To the Graduate Council:

I am submitting herewith a dissertation written by Thomas J. Fahey, Jr., entitled "Influences on Factorial Accuracy: A Monte Carlo Investigation." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Industrial and Organizational Psychology.

John W. Philpot, Major Professor

We have read this dissertation and recommend its acceptance:

Accepted for the Council:

Vice Chancellor
Graduate Studies and Research

4

INFLUENCES ON FACTORIAL ACCURACY:

A MONTE CARLO INVESTIGATION

A Dissertation

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Doctor of Philosophy

Degree

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ABSTRACT

Twenty-seven population correlation matrices were randomly generated with specified numbers of factors, proportions of common variance, ratios of the numbers of variables to the numbers of factors.

The correct numbers of factors in these 27 population correlation matrices were extracted via the principal axes factoring procedure using the true population communalities in the diagonals.

The correct numbers of factors were rotated via the normalized Varimax procedure.

Sixteen hundred and twenty sample correlation matrices with different ratios of observations per variable were randomly generated from the 27 population correlation matrices.

These 1,620 sample correlation matrices were factored via the principal axes procedure (no iterations) using squared multiple correlations, when the inverse of the matrix existed, as communality estimates; otherwise the absolute value of the highest off diagonal correlation was used. Factoring was carried to the point where the subsequent eigenvalue was less than or equal to zero, or to the value of n/2, where n is the number of variables, whichever was smaller.

Four separate Varimax rotations were performed to the following criteria:

A. Condition I--rotate one less factor in each sample than in the corresponding population.

B. Condition II--rotate the same number of factors in each sample that is contained in the corresponding population.

C. Condition III--rotate one more factor in each sample than in the corresponding population.

D. Condition IV--rotate the numbers of factors as determined by the scree tests.

The congruence coefficients and the means square errors between factor loadings on the rotated population factors and factor loadings on the corresponding rotated sample factors were computed. Corresponding factors were defined as those sample factors which had the highest congruence coefficient with the population factors. If a congruence coefficient were negative then all sample loadings on that factor were reflected and the congruence coefficient and mean square error recomputed.

The dependent variables (congruence coefficients or mean square errors) for each factor were separately regressed on the independent variables (number of factors, proportion of common variance, etc.) and their interactions under each of the four conditions previously discussed.

Mean accuracy indices were compared via dependent sample t tests for the conditions of overfactoring versus underfactoring.

Results indicate that far fewer observations are needed than were previously thought, and that overfactoring is superior to underfactoring; but, the issue is moot without more precise ways of determining the correct number of factors, and then there would be no logic to overfactoring.

When there are few factors, many variables, and a high proportion of common variance in the population, samples can be based on fewer observations than variables and still be highly replicable.

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CHAPTER I

INTRODUCTION

At a minimum, any field of study which endeavors to raise its status from that of a technology to that of a science must be capable of replicating its findings. The ability to replicate is the ability to generalize to some defined population. The viewpoint that a researcher performs a factor analysis on a number of variables characterizing some sample of people merely to describe that particular sample of people with fewer constructs is myopic. The information gained from such a study is limited since no other scientist will have that same sample of people to study. Ordinarily the purpose of studying any group of people is to allow generalization of the findings to other people who are "similar" to the ones in our sample.

The theory of statistical inference provides a vehicle for estimating the probability of replicating findings from a particular sample of observations. The method of factor analysis has been notably lacking in inferential techniques. Lawley's maximum likelihood (1940, 1953), Rao's canonical factor analysis (1955) and Jöreskog's maximum likelihood method (1962, 1963) are exceptions. These authors provide significance tests to test the hypothesis that a given number of factors account for the data. These significance tests, however, are appropriately applied only to the respective methods for factor estimation and only when the number of observations is large. "Moreover, these methods of estimation are either computationally arduous, as in the case of Lawley's (1953) or

Rao's (1955) method, or unfamiliar, as in the case of Jöreskog's* (1963) method. Consequently, they are rarely used and so the statistical tests are rarely applied." (Cliff and Hamburger, 1967)

Mathematically, the derivations of formulae for the generalization of factor analytic results (e.g., standard error of a factor loading, the number of factors to extract) are exceedingly complex and thus investigators have resorted to Monte Carlo studies for the determination of empirically based formulae of generalizability.

Related Research

Monte Carlo Studies of the Sampling Errors of Factor Loadings on Unrotated Common Factors

McNemar (1941) was probably the first to empirically investigate the sampling errors of factor loadings. Using centroid factor analysis and Pearson product moment correlations on data "which happened to be available" (p. 141), he randomly split 700 cases into 14 samples of 50 each and 7 samples of 100 each. The eight variables were always factored into three factors and the standard deviations of the first centroid loadings were about equal in magnitude to the standard error of a correlation coefficient based upon the respective number of cases.

The variability of the loadings on the second and third factors were about twice the first. He also reports the results of two other studies using tetrachoric correlation with median splits. The loadings

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*K. G. Jöreskog's 1963 Statistical Estimation in Factor Analysis is out of print and no longer available. Lawley and Maxwell's 1971 Factor Analysis as a Statistical Method (second edition) gives a full account of Jöreskog's computationally efficient procedures for maximum likelihood factor analysis and is still in print.

on the first centroid showed ". . . sampling fluctuations of about the same order as the chance errors of tetrachoric correlations . . . ." (p. 144). The loadings on the other two centroid factors exhibited much greater variability than did the loadings on the first factor.

Jöreskog (1963) was the only other investigator to test the sampling fluctuations of loadings on unrotated factors. Using maximum likelihood factor analysis he obtained mixed results in his tests. Some of his samples exhibited loading fluctuations around $1/\sqrt{N}$ (which is the approximate standard error of a zero product moment correlation) and others were considerably larger.

Since unrotated factors are rarely interpreted directly, the sampling variability of unrotated factor loadings is of interest only to the extent that one wishes to compare the efficiency of different methods of factoring the same data without regard to any rotational procedures which might subsequently be employed.

Monte Carlo Studies of the Sampling Errors of Factor Loadings on Rotated Common Factors

Jöreskog (1963) rotated the previously described factors to a least-squares fit of the population loadings and the obtained standard errors of the rotated loadings were somewhat smaller than $1/\sqrt{N}$. He also noted that the sampling errors of the non-zero loadings tended to be smaller than those for the zero loadings.

Hamburger (1965) as described in Cliff and Hamburger (1967) used four different population matrices which had twelve variables and four factors and varied in the degree of simple structure. Forty samples were generated from each of the four factor patterns. Twenty samples were based on 400 cases. All 160 matrices were factored by the

principal axes procedure using squared multiple correlations as the community estimates. The resulting factored matrices were then rotated by four procedures, the Varimax (Kaiser, 1958), the Equamax (Saunders, 1962), the Biquartmin (Harman, 1969), and the Procrustes (Hurley and Cattell, 1962). The standard error for each loading was computed and the average standard errors are presented in Example 1.

Example 1

Average Standard Errors for Each Condition

Types of Simple Structure	Method of Rotation			
	Equamax	Varimax	Biquartimin	Procrustes
N=100				
Univocal	.128	.128	.130	.118
Strong	.138	.148	.136	.115
Weak	.133	.138	.140	.112
None	.145	.142	.151	.113
N=400				
Univocal	.068	.068	.066	.061
Strong	.092	.095	.094	.079
Weak	.094	.098	.097	.077
None	.067	.092	.095	.057

The results of Hamburger's study are typical of the degree of preciseness with which the large sample standard error of a zero correlation approximates the standard error of a factor loading. All of the standard errors are overestimates of the standard error of a large sample correlation of zero, $1/\sqrt{N}$, and the overestimates range from 12 percent with the Procrustes rotation to 96 percent with the Varimax. The overestimates in Example 1 are themselves lower limits due to the use of standard errors of both zero and non-zero loadings which tends to deflate the estimate of the standard error of a zero loading.

Cliff and Pennell (1967) used the Bartlett decomposition of a Wishart matrix to generate random correlation matrices from defined population correlation matrices. The parameters varied in the two population matrices were the size of the individual loadings, the communalities of the variables and the strength of the factor (eigenvalue). Fifty samples for both population matrices were generated based on 15 variables and four factors. Each of the 100 matrices was factored to four factors using the principal axes method with squared multiple correlations as the communality. There were no iterations to improve the communality estimates and the sample factors were rotated to a least-squares fit (Cliff, 1966) of the appropriate population factors.

The average standard deviation of the loadings for the two sets of 50 sample correlation matrices was .12 and .16 as compared with $1/\sqrt{N}$ or .14. Two steps in the data analysis served to deflate the reported standard errors. First, the standard errors were computed only on those variables which had significant loadings in the population matrix. The factor loadings have limits of positive one and negative one.

As any population loading approaches either of these limits the sampling variability of that loading is restricted and thus the deflated standard error. Secondly, the root mean square (standard deviation) from the sample mean as opposed to the root mean square from the population value was used. It is well known that the sum of squares of deviations from some unknown point is a minimum when that point is the arithmetic mean. Any other point (such as the population loading) would inflate the sum of squared deviations. The use of the sample mean as the point about which squared deviations are taken assumes that the sample factor loadings are unbiased estimates of the population loadings. This has not yet been demonstrated.

As a means of summarizing their experiment, Cliff and Pennell (1967) used regression analysis to estimate the standard deviations of the significant factor loadings. Seven independent variables were used and they are:

1. size of the population factor loading
2. population communality of the variable
3. number of non-zero loadings on the factor
4. number of non-zero loadings on the variable
5. sum of the squared loadings on the factor
6. discrepancy between loading and other loadings on the factor
7. mean loading on the factor

Independent variable 6 is the difference between the loading and the mean of the other non-zero loadings on the factor.

Variables 1, 2, 5, and 7 correlated significantly and negatively with the standard deviations of the significant factor loadings. Two important points to be gained from this study (notwithstanding the

previous criticisms) are the reiteration of the importance of a precise communality estimate for small matrices (15 variables in this study) and the effect of factor strength. The correlation of variable 1 was explained as an artifact and variable 7 is highly correlated with variable 5 (r=.941).

Pennell (1968) defined his population matrices with univocal tests (each test had one positive loading on an axis with the loadings on other axes being zero) and varied the communalities and sample size. He also used the decomposition of a Wishart matrix to generate his sample correlation matrices. The principal axes factor method with squared multiple correlations was used. Only the correct number of factors was rotated and these were rotated to a least-squares fit (Cliff, 1966) of the appropriate population matrix.

For those variables with zero loadings Pennell found that the greatest influence on the size of the standard deviations was the number of observations upon which the various sample matrices were calculated. Using the data from Pennell's analysis of variance table (Pennell, 1968, p. 429) and Hays' formula for omega squared (Hays, 1963, p. 407), it can be shown that the number of observations accounts for 79.6 percent of the variance of the standard deviations of individual factor loadings. Even with this much explained variance there are other parameters influencing the size of a factor loading's standard error.

Two limitations of this study which Pennell points out were that only univocal tests were used and that the number of variables was not varied. One other limitation of this study, Jöreskog's (1963) study, and of the Cliff and Pennell (1967) study is that the rotation procedure used provided a least-squares fit to the population matrix. "This has

the advantage of showing the amount of sampling variation which is inherent in the data to which factor analysis is applied and the disadvantage of following a procedure which is rarely possible in practice." (Cliff and Hamburger, 1967, p. 436)

Mathematical Formulae of Generalizability

Girshick (1939) derived asymptotic standard errors of unrotated loadings on principal components while Lawley (1953, 1967) did the same for unrotated loadings produced in maximum likelihood factor analysis. The difficulty in extending this to rotated loadings is that the true transformation matrix M is unknown.

> If M is treated as a known matrix, it is easy to compute the sampling variances and covariances of the new loadings, since these are merely linear functions of the original loadings in which the coefficients are elements of M. This is not strictly justifiable as in practice M is usually derived from the data. It would be almost impossible to take sampling errors in the elements of M into account. The only course is, therefore, to ignore them in the hope that they are relatively small. (Lawley and Maxwell, 1971, p. 55)

Rippe (1953) presents a method to test the significance of common component loadings but it is based, as is Lawley's method, on large samples. Archer and Jennrich (1973) and Jennrich (1973) (despite Lawley's and Maxwell's statement) derived asymptotic standard errors of loadings for orthogonally and obliquely rotated factors, respectively. These formulae apply to maximum likelihood and principal component methods. Notwithstanding the mathematical ingenuity of these derivations, "Asymptotic results are hardly sufficient for the sample sizes which are customarily met with in practice." (Kendall, 1957, p. 104)

Monte Carlo studies of these mathematically derived results are needed for generalization to sample sizes researchers ordinarily use in their studies.

Criticisms of the Logic of Previous Monte Carlo Studies

A salient criticism of the previously described Monte Carlo studies is that all the studies assume a reasonably precise test for determining the completeness of factorization. By rotating only the actual number of factors they defined in their population matrices, the generalizability of their results is limited to the extent that variability due to estimating the number of significant factors inflates the standard error of the loadings on the retained factors. We still do not know the consequences of underfactoring and overfactoring, though Mosier (1939, p. 42) states, "On the whole, however, it is far safer to carry too many than too few factors." This is still the current position of many factor analysts. (Gorsuch, 1974, p. 160)

Another heretofore unmentioned criticism is the logic used in determining the standard errors. The methodology does not take into account the number of tests which are made. One could employ the Bonferroni inequality (Miller, 1966) as Cureton (1975, unpublished manuscript) suggests, which states that if a test of significance of n statistics (here factor loadings) whose sampling errors are correlated to unknown degrees is made at a level of α/n then the significance level that at least one of them is different from zero by chance is somewhere between α and α/n. This test is unduly conservative if a substantial number of the n null hypotheses are truly false (Miller, 1966). This conservativeness is due to the fact that the Bonferroni procedure

corrects for Type I errors in all n tests ". . . regardless of whether a Type I error is possible or likely for a given test." (Larzelere and Mulaik, 1977, p. 563)

A modification of the Bonferroni procedure has been suggested (Larzelere and Mulaik, 1977) and is called the multistage Bonferroni procedure. A family wise error rate α is chosen for the n statistics. The significance of each statistic is tested at the α/n level. If no statistic is significant the procedure ceases. If m (one or more) statistics are significant at the α/n level then the remaining n-m statistics are tested at the $\alpha/(n-m)$ level. This sequence continues until there is a stage where no statistics are found to be significant.

Apart from the lack of generalizability (see the criticisms and limitations mentioned above) of previous studies, a major limitation is the piecemeal nature of these studies. No one study simultaneously varies all parameters of interest. This limitation is to be expected given the magnitude of the problem and the monetary costs of generating the necessary data on computers.

In general the previously described research ". . . indicate that the loadings in a sample matrix should differ from a hypothesis matrix only by $1/\sqrt{N}$ on the average if the sample is rotated to hypothesis. The results, however, are [far] from being exact or extensive enough to warrant the routine reporting of 'significance' levels in the literature." (Cliff and Hamburger, 1967, p. 443)

And yet even this statement is too strong. Only Pennell (1968) and McNemar (1941) adequately describe the methods by which the population matrices were generated. McNemar (1941) used data which were

available and Pennell (1968) randomly generated factor loadings for a two-factor model. Pennell (1968) was not explicit beyond the two-factor model.

In addition, all the Monte Carlo studies have used population matrices with orthogonal factors. Admittedly, initial inquiry into any field requires boundary conditions and orthogonal factors which, as a first step, are by no means unreasonable. One should be reminded of Thurstone's (1947, p. vii) thoughts on the subject.

> If we impose the restriction that the reference frame shall be orthogonal, then we are imposing the condition that the factors or parameters shall be uncorrelated in the experimental population or in the general population. It is my own conviction that this restriction should not be imposed if we are looking for meaningful parameters. It seems just as unnecessary to require that mental traits shall be uncorrelated in the general population as to require that height and weight be uncorrelated in the general population.

Reformulation of the Problem

Generalizability is a function of invariance. The more invariant a statistic is across samples in the defined domain of interest, the more generalizable is the statistic. Thurstone (1947, Chap. XVI) distinguishes metric invariance from configural invariance. Metric invariance is concerned with the stability of individual factor loadings while configural invariance is concerned with the identification of the factor as a construct. Thurstone (1947, p. 365) states:

> The factor loadings might change markedly from one population to another; but, if the same test battery is used on both populations, the configuration should be invariant. As a matter of fact, this is a far more important consideration in factor analysis than the invariance of the factor loadings. If the object is to identify a set of significant parameters for the description and understanding of a domain, it is more important to be able to make the identification as to the nature of the parameters than to ascertain that, in this particular collection of individuals that we happen to be

> studying, the correlations between the parameters have certain numerical values and that the saturations of some measure j are such and such for this partiuclar group of individuals. Later, when the nature of each primary factor becomes well understood, it may be a matter of great importance to ascertain the numerical values for particular groups of subjects in the form of practical norms and the like, but not at the exploratory stage of investigation for which factorial methods have been devised. At this stage we are groping for some set of concepts or parameters in terms of which the domain may be understood." (Underlining added)

Cliff and Hamburger (1967, p. 443) agree with Thurstone though not as strongly. "The degree to which whole factors remain invariant across samples has not been studied from the person sampling approach. It, too, is worthy of study since the investigator is often more interested in this question than he is in single loadings."

The implied viewpoint of the previously described research is that if some number, n, of factor loadings is found to be statistically significant then the factor upon which they load is also significant. The viewpoint of this study, consistent with Thurstone's view, is that a step should logically precede the identification of significant factor loadings. That step is the identification of the whole factor as a significant construct before further study is justified such as the identification of individual variables which significantly load on it.

The general problem is then under what conditions could an investigator make a reasonable generalization of the dimensions or constructs found in a sample to the population from which the sample was drawn. The problem specifically addressed in this paper is under what conditions will a principal axes factor analysis using squared multiple correlations as communality estimates and no iterations with a normalized Varimax rotation on sample data produce factors <u>similar</u> to the factors in the population.

CHAPTER II

EXPERIMENTAL PROCEDURE

Generation of Sample Correlation Matrices

Browne (1968) has made use of Bartlett's (1933) decomposition of the Wishart matrix to generate random covariance or correlation matrices (see also Odell and Feiveson, 1966). A summary of Browne's (1968) explanation is given below.

It is known (Kshirsagar, 1959; Wijsman, 1959) that a random matrix from a Wishart distribution with an identity covariance matrix I with N-1 degrees of freedom can be expressed as

$$H = TT'$$

where T is a lower triangular matrix of order n whose elements are:

t_{ij} are random normal deviates distributed as N(0,1) $(i > j)$

t_{ij} are chi variates with N-i degrees of freedom

$t_{ij} = 0$ $(i < j)$

This random matrix, H, is used to construct a random matrix, A, distributed as $W(\Sigma,N-1)$, the Wishart distribution with population covariance matrix Σ, and (N-1) degrees of freedom. Σ, the population covariance matrix is of order n and is positive definite or semi-definite. There is therefore a matrix F, also of order n, such that

$$\Sigma = FF' \qquad (1)$$

Then

$$A = FHF' \qquad (2)$$

as was just stated is distributed as a Wishart distribution with population covariance matrix Σ, and (N-1) degrees of freedom.

Let,

$$\hat{R} = (\text{Diag } [A])^{-\frac{1}{2}} \; A \; (\text{Diag } [A]).^{-\frac{1}{2}} \qquad (3)$$

$\hat{R}$, of order n, is distributed as maximum likelihood estimates of the population correlation matrix, R, based on a sample of N observations from an n variate normal distribution with population covariance matrix, Σ.

Browne (1968, p. 277) states that examination of equations (1), (2), and (3) reveals that ". . . the population covariance matrix, Σ, may be set equal to the population correlation matrix . . . without affecting the distribution of . . ." $\hat{R}$. This results in the elements of F, in equation (1), which are population component loadings. In summary, the generation of sample correlation matrices requires population component loadings, random normal deviates, chi variates, and determination of N, the sample size.

The formulation of population factor loadings and the determination of sample sizes are discussed in subsequent sections in this chapter. Pseudorandom normal deviates were obtained by the following transformations (Box and Muller, 1958):

$$N_i = (-2 \log_e m_i)^{\frac{1}{2}} \text{ Cos } 2 \pi m_{i+1}$$

$$N_{i+1} = (-2 \log_c m_i)^{\frac{1}{2}} \text{ Sin } 2 \pi m_{i+1}$$

where m_i are random numbers from the unit interval.

This study used the multiplicative congruential generator, which is concisely described in Walker (p. 471, 1972a), to randomly generate numbers on the interval from zero to unity with a uniform density function. This is the most widely used procedure for generating random numbers on the unit interval with a uniform probability function via

computer (Walker, p. 471, 1972a). The actual FORTRAN program used is taken directly from Walker (p. 196, 1972b).

Kendall and Stuart (1969, Vol. I, p. 371) state that there are two approximations to the distribution of χ in common use. The first approximation is due to Fisher. He found that $\sqrt{2\chi^2}$ is approximately normally distributed with mean $\sqrt{2V-1}$ and unit variance. The second, and the one to be used in this study, is due to Wilson and Hilferty (1931). They found that $(\chi^2/V)^{1/3}$ is approximately normally distributed with mean 1-2/9V and variance 2/9V. Garwood (1936) made comparisons of both approximations with the exact values and found that Wilson and Hiferty's approximations were noticeably better than Fisher's.

Defining N_i as a random normal deviate, V=N-i degrees of freedom and remembering the equation for calculating a normal deviate is $N_i = \frac{X_i - \overline{X}}{\sigma_X}$ the chi approximation is determined by substituting $(\chi_i^2/V)^{1/3}$ for X_i, 1-2/9V for $\overline{X}$ and $(2/9V)^{\frac{1}{2}}$ for σ_X and solving for χ_i, not χ_i^2. After some algebra and recalling that V=N-i the resulting equation is $\chi_i = \{V\,[1-2/9V + N_i\,(2/9V)^{\frac{1}{2}}]^3\}^{\frac{1}{2}}$. This is the same equation Odell and Feiveson (1966) recommend.

The reader should note that this chi procedure and thus the correlational generation procedure breaks down when $V=N-i \leq 0$. This is equivalent to zero or negative degrees of freedom and occurs when the number of observations is equal to or less than the number of variables. Kaiser and Dickman (1962) have developed a procedure which circumvents this limitation. This procedure is based on the fundamental equation in component analysis which states that

$$Z = FX,$$

where F, of order n, represents any factoring of R, the population correlation matrix, and X of order n by N is the population component score matrix. Let these components, F, be uncorrelated (principal component analysis was used in this study). One then merely has to generate an "arbitrary $\hat{X}$, sampling randomly from uncorrelated populations with any distributions (normal distributions in this study), all with population mean zero and population variance one. Then the desired

$$\hat{Z} = F\hat{X}$$

may be found, where $\hat{Z}$ represents observations from a multivariate (normal) population with zero means, unit standard deviations, and correlations R."* (Kaiser and Dickman, 1962, pp. 179-180). It is a simple matter to compute $\hat{R}$, the sample correlation matrix, based on $\hat{Z}$, the multivariate normally distributed observations.

Measures of Factorial Similarity

One measure of accuracy of an estimate f of a population parameter θ, is $(f-\theta)^2$, which is the square of the difference between the estimate and the parameter. The expected value of this squared difference is the mean square error in the population. When this estimator is applied to random samples from the same population, the arithmetic average of the squared differences is used as an estimate of the mean square error. (Browne, 1969, p. 288)

The first measure of similarity is

$$d = \frac{\sum_{i=1}^{n} (f_i-\theta_i)^2}{n}$$

*Comments within the parentheses are mine.

where: f is a sample factor loading

θ is f's corresponding population factor loading

n is the number of variables.

A different d value is computed for each factor.

The second measure of factorial similarity, and the one most heavily advocated, is the congruence coefficient between the loadings on the sample factor and the loadings on the corresponding population factor. It is known by many names though. Burt (1948, p. 185) calls it the "unadjusted correlation," Tucker (1951, p. 43) the "coefficient of congruence," Wrigley and Neuhaus (1955) the "degree of factorial similarity," others call it "Tucker's Phi." The term "congruence coefficient" will be used in this paper. The equation for this index is

$$\emptyset = \frac{X'Y}{[X'XY'Y]^{\frac{1}{2}}}.$$

Where X is a vector of population factor loadings and Y is a vector of sample factor loadings for the corresponding sample factor. It is perhaps more easily understood in the following form,

$$\emptyset_{ij} = \frac{\sum_{k=1}^{n} X_{ki} Y_{kj}}{\left(\sum_{k=1}^{n} X_{ki}^2 \sum_{k=1}^{n} Y_{kj}^2 \right)^{\frac{1}{2}}}$$

where: n is the number of variables

i is the population factor

j is the corresponding sample factor

X and Y are factor loadings.

If the X's and Y's were deviations from their respective means, then $\emptyset_{ij}$ would equal r_{ij}, the Pearson product moment correlation.

This index is like the correlation coefficient "invariant to uniform multiplicative scaling except for a possible change in sign" (Anderberg, 1973, p. 73). Adding a constant to one or both vectors does change the value of the cosine, unlike the correlation coefficient. This index is more a measure of proportionality than similarity of absolute level. "In the case of orthogonal components where the factor scores have means of zero and variances of one, the result of calculating coefficients of congruence on the factor pattern is identical to correlating the exact factor scores and is, indeed, a simplified formula for that correlation." (Gorsuch, 1974, p. 253)

Though not mathematically correct, it might aid the reader to think of the congruence coefficients, in this study, as the estimated correlation coefficients between factor scores.

Korth and Tucker (1975) generated some baseline data and tested the magnitude of congruence coefficients for random factors. They found, as had been pointed out earlier (Nesselroade, Baltes, and Labouvie, 1971), that when the ratio of variables to factors in a matrix was 2 to 1 then at least one pair of perfectly matching factors can be found. This may be the reason for other researchers (Humphreys, 1964; Humphreys, Ilgen, McGrath, and Montanelli, 1969) noting that under proper conditions random factors could be endlessly duplicated. This was the basis for defining the population ratio of variables to factors at a minimum of four to one.

The first "corresponding sample factor" was determined by computing the congruence coefficients between the first rotated population factor and each rotated factor in the associated random sample matrix

and taking that sample factor with the highest congruence coefficient as the "corresponding sample factor." The second "corresponding sample factor" was determined by computing the congruence coefficients between the second rotated population factor and each rotated factor in the associated random sample matrix, excluding the previous "corresponding sample factor" and taking that sample factor associated with the highest congruence coefficient as the second "corresponding sample factor." This process continued for each of the rotated population factors each time excluding previously defined "corresponding sample factors."

Purposes of Investigation

The purpose in this study is to examine the influence of

1. the number of population factors, f
2. the ratio of number of variables (n) to number of factors, (f), n/f
3. the proportion of variance due to common factors, p
4. the number of observations (N) per variable (n), N/n as

they affect the similarity (accuracy) of sample and population factors.

Objective three requires further explanation since it has heretofore been unmentioned. The magnitude of correlation coefficients among variables composed of independent factors is a function of the magnitude of the common factor weights. Holding constant the variance of a variable, the magnitude of the common factor weights is an inverse function of the magnitude of the unique factor weights. As the proportion of variance in a matrix due to common factors is increased, the absolute values of the common factor loadings, on the average, are also increased (and vice versa). Ceteris Paribus, the larger a correlation

coefficient (and factor loadings are correlation coefficients), the greater the probability that it is statistically significant. A logical consequence of this is that given two factors comprised of the exact same variables, the factor having the largest loadings should emerge as a salient factor more often in samples and thus be a better estimate of the population factor.

In an effort to be as representative as possible of the kinds of factor analytic studies seen in the journals of the behavioral sciences (Psychology, Sociology, Economics, Anthropology, Political Science) and also to simulate as accurately as possible the situations which occur in non-published factor analytic studies (mainly the small sample sizes found in in-house studies in industry), the levels of the hypothesized influences of factorial accuracy are defined as follows:

1. the number of population factors (2, 4, 9)
2. the ratio of number of variables to number of factors (4, 6, 8)
3. the proportion of variance due to common factors (0.5, 0.7, 0.9)
4. the number of observations per variable (0.5, 1, 2, 4, 6, 12)

The number of observations per variable might seem excessively low, and yet it is not uncommon to find in-house factor analytic studies of 100-item questionnaires based on samples of 200, 100, or even fewer observations.

The levels within each hypothesized influence result in 162 cells. There will be ten replications per cell or a total of 1,620 correlation matrices to be factored.

The effects of these influences are examined under four conditions:

A. Extracting and rotating one less factor from the sample than that which exists in the corresponding population.

B. Extracting and rotating the exact number of factors from the sample which exists in the corresponding population.

C. Extracting and rotating one more factor from the sample than that which exists in the corresponding population.

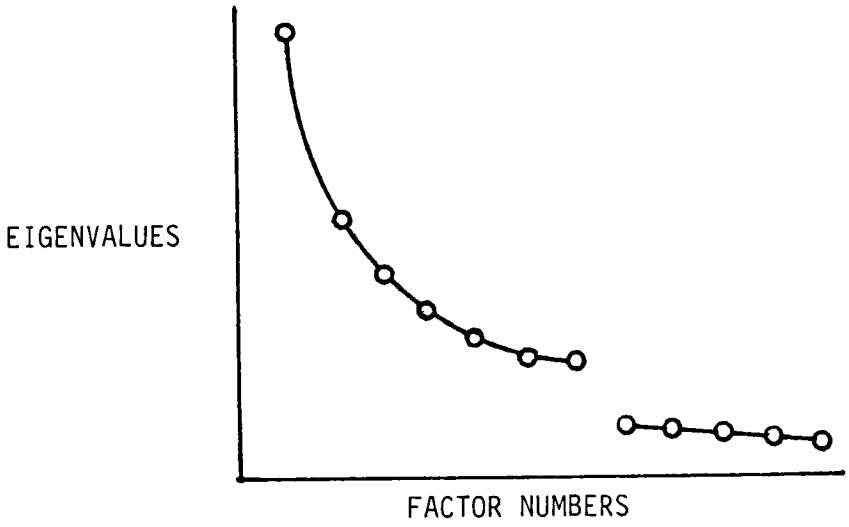
D. Extracting and rotating the number of factors in the sample as determined by the scree test. (Cattell, 1966; Cattell and Jaspers, 1967).

As was mentioned earlier, factor extraction is by the principal axes procedure using squared multiple correlations as the communality estimates with no iterations. Factors are rotated by the normalized Varimax procedure. If the squared multiple correlation cannot be calculated (the inverse of the matrix does not exist), then the absolute value of the highest off-diagonal correlation in each row is used as the communality estimates. Of the 1,620 sample correlations matrices 1,056 or 65% used the squared multiple correlation for communality estimates.

Scree Test

A test in common use to determine the number of factors to extract from a sample correlation matrix is the scree test (Cattell, 1966; Cattell and Jaspers, 1967). When the data have been initially overfactored a plot is made of the eigenvalues on the ordinate against the factor numbers on the abcissa. A smooth curve can usually be fitted

to the first m factors but cannot be extended to the remaining factors. These first m factors are considered significant. At the point where the smooth curve cannot be extended, a straight line is drawn through the remaining factors.



The remaining factors are considered to be error factors. The name "scree" test comes from the resemblance of such a plot to the rock slope of a mountain with rubble, called the scree, at the bottom.

Instead of examining the plots, one can examine the successive differences of the eigenvalues. The differences decrease regularly for those factors which correspond to the significant factors or the smooth curve on the plot. The beginning of the error factors is denoted by a substantial increase in the difference of the successive eigenvalues. The last significant factor is taken to be the factor associated with the larger of the two eigenvalues in which the increase in the differences occurred.

Example 2 contains an example of a matrix overfactored to twelve factors. The eigenvalue differences decrease regularly from 26.98 to .15; then there is a substantially larger difference, .32. The larger of the two eigenvalues associated with this difference is for factor 7 and so the scree test indicates that seven factors are significant.

EXAMPLE 2

Eigenvalues and Differences for a Hypothetical Correlation Matrix

Factor	Eigenvalue	Difference
1	33.28	
		26.98
2	6.30	
		1.99
3	4.31	
		1.77
4	2.54	
		.89
5	1.65	
		.28
6	1.37	
		.15
7	1.22	
		(.32)
8	.90	
		.08
9	.82	
		.12
10	.70	
		.07
11	.63	
		.06
12	.57	

Admittedly, the scree test provides only an approximation to the number of factors. There are times when an eigenvalue difference is larger than the preceding eigenvalue difference, but not "substantially" larger. This non-substantially larger difference is not interpreted to

be the break between real and error factors. There is no precise definition of "substantial." Cureton (1975) suggests that substantial be defined roughly as .15. For the purposes of the investigation, the break between significant and non-significant factors was taken to be at that point where an eigenvalue difference is larger than the preceding eigenvalue difference by .15 or greater.

There are also instances when more than one "substantially" larger difference occurs. In this study the difference between real and error factors was defined to be at that point where the first substantially larger difference occurred.

The Populations

The classical factor analysis model is:

$$Z_{1i} = x_{11}F_{1i} + x_{12}F_{2i} + \cdots + x_{1f}F_{fi} + u_1U_{1i}$$
$$Z_{2i} = x_{21}F_{1i} + x_{22}F_{2i} + \cdots + x_{2f}F_{fi} + u_2U_{2i}$$
$$\vdots$$
$$Z_{ni} = x_{n1}F_{1i} + x_{n2}F_{2i} + \cdots + x_{nf}F_{fi} + u_nU_{ni},$$
$$(i = 1,2,\ldots,N),$$

where N is the number of observations, n is the number of variables and f is the number of common factors. The F's and U's are common and unique factor scores respectively with their appropriate weights, x's and u's. The Z's are observed measurements hypothesized to be a weighted linear function of the orthogonal common and unique factors.

For convenience and without loss of generality it may be assumed that each Z, F, and U is in standard score form with zero mean and unit variance.

The variance, V_j, of any standardized variable Z_j is:

$$V_j = \frac{\sum_{i=1}^{N} Z_{ji}^2}{N} = 1 \qquad (1)$$

Upon substituting any variable Z_j from the classical factor analysis model we have

$$V_j = \frac{\sum_{i=1}^{N} (x_{j1}F_{1i} + x_{j2}F_{2i} + \ldots + x_{jf}F_{fi} + u_jU_{ji})^2}{N}.$$

Upon squaring the resultant equation is

$$V_j = \sum_{i=1}^{N} (x^2_{j1}F^2_{1i} + x^2_{j2}F^2_{2i} + \ldots + x^2_{j(f-1)}F^2_{(f-1)i} + x^2_{jf}F^2_{fi} + u^2_jU^2_{ji} + [x_{j1}x_{j2}F_{1i}F_{2i} + \ldots + x_{jf}u_jF_{fi}U_{ji}])/N.$$

The reader should recall that the correlation between any two variables in standard score form is

$$r_{z_jz_k} = \frac{\sum_{i=1}^{N} Z_{ji}Z_{ki}}{N}, \qquad (2)$$

and thus upon distributing the summation sign the terms involving F in the brackets are correlations between orthogonal factors and therefore equal zero. All other terms involving F^2 and U^2 are correlations between a factor and itself and therefore equal unity. All that remains in the equation for the variance are the squared common and unique factor loadings. Therefore

$$V_j = x^2_{j1} + x^2_{j2} + \ldots + x^2_{jf} + u^2_j = 1,$$

since Z_j is in standard score form.

This equation can be rearranged to

$$x^2_{j1} + x^2_{j2} +\ldots+ x^2_{jf} = 1-u^2_j$$

which is the equation for the communality of a variable. When variables are in standard score form the proportion of common variance in a matrix is merely the sum of the communalities divided by the total variance, n.

The reader should note the similarity between equations (1) and (2). If, in equation (2), j=k then we would have equation (1). Therefore, performing substitutions in equation (2) similar to those in equation (1), recalling that the unique factors are orthogonal and therefore the u's drop out of the equation, when $j \neq k$, and proceeding with the derivation in a similar manner results in

$$r_{z_j z_k} = x_{j1}x_{k1} + x_{j2}x_{k2} +\ldots+ x_{jf}x_{kf} .$$

The point of this short review of factor analysis is to demonstrate to the reader that all that is necessary for the generation of the population correlation matrices are the common and unique factor weights.

The task is to generate the x's and u's such that

$$\sum_{j=1}^{n}\sum_{k=1}^{f} x^2_{jk} \Big/ \left(\sum_{j=1}^{n}\sum_{k=1}^{f} [x^2_{jk} + u^2_j] \right)$$

equals the defined proportions of common variance, p.

The procedure is as follows:*

Generate n·f random numbers,

$$x'_{jk}(j=1,2,\ldots,n;k=1,2,\ldots,f) \quad \text{such that } -1<x'_{jk}<1,$$

---

*Thanks to Dr. John W. Philpot for suggesting this procedure.

and n random numbers u'_j, such that $-1<u'_j<1$, and $\sum_{j=1}^{n} u'^2_j > n(1-p)$

Find some factor c such that

$$\frac{c \sum_{j=1}^{n} u'^2_j}{n} = 1-p.$$

Since the variance due to the common factors by definition must account for some specified proportion, p, of total variance, the variance due to the unique factors must account for the remaining proportion of total variance, 1-p.

Solving for c in the above equation we have

$$c = \frac{n\,(1-p)}{\sum_{j=1}^{n} u'^2_j} \quad , c<1$$

and $u^2_j = c\, u'^2_j \quad (u^2_j < 1)$.

Since the matrix of interest is standardized, each variable, Z_j, has a variance of unity. We thus must find terms t_j for each row such that

$$t_j \sum_{k=1}^{f} x'^2_{jk} + cu'^2_j = 1$$

Solving for t_j we have

$$t_j = \frac{1 - cu'^2_j}{\sum_{k=1}^{f} x'^2_{jk}}$$

and therefore $x^2_{jk} = t_j\, x'^2_{jk}$.

Each x_{jk} uses the sign for x'_{jk} in the calculation of the population correlation matrices. Population correlation matrices are calculated by $R = XX' + U^2$.

Rotation

Notwithstanding Thurstone's (1947, p. vii) statement that there is no reason to impose the restriction of orthogonality on the factors the population factors are rotated by the normalized Varimax procedure. The researcher, by using the Varimax rotation on the sample, assumes the population factors are also orthogonal. One would have to presume that the investigator would have also used the Varimax rotation on the population. So the researcher's concern is, to what degree do the factors from sample data as located in the factor space by the Varimax procedure resemble the factors from the population of people also rotated by the Varimax procedure. There is no one best way to rotate the axes (nor to factor the data either). It is up to the researcher to analyze the data in a manner which yields as much useful information as possible. A comment by Rozeboom about factoring data is equally applicable to rotational procedures.

> If the data variables are to be analyzed as linear combinations of factors which are themselves, in turn, defined as composites of the data variables, aren't we just going in circles? Well, yes--in a way we are, but sometimes the view from one point on a circle is more interesting that from another. Complete analysis of a set of variables into defined factors is merely a linear transformation of the set, and a persons' scores on such factors, jointly contain exactly as much information about him, no more and no less, as do his scores on the original variables. But some ways to say the same thing are more illuminating than other ways . . . , and transformation of a set of data variables into a set of factors may very well reveal important relationships which are difficult to discern among the variables in their original form. (Rozeboom, 1966, p. 213)

Review

To briefly summarize, the following steps were followed:

Twenty-seven population correlation matrices were randomly generated with specified numbers of factors, proportions of common variance, and ratios of the numbers of variables to the numbers of factors.

The correct numbers of factors in these 27 population correlation matrices were extracted via the principal axes factoring procedure using the true population communalities in the diagonals.

The correct numbers of factors were rotated via the normalized Varimax procedure.

Sixteen hundred and twenty sample correlation matrices with different ratios of observations per variable were randomly generated (via the two procedures previously described) from the 27 population correlation matrices.

These 1,620 sample correlation matrices were factored via the principal axes procedure (no iterations) using squared multiple correlations, when the inverse of the matrix existed, as communality estimates; otherwise the absolute value of the highest off diagonal correlation was used. Factoring was carried to the point where the subsequent eigenvalue was less than or equal to zero, or to the value of n/2, whichever was smaller, where n is the number of variables.

Four separate Varimax rotations were performed to the following criteria:

A. Condition I - rotate one less factor in each sample than in the corresponding population.

B. Condition II - rotate the same number of factors in each sample that is contained in the corresponding population.

C. Condition III - rotate one more factor in each sample than in the corresponding population.

D. Condition IV - rotate the numbers of factors as determined by the scree tests.

The congruence coefficients and the means square errors between factor loadings on the rotated population factors and factor loadings on the corresponding rotated sample factors were computed. Corresponding factors were defined as those sample factors which had the highest congruence coefficient with the population factors. If a congruence coefficient were negative then all sample loadings on that factor were reflected and the congruence coefficient and mean square error recomputed.

CHAPTER III

DATA ANALYSIS METHODOLOGY

The purpose in this chapter is to provide the rationale which underlies the various data analytic methods used to determine what effects influence factorial accuracy and how. The major part of this chapter is devoted to the methods (analysis of variance and regression) involved when the sample correlation matrices were factored to the correct numbers of factors. The next part is devoted to the methods used when underfactoring was compared with overfactoring. The final part concerns the methods used when samples were factored to the numbers of factors as estimated by the scree test.

Correct Sample Factoring

The next three sections contain the basis for deciding which effects were subject to hypothesis testing (i.e., the model), the rationale for using regression analysis and the criteria by which the independent effects were chosen, and an evaluation of the resultant regression equations.

The Model

The dependent variables are measures of factorial accuracy, congruence coefficients, and mean square errors. Recalling the four main effects in this study:

1. number of observations (N) per variable (n) --N/n
2. number of variables (n) per factor (f) -- n/f

3. proportion of common variance in the population-- p

4. number of factors in the population -- f

it can be observed that some of the interactions could well have been used as main effects by other investigators. The interaction of 1 (N/n) with 2 (n/f) is the number of observations per factor. The interaction of 2 (n/f) with 4 (f) is the number of variables under consideration. The third order interaction of 1, 2, and 4 (N/n · n/f · f) is the number of observations upon which the sample correlation matrix is based. Due to the interpretability of many of the interactions they were included in this study (with the main effects) as effects worthy of investigation.

It should be noted that factors 5, 6, 7, 8, and 9 only emerge from populations with nine factors. The number of population factors in these instances is a constant. This results in linear constraints which artificially limit the full model from exhibiting significant effects. The number of factors, f, is obviously prohibited from having any effect. If the ratio of observations to variables, N/n, is a significant effect then its interaction with the number of factors is prohibited from being a significant effect since the interaction is perfectly correlated with N/n. Due to the above examples, only those effects which influenced the accuracy of the first four factors were considered.

Criteria for Independent Effects

The manner in which an investigator phrases the purpose of an investigation limits the available statistical analyses to those which properly address the stated purpose. Had the purpose of this investigation been to determine only which effects influenced factorial accuracy then a fixed effects analysis of variance would have been sufficient.

The purpose in this investigation, however, was to determine which effects and how these effects in the previously mentioned model influenced factorial accuracy. Since analysis of variance and regression analysis are both forms of the general linear model, the general linear model was used and the results were extracted in the most convenient form.

With four main effects, six second order interactions, four third order interactions, and one fourth order interaction, the number of independent variables was unwieldy. Coupled with the fact that some regression equations were based on 1,620 observations, statistical significance was virtually assured, though practical significance was not. In order to facilitate an understanding of the effects which impact factorial accuracy, a decision was made to use only the major pervasive effects in the regression analyses. The major pervasive effects are the ones that satisfied the following conditions: (1) the effect from a complete analysis (using all main effects and interactions) was significant at $p<.0001$ for factors one through four and (2) that effect entered the subsequent reduced regression equations for factors one through four at $p<.0001$.

These criteria were formulated in the interest of parsimony after preliminary data analyses. Preliminary analyses consisted of analyses of variance and stepwise regression equations. As many as ten variables were significant at $p<.05$. This large number in an exploratory study serves only to obfuscate, rather than enhance, the understanding of relationships. Also, those effects which entered the equations for predicting the congruence coefficient for factor one were not necessarily

the same ones which predicted the congruence coefficient (or mean square error) for factor two, etc.

The $p<.0001$ criteria is virtually tantamount, in this study, to preventing any variable from entering the regression equations without accounting for at least 2.5 to 3.0 percent of the variance (when the other variables are in the equation), thus the term, major. It is argued that effects which are more pervasive (i.e., generalizable in more situations) carry much more weight than do effects which meet only some statistical significance level, and thus the pervasive criterion.

Evaluation of Regression Equations

Each regression equation was examined to determine the goodness of fit of the effects, bias (residual analysis), and the degree of monotonicity for each effect.

Goodness of fit. The goodness of fit was examined to determine the utility of increasing the explanatory power of the equations through transformations of the effects. All analysis of variance tables in this paper represent a balanced design with equal numbers of observations in each cell. The sums of squares associated with any given effect are independent of the sums of squares associated with every other effect and are therefore additive. By adding the sums of squares in the analysis of variance tables associated with each of the major pervasive independent effects contained in the regression equations and then dividing by the total sum of squares, one achieves the upper limit of the proportion of variance that these effects explain in the sample. By comparing the proportion of variance that the regression equation

explains (R-square) with this upper limit, one then has a measure of the goodness of fit of the regression equation.

Residual analysis. The correlations of the residuals from the regression equations with the dependent variables are examined. Significant correlations are evidence of hetereoscedasticity; though nonsignificant correlations do not necessarily indicate homoscedasticity. Homoscedasticity is necessary for the appropriateness of the significance tests (F or t) of the regression coefficients.

Monotonicity. If an effect is nondecreasing monotonic, then increases in the effect, while holding constant the values of the other effects, result in increases in the dependent variable. If an effect is nonincreasing monotonic, then increases in the effect, while holding constant the values of the other effects, result in decreases in the dependent variable. Monotonicity can be determined algebraically or by the use of partial derivatives. To algebraically determine whether x in the following equation

$$y = b_1x + b_2z,$$

is nondecreasing monotonic, a positive constant, c, is added to x, which results in

$$y = b_1(x + c) + b_2z.$$

To be nondecreasing monotonic, the following inequality must hold

$$b_1(x + c) + b_2z > b_1 + b_2z.$$

The algebraic determination of monotonicity is used wherever possible. In more complex equations monotonicity is determined through the use of partial derivatives. A partial derivative is the slope of the dependent variable while holding constant all effects except the one

being examined for monotonicity. If the slope (i.e., partial derivative) is positive, the effect is nondecreasing monotonic. If the slope is negative, then the effect is nonincreasing monotonic. Monotonicity is examined to determine how the major pervasive effects influenced factorial accuracy.

Underfactoring Versus Overfactoring

The differences between the mean dependent variables, when the matrices are underfactored, and the mean dependent variables, when the matrices are overfactored, are compared using correlated t tests. The mean difference for each of the four factors was examined. Mean differences are examined only for those matrices which can be overfactored and underfactored. All the matrices can be underfactored by one factor; but not all matrices can be overfactored. The matrices which cannot be overfactored, and result in what seem to be meaningful factors, are the ones whose additional factors would be imaginary. An investigator would not be faced with the dilemma of whether to extract an additional factor if that factor resulted in a negative eigenvalue (imaginary factor), and therefore these matrices are excluded from these analyses.

By using the major pervasive effects (those determined for correct factoring) in computing regression equations for the conditons of under and overfactoring one can see other consequences of not knowing the correct number of factors to extract.

Scree Test

The last phase of this investigation examined the consequences of using one version of the scree test to estimate the number of

population factors. This version defined the break between significant and nonsignificant factors to be at that point where the difference between successive eigenvalues is .15 or greater than the difference between the previous two eigenvalues.

The consequences of the scree test are examined by regressing the dependent variables on the major pervasive effects and comparing the resultant equations with those obtained using exact factoring. The number of factors as determined by the scree test is used instead of the number of population factors.

CHAPTER IV

RESULTS AND DISCUSSION

This chapter follows the format established in Chapter III except that a discussion follows each major part, i.e., correct sample factoring, underfactoring versus overfactoring, and the scree test. Because there are two dependent variables, congruence coefficients and means square errors, both are presented and discussed within each major part before proceeding to the next part.

Correct Sample Factoring

Congruence Coefficients

Tables 1 through 4 contain the complete analyses of variance tables for the congruence coefficients associated with the first four population factors, when the samples were factored to the correct numbers of factors. The largest proportion of variance explained is 67.78 percent (factor 1). The other factors have even smaller proportions of variance explained.

The two most important effects, in terms of the proportions of sums of squares explained, are N/n and p. Note that the number of observations, $(N/n) \cdot (n/f) \cdot f$, accounts for only trivial sums of squares and in table 2 is not statistically significant by any significance level ordinarily acceptable to investigators.

If each main effect had been converted to the appropriate number of dummy variables, and regression analysis used on the complete model (all main effects and interactions) the results would be the same as

TABLE 1

Analysis of Variance of Congruence Coefficients for the First Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	161	14.7378	.09154	19.05	.0001	.6778
Error	1458	7.0064	.004805			
Grand Mean	1619	21.7442				
N/n	5	5.2366		217.94	.0001	
p	2	3.9702		413.09	.0001	
n/f	2	.2219		23.08	.0001	
f	2	1.5995		166.42	.0001	
(N/n)·p	10	.6828		14.21	.0001	
(N/n)·(n/f)	10	.2842		5.91	.0001	
(N/n)·f	10	.1521		3.16	.0005	
p·(n/f)	4	.1163		6.05	.0001	
p·f	4	1.3350		69.45	.0001	
(n/f)·f	4	.1297		6.75	.0001	
(N/n)·p·(n/f)	20	.0849		.88	.6091	
(N/n)·p·f	20	.1696		1.76	.0196	
(N/n)·(n/f)·f	20	.1974		2.05	.0040	
p·(n/f)·f	8	.3302		8.59	.0001	
(N/n)·p·(n/f)·f	40	.2272		1.18	.2036	

TABLE 2

Analysis of Variance of Congruence Coefficients for the Second Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	161	18.1293	.1126	15.22	.0001	.6269
Error	1458	10.7901	.007401			
Grand Mean	1619	28.9194				
N/n	5	7.6709		207.30	.0001	
p	2	3.6859		249.03	.0001	
n/f	2	1.4979		101.20	.0001	
f	2	1.0054		67.93	.0001	
(N/n)·p	10	1.1306		15.28	.0001	
(N/n)·(n/f)	10	.4135		5.59	.0001	
(N/n)·f	10	.1719		2.32	.0104	
p·(n/f)	4	.5037		17.02	.0001	
p·f	4	.1053		3.56	.0068	
(n/f)·f	4	.3212		10.85	.0001	
(N/n)·p·(n/f)	20	.1958		1.32	.1535	
(N/n)·p·f	20	.5132		3.47	.0001	
(N/n)·(n/f)·f	20	.1773		1.20	.1465	
p·(n/f)·f	8	.4011		6.78	.0001	
(N/n)·p·(n/f)·f	40	.3353		1.13	.2640	

TABLE 3

Analysis of Variance of Congruence Coefficients for the Third Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	107	16.2881	.1522	15.63	.0001	.6325
Error	972	9.4638	.0097			
Grand Mean	1079	25.7519				
N/n	5	7.0053		143.90	.0001	
p	2	3.5115		180.33	.0001	
n/f	2	.8730		44.83	.0001	
f	1	.8906		91.48	.0001	
(N/n)·p	10	.6951		7.14	.0001	
(N/n)·(n/f)	10	.3651		3.75	.0001	
(N/n)·f	5	.0216		.44	.8203	
p·(n/f)	4	.5712		14.67	.0001	
p·f	2	.0583		2.99	.0506	
(n/f)·f	2	.7124		36.58	.0001	
(N/n)·p·(n/f)	20	.3541		1.82	.0152	
(N/n)·p·f	10	.0609		.63	.7933	
(N/n)·(n/f)·f	10	.2427		2.49	.0059	
p·(n/f)·f	4	.7715		19.81	.0001	
(N/n)·p·(n/f)·f	20	.1548		.79	.7221	

TABLE 4

Analysis of Variance of Congruence Coefficients for the Fourth Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	107	26.3992	.2467	15.15	.0001	.6251
Error	972	15.8323	.01629			
Grand Mean	1079	42.2315				
N/n	5	11.6904		143.54	.0001	
p	2	5.7426		176.28	.0001	
n/f	2	1.7325		53.18	.0001	
f	1	.0421		2.58	.1084	
(N/n)·p	10	2.1568		13.24	.0001	
(N/n)·(n/f)	10	.4927		3.02	.0009	
(N/n)·f	5	.5948		7.30	.0001	
p·(n/f)	4	.1598		2.45	.0445	
p·f	2	.9063		27.82	.0001	
(n/f)·f	2	.6971		21.40	.0001	
(N/n)·p·(n/f)	20	.3961		1.22	.2321	
(N/n)·p·f	10	.2750		1.69	.0787	
(N/n)·(n/f)·f	10	.5451		3.35	.0003	
p·(n/f)·f	4	.5388		8.27	.0001	
(N/n)·p·(n/f)·f	20	.4291		1.32	.1582	

the analyses of variance, except that the analyses of variance tables do not contain regression coefficients. The resultant regression equations would contain 161 effects. The use of dummy variables in the regression equations would allow the estimation of factorial accuracy for only the levels of the effects contained in the study. Regression analysis, without dummy coding, allows the estimation of factorial accuracy for levels of effects not contained in the study and is therefore preferable.

Tables 5 through 8 contain the results of the regression equations, without dummy coding, from regressing the congruence coefficients for factors one through four on the independent effects. The independent effects satisfied the following conditions: (1) the effect from a complete analysis, i.e., analysis of variance or dummy coding regression analysis, using all main effects and interactions was significant at $p<.0001$ for factors one through four and (2) that effect entered the subsequent reduced regression equations, without dummy coding, for factors one through four at $p<.0001$.

As can be seen, the major pervasive effects are the ratio of the number of observations per variable, N/n (though a natural logarithm transformation was required), the proportion of common variance in the population, p, the ratio of the number of variables per factor in the population, n/f, and the interaction of the natural log of N/n with p. The natural logarithm transformation "linearized" N/n with respect to the congruence coefficients.

Evaluation of Regression Equations

Goodness of fit. Summing the sums of squares, in tables 1 through 4, associated with the effects used in the regression equations

TABLE 5

Regression Analysis of Congruence Coefficients for the First Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	9.1612	2.2903	F = 293.96	Pr>F = .0001
	Error	1615	12.5829	.0078		
	Total	1619	21.7441			
.5528	Intercept	1			t = 36.44	Pr>|t| = .0001
.1275	ln (N/n)	1			t = 14.19	Pr>|t| = .0001
.3943	P	1			t = 22.05	Pr>|t| = .0001
.006444	n/f	1			t = 4.80	Pr>|t| = .0001
-.1102	p · ln (N/n)	1			t = -8.81	Pr>|t| = .0001

R-square = .4213

TABLE 6

Regression Analysis of Congruence Coefficients for the Second Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	12.6112	3.1528	F = 312.22	Pr>F = .0001
	Error	1615	16.3082	.0101		
	Total	1619	28.9294			
.4576	Intercept	1			t = 26.50	Pr>|t| = .0001
.1608	ln (N/n)	1			t = 15.72	Pr>|t| = .0001
.4218	P	1			t = 20.73	Pr>|t| = .0001
.0152	n/f	1			t = 9.93	Pr>|t| = .0001
-.1421	p · ln (N/n)	1			t = -9.98	Pr>|t| = .0001

R-square = .4361

TABLE 7

Regression Analysis of Congruence Coefficients for the Third Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	10.7968	2.6992	F = 194.02	Pr>F = .0001
	Error	1075	14.9551	.0139		
	Total	1079	25.7519			
.3943	Intercept	1			t = 15.88	Pr>|t| = .0001
.1657	ln (N/n)	1			t = 11.27	Pr>|t| = .0001
.4764	P	1			t = 16.28	Pr>|t| = .0001
.01231	n/f	1			t = 5.60	Pr>|t| = .0001
-.1355	p · ln (N/n)	1			t = -6.62	Pr>|t| = .0001

R-square = .4193

TABLE 8

Regression Analysis of Congruence Coefficients for the Fourth Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	19.0554	4.7639	F = 220.97	Pr>F = .0001
	Error	1075	23.1761	.0216		
	Total	1079	42.2315			
.1906	Intercept	1			t = 6.17	Pr>|t| = .0001
.2347	ln (N/n)	1			t = 12.82	Pr>|t| = .0001
.6284	P	1			t = 17.25	Pr>|t| = .0001
.02014	n/f	1			t = 7.36	Pr>|t| = .0001
-.2010	p · ln (N/n)	1			t = -7.89	Pr>|t| = .0001

R-square = .4512

in tables 5 through 8 and dividing by the corrected total sum of squares results in the maximum proportion of variance which these effects could possibly account for in the sample. These proportions are used as upper limits to evaluate the goodness of fit of the regression equations presented in tables 5 through 8.

The following are the upper limits using the congruence coefficients as dependent variables for factors 1 through 4;

factor 1 - .4650

factor 2 - .4836

factor 3 - .4693

factor 4 - .5049

The proportions of variance accounted for by the same effects in the regression equations in tables 5 through 8 are the multiple R-square terms. These are:

factor 1 - .4213

factor 2 - .4361

factor 3 - .4193

factor 4 - .4512

Dividing the R-square terms from the regression analysis by the upper limits from the analysis of variance tables results in:

factor 1 - .9060

factor 2 - .9018

factor 3 - .8935

factor 4 - .8936

Of the maximum proportions of variance that could be accounted for using the effects in tables 5 through 8, those regression equations are accounting for the above proportions. In essence, no amount of curve

fitting beyond the logarithmic transformation that was performed, using the effects in tables 5 through 8, will result in substantial increases in the multiple R-squares.

Residual analysis. Ordinarily one expects the correlations between the dependent variables and the residuals from the regression equations to be near zero, otherwise the model is biased. This biasness violates the homoscedasticity assumption necessary for the F (and t) tests to be appropriate, and therefore statistical significance is questionable.

The congruence coefficients correlated against the residuals from the regression equations in tables 5 through 8 resulted in the following correlations:

factor 1 - .7607

factor 2 - .7509

factor 3 - .7621

factor 4 - .7409

Maximum possible prediction results from predicting every cell mean exactly. The following correlations resulted when congruence coefficients were correlated against the residuals from every cell mean:

factor 1 - .5676

factor 2 - .6108

factor 3 - .6062

factor 4 - .6123

It is therefore obvious that the high correlations between the congruence coefficients and the residuals from the regression equations

in tables 5 through 8 are a function of the data themselves and not the particular effects which entered the regression equations. Even so, the appropriateness of the normality assumption is questionable (though the test is robust especially with large N), and therefore also the statistical significance of the effects contained in the regression equations.

Since the normality assumption is questionable, it will suffice to use Tchebycheff's inequality.* The smallest t-ratio is 4.82, contained in table 5. The probability of obtaining a t-ratio is 4.82 or larger is at most .043, which is significant under most investigators rules of significance testing. All other t-ratios are even more significant.

Monotonicity. Each effect, n/f, N/n, and p, is examined for monotonicity. Two effects, n/f and N/n, involve more than one variable and therefore these variables are also examined for monotonicity.

The form of the regression equations in tables 5 through 8, pages 44 through 47, indicates that increasing the values of any effect, while holding constant the other effects, always increases the predicted congruence coefficients, within the limits of the data in this study (see Figures 1-4).

The general form of all four equations is

$$\hat{y} = b_0 + b_1 \ln(N/n) + b_2 p + b_3 n/f + b_4 p \cdot \ln(N/n),$$

where b_4 is always negative and the other regression coefficients are always positive.

---

*Also known as Chebychev's inequality. A variant of this inequality states that in random sampling the probability that a critical ratio of the type calculated in tables 5 through 8 (t-ratios) will exceed some constant, t, is always less than $1/t^2$, "irrespective of the shape of the population distribution." (Lord, 1953)

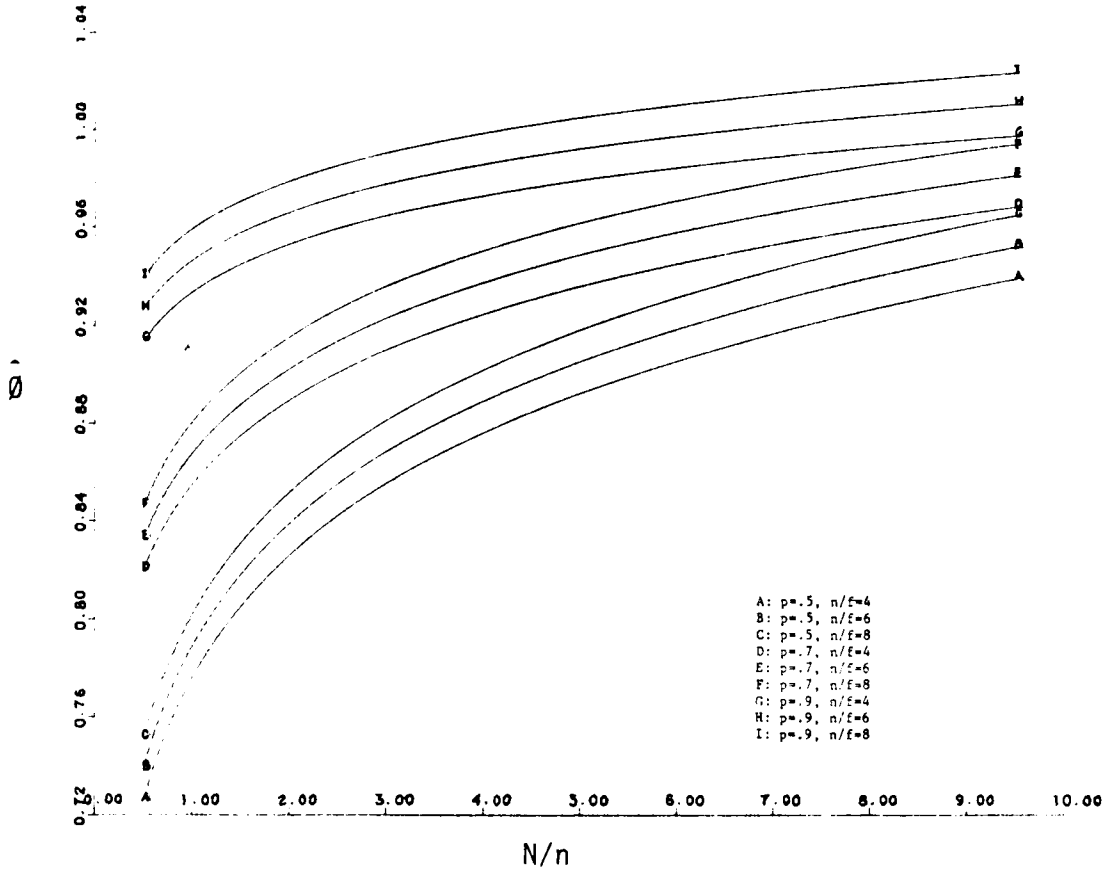


Figure 1. First factor predicted congruence coefficients, $\hat{\emptyset}$, plotted against the number of observations per variable, N/n, for various values of the populations' proportions of common variance, p, and numbers of variables per factor, n/f.

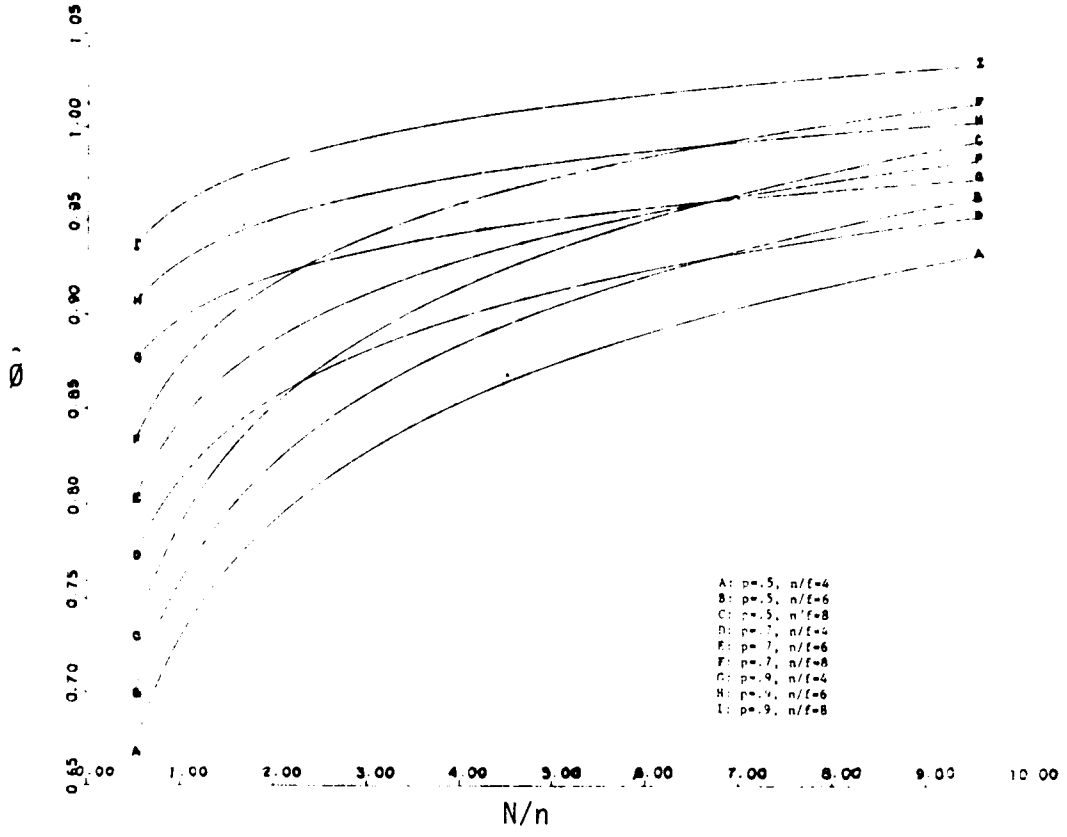


Figure 2. Second factor predicted congruence coefficients, $\hat{\emptyset}$, plotted against the number of observations per variable, N/n, for various values of the populations' proportions of common variance, p, and numbers of variables per factor, n/f.

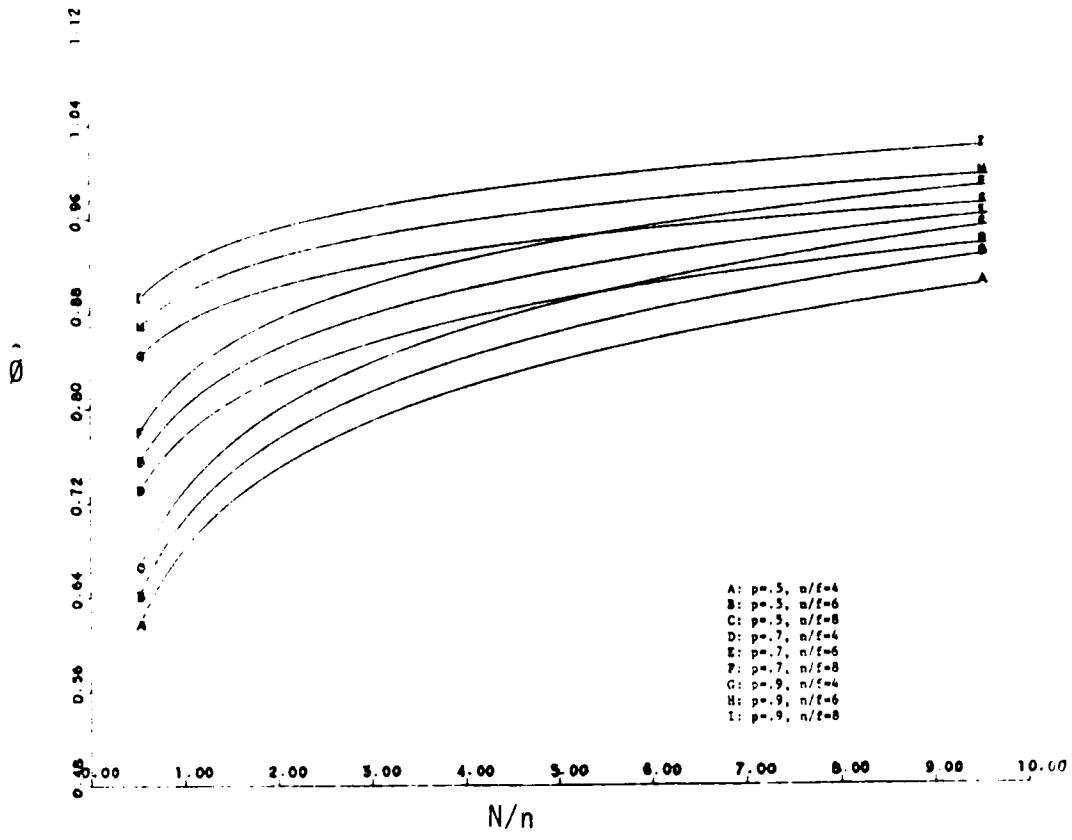


Figure 3. Third factor predicted congruence coefficients, $\hat{\emptyset}$, plotted against the number of observations per variable, N/n, for various values of the populations' proportions of common variance, p, and numbers of variables per factor, n/f.

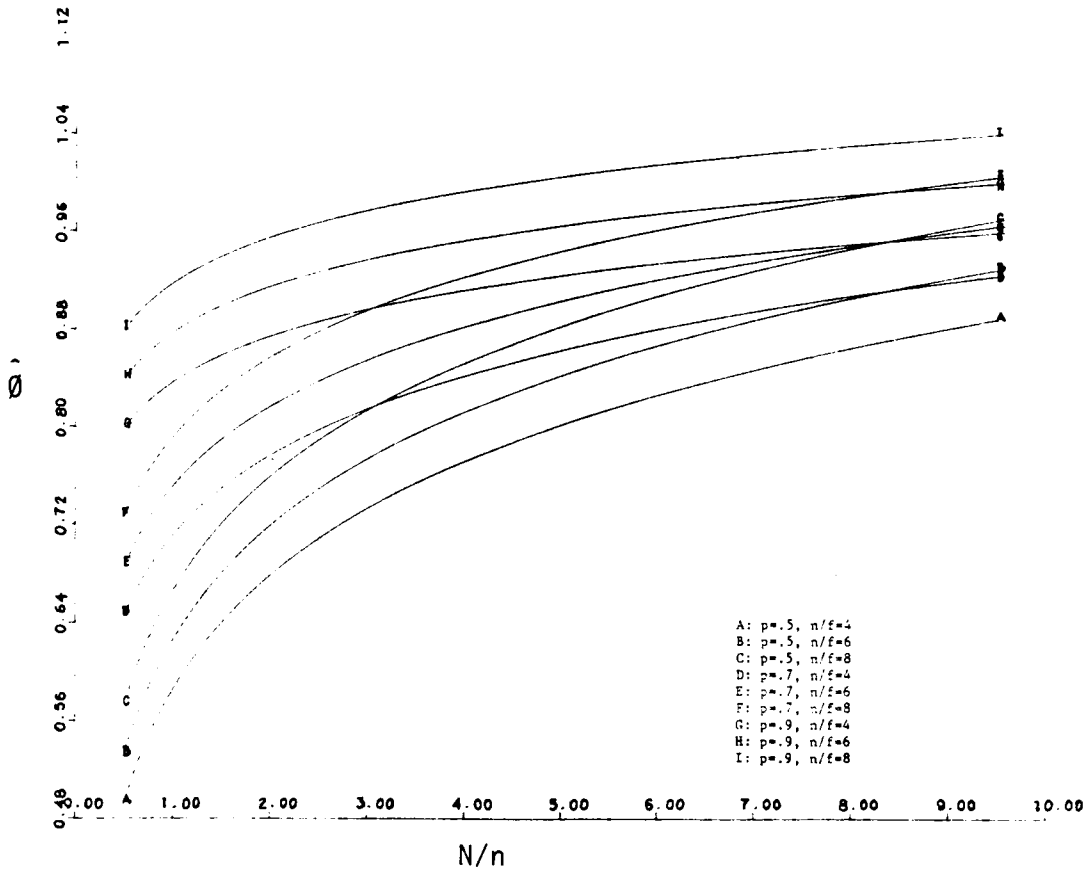


Figure 4. Fourth factor predicted congruence coefficients, $\hat{\emptyset}$, plotted against the number of observations per variable, N/n, for various values of the populations' proportions of common variance, p, and numbers of variables per factor, n/f.

The first effect examined is n/f. Since b_3, the regression coefficient for n/f, is positive and n/f must always be positive, increasing the number of variables per factor, n/f, while holding constant the other effects, increases the values of the predicted or expected congruence coefficients. But, n/f can be increased by increasing n, the number of variables, or by decreasing f, the number of population factors. Since decreasing f does not affect any other terms in the regression equations it can be concluded that, holding constant the other variables (i.e., N, p, and n), the fewer the number of factors in the population, the higher will be the values of the predicted congruence coefficients.

Since n is contained in three terms of the regression equations, the results of increasing n are not so readily apparent.

Taking the partial derivative of $\hat{y}$ with respect to n results in $b_3/f - (1/n)(b_1 + b_4p)$, which is the equation for the slope of $\hat{y}$ when holding constant all effects except n. The partial derivative is positive for $n > f \cdot (b_1 + b_4p)/b_3$. This positive slope indicates that increasing n will result in increased values of the expected congruence coefficients.

One caveat should be mentioned. Since p and f are held constant, any additional variables added to the sample should have, on the average, the same communalities as the original variables and not result in additional factors.

The next effect examined for monotonicity is N/n. The variable n was just demonstrated to be nondecreasing monotonic and therefore only N needs to be examined. Allowing only N to vary requires that

$$b_1 \ln(\frac{N+c}{n}) + b_4 p \cdot \ln(\frac{N+c}{n}) > b_1 \ln(N/n) + b_4 p \cdot \ln(N/n)$$

where $c > 0$; but practically c would be a positive integer.

Combining common terms results in

$$\ln(\frac{N+c}{n}) \cdot (b_1 + b_4 p) > \ln(N/n) \cdot (b_1 + b_4 p).$$

This inequality is true whenever

$$b_1 + b_4 p > 0$$

or $b_1 > -b_4 p$, since $\ln(\frac{N+c}{n}) > \ln(N/n)$. But $0 < p < 1$, therefore, it is sufficient, though not necessary that $b_1 > -b_4$, which was true in tables 5 through 8, pages 44 through 47.

The final effect to be examined for monotonicity is p. Holding constant all effects except p requires

$$b_2(p + c) + b_4(p + c)\ln(N/n) > b_2 p + b_4 p \cdot \ln(N/n),$$

where $0 < c < 1$, and $0 < p + c < 1$.

Combining common terms results in

$$(p + c) \cdot (b_2 + b_4 \ln(N/n)) > p(b_2 + b_4 \ln(n/n)).$$

For this inequality to hold

$$b_2 + b_4 \ln(N/n) > 0,$$

or $b_2 > -b_4 \ln(N/n)$, since $p + c > p$. Again this inequality holds true for every equation within the range of N/n. Thus, increasing the value of any effect, (i.e., N/n, p, n/f) while holding constant the others, results in increased expected values of the congruence coefficients. Also, within the previously defined limits, all variables (N, n, p, f) are monotonic. Decreases in f and increases in the other variables result in higher expected congruence coefficients.

The nondecreasing monotonic relationship of the proportion of common variance, p, to the congruence coefficients

> . . . confirms the theoretical expectation; the factor loadings become more . . . replicable as the communalities increased (e.g., Cliff and Pennell), 1967; Pennell, 1968). High communalities occur only when the reliabilities are high and the variables correlate with some of the other variables in the analysis. Therefore, variables without a prior history of good reliability estimates and good correlations with other variables in the analysis are not desired in a factor analysis. (Gorsuch, 1974, p. 295)

Table 9 contains the predicted congruence coefficients for factors 1 through 4 for various values of N/n, p, and n/f (see also Appendix A). Even though each investigator must decide the minimum expected level of the congruence coefficients before the investigator is willing to state with some degree of confidence that the sample reasonably maps the population, a congruence coefficient around 0.90 is certainly reasonable, and was used for discussion.

Discussion of Correct Sample Factoring

Because of the nondecreasing monotonicity of each effect, this discussion need only address the minimum values of N/n, p, and n/f, within the data limits, which result in expected congruence coefficients around .90. The reader may wish to use the regression equations in tables 5 through 8 or the expected values of the coefficients in table 9 to investigate the consequences of other values of these effects.

When N/n = .5, p = .9, and n/f = 5 the predicted congruence coefficients for the first two factors are .9202 and .8903. If n = 10 and f = 2 the number of observations required is only 5, a remarkably small sample size.

If there are four factors in the population and N/n = .5 then p must still be .9 while n/f must equal 8 before the predicted congruence

TABLE 9

Expected Congruence Coefficients for the First Four Population Factors With Correct Factoring for Various Values of the Number of Observations Per Variable, N/n, the Number of Variables Per Factor, n/f, and Proportion of Common Variance, p

p	n/f	N/n	Factor 1	Factor 2	Factor 3	Factor 4
.9	5	.5	.9202	.8903	.8544	.8196
.9	8	.5	.9395	.9359	.8913	.8801
.5	8	3.5	.8921	.9024	.8537	.8341
.5	7.5	5.5	.9216	.9354	.8918	.8846
.6	4	5.5	.9197	.9002	.8733	.8427
.8	4	5.5	.9610	.9362	.9223	.8998

coefficients are around .9 (.9395, .9359, .8913, .8801). This means that there are 32 variables and the sample size required is only 16 observations.

When p = .5 and there are two factors in the population, n/f must equal 8 and N/n must equal 3.5 to have expected congruence coefficients around .9 (.9821, .9024). These conditions would require 16 variables and 56 obervations, still a remarkably small sample.

When p = .5 and there are four population factors, n/f must equal 7.5 and N/n must equal 5.5. This results in expected congruence coefficients of .9216, .9354., .8918, and .8846. This means there should be 30 variables and 165 observations in the sample. More factors along with a small proportion of common variance drastically increases the requisite sample size; but, not to an unreasonable size.

When n/f equals 4 and there are two population factors, p must at least equal .6 and N/n must equal 5.5 to result in expected congruence coefficients of .9197 and .9002. This would require only eight variables and 44 observations.

If there are four population factors, with n/f = 4, p must be .8 and N/n remains at 5.5 to result in expected coefficients of .9610, .9362, .9223, and .8998. The number of variables doubles to 16 and the number of observations also doubles to 88. Again, the sample size required is smaller than the rules of thumb suggested in the literature (Gorsuch, 1974, p. 296).

Correct Sample Factoring

Means Square Errors

Tables 10 through 13 contain the analyses of variance tables for the means square errors associated with the first four population factors, when the samples are factored to the correct numbers of factors. The largest percent of variance explained is 65.62 percent (factor 2). Again, there are substantial percents of variance unexplained.

The most important variable, in terms of the proportions of sums of squares explained is N/n. As was the case with congruence coefficients, the number of observations, $(N/n) \cdot (n/f) \cdot f$, accounts for only trivial sums of squares.

Tables 14 through 17 contain the results of the regression equations from regressing the means square errors for factors 1 through 4 on the independent effects whose selection criteria were discussed in Chapter III.

As the reader will observe, these are the same effects which entered the regression equations when congruence coefficients were used as dependent variables. The interaction of the natural logarithm of N/n and p in table 17 violated the $p<.0001$ criterion for that regression equation only. The interaction term remains in that equation because:

1. it is still statistically significant,

2. it meets the $p<.0001$ criterion in the analysis of variance in table 13, and

3. it meets the other criteria in the other equations, for the congruence coefficients and means square errors.

TABLE 10

Analysis of Variance of Means Square Errors for the First Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	161	2.4316	.0151	14.85	.0001	.6212
Error	1458	1.4826	.0010			
Grand Mean	1619	3.9142				
N/n	5	1.0948		215.34	.0001	
p	2	.2187		107.55	.0001	
n/f	2	.0990		48.70	.0001	
f	2	.0765		37.63	.0001	
(N/n)·p	10	.1263		12.42	.0001	
(N/n)·(n/f)	10	.0991		9.75	.0001	
(N/n)·f	10	.1711		16.83	.0001	
p·(n/f)	4	.0407		10.02	.0001	
p·f	4	.0706		17.35	.0001	
(n/f)·f	4	.0902		22.17	.0001	
(N/n)·p·(n/f)	20	.0252		1.24	.2141	
(N/n)·p·f	20	.1126		5.54	.0001	
(N/n)·(n/f)·f	20	.0514		2.53	.0002	
p·(n/f)·f	8	.1007		12.38	.0001	
(N/n)·p·(n/f)·f	40	.0545		1.34	.0776	

TABLE 11

Analysis of Variance of Means Square Errors for the Second Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	161	3.4764	.0216	17.28	.0001	.6562
Error	1458	1.8218	.0012			
Grand Mean	1619	5.2982				
N/n	5	1.4929		238.95	.0001	
p	2	.2092		83.72	.0001	
n/f	2	.2521		100.86	.0001	
f	2	.1942		77.71	.0001	
(N/n)·p	10	.1203		9.63	.0001	
(N/n)·(n/f)	10	.1625		13.01	.0001	
(N/n)·f	10	.4224		33.81	.0001	
p·(n/f)	4	.0466		9.32	.0001	
p·f	4	.0770		15.40	.0001	
(n/f)·f	4	.0822		16.44	.0001	
(N/n)·p·(n/f)	20	.0308		1.23	.2181	
(N/n)·p·f	20	.1691		6.77	.0001	
(N/n)·(n/f)·f	20	.0666		2.67	.0001	
p·(n/f)·f	8	.0714		7.14	.0001	
(N/n)·p·(n/f)·f	40	.0791		1.58	.0123	

TABLE 12

Analysis of Variance of Means Square Errors for the Third Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	107	.9559	.0089	15.10	.0001	.6244
Error	972	.5750	.0006			
Grand Mean	1079	1.5309				
N/n	5	.5067		171.30	.0001	
p	2	.0798		67.43	.0001	
n/f	2	.0551		46.53	.0001	
f	1	.0158		26.78	.0001	
(N/n)·p	10	.0225		3.80	.0001	
(N/n)·(n/f)	10	.0323		5.45	.0001	
(N/n)·f	5	.0561		18.96	.0001	
p·(n/f)	4	.0373		15.76	.0001	
p·f	2	.0141		11.90	.0001	
(n/f)·f	2	.0312		26.35	.0001	
(N/n)·p·(n/f)	20	.0174		1.47	.0846	
(N/n)·p·f	10	.0119		2.01	.0291	
(N/n)·(n/f)·f	10	.0145		2.45	.0068	
p·(n/f)·f	4	.0449		18.99	.0001	
(N/n)·p·(n/f)·f	20	.0164		1.39	.1183	

TABLE 13

Analysis of Variance of Means Square Errors for the Fourth Population Factor When Samples are Factored to the Correct Number of Factors

Source	Degrees of Freedom	Sum of Squares	Mean Square	F	Pr.>F	R-Square
Total Model	107	1.3744	.0128	13.93	.0001	.6053
Error	972	.8963	.009			
Grand Mean	1079	2.2707				
N/n	5	.7201		156.19	.0001	
p	2	.0765		41.46	.0001	
n/f	2	.0923		50.06	.0001	
f	1	.0223		24.21	.0001	
(N/n)·p	10	.0354		3.84	.0001	
(N/n)·(n/f)	10	.0831		9.01	.0001	
(N/n)·f	5	.0389		8.44	.0001	
p·(n/f)	4	.0219		5.94	.0001	
p·f	2	.0462		25.07	.0001	
(n/f)·f	2	.0051		2.76	.0637	
(N/n)·p·(n/f)	20	.0392		2.12	.0028	
(N/n)·p·f	10	.0684		7.42	.0001	
(N/n)·(n/f)·f	10	.0107		1.16	.3136	
p·(n/f)·f	4	.0595		16.14	.0001	
(N/n)·p·(n/f)·f	20	.0546		2.96	.0001	

TABLE 14

Regression Analysis of Means Square Errors for the First
Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	1.3533	.3383	F = 213.35	Pr>F = .0001
	Error	1615	2.5610	.0016		
	Total	1619	3.9143			
.1563	Intercept	1			t = 22.84	Pr>|t| = .0001
-.05256	ln (N/n)	1			t = -12.96	Pr>|t| = .0001
-.1108	P	1			t = -13.74	Pr>|t| = .0001
-.004396	n/f	1			t = -7.26	Pr>|t| = .0001
.04265	p · ln (N/n)	1			t = 7.56	Pr>|t| = .0001

R-square = .3457

TABLE 15

Regression Analysis of Means Square Errors for the Second Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	1.8256	.4564	F = 212.27	Pr>F = .0001
	Error	1615	3.4725	.0022		
	Total	1619	5.2981			
.1805	Intercept	1			t = 22.66	Pr>|t| = .0001
-.0582	ln (N/n)	1			t = -12.33	Pr>|t| = .0001
-.1125	P	1			t = -11.98	Pr>|t| = .0001
-.007127	n/f	1			t = -10.10	Pr>|t| = .0001
.04555	p · ln (N/n)	1			t = 6.93	Pr>|t| = .0001

R-square = .3446

TABLE 16

Regression Analysis of Means Square Errors for the Third
Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.5868	.1467	F = 167.03	Pr>F = .0001
	Error	1075	.9441	.0009		
	Total	1079	1.5309			
.1235	Intercept	1			t = 19.81	Pr>|t| = .0001
-.0355	ln (N/n)	1			t = -9.61	Pr>|t| = .0001
-.07495	P	1			t = -10.20	Pr>|t| = .0001
-.003787	n/f	1			t = -6.86	Pr>|t| = .0001
.02366	p · ln (N/n)	1			t = 4.60	Pr>|t| = .0001

R-square = .3833

TABLE 17

Regression Analysis of Means Square Errors for the Fourth
Population Factor When Samples are Factored to the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.8100	.2025	F = 149.05	Pr>F = .0001
	Error	1075	1.4606	.0014		
	Total	1079	2.2706			
.1310	Intercept	1			t = 16.89	Pr>|t| = .0001
-.03499	ln (N/n)	1			t = -7.61	Pr>|t| = .0001
-.06691	P	1			t = -7.32	Pr>|t| = .0001
-.004976	n/f	1			t = -7.24	Pr>|t| = .0001
.01724	p · ln (N/n)	1			t = 2.70	Pr>|t| = .0071

R-square = .3567

Evaluation of Regression Equations

Goodness of fit. Adding the sums of squares in tables 10 through 13 associated with the effects in the regression equations in tables 14 through 17 and dividing by the respective corrected total sums of squares results in the following maximum proportions of variance which these effects could possibly account for in the samples:

factor 1 - .3931

factor 2 - .3915

factor 3 - .4337

factor 4 - .4071

The proportions of variance these effects (or transformations) account for in the regression equations in tables 14 through 17 are the following R-square terms:

factor 1 - .3457

factor 2 - .3446

factor 3 - .3833

factor 4 - .3567

Again, there is little variance left unexplained. Any additional curve fitting, beyond the natural logarithm transformation that was performed, would prove to be of little or no practical consequence.

Residual analysis. High correlations of the dependent variables with the residuals from the regression equations occur once again. These are:

factor 1 - .8089

factor 2 - .8096

factor 3 - .7853

factor 4 - .8020

Correlations of the means square errors with all cell mean deviations result in:

factor 1 - .6154

factor 2 - .5864

factor 3 - .6129

factor 4 - .6283

Again, the nature of the dependent variables is such that, given regression equations which predict every one of the 162 cell means exactly, substantial correlations between dependent variables and residuals will result.

Employing Tchebycheff's inequality results in statistical significance at $p<.05$ for every variable in every equation except the interaction term in table 17. Inclusion of this term has elsewhere been justified.

Monotonicity. Each effect, n/f, N/n, and p is again examined for monotonicity. Two effects, n/f and N/n, involve more than one variable and therefore these variables are also examined.

Unlike the congruence coefficient, where higher values are associated with greater accuracy, the mean square error requires smaller values for greater accuracy. Holding all but one of the effects in the regression equations constant, increasing the value of the effect free to vary results in decreased expected means square errors and thus improved accuracy. The proofs are similar to those used for the congruence coefficients.

The general form of the equations in tables 14 through 17 is

$$\hat{y} = b_o + b_1 \ln(N/n) + b_2 p + b_3 n/f + b_4 p \cdot \ln(N/n),$$

where $\hat{y}$ is the expected mean square error,

$$b_1, b_2, b_3 < 0 \text{ and } b_4 > 0.$$

The first effect to be examined is n/f. Since b_3, the regression coefficient is always negative, increasing the number of variables per factor, n/f, while holding constant the other effects, always results in decreased values of the expected means square errors. But, as was the case with congruence coefficients, n/f can be increased by increasing n, the number of variables, or by decreasing f, the number of population factors. Since decreasing f does not affect any other terms in the regression equations, it can be concluded that, holding constant the other variables (i.e., N, p, and n), the fewer the number of factors in the population, the lower will be the predicted means square errors.

Taking the partial derivative of $\hat{y}$ with respect to n results in $b_3/f - (1/n)(b_1 + b_4 p)$, which is the equation for the slope of y when holding constant all variables except n. When this equation is negative, that is for $n > f \cdot (b_1 + b_4 p)/b_3$, the equation is said to be non-increasing monotonic, and increasing n results in smaller expected values of the means square errors and therefore improved accuracy.

Once again, since p and f are held constant, any additional variables added to the sample should have, on the average, the same communalities as the original variables and not result in additional factors.

The next effect examined is N/n. The variable n was demonstrated to be nonincreasing monotonic and therefore only N needs to be examined.

Allowing only N to vary requires

$$b_1 \ln(\tfrac{N+c}{n}) + b_4 p \cdot \ln(\tfrac{N+c}{n}) < b_1 \ln(N/n) + b_4 p \cdot \ln(N/n),$$

where $c > 0$. Combining common terms results in

$$\ln(\tfrac{N+c}{n}) \cdot (b_1 + b_4 p) < \ln(N/n)\ (b_1 + b_4 p).$$

This inequality holds whenever $b_1 + b_4 p < 0$, since dividing both sides of an inequality by a negative number reverses the inequality sign. It should be noted that $b_1 + b_4 p < 0$ holds for the regression equations in tables 14 through 17.

The final effect to be examined for monotonicity is p. Holding constant all effects except p requires

$$b_2(p + c) + b_4(p + c) \cdot \ln(N/n) < b_2 p + b_4 p \cdot \ln(N/n)$$

where $c > 0$ and $p + c \leq 1$. Combining common terms results in

$$(p + c) \cdot (b_2 + b_4 \cdot \ln(N/n)) < p \cdot (b_2 + b_4 \cdot \ln(N/n)).$$

For this inequality to hold $b_2 + b_4 \ln(N/n) < 0$, which is true for each equation except the one in table 15. And in the equation in table 15 this inequality is true up to $N/n = 11.82$. Therefore, for all practical purposes, increasing the values of any effect (i.e., N/n, p, n/f) in tables 14 through 17, while holding constant the values of the other effects results in decreased values of the expected means square errors and thus improved accuracy.

Discussion of Correct Sample Factoring

Table 18 contains predicted or expected values of the means square errors for various values of the independent effects in tables 14 through 17 (see also Appendix B). A value of a mean square error which indicates that the sample factors reasonably map the population factors was even more difficult to define than was the value for the congruence

TABLE 18

Expected Means Square Errors for the First Four Population Factors With Correct Factoring for Various Values of the Number of Observations Per Variable, N/n, the Number of Variables Per Factor, n/f, and Proportion of Common Variance, p

p	n/f	N/n	Factor 1	Factor 2	Factor 3	Factor 4
.9	5	.5	.0444	.0555	.0470	.0594
.9	8	.5	.0312	.0342	.0356	.0445
.9	4	2.5	.0260	.0350	.0279	.0330
.9	6	2.5	.0172	.0207	.0203	.0231
.5	7	3	.0358	.0354	.0335	.0337
.7	7	3	.0230	.0230	.0237	.0241

coefficients. Due to this difficulty a mean square error of .035 was chosen. This is the expected value of the mean square error for factor 1 given a congruence coefficient of .9. This value of .035 is for discussion purposes only and is not meant to imply that this is or will be acceptable to investigators.

Because of the regression equations being nonincreasing monotonic only the minimum values of N/n, p, and n/f which result in expected means square errors of .035 will be discussed. Table 18 will be used for this discussion.

When N/n = .5 and f = 2, p must be equal to .9 and n/f must be equal to 8 to result in expected means square errors around .035 for the two factors. This would require 16 variables and only 8 observations. When the population has four factors the values of N/n, p, and n/f remain the same but the number of variables and observations required doubles.

If the proportion of common variance in the population is only .5 and there are two factors then three observations per variable and seven variables per factor are necessary. This would necessitate 14 variables and 42 observations. Having four population factors would require doubling the number of variables and observation required.

If, on the average, there are only four variables per factor then N/n must at least be 2.5 and p must be .9 to result in expected means square errors around .035 for two and four factors. For f = 2 there must be 8 variables and 20 observations. Again, when the population contains four factors the necessary number of variables and observations doubles. Once again the small sample size required to achieve reasonable accuracy is remarkable.

Underfactoring Versus Overfactoring

Tables 19 and 20 contain the results of correlated t tests used to determine the statistical significance of the difference between means for congruence coefficients and means square errors respectively for the conditions of underfactoring versus overfactoring. In every instance, overfactoring results in greater accuracy than does underfactoring.

Multivariate analysis of variance of repeated measurements was not used; but the significance levels are so small that the readers should feel assured that there was no capitalization on chance.

Even though overfactoring is superior to underfactoring the issue is moot because one needs to know the exact number of factors in the population before one can underfactor or overfactor; and then it makes no sense to do either. The safest advice for a researcher would be, "When in doubt extract one additional factor." A word of caution is in order here. This advice was not explicitly tested and is at best an inference.

Since a researcher may use the effects for exact factoring contained in tables 10 through 13 on pages 61 through 64 to estimate factorial accuracy, it might prove beneficial to observe the consequence of using these effects when in fact the researcher actually under- or overfactored. Tables 21 through 24 contain the regression equations for congruence coefficients for the condition of underfactoring while tables 25 through 28 contain the regression equations for congruence coefficients for the condition of overfactoring.

TABLE 19

T Tests for Correlated Mean Differences for the Conditions of Overfactoring Versus Underfactoring When the Dependent Variables are the Congruence Coefficients

	Overfactor		Underfactor					
Factor	Mean	Standard Deviation	Mean	Standard Deviation	N	Correlation	t	p
1	.9083	.1160	.8394	.1864	1316	.514	15.5	.0001
2	.8881	.1260	.8561	.1470	837	.726	9.0	.0001
3	.8634	.1573	.7946	.2252	837	.552	10.4	.0001
4	.8199	.1694	.7700	.2011	376	.571	5.6	.0001

TABLE 20

T Tests for Correlated Mean Differences for the Conditions of Overfactoring Versus Underfactoring When the Dependent Variables are the Means Square Errors

	Overfactor		Underfactor					
Factor	Mean	Standard Deviation	Mean	Standard Deviation	N	Correlation	t	p
1	.0338	.0482	.0736	.1134	1316	.489	-14.6	.0001
2	.0267	.0306	.0391	.0461	837	.668	-10.4	.0001
3	.0299	.0345	.0536	.0694	837	.550	-11.8	.0001
4	.0262	.0233	.0357	.0295	376	.581	-7.4	.0001

TABLE 21

Regression Analysis of Congruence Coefficients for the First Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	8.2101	2.0525	F = 77.02	Pr>F = .0001
	Error	1615	43.0389	.0266		
	Total	1619	51.2491			
.5272	Intercept	1			t = 18.80	Pr>|t| = .0001
.1003	ln (N/n)	1			t = 6.04	Pr>|t| = .0001
.3449	P	1			t = 10.43	Pr>|t| = .0001
.006139	n/f	1			t = 2.47	Pr>|t| = .0135
-.07327	p · ln (N/n)	1			t = -3.17	Pr>|t| = .0016

R-square = .1602

TABLE 22

Regression Analysis of Congruence Coefficients for the Second Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	7.5189	1.8797	F = 135.56	Pr>F = .0001
	Error	1075	14.9067	.0139		
	Total	1079	22.4256			
.5287	Intercept	1			t = 21.33	Pr>|t| = .0001
.1471	ln (N/n)	1			t = 10.02	Pr>|t| = .0001
.3781	P	1			t = 12.94	Pr>|t| = .0001
.0017	n/f	1			t = .80	Pr>|t| = .4264
-.1195	p · ln (N/n)	1			t = -5.85	Pr>|t| = .0001

R-square = .3353

TABLE 23

Regression Analysis of Congruence Coefficients for the Third Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	14.6101	3.6525	F = 117.01	Pr>F = .0001
	Error	1075	33.5567	.0312		
	Total	1079	48.1668			
.4171	Intercept	1			t = 11.22	Pr>|t| = .0001
.1606	ln (N/n)	1			t = 7.29	Pr>|t| = .0001
.5441	P	1			t = 12.41	Pr>|t| = .0001
-.0106	n/f	1			t = -3.23	Pr>|t| = .0013
-.1125	p · ln (N/n)	1			t = -3.67	Pr>|t| = .0003

R-square = .3033

TABLE 24

Regression Analysis of Congruence Coefficients for the Fourth Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	5.4876	1.3719	F = 49.85	Pr>F = .0001
	Error	535	14.7223	.0275		
	Total	539	20.2099			
.4257	Intercept	1			t = 8.62	Pr>|t| = .0001
.1786	ln (N/n)	1			t = 6.10	Pr>|t| = .0001
.5389	P	1			t = 9.26	Pr>|t| = .0001
-.0104	n/f	1			t = -2.38	Pr>|t| = .0176
-.1603	p · ln (N/n)	1			t = -3.94	Pr>|t| = .0001

R-square = .2715

TABLE 25

Regression Analysis of Congruence Coefficients for the First Population Factor
When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	7.7391	1.9348	F = 254.71	Pr>F = .0001
	Error	1311	9.9582	.0076		
	Total	1315	17.6973			
.5366	Intercept	1			t = 31.87	Pr>|t| = .0001
.1273	ln (N/n)	1			t = 12.50	Pr>|t| = .0001
.4294	P	1			t = 20.57	Pr>|t| = .0001
.0063	n/f	1			t = 4.14	Pr>|t| = .0001
-.1159	p · ln (N/n)	1			t = -7.80	Pr>|t| = .0001

R-square = .4373

TABLE 26

Regression Analysis of Congruence Coefficients for the Second Population Factor When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	8.7936	2.1984	F = 255.61	Pr>F = .0001
	Error	1311	11.2756	.0086		
	Total	1315	20.0692			
.4931	Intercept	1			t = 27.52	Pr>|t| = .0001
.1425	ln (N/n)	1			t = 13.16	Pr>|t| = .0001
.3957	P	1			t = 17.81	Pr>|t| = .0001
.0151	n/f	1			t = 9.28	Pr>|t| = .0001
-.1281	p · ln (N/n)	1			t = -8.10	Pr>|t| = .0001

R-square = .4382

TABLE 27

Regression Analysis of Congruence Coefficients for the Third Population Factor
When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	9.3062	2.3266	F = 120.27	Pr>F = .0001
	Error	832	11.3684	.0137		
	Total	836	20.6746			
.3314	Intercept	1			t = 11.23	Pr>|t| = .0001
.1739	ln (N/n)	1			t = 9.87	Pr>|t| = .0001
.5598	P	1			t = 14.59	Pr>|t| = .0001
.0156	n/f	1			t = 6.09	Pr>|t| = .0001
-.1555	p · ln (N/n)	1			t = -5.91	Pr>|t| = .0001

R-square = .4501

TABLE 28

Regression Analysis of Congruence Coefficients for the Fourth Population Factor When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	11.4566	2.8641	F = 143.48	Pr>F = .0001
	Error	832	16.6085	.0200		
	Total	836	28.0651			
.3327	Intercept	1			t = 9.33	Pr>|t| = .0001
.1597	ln (N/n)	1			t = 7.50	Pr>|t| = .0001
.4773	P	1			t = 10.29	Pr>|t| = .0001
.0161	n/f	1			t = 5.20	Pr>|t| = .0001
-.1049	p · ln (N/n)	1			t = -3.30	Pr>|t| = .0001

R-square = .4082

When the data are overfactored by one factor and the result is two factors, the minimum value of N/n to achieve expected congruence coefficients around .90 is .5. This would require p to be .9 and n/f to be 4. The expected congruence coefficients are .9323 and .8907 respectively for the two factors. The necessary number of observations is 4. Had the data been underfactored by one factor the lowest value of N/n to achieve expected congruence coefficients around .90 (.8983 and .9068) would be 2. This would require p to be .9 and n/f to be 6. This would necessitate 12 variables (only 8 were needed for overfactoring) and 24 observations. This is obviously a small number, but it is six times the number required for overfactoring.

Again, if there were two factors and the proportion of common variance in the population were .5 it would require N/n to be 5 and n/f to be 8 for the condition of overfactoring. The expected congruence coefficients would then be .8979 and .9204. The number of observations required for the 16 variables would be 64. Had the data been underfactored by one, given that p = .5, it would have required N/n to be 12 and n/f to be 6 to result in expected congruence coefficients of .8948 and .9452. Only 12 variables would be required but 144 observations would be necessary.

Tables 29 through 32 contain the regression equations for the means square errors for the condition of underfactoring while tables 33 through 36 contain the regression equations for the means square error for the condition of overfactoring. Without belaboring the point the results of these regression equations indicate a marked increase in necessary sample size to achieve expected means square errors around .035 when one happens to have underfactored rather than overfactored.

TABLE 29

Regression Analysis of Means Square Errors for the First Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	1.3369	.3342	F = 32.36	Pr>F = .0001
	Error	1615	16.6827	.0103		
	Total	1619	18.0196			
.1476	Intercept	1			t = 8.45	Pr>|t| = .0001
-.0333	ln (N/n)	1			t = -3.22	Pr>|t| = .0013
-.0303	P	1			t = -1.47	Pr>|t| = .1409
-.0061	n/f	1			t = -3.94	Pr>|t| = .0001
.0121	p · ln (N/n)	1			t = .84	Pr>|t| = .3991

R-square = .0742

TABLE 30

Regression Analysis of Means Square Errors for the Second Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.5486	.1372	F = 86.74	Pr>F = .0001
	Error	1075	1.6997	.0016		
	Total	1079	2.2483			
.0989	Intercept	1			t = 11.82	Pr>|t| = .0001
-.0347	ln (N/n)	1			t = -7.01	Pr>|t| = .0001
-.0517	P	1			t = -5.25	Pr>|t| = .0001
-.0011	n/f	1			t = -1.45	Pr>|t| = .1483
.0210	p · ln (N/n)	1			t = 3.04	Pr>|t| = .0024

R-square = .2440

TABLE 31

Regression Analysis of Means Square Errors for the Third Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.8698	.2174	F = 12.71	Pr>F = .0001
	Error	1075	3.7275	.0035		
	Total	1079	4.5973			
.1097	Intercept	1			t = 8.86	Pr>|t| = .0001
-.0351	ln (N/n)	1			t = -4.78	Pr>|t| = .0001
-.0706	P	1			t = -4.83	Pr>|t| = .0001
.0021	n/f	1			t = 1.88	Pr>|t| = .0607
.0149	p · ln (N/n)	1			t = 1.46	Pr>|t| = .1450

R-square = .1892

TABLE 32

Regression Analysis of Means Square Errors for the Fourth Population Factor
When Samples are Factored to One Less Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.1113	.0278	F = 41.19	Pr>F = .0001
	Error	535	.3612	.0007		
	Total	539	.4725			
.0791	Intercept	1			t = 10.23	Pr>|t| = .0001
-.0235	ln (N/n)	1			t = -5.13	Pr>|t| = .0001
-.0506	P	1			t = -5.55	Pr>|t| = .0001
.00001	n/f	1			t = .02	Pr>|t| = .9841
.0165	p · ln (N/n)	1			t = 2.59	Pr>|t| = .0098

R-square = .2355

TABLE 33

Regression Analysis of Means Square Errors for the First Population Factor
When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	1.0808	.2702	F =179.35	Pr>F = .0001
	Error	1311	1.9750	.0015		
	Total	1315	3.0558			
.1367	Intercept	1			t = 18.24	Pr>|t| = .0001
-.0384	ln (N/n)	1			t = -8.47	Pr>|t| = .0001
-.0736	P	1			t = -7.91	Pr>|t| = .0001
-.0051	n/f	1			t = -7.50	Pr>|t| = .0001
.0210	p · ln (N/n)	1			t = 3.17	Pr>|t| = .0016

R-square = .3537

TABLE 34

Regression Analysis of Means Square Errors for the Second Population Factor
When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	1.1367	.2842	F = 200.42	Pr>F = .0001
	Error	1311	1.8589	.0014		
	Total	1315	2.9956			
.1404	Intercept	1			t = 19.30	Pr>|t| = .0001
-.0394	ln (N/n)	1			t = -8.96	Pr>|t| = .0001
-.0665	P	1			t = -7.37	Pr>|t| = .0001
-.0066	n/f	1			t = -9.99	Pr>|t| = .0001
.0221	p · ln (N/n)	1			t = 3.44	Pr>|t| = .0006

R-square = .3795

TABLE 35

Regression Analysis of Means Square Errors for the Third Population Factor
When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.4328	.1082	F = 159.37	Pr>F = .0001
	Error	832	.5648	.0007		
	Total	836	.9976			
.1211	Intercept	1			t = 18.41	Pr>|t| = .0001
-.0312	ln (N/n)	1			t = -7.94	Pr>|t| = .0001
-.0714	P	1			t = -8.35	Pr>|t| = .0001
-.0043	n/f	1			t = -7.48	Pr>|t| = .0001
.0189	p · ln (N/n)	1			t = 3.22	Pr>|t| = .0013

R-square = .4338

TABLE 36

Regression Analysis of Means Square Errors for the Fourth Population Factor
When Samples are Factored to One More Than the Correct Numbers of Factors

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.3761	.0940	F = 134.28	Pr>F = .0001
	Error	832	.5826	.0007		
	Total	836	.9587			
.0959	Intercept	1			t = 14.36	Pr>|t| = .0001
-.0274	ln (N/n)	1			t = -6.87	Pr>|t| = .0001
-.0547	P	1			t = -6.30	Pr>|t| = .0001
-.0019	n/f	1			t = -3.31	Pr>|t| = .0010
.0130	p · ln (N/n)	1			t = 2.19	Pr>|t| = .0288

R-square = .3923

The key point to be made is that even though overfactoring is consistently statistically preferable to underfactoring the reader might not judge the actual mean differences contained in tables 19 and 20 to be of any practical significance. The drastic increase in sample sizes to achieve parity from underfactoring to overfactoring should be enough to justify the practical benefits of overfactoring.

Scree Test

The purpose of the scree test is to provide the researcher with criterion for distinguishing real, nonchance factors from chance or error factors. In other words, the scree test provides the researcher with a means for estimating the number of factors which exist in the population. When this estimate is used instead of the actual number of population factors, one obtains the equations contained in tables 37 through 44. Tables 37 through 40 use the congruence coefficients as the dependent variable while tables 41 through 44 use means square errors. The letter, f, in the tables is now used to represent the numbers of factors as determined by the scree test. Only tables 37 through 40 will be discussed since many of the following criticisms of the scree test are also applicable when the means square errors are the dependent variables.

Even though the squared multiple correlations (R-square) are high the use of the equations in this section yields results contrary to those equations when the correct number of factors are known (tables 5-8 and 14-17, pages 44-47 and 65-68). The regression coefficients for n/f, where f is the number of factors as determined by the scree test, are invariably negative. Therefore, increasing the number of factors,

TABLE 37

Regression Analysis of Congruence Coefficients for the First Population Factors When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	12.9678	3.2420	F =147.26	Pr>F = .0001
	Error	1368	30.1159	.0220		
	Total	1372	43.0837			
.7047	Intercept	1			t = 27.76	Pr>|t| = .0001
.1291	ln (N/n)	1			t = 7.69	Pr>|t| = .0001
.2702	P	1			t = 7.88	Pr>|t| = .0001
-.0101	n/f	1			t =-18.31	Pr>|t| = .0001
-.1181	p · ln (N/n)	1			t = -5.06	Pr>|t| = .0001

R-square = .3010

TABLE 38

Regression Analysis of Congruence Coefficients for the Second Population Factors When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	30.5949	7.6487	F =250.59	Pr>F = .0001
	Error	1368	41.7552	.0305		
	Total	1372	72.3501			
.8161	Intercept	1			t = 27.30	Pr>|t| = .0001
.1403	ln (N/n)	1			t = 7.10	Pr>|t| = .0001
.1603	P	1			t = 3.97	Pr>|t| = .0001
-.0172	n/f	1			t =-26.49	Pr>|t| = .0001
-.1107	p · ln (N/n)	1			t = -4.02	Pr>|t| = .0001

R-square = .4229

TABLE 39

Regression Analysis of Congruence Coefficients for the Third Population Factors When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	18.3858	4.5964	F = 89.97	Pr>F = .0001
	Error	752	38.4169	.0510		
	Total	756	56.8027			
.8070	Intercept	1			t = 15.97	Pr>|t| = .0001
.1419	ln (N/n)	1			t = 4.24	Pr>|t| = .0001
.1412	P	1			t = 2.07	Pr>|t| = .0391
-.0232	n/f	1			t =-16.18	Pr>|t| = .0001
-.1604	p · ln (N/n)	1			t = -2.23	Pr>|t| = .0262

R-square = .3237

TABLE 40

Regression Analysis of Congruence Coefficients for the Fourth Population Factors When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	8.2762	2.0690	F = 39.03	Pr>F = .0001
	Error	389	20.6191	.0530		
	Total	393	28.8953			
.7545	Intercept	1			t = 10.77	Pr>|t| = .0001
.1737	ln (N/n)	1			t = 3.70	Pr>|t| = .0002
.2481	P	1			t = 2.60	Pr>|t| = .0096
-.0332	n/f	1			t = -10.56	Pr>|t| = .0001
-.1811	p · ln (N/n)	1			t = -2.64	Pr>|t| = .0086

R-square = .2864

TABLE 41

Regression Analysis of Means Square Errors for the First Population Factors
When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.9388	.2347	F = 105.55	Pr>F = .0001
	Error	1368	3.0421	.0022		
	Total	1372	3.9809			
.0716	Intercept	1			t = 8.87	Pr>|t| = .0001
-.0464	ln (N/n)	1			t = -8.70	Pr>|t| = .0001
-.0237	P	1			t = -2.18	Pr>|t| = .0297
.0016	n/f	1			t = 9.29	Pr>|t| = .0001
.0381	p · ln (N/n)	1			t = 5.13	Pr>|t| = .0074

R-square = .2358

TABLE 42

Regression Analysis of Means Square Errors for the Second Population Factors
When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	2.0089	.5022	F = 154.54	Pr>F = .0001
	Error	1368	4.4458	.0032		
	Total	1372	6.4547			
.0395	Intercept	1			t = 4.05	Pr>|t| = .0001
-.0402	ln (N/n)	1			t = -6.24	Pr>|t| = .0001
.0120	P	1			t = .91	Pr>|t| = .3622
.0033	n/f	1			t = 15.55	Pr>|t| = .0001
.0217	p · ln (N/n)	1			t = 2.42	Pr>|t| = .0155

R-square = .3112

TABLE 43

Regression Analysis of Means Square Errors for the Third Population Factors When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.9969	.2492	F = 70.88	Pr>F = .0001
	Error	752	2.6439	.0035		
	Total	756	3.6408			
.0169	Intercept	1			t = 1.27	Pr>|t| = .2031
-.0316	ln (N/n)	1			t = -3.59	Pr>|t| = .0003
.0422	P	1			t = 2.35	Pr>|t| = .0189
.0040	n/f	1			t = 10.74	Pr>|t| = .0001
.0129	p · ln (N/n)	1			t = 1.03	Pr>|t| = .3030

R-square = .2738

TABLE 44

Regression Analysis of Means Square Errors for the Fourth Population Factors
When the Numbers of Factors in the Samples are Determined by the Scree Test

Regression Coefficient	Source	DF	SS	MS	Test Statistic	p-level
	Model	4	.3890	.0972	F = 40.16	Pr>F = .0001
	Error	389	.9418	.0024		
	Total	393	1.3308			
.0061	Intercept	1			t = .41	Pr>|t| = .6818
-.0367	ln (N/n)	1			t = -3.66	Pr>|t| = .0003
.0421	P	1			t = 2.07	Pr>|t| = .0392
.0052	n/f	1			t = 7.75	Pr>|t| = .0001
.0332	p · ln (N/n)	1			t = 2.27	Pr>|t| = .0240

R-square = .2923

while holding constant the other variables in the equations, results in lower expected congruence coefficients. This is the reverse of what occurred when the correct numbers of factors are known. Also, the equations are nonmonotonic with respect to p. Recall that

$$b_2 + b_4 \ln(N/n) > 0$$

must be satisfied, where b_2 is the regression coefficient for p and b_4 is the regression coefficient for $p \cdot \ln(N/n)$, for monotonicity with respect to p. The equations in tables 37 through 40 are nondecreasing monotonic with respect to p up to N/n equalling 9.85, 4.25, 3.77, and 3.94 respectively. When N/n is greater than these values, larger values of p yield smaller expected congruence coefficients. In effect, these equations are not monotonic with respect to p within the data limits of this study and; therefore, these equations yield results contrary to those results from the equations when the correct numbers of factors are known.

Due to the nonmonotonicity of p and monotonicity in the wrong direction for n/f, the use of this particular version of the scree test is not justified. These criticisms also add emphasis to the importance of finding a procedure which accurately estimates the correct number of population factors.

CHAPTER V

SUMMARY AND RECOMMENDATIONS

Population and Sample Generation

Twenty-seven population correlation matrices were randomly generated with specified numbers of factors (2, 4, 9), proportions of common variance (.5, .7, .9) and ratios of the numbers of variables to the numbers of factors (4, 6, 8). Using the true population communalities in the diagonals these 27 matrices were factored to the correct numbers of factors via the principal axes procedure. These factors were then rotated via the normalized Varimax procedure.

Sixteen hundred and twenty sample correlation matrices with different ratios of observations per variable (.5, 1, 2, 4, 6, 12) were randomly generated. These 1,620 sample matrices were factored using principal axes factor analysis (no iterations). Squared multiple correlations, when the inverse of the matrix existed, were used as communality estimates; otherwise the absolute value of the highest off diagonal correlation was used.

Four separate Varimax rotations were performed for each sample matrix. Each matrix was underfactored by one factor, factored correctly, overfactored by one factor, and factored to the number of factors as determined by a scree test.

Congruence coefficients and means square errors between factor loadings on the rotated population factors and factor loadings on the corresponding rotated sample factors were computed. Corresponding

factors were defined as those sample factors which had the highest congruence coefficients with the population factors.

Correct Sample Factoring

Analyses of variance and regression analyses were performed on the congruence coefficients and means square errors resulting from factoring to the correct numbers of factors. The only effects which were allowed to remain in the regression equations were the ones that were significant at $p < .0001$, across the factors in both the analyses of variance and the regression analyses. The following effects met the above criteria: the ratio of observations to variables (N/n) through a natural logarithm transformation, the proportion of common variance (p) in the population, the ratio of the number of variables per factor (n/f). and the interaction of the natural logarithm of N/n and p.

It was demonstrated that each effect (i.e., N/n, p, n/f) was monotonic for congruence coefficients and means square errors. Increasing the value of any of these effects results in improved accuracy. Also, for congruence coefficients and means square errors, it was demonstrated that each variable (i.e., N, n, p, f) was also monotonic. Increasing the values of N, n, or p result in improved accuracy. Decreasing the value of f also results in improved accuracy.

In order to achieve expected congruence coefficients of about .9 it was demonstrated that an investigator could have more variables than observations and still reach this expected level. For example, if $N/n = .5$, $p = .9$, and $n/f = .5$, the expected congruence coefficients for the first two factors are .9202 and .8903. This occurs when $n = 10$ and $N = 5$. If there are four population factors and $N/n = .5$, $p = .9$,

and n/f = 8 the resultant expected congruence coefficients for the four factors are .9395, .9359, .8913, and .8801. This will be achieved when there are 32 varaibles and only 16 observations. In essence, large numbers of variables, few factors, and high intercorrelations among the variables compensate for small sample sizes.

Underfactoring Versus Overfactoring

Overfactoring was demonstrated to consistently result in greater factorial accuracy than underfactoring. Sample sizes would have to be drastically increased when the data are underfactored to achieve equal levels of factorial accuracy with overfactoring.

Scree Test

Using the same effects as were used in the regression equations for correct factoring, substituting the number of factors as determined by the scree test for f, in regression equations on data from the scree test, resulted in equations whose effects were contrary to what is actually occurring when the correct numbers of factors are known. The effect, n/f, where f is now the number of factors determined by the scree test is monotonic in the wrong direction. Increasing this ratio results in less factorial accuracy. The effect p is no longer monotonic. Due to these criticisms the version of the scree test used in this investigation is not warranted.

Recommendations

Due to the failure of this version of the scree test and the fact that a researcher does not know whether the data are underfactored, overfactored, or factored to the correct number of factors the researcher is faced with the dilemma as to which sets of equations to use to estimate the required sample size. Quite obviously if an investigator knew that the data were overfactored or underfactored the investigator would refactor the data to the correct number of factors (or at least refactor to an estimate of the correct number of factors). It is therefore recommended that the equations for exact factoring in tables 5 through 8 on pages 44 through 47 or tables 14 through 17 on pages 65 through 68 be used to estimate a minimum sample size; but with certain conservative constraints.

It is suggested, until future research dictates otherwise, that a combination of procedures be used to estimate the correct numbers of factors. Using Harris' (1967) suggestion to factor and rotate the data set by three or four different procedures would be a good start. With each procedure the scree test should be applied visually, that is by plotting the eigenvalues and also ensuring that each factor has at least three or four salient loadings. The most conservative estimate of the correct number of factors would be the number of factors which are consistent across methods. This might prove to be overly conservative, since any incorrect estimations would probably be underestimations, and underestimating the correct numbers of factors results in less factorial accuracy than correctly estimating or overestimating. A less conservative

approach would be to use the modal number of factors as determined by visually applying the scree tests to the results of the various procedures.

The proportion of common variance in the sample reduced by .05 should not be an unduly conservative estimate of the proportion of common variance in the population. This suggestion, along with the ones for estimating the correct numbers of factors, are unsupported empirically; but it is thought that these suggestions are reasonable enough to allow investigators to make use of the results of this study until such time as more accurate parameter estimates are obtained.

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APPENDIXES

APPENDIX A

EXPECTED CONGRUENCE COEFFICIENTS FOR THE FIRST FOUR POPULATION FACTORS WITH CORRECT FACTORING FOR VARIOUS VALUES OF THE POPULATION PROPORTION OF COMMON VARIANCE, p, NUMBER OF VARIABLES PER FACTOR, n/f, AND NUMBER OF OBSERVATIONS PER VARIABLE, N/n

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.5	4	0.5	0.7255	0.6670	0.6139	0.4924
.5	4	1.0	0.7757	0.7293	0.6818	0.5854
.5	4	1.5	0.8050	0.7656	0.7215	0.6398
.5	4	2.0	0.8258	0.7915	0.7496	0.6784
.5	4	2.5	0.8420	0.8115	0.7715	0.7083
.5	4	3.0	0.8552	0.8279	0.7894	0.7328
.5	4	3.5	0.8663	0.8417	0.8044	0.7535
.5	4	4.0	0.8760	0.8537	0.8175	0.7714
.5	4	4.5	0.8845	0.8643	0.8291	0.7872
.5	4	5.0	0.8922	0.8737	0.8394	0.8013
.5	4	5.5	0.8991	0.8823	0.8487	0.8141
.5	4	6.0	0.9054	0.8901	0.8572	0.8258
.5	4	6.5	0.9112	0.8973	0.8651	0.8365
.5	4	7.0	0.9165	0.9039	0.8723	0.8465
.5	4	7.5	0.9215	0.9101	0.8791	0.8557
.5	4	8.0	0.9262	0.9159	0.8854	0.8644
.5	4	8.5	0.9306	0.9213	0.8913	0.8725
.5	4	9.0	0.9347	0.9265	0.8969	0.8802
.5	4	9.5	0.9386	0.9313	0.9022	0.8875
.5	5	0.5	0.7319	0.6822	0.6262	0.5125
.5	5	1.0	0.7821	0.7444	0.6941	0.6055
.5	5	1.5	0.8115	0.7808	0.7338	0.6599
.5	5	2.0	0.8323	0.8066	0.7620	0.6985
.5	5	2.5	0.8484	0.8267	0.7838	0.7285
.5	5	3.0	0.8616	0.8430	0.8017	0.7529
.5	5	3.5	0.8728	0.8569	0.8168	0.7736
.5	5	4.0	0.8825	0.8689	0.8298	0.7915
.5	5	4.5	0.8910	0.8794	0.8414	0.8073
.5	5	5.0	0.8986	0.8889	0.8517	0.8215
.5	5	5.5	0.9055	0.8974	0.8610	0.8343
.5	5	6.0	0.9118	0.9053	0.8695	0.8459
.5	5	6.5	0.9176	0.9124	0.8774	0.8567
.5	5	7.0	0.9230	0.9191	0.8846	0.8666
.5	5	7.5	0.9280	0.9253	0.8914	0.8759
.5	5	8.0	0.9326	0.9311	0.8977	0.8845
.5	5	8.5	0.9370	0.9365	0.9037	0.8927
.5	5	9.0	0.9412	0.9416	0.9093	0.9003
.5	5	9.5	0.9451	0.9465	0.9145	0.9076
.5	6	0.5	0.7384	0.6974	0.6385	0.5327
.5	6	1.0	0.7886	0.7596	0.7064	0.6257
.5	6	1.0	0.7886	0.7596	0.7064	0.6257

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.5	6	1.5	0.8179	0.7960	0.7461	0.6801
.5	6	2.0	0.8387	0.8218	0.7743	0.7187
.5	6	2.5	0.8549	0.8419	0.7961	0.7486
.5	6	3.0	0.8681	0.8582	0.8140	0.7731
.5	6	3.5	0.8792	0.8721	0.8291	0.7938
.5	6	4.0	0.8889	0.8840	0.8422	0.8117
.5	6	4.5	0.8974	0.8946	0.8537	0.8275
.5	6	5.0	0.9050	0.9041	0.8640	0.8416
.5	6	5.5	0.9119	0.9126	0.8733	0.8544
.5	6	6.0	0.9182	0.9204	0.8819	0.8661
.5	6	6.5	0.9240	0.9276	0.8897	0.8768
.5	6	7.0	0.9294	0.9343	0.8970	0.8868
.5	6	7.5	0.9344	0.9405	0.9037	0.8960
.5	6	8.0	0.9391	0.9463	0.9100	0.9047
.5	6	8.5	0.9435	0.9517	0.9160	0.9128
.5	6	9.0	0.9476	0.9568	0.9216	0.9205
.5	6	9.5	0.9515	0.9617	0.9269	0.9277
.5	7	0.5	0.7448	0.7126	0.6508	0.5528
.5	7	1.0	0.7950	0.7748	0.7187	0.6458
.5	7	1.5	0.8243	0.8112	0.7584	0.7002
.5	7	2.0	0.8452	0.8370	0.7866	0.7388
.5	7	2.5	0.8613	0.8570	0.8084	0.7688
.5	7	3.0	0.8745	0.8734	0.8263	0.7932
.5	7	3.5	0.8857	0.8872	0.8414	0.8139
.5	7	4.0	0.8953	0.8992	0.8545	0.8318
.5	7	4.5	0.9039	0.9098	0.8660	0.8476
.5	7	5.0	0.9115	0.9192	0.8763	0.8618
.5	7	5.5	0.9184	0.9278	0.8857	0.8746
.5	7	6.0	0.9247	0.9356	0.8942	0.8862
.5	7	6.5	0.9305	0.9428	0.9020	0.8970
.5	7	7.0	0.9358	0.9494	0.9093	0.9069
.5	7	7.5	0.9408	0.9556	0.9160	0.9162
.5	7	8.0	0.9455	0.9614	0.9223	0.9248
.5	7	8.5	0.9499	0.9669	0.9283	0.9330
.5	7	9.0	0.9540	0.9720	0.9339	0.9406
.5	7	9.5	0.9580	0.9769	0.9392	0.9479
.5	8	0.5	0.7513	0.7278	0.6631	0.5730
.5	8	1.0	0.8014	0.7900	0.7310	0.6660
.5	8	1.5	0.8308	0.8264	0.7707	0.7204
.5	8	2.0	0.8516	0.8522	0.7989	0.7590
.5	8	2.0	0.8516	0.8522	0.7989	0.7590

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.5	8	2.5	0.8678	0.8722	0.8208	0.7889
.5	8	3.0	0.8810	0.8886	0.8386	0.8134
.5	8	3.5	0.8921	0.9024	0.8537	0.8341
.5	8	4.0	0.9018	0.9144	0.8668	0.8520
.5	8	4.5	0.9103	0.9250	0.8783	0.8678
.5	8	5.0	0.9179	0.9344	0.8886	0.8819
.5	8	5.5	0.9248	0.9430	0.8980	0.8947
.5	8	6.0	0.9311	0.9508	0.9065	0.9064
.5	8	6.5	0.9369	0.9580	0.9143	0.9171
.5	8	7.0	0.9423	0.9646	0.9216	0.9271
.5	8	7.5	0.9473	0.9708	0.9283	0.9363
.5	8	8.0	0.9520	0.9766	0.9347	0.9450
.5	8	8.5	0.9563	0.9821	0.9406	0.9531
.5	8	9.0	0.9605	0.9872	0.9462	0.9608
.5	8	9.5	0.9644	0.9920	0.9515	0.9680
.5	9	0.5	0.7577	0.7429	0.6755	0.5931
.6	4	0.5	0.7726	0.7191	0.6709	0.5692
.6	4	1.0	0.8151	0.7714	0.7294	0.6482
.6	4	1.5	0.8400	0.8021	0.7636	0.6945
.6	4	2.0	0.8576	0.8238	0.7879	0.7273
.6	4	2.5	0.8713	0.8407	0.8067	0.7528
.6	4	3.0	0.8825	0.8544	0.8221	0.7736
.6	4	3.5	0.8920	0.8661	0.8351	0.7911
.6	4	4.0	0.9002	0.8762	0.8464	0.8064
.6	4	4.5	0.9074	0.8851	0.8563	0.8198
.6	4	5.0	0.9138	0.8930	0.8652	0.8318
.6	4	5.5	0.9197	0.9002	0.8733	0.8427
.6	4	6.0	0.9250	0.9068	0.8806	0.8526
.6	4	6.5	0.9299	0.9128	0.8874	0.8618
.6	4	7.0	0.9345	0.9184	0.8936	0.8702
.6	4	7.5	0.9387	0.9237	0.8994	0.8781
.6	4	8.0	0.9427	0.9285	0.9049	0.8854
.6	4	8.5	0.9464	0.9331	0.9100	0.8924
.6	4	9.0	0.9499	0.9374	0.9148	0.8989
.6	4	9.5	0.9532	0.9415	0.9194	0.9050
.6	5	0.5	0.7790	0.7343	0.6832	0.5893
.6	5	1.0	0.8215	0.7866	0.7417	0.6684
.6	5	1.5	0.8464	0.8172	0.7759	0.7146
.6	5	2.0	0.8641	0.8390	0.8002	0.7474
.6	5	2.5	0.8778	0.8558	0.8190	0.7729
.6	5	2.5	0.8778	0.8558	0.8190	0.7729

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.6	5	3.0	0.8889	0.8696	0.8344	0.7937
.6	5	3.5	0.8984	0.8813	0.8474	0.8113
.6	5	4.0	0.9066	0.8913	0.8587	0.8265
.6	5	4.5	0.9138	0.9002	0.8686	0.8399
.6	5	5.0	0.9203	0.9082	0.8775	0.8520
.6	5	5.5	0.9261	0.9154	0.8856	0.8628
.6	5	6.0	0.9315	0.9220	0.8929	0.8728
.6	5	6.5	0.9364	0.9280	0.8997	0.8819
.6	5	7.0	0.9409	0.9336	0.9059	0.8903
.6	5	7.5	0.9452	0.9388	0.9117	0.8982
.6	5	8.0	0.9491	0.9437	0.9172	0.9056
.6	5	8.5	0.9529	0.9483	0.9223	0.9125
.6	5	9.0	0.9564	0.9526	0.9271	0.9190
.6	5	9.5	0.9597	0.9567	0.9317	0.9252
.6	6	0.5	0.7854	0.7494	0.6956	0.6095
.6	6	1.0	0.8280	0.8018	0.7540	0.6885
.6	6	1.5	0.8525	0.8324	0.7883	0.7348
.6	6	2.0	0.8705	0.8542	0.8125	0.7676
.6	6	2.5	0.8842	0.8710	0.8314	0.7930
.6	6	3.0	0.8954	0.8848	0.8467	0.8138
.6	6	3.5	0.9049	0.8964	0.8597	0.8314
.6	6	4.0	0.9130	0.9065	0.8710	0.8467
.6	6	4.5	0.9203	0.9154	0.8810	0.8601
.6	6	5.0	0.9267	0.9234	0.8898	0.8721
.6	6	5.5	0.9326	0.9306	0.8979	0.8830
.6	6	6.0	0.9379	0.9372	0.9052	0.8929
.6	6	6.5	0.9428	0.9432	0.9120	0.9020
.6	6	7.0	0.9474	0.9488	0.9182	0.9105
.6	6	7.5	0.9516	0.9540	0.9241	0.9184
.6	6	8.0	0.9556	0.9589	0.9295	0.9257
.6	6	8.5	0.9593	0.9635	0.9346	0.9326
.6	6	9.0	0.9628	0.9678	0.9394	0.9392
.6	6	9.5	0.9661	0.9719	0.9440	0.9453
.6	7	0.5	0.7919	0.7646	0.7079	0.6296
.6	7	1.0	0.8344	0.8170	0.7664	0.7087
.6	7	1.5	0.8593	0.8476	0.8006	0.7549
.6	7	2.0	0.8770	0.8693	0.8248	0.7877
.6	7	2.5	0.8906	0.8862	0.8437	0.8132
.6	7	3.0	0.9018	0.9000	0.8591	0.8340
.6	7	3.5	0.9113	0.9116	0.8721	0.8516
.6	7	3.5	0.9113	0.9116	0.8721	0.8516

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.6	7	4.0	0.9195	0.9217	0.8833	0.8668
.6	7	4.5	0.9267	0.9306	0.8933	0.8802
.6	7	5.0	0.9332	0.9386	0.9022	0.8923
.6	7	5.5	0.9390	0.9458	0.9102	0.9031
.6	7	6.0	0.9444	0.9523	0.9175	0.9130
.6	7	6.5	0.9493	0.9584	0.9243	0.9222
.6	7	7.0	0.9538	0.9640	0.9305	0.9306
.6	7	7.5	0.9581	0.9692	0.9364	0.9385
.6	7	8.0	0.9620	0.9741	0.9418	0.9459
.6	7	8.5	0.9657	0.9787	0.9469	0.9528
.6	7	9.0	0.9693	0.9830	0.9517	0.9593
.6	7	9.5	0.9726	0.9871	0.9563	0.9655
.6	8	0.5	0.7983	0.7798	0.7202	0.6497
.6	8	1.0	0.8409	0.8322	0.7787	0.7288
.6	8	1.5	0.8657	0.8628	0.8129	0.7751
.6	8	2.0	0.8834	0.8845	0.8372	0.8079
.6	8	2.5	0.8971	0.9014	0.8560	0.8333
.6	8	3.0	0.9083	0.9152	0.8714	0.8541
.6	8	3.5	0.9177	0.9268	0.8844	0.8717
.6	8	4.0	0.9259	0.9369	0.8956	0.8869
.6	8	4.5	0.9332	0.9458	0.9056	0.9004
.6	8	5.0	0.9396	0.9537	0.9145	0.9124
.6	8	5.5	0.9455	0.9609	0.9225	0.9233
.6	8	6.0	0.9508	0.9675	0.9299	0.9332
.6	8	6.5	0.9557	0.9736	0.9366	0.9423
.6	8	7.0	0.9603	0.9792	0.9429	0.9508
.6	8	7.5	0.9645	0.9844	0.9487	0.9586
.6	8	8.0	0.9685	0.9893	0.9541	0.9660
.6	8	8.5	0.9722	0.9938	0.9592	0.9729
.6	8	9.0	0.9757	0.9981	0.9641	0.9794
.6	8	9.5	0.9790	1.0022	0.9686	0.9856
.7	4	0.5	0.8196	0.7711	0.7280	0.6460
.7	4	1.0	0.8545	0.8136	0.7771	0.7111
.7	4	1.5	0.8749	0.8385	0.8058	0.7492
.7	4	2.0	0.8894	0.8561	0.8262	0.7762
.7	4	2.5	0.9006	0.8698	0.8420	0.7972
.7	4	3.0	0.9098	0.8810	0.8549	0.8143
.7	4	3.5	0.9176	0.8905	0.8658	0.8288
.7	4	4.0	0.9243	0.8987	0.8752	0.8413
.7	4	4.5	0.9302	0.9059	0.8836	0.8524
.7	4	4.5	0.9302	0.9059	0.8836	0.8524

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.7	4	5.0	0.9355	0.9123	0.8911	0.8623
.7	4	5.5	0.9403	0.9182	0.8978	0.8713
.7	4	6.0	0.9447	0.9235	0.9040	0.8794
.7	4	6.5	0.9487	0.9284	0.9096	0.8870
.7	4	7.0	0.9525	0.9330	0.9149	0.8939
.7	4	7.5	0.9559	0.9372	0.9198	0.9004
.7	4	8.0	0.9592	0.9412	0.9243	0.9065
.7	4	8.5	0.9623	0.9449	0.9286	0.9122
.7	4	9.0	0.9651	0.9484	0.9327	0.9175
.7	4	9.5	0.9679	0.9517	0.9365	0.9226
.7	5	0.5	0.8261	0.7863	0.7403	0.6661
.7	5	1.0	0.8610	0.8288	0.7894	0.7312
.7	5	1.5	0.8814	0.8537	0.8181	0.7693
.7	5	2.0	0.8959	0.8713	0.8385	0.7964
.7	5	2.5	0.9071	0.8850	0.8543	0.8173
.7	5	3.0	0.9163	0.8962	0.8672	0.8345
.7	5	3.5	0.9240	0.9056	0.8781	0.8489
.7	5	4.0	0.9307	0.9138	0.8876	0.8615
.7	5	4.5	0.9367	0.9211	0.8959	0.8726
.7	5	5.0	0.9420	0.9275	0.9034	0.8825
.7	5	5.5	0.9468	0.9334	0.9101	0.8914
.7	5	6.0	0.9512	0.9387	0.9163	0.8996
.7	5	6.5	0.9552	0.9436	0.9219	0.9071
.7	5	7.0	0.9589	0.9482	0.9272	0.9141
.7	5	7.5	0.9624	0.9524	0.9321	0.9206
.7	5	8.0	0.9656	0.9564	0.9367	0.9266
.7	5	8.5	0.9687	0.9601	0.9409	0.9323
.7	5	9.0	0.9716	0.9636	0.9450	0.9377
.7	5	9.5	0.9743	0.9669	0.9488	0.9428
.7	6	0.5	0.8325	0.8015	0.7526	0.6862
.7	6	1.0	0.8674	0.8440	0.8017	0.7514
.7	6	1.5	0.8878	0.8689	0.8304	0.7895
.7	6	2.0	0.9023	0.8865	0.8508	0.8165
.7	6	2.5	0.9135	0.9002	0.8666	0.8375
.7	6	3.0	0.9227	0.9114	0.8795	0.8546
.7	6	3.5	0.9305	0.9208	0.8904	0.8691
.7	6	4.0	0.9372	0.9290	0.8999	0.8816
.7	6	4.5	0.9431	0.9362	0.9082	0.8927
.7	6	5.0	0.9484	0.9427	0.9157	0.9026
.7	6	5.5	0.9532	0.9485	0.9224	0.9116
.7	6	5.5	0.9532	0.9485	0.9224	0.9116

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.7	6	6.0	0.9576	0.9539	0.9286	0.9197
.7	6	6.5	0.9616	0.9588	0.9343	0.9272
.7	6	7.0	0.9654	0.9633	0.9395	0.9342
.7	6	7.5	0.9688	0.9676	0.9444	0.9407
.7	6	8.0	0.9721	0.9715	0.9490	0.9468
.7	6	8.5	0.9751	0.9753	0.9533	0.9525
.7	6	9.0	0.9780	0.9788	0.9573	0.9578
.7	6	9.5	0.9807	0.9821	0.9611	0.9629
.7	7	0.5	0.8390	0.8166	0.7649	0.7064
.7	7	1.0	0.8738	0.8592	0.8140	0.7715
.7	7	1.5	0.8943	0.8840	0.8427	0.8096
.7	7	2.0	0.9087	0.9017	0.8631	0.8366
.7	7	2.5	0.9200	0.9154	0.8789	0.8576
.7	7	3.0	0.9292	0.9265	0.8918	0.8747
.7	7	3.5	0.9369	0.9360	0.9027	0.8892
.7	7	4.0	0.9436	0.9442	0.9122	0.9018
.7	7	4.5	0.9496	0.9514	0.9205	0.9128
.7	7	5.0	0.9549	0.9579	0.9280	0.9227
.7	7	5.5	0.9597	0.9637	0.9347	0.9317
.7	7	6.0	0.9640	0.9691	0.9409	0.9399
.7	7	6.5	0.9681	0.9740	0.9466	0.9474
.7	7	7.0	0.9718	0.9785	0.9518	0.9544
.7	7	7.5	0.9753	0.9828	0.9567	0.9608
.7	7	8.0	0.9785	0.9867	0.9613	0.9669
.7	7	8.5	0.9816	0.9904	0.9656	0.9726
.7	7	9.0	0.9845	0.9939	0.9696	0.9780
.7	7	9.5	0.9872	0.9973	0.9735	0.9830
.7	8	0.5	0.8454	0.8318	0.7772	0.7265
.7	8	1.0	0.8803	0.8743	0.8263	0.7917
.7	8	1.5	0.9007	0.8992	0.8550	0.8298
.7	8	2.0	0.9152	0.9169	0.8754	0.8568
.7	8	2.5	0.9264	0.9305	0.8912	0.8777
.7	8	3.0	0.9356	0.9417	0.9041	0.8949
.7	8	3.5	0.9434	0.9512	0.9150	0.9094
.7	8	4.0	0.9501	0.9594	0.9245	0.9219
.7	8	4.5	0.9560	0.9666	0.9328	0.9330
.7	8	5.0	0.9613	0.9731	0.9403	0.9429
.7	8	5.5	0.9661	0.9789	0.9471	0.9518
.7	8	6.0	0.9705	0.9842	0.9532	0.9600
.7	8	6.5	0.9745	0.9892	0.9589	0.9675
.7	8	6.5	0.9745	0.9892	0.9589	0.9675

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.7	8	7.0	0.9783	0.9937	0.9641	0.9745
.7	8	7.5	0.9817	0.9979	0.9690	0.9810
.7	8	8.0	0.9850	1.0019	0.9736	0.9870
.7	8	8.5	0.9880	1.0056	0.9779	0.9927
.7	8	9.0	0.9909	1.0091	0.9819	0.9981
.7	8	9.5	0.9936	1.0124	0.9858	1.0032
.8	4	0.5	0.8667	0.8231	0.7850	0.7227
.8	4	1.0	0.8939	0.8558	0.8247	0.7739
.8	4	1.5	0.9099	0.8749	0.8479	0.8039
.8	4	2.0	0.9212	0.8885	0.8644	0.8251
.8	4	2.5	0.9300	0.8990	0.8772	0.8416
.8	4	3.0	0.9371	0.9076	0.8876	0.8551
.8	4	3.5	0.9432	0.9149	0.8965	0.8665
.8	4	4.0	0.9485	0.9211	0.9041	0.8763
.8	4	4.5	0.9531	0.9267	0.9109	0.8850
.8	4	5.0	0.9572	0.9317	0.9169	0.8928
.8	4	5.5	0.9610	0.9362	0.9223	0.8998
.8	4	6.0	0.9644	0.9403	0.9273	0.9063
.8	4	6.5	0.9675	0.9440	0.9319	0.9122
.8	4	7.0	0.9705	0.9475	0.9362	0.9176
.8	4	7.5	0.9732	0.9508	0.9401	0.9227
.8	4	8.0	0.9757	0.9538	0.9438	0.9275
.8	4	8.5	0.9781	0.9567	0.9473	0.9320
.8	4	9.0	0.9803	0.9594	0.9506	0.9362
.8	4	9.5	0.9825	0.9619	0.9536	0.9402
.8	5	0.5	0.8731	0.8383	0.7973	0.7429
.8	5	1.0	0.9004	0.8710	0.8370	0.7941
.8	5	1.5	0.9163	0.8901	0.8602	0.8240
.8	5	2.0	0.9276	0.9037	0.8767	0.8453
.8	5	2.5	0.9364	0.9142	0.8895	0.8617
.8	5	3.0	0.9436	0.9228	0.8999	0.8752
.8	5	3.5	0.9496	0.9300	0.9088	0.8866
.8	5	4.0	0.9549	0.9363	0.9164	0.8965
.8	5	4.5	0.9595	0.9419	0.9232	0.9052
.8	5	5.0	0.9637	0.9468	0.9292	0.9129
.8	5	5.5	0.9674	0.9513	0.9347	0.9200
.8	5	6.0	0.9708	0.9554	0.9396	0.9264
.8	5	6.5	0.9740	0.9592	0.9442	0.9323
.8	5	7.0	0.9769	0.9627	0.9485	0.9378
.8	5	7.5	0.9796	0.9660	0.9524	0.9429
.8	5	7.5	0.9796	0.9660	0.9524	0.9429

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.8	5	8.0	0.9822	0.9690	0.9561	0.9477
.8	5	8.5	0.9845	0.9719	0.9596	0.9521
.8	5	9.0	0.9868	0.9745	0.9629	0.9564
.8	5	9.5	0.9889	0.9771	0.9660	0.9603
.8	6	0.5	0.8796	0.8535	0.8096	0.7630
.8	6	1.0	0.9068	0.8862	0.8493	0.8142
.8	6	1.5	0.9228	0.9053	0.8726	0.8442
.8	6	2.0	0.9341	0.9188	0.8890	0.8654
.8	6	2.5	0.9429	0.9294	0.9018	0.8819
.8	6	3.0	0.9500	0.9379	0.9123	0.8954
.8	6	3.5	0.9561	0.9452	0.9211	0.9067
.8	6	4.0	0.9613	0.9515	0.9287	0.9166
.8	6	4.5	0.9660	0.9571	0.9355	0.9253
.8	6	5.0	0.9701	0.9620	0.9415	0.9331
.8	6	5.5	0.9739	0.9665	0.9470	0.9401
.8	6	6.0	0.9773	0.9706	0.9520	0.9465
.8	6	6.5	0.9804	0.9744	0.9565	0.9525
.8	6	7.0	0.9833	0.9779	0.9608	0.9579
.8	6	7.5	0.9861	0.9811	0.9647	0.9630
.8	6	8.0	0.9886	0.9842	0.9684	0.9678
.8	6	8.5	0.9910	0.9870	0.9719	0.9723
.8	6	9.0	0.9932	0.9897	0.9752	0.9765
.8	6	9.5	0.9954	0.9923	0.9783	0.9805
.8	7	0.5	0.8860	0.8687	0.8219	0.7832
.8	7	1.0	0.9133	0.9013	0.8616	0.8344
.8	7	1.5	0.9292	0.9205	0.8849	0.8643
.8	7	2.0	0.9405	0.9340	0.9013	0.8855
.8	7	2.5	0.9493	0.9445	0.9141	0.9020
.8	7	3.0	0.9565	0.9531	0.9246	0.9155
.8	7	3.5	0.9625	0.9604	0.9334	0.9269
.8	7	4.0	0.9678	0.9667	0.9410	0.9367
.8	7	4.5	0.9724	0.9722	0.9478	0.9454
.8	7	5.0	0.9766	0.9772	0.9538	0.9532
.8	7	5.5	0.9803	0.9817	0.9593	0.9603
.8	7	6.0	0.9837	0.9858	0.9643	0.9667
.8	7	6.5	0.9869	0.9896	0.9689	0.9726
.8	7	7.0	0.9898	0.9931	0.9731	0.9781
.8	7	7.5	0.9925	0.9963	0.9771	0.9832
.8	7	8.0	0.9950	0.9994	0.9807	0.9879
.8	7	8.5	0.9974	1.0022	0.9842	0.9924
.8	7	8.5	0.9974	1.0022	0.9842	0.9924

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.8	7	9.0	0.9997	1.0049	0.9875	0.9966
.8	7	9.5	1.0018	1.0075	0.9906	1.0006
.8	8	0.5	0.8925	0.8838	0.8343	0.8033
.8	8	1.0	0.9197	0.9165	0.8740	0.8545
.8	8	1.5	0.9357	0.9356	0.8972	0.8844
.8	8	2.0	0.9470	0.9492	0.9137	0.9057
.8	8	2.5	0.9557	0.9597	0.9264	0.9222
.8	8	3.0	0.9629	0.9683	0.9369	0.9356
.8	8	3.5	0.9690	0.9756	0.9457	0.9470
.8	8	4.0	0.9742	0.9819	0.9534	0.9569
.8	8	4.5	0.9789	0.9874	0.9601	0.9656
.8	8	5.0	0.9830	0.9924	0.9661	0.9734
.8	8	5.5	0.9867	0.9969	0.9716	0.9804
.8	8	6.0	0.9902	1.0010	0.9766	0.9868
.8	8	6.5	0.9933	1.0047	0.9812	0.9927
.8	8	7.0	0.9962	1.0082	0.9854	0.9982
.8	8	7.5	0.9989	1.0115	0.9894	1.0033
.8	8	8.0	1.0015	1.0145	0.9931	1.0081
.8	8	8.5	1.0039	1.0174	0.9965	1.0126
.8	8	9.0	1.0061	1.0201	0.9998	1.0168
.8	8	9.5	1.0082	1.0226	1.0029	1.0208
.9	4	0.5	0.9137	0.8752	0.8420	0.7995
.9	4	1.0	0.9334	0.8980	0.8723	0.8368
.9	4	1.5	0.9448	0.9113	0.8901	0.8586
.9	4	2.0	0.9530	0.9208	0.9027	0.8740
.9	4	2.5	0.9593	0.9282	0.9124	0.8860
.9	4	3.0	0.9645	0.9342	0.9204	0.8958
.9	4	3.5	0.9688	0.9392	0.9271	0.9041
.9	4	4.0	0.9726	0.9436	0.9330	0.9113
.9	4	4.5	0.9759	0.9475	0.9381	0.9176
.9	4	5.0	0.9789	0.9510	0.9427	0.9233
.9	4	5.5	0.9816	0.9541	0.9469	0.9284
.9	4	6.0	0.9841	0.9570	0.9507	0.9331
.9	4	6.5	0.9863	0.9596	0.9542	0.9374
.9	4	7.0	0.9884	0.9621	0.9574	0.9414
.9	4	7.5	0.9904	0.9643	0.9605	0.9451
.9	4	8.0	0.9922	0.9665	0.9633	0.9485
.9	4	8.5	0.9939	0.9685	0.9659	0.9518
.9	4	9.0	0.9955	0.9703	0.9684	0.9549
.9	4	9.5	0.9971	0.9721	0.9708	0.9578

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.9	5	0.5	0.9202	0.8903	0.8544	0.8196
.9	5	1.0	0.9398	0.9132	0.8847	0.8569
.9	5	1.5	0.9513	0.9265	0.9024	0.8787
.9	5	2.0	0.9594	0.9360	0.9150	0.8942
.9	5	2.5	0.9657	0.9433	0.9247	0.9062
.9	5	3.0	0.9709	0.9493	0.9327	0.9160
.9	5	3.5	0.9753	0.9544	0.9394	0.9243
.9	5	4.0	0.9790	0.9588	0.9453	0.9314
.9	5	4.5	0.9824	0.9627	0.9504	0.9378
.9	5	5.0	0.9854	0.9662	0.9550	0.9434
.9	5	5.5	0.9881	0.9693	0.9592	0.9485
.9	5	6.0	0.9905	0.9722	0.9630	0.9532
.9	5	6.5	0.9928	0.9748	0.9665	0.9575
.9	5	7.0	0.9949	0.9772	0.9697	0.9615
.9	5	7.5	0.9968	0.9795	0.9728	0.9652
.9	5	8.0	0.9987	0.9816	0.9756	0.9687
.9	5	8.5	1.0004	0.9836	0.9782	0.9720
.9	5	9.0	1.0020	0.9855	0.9807	0.9750
.9	5	9.5	1.0035	0.9873	0.9831	0.9779
.9	6	0.5	0.9266	0.9055	0.8667	0.8398
.9	6	1.0	0.9463	0.9283	0.8970	0.8770
.9	6	1.5	0.9577	0.9417	0.9147	0.8988
.9	6	2.0	0.9659	0.9512	0.9273	0.9143
.9	6	2.5	0.9722	0.9585	0.9370	0.9263
.9	6	3.0	0.9773	0.9645	0.9450	0.9361
.9	6	3.5	0.9817	0.9696	0.9518	0.9444
.9	6	4.0	0.9855	0.9740	0.9576	0.9516
.9	6	4.5	0.9888	0.9779	0.9627	0.9579
.9	6	5.0	0.9918	0.9813	0.9674	0.9636
.9	6	5.5	0.9945	0.9845	0.9715	0.9687
.9	6	6.0	0.9970	0.9873	0.9753	0.9734
.9	6	6.5	0.9992	0.9900	0.9788	0.9777
.9	6	7.0	1.0013	0.9924	0.9821	0.9817
.9	6	7.5	1.0033	0.9947	0.9851	0.9854
.9	6	8.0	1.0051	0.9968	0.9879	0.9888
.9	6	8.5	1.0068	0.9988	0.9906	0.9921
.9	6	9.0	1.0084	1.0007	0.9931	0.9952
.9	6	9.5	1.0100	1.0025	0.9954	0.9981
.9	7	0.5	0.9331	0.9207	0.8790	0.8599
.9	7	1.0	0.9527	0.9435	0.9093	0.8972
.9	7	1.0	0.9527	0.9435	0.9093	0.8972

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.9	7	1.5	0.9642	0.9569	0.9270	0.9190
.9	7	2.0	0.9723	0.9663	0.9396	0.9345
.9	7	2.5	0.9786	0.9737	0.9494	0.9464
.9	7	3.0	0.9838	0.9797	0.9573	0.9562
.9	7	3.5	0.9882	0.9848	0.9641	0.9645
.9	7	4.0	0.9919	0.9892	0.9699	0.9717
.9	7	4.5	0.9953	0.9931	0.9751	0.9780
.9	7	5.0	0.9982	0.9965	0.9797	0.9837
.9	7	5.5	1.0009	0.9997	0.9838	0.9888
.9	7	6.0	1.0034	1.0025	0.9876	0.9935
.9	7	6.5	1.0057	1.0052	0.9911	0.9978
.9	7	7.0	1.0078	1.0076	0.9944	1.0018
.9	7	7.5	1.0097	1.0099	0.9974	1.0055
.9	7	8.0	1.0115	1.0120	1.0002	1.0090
.9	7	8.5	1.0133	1.0140	1.0029	1.0122
.9	7	9.0	1.0149	1.0159	1.0054	1.0153
.9	7	9.5	1.0164	1.0177	1.0077	1.0182
.9	8	0.5	0.9395	0.9359	0.8913	0.8801
.9	8	1.0	0.9591	0.9587	0.9216	0.9173
.9	8	1.5	0.9706	0.9721	0.9393	0.9391
.9	8	2.0	0.9788	0.9815	0.9519	0.9546
.9	8	2.5	0.9851	0.9889	0.9617	0.9666
.9	8	3.0	0.9902	0.9949	0.9696	0.9764
.9	8	3.5	0.9946	1.0000	0.9764	0.9847
.9	8	4.0	0.9984	1.0044	0.9822	0.9919
.9	8	4.5	1.0017	1.0082	0.9874	0.9982
.9	8	5.0	1.0047	1.0117	0.9920	1.0039
.9	8	5.5	1.0074	1.0148	0.9961	1.0090
.9	8	6.0	1.0099	1.0177	0.9999	1.0137
.9	8	6.5	1.0121	1.0203	1.0034	1.0180
.9	8	7.0	1.0142	1.0228	1.0067	1.0219
.9	8	7.5	1.0162	1.0251	1.0097	1.0256
.9	8	8.0	1.0180	1.0272	1.0125	1.0291
.9	8	8.5	1.0197	1.0292	1.0152	1.0324
.9	8	9.0	1.0213	1.0311	1.0177	1.0354
.9	8	9.5	1.0229	1.0328	1.0200	1.0384
.9	8	9.5	1.0229	1.0328	1.0200	1.0384

APPENDIX B

EXPECTED MEANS SQUARE ERRORS FOR THE FIRST FOUR POPULATION FACTORS WITH CORRECT FACTORING FOR VARIOUS VALUES OF THE POPULATION PROPORTION OF COMMON VARIANCE, p, NUMBER OF VARIABLES PER FACTOR, n/f, AND NUMBER OF OBSERVATIONS PER VARIABLE, N/n

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.5	4	0.5	0.1050	0.1203	0.0873	0.0959
.5	4	1.0	0.0833	0.0957	0.0709	0.0776
.5	4	1.5	0.0707	0.0814	0.0613	0.0669
.5	4	2.0	0.0617	0.0712	0.0545	0.0594
.5	4	2.5	0.0547	0.0633	0.0492	0.0535
.5	4	3.0	0.0490	0.0568	0.0449	0.0487
.5	4	3.5	0.0442	0.0514	0.0412	0.0446
.5	4	4.0	0.0400	0.0466	0.0381	0.0411
.5	4	4.5	0.0363	0.0425	0.0353	0.0380
.5	4	5.0	0.0330	0.0387	0.0328	0.0352
.5	4	5.5	0.0301	0.0354	0.0305	0.0327
.5	4	6.0	0.0274	0.0323	0.0285	0.0304
.5	4	6.5	0.0249	0.0294	0.0266	0.0283
.5	4	7.0	0.0225	0.0268	0.0248	0.0263
.5	4	7.5	0.0204	0.0244	0.0232	0.0245
.5	4	8.0	0.0184	0.0221	0.0217	0.0228
.5	4	8.5	0.0165	0.0199	0.0202	0.0212
.5	4	9.0	0.0147	0.0179	0.0189	0.0197
.5	4	9.5	0.0130	0.0160	0.0176	0.0183
.5	5	0.5	0.1006	0.1132	0.0835	0.0909
.5	5	1.0	0.0789	0.0886	0.0671	0.0727
.5	5	1.5	0.0663	0.0743	0.0575	0.0620
.5	5	2.0	0.0573	0.0641	0.0507	0.0544
.5	5	2.5	0.0503	0.0562	0.0454	0.0485
.5	5	3.0	0.0446	0.0497	0.0411	0.0437
.5	5	3.5	0.0398	0.0442	0.0374	0.0396
.5	5	4.0	0.0356	0.0395	0.0343	0.0361
.5	5	4.5	0.0319	0.0353	0.0315	0.0330
.5	5	5.0	0.0286	0.0316	0.0290	0.0302
.5	5	5.5	0.0257	0.0282	0.0267	0.0277
.5	5	6.0	0.0230	0.0251	0.0247	0.0254
.5	5	6.5	0.0205	0.0223	0.0228	0.0233
.5	5	7.0	0.0181	0.0197	0.0210	0.0214
.5	5	7.5	0.0160	0.0172	0.0194	0.0195
.5	5	8.0	0.0140	0.0150	0.0179	0.0178
.5	5	8.5	0.0121	0.0128	0.0164	0.0162
.5	5	9.0	0.0103	0.0108	0.0151	0.0147
.5	5	9.5	0.0086	0.0089	0.0138	0.0133
.5	6	0.5	0.0962	0.1060	0.0797	0.0860
.5	6	1.0	0.0745	0.0815	0.0633	0.0677
.5	6	1.0	0.0745	0.0815	0.0633	0.0677

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.5	6	1.5	0.0619	0.0671	0.0537	0.0570
.5	6	2.0	0.0529	0.0569	0.0469	0.0494
.5	6	2.5	0.0459	0.0490	0.0416	0.0435
.5	6	3.0	0.0402	0.0426	0.0373	0.0387
.5	6	3.5	0.0354	0.0371	0.0337	0.0347
.5	6	4.0	0.0312	0.0324	0.0305	0.0311
.5	6	4.5	0.0275	0.0282	0.0277	0.0280
.5	6	5.0	0.0243	0.0245	0.0252	0.0252
.5	6	5.5	0.0213	0.0211	0.0230	0.0227
.5	6	6.0	0.0186	0.0180	0.0209	0.0204
.5	6	6.5	0.0161	0.0152	0.0190	0.0183
.5	6	7.0	0.0137	0.0126	0.0172	0.0164
.5	6	7.5	0.0116	0.0101	0.0156	0.0146
.5	6	8.0	0.0096	0.0078	0.0141	0.0129
.5	6	8.5	0.0077	0.0057	0.0126	0.0113
.5	6	9.0	0.0059	0.0037	0.0113	0.0097
.5	6	9.5	0.0042	0.0017	0.0100	0.0083
.5	7	0.5	0.0918	0.0989	0.0759	0.0810
.5	7	1.0	0.0701	0.0744	0.0595	0.0627
.5	7	1.5	0.0575	0.0600	0.0499	0.0520
.5	7	2.0	0.0485	0.0498	0.0431	0.0444
.5	7	2.5	0.0415	0.0419	0.0378	0.0386
.5	7	3.0	0.0358	0.0354	0.0335	0.0337
.5	7	3.5	0.0310	0.0300	0.0299	0.0297
.5	7	4.0	0.0268	0.0253	0.0267	0.0262
.5	7	4.5	0.0231	0.0211	0.0239	0.0231
.5	7	5.0	0.0199	0.0173	0.0214	0.0203
.5	7	5.5	0.0169	0.0140	0.0192	0.0178
.5	7	6.0	0.0142	0.0109	0.0171	0.0155
.5	7	6.5	0.0117	0.0081	0.0152	0.0134
.5	7	7.0	0.0093	0.0054	0.0135	0.0114
.5	7	7.5	0.0072	0.0030	0.0118	0.0096
.5	7	8.0	0.0052	0.0007	0.0103	0.0079
.5	7	8.5	0.0033	-.0015	0.0089	0.0063
.5	7	9.0	0.0015	-.0035	0.0075	0.0048
.5	7	9.5	-.0002	-.0054	0.0062	0.0033
.5	8	0.5	0.0874	0.0918	0.0721	0.0760
.5	8	1.0	0.0657	0.0672	0.0557	0.0577
.5	8	1.5	0.0531	0.0529	0.0461	0.0470
.5	8	2.0	0.0441	0.0427	0.0393	0.0395
.5	8	2.0	0.0441	0.0427	0.0393	0.0395

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.5	8	2.5	0.0371	0.0348	0.0340	0.0336
.5	8	3.0	0.0314	0.0283	0.0297	0.0288
.5	8	3.5	0.0266	0.0229	0.0261	0.0247
.5	8	4.0	0.0224	0.0181	0.0229	0.0212
.5	8	4.5	0.0188	0.0140	0.0201	0.0181
.5	8	5.0	0.0155	0.0102	0.0176	0.0153
.5	8	5.5	0.0125	0.0068	0.0154	0.0128
.5	8	6.0	0.0098	0.0038	0.0133	0.0105
.5	8	6.5	0.0073	0.0009	0.0114	0.0084
.5	8	7.0	0.0050	-.0017	0.0097	0.0064
.5	8	7.5	0.0028	-.0041	0.0080	0.0046
.5	8	8.0	0.0008	-.0064	0.0065	0.0029
.5	8	8.5	-.0011	-.0086	0.0051	0.0013
.5	8	9.0	-.0029	-.0106	0.0037	-.0002
.5	8	9.5	-.0046	-.0125	0.0024	-.0016
.5	9	0.5	0.0830	0.0847	0.0683	0.0710
.6	4	0.5	0.0909	0.1059	0.0781	0.0880
.6	4	1.0	0.0722	0.0845	0.0634	0.0709
.6	4	1.5	0.0613	0.0720	0.0547	0.0610
.6	4	2.0	0.0535	0.0631	0.0486	0.0539
.6	4	2.5	0.0475	0.0562	0.0439	0.0484
.6	4	3.0	0.0426	0.0506	0.0400	0.0439
.6	4	3.5	0.0384	0.0458	0.0367	0.0401
.6	4	4.0	0.0348	0.0417	0.0338	0.0368
.6	4	4.5	0.0317	0.0381	0.0313	0.0339
.6	4	5.0	0.0288	0.0348	0.0291	0.0313
.6	4	5.5	0.0263	0.0319	0.0271	0.0289
.6	4	6.0	0.0239	0.0292	0.0252	0.0268
.6	4	6.5	0.0218	0.0267	0.0235	0.0248
.6	4	7.0	0.0198	0.0244	0.0219	0.0230
.6	4	7.5	0.0179	0.0223	0.0205	0.0213
.6	4	8.0	0.0162	0.0203	0.0191	0.0197
.6	4	8.5	0.0145	0.0184	0.0178	0.0182
.6	4	9.0	0.0130	0.0167	0.0166	0.0168
.6	4	9.5	0.0115	0.0150	0.0154	0.0155
.6	5	0.5	0.0865	0.0988	0.0744	0.0831
.6	5	1.0	0.0678	0.0774	0.0596	0.0660
.6	5	1.5	0.0569	0.0648	0.0510	0.0560
.6	5	2.0	0.0491	0.0560	0.0448	0.0489
.6	5	2.5	0.0431	0.0491	0.0401	0.0434
.6	5	2.5	0.0431	0.0491	0.0401	0.0434

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.6	5	3.0	0.0382	0.0435	0.0362	0.0389
.6	5	3.5	0.0341	0.0387	0.0329	0.0351
.6	5	4.0	0.0305	0.0346	0.0301	0.0318
.6	5	4.5	0.0273	0.0309	0.0276	0.0289
.6	5	5.0	0.0244	0.0277	0.0253	0.0263
.6	5	5.5	0.0219	0.0247	0.0233	0.0240
.6	5	6.0	0.0195	0.0221	0.0214	0.0218
.6	5	6.5	0.0174	0.0196	0.0197	0.0198
.6	5	7.0	0.0154	0.0173	0.0181	0.0180
.6	5	7.5	0.0135	0.0152	0.0167	0.0163
.6	5	8.0	0.0118	0.0132	0.0153	0.0147
.6	5	8.5	0.0101	0.0113	0.0140	0.0132
.6	5	9.0	0.0086	0.0095	0.0128	0.0118
.6	5	9.5	0.0071	0.0079	0.0116	0.0105
.6	6	0.5	0.0821	0.0916	0.0706	0.0781
.6	6	1.0	0.0634	0.0702	0.0558	0.0610
.6	6	1.5	0.0525	0.0577	0.0472	0.0510
.6	6	2.0	0.0447	0.0488	0.0410	0.0439
.6	6	2.5	0.0387	0.0420	0.0363	0.0384
.6	6	3.0	0.0338	0.0363	0.0324	0.0339
.6	6	3.5	0.0297	0.0316	0.0291	0.0301
.6	6	4.0	0.0261	0.0274	0.0263	0.0268
.6	6	4.5	0.0229	0.0238	0.0238	0.0239
.6	6	5.0	0.0200	0.0206	0.0215	0.0213
.6	6	5.5	0.0175	0.0176	0.0195	0.0190
.6	6	6.0	0.0151	0.0149	0.0176	0.0168
.6	6	6.5	0.0130	0.0125	0.0159	0.0149
.6	6	7.0	0.0110	0.0102	0.0144	0.0130
.6	6	7.5	0.0091	0.0080	0.0129	0.0113
.6	6	8.0	0.0074	0.0060	0.0115	0.0097
.6	6	8.5	0.0057	0.0042	0.0102	0.0083
.6	6	9.0	0.0042	0.0024	0.0090	0.0068
.6	6	9.5	0.0027	0.0007	0.0078	0.0055
.6	7	0.5	0.0777	0.0845	0.0668	0.0731
.6	7	1.0	0.0590	0.0631	0.0520	0.0560
.6	7	1.5	0.0481	0.0506	0.0434	0.0460
.6	7	2.0	0.0404	0.0417	0.0373	0.0389
.6	7	2.5	0.0343	0.0348	0.0325	0.0334
.6	7	3.0	0.0294	0.0292	0.0286	0.0289
.6	7	3.5	0.0253	0.0244	0.0253	0.0251
.6	7	3.5	0.0253	0.0244	0.0253	0.0251

n	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.6	7	4.0	0.0217	0.0203	0.0225	0.0219
.6	7	4.5	0.0185	0.0167	0.0200	0.0190
.6	7	5.0	0.0156	0.0134	0.0177	0.0164
.6	7	5.5	0.0131	0.0105	0.0157	0.0140
.6	7	6.0	0.0107	0.0078	0.0138	0.0119
.6	7	6.5	0.0086	0.0053	0.0121	0.0099
.6	7	7.0	0.0066	0.0030	0.0106	0.0081
.6	7	7.5	0.0047	0.0009	0.0091	0.0064
.6	7	8.0	0.0030	-.0011	0.0077	0.0048
.6	7	8.5	0.0013	-.0030	0.0064	0.0033
.6	7	9.0	-.0002	-.0047	0.0052	0.0019
.6	7	9.5	-.0017	-.0064	0.0041	0.0005
.6	8	0.5	0.0733	0.0774	0.0630	0.0681
.6	8	1.0	0.0547	0.0560	0.0482	0.0510
.6	8	1.5	0.0437	0.0435	0.0396	0.0411
.6	8	2.0	0.0360	0.0346	0.0335	0.0340
.6	8	2.5	0.0299	0.0277	0.0287	0.0285
.6	8	3.0	0.0250	0.0221	0.0248	0.0240
.6	8	3.5	0.0209	0.0173	0.0215	0.0202
.6	8	4.0	0.0173	0.0132	0.0187	0.0169
.6	8	4.5	0.0141	0.0096	0.0162	0.0140
.6	8	5.0	0.0112	0.0063	0.0139	0.0114
.6	8	5.5	0.0087	0.0034	0.0119	0.0090
.6	8	6.0	0.0063	0.0007	0.0101	0.0069
.6	8	6.5	0.0042	-.0018	0.0084	0.0049
.6	8	7.0	0.0022	-.0041	0.0068	0.0031
.6	8	7.5	0.0003	-.0062	0.0053	0.0014
.6	8	8.0	-.0014	-.0082	0.0039	-.0002
.6	8	8.5	-.0031	-.0101	0.0026	-.0017
.6	8	9.0	-.0046	-.0118	0.0014	-.0031
.6	8	9.5	-.0061	-.0135	0.0003	-.0044
.7	4	0.5	0.0769	0.0915	0.0690	0.0801
.7	4	1.0	0.0612	0.0732	0.0559	0.0643
.7	4	1.5	0.0519	0.0626	0.0482	0.0550
.7	4	2.0	0.0454	0.0550	0.0428	0.0484
.7	4	2.5	0.0404	0.0491	0.0385	0.0433
.7	4	3.0	0.0362	0.0443	0.0351	0.0391
.7	4	3.5	0.0327	0.0403	0.0322	0.0355
.7	4	4.0	0.0297	0.0368	0.0296	0.0325
.7	4	4.5	0.0270	0.0337	0.0274	0.0298
.7	4	4.5	0.0270	0.0337	0.0274	0.0298

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.7	4	5.0	0.0246	0.0309	0.0254	0.0274
.7	4	5.5	0.0224	0.0284	0.0236	0.0252
.7	4	6.0	0.0205	0.0261	0.0220	0.0232
.7	4	6.5	0.0187	0.0240	0.0204	0.0214
.7	4	7.0	0.0170	0.0220	0.0190	0.0197
.7	4	7.5	0.0154	0.0202	0.0177	0.0181
.7	4	8.0	0.0139	0.0185	0.0165	0.0166
.7	4	8.5	0.0126	0.0169	0.0154	0.0152
.7	4	9.0	0.0113	0.0154	0.0143	0.0139
.7	4	9.5	0.0100	0.0140	0.0133	0.0127
.7	5	0.5	0.0725	0.0844	0.0652	0.0752
.7	5	1.0	0.0568	0.0661	0.0521	0.0593
.7	5	1.5	0.0476	0.0554	0.0444	0.0500
.7	5	2.0	0.0410	0.0479	0.0390	0.0434
.7	5	2.5	0.0360	0.0420	0.0347	0.0383
.7	5	3.0	0.0318	0.0372	0.0313	0.0341
.7	5	3.5	0.0283	0.0331	0.0284	0.0306
.7	5	4.0	0.0253	0.0296	0.0258	0.0275
.7	5	4.5	0.0226	0.0265	0.0236	0.0248
.7	5	5.0	0.0202	0.0238	0.0216	0.0224
.7	5	5.5	0.0181	0.0213	0.0198	0.0202
.7	5	6.0	0.0161	0.0190	0.0182	0.0182
.7	5	6.5	0.0143	0.0169	0.0167	0.0164
.7	5	7.0	0.0126	0.0149	0.0152	0.0147
.7	5	7.5	0.0110	0.0131	0.0139	0.0131
.7	5	8.0	0.0095	0.0114	0.0127	0.0116
.7	5	8.5	0.0082	0.0098	0.0116	0.0102
.7	5	9.0	0.0069	0.0083	0.0105	0.0089
.7	5	9.5	0.0056	0.0069	0.0095	0.0077
.7	6	0.5	0.0681	0.0772	0.0614	0.0702
.7	6	1.0	0.0524	0.0590	0.0483	0.0543
.7	6	1.5	0.0432	0.0483	0.0406	0.0450
.7	6	2.0	0.0366	0.0407	0.0352	0.0384
.7	6	2.5	0.0316	0.0349	0.0310	0.0333
.7	6	3.0	0.0274	0.0301	0.0275	0.0291
.7	6	3.5	0.0239	0.0260	0.0246	0.0256
.7	6	4.0	0.0209	0.0225	0.0221	0.0225
.7	6	4.5	0.0182	0.0194	0.0198	0.0198
.7	6	5.0	0.0158	0.0166	0.0178	0.0174
.7	6	5.5	0.0137	0.0141	0.0160	0.0152
.7	6	5.5	0.0137	0.0141	0.0160	0.0152

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.7	6	5.0	0.0117	0.0118	0.0144	0.0132
.7	6	5.5	0.0099	0.0097	0.0129	0.0114
.7	6	7.0	0.0082	0.0078	0.0115	0.0097
.7	6	7.5	0.0066	0.0060	0.0102	0.0081
.7	6	8.0	0.0052	0.0043	0.0089	0.0066
.7	6	8.5	0.0038	0.0027	0.0078	0.0053
.7	6	9.0	0.0025	0.0012	0.0067	0.0039
.7	6	9.5	0.0012	-.0003	0.0057	0.0027
.7	7	0.5	0.0637	0.0701	0.0577	0.0652
.7	7	1.0	0.0480	0.0519	0.0445	0.0493
.7	7	1.5	0.0388	0.0412	0.0368	0.0400
.7	7	2.0	0.0322	0.0336	0.0314	0.0334
.7	7	2.5	0.0272	0.0277	0.0272	0.0283
.7	7	3.0	0.0230	0.0230	0.0237	0.0241
.7	7	3.5	0.0195	0.0189	0.0208	0.0206
.7	7	4.0	0.0165	0.0154	0.0183	0.0176
.7	7	4.5	0.0138	0.0123	0.0160	0.0149
.7	7	5.0	0.0114	0.0095	0.0140	0.0124
.7	7	5.5	0.0093	0.0070	0.0122	0.0103
.7	7	6.0	0.0073	0.0047	0.0106	0.0083
.7	7	6.5	0.0055	0.0026	0.0091	0.0064
.7	7	7.0	0.0038	0.0007	0.0077	0.0047
.7	7	7.5	0.0022	-.0012	0.0064	0.0031
.7	7	8.0	0.0008	-.0029	0.0051	0.0017
.7	7	8.5	-.0006	-.0045	0.0040	0.0003
.7	7	9.0	-.0019	-.0060	0.0029	-.0010
.7	7	9.5	-.0031	-.0074	0.0019	-.0023
.7	8	0.5	0.0593	0.0630	0.0539	0.0602
.7	8	1.0	0.0436	0.0447	0.0407	0.0444
.7	8	1.5	0.0344	0.0341	0.0331	0.0351
.7	8	2.0	0.0278	0.0265	0.0276	0.0285
.7	8	2.5	0.0228	0.0206	0.0234	0.0234
.7	8	3.0	0.0186	0.0158	0.0199	0.0192
.7	8	3.5	0.0151	0.0118	0.0170	0.0156
.7	8	4.0	0.0121	0.0083	0.0145	0.0126
.7	8	4.5	0.0094	0.0052	0.0123	0.0099
.7	8	5.0	0.0070	0.0024	0.0103	0.0075
.7	8	5.5	0.0049	-.0001	0.0085	0.0053
.7	8	6.0	0.0029	-.0024	0.0068	0.0033
.7	8	6.5	0.0011	-.0045	0.0053	0.0014
.7	8	6.5	0.0011	-.0045	0.0053	0.0014

D	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.7	8	7.0	-.0006	-.0065	0.0039	-.0002
.7	8	7.5	-.0022	-.0083	0.0026	-.0018
.7	8	8.0	-.0036	-.0100	0.0014	-.0033
.7	8	8.5	-.0050	-.0116	0.0002	-.0047
.7	8	9.0	-.0063	-.0131	-.0009	-.0060
.7	8	9.5	-.0075	-.0145	-.0019	-.0072
.8	4	0.5	0.0629	0.0771	0.0599	0.0723
.8	4	1.0	0.0501	0.0620	0.0484	0.0576
.8	4	1.5	0.0426	0.0532	0.0417	0.0490
.8	4	2.0	0.0373	0.0469	0.0369	0.0429
.8	4	2.5	0.0332	0.0421	0.0332	0.0381
.8	4	3.0	0.0298	0.0381	0.0302	0.0343
.8	4	3.5	0.0270	0.0347	0.0276	0.0310
.8	4	4.0	0.0245	0.0318	0.0254	0.0282
.8	4	4.5	0.0223	0.0293	0.0235	0.0257
.8	4	5.0	0.0204	0.0270	0.0217	0.0235
.8	4	5.5	0.0186	0.0249	0.0201	0.0214
.8	4	6.0	0.0170	0.0230	0.0187	0.0196
.8	4	6.5	0.0156	0.0213	0.0174	0.0179
.8	4	7.0	0.0142	0.0196	0.0161	0.0163
.8	4	7.5	0.0129	0.0181	0.0150	0.0149
.8	4	8.0	0.0117	0.0167	0.0139	0.0135
.8	4	8.5	0.0106	0.0154	0.0129	0.0122
.8	4	9.0	0.0096	0.0142	0.0120	0.0110
.8	4	9.5	0.0086	0.0130	0.0111	0.0098
.8	5	0.5	0.0585	0.0699	0.0561	0.0673
.8	5	1.0	0.0457	0.0549	0.0446	0.0526
.8	5	1.5	0.0382	0.0460	0.0379	0.0440
.8	5	2.0	0.0329	0.0398	0.0331	0.0379
.8	5	2.5	0.0288	0.0349	0.0294	0.0332
.8	5	3.0	0.0254	0.0310	0.0264	0.0293
.8	5	3.5	0.0226	0.0276	0.0238	0.0260
.8	5	4.0	0.0201	0.0247	0.0216	0.0232
.8	5	4.5	0.0179	0.0221	0.0197	0.0207
.8	5	5.0	0.0160	0.0198	0.0179	0.0185
.8	5	5.5	0.0142	0.0178	0.0164	0.0165
.8	5	6.0	0.0126	0.0159	0.0149	0.0146
.8	5	6.5	0.0112	0.0141	0.0136	0.0129
.8	5	7.0	0.0098	0.0125	0.0124	0.0113
.8	5	7.5	0.0085	0.0110	0.0112	0.0099
.8	5	7.5	0.0085	0.0110	0.0112	0.0099

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.8	5	8.0	0.0073	0.0096	0.0101	0.0085
.8	5	8.5	0.0062	0.0083	0.0091	0.0072
.8	5	9.0	0.0052	0.0071	0.0082	0.0060
.8	5	9.5	0.0042	0.0059	0.0073	0.0049
.8	6	0.5	0.0541	0.0628	0.0523	0.0623
.8	6	1.0	0.0413	0.0477	0.0408	0.0476
.8	6	1.5	0.0338	0.0389	0.0341	0.0390
.8	6	2.0	0.0285	0.0327	0.0293	0.0329
.8	6	2.5	0.0244	0.0278	0.0256	0.0282
.8	6	3.0	0.0210	0.0238	0.0226	0.0243
.8	6	3.5	0.0182	0.0205	0.0201	0.0211
.8	6	4.0	0.0157	0.0176	0.0178	0.0182
.8	6	4.5	0.0135	0.0150	0.0159	0.0157
.8	6	5.0	0.0116	0.0127	0.0141	0.0135
.8	6	5.5	0.0098	0.0106	0.0126	0.0115
.8	6	6.0	0.0082	0.0087	0.0111	0.0096
.8	6	6.5	0.0068	0.0070	0.0098	0.0079
.8	6	7.0	0.0054	0.0054	0.0086	0.0064
.8	6	7.5	0.0041	0.0039	0.0074	0.0049
.8	6	8.0	0.0029	0.0025	0.0064	0.0035
.8	6	8.5	0.0018	0.0012	0.0054	0.0023
.8	6	9.0	0.0008	-.0001	0.0044	0.0010
.8	6	9.5	-.0002	-.0013	0.0035	-.0001
.8	7	0.5	0.0497	0.0557	0.0485	0.0573
.8	7	1.0	0.0369	0.0406	0.0370	0.0426
.8	7	1.5	0.0294	0.0318	0.0303	0.0340
.8	7	2.0	0.0241	0.0255	0.0255	0.0279
.8	7	2.5	0.0200	0.0207	0.0218	0.0232
.8	7	3.0	0.0166	0.0167	0.0188	0.0194
.8	7	3.5	0.0138	0.0134	0.0163	0.0161
.8	7	4.0	0.0113	0.0104	0.0141	0.0133
.8	7	4.5	0.0092	0.0079	0.0121	0.0108
.8	7	5.0	0.0072	0.0056	0.0104	0.0085
.8	7	5.5	0.0055	0.0035	0.0088	0.0065
.8	7	6.0	0.0038	0.0016	0.0073	0.0047
.8	7	6.5	0.0024	-.0001	0.0060	0.0030
.8	7	7.0	0.0010	-.0017	0.0048	0.0014
.8	7	7.5	-.0003	-.0032	0.0036	-.0001
.8	7	8.0	-.0015	-.0046	0.0026	-.0014
.8	7	8.5	-.0026	-.0060	0.0016	-.0027
.8	7	8.5	-.0026	-.0060	0.0016	-.0027

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.8	7	9.0	-.0036	-.0072	0.0006	-.0039
.8	7	9.5	-.0046	-.0084	-.0003	-.0051
.8	8	0.5	0.0453	0.0486	0.0447	0.0524
.8	8	1.0	0.0325	0.0335	0.0332	0.0377
.8	8	1.5	0.0250	0.0247	0.0265	0.0291
.8	8	2.0	0.0197	0.0184	0.0218	0.0230
.8	8	2.5	0.0156	0.0135	0.0181	0.0182
.8	8	3.0	0.0122	0.0096	0.0150	0.0144
.8	8	3.5	0.0094	0.0062	0.0125	0.0111
.8	8	4.0	0.0069	0.0033	0.0103	0.0083
.8	8	4.5	0.0048	0.0008	0.0083	0.0058
.8	8	5.0	0.0028	-.0015	0.0066	0.0035
.8	8	5.5	0.0011	-.0036	0.0050	0.0015
.8	8	6.0	-.0005	-.0055	0.0036	-.0003
.8	8	6.5	-.0020	-.0072	0.0022	-.0020
.8	8	7.0	-.0034	-.0089	0.0010	-.0036
.8	8	7.5	-.0047	-.0104	-.0001	-.0050
.8	8	8.0	-.0059	-.0118	-.0012	-.0064
.8	8	8.5	-.0070	-.0131	-.0022	-.0077
.8	8	9.0	-.0080	-.0143	-.0032	-.0089
.8	8	9.5	-.0090	-.0155	-.0041	-.0101
.9	4	0.5	0.0488	0.0627	0.0507	0.0644
.9	4	1.0	0.0390	0.0507	0.0409	0.0509
.9	4	1.5	0.0332	0.0438	0.0351	0.0430
.9	4	2.0	0.0292	0.0388	0.0311	0.0374
.9	4	2.5	0.0260	0.0350	0.0279	0.0330
.9	4	3.0	0.0234	0.0318	0.0253	0.0295
.9	4	3.5	0.0212	0.0292	0.0231	0.0265
.9	4	4.0	0.0193	0.0269	0.0212	0.0239
.9	4	4.5	0.0177	0.0249	0.0195	0.0216
.9	4	5.0	0.0162	0.0231	0.0180	0.0195
.9	4	5.5	0.0148	0.0214	0.0167	0.0177
.9	4	6.0	0.0136	0.0199	0.0154	0.0160
.9	4	6.5	0.0125	0.0185	0.0143	0.0144
.9	4	7.0	0.0114	0.0173	0.0133	0.0130
.9	4	7.5	0.0104	0.0161	0.0123	0.0116
.9	4	8.0	0.0095	0.0150	0.0114	0.0104
.9	4	8.5	0.0087	0.0139	0.0105	0.0092
.9	4	9.0	0.0079	0.0129	0.0097	0.0081
.9	4	9.5	0.0071	0.0120	0.0089	0.0070
.9	4	2.5	0.0071	0.0120	0.0089	0.0070

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.9	5	0.5	0.0444	0.0555	0.0470	0.0594
.9	5	1.0	0.0346	0.0436	0.0371	0.0459
.9	5	1.5	0.0289	0.0366	0.0313	0.0380
.9	5	2.0	0.0248	0.0317	0.0273	0.0324
.9	5	2.5	0.0216	0.0279	0.0241	0.0281
.9	5	3.0	0.0190	0.0247	0.0215	0.0245
.9	5	3.5	0.0168	0.0221	0.0193	0.0215
.9	5	4.0	0.0149	0.0198	0.0174	0.0189
.9	5	4.5	0.0133	0.0177	0.0157	0.0166
.9	5	5.0	0.0118	0.0159	0.0142	0.0146
.9	5	5.5	0.0104	0.0143	0.0129	0.0127
.9	5	6.0	0.0092	0.0128	0.0117	0.0110
.9	5	6.5	0.0081	0.0114	0.0105	0.0094
.9	5	7.0	0.0070	0.0101	0.0095	0.0080
.9	5	7.5	0.0060	0.0089	0.0085	0.0067
.9	5	8.0	0.0051	0.0078	0.0076	0.0054
.9	5	8.5	0.0043	0.0068	0.0067	0.0042
.9	5	9.0	0.0035	0.0058	0.0059	0.0031
.9	5	9.5	0.0027	0.0049	0.0051	0.0021
.9	6	0.5	0.0400	0.0484	0.0432	0.0544
.9	6	1.0	0.0302	0.0365	0.0333	0.0409
.9	6	1.5	0.0245	0.0295	0.0276	0.0330
.9	6	2.0	0.0204	0.0246	0.0235	0.0274
.9	6	2.5	0.0172	0.0207	0.0203	0.0231
.9	6	3.0	0.0146	0.0176	0.0177	0.0195
.9	6	3.5	0.0124	0.0149	0.0155	0.0165
.9	6	4.0	0.0106	0.0126	0.0136	0.0139
.9	6	4.5	0.0089	0.0106	0.0120	0.0116
.9	6	5.0	0.0074	0.0088	0.0105	0.0096
.9	6	5.5	0.0060	0.0072	0.0091	0.0077
.9	6	6.0	0.0048	0.0057	0.0079	0.0060
.9	6	6.5	0.0037	0.0043	0.0067	0.0045
.9	6	7.0	0.0026	0.0030	0.0057	0.0030
.9	6	7.5	0.0016	0.0018	0.0047	0.0017
.9	6	8.0	0.0007	0.0007	0.0038	0.0004
.9	6	8.5	-.0001	-.0003	0.0029	-.0008
.9	6	9.0	-.0009	-.0013	0.0021	-.0019
.9	6	9.5	-.0017	-.0022	0.0013	-.0029
.9	7	0.5	0.0356	0.0413	0.0394	0.0494
.9	7	1.0	0.0258	0.0294	0.0295	0.0359
.9	7	1.0	0.0258	0.0294	0.0295	0.0359

p	n/f	N/n	FACTOR 1	FACTOR 2	FACTOR 3	FACTOR 4
.9	7	1.5	0.0201	0.0224	0.0238	0.0281
.9	7	2.0	0.0160	0.0174	0.0197	0.0225
.9	7	2.5	0.0128	0.0136	0.0165	0.0181
.9	7	3.0	0.0102	0.0105	0.0139	0.0146
.9	7	3.5	0.0081	0.0078	0.0117	0.0116
.9	7	4.0	0.0062	0.0055	0.0098	0.0090
.9	7	4.5	0.0045	0.0035	0.0082	0.0067
.9	7	5.0	0.0030	0.0017	0.0067	0.0046
.9	7	5.5	0.0016	0.0000	0.0053	0.0028
.9	7	6.0	0.0004	-.0015	0.0041	0.0011
.9	7	6.5	-.0007	-.0028	0.0029	-.0005
.9	7	7.0	-.0018	-.0041	0.0019	-.0019
.9	7	7.5	-.0028	-.0053	0.0009	-.0033
.9	7	8.0	-.0037	-.0064	-.0000	-.0045
.9	7	8.5	-.0045	-.0075	-.0009	-.0057
.9	7	9.0	-.0053	-.0084	-.0017	-.0068
.9	7	9.5	-.0061	-.0094	-.0024	-.0079
.9	8	0.5	0.0312	0.0342	0.0356	0.0445
.9	8	1.0	0.0214	0.0222	0.0257	0.0310
.9	8	1.5	0.0157	0.0153	0.0200	0.0231
.9	8	2.0	0.0116	0.0103	0.0159	0.0175
.9	8	2.5	0.0084	0.0065	0.0127	0.0131
.9	8	3.0	0.0058	0.0033	0.0101	0.0096
.9	8	3.5	0.0037	0.0007	0.0080	0.0066
.9	8	4.0	0.0018	-.0016	0.0061	0.0040
.9	8	4.5	0.0001	-.0036	0.0044	0.0017
.9	8	5.0	-.0014	-.0055	0.0029	-.0004
.9	8	5.5	-.0028	-.0071	0.0015	-.0022
.9	8	6.0	-.0040	-.0086	0.0003	-.0039
.9	8	6.5	-.0051	-.0100	-.0008	-.0055
.9	8	7.0	-.0062	-.0112	-.0019	-.0069
.9	8	7.5	-.0071	-.0124	-.0029	-.0083
.9	8	8.0	-.0081	-.0135	-.0038	-.0095
.9	8	8.5	-.0089	-.0146	-.0047	-.0107
.9	8	9.0	-.0097	-.0156	-.0055	-.0118
.9	8	9.5	-.0105	-.0165	-.0062	-.0129
.9	8	9.5	-.0105	-.0165	-.0062	-.0129

VITA

Thomas Joseph Fahey, Jr., was born in Savannah, Georgia, on August 3, 1948. He attended elementary schools in Atlanta, Georgia, and Greensboro, North Carolina. He was graduated from Grimsley Senior High School in Greensboro in May 1966. He was awarded a Bachelor of Science degree with a major in Personnel Management from The University of Tennessee in December 1970.

He began work at the Tennessee Valley Authority in 1974 where he is still employed. He was awarded a Doctor of Philosophy degree with a major in Industrial and Organizational Psychology in June 1981.