To the Graduate Council:

I am submitting herewith a thesis written by James A. Nickerson, Jr. entitled "Ultraproducts in Commutative Algebra." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

David F. Anderson

David F. Anderson, Major Professor

We have read this thesis
and recommend its acceptance:

Robert M. McConnel

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Accepted for the Council:

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Graduate Studies and Research

ULTRAPRODUCTS IN COMMUTATIVE ALGEBRA

A Thesis

Presented for the

Master of Science

Degree

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James A. Nickerson, Jr.

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DEDICATION

To my wife Laura, with love and gratitude.

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ABSTRACT

Since 1966, when Abraham Robinson [5] used ultrapowers to defend Liebniz' disreputed theory of infinitesimals in analysis, interest in ultrafilters and ultrapowers, in both analysis and algebra, has grown markedly. This paper will explore some general properties of ultrapowers of commutative rings.

Chapter 1 consists of the basic definitions and characteristics of filters, ultrafilters and ultraproducts. In particular, we will examine the conditions under which a given ring is isomorphic to its ultrapower.

In Chapter 2, we find necessary and sufficient conditions on a ring R under which certain ring-theoretic properties of R will be shared by its ultrapower.

The dimension of a commutative ring with identity is defined in Chapter 3, and the relationships between the dimension of a ring R and that of its ultrapower are discussed.

Chapter 4 states some results comparing two functors from R-modules to $R_{\mathcal{F}}$-modules.

It is assumed that the reader has a working knowledge of commutative ring theory. Our basic reference for undefined terms will be Hungerford [4].

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CHAPTER 1

INTRODUCTORY CONCEPTS AND PROPERTIES OF FILTERS, ULTRAFILTERS, AND ULTRAPRODUCTS

The study of ultraproducts in commutative algebra begins with presenting the basic definitions and properties of filters, ultrafilters, and ultraproducts. We will set the foundations for our work in the remaining chapters by describing the conditions under which a commutative ring with identity is isomorphic to its ultrapower.

Definition 1.1. Let X be a non-empty set. A collection F of subsets of X is called a filter if

(a) $\emptyset \notin F$.

(b) $A, B \in F$ imply $A \cap B \in F$.

(c) $A \in F$ and $A \subset D \subset X$ imply that $D \in F$.

Note that (a) and (b) together imply that the intersection of any two elements of F is non-empty; (b) implies, by induction, that the intersection of a finite number of elements of F is in F; and (c) implies that $X \in F$.

We next give several examples of filters.

Examples of Filters

1. If $\emptyset \neq A \subset X$, then $\{D | A \subset D \subset X\}$ is a filter:

   (a) $\emptyset \notin F$ because $A \not\subset \emptyset$.

   (b) For $B, C \in F$, $A \subset B$ and $A \subset C$ imply that $A \subset B \cap C$, and thus $B \cap C \in F$.

   (c) For $B \in F$ and $B \subset C \subset X$, $A \subset B \subset C$ implies that $A \subset C$, and thus $C \in F$.

2. Let X be an infinite set. A subset A of X is called <u>cofinite</u> if its complement A^c in X is finite. Then $F = \{A \mid A \subset X$ and A is cofinite$\}$ is a filter:

(a) $\emptyset \notin F$ because $\emptyset$ is not cofinite .

(b) For $A, B \in F$, $(A \cap B)^c = A^c \cup B^c$. Hence, $(A \cap B)^c$ is the union of two finite sets, and thus $A \cap B$ is cofinite and in F .

(c) If $A \in F$ and $A \subset D \subset X$, then we necessarily have $D^c \subset A^c$, which implies that D^c is finite because A^c is finite. Thus D is cofinite and in F .

3. Let X be an infinite set with $|X| = \alpha$, and let $\beta \leq \alpha$ be an infinite cardinal number. Then $F = \{A \mid A \subset X$ and $|A^c| < \beta\}$ is a filter:

(a) $\emptyset \notin F$ because $|\emptyset^c| = |X| = \alpha \geq \beta$.

(b) For $A, B \in F$, $|(A \cap B)^c| = |A^c \cup B^c| \leq |A^c| + |B^c| < \beta + \beta = \beta$ because β is infinite.

(c) For $A \in F$, if $A \subset D \subset X$, then $D^c \subset A^c$ and thus $|D^c| \leq |A^c| < \beta$. Hence $D \in F$.

Note that Example 3 is a generalization of Example 2.

If X is a non-empty set, and $\{R_\alpha \mid \alpha \in X\}$ is a collection of rings, then $\prod_{\alpha \in X} R_\alpha$ is a ring. This is the first step in the process of using filters to construct new rings. Propositions 1.2 and 1.3 allow us to define an ideal I of $\prod_{\alpha \in X} R_\alpha$.

<u>Proposition 1.2.</u> Let X be a non-empty set, $\{R_\alpha \mid \alpha \in X\}$ a collection of rings, and F a filter on X . The relation on $\prod_{\alpha \in X} R_\alpha$

defined by $f \sim g$ if and only if $\{\alpha | f(\alpha) = g(\alpha)\} \in F$ is a congruence relation.

Proof. Fix $f, g, h \in \prod_{\alpha \in X} R_\alpha$.

(a) $\{\alpha | f(\alpha) = f(\alpha)\} = X \in F$, thus $f \sim f$.

(b) $f \sim g$ implies $\{\alpha | f(\alpha) = g(\alpha)\} = \{\alpha | g(\alpha) = f(\alpha)\} \in F$, and so $g \sim f$.

(c) $f \sim g$ and $g \sim h$ imply $\{\alpha | f(\alpha) = g(\alpha)\}$, $\{\alpha | g(\alpha) = h(\alpha)\} \in F$. The intersection $\{\alpha | f(\alpha) = g(\alpha) = h(\alpha)\}$ of these two sets is in F, and $\{\alpha | f(\alpha) = h(\alpha)\} \supset \{\alpha | f(\alpha) = g(\alpha) = h(\alpha)\}$ implies $\{\alpha | f(\alpha) = h(\alpha)\} \in F$. Thus $f \sim h$.

(d) $f \sim h$ implies $\{\alpha | f(\alpha) + g(\alpha) = h(\alpha) + g(\alpha)\} = \{\alpha | f(\alpha) = h(\alpha)\} \in F$, and thus $f + g \sim h + g$.

(e) $f \sim h$ implies $\{\alpha | f(\alpha) = h(\alpha)\} \in F$. Then $\{\alpha | f(\alpha)\, g(\alpha) = h(\alpha)\, g(\alpha)\} \supset \{\alpha | f(\alpha) = h(\alpha)\} \in F$ implies $fg \sim hg$. Similarly, $gf \sim gh$. ∎

Proposition 1.3. With X, F and $\{R_\alpha | \alpha \in x\}$ as in Proposition 1.2, define $I = \{f \in \prod_{\alpha \in X} R_\alpha | f \sim 0\}$. Then I is an ideal of $R = \prod_{\alpha \in X} R_\alpha$.

Proof. Since $X \in F$, $0 \in I$ because $\{\alpha | 0(\alpha) = 0\} = X \in F$. Thus I is non-empty. Fix $f, g \in I$ and $h \in R$.

(a) By definition, $A = \{\alpha | f(\alpha) = 0\}$, $B = \{\alpha | g(\alpha) = 0\} \in F$. Thus $\{\alpha | f(\alpha) - g(\alpha) = 0\} = \{\alpha | f(\alpha) = g(\alpha)\} \supset \{\alpha | f(\alpha) = g(\alpha) = 0\} = A \cap B \in F$. Hence, $f - g \in I$.

(b) Because $\{\alpha | h(\alpha)\, f(\alpha) = 0\} \supset \{\alpha | f(\alpha) = 0\} \in F$, we have that $\{\alpha | h(\alpha)\, f(\alpha) = 0\} \in F$ and hence $hf \in I$. Similarly, $fh \in I$. Thus, I is an ideal of R. ∎

Unless otherwise noted, we will hereafter consider X to be a non-empty set, $\mathcal{F}$ a filter on X, and R_α a commutative ring with identity.

With Proposition 1.3, we are ready to obtain a new ring R^* from a collection $\{R_\alpha \mid \alpha \in X\}$ of rings.

Definition 1.4. Define R^* to be the commutative ring with identity $\prod_{\alpha \in X} R_\alpha / I$, where $I = \{f \in \prod_{\alpha \in X} R_\alpha \mid f \sim 0\}$. If $R_\alpha = R$ for each $\alpha \in X$, then we write $R_{\mathcal{F}} = R^*$. Thus R^* is the set of equivalence classes of $\prod_{\alpha \in X} R_\alpha$ with respect to "$\sim$" described in Proposition 1.2, with

$$[f] + [g] = [f + g] \quad \text{and} \quad [f][g] = [fg] .$$

Notation. We will denote elements of R^* as a^* with the understanding that $a^* = a + I = \overline{\{a_\alpha\}} \in R^*$ for some $a = \{a_\alpha\} \in \prod_{\alpha \in X} R_\alpha$. Also, if $a \in R$, we will often write $a^* = \overline{\{a\}} \in R_{\mathcal{F}}$. It will be clear in context which is meant.

In the following chapters, we will be studying some properties of the ring $R_{\mathcal{F}}$ that we obtain from the ring R by means of Definition 1.4. The main inspiration for proceeding in this manner comes from Remark 1.25, which states that the mapping $\Phi\colon R \to R_{\mathcal{F}}$ defined by $\Phi(r) = r^*$ is a monomorphism. Until then, it is instructive to be as general as possible, so we will be examining the ring R^* generated by the collection $\{R_\alpha \mid \alpha \in X\}$ of rings.

A broader generalization is introduced here. If, for each $\alpha \varepsilon X$, M_α is an R_α-module, then define $M^* = \prod_{\alpha\varepsilon X} M_\alpha/N$, where $N = \{f \varepsilon \prod_{\alpha\varepsilon X} M_\alpha | f \sim 0\}$. The proof of Proposition 1.3 shows that N is an $\prod_{\alpha\varepsilon X} R_\alpha$-submodule of $\prod_{\alpha\varepsilon X} M_\alpha$. Thus M^* is well-defined.

<u>Proposition 1.5.</u> If each M_α is an R_α-module, then M^* is an R^*-module.

<u>Proof</u>. Define a function $\delta: R^* \times M^* \to M^*$ by $\delta(r^*, m^*) = r^*m^* = (rm)^*$. This is a well-defined function because for each $\alpha \varepsilon X$, $r_\alpha m_\alpha \varepsilon M_\alpha$, and thus $rm \varepsilon \prod_{\alpha\varepsilon X} M_\alpha$. In addition, if $r^* = s^* \varepsilon R^*$ and $m^* = n^* \varepsilon M^*$, then $(rm)^* = (sn)^*$, for $\{\alpha | r_\alpha m_\alpha = s_\alpha n_\alpha\} \supset \{\alpha | r_\alpha = s_\alpha$ and $m_\alpha = n_\alpha\} \supset \{\alpha | r_\alpha = s_\alpha\} \cap \{\alpha | m_\alpha = n_\alpha\}$, which is in F because both $\{\alpha | r_\alpha = s_\alpha\}$ and $\{\alpha | m_\alpha = n_\alpha\}$ are in F. Thus $\{\alpha | r_\alpha m_\alpha = s_\alpha n_\alpha\} \varepsilon F$, so $(rm)^* = (sn)^*$ and δ is well-defined. Let $r, s \varepsilon \prod_{\alpha\varepsilon X} R_\alpha$ and $m, n \varepsilon \prod_{\alpha\varepsilon X} M_\alpha$. For each $\alpha \varepsilon X$, $r_\alpha(m_\alpha + n_\alpha) = r_\alpha m_\alpha + r_\alpha n_\alpha$; this implies that $r(m + n) = rm + rn$, and thus $r^*(m^* + n^*) = r^*m^* + r^*n^* = (rm)^* + (rn)^*$. In addition, for each $\alpha \varepsilon X$, $(r_\alpha + s_\alpha)m_\alpha = r_\alpha m_\alpha + s_\alpha m_\alpha$; this implies that $(r + s)m = rm + sm$, and thus $(r^* + s^*)m^* = r^*m^* + s^*m^* = (rm)^* + (sm)^*$. Finally, for each α, $(r_\alpha s_\alpha)m_\alpha = r_\alpha(s_\alpha m_\alpha)$. This implies that $(rs)m = r(sm)$, and thus $(r^*s^*)m^* = r^*(s^* m^*) = r^*(sm)^*$. Hence, M^* is an R^*-module. ∎

Note that if, in addition, each M_α is a unitary R_α-module, then M^* is a unitary R^*-module: $1m^* = m^*$ because $1_{R_\alpha} m_\alpha = m_\alpha$ for each $\alpha \varepsilon X$.

The following proposition, though trivial, is a useful tool in our study of M^* .

<u>Proposition 1.6</u>. If for each $\alpha \in X$, N_α is a submodule of an R_α-module M_α , then there exists an R^*-module monomorphism $\delta\colon N^* \to M^*$.

<u>Proof</u>. If we set $I_N = \{f \in \prod_{\alpha \in X} N_\alpha | f \sim 0\}$ and $I_M = \{g \in \prod_{\alpha \in X} M_\alpha | g \sim 0\}$, then we may define $\delta\colon N^* \to M^*$ by $\delta(n + I_N) = n + I_M$. This is clearly a well-defined R^*-module homomorphism. It is injective for if $n + I_N \neq 0$, then $\{\alpha | n_\alpha = 0\} \notin F$, which is enough to ensure that $n + I_M \neq 0$. ∎

This proposition enables us to consider N^* to be a subset of M^* ; this quickly inspires the following three corollaries.

<u>Corollary 1.7.</u> If each N_α is a submodule of an R_α-module M_α , then N^* is a submodule of M^* .

<u>Proof</u>. By Proposition 1.5, N^* and M^* are R^*-modules and $N^* \subset M^*$ by Proposition 1.6 above. ∎

<u>Corollary 1.8.</u> If each I_α is an ideal of R_α , then I^* is an ideal of R^* .

<u>Proof</u>. By Corollary 1.7, I^* is a subgroup of R^* . For all $a^* \in I^*$ and $r^* \in R^*$, $a^*r^* \in I^*$ because $a_\alpha r_\alpha \in I_\alpha$ for each $\alpha \in X$. Thus I^* is an ideal of R^* . ∎

An obvious consequence of Corollary 1.8 is the following corollary.

<u>Corollary 1.9</u>. If I is an ideal of R , then I_F is an ideal of R_F .

The importance of Proposition 1.6 and its corollaries may be shown by considering the fact that if R is an integral domain with quotient field K, then we may consider $R_{\mathcal{F}}$ to be a subring of $K_{\mathcal{F}}$.

Proposition 1.10 is an extension of Corollary 1.7; it is an extremely useful result which will be continuously referred to throughout this paper.

<u>Proposition 1.10</u>. Let N be a submodule of an R-module M, and let $a^* \in M_{\mathcal{F}}$. Then $a^* \in N_{\mathcal{F}}$ if and only if $\{\alpha | a_\alpha \in N\} \in \mathcal{F}$.

<u>Proof</u>. Suppose that $a^* \in N_{\mathcal{F}}$. Then there exists some $b \in \prod_{\alpha \in X} N_\alpha$ such that $a^* = b^*$. This implies that $\{\alpha | a_\alpha \in N\} \supset \{\alpha | a_\alpha = b_\alpha\} \in \mathcal{F}$, and thus $\{\alpha | a_\alpha \in N\} \in \mathcal{F}$. Conversely, suppose that $\{\alpha | a_\alpha \in N\} \in \mathcal{F}$. Define $b \in \prod_{\alpha \in X} N$ by $b_\alpha = \begin{cases} a_\alpha & \text{if } a_\alpha \in N \\ 0 & \text{if } a_\alpha \notin N \end{cases}$. Clearly $b^* \in N_{\mathcal{F}}$ and $\{\alpha | a_\alpha = b_\alpha\} = \{\alpha | a_\alpha \in N\} \in \mathcal{F}$. Thus $a^* = b^*$ and $a^* \in N_{\mathcal{F}}$. ∎

The proof of Proposition 1.10 enables us to hereafter assume that if $a^* \in N_{\mathcal{F}} \subset M_{\mathcal{F}}$, then $a_\alpha \in N_\alpha$ for each $\alpha \in X$.

The following two results will probably be anticipated by those familiar with algebra or topology.

<u>Proposition 1.11</u>. Let N_1 and N_2 be submodules of an R-module M. Then $(N_1 \cap N_2)_{\mathcal{F}} = N_{1\mathcal{F}} \cap N_{2\mathcal{F}}$.

<u>Proof</u>. Pick $x^* \in (N_1 \cap N_2)_{\mathcal{F}}$. Then, by definition, $\{\alpha | x_\alpha \in N_1 \cap N_2\} \in \mathcal{F}$. Since $\{\alpha | x_\alpha \in N_1\} \supset \{\alpha | x_\alpha \in N_1 \cap N_2\} \in \mathcal{F}$, this implies that $x^* \in N_{1\mathcal{F}}$; similarly $x^* \in N_{2\mathcal{F}}$, and so $x^* \in N_{1\mathcal{F}} \cap N_{2\mathcal{F}}$.

Conversely, pick $y^* \in N_{1F} \cap N_{2F}$. We have $y^* \in N_{1F}$ and $y^* \in N_{2F}$, so $\{\alpha | y_\alpha \in N_1\}$ and $\{\alpha | y_\alpha \in N_2\}$ are both elements of F. Hence $\{\alpha | y_\alpha \in N_1\} \cap \{\alpha | y_\alpha \in N_2\} = \{\alpha | y_\alpha \in N_1 \cap N_2\}$ is in F. Thus, we have $y^* \in (N_1 \cap N_2)_F$. Hence $(N_1 \cap N_2)_F = N_{1F} \cap N_{2F}$. ∎

An immediate consequence of this result is the following corollary.

Corollary 1.12. If $N_1, N_2, \ldots, N_k$ are submodules of an R-module M, then $(\bigcap_{i=1}^{k} N_i)_F = \bigcap_{i=1}^{k} (N_{iF})$.

With Corollary 1.12 we have another relatively trivial result which will prove to be quite useful in the pages to come.

We next investigate the functorial aspects in the construction of the R*-module M^*. In the same way we construct M^* from a collection $\{M_\alpha | \alpha \in X\}$ of ring modules, we can construct an R*-module homomorphism f^* from a collection of module homomorphisms.

Proposition 1.13. For each $\alpha \in X$, let A_α and B_α be R_α-modules and $f_\alpha\colon A_\alpha \to B_\alpha$ be an R_α-module homomorphism. The function $f^*\colon A^* \to B^*$ defined by $f^*(a^*) = \{\overline{f_\alpha(a_\alpha)}\} \in B^*$ is an R*-module homomorphism.

Proof. If $a^* = c^* \in A^*$, then $W = \{\alpha | a_\alpha = c_\alpha\} \in F$. For each $\alpha \in W$, $f_\alpha(a_\alpha) = f_\alpha(c_\alpha)$, thus $\{\alpha | f_\alpha(a_\alpha) = f_\alpha(c_\alpha)\} \supset W \in F$. This implies that $\{\alpha | f_\alpha(a_\alpha) = f_\alpha(c_\alpha)\} \in F$ and so $f^*(a^*) = f^*(c^*)$ and f^* is thus well-defined. Fix $a^*, c^* \in A^*$ and $r^* \in R^*$. Since $f^*(a^* + c^*) = \{\overline{f_\alpha(a_\alpha + c_\alpha)}\} = \{\overline{f_\alpha(a_\alpha) + f_\alpha(c_\alpha)}\} = \{\overline{f_\alpha(a_\alpha)}\} + \{\overline{f_\alpha(c_\alpha)}\} = f^*(a^*) + f^*(c^*)$, f^* is a group homomorphism. Furthermore, because $f^*(r^*a^*) = \{\overline{f_\alpha(r_\alpha a_\alpha)}\} = \{\overline{r_\alpha f_\alpha(a_\alpha)}\} = \{\overline{r_\alpha}\}\{\overline{f_\alpha(a_\alpha)}\} = r^*f^*(a^*)$, f^* is an R*-module homomorphism. ∎

Proposition 1.13 is another straightforward result with many applications, among them the following four results.

Proposition 1.14. For each $\alpha \in X$, let A_α, B_α, and C_α be R_α-modules and $A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha$ an exact sequence of R_α-modules. Then $A^* \xrightarrow{f^*} B^* \xrightarrow{g^*} C^*$ is an exact sequence of R^*-modules.

Proof. First, if for each $\alpha \in X$, $A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha$ is an exact sequence, then $\prod_{\alpha\in X} A_\alpha \xrightarrow{f} \prod_{\alpha\in X} B_\alpha \xrightarrow{g} \prod_{\alpha\in X} C_\alpha$, where $f(a) = \{f_\alpha(a_\alpha)\}$ and $g(b) = \{g_\alpha(b_\alpha)\}$, is also an exact sequence. To show this, pick $b \in \operatorname{Im} f = \prod_{\alpha\in X} (\operatorname{Im} f_\alpha)$. Then for each α, $b_\alpha \in \operatorname{Im} f_\alpha = \operatorname{Ker} g_\alpha$; this implies $g_\alpha(b_\alpha) = 0$. Thus $b \in \operatorname{Ker} g$ and $\operatorname{Im} f \subset \operatorname{Ker} g$. Fix $k \in \operatorname{Ker} g$. Because $g(k) = 0$, for each $\alpha \in X, g_\alpha(k_\alpha) = 0$, and so $k_\alpha \in \ker g_\alpha = \operatorname{Im} f_\alpha$. For each α, pick an $a_\alpha \in A_\alpha$ such that $f_\alpha(a_\alpha) = k_\alpha$. If we then define $a = \{a_\alpha\}$, then $f(a) = \{f_\alpha(a_\alpha)\} = \{k_\alpha\} = k$. Thus $\ker g = \operatorname{Im} f$ and

$$\prod_{\alpha\in X} A_\alpha \xrightarrow{f} \prod_{\alpha\in X} B_\alpha \xrightarrow{g} \prod_{\alpha\in X} C_\alpha$$ is exact.

Now, pick $b^* \in \operatorname{Im} f^*$. Then there exists an $a \in \operatorname{Im} f$ such that $a^* = b^*$. Because $a \in \ker g$, $g^*(b^*) = g^*(a^*) = \overline{\{g_\alpha(a_\alpha)\}} = 0$. Thus $b^* \in \ker g^*$ and $\operatorname{Im} f^* \subset \ker g^*$. To finish, pick $b^* \in \ker g^*$. Define $a \in \prod_{\alpha\in X} B_\alpha$ as $a_\alpha = \begin{cases} b_\alpha & \text{if } b_\alpha \in \ker g \\ 0 & \text{otherwise.} \end{cases}$. Clearly $a \in \ker g$ and $a^* = b^*$ because $\{\alpha | a_\alpha = b_\alpha\} = \{\alpha | b_\alpha \in \ker g\} \in \mathcal{F}$. Since

$a \in \ker g = \operatorname{Im} f$, $a^* = b^* \in \operatorname{Im} f^*$. Thus $\ker g^* = \operatorname{Im} f^*$ and

$A^* \xrightarrow{f^*} B^* \xrightarrow{g^*} C^*$ is an exact sequence. ■

Corollary 1.15. Let R be a commutative ring with identity and M and N R-modules. Then $(M \oplus N)_{\mathcal{F}} \approx M_{\mathcal{F}} \oplus N_{\mathcal{F}}$ as $R_{\mathcal{F}}$-modules.

Proof. Clearly $0 \to M \xrightarrow{f} M \oplus N \xrightarrow{g} N \to 0$, where $f(m) = (m, 0)$ and $g((m, n)) = n$ is a short exact sequence. By Proposition 1.14, $0 \to M_{\mathcal{F}} \xrightarrow{f^*} (M \oplus N)_{\mathcal{F}} \xrightarrow{g^*} N_{\mathcal{F}} \to 0$ is a short exact sequence. Define a function $k\colon M \oplus N \to M$ by $k((m, n)) = m$. Clearly, k is well-defined and $k^* f^* = 1_{M_{\mathcal{F}}}$. Hence the sequence is split exact, and so

$M_{\mathcal{F}} \oplus N_{\mathcal{F}} \approx (M \oplus N)_{\mathcal{F}}$. ■

Remark. Corollary 1.15 implies that if $\{M_n\}$ is a finite collection of R-modules, then $(\bigoplus_n M_n)_{\mathcal{F}} \approx \bigoplus_n (M_{n_{\mathcal{F}}})$. In general, however, $(\bigoplus_k M_k)_{\mathcal{F}}$ is not isomorphic to $\bigoplus_k (M_{k_{\mathcal{F}}})$ when $\{M_k\}$ is an infinite collection of R-modules (see Remark 4.9).

Propositions 1.16 and 1.17 anticipate our work in the remaining chapters.

Proposition 1.16. The function from the category of commutative rings with identity to itself given by $R \to R_{\mathcal{F}}$ is a covariant functor.

Proof. If R is a commutative ring with identity, then so is $R_{\mathcal{F}}$. Let $f\colon A \to B$, where A and B are commutative rings with identity, be a ring homomorphism. Define $f^*\colon A_{\mathcal{F}} \to B_{\mathcal{F}}$ by $f^*(a^*) = \overline{\{f(a_\alpha)\}} \in B_{\mathcal{F}}$. By Proposition 1.13, f^* is a group homomorphism, and because, for each $a^*, c^* \in A_{\mathcal{F}}$,

$f^*(a^*c^*) = \{\overline{f(a_\alpha c_\alpha)}\} = \{\overline{f(a_\alpha)\ f(c_\alpha)}\} = \{\overline{f(a_\alpha)}\}\{\overline{f(c_\alpha)}\} = f^*(a^*)\ f^*(c)^*$, f^* is a ring homomorphism. Let $i\colon A \to A$ be the identity map. Then $i^*\colon A_F \to A_F$ is the identity map because $i^*(a^*) = \{\overline{i(a_\alpha)}\} = \{\overline{a_\alpha}\} = a^*$. Let $f\colon A \to B$ and $g\colon B \to C$ be ring homomorphisms. We see that $(g \circ f)^* = g^* \circ f^*$ because $(g \circ f)^*(a^*) = \{\overline{(g \circ f)(a_\alpha)}\} = \{\overline{g(f(a_\alpha))}\} = g^*\{\overline{f(a_\alpha)}\} = g^* \circ f^*(a^*)$ for each $a^* \in A_F$. This shows that we have a covariant functor. ∎

<u>Proposition 1.17</u>. Let R be a commutative ring with identity. The function from the category of R-modules to the category of R_F-modules defined by $M \to M_F$ is an additive, exact covariant functor.

<u>Proof</u>. By our work in Proposition 1.13 and Proposition 1.16, this is a covariant functor. Proposition 1.14 tells us that it is an exact functor. It is pre-additive because each $\mathrm{Hom}_R(A, B)$ is an additive abelian group, and $(f + g)^*(a^*) = \{\overline{(f + g)(a_\alpha)}\} = \{\overline{f(a_\alpha) + g(a_\alpha)}\} = \{\overline{f(a_\alpha)}\} + \{\overline{g(a_\alpha)}\} = f^*(a^*) + g^*(a^*)$ shows that it is an additive functor. ∎

We now define the ultrafilter and the ultraproduct of a collection of rings.

<u>Definition 1.18</u>. An <u>ultrafilter</u> is a filter F which is maximal in the natural ordering of filters on a set X given by $F_1 \leq F_2$ if and only if $F_1 \subseteq F_2$. If F is an ultrafilter, then R^* is called an <u>ultraproduct</u> of the R_α's . If $R_\alpha = R$ for each $\alpha \in X$, then we call R_F an <u>ultrapower</u> of R .

From this point on, we will be studying ultraproducts almost exclusively, for the stability of ultraproducts as opposed to non-ultraproducts provides a solid basis for research. Propositions 1.21 and 1.22 are the prime examples of what we mean.

The following proposition is important because it implies that there exists an ultrafilter $\mathcal{F}$ on every non-empty set X.

Proposition 1.19. Every filter $\mathcal{F}$ on a set X is contained in an ultrafilter $\mathcal{F}'$ on X.

Proof. Let $\mathcal{F}$ be a filter on X. Let $\mathcal{C} = \{A \mid A$ is a filter on X and $\mathcal{F} \subseteq A\}$. As it contains $\mathcal{F}$, $\mathcal{C}$ is non-empty. Partially order $\mathcal{C}$ by inclusion, and let $\mathcal{B} \subset \mathcal{C}$ be a chain. Then $K = \bigcup_{W \varepsilon \mathcal{B}} W$ is a filter on X: fix $S_1, S_2 \varepsilon K$ and $S_1 \subset D \subset X$.

(a) $\emptyset \notin K$ because $\emptyset \notin W$ for each $W \varepsilon \mathcal{B}$.

(b) There exist $W_1, W_2 \varepsilon \mathcal{B}$ such that $S_1 \varepsilon W_1$ and $S_2 \varepsilon W_2$. $\mathcal{B}$ is totally ordered, so suppose that $W_1 \subseteq W_2$. Then $S_1 \varepsilon W_2$ and thus $S_1 \cap S_2 \varepsilon W_2 \subset K$.

(c) If $S_1 \subset D$, then $D \varepsilon W_1 \subset K$ by definition of a filter. Thus, K is a filter. Clearly K is an upper bound of $\mathcal{B}$ in $\mathcal{C}$ and it follows from Zorn's Lemma that $\mathcal{C}$ has a maximal element $\mathcal{F}'$, which will clearly be an ultrafilter that contains $\mathcal{F}$. ∎

We now present some equivalent conditions for a filter to be an ultrafilter. Proposition 1.20 is a generalization of a result by Herstein [3, Lemma 7.3.1].

Proposition 1.20. For a filter $\mathcal{F}$ on a set X, the following statements are equivalent.

(a) $\mathcal{F}$ is an ultrafilter.

(b) $A \cup B \in \mathcal{F}$ implies that $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(c) For $A \subset X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

<u>Proof</u>. (a) $\Rightarrow$ (b) Suppose that $A \cup B = C \in \mathcal{F}$. Without loss of generality, let us suppose that $B \notin \mathcal{F}$. For $S \in \mathcal{F}$, $\emptyset \neq S \cap C = S \cap (A \cup B) = (S \cap A) \cup (S \cap B) \in \mathcal{F}$. This implies that we cannot have $S \cap A = \emptyset$, for otherwise $S \cap B = S \cap C \in \mathcal{F}$, and $S \cap B \subset B$ would imply $B \in \mathcal{F}$, a contradiction. Define $\mathcal{F}' = \mathcal{F} \cup \{D \mid S \cap A \subset D \text{ for some } S \in \mathcal{F}\}$. Clearly $\mathcal{F} \subseteq \mathcal{F}'$, and $X \in \mathcal{F}$ implies that $X \cap A = A \in \mathcal{F}'$. We claim that $\mathcal{F}'$ is a filter.

(a) Since for each $S \in \mathcal{F}$, $S \cap A \neq \emptyset$, we conclude $\emptyset \notin \mathcal{F}'$.

(b) Fix $S_1, S_2 \in \mathcal{F}'$.

(1) If $S_1, S_2 \in \mathcal{F}$, then $S_1 \cap S_2 \in \mathcal{F} \subseteq \mathcal{F}'$.

(2) If $S_1 \supset W_1 \cap A$ and $S_2 \supset W_2 \cap A$ for some W_1 , $W_2 \in \mathcal{F}$, then $S_1 \cap S_2 \supset (W_1 \cap A) \cap (W_2 \cap A) = (W_1 \cap W_2) \cap A \in \mathcal{F}'$. Thus, $S_1 \cap S_2 \in \mathcal{F}'$.

(3) If $S_1 \in \mathcal{F}$ and $S_2 = W_2 \cap A$ for some $W_2 \in \mathcal{F}$, then $S_1 \cap S_2 \supset S_1 \cap (W_2 \cap A) = (S_1 \cap W_2) \cap A \in \mathcal{F}'$. Thus $S_1 \cap S_2 \in \mathcal{F}'$.

(c) For $S \in \mathcal{F}'$, suppose $S \subset D \subset X$.

(1) If $S \in \mathcal{F}$, then $D \in \mathcal{F} \subseteq \mathcal{F}'$.

(2) If $S \supset W \cap A$ for some $W \in \mathcal{F}$, then $D \supset S \supset W \cap A$ and so $D \in \mathcal{F}'$.

This shows that $\mathcal{F}'$ is a filter that contains A . Since $\mathcal{F}$ is an ultrafilter and $\mathcal{F} \subseteq \mathcal{F}'$, we must have $\mathcal{F} = \mathcal{F}'$, and thus $A \in \mathcal{F}'$.

(b) $\Rightarrow$ (c) For each $A \subset X$, $A \cup A^c = X \in \mathcal{F}$, and thus either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

(c) $\Rightarrow$ (a) Suppose there exists a filter F' properly containing F. Pick any $T \in F' - F$ and $A \in F \subseteq F'$. Set $\emptyset \neq A \cap T = B \in F'$. Now, $B \notin F$ because $B \subset T$ and otherwise $T \in F$; thus $B^c \in F \subseteq F'$. But we had $B \in F'$, and then $\emptyset = B \cap B^c \in F'$, a contradicition. Hence, F is an ultrafilter. ∎

Now, some examples of ultrafilters. It is instructive to compare these with the examples of filters presented following Definition 1.1.

Examples of Ultrafilters

(a) A principal ultrafilter is a filter of the form $\{P | P \subset X$ and $a \in P$ for some fixed $a \in X\}$. By Proposition 1.20(b), any ultrafilter that contains a finite subset of X contains a singleton, and is thus principal. In particular, if X is a finite set, then any ultrafilter on X is a principal ultrafilter.

(b) A non-principal ultrafilter is any ultrafilter which is not principal. By our remark above, a non-principal ultrafilter on a set X may not contain a finite subset of X; hence, Proposition 1.20(c) implies that a non-principal ultrafilter on X contains every cofinite subset of X.

We next give two ring-theoretic conditions on R^* which are equivalent to the filter F being an ultrafilter. Proposition 1.21 gives us our first concrete idea of the significant differences between ultraproducts and non-ultraproducts.

Proposition 1.21. A filter F is an ultrafilter if and only if for each $0 \neq a^* \in R^*$ there exists a $b^* \in R^*$ such that $a^* = b^*$ and $b_\alpha \neq 0$ for each $\alpha \in X$.

Proof. (⇒) Pick $0 \neq a^* \in R^*$. Since $\{\alpha | a_\alpha = 0\} \notin F$ and F is an ultrafilter, we have $\{\alpha | a_\alpha = 0\}^c = \{\alpha | a_\alpha \neq 0\} \in F$. Define $b^* \in R^*$ by $b_\alpha = \begin{cases} a_\alpha & \text{if } a_\alpha \neq 0 \\ 1_{R_\alpha} & \text{otherwise} \end{cases}$. Then $\{\alpha | a_\alpha = b_\alpha\} = \{\alpha | a_\alpha \neq 0\} \in F$ and so $a^* = b^*$.

(⇐) Let $A \subset X$. Without loss of generality, we may assume that $A^c \notin F$. Define $a^* \in R^*$ by $a_\alpha = \begin{cases} 1_{R_\alpha} & \text{if } \alpha \in A \\ 0 & \text{otherwise} \end{cases}$. Since $\{\alpha | a_\alpha = 0\} = A^c \notin F$, $a^* \neq 0$ and there exists, by hypothesis, a $b^* \in R^*$ such that $a^* = b^*$ and $b_\alpha \neq 0$ for each $\alpha \in X$. Because $a^* = b^*$, $A = \{\alpha | a_\alpha \neq 0\} \supset \{\alpha | a_\alpha = b_\alpha\} \in F$, and thus $A \in F$. This implies, by Proposition 1.20, that F is an ultrafilter. ∎

If F is an ultrafilter, then for $0 \neq a^* \in R^*$, we may now assume that $a_\alpha \neq 0$ for each $\alpha \in X$.

Proposition 1.21 gives our future research a tremendous push; without this result, the proofs of numerous propositions in the following pages are almost unbearably complicated. The next proposition by itself is justification enough for discontinuing any consideration of non-ultraproducts.

Proposition 1.22. Let each R_α be an integral domain. Then R^* is an integral domain if and only if F is an ultrafilter.

Proof. First note that for each $a, b \in \prod_{\alpha \in X} R_\alpha$, $\{\alpha | a_\alpha = 0\} \cup \{\alpha | b_\alpha = 0\} = \{\alpha | a_\alpha b_\alpha = 0\}$. If α is such that $a_\alpha = 0$

or $b_\alpha = 0$, then $a_\alpha b_\alpha = 0$. Conversely, if α is such that $a_\alpha b_\alpha = 0$, then, since R_α is, by hypothesis, an integral domain, $a_\alpha = 0$ or $b_\alpha = 0$.

($\Rightarrow$) Suppose that F is not an ultrafilter. Then by Proposition 1.20, there exists an $A \cup B = C \in F$ such that neither A nor B is an element of F . Fix $a^*, b^* \in R^*$ such that $\{\alpha | a_\alpha = 0\} = A$ and $\{\alpha | b_\alpha = 0\} = B$ (we can find such elements of R^*: consider $a \in \prod_{\alpha \in X} R_\alpha$ where $a_\alpha = \begin{cases} 1_{R_\alpha} & \alpha \notin A \\ 0 & \alpha \in A \end{cases}$) . Clearly both a^* and b^* are non-zero. Since $A \cup B = \{\alpha | a_\alpha = 0\} \cup \{\alpha | b_\alpha = 0\} = \{\alpha | a_\alpha b_\alpha = 0\} = C \in F$, we have $a^*b^* = 0$, a contradiction of our hypothesis that R^* is an integral domain. Thus, F is an ultrafilter.

($\Leftarrow$) Let F be an ultrafilter. Suppose that $a^*, b^* \in R^*$ are such that $a^*b^* = 0$, which implies that $\{\alpha | a_\alpha b_\alpha = 0\} \in F$. By our remark above, $\{\alpha | a_\alpha = 0\} \cup \{\alpha | b_\alpha = 0\} = \{\alpha | a_\alpha b_\alpha = 0\} \in F$, and since F is an ultrafilter, we must have $\{\alpha | a_\alpha = 0\} \in F$ or $\{\alpha | b_\alpha = 0\} \in F$ by Proposition 1.20. This implies that $a^* = 0$ or $b^* = 0$. Thus R^* is a commutative ring with identity with no non-zero zero divisors; hence R^* is an integral domain. ∎

The following proposition is an interesting variation of Proposition 1.22. First, a lemma.

Lemma 1.23. If F is an ultrafilter and $A \in F$, then there exists an ultrafilter F' on A such that $F' \subseteq F$.

Proof. Define $F' = \{B \in F | B \subset A\}$. This is a collection of subsets of A and $A \in F'$ since $A \in F$. Clearly $F' \subseteq F$. F' is a filter:

(a) $\emptyset \notin F'$, for $F' \subseteq F$ and $\emptyset \notin F$.

(b) For $D, C \in F' \subseteq F$, $D \cap C \in F$ and clearly $D \cap C \subset A$, hence $D \cap C \in F'$.

(c) If $D \in F'$ and $D \subset G \subset A \in F$, then $G \in F$ and thus $G \in F'$.

Hence, F' is a filter. It is an ultrafilter: suppose $W \in F'$ and $C \cup D = W$. Since $W \in F$ and F is an ultrafilter, either $C \in F$ or $D \in F$ by Proposition 1.20. Since $W \subset A$, we must have $C \subset A$ and $D \subset A$; thus $C \in F'$ or $D \in F'$. By Proposition 1.20, this implies that F' is an ultrafilter. ∎

<u>Proposition 1.24</u>. Let F be an ultrafilter. The following statements are equivalent.

(a) R^* is an integral domain.

(b) $\{\alpha | R_\alpha$ is an integral domain$\} \in F$.

(c) R^* is isomorphic to an ultraproduct of integral domains.

<u>Proof</u>. (a) ⇒ (b) . Patently $\{\alpha | R_\alpha$ is an integral domain$\} \cup \{\alpha | R_\alpha$ is not an integral domain$\} = X \in F$, thus, since F is an ultrafilter, Proposition 1.20 implies that if $A = \{\alpha | R_\alpha$ is an integral domain$\}$, then either $A \in F$ or $A^c \in F$. Without loss of generality, suppose that $A^c = \{\alpha | R_\alpha$ is not an integral domain$\} \in F$. For each $\alpha \in A^c$, there exist non-zero $b_\alpha, c_\alpha \in R$ such that $b_\alpha c_\alpha = 0$.

Define $b^* = \{\overline{b(\alpha)}\}$ where $b(\alpha) = \begin{cases} b_\alpha & \alpha \in A^c \\ 1_{R_\alpha} & \alpha \in A \end{cases}$. Define c^* similarly.

Clearly b^* and c^* are non-zero but $b^*c^* = 0$ because $\{\alpha | b(\alpha)c(\alpha) = 0\} = A^c \in F$. This contradicts our assumption that R^* is an integral domain. Thus $A^c \notin F$ and $A = \{\alpha | R_\alpha$ is an integral domain$\} \in F$.

(b) $\Rightarrow$ (c) Set $A = \{\alpha | R_\alpha$ is an integral domain$\}$. Since, by hypothesis, $A \in F$, Lemma 1.23 implies that there exists an ultrafilter $F' = \{B \in F | B \subset A\}$ on A such that $F' \subseteq F$. For each $\alpha \in A \subset X$, set $S_\alpha = R_\alpha$; each S_α is an integral domain. Let S^* be the ultraproduct of the S_α , $\alpha \in A$, on F' . Define a mapping δ: $R^* \to S^*$ by $\{\overline{a_\alpha}\}_{\alpha \in X} \to \{\overline{a_\alpha}\}_{\alpha \in A}$. If $\{\overline{a_\alpha}\}_{\alpha \in X} = \{\overline{c_\alpha}\}_{\alpha \in X}$, then $\{\alpha | a_\alpha = c_\alpha\} \in F$ and thus $A \cap \{\alpha | a_\alpha = c_\alpha\} \in F'$. Thus $\{\overline{a_\alpha}\}_{\alpha \in A} = \{\overline{c_\alpha}\}_{\alpha \in A}$, and the map is well-defined. It is clearly a surjective homomorphism. Since $F' \subseteq F$, if $A \cap \{\alpha \in X | a_\alpha = 0\} \in F'$, then $\{\alpha \in X | a_\alpha = 0\} \supset \{\alpha \in A | a_\alpha = 0\} \in F$; this implies that the map is injective. Thus R^* is isomorphic to S^* , which is an ultraproduct of integral domains.

(c) $\Rightarrow$ (a) This is a straightforward application of Proposition 1.22. ∎

On the following remarks rest the foundations for the remainder of this paper.

Remark 1.25. Let R be a commutative ring with identity and F an ultrafilter. Define a homomorphism Φ: $R \to R_F$ by $\Phi(r) = \{\overline{r}\} = r^*$.

(a) Φ is always injective. For if $0 \neq r \in R$, then r^* is such that $\{\alpha | r_\alpha = 0\} = \emptyset \notin F$; thus $r^* \neq 0$. Hence $\ker \Phi = \{0\}$ and Φ is injective. Thus we may identify R as a subring of R_F .

(b) Let F be a principal ultrafilter, that is, there exists a $\beta \in X$ such that F consists of each subset of X that contains β . Pick $a^* \in R_F$, and suppose that $a_\beta = b \in R$. Then $a^* = b^*$ because $\{\alpha | a_\alpha = b\} \supset \{\beta\} \in F$. Thus Φ will be surjective and an isomorphism. Any ultrafilter F on a finite set X is necessarily principal and thus $R \approx R_F$.

We will hereafter assume that F is a non-principal ultrafilter on (a necessarily infinite set) X . To avoid set-theoretic complications we will assume that $X = \{1, 2, \ldots\} = \mathbb{N}$, although most of the remaining results are valid when X is any infinite set.

(c) Suppose that $R = \{r_1, \ldots, r_n\}$. For $b^* \varepsilon R$, $\bigcup_{i=1}^{n} \{\alpha | b_\alpha = r_i\} = X$. Since F is an ultrafilter, we may apply Proposition 1.20(b) $n - 1$ times and conclude that for some $1 \leq i \leq n$, $\{\alpha | b_\alpha = r_i\} \varepsilon F$, and thus $b^* = r_i^*$ and Φ is surjective.

Hereafter, we will assume that R is an infinite commutative ring with identity.

(d) Let K be a countably infinite proper subset of R: $K = \{k_1, k_2, \ldots\} \varepsilon R$. Define $a^* \varepsilon R$ by $a^* = \{\overline{k_n}\}$. Now for $b \varepsilon R - K$, $\{n | a_n = b\} = \emptyset \notin F$, which implies $a^* \neq b^*$; for $k_n \varepsilon K$, $\{n | a_n = k_n\} = \{n\} \notin F$ because F is a non-principal ultrafilter, and hence $a^* \neq k_n^*$. This implies that Φ is not surjective.

The next proposition is a continuation of Proposition 1.24.

<u>Proposition 1.26</u>. If each R_n is an integral domain with quotient field K_n , then R^* is an integral domain with quotient field K^* .

<u>Proof</u>. By Proposition 1.22, R^* is an integral domain and thus has a quotient field $L = \{a^*b^{*-1} | a^*, b^* \varepsilon R^* \text{ and } b^* \neq 0\}$. By Proposition 1.21, if $b^* \neq 0$, then we may assume $b_n \neq 0$ for each $n \varepsilon \mathbb{N}$. Thus $L = \{a^*b^{*-1} | a^*, b^* \varepsilon R^* \text{ and } b_n \neq 0 \text{ for each } n \varepsilon \mathbb{N}\}$. By definition, $K^* = \{(ab^{-1})^* | a, b \varepsilon \prod_{n \varepsilon \mathbb{N}} R_n \text{ and } b_n \neq 0 \text{ for each } n \varepsilon \mathbb{N}\}$. Define a mapping $\delta\colon L \to K^*$ by $\delta(a^*b^{*-1}) \to (ab^{-1})^*$. This map is

well-defined, for if $a^*b^{*-1} = c^*d^{*-1}$, then, since R^* is an integral domain, we must have that $a^*d^* - b^*c^* = 0$, and thus $\{n | a_n d_n - b_n c_n = 0\} \in F$. This implies that $\{n | \frac{a_n d_n - b_n c_n}{b_n d_n} = 0\} =$ $= \{n | (ab^{-1})_n - (cd^{-1})_n = 0\} \in F$; hence $(ab^{-1})^* = (cd^{-1})^*$ and δ is well-defined. The map is a homomorphism, for $\delta(a^*b^{*-1} + c^*d^{*-1}) =$ $\delta((a^*d^* + b^*c^*)((b^*d^*)^{-1}) = \delta((ad + bc)^*(bd)^{*-1}) = [(ad + bc)(bd)^{-1}]^* =$ $(ab^{-1} + cd^{-1})^* = (ab^{-1})^* + (cd^{-1})^*$. Furthermore, $\delta(a^*b^{*-1} \cdot c^*d^{*-1}) =$ $\delta((ac)^* (bd)^{*-1}) = [(ac)(bd)^{-1}]^* = [(ac)(b^{-1}d^{-1})]^* = [(ab^{-1})(cd^{-1})]^* =$ $(ab^{-1})^*(cd^{-1})^*$. It is an injection, for if $a^*b^{*-1} \neq 0$, then, since $b^* \neq 0$, we must have $a^* \neq 0$. By Proposition 1.21, we may assume that $a_n \neq 0$ for each $n \in \mathbb{N}$. Then $\delta(a^*b^{*-1}) = (ab^{-1})^*$ and $\{n | (ab^{-1})_n \neq 0\} = \{n | a_n \neq 0\} = \mathbb{N} \in F$; this implies that $(ab^{-1})^* \neq 0$ and so the mapping is injective. It is a surjection. Let $(ab^{-1})^* \in K^*$, with $a, b \in \prod_{n \in \mathbb{N}} R_n$ and $b_n \neq 0$ for each $n \in \mathbb{N}$. By the definition of R^* , clearly $\delta(a^*b^{*-1}) = (ab^{-1})^*$. Thus $L \approx K^*$ and R^* has K^* as its quotient field. ∎

We thus have the following diagram of inclusions, where K is the quotient field of R and K_F the quotient field of R_F .

$$\begin{array}{ccc} K & \subset & K_F \\ \cup & & \cup \\ R & \subset & R_F \end{array}$$

The final result of this chapter is a generalization of Proposition 1.26.

<u>Proposition 1.27</u>. If S is the set of non-zero divisors of R, then the total quotient ring of $R_{\mathcal{F}}$ is $(S^{-1}R)_{\mathcal{F}}$, where $S^{-1}R$ is the total quotient ring of R.

<u>Proof</u>. We must first establish the following claim: for $0 \neq a^* \in R$, a^* is not a zero divisor if and only if $\{n | a_n \in S\} \in \mathcal{F}$.

<u>Proof of claim</u>. ($\Rightarrow$) Suppose $a^* \in R$ is not a zero divisor. Clearly $\{n | a_n \in S\} \cup \{n | a_n \notin S\} = \mathbb{N} \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, Proposition 1.20 implies that either $\{n | a_n \in S\}$ or $\{n | a_n \notin S\}$ is contained in $\mathcal{F}$. Let us suppose that $D = \{n | a_n \notin S\} \in \mathcal{F}$. For each $n \in D$ there exists an $0 \neq b_n \in R$ such that $a_n b_n = 0$. Define

$c^* \in R$ by $c_n = \begin{cases} b_n & \text{if } n \in D \\ 1 & \text{otherwise} \end{cases}$. Clearly $c^* \neq 0$ and $a^*c^* = 0$

because $\{n | a_n c_n = 0\} = \{n | a_n \notin S\} = D \in \mathcal{F}$. This contradicts our assumption that a^* is not a zero divisor. Thus we must conclude $D \notin \mathcal{F}$ and hence $\{n | a_n \in S\} \in \mathcal{F}$.

($\Leftarrow$) Suppose that $W = \{n | a_n \in S\} \in \mathcal{F}$. Define $b^* \in R_{\mathcal{F}}$ by

$b_n = \begin{cases} a_n & n \in W \\ 1 & \text{otherwise} \end{cases}$. Clearly $b^* \neq 0$ and b^* is not a zero

divisor because b_n is not a zero divisor for each $n \in \mathbb{N}$; furthermore $b^* = a^*$ because $\{n | a_n = b_n\} = \{n | a_n \in S\} \in \mathcal{F}$. Thus a^* is not a zero divisor and our claim is proved.

Now, let T be the total quotient ring of $R_{\mathcal{F}}$: $T = \{r^*s^{*-1} | r^*, s^* \in R_{\mathcal{F}}$ and s^* is not a zero divisor$\}$. If $s^* \in R_{\mathcal{F}}$ is not a zero divisor, then by our claim above, $\{n | s_n \in S\} \in \mathcal{F}$ and thus $s^* \in S_{\mathcal{F}}$. By Proposition 1.10, this enables us to assume that $s_n \in S$ for each $n \in \mathbb{N}$. Recalling that

$(S^{-1}R)_{\mathcal{F}} = \{(rs^{-1})^* \mid r \in \prod_{n\in\mathbb{N}} R\ ,\ \ s \in \prod_{n\in\mathbb{N}} S\}$, we can define a mapping $\delta\colon\ T \to (S^{-1}R)_{\mathcal{F}}$ by $\delta(r^*\, s^{*-1}) \to (rs^{-1})^*$. That δ is a well-defined isomorphism follows from the proof of Proposition 1.26. Hence, the total quotient field of $R_{\mathcal{F}}$ is $(S^{-1}R)_{\mathcal{F}}$. ∎

CHAPTER 2

RING-THEORETIC PROPERTIES OF ULTRAPOWERS

By Remark 1.25, when $\mathcal{F}$ is a non-principal ultrafilter and R an infinite commutative ring with identity, the mapping $\Phi: R \to R_{\mathcal{F}}$ defined by $\Phi(r) = r^*$ is a non-surjective monomorphism; thus we may consider R to be a proper subset of $R_{\mathcal{F}}$. In this chapter, we will consider certain ring-theoretic properties and try to find necessary and sufficient conditions on R for $R_{\mathcal{F}}$ to have these properties. Proposition 2.1, which states that $R_{\mathcal{F}}$ is a field if and only if R is a field, is the key to the solution of this problem, for the ring-theoretic properties we will consider here are all such that a field will trivially possess these properties. Proposition 2.1 thus gives us an intuitive idea of how to proceed: we assume that R (and thus $R_{\mathcal{F}}$) is not a field and that $R_{\mathcal{F}}$ has some specific ring-theoretic property; we will either reach a contradiction, or, because $R \subset R_{\mathcal{F}}$, show that R also has this property. If a contradiction is reached, then Proposition 2.1 implies that $R_{\mathcal{F}}$ will have this specific property if and only if R is a field. On the other hand, if we show that R has this property, then, because $R_{\mathcal{F}}$ was originally not a field, it should be enough to assume that R has this property for $R_{\mathcal{F}}$ to possess it also. Thus R will have this property if and only if $R_{\mathcal{F}}$ does. It will be seen that things work out in exactly this manner.

In the latter part of this chapter, we will examine some relationships between the ideals of R and those of $R_{\mathcal{F}}$; while these results are of sufficient interest to stand by themselves, they are crucial to the development of Chapter 3.

Proposition 2.1. R is a field if and only if $R_{\mathcal{F}}$ is a field.

Proof ($\Rightarrow$) By Proposition 1.22, $R_{\mathcal{F}}$ is an integral domain. Fix any $0 \neq a^* \in R_{\mathcal{F}}$. We may assume each $a_n \neq 0$. Define $b \in \prod_{n\in\mathbb{N}} R$ by $b_n = a_n^{-1}$. Clearly $a_n b_n = 1$ for each n and thus $a^*b^* = 1$. Hence, $R_{\mathcal{F}}$ is a field.

($\Leftarrow$) By Proposition 1.24, R is an integral domain. Pick any $0 \neq a \in R$. Because $R_{\mathcal{F}}$ is a field, a^* has an inverse and so there exists a $b^* \in R_{\mathcal{F}}$ such that $a^*b^* = 1$, and thus $\{n \mid ab_n = 1\} \in \mathcal{F}$. Since $\emptyset \notin \mathcal{F}$, we must conclude that there exists an element c of R such that $ac = 1$. Thus R is a field. ∎

Note that Propositions 2.1 and 1.26 together imply that if R is an integral domain with quotient field K, then $R_{\mathcal{F}}$ is a field if and only if $R_{\mathcal{F}} = K_{\mathcal{F}}$.

Proposition 2.1 is a double edged sword. Firstly, if each $0 \neq r \in R$ has an inverse, then so will each $0 \neq s^* \in R_{\mathcal{F}}$. Secondly, and perhaps more importantly, if there is a certain ring theoretic property that $R_{\mathcal{F}}$ will not possess unless it is a field, then Proposition 2.1 implies that $R_{\mathcal{F}}$ will possess this property if and only if R is a field.

We next present a very useful lemma.

Lemma 2.2. For $f^*, g^* \in R_{\mathcal{F}}$, f^* divides g^* if and only if $\{n\colon\ f_n | g_n\} \in \mathcal{F}$.

Proof. ($\Rightarrow$) If f^* divides g^*, then there exists a $h^* \in R_{\mathcal{F}}$ such that $f^*h^* = g^*$. This implies that $\{n \mid f_n h_n = g_n\} \in \mathcal{F}$, and thus $\{n\colon\ f_n | g_n\} \supset \{n \mid f_n h_n = g_n\} \in \mathcal{F}$. Hence $\{n\colon\ f_n | g_n\} \in \mathcal{F}$.

($\Leftarrow$) Suppose $A = \{n\colon\ f_n | g_n\} \in \mathcal{F}$. For each $n \in A$ there exists a $k_n \in R$ such that $f_n k_n = g_n$. Define $h^* \in R_{\mathcal{F}}$ by $h_n = \begin{cases} k_n & n \in A \\ 1 & \text{otherwise} \end{cases}$. Then $\{n | f_n h_n = g_n\} = A \in \mathcal{F}$ and thus $f^* h^* = g^*$. ∎

This lemma is similar to many results in Chapter 1, such as Proposition 1.10. Like so many simple results in this paper, it will be referred to again and again in the following pages.

Our next result is our first proposition of the form "$R_{\mathcal{F}}$ possesses a certain ring-theoretic property if and only if R is a field." Note that, as discussed in the introduction, we first assume that R is not a field and reach a contradiction.

<u>Proposition 2.3</u>. Let R be an integral domain. Then $R_{\mathcal{F}}$ satisfies the ascending chain condition on principal ideals if and only if R is a field.

<u>Proof</u>. ($\Rightarrow$) Suppose that R is not a field. Pick $0 \neq d \notin U(R)$, where $U(R)$ is the multiplicative group of units of R . For $n = 0, 1, 2, \ldots$, define $k_n^* \in R_{\mathcal{F}}$ by $k_n(m) = \begin{cases} 1 & m \leq n \\ d^{m-n} & m > n \end{cases}$. If we define for each n , a $z_n^* \in R_{\mathcal{F}}$ by $z_n(m) = \begin{cases} 1 & m \leq n \\ d & m > n \end{cases}$, we see that $z_n^* k_{n-1}^* = k_n^*$. When $m \leq n$, $k_{n+1}(m) = 1$, $z_n(m) = 1$ and $k_n(m) = k_{m+1}(m)\ z_n(m) = 1$; when $m > n$ $k_{n+1}(m) = d^{m-(n+1)}$ and so $k_m(m) = k_{n+1}(m)\ z_n(m) = d^{m-(n+1)} \cdot d = d^{m-n}$. Hence, $k_{n+1}^* | k_n^*$. Since $\{m | z_0(m) \in U(R)\} = \emptyset$ and for $n = 1,2,\ldots, \{m | z_n(m) \in U(R)\} = \{1, 2, \ldots, n\} \notin \mathcal{F}$, $z_n^* \notin U(R_{\mathcal{F}})$ for each n , and so k_n^* does not

divide k^*_{n+1} because R_F is an integral domain. Thus we have an infinite sequence of distinct principal ideals

$(k^*_0) \subsetneqq (k^*_1) \subsetneqq \ldots \subsetneqq (k^*_n) \subsetneqq \ldots$, which shows that R_F does not satisfy the ascending chain condition on principal ideals.

($\Leftarrow$) If R is a field, then, by Proposition 2.1, R_F is a field and thus satisfies the ascending chain condition on principal ideals. ∎

Corollary 2.4. Let R be an integral domain. Then R_F is Noetherian if and only if R is a field.

Proof. ($\Rightarrow$) If R_F is Noetherian, then it satisfies the ascending chain condition on principal ideals. Proposition 2.3 implies that R is a field.

($\Leftarrow$) If R is a field, then R_F is a field by Proposition 2.1 and thus R_F is Noetherian. ∎

Note that since every principal ideal domain R is Noetherian, this corollary implies that R_F is a principal ideal domain if and only if R is a field; the same reasoning holds for Dedekind domains.

Definition 2.5. Let R be an integral domain with quotient field K . Then R is integrally closed if whenever $x \in K$ is such that $x^n + a_1x^{n-1} + \ldots + a_n = 0$ for some $n \geq 1$ and $a_i \in R$, $1 \leq i \leq n$, then $x \in R$.

The following proposition, our first example of a result of the form "R has a certain property if and only if R_F does," is typical of results of this type. The proof is straightforward and uncomplicated.

Proposition 2.6. R is integrally closed if and only if $R_{\mathcal{F}}$ is integrally closed.

Proof. ($\Rightarrow$) Suppose there exists an $n \geq 1$, $a_i^* \in R_{\mathcal{F}}$ for $1 \leq i \leq n$, and $x^* \in K_{\mathcal{F}}$ such that $x^{*n} + a_1^* x^{*n-1} + \ldots + a_n^* = 0$. Then $A = \{m | x(m)^n + a_1(m)x(m)^{n-1} + \ldots + a_n(m) = 0\} \in \mathcal{F}$. For each $m \in A$, $x(m) \in R$ because R is, by hypothesis, integrally closed. Then $\{m | x(m) \in R\} \supset A \in \mathcal{F}$, which implies that $x^* \in R$, and thus $R_{\mathcal{F}}$ is integrally closed.

($\Leftarrow$) Fix $x \in K$, $n \geq 1$, and $a_i \in R$, $1 \leq i \leq n$. Suppose that $x^n + a_1 x^{n-1} + \ldots + a_n = 0$. This implies that $x^{*n}\ a_1^*\ x^{*n-1} + \ldots + a_n^* = 0$ in $R_{\mathcal{F}}$. By hypothesis, $R_{\mathcal{F}}$ is integrally closed and thus $x^* \in R_{\mathcal{F}}$, which implies that $x \in R$. ∎

Definition 2.7. Let R be an integral domain with quotient field K. Then R is <u>completely integrally closed</u> if for $x \in K$, whenever there exists an $0 \neq d \in R$ such that $dx^n \in R$ for all $n \geq 1$, then $x \in R$.

It is well known that if R is completely integrally closed then R is integrally closed.

Proposition 2.8. $R_{\mathcal{F}}$ is completely integrally closed if and only if R is a field.

Proof. ($\Rightarrow$) By Proposition 1.24, $R_{\mathcal{F}}$ is an integral domain. Suppose that R is not a field. Fix $0 \neq d \notin U(R)$ and define $k^* \in K_{\mathcal{F}}$ by $k_m = d^{-1}$. Define $x^* \in K_{\mathcal{F}}$ by $x_m = d^m$. Clearly $x^* \in R_{\mathcal{F}}$ and $k^* \notin R_{\mathcal{F}}$. For $n \geq 1$, $k_m^n = d^{-n}$ and so $x_m\, k_m^n = d^m d^{-n} = d^{m-n}$. Thus when $m \geq n$, $x_m\, k_m^n \in R$. This implies that

$\{m | k_m^n x_m \in R\} = \{m | m \geq n\} \in \mathcal{F}$ because $\{m | m \geq n\}$ is a cofinite set. Thus for each $n \geq 1$, $k^{*n} x^* \in R_{\mathcal{F}}$ but $k^* \notin R_{\mathcal{F}}$, and hence $R_{\mathcal{F}}$ is not completely integrally closed, a contradiction of our hypothesis. Thus, R must be a field.

($\Leftarrow$) If R is a field, then $R_{\mathcal{F}}$ is a field and hence is completely integrally closed. ■

If R is a completely integrally closed ring which is not a field, then $R_{\mathcal{F}}$ will be integrally closed but not completely integrally closed.

Proposition 2.9. $R_{\mathcal{F}}$ is a unique factorization domain if and only if R is a field.

Proof. ($\Rightarrow$) If $R_{\mathcal{F}}$ is a UFD, then $R_{\mathcal{F}}$ satisfies the ascending chain condition on principal ideals; by Proposition 2.3, $R_{\mathcal{F}}$ is a field.

($\Leftarrow$) If R is a field, then by Proposition 2.1, $R_{\mathcal{F}}$ is a field and thus a UFD. ■

An alternative proof of Proposition 2.9 is obtained by noting that unique factorization domains are completely integrally closed and applying Proposition 2.8.

Definition 2.10. An integral domain R with quotient field K is seminormal if for $x \in K$, whenever $x^2, x^3 \in R$, then $x \in R$.

Seminormal integral domains are significant because they are precisely the integral domains R for which the natural map $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X])$ is an isomorphism [1, Theorem 1.6].

Proposition 2.11. R is seminormal if and only if $R_{\mathcal{F}}$ is seminormal.

Proof. ($\Rightarrow$) Pick $x^* \varepsilon K_{\mathcal{F}}$. Suppose that ${x^*}^2, {x^*}^3 \varepsilon R_{\mathcal{F}}$. Then $\{n|x_n^2 \varepsilon R\}$ and $\{n|x_n^3 \varepsilon R\} \varepsilon \mathcal{F}$, and thus so is their intersection $A = \{n|x_n^2, x_n^3 \varepsilon R\}$. For each $n \varepsilon A$, $x_n \varepsilon R$ because R is seminormal. Thus $\{n|x_n \varepsilon R\} \supset A \varepsilon \mathcal{F}$ and so $x^* \varepsilon R_{\mathcal{F}}$. Thus $R_{\mathcal{F}}$ is seminormal.

($\Leftarrow$) Suppose $R_{\mathcal{F}}$ is seminormal. Pick $x \varepsilon K$ such that $x^2, x^3 \varepsilon R$. We have $x^* \varepsilon K_{\mathcal{F}}$ and ${x^*}^2, {x^*}^3 \varepsilon R_{\mathcal{F}}$. Since $R_{\mathcal{F}}$ is, by hypothesis, seminormal, $x^* \varepsilon R_{\mathcal{F}}$. Because $\emptyset \neq \{n|x \varepsilon R\} \varepsilon \mathcal{F}$, we have $x \varepsilon R$, and thus R is seminormal. ∎

Definition 2.12. An ideal I of an integral domain R with quotient field K is invertible if there are elements $a_1, \ldots, a_n \varepsilon I$ and $q_1, \ldots, q_n \varepsilon K$ such that

(a) $q_i I \subset R$ for each q_i, $1 \leq i \leq n$.

(b) $\sum_{i=1}^{n} q_i a_i = 1$.

If I is invertible as above, then $I = (a_1, \ldots, a_n)$. If we let $J = (q_1, \ldots, q_n)$, then $IJ = R$ and $J = I^{-1} = \{x \varepsilon K|xI \subset R\}$.

Definition 2.13. An integral domain R is a Prüfer domain if every finitely generated non-zero ideal of R is invertible.

Note that a Prüfer domain is a Dedekind domain if and only if it is Noetherian.

The following result is due to Heitmann and Levy [2].

Proposition 2.14. Let R be an integral domain with quotient field K. Then R is a Prüfer domain if and only if $R_{\mathcal{F}}$ is a Prüfer domain.

Proof. ($\Rightarrow$) By Proposition 1.26, $R_{\mathcal{F}}$ is an integral domain with quotient field K. Let $x_1^*, \ldots, x_m^* \in R_{\mathcal{F}}$ be non-zero. Because R is a Prüfer domain, for each $n \in \mathbb{N}$ there exist $y_{1n}, y_{2n}, \ldots, y_{mn} \in K$ such that $\sum_{i=1}^{m} x_{in}\, y_{in} = 1$ and $y_{in}\, x_{jn} \in R$ for $1 \leq i, j \leq m$. For each i, $1 \leq i \leq m$, set $y_i^* = \{\overline{y_{in}}\} \in K_{\mathcal{F}}$. Then

$\sum_{i=1}^{m} x_i^*\, y_i^* = 1$ because $\sum_{i=1}^{m} x_{in}\, y_{in} = 1$ for each $n \in \mathbb{N}$. Furthermore, $y_i^*\, x_j^* \in R_{\mathcal{F}}$, $1 \leq i, j \leq m$, because $y_{in}\, x_{jn} \in R$ for each n. Hence, by definition, $I = (x_1^*, \ldots, x_m^*)$ is an invertible ideal, and thus $R_{\mathcal{F}}$ is a Prüfer domain.

($\Leftarrow$) Pick non-zero $x_1, \ldots, x_m \in R$. Since $R_{\mathcal{F}}$ is, by hypothesis, a Prüfer domain, there exist $y_1^*, \ldots, y_m^* \in K_{\mathcal{F}}$ such that

$\sum_{i=1}^{m} x_i^*\, y_i^* = 1$ and $y_i^*\, x_j^* \in R_{\mathcal{F}}$, $1 \leq i, j \leq m$. Hence

$\{n \mid \sum_{i=1}^{m} x_i y_{in} = 1\}$ and $\{n \mid y_{in}\, x_j \in R\}$, $1 \leq i, j \leq m$, are contained in $\mathcal{F}$; and thus $D = \{n \mid \sum_{i=1}^{m} x_i y_{in} = 1\} \cap [\bigcap_{\substack{i=1\\ j=1}}^{m} \{n \mid y_{in}\, x_j \in R\}] \in \mathcal{F}$.

Pick any $n \in D$ and define $k_i \in K$, $1 \leq i \leq m$, as $k_i = y_{in}$. Clearly $\sum_{i=1}^{m} x_i k_i = 1$ and $k_i x_j \in R$, $1 \leq i, j \leq m$. Thus $(x_1, \ldots, x_m)$ is invertible and so R is a Prüfer domain. ∎

Since each Dedekind domain is a Prüfer domain, if R is a Dedekind domain which is not a field, then $R_{\mathcal{F}}$ is a Prüfer domain but not Dedekind.

<u>Definition 2.15</u>. An integral domain R is a <u>Bézout domain</u> if every finitely generated ideal in R is a principal ideal.

A Bézout domain is Noetherian if and only if it is a principal ideal domain.

<u>Proposition 2.16</u>. R is a Bezout domain if and only if R_F is a Bézout domain.

<u>Proof</u>. ($\Rightarrow$) For $1 \leq i \leq m$, pick $a_i^* \in R_F$. Since R is Bézout, for each $n \in \mathbb{N}$ there exists a $c_n \in R$ and k_{in}, $1 \leq i \leq m$, such that $\sum_{i=1}^{m} a_{in} k_{in} = c_n$ and for any $r_{1n}, r_{2n}, \ldots, r_{mn} \in R$ there corresponds an $r_n \in R$ such that $\sum_{i=1}^{m} a_{in} r_{in} = r_n c_n$. Defining $c^* = \{\overline{c_n}\}$ and $k_i^* = \{\overline{k_{in}}\}$, $1 \leq i \leq m$, we claim $(a_1^*, \ldots, a_m^*) = (c^*)$. Pick $r_1^*, \ldots, r_m^* \in R_F$. For each $n \in \mathbb{N}$, there exists an $r_n \in R$ such that $\sum_{i=1}^{m} a_{in} r_{in} = r_n c_n$. Defining $r^* = \{\overline{r_n}\} \in R_F$, we clearly have $\sum_{i=1}^{m} a_i^* r_i^* = r^* c^*$, and hence $(a_1^*, a_2^*, \ldots, a_m^*) \subset (c^*)$. Conversely, since $\sum_{i=1}^{m} a_{in} k_{in} = c_n$ for each $n \in \mathbb{N}$, we clearly have $\sum_{i=1}^{m} a_i^* k_i^* = c^*$, and hence $(c^*) = (a_1^*, \ldots, a_m^*)$. Since the a_i^*'s were arbitrarily selected, this implies that every finitely generated ideal of R_F is a principal ideal; hence R_F is a Bézout domain.

($\Rightarrow$) Pick $x_1, \ldots, x_m, r_1, \ldots, r_m \in R$. Since, by hypothesis, R is Bézout, there exist $c^*, r^* \in R_F$ and $k_i^* \in R_F$, $1 \leq i \leq m$, such that $\sum_{i=1}^{m} x_i^* k_i^* = c^*$ and $\sum_{i=1}^{m} x_i^* r_i^* = r^* c^*$. This implies that both $\{n \mid \sum_{i=1}^{m} x_i k_{in} = c_n\}$ and $\{n \mid \sum_{i=1}^{m} x_i r_i = r_n c_n\}$ are in F ; hence their intersection W is an element of F . Fixing $n \in W$, let $c' = c_n$, $r' = r_n$ and $k_i' = k_{in}$, $1 \leq i \leq m$. Then $\sum_{i=1}^{m} x_i r_i = r'c'$ and $\sum_{i=1}^{m} x_i k_i' = c'$. Because the r_i's were selected arbitrarily, this implies that $(x_1, \ldots, x_m) = (c')$; hence, every finitely generated ideal in R is a principal ideal, and thus R is a Bézout domain. ■

Note that if R is a principal ideal domain, then it is a Bézout domain. This proposition implies that if R is a principal ideal domain that is not a field, then R_F is a Bézout domain but not a principal ideal domain.

Notice that the Propositions 2.14 and 2.16 indicate that there exists a stable relationship between the finitely generated ideals of R and the finitely generated ideals of R_F .

The remaining results of this chapter will figure prominently in Chapter 3.

Definition 2.17. R is a Boolean ring if for each $a \in R$, $a^2 = a$.

Proposition 2.18. R_F is a Boolean ring if and only if R is a Boolean ring.

Proof. Pick $a^* \in R_F$. For each $n \in \mathbb{N}$, $a_n^2 = a_n$, thus $\{n \mid a_n^2 = a_n\} = \mathbb{N} \in F$. Hence, $a^{*2} = a^*$, and R_F is a Boolean ring.

($\Rightarrow$) Pick $a \in R$. Since R_F is Boolean by hypothesis, $a^{*2} = a^*$ and so $\{n \mid a_n^2 = a_n\} = \{n \mid a^2 = a\} \in F$. This implies, since F

does not contain the null set, that $a^2 = a$. Hence R is a Boolean ring. ∎

<u>Definition 2.19</u>. R is <u>Von Neumann regular</u> if for each $a \in R$ there exists a $b \in R$ such that $a^2 b = a$.

Notice that a Boolean ring is Von Neumann regular. A Von Neumann regular ring that is an integral domain is a field because if $a \neq 0$ and $a^2 b = a$, then $ab = 1$.

<u>Proposition 2.20</u>. $R_\mathcal{F}$ is Von Neumann regular if and only if $R_\mathcal{F}$ is Von Neumann regular.

<u>Proof</u>. ($\Rightarrow$) Pick $a^* \in R_\mathcal{F}$. For each $n \in \mathbb{N}$, there exists a $b_n \in R$ such that $a_n^2 b_n = a_n$. Define $b^* = \overline{\{b_n\}} \in R_\mathcal{F}$. Clearly ${a^*}^2 b^* = a^*$, and therefore $R_\mathcal{F}$ is Von Neumann regular.

($\Leftarrow$) Pick $a \in R$. Since $R_\mathcal{F}$ is, by hypothesis, Von Neumann regular, there exists a $b^* \in R_\mathcal{F}$ such that ${a^*}^2 b^* = a^*$. This implies that $A = \{n | a^2 b_n = a\} \in \mathcal{F}$. Pick any $n \in A$ and set $c = b_n$. Then $a^2 c = a$, and R is Von Neumann regular. ∎

<u>Definition 2.21</u>. An integral domain R is a <u>valuation domain</u> if and only if for any $a, b \in R$, then $a|b$ or $b|a$.

Some well-known equivalent conditions for an integral domain R to be a valuation domain are a) R has quotient field K and for $x \in K$, either $x \in R$ or $x^{-1} \in R$; and b) the ideals of R are totally ordered. Note that the latter alternate condition implies that a valuation domain is a local ring.

An interesting property of a completely integrally closed valuation domain which is not a field is that it contains exactly one non-trivial prime ideal (See Proposition 3.6).

Proposition 2.22. R is a valuation domain if and only if R_F is a valuation domain.

Proof. ($\Rightarrow$) By Proposition 1.22, R_F is an integral domain. Pick $h^*, k^* \in R_F$. Because R is a valuation domain, for each $n \in \mathbb{N}$, $h_n | k_n$ or $k_n | h_n$. Thus $\{n: h_n | k_n\} \cup \{n: k_n | h_n\} = \mathbb{N} \in F$, and since F is an ultrafilter, Proposition 1.20 implies that $\{n: h_n | k_n\} \in F$ or $\{n: k_n | h_n\} \in F$. Lemma 2.2 implies that $h^* | k^*$ or $k^* | h^*$, and hence R_F is a valuation domain.

($\Leftarrow$) Pick $a, b \in R$. Because R_F is a valuation domain, a^* divides b^* or b^* divides a^*. Without loss of generality, suppose that $a^* | b^*$. Then, by Lemma 2.2, $\emptyset \neq \{n: a | b\} \in F$, which implies that $a | b$, and hence R is a valuation domain. ∎

We now present some results that provide the foundations for our work in Chapter 3.

Proposition 2.23. If I is an ideal of R, then $(R/I)_F \approx R_F/I_F$.

Proof. It is clear that $0 \to I \to R \to R/I \to 0$ is an exact sequence of R-modules, thus, by Proposition 1.14, $0 \to I_F \to R_F \to (R/I)_F \to 0$ is an exact sequence of R_F-modules. Hence $R_F/I_F \approx (R/I)_F$ as R_F-modules. Patently, this is also a ring isomorphism, and so $(R/I)_F \approx R_F/I_F$. ∎

The importance of the following two corollaries is immediately apparent.

Corollary 2.24. If P is a prime ideal of R, then P_F is a prime ideal of R_F.

Proof. Since P is prime, R/P is an integral domain. Proposition 1.22 implies that $(R/P)_F \approx R_F/P_F$ is an integral domain, and thus P_F is a prime ideal of R_F. ∎

Corollary 2.25. If M is a maximal ideal of R, then M_F is a maximal ideal of R_F.

Proof. Since M is maximal, R/M is a field and Proposition 2.1 implies that $(R/M)_F \approx R_F/M_F$ is a field. Thus M_F is a maximal ideal of R_F. ∎

Definition 2.26. If I is an ideal of R, then we define the radical of I, denoted $\mathrm{Rad}\ I$, to be the intersection of all the prime ideals P which contain I. If $\{P \mid P$ is a prime ideal of R and $I \subset P\} = \emptyset$, then $\mathrm{Rad}\ I$ is defined to be R. The radical of the zero ideal is called the nilradical of R and is denoted $\mathrm{nil}\ R$. It is the intersection of all the prime ideals of R. By [4, VIII 2.6], $\mathrm{nil}\ R = \{x \in R \mid x^n = 0$ for some $n > 0\}$.

The relationship between the nilradical of R and the nilradical of R_F is described in Proposition 2.28.

Definition 2.27. The Jacobson radical of R, denoted $J(R)$, is defined to be the intersection of all maximal ideals of R. It is well known that $J(R) = \{r \in R \mid 1 - rx \in U(R)$ for each $x \in R\}$.

Note that $\mathrm{nil}\ R \subset J(R)$ for any ring R.

It is instructive to compare the differences in Propositions 2.28 and 2.29.

<u>Proposition 2.28</u>. nil $R_F \subset (\text{nil } R)_F$. Moreover, equality holds if and only if there exists a fixed n such that $x^n = 0$ for all $x \in \text{nil } R$.

<u>Proof</u>. nil $R = \{x | x^n = 0$ for some $n > 0\}$ and nil $R_F = \{x^* | x^{*n} = 0$ for some $n > 0\}$. Pick $x^* \in \text{nil } R_F$. Then $x^{*n} = 0$ for some n implies that $\{m | x_m^n = 0\} \in F$. Since $\{m | x_m \in \text{nil } R\} \supset \{m | x_m^n = 0\} \in F$, $x^* \in (\text{nil } R)_F$. Thus nil $R \subset (\text{nil } R)_F$.

($\Rightarrow$) Suppose equality holds. Suppose there exists no fixed n such that $x^n = 0$ for each $x \in \text{nil } R$. This implies that if $a \in \text{nil } R$ and $n > 1$ is such that $a^n = 0$, but $a^{n-1} \neq 0$, then there exists a $b \in \text{nil } R$ and $m > n$ such that $b^m = 0$, $b^{m-1} \neq 0$. Pick $a_1 \in \text{nil } R$ and suppose $n_1 > 0$ is such that $a_1^{n_1} = 0$ but $a_1^{n_1-1} \neq 0$. We can pick $a_2 \in \text{nil } R$ and $n_2 > n_1$ such that $a_n^{n_2} = 0$ but $a_2^{n_2-1} \neq 0$. Repeating this for $m = 2, 3, \ldots$ we can define $a^* = \{\overline{a_m}\} \in (\text{nil } R)_F$. Now, for any $k > 0$, $a^{*k} \neq 0$ because at most k of the a_m^k's will be zero; this implies that $a^* \notin \text{nil } R_F$. But we had nil $R_F = (\text{nil } R)$, so we have reached a contradiction. Thus there exists an $n \geq 1$ such that $x^n = 0$ for each $x \in \text{nil } R$.

($\Leftarrow$) Pick $x^* \in (\text{nil } R)_F$. We may assume $x_m \in \text{nil } R$ for each $m \in \mathbb{N}$. Pick an n such that $x^n = 0$ for each $x \in \text{nil } R$. Then $x^{*n} = 0$ because $x_m^n = 0$ for each m . Thus $x^* \in \text{nil } R_F$, and we have equality. ∎

<u>Proposition 2.29</u>. $J(R)_F = J(R_F)$.

<u>Proof</u>. We first show that $U(R_F) = U(R)_F$. Pick $x^* \in U(R_F)$. Then, by definition, there exists a $y^* \in R_F$ such that $x^*y^* = 1$, which implies that $W = \{n | x_n y_n = 1\} \in F$. Define a $w^* \in R_F$ by

$$w_n = \begin{cases} x_n & \text{if } n \in W \\ 1 & \text{otherwise} \end{cases}$$

. Clearly $w^* \in U(R)$ and $w^* = x^*$ for $\{n | w_n = x_n\} \supset W \in F$. Thus $U(R_F) \subset U(R)_F$; the converse is obvious. To show $J(R)_F = J(R_F)$, pick $r^* \in J(R_F)$. For each $x^* \in R_F$, $1 - r^*x^* \in U(R_F)$, by definition. Suppose $r^* \notin J(R)_F$; then $W = \{n | r_n \notin J(R)\} \in F$ by Proposition 1.10. For each $n \in W$, there exists an $x_n \in R$ such that $1 - r_n x_n \notin U(R)$. Define $y^* \in R_F$ by

$$y_n = \begin{cases} x_n & n \in W \\ 1 & \text{otherwise} \end{cases}$$

. Now, $\{n | 1 - r_n y_n \notin U(R)\} \supset W \in F$, which implies that $1 - r^*y^* \notin U(R)_F = U(R_F)$, which is a contradiction of our assumption that $r^* \in J(R_F)$. Thus, we have $r^* \in J(R)_F$, and hence, $J(R_F) \subset J(R)_F$. Conversely, let $r^* \in J(R)_F$. We may assume $r_n \in J(R)$ for each $n \in \mathbb{N}$. Pick $x^* \in R_F$. For each n , $1 - r_n x_n \in U(R)$, which implies that $1 - r^*x^* \in U(R)_F = U(R_F)$, and thus $r^* \in J(R_F)$ by definition. Hence, $J(R_F) = J(R)_F$. ∎

This result plays an important role in the proof of the next proposition.

<u>Proposition 2.30</u>. If R is semilocal with maximal ideals $M_1, M_2, \ldots, M_n$, then R_F is semilocal with maximal ideals $M_{1F}, M_{2F}, \ldots, M_{nF}$.

<u>Proof</u>. By Corollary 2.16, $M_{1F}, \dots, M_{nF}$ are maximal ideals in R_F. Let N be any maximal ideal in R_F; we will show that $N = M_{iF}$ for some i, $1 \leq i \leq n$. By Proposition 2.29, $J(R_F) = J(R)_F$, so we have $J(R)_F = (\bigcap_{i=1}^{n} M_i)_F = \bigcap_{i=1}^{n} M_{iF} = J(R_F)$. Thus, by definition, $\bigcap_{i=1}^{n} M_{iF} \subset N$. Since N is maximal and thus prime, this implies that $M_{iF} \subset N$ for some i, $1 \leq i \leq n$; thus $N = M_{iF}$. Hence, R_F is semilocal with maximal ideals $M_{1F}, M_{2F}, \dots, M_{nF}$. ■

<u>Corollary 2.31</u>. If R is a local ring with maximal ideal M, then R_F is local with maximal ideal M_F.

The following remark shows that Proposition 2.30 cannot be generalized.

<u>Remark</u>. Notice that if M_n, $n = 1, 2, \dots$, are distinct maximal ideals of R, then M^* is an ideal of R_F by Corollary 1.8. Fix $r^* \varepsilon R_F - M^*$. Assuming that $r_n \varepsilon M_n$ for each n, there exists an $x_n \varepsilon R$ such that $1 - r_n x_n \varepsilon M_n$ because each M_n is a maximal ideal of R. If we set $x^* = \{\overline{x_n}\} \varepsilon R_F$, then clearly $1 - r^*x^* \varepsilon M^*$, which implies that M^* is a maximal ideal. Furthermore, if we fix an n, for each $m \neq n$ there exists an $x_m \varepsilon M_m - M_n$. Defining $y^* \varepsilon M^*$ by $y_m = \begin{cases} x_m & m \neq n \\ 0 & m = n \end{cases}$, we see that $y^* \notin M_{nF}$ by Proposition 1.10, for $\{m | y_m \varepsilon M_n\} = \{n\} \notin F$. This implies that $M^* \neq M_{nF}$ for each n.

CHAPTER 3

THE DIMENSION OF ULTRAPOWERS

In this chapter we will examine some relationships between the prime ideals of R and the prime ideals of R_F. We first define the dimension of a ring R and note that the dimension of a given ring is always less than or equal to the dimension of its ultrapower. We then give several examples of rings R for which $\dim R = \dim R_F$, and, finally, we show that if $0 \neq \dim R < \infty$, then $\dim R < \dim R_F$.

Definition 3.1. Let P be a prime ideal of R. Define the height of P as $\operatorname{ht} P = \sup\{n \mid \text{there exist prime ideals } P_0 \subsetneqq P_1 \subsetneqq \cdots P_n = P\}$, and the dimension of R as $\dim R = \sup\{\operatorname{ht} P \mid P \text{ is a prime ideal of } R\}$.

Remark 3.2. Since a field contains only the trivial prime ideal 0, $\dim R = 0$ if R is a field. If R is an integral domain which is not a field, then $\dim R \geq 1$ because R contains a non-trivial maximal ideal M, which will be prime.

Furthermore, if $P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$ are distinct prime ideals in R, then by Corollary 2.24, $P_{0_F} \subsetneqq P_{1_F} \subsetneqq \cdots \subsetneqq P_{n_F}$ are distinct prime ideals in R_F. This implies that for any ring R, $\dim R \leq \dim R_F$.

If P is any prime ideal in R, then there will exist a maximal ideal M containing P, whence $\operatorname{ht} M \geq \operatorname{ht} P$. Thus, if $\dim R = n$, then there exists a maximal ideal M of R such that $\operatorname{ht} M = \dim R = n$.

Since $\dim R \leq \dim R_{\mathcal{F}}$, the natural question to ask is: When does $\dim R = \dim R_{\mathcal{F}}$? As a first example, if R is a field, then $R_{\mathcal{F}}$ is a field by Proposition 2.1. Hence, if R is a field, then $\dim R = \dim R_{\mathcal{F}}$. Proposition 3.3 gives us two more examples in which equality is attained.

Proposition 3.3. If R is Von Neumann regular, then $\dim R = 0$.

Proof. We need to show that every maximal ideal in R contains no prime ideals other than itself. Let M be a maximal ideal in R and $P \subset M$ a prime ideal. Now, R/P is an integral domain and clearly it is Von Neumann regular. Since every Von Neumann regular integral domain is a field, R/P is a field and hence P is a maximal ideal in R . Thus $P = M$ and $\dim R = 0$. ∎

Since every Boolean ring is Von Neumann regular, if R is a Boolean ring, Proposition 3.3 implies that $\dim R = 0$.

If R is Von Neumann regular, then Proposition 2.20 implies that $R_{\mathcal{F}}$ is Von Neumann regular; in addition, if R is Boolean, $R_{\mathcal{F}}$ is Boolean by Proposition 2.18. Hence, we have two more examples where $\dim R = \dim R_{\mathcal{F}}$.

We have so far presented three examples of a ring R where $\dim R = \dim R_{\mathcal{F}}$. In each of these cases R is a zero-dimensional ring. Our next proposition shows that the fact that $\dim R = 0$ does not guarantee that $\dim R_{\mathcal{F}} = 0$; an example of a zero-dimensional ring with a non-zero dimensional ultrapower follows. First, we present a lemma.

Lemma 3.4. A local ring R has dimension zero if and only if each non-unit of R is nilpotent.

Proof. Let M be the unique maximal ideal of R.

($\Rightarrow$) M is prime and $\dim R = 0$ implies that M is the only prime ideal in R; thus $\text{nil } R = \{r \mid r^n = 0 \text{ for some } n \geq 1\} = M$. Therefore, every $m \in M$ is nilpotent. Since R is a local ring, this implies that every non-unit is nilpotent.

($\Leftarrow$) Suppose that M is a nil ideal. Then $\text{nil } R = \{r \mid r^n = 0 \text{ for some } n \geq 1\} = M$ because M consists of all non-units and a unit of R is never a zero divisor and thus never nilpotent. If Q were any other prime ideal of R, then we would have $\text{nil } R = \bigcap\{P \mid P \text{ is a prime ideal of } R\} \subset Q \subsetneqq M$, a contradiction. Thus $Q = M$ and $\dim R = 0$. ∎

Because the maximal ideal M of a local ring R consists precisely of the non-units of R, then $\dim R = 0$ if and only if $M = \text{nil } R$.

Note that this lemma implies that a Boolean ring R which is not a field is not local. If R is Boolean, then Propositions 3.3 and 2.18 imply that $\dim R = \dim R_{\mathcal{F}} = 0$. Thus, if $0 \neq a \in R$ is a non-unit, then $a^n = a$ for any $n \geq 1$, which implies that a is not nilpotent.

Proposition 3.5. Let R be a zero-dimensional local ring with maximal ideal M. Then $\dim R_{\mathcal{F}} = 0$ if and only if $(\text{nil } R)_{\mathcal{F}} = \text{nil } R_{\mathcal{F}}$. Thus $\dim R_{\mathcal{F}} = 0$ if and only if there is a fixed n such that $x^n = 0$ for each $x \in M$.

Proof. ($\Rightarrow$) By Corollary 2.31, $R_{\mathcal{F}}$ is a local ring with maximal ideal $M_{\mathcal{F}}$. By Lemma 3.4 $M_{\mathcal{F}} = \text{nil } R_{\mathcal{F}}$. Because R is local and $\dim R = 0$, $\text{nil } R = M$ and $(\text{nil } R)_F = M_F = \text{nil } R_F$.

($\Leftarrow$) Since $\dim R = 0$ and R is local, $M = \text{nil } R$ by Lemma 3.4 and hence $M_F = (\text{nil } R)_F = \text{nil } R_F$ by hypothesis. Since R_F is local, this implies that $\dim R_F = 0$. The last statement of the proposition follows from Proposition 2.28. ∎

As an example of a zero-dimensional local ring R such that $\dim R_F > 0$, consider the ring $R = K[x_1, x_2, \ldots]/(x_1^2, x_2^3, \ldots, x_n^{n+1}, \ldots)$ where K is a field. Now, R is local. For if P is any prime ideal of R containing $I = (x_1^2, x_2^3, \ldots, x_n^{n+1}, \ldots)$, then we have $x_n^{n+1} \varepsilon P$ for each n . Since P is prime, this implies that $x_n \varepsilon P$ for each n , and so $(x_1, \ldots, x_n, \ldots) \subset P$. Since K is a field, $(x_1, \ldots, x_n, \ldots)$ is a maximal ideal of $K[x_1, \ldots, x_n, \ldots]$, and so $P = (x_1, \ldots, x_n, \ldots)$. This implies that R is a local ring and $\dim R = 0$. Now, for each n , $x_n^n + I \neq 0$ but $x_n^{n+1} + I = 0$, thus there does not exist a fixed $n > 0$ such that $x^n = 0$ for each $x \varepsilon \text{nil } R$. By Proposition 3.5, this implies that $\dim R_F > 0$.

We now switch our attention to rings R where the dimension of R is finite but non-zero. Again, we are concerned about whether or not $\dim R = \dim R_F$. Our next proposition, with the aid of two results from Chapter 2, gives us an example of a ring R where $0 \neq \dim R < \infty$ and $\dim R < \dim R_F$.

<u>Proposition 3.6</u>. Let R be a valuation domain. Then R is completely integrally closed if and only if $\dim R \leq 1$.

<u>Proof</u>. ($\Rightarrow$) If R is a field (and thus trivally completely integrally closed), then $\dim R = 0$. So suppose that R is not a

field. As noted in Remark 3.2, if R is a domain but not a field, then $\dim R \geq 1$. By a remark following Definition 2.21, R is local; let us say M is the unique non-zero maximal ideal. Let P be any non-trivial prime ideal contained in M. We will show $P = M$. Pick $0 \neq x \in M$, $0 \neq z \in P$. Since $x \in M$, $x^{-1} \notin R$, for M consists of all the non-units of R. R is completely integrally closed, so there is an $n \geq 1$ such that $z(x^{-1})^n = zx^{-n} \notin R$. Since R is a valuation domain, this implies that $(zx^{-n})^{-1} = z^{-1}x^n \in R$. Because $z \in P$, $zz^{-1}x^n = x^n \in P$, which implies that $x \in P$ since P is a prime ideal. Thus $M = P$ and $\dim R = 1$.

($\Leftarrow$) By hypothesis, R is a valuation domain, and thus local with maximal ideal M. If $\dim R = 0$, them $M = 0$ because 0 is prime in an integral domain, and so R is a field and is thus trivially completely integrally closed. So suppose that $\dim R = 1$ and that R is not completely integrally closed. Then there exists an $x^{-1} \in K - R$ (hence $x \in M$) and $0 \neq d \in R$ such that $dx^{-n} \in R$ for every $n \geq 1$. Now $dx^{-1} \in R$ implies that $d = dx^{-1} \cdot x \in M$. Since M is the only non-trivial prime ideal of R, $\operatorname{Rad} I = M$ for any non-trivial ideal $I \neq R$ of R. Now $d \in M$ implies that $(d) \subset M$, so $\operatorname{Rad}(d) = M$. Hence there exists an $n \geq 1$ and $0 \neq c \in R$ such that $cd = x^n$, which implies that $d = c^{-1}x^n$. Now, by hypothesis, $dx^{-(n+1)} = k \in R$. Then $dx^{-(n+1)} = c^{-1}x^n x^{-(n+1)} = c^{-1}x^{-1} = k \in R$. This implies that $x^{-1} = ck \in R$, a contradiction because we had $x \in M$. Thus R is completely integrally closed. ∎

Note that Proposition 3.6 implies that if R is a completely integrally closed valuation domain which is not a field, then $\dim R = 1$.

By Propositions 2.8 and 2.11, $R_{\mathcal{F}}$ is a valuation domain that is not completely integrally closed, and thus $\dim R_{\mathcal{F}} > \dim R = 1$.

We are now ready to present the major result of this chapter. First, two lemmas, the second of which can be used with Proposition 3.6 to show that if R is a completely integrally closed valuation domain which is not a field, then $\dim R < \dim R_{\mathcal{F}}$.

Lemma 3.7. If P is a prime ideal of R , then $(R_P)_{\mathcal{F}} \approx {R_{\mathcal{F}}}_{P_{\mathcal{F}}}$.

Proof. First, by Corollary 2.15, $P_{\mathcal{F}}$ is a prime ideal of $R_{\mathcal{F}}$ and so ${R_{\mathcal{F}}}_{P_{\mathcal{F}}}$ is a commutative ring with identity. By definition, $R_P = S^{-1} R$ where $S = R - P$, hence $(R_P)_{\mathcal{F}} = \{(rs^{-1})^* | r \in R ,$ $s \in R - P\}$. Similarly, ${R_{\mathcal{F}}}_{P_{\mathcal{F}}} = \{r^*s^{*-1} | r^* \in R_{\mathcal{F}} , s^* \in R_{\mathcal{F}} - P_{\mathcal{F}}\}$. Since $s^* \notin P_{\mathcal{F}}$, Proposition 1.10 implies that $\{n | s_n \in P\} \notin \mathcal{F}$. Thus, since $\mathcal{F}$ is an ultrafilter, $\{n | s_n \notin P\} \in \mathcal{F}$. Define $t^* \in R_{\mathcal{F}} - P_{\mathcal{F}}$ by

$$t_n = \begin{cases} s_n & \text{if } s_n \notin P \\ 1 & \text{otherwise} \end{cases}$$

. Clearly $t^* = s^*$ and for each n , $t_n \in R - P$. Hence for each $s^* \in R_{\mathcal{F}} - P_{\mathcal{F}}$, we may assume that $s_n \in R - P$ for each n . Define a function δ: ${R_{\mathcal{F}}}_{P_{\mathcal{F}}} \to (R_P)_{\mathcal{F}}$ by $\delta(r^*s^{*-1}) = (rs^{-1})^*$. Following the proof of Proposition 1.26, we see that this is a well-defined ring isomorphism; thus $(R_P)_{\mathcal{F}} \approx {R_{\mathcal{F}}}_{P_{\mathcal{F}}}$. ∎

Lemma 3.8. If R is a one-dimensional local integral domain, then $\dim R_{\mathcal{F}} > 1$.

Proof. R has a maximal ideal $M \neq 0$ and no other non-trivial prime ideals. By Corollary 2.22, $R_{\mathcal{F}}$ has maximal ideal $M_{\mathcal{F}} \neq 0$ and

0 is prime in $R_{\mathcal{F}}$ because $R_{\mathcal{F}}$ is an integral domain; thus $\dim R_{\mathcal{F}} \geq 1$. Suppose that $\dim R_{\mathcal{F}} = 1$. Then for any $0 \neq I \neq R_{\mathcal{F}}$ an ideal of $R_{\mathcal{F}}$, $\mathrm{Rad}\, I = M_{\mathcal{F}}$. Since $R_{\mathcal{F}}$ is a local ring, $M_{\mathcal{F}}$ consists of all the non-units of $R_{\mathcal{F}}$. Pick $0 \neq x \in M$ (R is not a field). Define $y^* = \overline{\{x^m\}} \in R_{\mathcal{F}}$. Clearly $y^* \in M_{\mathcal{F}}$. Thus $\mathrm{Rad}(y^*) = M_{\mathcal{F}}$, and hence there exists a $k^* \in R_{\mathcal{F}}$ and an $n > 0$ such that $k^* y^* = x^{*n}$. Thus $\{m \mid k_m x^m = x^n\} \in \mathcal{F}$. Since $\mathcal{F}$ is not a principal ultrafilter, this set cannot be finite and hence there exists an $m > n$ such that $k_m x^m = x^n$; say $m = n + w$ for some $w \geq 1$. Then $k_m x^m = k_m x^{w+n} = k_m x^w x^n = x^n$. Since R is an integral domain, we get $k_m x^w = 1$, which implies that x is a unit in R, contradicting our assumption that $x \in M$. Hence, $\dim R_{\mathcal{F}} > 1$. ∎

If R is a completely integrally closed valuation domain which is not a field, then it is a local ring with, by Proposition 3.6, dimension one. Lemma 3.8 then implies that $\dim R > \dim R_{\mathcal{F}}$.

It is interesting to note how much power this rather innocuous looking lemma wields. It is the main tool we use to prove the following theorem, the major result of Chapter 3.

Theorem 3.9. If R is a commutative ring with identity and $0 \neq \dim R = n < \infty$, then $\dim R_{\mathcal{F}} > \dim R$.

Proof. We will consider separately the cases where R is an integral domain and where it is not. So, first, let us assume that $R_{\mathcal{F}}$ is an integral domain. Pick a maximal ideal M of R such that $\mathrm{ht}\, M = n$. We then have $n + 1$ distinct prime ideals of R that form a chain: $0 = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = M$. Clearly $\mathrm{ht}\, P_1 = 1$.

By [4, Theorems III 4.3(ii) and III 4.11(ii)], R_{P_1} is a local integral domain and $\dim R_{P_1} = 1$. Hence, by our preceding lemma, $\dim (R_{P_1})_F > 1$. Since by Lemma 3.7, $(R_{P_1})_F \approx R_{F_{P_{1_F}}}$, we have $\dim R_{F_{P_{1_F}}} > 1$. This implies that there exists a prime ideal Q of R_F such that $0 \subsetneqq Q \subsetneqq P_{1_F} \subsetneqq \cdots \subsetneqq P_{n_F} = M_F$. Hence $\operatorname{ht} M_F > n$ and thus $\dim R_F > \dim R = n$.

Now, let us assume that R is not an integral domain. Suppose $0 \neq \dim R = n < \infty$, and let M be a maximal ideal of R such that $\operatorname{ht} M = n$. As before, we have $n + 1$ distinct prime ideals $0 \neq P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = M$. The commutative ring with identity R/P_0 is an integral domain. Because the ideals of R/P_0 are of the form I/P_0 where I is an ideal of R that contains P_0, and because $(R/I) \approx (R/P_0/I/P_0)$, we can say that P/P_0 is a prime ideal of R/P_0 if and only if P is a prime ideal of R and $P_0 \subset P$. Thus $\operatorname{ht} M/P_0 = n$. Since R/P_0 is an integral domain, our work above allows us to conclude that $\operatorname{ht}(M/P_0)_F > n$. Since, by Proposition 2.14, $(M/P_0)_F \approx M_F/P_{0F}$, we have that $\operatorname{ht}(M_F/P_{0F}) > n$. This implies that there exists a prime ideal I of R_F such that $P_{iF} \subsetneqq I \subsetneqq P_{i+1F}$ for some $0 \leq i \leq n$, and hence $\operatorname{ht} M_F > n$. Thus $\dim R_F > \dim R = n$. ∎

CHAPTER 4

COMPARING TWO FUNCTORS FROM R-MODULES TO R_F-MODULES

In this chapter we will show that the function from the category of R-modules to the category of R_F-modules given by $M \to R_F \otimes_R M$ is a covariant functor. Noting that the function from the category of R-modules to the category of R_F-modules given by $M \to M_F$ is also a covariant functor (Proposition 1.17), we can define a function ψ_M: $R_F \otimes_R M \to M_F$ and compare the two functors by examining the behavior of this function.

We first present a proposition which allows us to define ψ_M as an R_F-module homomorphism.

Proposition 4.1. If M is an R-module, then $R_F \otimes_R M$ is an R_F-module.

Proof. Pick $a^*, r^*, s^* \in R_F$ and $m, n \in M$. Define a function from $R_F \times (R_F \otimes_R M) \to R_F \otimes_R M$ by $a^*(r^* \otimes m) = (ar)^* \otimes m$. Clearly, this map is well defined. Since (a) $a^*(r^* \otimes m + s^* \otimes n) = a^*[(r^* + s^*) \otimes (m + n)] = ((ar)^* + (as)^*) \otimes (m + n) = (ar)^* \otimes m + (as^*) \otimes n = a^*(r^* \otimes m) + a^*(s^* \otimes n)$; (b) $(a^* + r^*)(s^* \otimes m) = (a^*s^* + r^*s^*) \otimes m = a^*s^* \otimes m + r^*s^* \otimes m = a^*(s^* \otimes m) + r^*(s^* \otimes m)$; and (c) $r(s^*a^* \otimes m) = (r^*s^*a^* \otimes m) = (rs)^* (a^* \otimes m)$, $R_F \otimes_R M$ is an R_F-module. ■

The next proposition is similar to Proposition 1.17 and will be referred to later in this chapter.

Proposition 4.2. The function from the category of R-modules to the category of R_F-modules given by $M \to R_F \otimes_R M$ and $f \to f' = 1_{R_F} \otimes_R f$ for $f \in \mathrm{Hom}_R(A, B)$ is a right exact covariant functor.

Proof. Let M be an R-module. By Proposition 4.1, $R_F \otimes_R M$ is an R_F-module. Clearly $1'_M = 1_{R_F} \otimes_R 1_M = 1_{R_F \otimes_R M}$. Pick $f \in \mathrm{Hom}_R(A, B)$ and $g \in \mathrm{Hom}_R(B, C)$. Then $(g \circ f)' = 1_{R_F} \otimes_R (g \circ f)$ $= (1_{R_F} \otimes_R g) \circ (1_{R_F} \otimes_R f) = g' \circ f'$. Thus, this operation is a covariant functor. Since R_F is patently an R-module, the functor is a right exact functor by [4, Proposition IV 5.4]. ∎

We now define ψ_M for an R-module M .

Proposition 4.3. If M is an R-module, then the function $\psi_M\colon R_F \otimes_R M \to M_F$ defined by $\overline{\{r_n\}} \otimes m \to \overline{\{r_n m\}}$ is an R_F-module homomorphism.

Proof. Define a mapping $f\colon R_F \times M \to M_F$ by $f(\overline{\{r_n\}}, m) = \overline{\{r_n m\}}$. This is clearly well-defined. Pick $\overline{\{r_n\}}, \overline{\{s_n\}} \in R_F$, $m, m_1 \in M$ and $r \in R$. The function is a middle linear map: $f(\overline{\{r_n + s_n\}}, m) =$ $\overline{\{(r_n + s_n)m\}} = \overline{\{r_n m\}} + \overline{\{s_n m\}} = f(\overline{\{r_n\}}, m) + f(\overline{\{s_n\}}, m)$; and $f(\overline{\{rs_n\}}, m) = \overline{\{rs_n m\}} = f(\overline{\{s_n\}}, rm)$. Thus, by [4, Theorem IV 5.2], there exists a unique group homomorphism $\psi_M\colon R_F \otimes_R M \to M_F$ such that $\psi_M(\overline{\{r_n\}} \otimes m) = \overline{\{r_n m\}}$. In addition, ψ_M is an R_F-module homomorphism, for $\overline{\{s_n\}}\, \psi_M(\overline{\{r_n\}} \otimes m) = \overline{\{s_n\}}\,\overline{\{r_n m\}} = \overline{\{s_n r_n m\}} = \psi_M(\overline{\{s_n r_n\}} \otimes m) =$ $\psi_m(s^*(\overline{\{r_n\}} \otimes m))$, where $s^* = \overline{\{s_n\}}$. ∎

We are ready to present sufficient conditions on M for ψ_M to be injective, surjective, or bijective. First, a lemma.

Lemma 4.4. ψ_{R^n} is an isomorphism.

Proof. We have a sequence of isomorphisms $R_F \otimes_R R^n \approx \bigoplus_{i=1}^{n} (R_F \otimes_R R) \approx \bigoplus_{i=1}^{n} R_F \approx (\bigoplus_{i=1}^{n} R)_F = R^n{}_F$. Thus $R_F \otimes_R R^n \approx R^n{}_F$. Now,

$\psi_{R^n}(\{\overline{r_m}\} \otimes (r_1,\ldots, r_n)) = \{\overline{r_m(r_1,\ldots, r_n)}\} = \{\overline{(r_m r_1,\ldots, r_m r_n)}\}$.

If we send $(\{\overline{r_m}\} \otimes (r_1,\ldots, r_n))$ through the sequence of isomorphisms described above, we have $(\{\overline{r_m}\} \otimes (r_1,\ldots, r_n)) \to \bigoplus_{i=1}^{n} (\{\overline{r_m}\} \otimes r_i) \to \bigoplus_{i=1}^{n} \{\overline{r_m r_i}\} \to \{\overline{(r_m r_1,\ldots, r_m r_n)}\}$. This implies that ψ_{R^n} follows this sequence of isomorphisms and thus is an isomorphism. ∎

Proposition 4.5. If M is finitely generated, then ψ_M is an epimorphism.

Proof. Since M is finitely generated, there exists an exact sequence $R^n \xrightarrow{f} M \to 0$, where n is the number of generators of M . By Propositions 4.2 and 1.17, both $R_F \otimes_R R^n \to R_F \otimes_R M \to 0$ and $R^n{}_F \to M_F \to 0$ are exact sequences. We thus have the following commutative diagram

$$\begin{tikzcd}
R_F \otimes_R R^n \arrow[r, "1 \otimes f"] \arrow[d, "\psi_{R^n}"', "\approx"] & R_F \otimes_R M \arrow[r] \arrow[d, "\psi_M"] & 0 \arrow[d] \\
R^n{}_F \arrow[r, "f^*"] & M_F \arrow[r] & 0
\end{tikzcd}$$

Since f^* is surjective, ψ_M is surjective. ∎

The following definition gives us a generalization of finitely generated modules.

<u>Definition 4.6</u>. An R-module M is <u>finitely presented</u> if there exists an exact sequence $R^m \to R^n \to M \to 0$.

Clearly a finitely presented R-module M is also finitely generated. If R is Noetherian, then a finitely generated R-module M is also finitely presented. Note that if M is a finitely presented R_F-module, then M_F is a finitely presented R_F-module.

As an example of a finitely generated module which is not finitely presented, consider $K[x_1, x_2,\ldots]/(x_1, x_2,\ldots)$, where K is a field. If we set $R = K[x_1, x_2,\ldots]$ and $M = (x_1, x_2,\ldots)$, then R/M is patently an R-module, and we have an exact sequence $M \to R \to R/M \to 0$. This implies that R/M is finitely generated. Because M is not finitely generated, we may conclude that there exists no exact sequence $R^n \to R \to R/M \to 0$, which implies that $R/M = K[x_1, x_2,\ldots]/(x_1, x_2,\ldots)$ is not finitely presented.

<u>Proposition 4.7</u>. If M is finitely presented, then ψ_M is an isomorphism.

<u>Proof</u>. By definition, there exists an exact sequence $R^m \to R^n \to M \to 0$. By Propositions 4.2 and 1.17, we obtain a diagram of exact sequences which commutes:

$$\begin{tikzcd}
R_F \otimes_R R^m \arrow[r] \arrow[d] & R_F \otimes_R R^n \arrow[r] \arrow[d] & R_F \otimes_R M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, dash] & 0 \arrow[d, dash] \\
R^m{}_F \arrow[r] & R^n{}_F \arrow[r] & M_F \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

By the 5-Lemma [6, Lemma 6.15], ψ_M is an isomorphism. ∎

Hence, if M is finitely presented, we can consider the functors described in Proposition 1.17 and Proposition 4.2 to be equivalent.

Our next proposition gives us an example of an R-module M which is not finitely presented, but where ψ_M is a monomorphism.

Proposition 4.8. If V is a free R-module, then ψ_V is a monomorphism.

Proof. Let $\{e_k\}_{k\varepsilon K}$ be a basis for V over R. For each $k \varepsilon K$, define $e^*_k = \{\overline{e_k}\} \varepsilon V_F$. The set $\{e^*_k | k \varepsilon K\}$ is linearly independent over R_F. For, if there exist $r^*_i \varepsilon R_F$, $1 \leq i \leq n$, such that $\sum_{i=1}^{n} r^*_i e^*_i = 0$, then $\{m | \sum_{i=1}^{n} r_{im} e_i = 0\} \varepsilon F$. Since $\{e_k | k \varepsilon K\}$ is linearly independent over R, this implies that $\{m | r_{im} = 0\} \supset \{m | \sum_{i=1}^{n} r_{im} e_i = 0\} \varepsilon F$ for each i, $1 \leq i \leq n$, and so $r^*_i = 0$ for each i. Hence $\{e^*_k | k \varepsilon K\}$ is linearly independent over R_F and because $R_F \otimes_R V$ is free on $\{1 \otimes e_k\}_{k\varepsilon K}$ and $\psi_V(1 \otimes e_k) = e^*_k$, this implies that $\psi_V\colon R_F \otimes_R V \to V_F$ is an R_F-module monomorphism. ∎

The following remark gives us an example of an R-module M where ψ_M is injective but not surjective and also shows that we cannot generalize the remarks that follow Proposition 1.15.

Remark 4.9. If V has an infinite basis, then ψ_V is not surjective. Let $\{e_i\}_{i=1}^{\infty}$ be a countably infinite subset of a basis for V. If we define $e^* \varepsilon V_F$ as $e^* = \{\overline{e_m}\}$, then clearly e^* is not in the subset of V_F spanned by $\{e_i^*\}_{i=1}^{\infty}$. This implies, because $R_F \otimes_R V$ is free on $\{1 \otimes e_k\}_{k \varepsilon K}$ and $\psi_V(1 \otimes e_k) = e_k^*$, that ψ_V is not surjective. Hence, in general, $(\bigoplus_{\alpha} M_\alpha)_F \not\approx \bigoplus_{\alpha}(M_{\alpha F})$ over infinitely indexed sets, in contrast to the remarks following Proposition 1.15.

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