 1 2 3 1 2 6 1 2 3 4 5 6 1 1 1 1 1 1 2 1 1 1 2 1 2 3 1 1 1 1 1 1 4 1 1 3 4 3 4 5 1 1 1 2 1 2 6 1 1 3 6 3 6

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

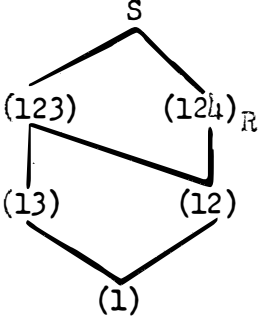
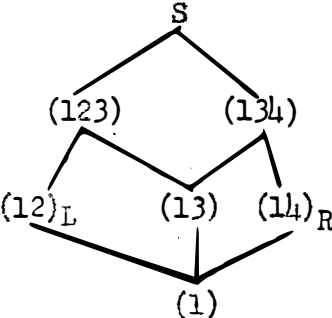
Subsemigroups	Semilattice	Ideals	Ideal Table
4.112	(12)--2.4	Two Sided	1 2 3 4 5
(13)--2.4		*(1)	1
(14)--2.2		*(12)	2
(123)--3.18		*(13)	3
(124)--3.8		*(123)	4
(134)--3.7		*S	5
		Right	1 2 3 4 5 6
		*(1)	1
		*(12)	2
		*(13)	3
		*(123)	4
		*(124)	5
		*S	6
4.113	(12)--2.2	Two Sided	1 2 3 4 5
(14)--2.2		*(1)	1
(13)--2.4		*(13)	2
(123)--3.7		*(123)	3
(134)--3.7		*(134)	4
		*S	5
		Left	1 2 3 4 5 6
		*(1)	1
		*(13)	2
		*(12)	3
		*(123)	4
		*(134)	5
		*S	6
		Right	1 2 3 4 5 6
		*(1)	1
		*(13)	2
		*(14)	3
		*(123)	4
		*(134)	5
		*S	6

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table																																																																																																																																						
4.114 (12)--2.2 (14)--2.2 (13)--2.4 (123)--3.7 (134)--3.7 (124)--3.17	<pre>graph TD S((S)) --> 123((123)) S --> 134((134)) 123 --> 12L((12)_L) 134 --> 14R((14)_R) 12L --> 13((13)) 14R --> 13 13 --> 1((1))</pre>	Two Sided *(1) *(13) *(123) *(134) * S Left *(1) *(13) *(12) *(123) *(134) * S Right *(1) *(13) *(14) *(123) *(134) * S	<table><tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>1</td><td>1</td><td>2</td><td>1</td><td>2</td></tr><tr><td>3</td><td>1</td><td>1</td><td>3</td><td>1</td><td>3</td></tr><tr><td>4</td><td>1</td><td>2</td><td>2</td><td>4</td><td>4</td></tr><tr><td>5</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr></table> <table><tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>1</td><td>1</td><td>2</td><td>2</td><td>1</td><td>2</td></tr><tr><td>3</td><td>1</td><td>1</td><td>3</td><td>3</td><td>1</td><td>3</td></tr><tr><td>4</td><td>1</td><td>1</td><td>4</td><td>4</td><td>1</td><td>4</td></tr><tr><td>5</td><td>1</td><td>2</td><td>2</td><td>2</td><td>5</td><td>5</td></tr><tr><td>6</td><td>1</td><td>2</td><td>4</td><td>4</td><td>5</td><td>6</td></tr></table> <table><tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>1</td><td>1</td><td>1</td><td>2</td><td>1</td><td>2</td></tr><tr><td>3</td><td>1</td><td>2</td><td>3</td><td>2</td><td>5</td><td>5</td></tr><tr><td>4</td><td>1</td><td>1</td><td>1</td><td>4</td><td>1</td><td>4</td></tr><tr><td>5</td><td>1</td><td>2</td><td>3</td><td>2</td><td>5</td><td>5</td></tr><tr><td>6</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr></table>		1	2	3	4	5	1	1	1	1	1	1	2	1	1	2	1	2	3	1	1	3	1	3	4	1	2	2	4	4	5	1	2	3	4	5		1	2	3	4	5	6	1	1	1	1	1	1	1	2	1	1	2	2	1	2	3	1	1	3	3	1	3	4	1	1	4	4	1	4	5	1	2	2	2	5	5	6	1	2	4	4	5	6		1	2	3	4	5	6	1	1	1	1	1	1	1	2	1	1	1	2	1	2	3	1	2	3	2	5	5	4	1	1	1	4	1	4	5	1	2	3	2	5	5	6	1	2	3	4	5	6
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4.115 (12)--2.2 (14)--2.2 (24)--2.2 (13)--2.4 (123)--3.7 (124)--3.16 (134)--3.16	<pre>graph TD S((S)) --> 123((123)) S --> 124L((124)_L) 123 --> 13((13)) 124L --> 12L((12)_L) 13 --> 1((1)) 12L --> 1</pre>	Two Sided *(1) *(13) *(123) * S Left *(1) *(13) *(12) *(124) *(123) * S	<table><tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>1</td><td></td><td></td><td></td><td></td></tr><tr><td>2</td><td>=4.115</td><td></td><td></td><td></td></tr><tr><td>3</td><td></td><td></td><td></td><td></td></tr><tr><td>4</td><td></td><td></td><td></td><td></td></tr></table> <table><tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td>2</td><td>1</td><td>1</td><td>2</td><td>2</td><td>2</td><td>2</td></tr><tr><td>3</td><td>1</td><td>1</td><td>3</td><td>3</td><td>3</td><td>3</td></tr><tr><td>4</td><td>1</td><td>1</td><td>3</td><td>4</td><td>3</td><td>4</td></tr><tr><td>5</td><td>1</td><td>1</td><td>5</td><td>5</td><td>5</td><td>5</td></tr><tr><td>6</td><td>1</td><td>1</td><td>5</td><td>6</td><td>5</td><td>6</td></tr></table>		1	2	3	4	1					2	=4.115				3					4						1	2	3	4	5	6	1	1	1	1	1	1	1	2	1	1	2	2	2	2	3	1	1	3	3	3	3	4	1	1	3	4	3	4	5	1	1	5	5	5	5	6	1	1	5	6	5	6																																																												
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6	1	1	5	6	5	6																																																																																																																																			

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table
4.116 (12) -2.2 (14) -2.2 (24) -2.2 (13) -2.4 (123) -3.7 (124) -3.16 (134) -3.10	S (124)_L (134) (14) (4)	* (4), * (14), * (134), * S=4.116 Left 1 2 3 4 5 1 1 1 1 1 1 2 2 2 2 1 2 3 2 3 1 2 4 2 4 1 2 5 2 5	
4.117 (12) -2.2 (14) -2.2 (13) -2.4 (123) -3.7 (124) -3.17 (134) -3.6	S (124)_L (123) (134) (12)_L (14) (13) (1)	Two Sided 1 2 3 4 5 6 1 1 1 1 1 1 1 1 1 2 1 2 1 1 3 1 3 3 1 1 1 4 1 4 1 1 3 2 3 5 1 1 3 4 3 6	
4.118 (12) -2.2 (14) -2.2 (24) -2.2 (13) -2.4 (123) -3.7 (124) -3.16 (134) -3.8	S (123) (12)_L (13) (1)	* (1), * (13), * (123), * S=4.118 Left 1 2 3 4 5 1 1 1 1 1 1 2 1 2 2 1 3 1 3 3 1 4 1 4 4 1 4 3 4 5	

TABLE III

IDEAL ANALYSIS 4×4 SEMIGROUPS (Continued)

Subsemigroups	Semilattice	Ideals	Ideal Table																																																																								
4.119 (12) -- 2.4 (13) -- 2.2 <u>1111</u> (14) -- 2.2 1112 (34) -- 2.2 1133 (123) -- 3.6 1134 (124) -- 3.7 (134) -- 3.16		Two Sided *(1) *(12) *(13) *(123) * S Left *(1) *(12) *(13) *(123) *(134) * S	<table> <tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr> <tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr> <tr><td>2</td><td>1</td><td>2</td><td>2</td><td>2</td></tr> <tr><td>3</td><td>1</td><td>2</td><td>3</td><td>2</td></tr> <tr><td>4</td><td>1</td><td>1</td><td>4</td><td>1</td></tr> <tr><td>5</td><td>1</td><td>2</td><td>5</td><td>2</td></tr> </table> <table> <tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr> <tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr> <tr><td>2</td><td>1</td><td>1</td><td>1</td><td>2</td><td>2</td></tr> <tr><td>3</td><td>1</td><td>1</td><td>3</td><td>3</td><td>3</td></tr> <tr><td>4</td><td>1</td><td>1</td><td>3</td><td>3</td><td>4</td></tr> <tr><td>5</td><td>1</td><td>1</td><td>3</td><td>3</td><td>5</td></tr> <tr><td>6</td><td>1</td><td>1</td><td>3</td><td>3</td><td>6</td></tr> </table>	1	2	3	4	5	1	1	1	1	1	2	1	2	2	2	3	1	2	3	2	4	1	1	4	1	5	1	2	5	2	1	2	3	4	5	6	1	1	1	1	1	1	2	1	1	1	2	2	3	1	1	3	3	3	4	1	1	3	3	4	5	1	1	3	3	5	6	1	1	3	3	6
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5	1	1	3	3	5																																																																						
6	1	1	3	3	6																																																																						
4.120 (12) -- 2.1 (14) -- 2.2 <u>1221</u> (123) -- 3.1 2112 (124) -- 3.5 2112 1234		*(12), *(123), *S = 3.7 Right *(12) *(123) *(124) * S	<table> <tr><th>1</th><th>2</th><th>3</th><th>4</th></tr> <tr><td>1</td><td></td><td></td><td></td></tr> <tr><td>2</td><td></td><td></td><td></td></tr> <tr><td>3</td><td></td><td></td><td></td></tr> <tr><td>4</td><td></td><td></td><td></td></tr> </table> <table> <tr><th>1</th><th>2</th><th>3</th><th>4</th></tr> <tr><td>1</td><td></td><td></td><td></td></tr> <tr><td>2</td><td></td><td></td><td></td></tr> <tr><td>3</td><td></td><td></td><td></td></tr> <tr><td>4</td><td></td><td></td><td></td></tr> </table>	1	2	3	4	1				2				3				4				1	2	3	4	1				2				3				4																																			
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TABLE III

IDEAL ANALYSIS Lpd_4 SEMIGROUPS (Continued)

4.121	(12) $\rightarrow$ 2.4		Two Sided	1 2 3 4 5
<u>1111</u>	(13) $\rightarrow$ 2.4		*(1)	1 1 1 1 1 1
<u>1111</u>	(14) $\rightarrow$ 2.4		(12)	2 1 1 1 1 1
<u>1112</u>	(123) $\rightarrow$ 3.18		(123)	3 1 1 1 2 2
<u>1111</u>	(124) $\rightarrow$ 3.18		(124)	4 1 1 1 1 1
			S	5 1 1 1 2 2
			Left	1 2 3 4 5 6
			*(1)	1 1 1 1 1 1 1
			(12)	2 1 1 1 1 1 1
			(14)	3 1 1 1 1 1 1
			(123)	4 1 1 2 1 2 2
			(124)	5 1 1 1 1 1 1
			S	6 1 1 2 1 2 2
			Right	1 2 3 4 5 6
			*(1)	1 1 1 1 1 1 1
			(12)	2 1 1 1 1 1 1
			(13)	3 1 1 1 1 2 2
			(123)	4 1 1 1 1 2 2
			(124)	5 1 1 1 1 1 1
			S	6 1 1 1 1 2 2

TABLE IV

LATTICE ISOMORPHISMS

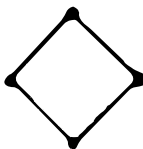
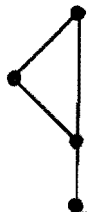
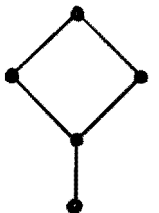
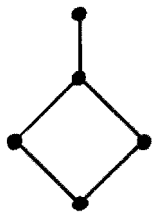
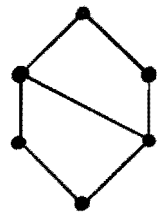
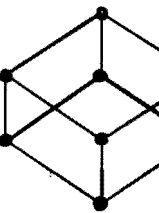
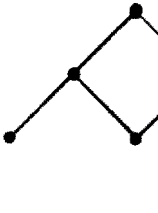
 L1 4.10 4.11	 L2 4.12 4.13 4.14 4.15 4.29 4.35	 L3 4.17 4.18 4.20 4.21 4.24 4.25 4.26 4.28 4.33 4.34	 L4 4.1 4.8 4.27 4.38 4.39 4.41 4.42 4.45 4.47 4.106	 L5 4.16 4.22 4.30 4.31 4.32 4.103 4.107 4.120	 L6 4.19
 L7 4.3 4.7 4.23 4.36 4.40 4.46 4.53 4.69 4.104 4.116	 L8 4.37 4.49 4.99 4.118 4.44	 L9 4.4 4.5 4.6 4.9 4.43 4.51 4.52 4.54 4.76 4.90 4.105 4.112 4.115 4.119	 L10 4.2 4.48 4.50 4.55 4.68 4.73 4.75 4.102 4.110 4.117	 L11 4.66	

TABLE IV

LATTICE ISOMORPHISMS (Continued)

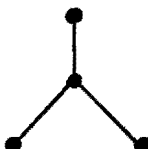
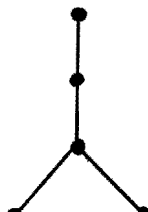
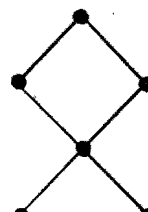
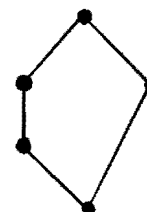
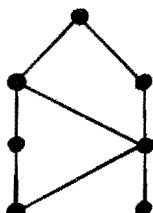
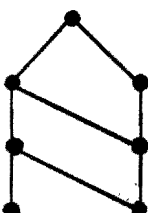
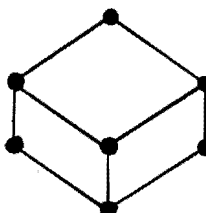
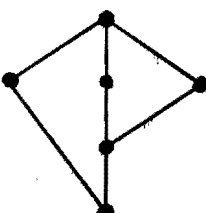
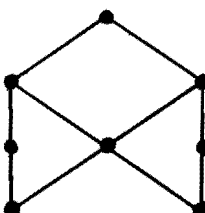
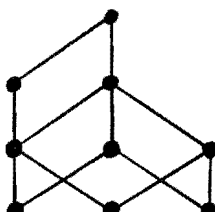
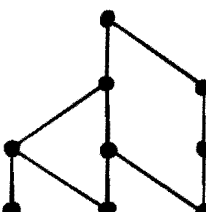
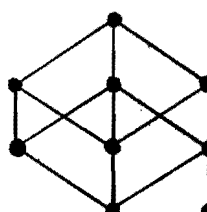
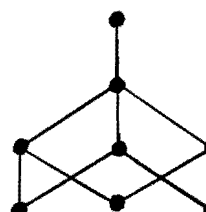
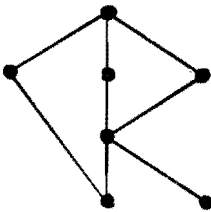
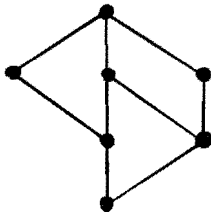
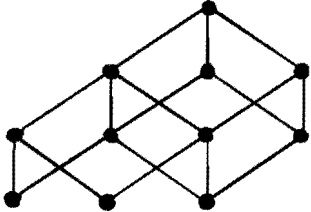
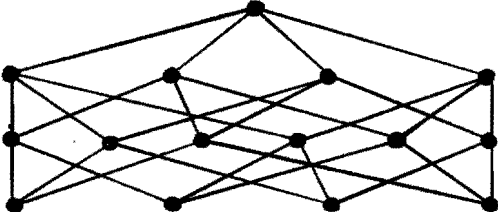
 L12 4.65	 L13 4.60 4.63	 L14 4.57 4.61 4.97	 L15 4.98 4.100	 L16 4.88 4.95	 L17 4.86
 L18 4.82 4.83 4.87 4.93	 L19 4.56 4.58 4.84 4.85 4.94	 L20 4.108 4.109 4.113 4.114 4.121	 L21 4.74	 L22 4.78 4.80 4.81 4.92	
 L23 4.59	 L24 4.62	 L25 4.67 4.77 4.79 4.89 4.91	 L26 4.96		

TABLE IV

LATTICE ISOMORPHISMS (Continued)

 L27 4.64	 L28 4.72	 L29 4.111
 L30 4.70 4.71	 L31 4.101	

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- Dubreil , Paul. Algèbre. Paris: Gauthier-Villars, 1946. Vol. 1
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- _____. "Finite Ova," The American Journal of Mathematics, 59:23-32, 1937.

APPENDIX

APPENDIX

In this appendix, compiled jointly by Messrs. K. S. Carman, J. C. Harden, and E. E. Posey, are listed what the authors believe to be all distinct semigroups of orders two, three, and four, and their proper sub-semigroups. Two semigroups are regarded as distinct if no isomorphism (or anti-isomorphism) can be set up between them.

Commutative semigroups of these orders (under the name of "finite ova") were listed by A. R. Poole in his dissertation,¹ but were omitted from the published version.² We have constructed independently all semigroups of orders two and three (non-commutative as well as commutative), and our list of commutative semigroups of those orders agrees perfectly with Poole's.

In addition to constructing the non-commutative semigroups of order four, we have examined Poole's list of commutative semigroups of that order, and have found what appear to be a few errors. It should be pointed out that the copy of Poole's thesis available to us was a carbon copy borrowed from the library of the California Institute of Technology, with the subscripts in the semigroup multiplication tables filled in by hand, so that it is entirely possible that the errors noted do not occur in the original dissertation; indeed, some of these errors are obviously merely mistakes in copying. We did not construct the commutative

¹A. R. Poole, "Finite Ova," (Doctoral Dissertation, California Institute of Technology, 1935).

²A. R. Poole, "Finite Ova," The American Journal of Mathematics, 59:23-32, 1937.

semigroups of order four independently, and cannot claim that our examination of Poole's list was sufficiently exhaustive to preclude the possibility that it contains errors other than those noted below.

Poole's R 421 appears as:

$$u_1 \quad u_1 \quad u_1 \quad u_1$$

$$u_1 \quad u_2 \quad u_3 \quad u_2$$

$$u_1 \quad u_3 \quad u_1 \quad u_3$$

$$u_1 \quad u_2 \quad u_1 \quad u_2$$

which is non-commutative since $u_4u_3 \neq u_3u_4$. An examination of associativity shows that if $u_4u_3 = u_1$ then the system is not a semigroup regardless of what the product u_3u_4 may be. Changing u_4u_3 from u_1 to u_3 , we have a commutative semigroup, distinct from all others in Poole's list, which is given in our list as No. 4.38.

The multiplication table of Poole's R 430 is identical with that of his R 417; this semigroup appears in our list as No. 4.9. Changing a single subscript yields a new commutative semigroup, No. 4.5 in our list:

R 417, R 430

4.5

$$u_1 \quad u_1 \quad u_1 \quad u_1$$

$$u_1 \quad u_1 \quad u_1 \quad u_1$$

$$u_1 \quad u_2 \quad u_1 \quad u_2$$

$$u_1 \quad u_2 \quad u_1 \quad u_2$$

$$u_1 \quad u_1 \quad u_1 \quad u_1$$

$$u_1 \quad u_1 \quad u_1 \quad u_1$$

$$u_1 \quad u_2 \quad u_1 \quad u_2$$

$$u_1 \quad u_2 \quad u_1 \quad u_4$$

Poole's T 42 and T 43 are identical, and appear in our list as No. 4.22. Two changes produce a new commutative semigroup, our No. 4.23:

T 4 ₂ , T 4 ₃				4.23			
u ₁	u ₁	u ₁	u ₄	u ₁	u ₁	u ₁	u ₄
u ₁	u ₂	u ₁	u ₄	u ₁	u ₂	u ₁	u ₄
u ₁	u ₁	u ₁	u ₄	u ₁	u ₁	u ₃	u ₄
u ₄	u ₄	u ₄	u ₁	u ₄	u ₄	u ₄	u ₄

Multiplication table T 4₁₅ in Poole's list fails to satisfy the associativity condition, but can be so modified as to become a commutative semigroup apparently distinct from all others in that list; we include it as No. 4.3:

R 4 ₁₅				4.3			
u ₁	u ₁	u ₁	u ₁	u ₁	u ₁	u ₁	u ₁
u ₁	u ₂	u ₁	u ₁	u ₁	u ₁	u ₁	u ₁
u ₁	u ₁	u ₁	u ₂	u ₁	u ₁	u ₂	u ₂
u ₁	u ₁	u ₂	u ₁	u ₁	u ₁	u ₂	u ₂

Poole's semigroups R 4₂ and R 4₈ seem to be isomorphic; this semigroup appears in our list as No. 4.43. Poole's tables and an isomorphism between them are exhibited below:

R 4 ₂					R 4 ₈			
u ₁	u ₁	u ₁	u ₁	u ₁ $\longleftrightarrow$ u ₁	u ₁	u ₁	u ₁	u ₁
u ₁	u ₂	u ₁	u ₁	u ₂ $\longleftrightarrow$ u ₁₁	u ₁	u ₂	u ₂	u ₁
u ₁	u ₁	u ₃	u ₃	u ₃ $\longleftrightarrow$ u ₂	u ₁	u ₂	u ₃	u ₃
u ₁	u ₁	u ₃	u ₄	u ₄ $\longleftrightarrow$ u ₃	u ₁	u ₁	u ₂	u ₄

In Table Ia are listed the four semigroups of order two, of which only No. 2.3 is non-commutative. Table IIa contains the six non-commutative semigroups of order three (Nos. 3.7, 3.9, 3.12, 3.13, 3.14, 3.15).

and the twelve commutative semigroups of order three, the latter being identical (to within isomorphism) with Poole's ova of order three. In Table IIIa we list the 55 commutative semigroups of order four (Nos. 4.1-4.55) and the 66 non-commutative semigroups of order four (Nos. 4.56 - 4.121).

In Table IVa, the columns are headed by the reference numbers of the semigroups of order two, and the rows are labeled with the reference numbers of the semigroups of order three. If No. 3.z ($z = 1, \dots, 18$) contains k subsemigroups isomorphic to No. 2.y ($y = 1, \dots, 4$), the letter k is placed at the intersection of the row 3.z and the column 2.y. Table Va exhibits similarly the subsemigroups of orders two and three of the semigroups of order four. Subsemigroups of order one are simply the idempotent elements of a semigroup, and are not listed separately since they are visible immediately upon inspection of the multiplication table of the semigroup.

It would be futile to expect tables of this sort to be entirely free from errors. The authors have checked their work repeatedly, and have rectified several mistakes, but others have doubtless escaped their notice. They will appreciate receiving corrections from users of the tables.

TABLE Ia

SEMIGROUPS OF ORDER TWO

2.1	2.2	2.3	2.4
1 2 2 1	1 1 1 2	1 1 2 2	1 1 1 1

TABLE IIa

SEMIGROUPS OF ORDER THREE

3.1	3.2	3.3	3.4	3.5	3.6
1 2 2 2 1 1 2 1 1	1 1 1 1 1 1 1 1 2	1 2 3 2 3 1 3 1 2	1 1 3 1 1 3 3 3 1	1 1 3 1 2 3 3 3 1	1 1 1 1 1 1 1 1 3
3.7	3.8	3.9	3.10	3.11	3.12
1 1 1 1 2 1 1 3 1	1 1 1 1 1 2 1 2 3	1 1 1 2 2 2 1 1 1	1 1 3 1 1 3 3 3 3	1 2 3 2 1 3 3 3 3	1 1 1 2 2 2 1 1 3
3.13	3.14	3.15	3.16	3.17	3.18
1 1 1 2 2 2 1 2 3	1 1 1 2 2 2 3 3 3	1 1 3 2 2 3 3 3 3	1 1 1 1 2 2 1 2 3	1 1 1 1 2 1 1 1 3	1 1 1 1 1 1 1 1 1

TABLE IIIa

SEMIGROUPS OF ORDER FOUR

4.1	4.2	4.3	4.4	4.5	4.6	4.7
1 1 1 1 1 2 2 2 1 2 3 3 1 2 3 3	1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1 2 2 1 1 2 2	1 1 1 1 1 2 1 4 1 1 1 1 1 4 1 1	1 1 1 1 1 2 1 2 1 1 1 1 1 2 1 4	1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 3	1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2
4.8	4.9	4.10	4.11	4.12	4.13	4.14
1 1 1 1 1 1 1 3 1 1 1 1 1 3 1 2	1 1 1 1 1 2 1 2 1 1 1 1 1 2 1 2	1 2 3 4 2 1 4 3 3 4 2 1 4 3 1 2	1 2 3 4 2 1 4 3 3 4 1 2 4 3 2 1	1 2 3 4 2 3 1 4 3 1 2 4 4 4 4 4	1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 4	1 2 1 1 2 1 2 2 1 2 3 4 1 2 4 3
4.15	4.16	4.17	4.18	4.19	4.20	4.21
1 2 1 2 2 1 2 1 1 2 3 4 2 1 4 3	1 2 1 1 2 1 2 2 1 2 3 1 1 2 1 4	1 2 1 1 2 1 2 2 1 2 3 4 1 2 4 4	1 2 3 1 2 1 3 2 3 3 3 3 1 2 3 4	1 2 3 3 2 1 3 3 3 3 3 3 3 3 3 4	1 2 3 4 2 1 3 4 3 3 3 4 4 4 4 4	1 1 1 4 1 2 3 4 1 3 1 4 4 4 4 1
4.22	4.23	4.24	4.25	4.26	4.27	4.28
1 1 1 4 1 2 1 4 1 1 1 4 4 4 4 1	1 1 1 4 1 2 1 4 1 1 3 4 4 4 4 4	1 1 3 3 1 2 3 4 3 3 1 1 3 4 1 1	1 2 1 4 2 2 2 2 1 2 1 4 4 2 4 1	1 2 4 4 2 2 2 2 4 2 1 1 4 2 1 1	1 1 1 1 1 2 1 4 1 1 1 1 1 4 1 2	1 1 1 1 1 2 3 4 1 3 1 3 1 4 3 2
4.29	4.30	4.31	4.32	4.33	4.34	4.35
1 1 3 1 1 2 3 2 3 3 1 3 1 2 3 2	1 2 1 1 2 1 2 2 1 2 1 1 1 2 1 1	1 2 1 2 2 1 2 1 1 2 1 2 2 1 2 1	1 2 2 2 2 1 1 1 2 1 1 1 2 1 1 1	1 1 3 1 1 1 3 1 3 3 1 3 1 1 3 2	1 1 3 3 1 1 3 3 3 3 1 1 3 3 1 2	1 2 3 3 2 3 1 1 3 1 2 2 3 1 2 2
4.36	4.37	4.38	4.39	4.40	4.41	4.42
1 1 1 1 1 2 3 3 1 3 1 1 1 3 1 1	1 1 1 1 1 2 3 4 1 3 1 1 1 4 1 1	1 1 1 1 1 2 3 2 1 3 1 3 1 2 3 2	1 1 1 1 1 2 3 4 1 3 1 1 1 4 1 3	1 1 1 1 1 2 2 2 1 2 2 2 1 2 2 2	1 1 1 1 1 2 2 2 1 2 2 2 1 2 2 3	1 1 1 1 1 2 2 2 1 2 3 3 1 2 3 4

TABLE IIIa

SEMI-GROUPS OF ORDER FOUR (Continued)

4.43	4.44	4.45	4.46	4.47	4.48	4.49
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 2 2 1	1 2 2 1	1 2 2 4	1 2 2 2	1 2 2 2	1 2 1 1	1 2 1 2
1 2 3 1	1 2 3 4	1 2 3 4	1 2 3 2	1 2 3 4	1 1 3 1	1 1 3 3
1 1 1 4	1 1 4 1	1 4 4 1	1 2 2 2	1 2 4 2	1 1 1 4	1 2 3 4
4.50	4.51	4.52	4.53	4.54	4.55	4.56
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 4
1 2 1 1	1 2 1 1	1 2 1 2	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 4
1 1 3 1	1 1 3 4	1 1 3 1	1 1 1 2	1 1 1 1	1 1 1 1	1 1 2 4
1 1 1 1	1 1 4 1	1 2 1 2	1 1 2 1	1 1 1 2	1 1 1 1	1 1 1 4
4.57	4.58	4.59	4.60	4.61	4.62	4.63
1 1 1 4	1 1 3 4	1 1 1 1	1 2 3 4	1 1 3 4	1 1 1 1	1 1 1 1
1 2 3 4	1 1 3 4	2 2 2 2	2 1 3 4	1 1 3 4	2 2 2 2	2 2 2 2
1 3 2 4	3 3 3 4	1 1 1 2	3 3 3 4	3 3 3 4	3 3 3 3	1 2 3 4
1 1 1 4	3 3 3 4	4 4 4 4	4 4 3 4	4 4 3 4	1 2 2 4	2 1 4 3
4.64	4.65	4.66	4.67	4.68	4.69	4.70
1 1 3 3	1 2 3 4	1 1 1 1	1 1 3 3	1 1 3 3	1 1 1 1	1 1 1 1
2 2 4 4	2 1 4 3	2 2 2 2	2 2 3 3	2 2 3 3	1 1 2 2	2 2 2 2
1 1 3 3	1 2 3 4	1 1 3 3	3 3 3 3	3 3 3 3	1 2 3 4	3 3 3 3
2 2 4 4	2 1 4 3	2 2 4 4	4 4 4 4	4 4 3 3	1 2 3 4	3 3 3 4
4.71	4.72	4.73	4.74	4.75	4.76	4.77
1 1 1 1	1 1 3 4	1 1 3 3	1 1 3 4	1 1 3 3	1 1 1 1	1 1 1 1
2 2 2 2	2 2 3 4	2 2 3 3	2 2 3 4	2 2 3 3	1 1 1 3	2 2 2 2
1 1 1 1	3 3 3 4	3 3 3 3	3 3 3 3	3 3 3 3	1 1 1 3	1 1 1 1
4 4 4 4	3 3 3 4	3 3 3 4	3 3 3 3	3 3 3 3	1 2 3 4	1 1 1 1
4.78	4.79	4.80	4.81	4.82	4.83	4.84
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 4
2 2 2 2	2 2 2 2	2 2 2 2	2 2 2 2	2 2 2 2	2 2 2 2	1 2 1 4
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 3	1 2 3 1	1 1 1 3	1 3 1 4
2 2 2 2	1 1 1 4	2 2 2 4	2 2 2 4	1 1 1 1	1 2 1 4	1 1 1 4

TABLE IIIa

SEMIGROUPS OF ORDER FOUR (Continued)

4.85	4.86	4.87	4.88	4.89	4.90	4.91
1 1 1 1 2 2 2 2 1 1 1 3 1 1 3 4	1 1 1 4 2 2 2 4 1 1 1 4 4 4 4 4	1 1 1 1 2 2 2 2 1 1 1 1 1 2 3 4	1 1 1 1 2 2 2 2 1 1 1 3 1 2 3 4	1 1 1 1 2 2 2 2 1 1 1 3 1 1 1 4	1 1 1 4 2 2 2 4 1 1 3 4 4 4 4 4	1 1 1 1 2 2 2 2 1 1 3 1 1 1 1 4
4.92	4.93	4.94	4.95	4.96	4.97	4.98
1 1 1 1 2 2 2 2 1 1 3 1 2 2 2 4	1 1 1 1 2 2 2 2 1 2 3 1 1 1 1 4	1 1 1 1 2 2 2 2 1 1 3 3 1 1 3 4	1 1 1 1 2 2 2 2 1 1 3 3 1 2 3 4	1 1 1 1 2 2 2 2 3 3 3 3 1 2 3 4	1 1 1 1 2 2 2 2 1 2 3 3 1 2 3 4	1 1 3 4 2 2 3 4 3 3 3 4 4 4 3 4
4.99	4.100	4.101	4.102	4.103	4.104	4.105
1 1 1 4 2 2 2 4 1 2 3 4 4 4 4 4	1 1 3 4 2 2 3 4 3 3 3 3 4 4 4 4	1 1 1 1 2 2 2 2 3 3 3 3 4 4 4 4	1 1 1 4 2 2 2 4 3 3 3 4 4 4 4 4	1 1 3 4 2 2 3 4 3 3 3 4 4 4 4 3	1 1 3 4 2 2 3 4 3 3 3 4 4 4 4 4	1 1 3 4 1 1 3 4 3 3 3 3 3 3 3 3
4.106	4.107	4.108	4.109	4.110	4.111	4.112
1 2 3 4 2 1 3 4 3 3 3 3 3 3 3 3	1 1 1 4 1 2 1 4 1 3 1 4 4 4 4 1	1 1 1 1 1 2 1 1 1 3 1 1 1 3 1 1	1 1 1 1 1 2 1 1 1 3 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1 1 1 1 2 3 4	1 1 1 1 1 2 1 4 1 3 1 1 1 1 1 1	1 1 1 1 1 1 1 2 1 1 1 1 1 2 3 4
4.113	4.114	4.115	4.116	4.117	4.118	4.119
1 1 1 1 1 2 1 1 1 3 1 1 1 3 3 4	1 1 1 1 1 2 1 1 1 3 1 1 1 1 3 4	1 1 1 1 1 2 1 2 1 3 1 3 1 2 1 4	1 1 1 4 1 2 1 4 1 3 1 4 4 4 4 4	1 1 1 1 1 2 1 1 1 3 1 1 1 1 1 4	1 1 1 1 1 2 1 2 1 3 1 3 1 2 3 4	1 1 1 1 1 1 1 2 1 1 3 3 1 1 3 4
4.120	4.121					
1 2 2 1 2 1 1 2 2 1 1 2 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1					

TABLE IVa

PROPER SUBSEMIGROUPS OF SEMIGROUPS OF ORDER THREE

	2.1	2.2	2.3	2.4
3.1	1			
3.2				1
3.3				
3.4	1			1
3.5	1	1		
3.6		1		1
3.7		1		1
3.8		1		1
3.9			1	1
3.10		1		1
3.11	1	1		
3.12		1	1	
3.13		2	1	
3.14			3	
3.15		2	1	
3.16		3		
3.17		2		
3.18				2

TABLE Va

PROPER SUBSEMIGROUPS OF SEMIGROUPS OF ORDER FOUR

4.	2.				3.																	
	1	2	3	4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1		3		1										2						1		
2		1		2						2												1
3				1		2																
4		1		2						1		1										1
5		3		1						2										1		
6		1		1		1				1												
7				2		1											1					1
8				2																		1
9		1		2						1				1								
10	1																					
11	2																					
12		1					1															
13		1					1															
14	2								1						1							
15	2								1													
16	1	2							2												1	
17	1	3							2											1		
18	1	3							1						1					1		
19	1	3													1						1	
20	1	3													2					1		
21	1	1		1				1	1			1										
22	1	1		1				1	1	1												
23		5																		2	1	
24	1	1		1					1													
25	1	1		1				1						1	1							
26	1	1		1											1							
27		1		1						1					1							
28		1		1							1				1							
29	1	1							1					1								
30	1			2				2														1
31	1			1	1			1														
32	1				2																	
33	1			1		1		1														
34	1			1				1														
35							1															
36		1		2								1										1
37		1		2								1										1
38		1		2								1		1								
39		1		1		1						1										
40		1		2										2								1
41		1		1		1								1								

TABLE Va

PROPER SUBSEMIGROUPS OF SEMIGROUPS OF ORDER FOUR
(Continued)

4.	2.				3.																		
	1	2	3	4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
42		6																		4			
43		4																		1	2		
44		3								1		1								1			
45		3										2								1			
46		3								1				1						1			
47		3		1								1		1						1			
48		3																			3		
49		4																		2	1		
50		2		1						2											1		
51		2		1						1		1									1		
52		2												1							1		
53				3																			2
54				2		1																1	
55				3																		3	
56			1	1		1							1										
57	1	1	1												1	1							
58		1	1	1										1		1							
59			3	1									1					1					
60	1	1	1												2		1						
61		2	1	1										2			1						
62		2	3													1	1	1					
63	1	2	1														1						
64			3																				
65	2		1																				
66		1	2													2							
67		2	2													2				1			
68		2	1	1							2									1			
69		2	1	1								2								1			
70		1	3													2		1					
71			3	1									2					1					
72		1	2													2			1				
73		3	1																1		2		
74		2	1	1							2								1				
75		2	1	1						2									1				
76		1		2								1										1	
77			1	2									2									1	
78			1	2									2										
79		1	1	1						1			1			1							
80		1	1	1									1			1							
81		1	1	1									1			1							
82		2	1	1		1							1				1						

