To the Graduate Council:

I am submitting herewith a dissertation written by Mary Vann Eberlein entitled "The GA Heuristic Generically Has Hyperbolic Fixed Points." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Computer Science.

Michael D. Vose, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Associate Vice Chancellor and
Dean of the Graduate School

The GA Heuristic Generically Has Hyperbolic Fixed Points

A Dissertation

Presented for the

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Degree

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Mary Vann Eberlein

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DEDICATION

To Scott, Bonnie, Cosa, Otis, and Ripley

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Abstract

A continuously differentiable function $\mathcal{G}$ models the behavior of a simple genetic algorithm, incorporating the genetic operators selection, mutation and crossover. In this model, populations are represented by points in $\Re^n$. (Typically, in genetic search the populations correspond to points in $\Lambda = \{x \in \Re^n : \forall i.x_i \geq 0, \sum x_i = 1\}$). Given population p the expected next population of the GA is $\mathcal{G}(p)$. Several facts are known about the relationship between the GA and $\mathcal{G}$. In particular, the GA spends much of its time near fixed points of $\mathcal{G}$ in the large population case. Consequently, it is desirable to understand the behavior of $\mathcal{G}$ near its fixed points.

A fixed point x of $\mathcal{G}$ is called *hyperbolic* if the differential $d\mathcal{G}_x$ of $\mathcal{G}$ at x has no eigenvalues of modulus one. If x is a hyperbolic fixed point of $\mathcal{G}$, then $d\mathcal{G}_x$ sheds light on the behavior of the sequence $p, \mathcal{G}(p), \mathcal{G}^2(p), ...$ in a neighborhood of x. Therefore, the differential of $\mathcal{G}$ can be used to explain the behavior of $\mathcal{G}$ near its hyperbolic fixed points. This report shows that in the positive mutation case, there is a dense open set in $(\Re^+)^n$ such that if the fitness parameter of $\mathcal{G}$ belongs to this set, then all fixed points of $\mathcal{G}$ in Λ are hyperbolic.

Contents

1 Introduction **1**

1.1 Background Information 1

1.2 Survey of GA Theory 2

1.2.1 Early Work on Genetic Algorithms 2

1.2.2 GA-Hardness and Performance Prediction . . 9

1.2.3 Implementation Choices 13

1.2.4 Mathematical Models of GAs 20

2 Motivation **24**

2.1 Behavior In The Hyperbolic Case 24

2.1.1 Hyperbolicity and The Stable Manifold Theorem 26

2.2 Significance of Results 27

2.3 Examples 34

3 Preliminary Definitions **38**

4 Goals and Outline of the Proof **41**

5 Notation and Definitions **49**

6 A Dense Open Subset of $\phi(Z)^c \cap C$ **54**

6.1 Preliminary Results 55

7 Really Nice Fixed Points **62**

7.1 Preliminary Results 62

8 Hyperbolic Fixed Points of $\mathcal{G}$ **70**

9 Future Work **75**

Appendix **87**

A.1 Some Results about Continuous Functions 88

A.2 Algebra 92

A.2.1 Definitions and Helpful Facts 92

A.2.2 Proof of Mulay's Theorem 101

A.3 Useful Theorems 113

List of Figures

Typical Convergence 28

Nonexponential Convergence 31

f_0 is not persistent 36

The Inverse Function Theorem 45

Relationship between Z, W, and $\phi(Z)$ 52

Nice inverse images 61

$\phi(z)$ has more nice inverse images 69

Chapter 1

Introduction

1.1 Background Information

A continuously differentiable heuristic $\mathcal{G}$ models the evolution of a simple GA; $\mathcal{G}$ encompasses the genetic operators fitness, mutation, and crossover. We assume throughout this report that the mutation rate is positive. In this model, populations (sub-collections of the search space) are represented by vectors (called *population vectors*) in $\Re^n$. (In genetic search, it is typically the case that populations correspond to vectors in $\Lambda = \{x \in \Re^n : \forall i.x_i \geq 0, \sum x_i = 1\}$).

The dynamical system obtained by iterating $\mathcal{G}$ is complicated. Considering an approximation may be useful. Locally, the differential is the best linear estimate of $\mathcal{G}$, so $d\mathcal{G}$ is the natural choice for a simplifying approximation. Assuming a coordinate system with fixed

point x at the origin,

$$\mathcal{G}(y) = d\mathcal{G}_x y + o(y).$$

Hence $d\mathcal{G}_x$ yields an excellent estimate of $\mathcal{G}$ at points which are close to x. The hope is that the dynamics of $d\mathcal{G}_x$ yield information about trajectories under $\mathcal{G}$ of points near x, and, under certain conditions, this is the case.

If $d\mathcal{G}_x$ has an eigenvalue of modulus one (i.e., x is nonhyperbolic), then there is some direction in which $d\mathcal{G}_x$ does not affect movement either towards or away from the origin. In this case, the differential does not provide much insight into the behavior of $\mathcal{G}$'s iterates near x. This is an atypical situation however; this report shows that when mutation is positive, there exists a dense open set in $(\Re^+)^n$ such that if the fitness vector belongs to this dense open set, then all fixed points of $\mathcal{G}$ in Λ are hyperbolic.

1.2 Survey of GA Theory

1.2.1 Early Work on Genetic Algorithms

Genetic algorithms (GAs) were developed by John Holland in the 1960's and 1970's, as a means of imitating a "robust, adaptive sys-

tem" [30]. GAs are search algorithms which are based on a simple model of population genetics. Elements of the search space, which are candidate solutions for the given problem, are usually represented by fixed-length strings, or *chromosomes*, over some finite alphabet; traditionally the chromosomes are strings of zeros and ones. Each chromosome has an associated fitness, and strings of high fitness represent good solutions. A genetic algorithm is a means of moving from one population of chromosomes to a new population by way of the genetics-based operators selection, mutation, and crossover. Based on the fitness values of elements in the population, two "parents" (e.g., bit strings) are chosen by the selection operator to reproduce. The crossover operator combines the parents by exchanging segments of the two bit strings, and mutation randomly flips bits in the two strings. One of the resulting strings (or "children") is chosen at random to be a member of the next population. The populations, which are also called generations, are of fixed size.

Holland's book [30] set forth the notion that the power of GAs is a result of the way in which a GA processes *schemas*, which are subsets of the search space. Specifically, a schema is a set of bit strings that can be represented by a string over the alphabet $\{0, 1, *\}$, where the

asterisk denotes a "don't care" position. For example, the schema $H = 0**1$ represents all bit strings of length four that begin with zero and end in one.

Holland's idea was that GAs work by discovering and combining good *building blocks* (schemas such that the average fitness of the schema's elements is high). Holland noted that any bit string of length l is an instance of 2^l different schemas, and thus a population of size N contains instances of between 2^l and $N \times 2^l$ schemas. Therefore when the GA is computing the fitnesses of the N strings in a given population, it is implicitly estimating the average fitness of a large number of schemas. This implicit evaluation of a large number of schema fitnesses, which is claimed to explain (in part) the power of GAs, is known as *implicit parallelism* [30].

With respect to the notion of implicit parallelism, we should note that there has been no verification that schema processing is an important factor in GA performance. This is partially due to a lack of agreement on what role schema processing plays in GA behavior (see [22] and [27]). It is doubtful that the performance of a GA can depend on implicit parallelism, because the average fitness of schemas does not contain any more information than the fitness of the pop-

ulation elements. Given the fitness of each string in the current population, we can compute a variety of statistics – mean population fitness, the observed average fitness of each subset of the search space, the average schema fitness, etc. – but without justification, there is no reason to believe any one of these statistics is relevant to the GA's ability to search. Furthermore, the idea that implicit parallelism somehow implies superior performance in GAs is invalidated by a result called the No Free Lunch Theorem [62]: the No Free Lunch Theorem says that, given any reasonable performance measure, no search method out-performs any other when a sufficiently large class of optimization problems is considered. Hence a GA does not, on average, out-perform other search strategies. The significance of implicit parallelism was suggested by Holland [30], and has been widely accepted without justification. The repeated references (e.g., [22], [23]) in the literature to the unsubstantiated power of implicit parallelism is an indication of the need for rigorous methods of study in GA research.

Holland formalized his view of schema processing in a theorem which provides a lower bound on the expected number of instances of a given schema in a population (assuming it was represented in the previous population). Let m(H,t) be the number of instances of

schema H in the t^{th} population, and let f(H,t) be the average fitness of H's instances in the t^{th} population. Define o(H) to be the number of "defined" or non-asterisk bits in H, and let d(H) be the *defining length* of H, or distance between the outermost defined bits of H. Then Holland's Schema Theorem says

$$\mathcal{E}(\mathrm{m(H,t+1)}) \geq \frac{\mathrm{f(H,t)}}{\bar{\mathrm{f}}(\mathrm{t})}\mathrm{m(H,t)}(1 - p_c\frac{\mathrm{d(H)}}{l-1})(1-p_m)^{\mathrm{o(H)}}.$$

In this inequality, p_c is the probability of crossover and p_m is the probability of mutation (also known as the crossover rate and the mutation rate, respectively). The Schema Theorem describes the growth of schemas from one generation to the next. The Schema Theorem is often (incorrectly) said to imply that the number of instances of short, low-order, highly fit building blocks in the population will increase exponentially over time (see, for example [39] or [22]). This interpretation assumes that schema are not significantly *disrupted* (destroyed) by crossover and mutation. This view of the Schema Theorem also assumes that, for a schema H of above-average fitness, the ratio $\frac{\mathrm{f(H,t)}}{\bar{\mathrm{f}}(\mathrm{t})}$ is relatively constant, and in particular, bounded away from one. (Otherwise exponential growth does not follow from the Schema Theorem). However, this assumption is actually contradicted by the simultaneous claim that the number of instances of schema H in the population grows exponentially over time.

Recently, a number of researchers have questioned the relevance of the Schema Theorem in understanding the behavior of GAs (see, for example, [2], [29], [40]). First of all, the Schema Theorem does not take into account the ability of genetic operators to **create** instances of schema [23]. Additionally, as Radcliffe and Altenberg (see [2]) both demonstrate, the Schema Theorem holds even for a GA that carries out random search. So how can the Schema Theorem explain the power of GAs? The Schema Theorem applies to all GAs, regardless of performance. As we noted earlier, the Schema Theorem has also been said to prove that GAs work by effectively processing schemas. However, the Schema Theorem has been extended to arbitrary subsets of the search space [45], and is a weak statement at that. At the present time, there is no conclusive evidence that schema processing or the Schema Theorem is relevant to the performance – good or bad – of GAs. Hopefully these recent valid criticisms will alert GA researchers to the need for rigorous theoretical work.

Holland's goal in developing genetic algorithms was the design and implementation of "robust, adaptive systems" [30]. The robustness of GAs is claimed and referred to frequently in GA research papers

(e.g., [15]), but no formal definition of robustness has been given, and no notion of robustness in GAs has previously been proved. We propose the following definition of robustness: a GA is *robust* if small changes in the GA's parameters produce only small changes in the GA's behavior. (In other words, a GA is robust if it is not overly sensitive to small changes in parameters). The result of this report is, in part, a robustness result for GAs: because of the generic hyperbolicity of fixed points (which is proved in this paper), it can be rigorously verified that small changes in fitness produce only small changes in behavior (see section 2.1.1). There is a serious need for more formal and rigorous analysis of this kind; the GA research community has too often used poorly defined concepts (or concepts not defined at all, such as robustness) and ill-justified claims as a basis for research.

The belief that a GA works by combining instances of good (highly fit), low-order schemas to form instances of highly fit higher-order schemas is known as the Building Block Hypothesis [22]. Crossover is thought to have the ability to form good high-order schemas by recombining good low-order schemas. As we will discuss in the next section, much of the research devoted to answering the question "what makes a problem hard for a GA?" is centered around the

Building Block Hypothesis.

1.2.2 GA-Hardness and Performance Prediction

Characterizing optimization problems which are difficult for a GA to solve is a central area of focus in theoretical GA research. Problems which are difficult for a GA are called *GA-hard.* One common approach to analyzing GA difficulty is the study of *deceptive* problems [22], which are problems that violate the Building Block Hypothesis "in the extreme" [23]. More specifically, a problem is deceptive if certain hyperplanes (subspaces represented by schemas) guide the search toward a solution which is not globally competitive [23]. It is assumed that deception in fitness functions makes it hard for a GA to find a global optimum.

A great deal of published research has focused on deception. Early work on deception focused on the construction of deceptive problems (e.g., [23], [37]). More recently, Liepins and Vose [37] specified transformations in problem representation that make deceptive problems easy. Goldberg, Korb and Deb [25] defined a GA-based optimizer called messy GAs in order to deal with deceptive problems more effectively. In [60], Whitley claimed that only deceptive problems

are challenging optimization problems for GAs.

Recent research reveals dissenting views on the importance of deception ([27], [29], [21]). Grefenstette [27] pointed out that the Building Block Hypothesis underlies the assumption that deception is responsible for GA difficulty. He argued that the Building Block Hypothesis is invalid, and gave examples that illustrate that deception is neither necessary nor sufficient to cause difficulty for GAs.

Grefenstette [27] criticized the blind acceptance of deception as a central factor in GA-hard problems. As he pointed out, the Building Block Hypothesis (BBH) does not, as is commonly asserted, follow from the Schema Theorem, and there is little justification or basis for the importance it is assigned by many GA researchers. On the contrary, for years there has been empirical evidence that the notion of deception and the Building Block Hypothesis are not the answer to characterizing GA-hardness [23]. The quantity of published research on deception and GA-hardness, which relies implicitly on the Building Block Hypothesis, is consequently an embarrassment. The BBH should never have been viewed as anything more than a potential explanation of a GA's power; that it has been so widely accepted without proof or verification (and in spite of counterexam-

ples) points to a serious error on the part of the GA community. Hopefully, by bringing rigorous analysis to the investigation of GA behavior (as is done in this paper), intuitions and folklore can be replaced by mathematical certainty.

Another focus in work on GA-hard problems is *epistasis*; epistasis measures the degree of non-linearity in fitness functions. Davidor [13] suggests that highly epistatic fitness functions (i.e., fitness functions with a high degree of non-linearity) are GA-hard. In [12] and [13], Davidor investigates means of determining the degree of epistasis of a given problem.

Other researchers have analyzed the effect of noncoding segments (segments of bits which do not affect a string's fitness) on performance. Wu and Lindsay [65] were motivated to study noncoding segments because such segments occur in DNA; however, their results were mixed. Levenick's [36] experimental work indicated that including noncoding segments can improve the GA's ability to find optima in a fixed number of generations. However, Forrest and Mitchell [20] ran a series of experiments investigating the effect of noncoding segments, and found no performance improvement when noncoding segments were included.

Some work on GA-hardness deals with discovering ways of measuring the degree of difficulty a problem will pose for a GA. Jones and Forrest [33] introduced the *fitness distance correlation* (FDC) as a measure of search performance. They showed, on a number of well-studied problems, that the correlation between fitness and distance from the goal (optima) is a useful predictor of performance. Their research resulted from considering the connection between fitness functions of GAs and heuristic functions in heuristic state-space search. (In heuristic search, it is generally thought that the values of heuristic functions should correlate well with distance to the search goal [33]). A number of other measures of GA difficulty have been considered as well. Other researchers have studied the connection between GA performance and the fitness ratio (average fitness of population elements with a fixed bit in a given location over average fitness in population) [3], and the connection between performance and fitness variance in formae (generalized schemas) [45].

Some recent work on GA-hardness often involves an interesting hypothesis (e.g., FDC is a good predictor of performance) and testing of that hypothesis through experiments on a relatively small suite

of test functions. This type of work is quite practical in that we want to know how GAs work on specific types of problems. However, it also illustrates the need for a means of carrying out more general analysis of GA behavior. The question arises as to whether these studies speak to the particular set of test functions used, or whether they have a message which applies to genetic search in general. Exact mathematical models of GAs provide a framework in which questions about GA behavior can be approached analytically in a general context rather than empirically.

1.2.3 Implementation Choices

Encoding the Problem

The manner in which search space elements (potential solutions) are encoded is a central factor in the success of a GA. A change in encoding can transform a difficult problem into an easy one [6]. Liepins and Vose (see [6]) showed that a class of deceptive problems becomes easy for GAs with a change in representation, and that the necessary change is induced by a matrix. Binary codings are the most common, in large part because most theoretical work assumes that candidate solutions are represented by bit strings. The use of gray coding instead of the natural binary coding has been explored

(see [39]), as well as the use of strings over an alphabet of large size, including floating-point encodings [63]. Currently there are no general guidelines for choosing an encoding. Most results on encoding involve comparing various encodings on specific types of problems.

Selection Operators

The selection method also has an impact on GA performance. Several types of selection are commonly used.

Proportional selection is the simplest selection scheme, and was proposed by Holland [30]. In proportional selection, the probability of a string being selected is the ratio of the string's fitness to average population fitness. Holland's Schema Theorem assumes the use of proportional selection, though this restriction is sometimes ignored.

Ranking selection, suggested by Baker [5], ranks the chromosomes in the population 1 to N (where N is the population size) according to fitness. The user chooses a value max to be the expected number of times the most fit string is chosen to reproduce. The other strings are chosen to reproduce with probability that decreases monotonically with decreasing string fitness. Baker recommended setting

max to 1.1, and his empirical results indicated improvement over proportional selection. DeJong [15] indicated that ranking selection can prevent what is known as *premature convergence*, quick convergence which does not allow much time for exploration of the search space.

In tournament selection, t strings are chosen at random from the population, and the most fit of these t strings is selected to reproduce [8]. The integer t is called the tournament size. Goldberg and Deb [24] carried out an analysis of tournament selection which considered the behavior of the best individual. They also researched the *takeover time* – the number of generations computed before one optimal string comprises the entire population – under tournament selection. In [8], Blickle and Thiele showed that, under tournament selection, the least fit strings had the lowest reproduction rate and the strings with highest fitness had the highest reproduction rate. Goldberg and Deb's [24] comparison of ranking and tournament selection found that they yielded the same expected performance.

Sometimes a "steady state" strategy is used in conjunction with a selection scheme. When a steady state method is used, only a few strings are replaced from one generation to the next. Typically a few

of the least-fit strings are replaced by applying selection, mutation and crossover to the fittest strings. Steady state selection has been studied by Syswerda [50, 51] and Whitley [59]; they concluded that the steady state method can improve performance. However, as is the case for encoding choices, there are no broad criteria for deciding whether or not to use a steady state strategy. Currently most results on steady state schemes involve empirical tests on small suites of problems; such work yields useful results on particular problem classes, but no broad insights about when such strategies are useful.

Crossover

The crossover operator, which is typically applied with high probability (i.e., $p_c \geq 0.6$), is thought to increase GA performance by combining good, short-order building blocks to form good, long building blocks (BBH). However, crossover also destroys strings which belong to highly-fit schemas. One-point crossover, which involves choosing a bit position (*crossover point*) at random and exchanging the substrings of the parents after that point, is likely to disrupt schemas with long defining lengths. For example, the schema 0****1 is more likely to be destroyed by one-point crossover than the schema 01****. This problem is somewhat alleviated by *in-*

version, an operator that reorders the bits in a string; inversion can shorten the defining lengths of important schemas. Improvements have been reported with the use of inversion [58]. Bridges and Goldberg [9] analyzed the disruptive effects of one-point crossover, achieving tighter bounds than those in the Schema Theorem, by considering increases in instances of a given schema due to disruption of other schemas. Empirical studies suggest that the two-point crossover operator, which chooses two bit positions at random and exchanges the bit segments between them, is best at minimizing disruption (see [49]). Using two crossover points results in a smaller probability of disrupting schemas with large defining lengths.

Not all researchers believe that preventing disruption is of primary importance when choosing a crossover operator [47]. Less disruptive operators like one-point and two-point crossover can permit *spurious correlations* [47], which result when low-fitness schemas reproduce frequently because their defined bits are close to the defined bits of highly-fit schemas. Spurious correlations can lead to premature convergence. Several researchers have produced results indicating that more disruptive crossover operators can reduce the risk of spurious correlations, thereby improving GA performance [50]. In particular, promising increases in performance have been demonstrated with

uniform crossover, which exchanges every bit with probability p_c [49].

Terry Jones [32] questioned the usefulness of crossover. In an effort to determine if the crossover's exchange of information between strings is truly beneficial, he compared the performance of GAs with crossover to that of GAs with random crossover. (The random crossover operator crosses each parent with a random string). Jones' results, along with those of Eshelman and Schaffer[19], imply that the class of problems on which crossover improves performance may be small. Likewise, Mühlenbein (see [39]) claimed that the power of crossover is over-estimated. While this work is very interesting, the results are "local" in the sense that they have only been verified on a small number of problems. The results could potentially be extended by modeling a GA with random crossover and studying its behavior analytically.

Parameter Values

Choosing good parameter values (e.g., population size, crossover rate, mutation rate) has been another focus in GA research. DeJong (see [39]) studied the performance of GAs under various pa-

rameter values; his results suggested choosing a population of size 50-100, and setting the one-point crossover rate and mutation rate to about 0.6 and .001 respectively. Grefenstette (see [39]) used a GA to optimize parameter values for another GA. In his "meta-level" GA, each chromosome encoded population size, crossover rate and mutation rate. Grefenstette's results indicated a population size of 30, a crossover rate of 0.95 and a mutation rate of .01. His results showed some improvement over DeJong's parameter choices. Baker [5] tried changing the crossover rate dynamically; he increased the crossover rate whenever the percentage involvement (percentage of the current population which is reproducing) decreased. His studies were aimed at preventing premature crossover, and showed significantly improved performance.

The parameter settings that resulted from the studies by Grefenstette and DeJong have been widely used as "rules of thumb" for practitioners. However, these results really involve fine-tuning parameter settings for a set of test functions. The parameters given by Grefenstette and DeJong do not necessarily provide a better search algorithm in general. In order to make more broad-based statements about parameter choices, rigorous analysis is required. Like the work on encodings, selection strategies and crossover operators

which we reviewed, impressive results have been obtained for a variety of practical applications, and yet these results are narrow in their focus. There is a need for improved theoretical understanding of the ways in which GAs are affected by changes in parameter values.

1.2.4 Mathematical Models of GAs

The first mathematical model of a GA was developed by Goldberg [23]. Given any population of a GA, Goldberg derived a formula for the expected next generation. His model was limited to a two-bit GA under proportional selection and crossover. Goldberg used his model to analyze deceptive problems. Papers by Vose [53] and Vose and Liepins [56] extended Goldberg's model to incorporate general selection, binary strings of any length, and mutation. This augmented model is defined in terms of a continuously differentiable function $\mathcal{G} : \Re^n \rightarrow \Re^n$; in this model, the t^{th} population is represented by a vector $p^t \in \Re^n$, where p_i^t is the proportion of the t^{th} population equal to string i. Given a population represented by the vector p, $G(p)$ is the expected next population. Further extensions of this model can be found in [57] and [52]. Because in the limit, as population size grows to infinity, $\mathcal{G}$ is the transition function from

one generation to the next [55], $\mathcal{G}$ is sometimes called the infinite population model. Vose [55] tied the behavior of the GA to that of the infinite population model by proving that, for sufficiently large population size, the GA remains close to the path taken by $\mathcal{G}$ for a large number of iterations and with high probability.

An exact Markov chain model of a GA was presented in a paper by Nix and Vose [44]. Considering population vectors as states, their Markov chain model is described by the transition matrix Q, given by

$$Q_{s,s'} = r! \prod_{j=0}^{n-1} \frac{(\mathcal{G}(x)_j)^{rs'_j}}{(rs'_j)},$$

where r is the size of the population. Using this model it was shown that the GA spends most of its time near fixed points of $\mathcal{G}$ with high probability [44]. Davis [14] also devised a Markov chain model of GAs; his work focused on applying methods from simulated annealing to GAs.

DeJong, Spears and Gordon [16] used the Markov chain model to study the short-term behavior of genetic algorithms used as function optimizers. They computed, for the t^{th} generation, the expected best fitness value and the expected average fitness. DeJong et. al. also studied expected "first passage times": the first time an

optimum is encountered and the first time the average population fitness exceeds some given threshold. Their work extended that of Goldberg and Segrest [26].

Besides the foundational work that was described above, other research has relied on mathematical modeling of GAs. Whitley and Yoo [61] developed an exact model of a GA which uses permutation-based representations (e.g., tours in the traveling sales-rep problem are often represented by permutations). In [28], Grefenstette developed a predictive model of GAs based on characteristics of the fitness distributions of genetic operators. (The fitness distribution of an operator is the distribution of fitness values resulting from an application of the given operator). Grefenstette's work was aimed towards predicting mean population fitness under proportional selection. Altenberg [2] computed the change in fitness distribution from one generation to the next under selection and crossover; Altenberg argued that GA performance is dependent on the GA's ability to increase the upper tail of the fitness distribution.

Some researchers have used mathematical models to study convergence in GAs. Goldberg and Segrest [26] analyzed a Markov chain model for a one-bit GA with selection and mutation in order to cal-

culate the expected time to various levels of convergence. Assuming proportional selection, Ankenbrandt [3] derived estimates of convergence time which depended on fitness ratios (ratio of average fitness of a given schema to the average population fitness). Analysis of convergence with other selection methods was carried out by Goldberg and Deb [24]. Bäck [4] analyzed convergence rates as well.

The models mentioned in this section provide a means for establishing rigorous and general results about GA behavior. The results of DeJong et. al. [16], Vose [53] and Bäck [4] (and others) are strong results which were verified analytically, and which are not limited to very restrictive problem classes. These results are more powerful and far-reaching than results that are verified empirically on small test suites.

Chapter 2

Motivation

2.1 Behavior In The Hyperbolic Case

Our goal in this section is to explain the dynamics of $\mathcal{G}$ near a hyperbolic fixed point. In the discussion which follows, we choose (for the sake of simplicity) our coordinate system so that the fixed point x is at the origin. The results which are described here are examined in great detail in [1].

Because $x = \mathbf{0}$ is a hyperbolic fixed point, $\Re^n$ is the direct sum of subspaces E^+ and E^-, where points in E^+ and E^- can be characterized by their behavior under iteration of $d\mathcal{G}_\mathbf{0}$. For $z \in E^+$, the *stable space*, $d\mathcal{G}_\mathbf{0}^n z \to \mathbf{0}$ as $n \to +\infty$, and for $z \in E^-$, the *unstable space*, $d\mathcal{G}_\mathbf{0}^{-n} z \to \mathbf{0}$ as $n \to +\infty$. So the subspace E^+ consists of points which converge to the fixed point $\mathbf{0}$ under iteration of $d\mathcal{G}_\mathbf{0}$, and E^-

is the set of points which converge to the fixed point under iteration of $d\mathcal{G}_{\mathbf{0}}^{-1}$. The dimension of E^+ equals the number of eigenvalues of $d\mathcal{G}_{\mathbf{0}}$ which lie inside the unit circle (counted with multiplicity), and the dimension of E^- is equal to the number of eigenvalues of $d\mathcal{G}_{\mathbf{0}}$ which lie outside the unit circle (counted with multiplicity).

Let $V \subset \Re^n$ be a small open set containing $x = \mathbf{0}$. Analogous to the stable and unstable spaces corresponding to $d\mathcal{G}_{\mathbf{0}}$, hyperbolicity guarantees the local existence of two surfaces W^+ and W^-, such that

$$W^+ = \{z \in V : \mathcal{G}^n(z) \to \mathbf{0} \text{ as } n \to +\infty\}$$

$$W^- = \{z \in V : \mathcal{G}^{-n}(z) \to \mathbf{0} \text{ as } n \to +\infty\}.$$

In other words, points on W^+ move towards $\mathbf{0}$ under iteration of $\mathcal{G}$, and points on W^- move towards $\mathbf{0}$ under iteration of $\mathcal{G}^{-1}$. Provided a suitable norm is used to measure distance, $\mathcal{G}$ (respectively $\mathcal{G}^{-1}$) is a contraction mapping on W^+ (respectively W^-). The surface W^+ is called the *stable manifold* and has the same dimension as E^+. Similarly, W^- is called the *unstable manifold*, and has the same dimension as E^-. The stable and unstable manifolds have global analogs, the *global stable manifold* and the *global unstable manifold* respectively, given by

$$W^+_{glob} = \bigcup_{n>0} \mathcal{G}^{-n}(W^+)$$

and

$$W^-_{glob} = \bigcup_{n>0} \mathcal{G}^{n}(W^-).$$

The global stable manifold, which is also called the *basin of attraction* of the fixed point $x = \mathbf{0}$, is the set of points which converge to x under iteration of $\mathcal{G}$.

2.1.1 Hyperbolicity and The Stable Manifold Theorem

A dynamical system, such as that obtained by iterating some map g, is considered *stable* if its dynamics are not significantly altered by small changes or perturbations of g. In this context, we consider g and g' to be "close" if both the difference between g and g' and the difference between their differentials is small. Stability near a fixed point x of g guarantees the following :

- Preservation of a fixed point under small perturbation of g (that is, a map g' which is close to g has a fixed point x' near x),
- Preservation of the map's behavior near the fixed point under small perturbation (i.e., the dimension of the stable (unstable) manifold of g at x is the same as the dimension of the stable (unstable) manifold of g' at x').

We will say that a map g is *persistent* if it satisfies the two properties listed above. A theorem from Dynamical Systems called the Stable Manifold Theorem (see [1] and [17]) states that the hyperbolicity of g's fixed points guarantees the persistence of g.

Before a GA can be used to solve a problem, GA parameters corresponding to the problem must be chosen. Usually it is **not** the case that there is one obvious choice for parameters (such as fitness, mutation or crossover); that is, the GA which is being used is an approximate model. It is desirable that heuristics with similar parameters behave in a similar manner, and it is of interest and useful in terms of theoretical consequences to know if the GA's heuristic is persistent. When the heuristic $\mathcal{G}$ has hyperbolic fixed points, this is the case.

2.2 Significance of Results

In the theory of genetic algorithms, it has long been an open question as to whether the fixed points of $\mathcal{G}$ are generically hyperbolic ([54], [55]). This report answers this open question by showing that it is indeed the case that the GA heuristic typically has hyperbolic fixed points. Some important results in genetic algorithms have

been previously proved based on the assumption of hyperbolic fixed points ([54], [55], [57]); this paper therefore completes the foundations for those results. To provide an introduction to theoretical consequences of hyperbolicity, some results requiring hyperbolicity of fixed points are discussed below.

1. Practitioners have commonly observed convergence of the form in Figure 1 for genetic algorithms:

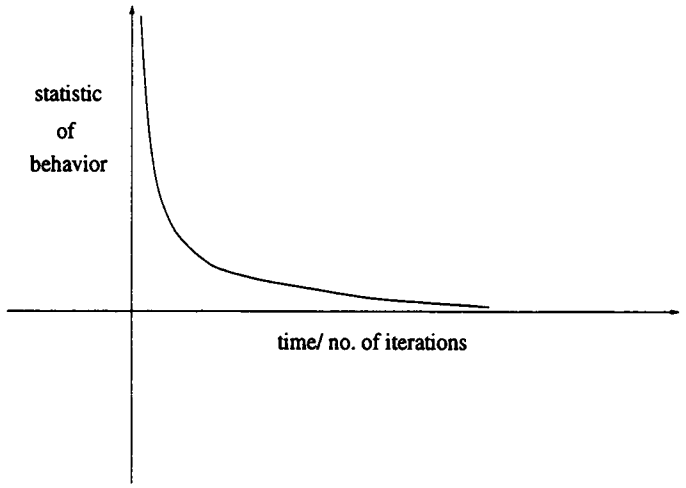


Figure 1: Typical Convergence

The results in [54] explain this type of phenomenon. To illustrate the basic idea, consider a heuristic function $\mathcal{G}$ which has a fixed point x such that the largest eigenvalue (in modulus) of $d\mathcal{G}_x$ is $\frac{1}{2}$. Since, in the hyperbolic case, $d\mathcal{G}_x$ is the dominating influence in the movement of iterates of $\mathcal{G}$ near x, consider

the movement induced by $d\mathcal{G}_x$. Starting at population p, after one iteration of $d\mathcal{G}_x$, the new population has moved at least half the distance to the limit point. After two iterations, the new population is four times as close as p to the limit. After five iterations, the new population is thirty-two times as close to the limit as p, and so forth. Therefore, in the hyperbolic case, the differential indicates an exponentially fast approach to the limit. So the type of convergence behavior observed in practice is not surprising, since fixed points of the GA heuristic are generically hyperbolic (as proved in this paper). Although there is significantly more involved, in [54] Vose proves formally that if iterates of the GA heuristic converge, and if the heuristic has hyperbolic fixed points, then the heuristic converges in logarithmic time; an important underlying principle in his proof is precisely the rapid approach to the limit which hyperbolicity implies.

Note that for functions with nonhyperbolic fixed points, logarithmic convergence is not the norm. Consider, for example, the function $g : \Re \to \Re$ given by

$$g(x) = x - x^3.$$

The differential of g is given by

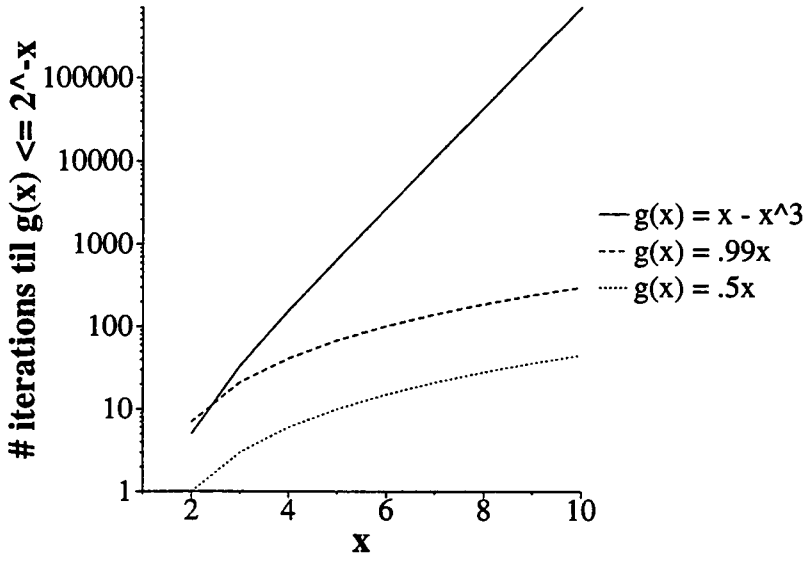
$$dg_x = 1 - 3x^2.$$

Note that 0 is a nonhyperbolic fixed point of g. Let r be the distance of x from 0. The square of the distance of $g(x)$ from the origin is

$$|g(x) - 0|^2 = x^2(1 - 2x^2 + x^4) = r^2(1 - 2r^2 + r^4) = r^2(1 - r^2)^2.$$

So for $x \in (0,1)$, $g(x)$ is closer to 0 than x. Suppose by way of contradiction that iterates of g starting at some $p \in (0,1)$ do not converge to 0; then there is some $k \in (0,1)$ such that $g^i(p) > k$ for all i. Therefore if x is any iterate of g, $| g(x) |^2 = r^2(1 - r^2)^2 \leq r^2 c_k$ where $c_k = (1 - k^2)^2$. So each iterate of g moves closer to 0 by a factor of $\sqrt{c_k}$, and $c_k \in (0,1)$. Hence $| g^i(p) - 0 | \leq (\sqrt{c_k})^i p$, and $(\sqrt{c_k})^i p \to 0$ as $i \to \infty$. So $g^i(p)$ is not always greater than k, and we've reached a contradiction. So iterates of g do converge, provided we start at some $p \in (0,1)$.

Note, however, that iterates of g (starting at $p \in (0,1)$) do not converge exponentially fast, since $| g(x) |^2 = r^2(1 - 2r^2 + r^4)$ and $(1 - 2r^2 + r^4) \to 1$ as $r \to 0$. The nonexponential convergence of g is illustrated in Figure 2 below.



The functions f and h converge exponentially fast to 0, whereas g does not.

Figure 2: Nonexponential Convergence

2. If we think of population vectors as states, an exact model of the genetic algorithm is given by a Markov chain. Let r be the population size, and n be the search space size. For states s and s' (i.e., population vectors s and s'), the transition probabilities are given by the matrix Q defined by

$$Q_{s,s'} = r! \prod_{j=0}^{n-1} \frac{(\mathcal{G}(x)_j)^{rs'_j}}{(rs'_j)}.$$

In [55], Vose makes use of the rapid convergence that occurs in the hyperbolic case to analyze the steady state distribution of the GA's Markov chain. The transition probabilities are

extremely complicated, and involve exponentially many equations, each involving exponentially many variables (i.e., exponential in the string length). However, computing the steady state distribution becomes manageable based on the following principle:

When analyzing a GA, both a signal component and a noise component can be discerned in the movement from one population to the next. The signal component is determined by $\mathcal{G}$, which computes the expected next population; the noise component is produced by the error term we get when approximating the GA by $\mathcal{G}$, and is asymptotic to a multinomial distribution as $r \to \infty$. If we iterate $\mathcal{G}$ to predict the path taken by the GA, then after a number of iterations, noise accumulates, degrading the probability that the prediction is accurate. However, in the hyperbolic case, iterates of $\mathcal{G}$ converge quickly. Hence the noise may be negligible for the reason that there is insufficient time for it to accumulate to unmanageable size. The steady state distribution can be understood via analyzing the proportion of time that the Markov chain spends in various states. In the hyperbolic case, this analysis can therefore be carried out by using $\mathcal{G}$ to reliably predict the relevant fea-

tures of the genetic algorithm's stochastic dynamics. See [55] for details, but the point here is how the determination of the steady state behavior is dependent on hyperbolicity.

3. Currently, it is conjectured that iterates of $\mathcal{G}$ converge if mutation is in the range $[0, \frac{1}{2}]$. If iterates converge, then the signal component, $\mathcal{G}$, acts as an underlying bias moving the GA towards some particular answer, or state, in the search space. In practice, cycling and chaotic behavior are not observed in the types of GAs that are modeled by $\mathcal{G}$.

   Wright and Vose proved in [64] that, given zero mutation and a positive linear fitness function, iterates of $\mathcal{G}$ converge. Additionally they used the generic hyperbolicity of $\mathcal{G}$'s fixed points to show that their result still holds for small mutation rates and fitness functions which are nearly linear. That is, in [64], hyperbolicity was used to settle an open problem for a subset of heuristics: namely that iterates of $\mathcal{G}$ converge. The proof in [64], which is the only known proof of convergence for non-trivial fitness functions, relies on the notion of persistence of the GA heuristic. Recall that persistence is a consequence of hyperbolicity.

2.3 Examples

In the following example, we examine a function g at fixed point $\mathbf{0}$. We will see that the moduli of the eigenvalues of $dg_{\mathbf{0}}$ do indeed influence how accurately dg predicts the behavior of iterates of g. Consider the function $g : \Re^2 \to \Re^2$ given by

$$g(x) = \begin{pmatrix} x_1 - (x_1^2 + x_2^2)x_2 \\ x_2 + (x_1^2 + x_2^2)x_1 \end{pmatrix}.$$

Note that

$$dg = \begin{pmatrix} 1 - 2x_1x_2 & -x_1^2 - 3x_2^2 \\ 3x_1^2 + x_2^2 & 1 + 2x_1x_2 \end{pmatrix},$$

and in particular,

$$dg_{\mathbf{0}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ which is the identity matrix.}$$

Hence the differential does not account for change. Let the distance r of x from the origin be given by $r = \sqrt{x_1^2 + x_2^2}$. The square of the distance of $g(x)$ from the origin is

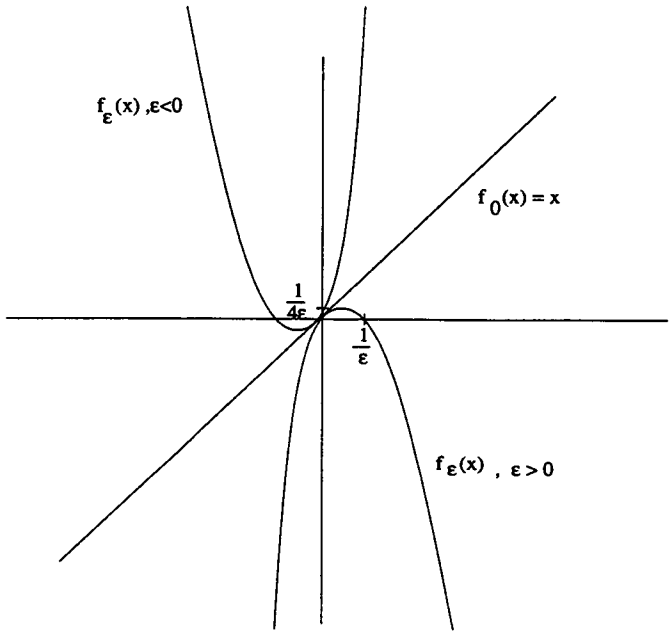
$$\| g(x) \|^2 = x_1^2 + x_2^2(x_1^2 + x_2^2)^2 + x_2^2 + x_1^2(x_1^2 + x_2^2)^2$$

$$= (x_1^2 + x_2^2) + (x_1^2 + x_2^2)^3 = r^2 + r^6 = r^2(r^4 + 1).$$

Since $r\sqrt{r^4 + 1}$ is always greater than r when $x \neq \mathbf{0}$, iterates of g move away from the origin. Since the differential does not move

points at all, dg does not shed light on trajectories under g.

Let $f_\epsilon : \Re \to \Re$ be given by $f_\epsilon(x) = x - \epsilon x^2$, where ϵ is a real parameter. Note that $f_0(x) = x$, and $f_0' = 1$. Therefore f_0 has a nonhyperbolic fixed point at **every** $x \in \Re$. For arbitrarily small (in modulus) $\epsilon \neq 0$, the only fixed point of f_ϵ is 0. The map f_0 is not persistent, since (as we see in Figure 3) arbitrarily close maps do not even have the same number of fixed points.



f_0, which fixes every point in $\Re$, has arbitrarily close maps which have only a single fixed point. This sensitivity to small changes does not occur in maps with hyperbolic fixed points.

Figure 3: f_0 is not persistent.

Consider the function $g : \Re^2 \to \Re^2$ given by

$$g(x) = \begin{pmatrix} x_1 - \epsilon(x_1^2 + x_2^2)x_2 \\ x_2 + \epsilon(x_1^2 + x_2^2)x_1 \end{pmatrix} \text{ where } \epsilon \text{ is a real parameter.}$$

Note that

$$dg = \begin{pmatrix} 1 - 2\epsilon x_1 x_2 & -\epsilon x_1^2 - 3\epsilon x_2^2 \\ 3\epsilon x_1^2 + \epsilon x_2^2 & 1 + 2\epsilon x_1 x_2 \end{pmatrix},$$

and

$$dg_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore g has a nonhyperbolic fixed point at $\mathbf{0}$ for all ϵ. For all $\epsilon > 0$, iterates of g move points away from $\mathbf{0}$. However, when $\epsilon = 0$, $g(x) = x$, so iterates of g do not move points at all. Hence the behavior of g near fixed point $\mathbf{0}$ is not preserved by small perturbations of g.

For the function in the next example, we compute the stable and unstable manifolds.

Define the function $g : \Re^2 \to \Re^2$ by

$$g(x) = \begin{pmatrix} \frac{1}{3}x_1 \\ 3x_2 - \frac{26}{9}x_1^2 \end{pmatrix}.$$

Then

$$dg = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{52}{9}x_1 & 3 \end{pmatrix},$$

and

$$dg_{\mathbf{0}} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}.$$

So g has a hyperbolic fixed point at $\mathbf{0}$.

Note that $g \begin{pmatrix} t \\ t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}t \\ 3t^2 - \frac{26}{9}t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}t \\ (\frac{1}{3}t)^2 \end{pmatrix}$.

Therefore the curve $y = x^2$ is the global stable manifold.

Note also that $g \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 3t \end{pmatrix}$.

So the y-axis serves as the global unstable manifold.

Chapter 3

Preliminary Definitions

Let $\delta_{i,j}$ be given by

$$\delta_{i,j} = \begin{cases} 0 & \text{if i} \neq \text{j} \\ 1 & \text{if i} = \text{j} \end{cases}$$

For $x \in \Re^n$ let x^{-1} be the vector in $\Re^n$ with i^{th} component $\frac{1}{x_i}$.

Let the set Λ be defined by

$$\Lambda = \{x \in \Re^n : \forall i. x_i \geq 0, \sum x_i = 1\}.$$

A column vector with entries $e_1, e_2, \ldots, e_n$ is denoted by

$$< e_1, e_2, \ldots, e_n > .$$

Denote the diagonal matrix having the entries of the vector

$$l =< l_0, l_1, ...l_{n-1} >$$

on its diagonal by diag(l); that is, diag$(l)_{i,i} = l_i$.

The determinant of a matrix L is denoted by det(L).

The ϵ-*neighborhood* of $t \in \Re^n$ is the set

$$\mathcal{B}_\epsilon(t) = \{x : \| x - t \| < \epsilon\}.$$

(Unless otherwise noted, $\| x \| = \sqrt{x^T x}$ for vectors x.)

For any set $T \subset \Re^n$ and $\epsilon > 0$, the ϵ-*neighborhood* of T is

$$\mathcal{B}_\epsilon(T) = \cup_{t \in T} \mathcal{B}_\epsilon(t).$$

If the value of $\epsilon > 0$ is not important, the term *neighborhood* is sometimes used to refer to an ϵ-neighborhood.

For any set A, the set-theoretic complement of A is denoted A^c.

For any set A, the *interior* of A is given by

$$\text{Int}(A) = \{p : \exists \text{ neighborhood } N \text{ of } p \text{ such that } N \subset A\}.$$

A set $A \subset \Re^n$ is *dense* in $\Re^n$ if, given any $x \in \Re^n$ and any neighborhood N of x, $N \cap A \neq \emptyset$.

For any set A, the *closure* of A is given by

$$\text{Cl}(A) = \{p : \forall \text{ neighborhood } N \text{ of } p,\ N \cap A \neq \emptyset\}.$$

A set A is *nowhere dense* if the interior of the closure of A is $\emptyset$.

Chapter 4

Goals and Outline of the Proof

A continuously differentiable function $\mathcal{G} : \Lambda \to \Lambda$ is used to model a simple genetic algorithm. This function is the composition of two functions $\mathcal{M}$ and $\mathcal{F}$; the mixing function $\mathcal{M}$ encodes information about crossover and mutation, and $\mathcal{F}$ contains information about selection. The function $\mathcal{F}$ can be written in terms of a fitness matrix $F = \text{diag}(f)$ where f is the fitness vector.

The goal of this paper is to prove that for all fitness vectors f in some dense open set of $(\Re^+)^n$, every fixed point of $\mathcal{G} = \mathcal{M} \circ \mathcal{F}$ in Λ is hyperbolic. However, the function $\mathcal{G}' = F\mathcal{M}$ is easier to work with since, unlike $\mathcal{G}$, it does not involve division. A rational function

ϕ will be defined and shown to have the property that, for x in the dense and open domain S of ϕ, x is a fixed point of $\mathcal{G}'$ if and only if $f = \phi(x)$. Moreover, if for fixed fitness f $\mathcal{G}'$ has only hyperbolic fixed points in S, and if the fitness vector is chosen from $(\Re^+)^n$, then $\mathcal{G}$ has only hyperbolic fixed points (in Λ) as well. Hence the focus here is on proving Theorem 7.1 of this report, which says that for all fitness vectors f in some dense open set of $\Re^n$, all fixed points of $\mathcal{G}'$ in ϕ's domain are hyperbolic.

A vector x is said to be *nice* if it is a hyperbolic fixed point of $\text{diag}(\phi(x))\mathcal{M}$. That is, $x \in S$ is nice if it is a hyperbolic fixed point of $\mathcal{G}'$. Using this terminology, Theorem 7.1 says that there exists a dense open set in $\Re^n$ such that given any fitness vector f in that set, all elements of $\phi^{-1}(f)$ are nice.

Our proof of Theorem 7.1 depends largely on the following facts, which are proved later in the paper:

(F1) The set $\mathcal{N}$ of nice vectors is open (Lemma 7.6).

(F2) There is a dense open set U such that for all fitness vectors f in U, the following properties hold:

1. The number of elements in $\phi^{-1}(f)$ is is uniformly bounded.

2. For every $x \in \phi^{-1}(f)$, $\det(d\phi_x) \neq 0$.

   (Recall that if x is in S (the domain of ϕ), x is a fixed point of $\mathcal{G}'$ if and only if $f = \phi(x)$.)

(F3) Given a vector $y \in U$ such that m elements of $\phi^{-1}(y)$ are nice, either all elements of $\phi^{-1}(y)$ are nice, or there exists y' arbitrarily close to y such that $\phi^{-1}(y')$ has at least $m+1$ nice elements (Lemma 7.7).

Theorem 7.1 is proved by establishing the existence of a dense open subset D of U such that all members of $\phi^{-1}(D)$ are nice. An outline of the proof follows. It is an outline only in the sense that it is based on the stated properties of ϕ and the three facts enumerated above (these key facts will be established later in chapters 7, 6 and 7 respectively); in all other respects the proof is complete.

Let $D = \{f \in U : \phi^{-1}(V) \subset \mathcal{N}$ for some neighborhood V of $f\}$. Note that $\phi^{-1}(D) \subset \mathcal{N}$. Our objective is to show that D is dense and open in $\Re^n$. This is achieved in three steps.

1. Show that D is dense in U: given $y \in U$ and $\alpha > 0$, prove that $\mathcal{B}_\alpha(y) \cap D \neq \emptyset$.

(a) First choose appropriate $f \in \mathcal{B}_\alpha(y) \cap U$.

Define the integer m by

$$m = \max_{f \in \mathcal{B}_\alpha(y) \cap U} |\phi^{-1}(f)| .$$

Note that this maximum exists (by F2) since it is taken over a subset of U. Let

$$T = \{f \in \mathcal{B}_\alpha(y) \cap U : |\phi^{-1}(f)| = m\}.$$

and define

$$p = \max_{f \in T} \{\text{number of nice elements in } \phi^{-1}(f)\}.$$

Choose $f \in T$ such that $\phi^{-1}(f)$ has p nice elements. That is, among the elements of $\mathcal{B}_\alpha(y) \cap U$ with inverse images of maximal size, f's inverse image contains the maximal number of nice elements.

(b) Next show all elements of $\phi^{-1}(f)$ are nice.

Since the result is vacuously true if $\phi^{-1}(f) = \emptyset$, assume that $\phi^{-1}(f)$ is nonempty. Let $\phi^{-1}(f) = \{x^1, x^2, ..., x^m\}$. Since $f \in U$, $\det(d\phi_{x^i}) \neq 0$ for $i = 1, 2, ..., m$ (by F2). By the Inverse Function Theorem (see appendix), there

are open sets $\mathcal{O}^i$ and V such that $x^i \in \mathcal{O}^i$, $f \in V$, and $\phi_i = \phi|_{\mathcal{O}^i}: \mathcal{O}^i \to V$ are continuous bijections. We can assume without loss of generality that $V \subset \mathcal{B}_\alpha(y) \cap U$ since $\mathcal{B}_\alpha(y) \cap U$ is an open neighborhood of f and U is open. We can also assume that the $\mathcal{O}^i$'s are disjoint since m is finite. (See Figure 4).

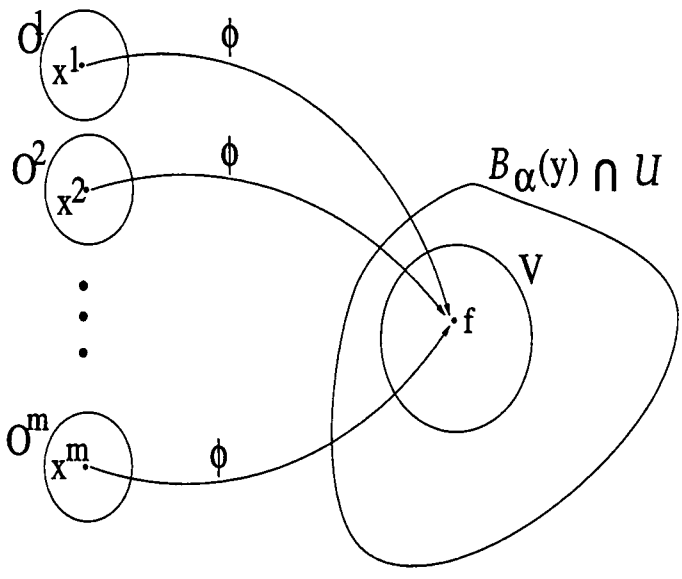


Figure 4: The Inverse Function Theorem.

Suppose that not all of the x^i are nice. By fact F3, there exists $f' \in V$ such that $\phi^{-1}(f')$ has at least $p+1$ nice elements. Since $\phi_i : \mathcal{O}^i \to V$ are bijections, $|\phi^{-1}(f')| \geq m$. But by the definition of m, $|\phi^{-1}(f')| \leq m$. Hence $|\phi^{-1}(f')| = m$. So $f' \in T$, which contradicts the maximality of p. Hence all elements of $\phi^{-1}(f)$ are nice.

(c) Finally show $f \in D$ (that is, find a neighborhood V of f such that $\phi^{-1}(V) \subset \mathcal{N}$).

Suppose first that $\phi^{-1}(f)$ is nonempty. We have shown that $x^i \in \mathcal{N}$ for all $i = 1, ..., m$. Since $\mathcal{N}$ is open (by F1), we can assume $\mathcal{O}^i \subset \mathcal{N}$. By the definition of m, any element of V has at most m inverse images. Since ϕ_i^{-1} is a bijection for all i, it follows that all elements of V have exactly m inverse images, and these are contained in $\bigcup_i \mathcal{O}^i$. Since $\mathcal{O}^i \subset \mathcal{N}$, it follows that all elements of $\phi^{-1}(V)$ are nice.

If $\phi^{-1}(f) = \emptyset$, then $\phi^{-1}(f') = \emptyset$ for all $f' \in \mathcal{B}_\alpha(y) \cap U$ by definition of m. Therefore $\phi^{-1}(\mathcal{B}_\alpha(y) \cap U) = \emptyset \subset \mathcal{N}$, so $f \in D$.

2. Show D is dense in $\Re^n$.

Because D is dense in U and U is dense in $\Re^n$,

$$\mathrm{Cl}(D) = \mathrm{Cl}(\mathrm{Cl}(D)) = \mathrm{Cl}(U) = \Re^n.$$

3. Show D is open.

The set D is given by

$$D = \{y \in U : \phi^{-1}(V) \subset \mathcal{N} \text{ for some neighborhood } V \text{ of } y\}.$$

Suppose $y \in D$. We must show that some neighborhood of y is contained in D. By definition of D, there is a neighborhood V of y such that $\phi^{-1}(V) \subset \mathcal{N}$. For any $f \in V$, V is an open set containing f such that $\phi^{-1}(V) \subset \mathcal{N}$; hence $f \in D$. Therefore $V \subset D$, and it follows that D is open.

Some consideration should be given to the possibility that the above proof may be vacuous; i.e., it's possible that for some fitness vectors in D, $\mathcal{G}$ may not have any fixed points. However, Brouwer's Fixed Point Theorem (see appendix) tells us that $\mathcal{G}$ has fixed points in Λ whenever all entries of the fitness vector are positive. Therefore there is no reason for concern, since in practice fitness vectors with positive entries are typically chosen. In this report, we are only interested in fixed points of $\mathcal{G}$ which correspond to fitness vectors with positive entries; note that this report shows that, given any fitness vector in $D \cap (\Re^+)^n$, all fixed points of $\mathcal{G}$ in Λ are hyperbolic. The

remainder of this report is devoted to establishing the claims made about ϕ and proving the key facts F1, F2, and F3.

Chapter 5

Notation and Definitions

For vectors $x, y \in \Re^n$, x and y are *perpendicular* if the dot product $x \cdot y$ is zero. The set $x^\perp$ is defined by

$$x^\perp = \{y \in \Re^n : x \cdot y = 0\}.$$

For an arbitrary integer i, the vector $< i, i, \ldots, i >$ is denoted by $\mathbf{i}$.

For integers i and j with base 2 representations i_2 and j_2, we will use $i \oplus j$ to denote the exclusive-or of i_2 and j_2.

Let the set Λ be given by

$$\Lambda = \{x \in \Re^n : \forall i. x_i \geq 0, \mathbf{1}^T x = 1\}.$$

The elements of Λ are sometimes called *population vectors.*

A symmetric $n \times n$ matrix M, called the *mixing matrix*, incorporates information about mutation and crossover. The matrix M satisfies the property $\sum_{u,v} M_{u,v} = n$. For a thorough description of M and its properties, see [52].

The *mixing function* $\mathcal{M}$ is defined by

$$\mathcal{M}(x)_i = \sum_{u,v} x_{i \oplus u} x_{i \oplus v} M_{u,v}.$$

Define the GA heuristic $\mathcal{G} : \Lambda \to \Lambda$ by

$$\mathcal{G} = \mathcal{M} \circ \mathcal{F}$$

where

$$F = \text{diag}(f),$$

$$\text{and } \mathcal{F}(x) = \frac{Fx}{\mathbf{1}^T F x}.$$

Define the function $\mathcal{G}'$ by $\mathcal{G}' = F\mathcal{M}$. As we noted in the previous chapter, $\mathcal{G}'$ is easier to work with than $\mathcal{G}$ because $\mathcal{G}'$ does not involve division.

Recall that a fixed point x of $\mathcal{G}$ is *hyperbolic* if none of the eigenvalues of $d\mathcal{G}_x$ have modulus one.

Define the set S by

$$S = \{x \in \Re^n : \forall i. x_i \mathcal{M}(x)_i \neq 0\}.$$

The function ϕ which was introduced earlier has domain S and is defined by

$$\phi(x) = \text{diag}(\mathcal{M}(x)^{-1})x.$$

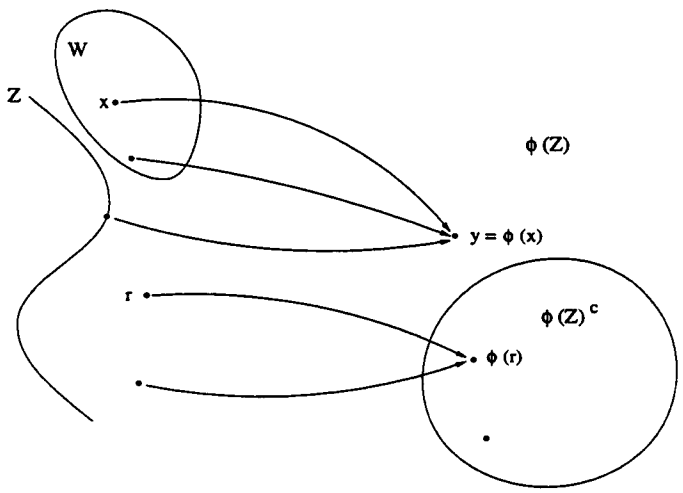
An *associate* of a vector x is an element of $\phi^{-1}(\phi(x))$.

Let $Z = \{x \in S : \det(d\phi_x) = 0\}$, the set of points at which the Inverse Function Theorem does not guarantee that ϕ is locally invertible.

The set $\phi(Z)$ contains the images of vectors at which ϕ might not be locally invertible. (We will show later that this set is "small" in some sense.) Note that $\phi(Z)^c = \{y \in \Re^n : \phi^{-1}(y) \cap Z = \emptyset\}$ is the set of points y such that either y is not in the range of ϕ, or ϕ is locally invertible at every element of y's inverse image.

Let $W = \phi^{-1}(\phi(Z)) \cap Z^c$, a set of points x at which ϕ is locally invertible, though potentially ϕ is not locally invertible at one (or more) of x's associates.

In Figure 5, the relationship between points in the sets Z, W, and $\phi(Z)$ is depicted. Note that if a vector x is in the set W, then at least one of x's associates lies in Z. So the image of W under ϕ does not intersect $\phi(Z)^c$.



Since $\phi(r) \in \phi(Z)^c$, all of r's associates avoid Z, and consequently ϕ is locally invertible at r and its associate. Although ϕ is locally invertible at x, one of x's associates is in Z and so ϕ might not be locally invertible there.

Figure 5: Relationship between Z, W, and $\phi(Z)$.

In the appendix, we will prove that there exists a positive integer k

such that the set $\{y \in \Re^n : |\phi^{-1}(y)| \leq k\}$ is dense and open. Let

$C = \{y \in \Re^n : |\phi^{-1}(y)| \leq k\}$.

Chapter 6

A Dense Open Subset of

$\phi(Z)^c \cap C$

In order to prove the main result of this report, Theorem 7.1, we assumed the existence of a dense open subset U of $\phi(Z)^c \cap C$. The goal of this chapter is to prove Theorem 6.1, which establishes fact F2. Three results are key to this proof:

(R1) Given $y \in \phi(Z)^c \cap C$, there exists arbitrarily close y' such that some neighborhood of y' is contained in $\phi(Z)^c \cap C$ (Lemma 6.7).

(R2) $\phi(Z)^c \cap C$ is dense (Lemmas 6.3 and 6.5).

(R3) There exists a positive integer k such that the set $C = \{f \in \Re^n : |\phi^{-1}(f)| \leq k\}$ is dense and open in $\Re^n$. (This theorem and an outline of its proof were provided by S. Mulay. A detailed proof of this result is contained in the appendix.)

Given these three facts, the proof of Theorem 6.1 is easy.

Theorem 6.1 *There exists a positive integer k and a dense open set U in $\Re^n$ such that for every f in U, $|\phi^{-1}(f)| \leq k$ and $det(d\phi_x) \neq 0$ for all $x \in \phi^{-1}(f)$.*

Proof of Theorem 6.1 Let $U = \text{Int}(\phi(Z)^c \cap C)$. The interior of any set is open, so U is open. We need only to prove that U is dense.

Since $\phi(Z)^c \cap C$ is dense (by R2), given any nonempty open set $P \subset \Re^n$ there exists $y \in P \cap (\phi(Z)^c \cap C)$. By R1, there exists $y' \in U \cap P$. So we have shown that U intersects all nonempty open sets in $\Re^n$. Therefore U is dense. ■

6.1 Preliminary Results

In this section, we prove results R1 and R2. As we noted above, fact R2 is a consequence of Lemmas 6.3 and 6.5. In order to prove result R1, which is Lemma 6.7, we use Lemmas 6.2 - 6.5.

Lemma 6.2 *The function $f(x) = det(d\phi_x)$ is rational in x and continuous on S.*

Proof Let $A = d\phi_x$. Note that

$$A_{i,j} = \frac{\delta_{i,j}\mathcal{M}(x)_i - 2x_i \sum_s x_s M_{i\oplus j, s\oplus i}}{(\sum_{u,v} x_{i\oplus u} x_{i\oplus v} M_{u,v})^2}.$$

By definition, $\det(A) = \sum \epsilon(\sigma) A_{\sigma(0),0} \ldots A_{\sigma(n-1),n-1}$ where the sum is taken over all permutations σ of $\{0, 1, \ldots n-1\}$ and $\epsilon(\sigma) \in \{-1, 1\}$. It follows that $\det(A)$ is a rational function in x.

We noted above that $f(x)$ is polynomial in the entries of $d\phi_x$. Since the denominator of $(d\phi_x)_{i,j}$ is

$$(\sum_{u,v} x_{i\oplus u} x_{i\oplus v} M_{u,v})^2 = (\mathcal{M}(x)_i)^2,$$

it follows from the definition of S that the denominator of $(d\phi_x)_{i,j}$ is nonzero on S for all i and j. Therefore the denominator of f is nonzero on S. ■

Lemma 6.3 *If T is a dense open set in $\Re^n$ and E is a dense set in $\Re^n$, then $T \cap E$ is dense in $\Re^n$.*

Proof Suppose by way of contradiction that $(T \cap E)^c = T^c \cup E^c$ contains a nonempty open set $\mathcal{O}$. Because T is dense, there exists some $x \in \mathcal{O} \cap T$. Since T and $\mathcal{O}$ are open, $\mathcal{O}_1 = \mathcal{O} \cap T$ is an open set about x. The set E is dense and $\mathcal{O}_1$ is open, so $E \cap \mathcal{O}_1 \neq \emptyset$. Let $t \in E \cap \mathcal{O}_1$. Since $\mathcal{O}_1 = \mathcal{O} \cap T \subset T$, it follows that $t \in T \cap E$. But

$t \in \mathcal{O}_1 \subset \mathcal{O} \subset (T \cap E)^c$. Contradiction! ■

Lemma 6.4 *The set S is dense and open in $\Re^n$.*

Proof Consider the continuous function $r(x) = \prod x_i \mathcal{M}(x)_i$. Since $\{0\}$ is closed, $S^c = r^{-1}(\{0\})$ is closed (by continuity). Thus S is open. We now show that S is dense. Suppose by way of contradiction that S^c contains a nonempty open set $\mathcal{O}$. Then r is a polynomial which is 0 on $\mathcal{O}$, and hence $r \equiv 0$. But this contradicts the fact that r is not the zero polynomial. ■

Lemma 6.5 *The set $\phi(Z)^c$ is dense in $\Re^n$.*

Proof Sard's Theorem (see appendix) implies that $\phi(Z)$ has measure 0. Since a set of measure 0 cannot contain a nonempty open set, it follows that $\phi(Z)^c$ is dense. ■

Lemma 6.6 *The set $Z^c \cap S$ is dense and open in $\Re^n$ when mutation is positive.*

Proof Let $f(x) = \det(d\phi_x)$. Note that $Z = f^{-1}(\{0\})$. By Lemma 6.2, f is continuous. So it follows that Z is closed in S. Since S is open, $Z^c \cap S$ is open in $\Re^n$. Next we show that $Z^c \cap S$ is dense. By Lemmas 6.3 and 6.4, it suffices to show that Z^c is dense. Suppose by

way of contradiction that Z contains a nonempty open set $\mathcal{O}$. Since f is a continuous function and f is 0 on a nonempty open set, it follows that $f \equiv 0$. We reach a contradiction by showing $f(\mathbf{1}) \neq 0$. Recall that

$$(d\phi_x)_{i,j} = \frac{\delta_{i,j}\mathcal{M}(x)_i - 2x_i \sum_s x_s M_{i\oplus j, s\oplus i}}{(\sum_{u,v} x_{i\oplus u} x_{i\oplus v} M_{u,v})^2}.$$

Therefore

$$(d\phi_1)_{i,j} = \frac{\delta_{i,j} n - 2\sum_s 1 M_{i\oplus j, s\oplus i}}{n^2} = \frac{\delta_{i,j} n - (d\mathcal{M}_1)_{i,j}}{n^2}.$$

So $n(d\phi_1)_{i,j} = \delta_{i,j} - \frac{(d\mathcal{M}_1)_{i,j}}{n}$, and hence $nd\phi_1 = I - \frac{d\mathcal{M}_1}{n}$. Let $A = \frac{d\mathcal{M}_1}{n}$. Since n is nonzero, $f(\mathbf{1}) \neq 0$ if and only if $\det(nd\phi_1) \neq 0$. Since $\det(nd\phi_1) = \det(I - A)$, we will prove that $f(\mathbf{1}) \neq 0$ by showing that 1 is not an eigenvalue of A. Note that $Ay = y$ implies $d\mathcal{M}_1 y = ny$. However, Vose has proved in [52] that $\text{spec}(d\mathcal{M}_x) = 2(\mathbf{1}^T x)\text{spec}(M^*)$, and that 1 is the only eigenvalue of M^* with modulus greater than or equal to $\frac{1}{2}$ when mutation is positive. Therefore $\text{spec}(d\mathcal{M}_1) = 2n\text{spec}(M^*)$, $2n$ is an eigenvalue of $d\mathcal{M}_1$, and all other eigenvalues of $d\mathcal{M}_1$ have modulus less than n. Hence n is not an eigenvalue of $d\mathcal{M}_1$, and so 1 is not an eigenvalue of A. It follows that $f(\mathbf{1}) \neq 0$. ■

Lemma 6.7 *Given $y \in \phi(Z)^c \cap C$, there exists arbitrarily close $y' \in \phi(Z)^c \cap C$ such that some neighborhood of y' is contained in*

$\phi(Z)^c \cap C$.

Proof If there is a neighborhood of y which is contained in $\phi(Z)^c \cap C$, there is nothing to prove. So assume no neighborhood of y is contained in $\phi(Z)^c \cap C$.

Since C is an open set and $y \in C$, choose arbitrarily small $\alpha > 0$ such that $\mathcal{B}_\alpha(y) \subset C$. We will show that for some $y' \in \mathcal{B}_\alpha(y)$, a neighborhood of y' is contained in $\phi(Z)^c \cap C$.

Let $m = \max_{f \in \mathcal{B}_\alpha(y)} |\phi^{-1}(f) \cap Z^c|$, and choose $f \in \mathcal{B}_\alpha(y)$ such that $|\phi^{-1}(f) \cap Z^c| = m$.

In the proof that follows, we will need to assume that $m > 0$. So first suppose by way of contradiction that $m = 0$. Then $\phi^{-1}(\mathcal{B}_\alpha(y) \cap \phi(S)) \subset Z$. Note that $\phi : S \to \phi(S)$ is continuous and $\mathcal{B}_\alpha(y) \cap \phi(S)$ is open in $\phi(S)$; hence $\phi^{-1}(\mathcal{B}_\alpha(y) \cap \phi(S))$ is an open subset of S. Since Z contains no nonempty open sets (Lemma 6.6), $\phi^{-1}(\mathcal{B}_\alpha(y) \cap \phi(S)) = \emptyset$. Therefore $\phi^{-1}(f) = \emptyset$ for all $f \in \mathcal{B}_\alpha(y)$, and hence $\mathcal{B}_\alpha(y) \subset \phi(Z)^c \cap C$. This contradicts our assumption that no neighborhood of y is contained in $\phi(Z)^c \cap C$. Hence $m > 0$.

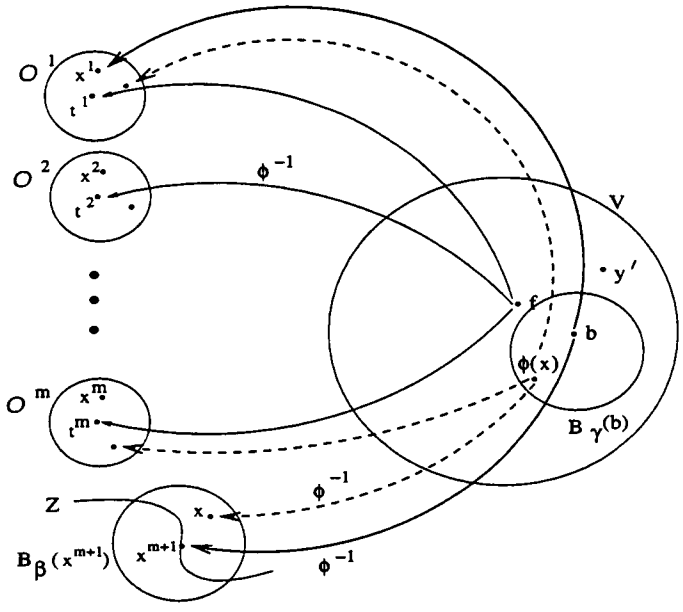
Denote the elements in $\phi^{-1}(f) \cap Z^c$ by $t^1, t^2, ..., t^m$. By the Inverse

Function Theorem, there are open sets $\mathcal{O}^i$ and V such that $t^i \in \mathcal{O}^i$, $f \in V$, $\mathcal{O}^i \subset Z^c$, and $\phi^{-1} : V \to \mathcal{O}^i$ is a differentiable bijection for all i. Since $f \in \mathcal{B}_\alpha(y)$, we can assume that $V \subset \mathcal{B}_\alpha(y) \subset C$.

By Lemmas 6.3 and 6.5, $\phi(Z)^c \cap C$ is dense, and therefore there exists $y' \in (\phi(Z)^c \cap C) \cap V$. Since $y' \in V$, the Inverse Function Theorem implies that $|\phi^{-1}(y')| \geq m$. Hence by the definition of m, $|\phi^{-1}(y')| = m$.

Suppose by way of contradiction that no neighborhood of y' is contained in $\phi(Z)^c \cap C$. Then since V is open and $y' \in V \subset C$, there exists $b \in V$ such that $\phi^{-1}(b) \cap Z \neq \emptyset$. Let $x^{m+1} \in \phi^{-1}(b) \cap Z$. By the Inverse Function Theorem, we know that there is $x^i \in \mathcal{O}^i \cap \phi^{-1}(b)$, for $i = 1, ..., m$, as well.

Choose $\gamma > 0$ such that $\mathcal{B}_\gamma(b) \subset V$. The function ϕ is continuous on S, so there exists $\beta > 0$ such that if $x \in \mathcal{B}_\beta(x^{m+1}) \cap S$, then $\phi(x) \in \mathcal{B}_\gamma(b)$. By Lemma 6.6, $Z^c \cap S$ is dense, so we can choose $x \in \mathcal{B}_\beta(x^{m+1}) \cap (Z^c \cap S)$. Note that $\phi^{-1}(\phi(x))$ contains an element in each $\mathcal{O}^i$, $i = 1, ..., m$, as well as x. Hence $\phi^{-1}(\phi(x))$ contains at least $m + 1$ elements in Z^c (see Figure 6).



Since $b \notin \phi(Z)^c$, there is $x^{m+1} \in Z$ such that $b = \phi(x^{m+1})$. The bijections $\phi^{-1} : V \to \mathcal{O}^i$ give us $x^i \in \mathcal{O}^i$ such that $b = \phi(x^i)$. By continuity of ϕ, if $x \in \mathcal{B}_\beta(x^{m+1})$ then $\phi(x) \in \mathcal{B}_\gamma(b)$. Choose $x \in \mathcal{B}_\beta(x^{m+1}) \cap (Z^c \cap S)$; then $| \phi^{-1}(\phi(x)) \cap Z^c | > m$. Contradiction!

Figure 6: Nice inverse images.

But this contradicts the definition of m. Therefore some neighborhood of y' is contained in $\phi(Z)^c \cap C$. ■

Chapter 7

Really Nice Fixed Points

Theorem 7.1 *The set*

$$D = \{y \in U : \phi^{-1}(V) \subset \mathcal{N} \text{ for some neighborhood } V \text{ of } y\}$$

is dense and open in $\Re^n$.

Theorem 7.1 was proved in chapter 4. What follows are proofs of the remaining facts that were referenced in the proof of theorem 7.1.

7.1 Preliminary Results

Lemma 7.2 *For all* $x \in S$, $F\mathcal{M}(x) = x$ *if and only if* $F = diag(\phi(x))$.

Proof Suppose that $F\mathcal{M}(x) = x$. Then $F_{i,i}\mathcal{M}(x)_i = x_i$. For all $x \in S$, $\mathcal{M}(x)_i \neq 0$; therefore $F_{i,i} = \frac{x_i}{\mathcal{M}(x)_i}$. Recall that

$\phi(x)_i = (\text{diag}(\mathcal{M}(x)^{-1})x)_i = \frac{x_i}{\mathcal{M}(x)_i}$. It follows that $F = \text{diag}(\phi(x))$.

Next assume that $x \in S$ and $F = \text{diag}(\phi(x))$. Then $F_{i,i} = \frac{x_i}{\mathcal{M}(x)_i}$, which implies that $(F\mathcal{M}(x))_i = x_i$. Therefore $F\mathcal{M}(x) = x$. ■

Given a matrix A, let the *twist* of A be given by $A^*_{i,j} = A_{i\oplus j,i}$. Given $x \in S$, let $E = \text{diag}(\phi(x))d\mathcal{M}_x$ and $P = I - \frac{xx^T}{x^Tx}$.

Lemma 7.3 *Let $x \in S$. Then spec(EP) = spec$(E) \setminus \{2\} \cup \{0\}$.*

Proof Note that

$$EPx = E(I - \frac{xx^T}{x^Tx})x = E(x - x) = 0.$$

So 0 is an eigenvalue of EP with corresponding eigenvector x. Vose has proved in [52] that $d\mathcal{M}_x x = 2\mathcal{M}(x)$, and Lemma 7.2 tells us that $\text{diag}(\phi(x))\mathcal{M}(x) = x$. So

$$\begin{aligned} Ex &= \text{diag}(\phi(x))d\mathcal{M}_x x = \text{diag}(\phi(x))(2\mathcal{M}(x)) \\ &= 2\text{diag}(\phi(x))\mathcal{M}(x) = 2x. \end{aligned}$$

Therefore two is an eigenvalue of E with corresponding eigenvector x. Note that if $z \in x^\perp$, then $x^Tz = 0$, and therefore $x^T(E^Tz) = (Ex)^Tz = (2x)^Tz = 2x^Tz = 0$. Hence for any $z \in x^\perp$, $E^Tz \in x^\perp$ also.

Next we will show that if $z \in x^{\perp}$, then $P^T z = z$. If $z \in x^{\perp}$, then $x^T z = 0$. So $P^T z = (I - \frac{xx^T}{x^T x})^T z = (I - \frac{xx^T}{x^T x})z = z - \frac{xx^T z}{x^T x} = z - 0 = z$. Therefore $P^T \mid_{x^{\perp}} = I$.

Since $EPx = 0x$, Theorem 6.12 of [52] (see appendix) tells us that

$$\text{spec}(EP) = \text{spec}((EP)^T \mid_{x^{\perp}}) \cup \{0\} \equiv \text{spec}(P^T E^T \mid_{x^{\perp}}) \cup \{0\}.$$

Note that $P^T E^T \mid_{x^{\perp}} = E^T \mid_{x^{\perp}}$ since $P^T z = z$ for all $z \in x^{\perp}$ and $E^T : x^{\perp} \to x^{\perp}$. Thus $\text{spec}(EP) = \text{spec}(E^T \mid_{x^{\perp}}) \cup \{0\}$.

Since $Ex = 2x$, Theorem 6.12 of [52] tells us that

$$\text{spec}(E) = \text{spec}(E^T \mid_{x^{\perp}}) \cup \{2\},$$

and hence

$$\text{spec}(E^T \mid_{x^{\perp}}) = \text{spec}(E) \setminus \{2\}.$$

Therefore $\text{spec}(EP) = \text{spec}(E) \setminus \{2\} \cup \{0\}$. ■

Define a function $q : S \to \mathbb{C}$ by $q(x) = \prod_{i,j}(\lambda_i \lambda_j - 1)$ where the λ's are the roots of the characteristic polynomial

$$c(y) = \left| yI - \text{diag}(\phi(x)) d\mathcal{M}_x (I - \frac{xx^T}{x^T x}) \right| .$$

Lemma 7.4 *The function q is not identically zero.*

Proof Consider $x =< \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} >$. Note that since $\sum_{u,v} M_{u,v} = n$, it follows that

$$\mathcal{M}(x)_i = \sum_{u,v} x_{i\oplus u} x_{i\oplus v} M_{u,v} = \sum_{u,v} \frac{1}{n^2} M_{u,v} = \frac{1}{n^2} \sum_{u,v} M_{u,v} = \frac{1}{n}.$$

Hence $x \in S$.

We will show that $q(x) \neq 0$. It suffices to show that all roots of $c(y)$ have modulus less than one. Equivalently, we need to show that all eigenvalues of EP have modulus strictly less than one. By Lemma 7.3, it is sufficient to show that for $x =< \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} >$, all the eigenvalues of E have modulus less than one except for the simple eigenvalue 2.

Note that for our choice of x, $\phi(x)_i = 1$ for all i. Therefore

$$E = \text{diag}(\phi(x)) d\mathcal{M}_x = I d\mathcal{M}_x = d\mathcal{M}_x.$$

Theorem 6.13 of [52] (see appendix) shows that

$$\text{spec}(d\mathcal{M}_x) = 2(\mathbf{1}^T x)\text{spec}(M^*).$$

Additionally, Vose has proved in [52] that 1 is a simple eigenvalue of M^*, and all other eigenvalues of M^* have modulus less than $\frac{1}{2}$. Since $\mathbf{1}^T x = 1$, $\text{spec}(E) = \text{spec}(d\mathcal{M}_x) = 2\text{spec}(M^*)$. Therefore all eigenvalues of E have modulus less than one except for the simple

eigenvalue 2. Hence all eigenvalues of EP have modulus less than one, and so for $x =< \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} >$, $q(x) \neq 0$. ■

Lemma 7.5 *The set* $R = \{x \in S : q(x) \neq 0\}$ *is dense and open in* $\Re^n$. *Moreover, for all* $x \in R$ *no eigenvalue of* $diag(\phi(x))d\mathcal{M}_x$ *has modulus one.*

Proof First we will show that $q(x)$ is a rational function in x. Recall that $q(x) = \prod_{i,j}(\lambda_i\lambda_j - 1)$ where the λ's are the roots of the characteristic polynomial $c(y) = |yI - EP|$. The characteristic polynomial c is given by

$$c(y) = (y - \lambda_0)(y - \lambda_1)...(y - \lambda_{n-1}) = \sum_{i=0}^{n-1} y^i(-1)^{n-i} f_{n-i}$$

where the f's are the elementary symmetric functions (see chapter 5 of [31]) in the λ's. Since q is symmetric in the λ's, the Symmetric Function Theorem (see appendix) tells us that q is a polynomial in the elementary symmetric functions in $\lambda_0, \lambda_1, ...\lambda_{n-1}$; therefore q is a polynomial in the coefficients of c, the characteristic polynomial of $EP = \text{diag}(\phi(x))d\mathcal{M}_x(I - \frac{xx^T}{x^Tx})$.

Note that

$$(d\mathcal{M}_x)_{i,j} = 2\sum_s x_s M_{s\oplus i, i\oplus j}$$

and

$$(\text{diag}(\phi(x)))_{i,i} = \phi(x)_i = \frac{x_i}{\mathcal{M}(x)_i} = \frac{x_i}{\sum_{u,v} x_{u\oplus i} x_{v\oplus j} M_{u,v}}.$$

So the matrices $\text{diag}(\phi(x))$, $d\mathcal{M}_x$, and $I - \frac{xx^T}{x^T x}$ have entries which are rational functions in x. Hence for any i and j, $(EP)_{i,j}$ is a rational function in x, and consequently the coefficients of $c(y) = |yI - EP|$ are rational functions in x. Therefore q is a rational function in x. So we have showed that $q(x) = \frac{p(x)}{d(x)}$ for some polynomials p and d. (Note that the entries of EP are defined for all $x \in S$, and therefore $x \in S$ implies that $d(x) \neq 0$.)

By Lemma 7.4, $q \not\equiv 0$. Theorem A.2 in the appendix tells us that a rational function is either identically zero or nonzero on a dense open set. Hence R is dense and open.

Clearly if $x \in R$, then no eigenvalue of EP has modulus one. It follows from Lemma 7.3 that, for $x \in R$, no eigenvalue of $E = \text{diag}(\phi(x))d\mathcal{M}_x$ has modulus one either. ■

Note: We have proved that R is a dense open set of nice vectors. In the next lemma, we show that the set $\mathcal{N}$ of nice vectors is open, and therefore dense and open.

Lemma 7.6 *The set $\mathcal{N}$ of nice vectors is dense and open.*

Proof Note that

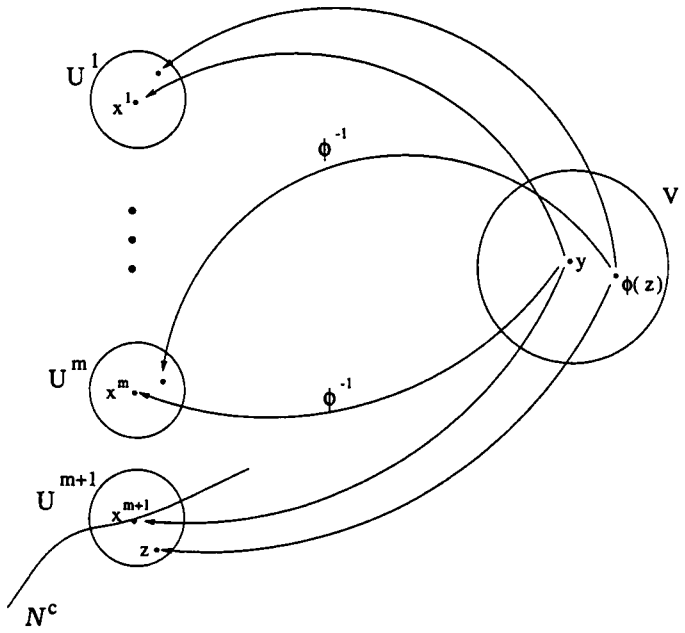
$$\mathcal{N} = \{x \in S\text{: no root of } |yI - \text{diag}(\phi(x))d\mathcal{M}_x| \text{ has modulus one}\}.$$

The coefficients of the polynomial $|yI - \text{diag}(\phi(x))d\mathcal{M}_x|$ are continuous functions of x on S. Consider $x \in \mathcal{N}$. It follows from Theorem A.3 (see appendix) that any polynomial which is sufficiently close to $|yI - \text{diag}(\phi(x))d\mathcal{M}_x|$ in terms of coefficients will not have any roots of modulus one either. Since the coefficients of $|yI - \text{diag}(\phi(x))d\mathcal{M}_x|$ are continuous functions in x, there is some neighborhood of a nice vector x such that all elements of the neighborhood are also nice. Therefore $\mathcal{N}$ is open. ■

Lemma 7.7 *Let $y \in U$ and $\epsilon > 0$ be given, and let $x^1, x^2, ..., x^m$ be the nice elements of $\phi^{-1}(y)$. Either every element of $\phi^{-1}(y)$ is nice, or $\exists\ y' \in \mathcal{B}_\epsilon(y)$ such that $\phi^{-1}(y')$ contains at least $m+1$ nice elements.*

Proof Let $y \in U$ and suppose $x^1, ..., x^m$ are the nice elements of $\phi^{-1}(y)$, and x^{m+1} is an element of $\phi^{-1}(y)$ which isn't nice. By definition of U and the Inverse Function Theorem, there are continuous bijections $\phi^{-1} : V \to U^i$ where V and U^i are open, $y \in V$, and

$x^i \in U^i$. Since the set $\mathcal{N}$ of nice vectors is open and $x^1, ..., x^m$ are nice, we can assume $U^1, ..., U^m \subset \mathcal{N}$. Since $\mathcal{N}$ is dense (Lemma 7.6), there is $z \in U^{m+1}$ which is nice (see Figure 7). It follows that $\phi^{-1}(\phi(z))$ has at least $m+1$ nice elements. ■



Since we can choose V to be arbitrarily small, $\phi(z)$ is arbitrarily close to y and has more nice inverse images.

Figure 7: $\phi(z)$ has more nice inverse images.

Chapter 8

Hyperbolic Fixed Points of $\mathcal{G}$

The goal of this chapter is to prove that for all fitness vectors in a dense open set in $(\Re^+)^n$, the map $\mathcal{G} = \mathcal{M}\mathcal{F} : \Lambda \to \Lambda$ has hyperbolic fixed points. The following lemmas prepare the way.

Lemma 8.1 *If* $\mathcal{G}(x) = x$ *and* $\mathbf{1}^T Fx \neq 0$, *then* $\mathcal{G}'(y) = y$ *where* $y = \frac{Fx}{(\mathbf{1}^T Fx)^2}$.

Proof Note that for scalar c and vector z,

$$\mathcal{M}(cz) = \sum_{u,v}(cz)_{i\oplus u}(cz)_{i\oplus v}M_{u,v} = \sum_{u,v} c^2 z_{i\oplus u}z_{i\oplus v}M_{u,v} = c^2\mathcal{M}(z).$$

So since $\mathcal{G}(x) = x$,

$$x = \mathcal{M}(\frac{Fx}{(\mathbf{1}^T Fx)}) = \mathcal{M}(\mathbf{1}^T Fxy) = (\mathbf{1}^T Fx)^2\mathcal{M}(y).$$

So $\mathcal{M}(y) = \frac{x}{(1^T Fx)^2}$. Therefore $\mathcal{G}'(y) = F\mathcal{M}(y) = \frac{Fx}{(1^T Fx)^2} = y$. ■

If x is a fixed point of $\mathcal{G}$ and $1^T Fx \neq 0$, we call $y = \frac{Fx}{(1^T Fx)^2}$ the *corresponding fixed point* of $\mathcal{G}'$.

Lemma 8.2 *If* $Ay = \lambda y$ *then* $A^T : y^\perp \to y^\perp$.

Proof Let $z \in y^\perp$, that is, assume that $y^T z = 0$. We want to show that $y^T(A^T z) = 0$. We have $y^T A^T z = (Ay)^T z = (\lambda y)^T z = \lambda y^T z = 0$.

■

Lemma 8.3 *If* x *is a fixed point of* $\mathcal{G}$ *and* $y = \frac{Fx}{(1^T Fx)^2}$ *is the corresponding fixed point of* $\mathcal{G}'$, *then* $spec(d\mathcal{G}_x) = spec(d\mathcal{G}'_y) \setminus \{2\} \cup \{0\}$.

Proof Assume that $\mathcal{G}(x) = x$ and let $y = \frac{Fx}{(1^T Fx)^2}$, the corresponding fixed point of $\mathcal{G}'$. Note that $\mathcal{F}(x) = \frac{Fx}{1^T Fx}$, so

$$d\mathcal{F}_x = \frac{F}{1^T Fx} - Fx\frac{1^T}{(1^T Fx)^2}F$$

by the product rule. From [52] we know that $d\mathcal{M}_y c$ is linear in y and c. Since $d\mathcal{G}'_y = Fd\mathcal{M}_y$, it follows that $d\mathcal{G}'_y c$ is also linear in y and c. Since $\mathcal{G} = F^{-1} \circ \mathcal{G}' \circ \mathcal{F}$, the chain rule gives

$$\begin{aligned} d\mathcal{G}_x &= F^{-1} d\mathcal{G}'_{\mathcal{F}x}\left(\frac{F}{1^T Fx} - Fx\frac{1^T}{(1^T Fx)^2}F\right) \\ &= F^{-1} d\mathcal{G}'_{\frac{Fx}{1^T Fx}} \frac{1}{1^T Fx}\left(F - Fx\frac{1^T}{1^T Fx}F\right) \end{aligned}$$

$$= F^{-1} d\mathcal{G}'_y (I - Fx\tfrac{\mathbf{1}^T}{\mathbf{1}^T Fx})F$$

Therefore $\text{spec}(d\mathcal{G}_x) = \text{spec}(Fd\mathcal{G}_x F^{-1}) = \text{spec}(d\mathcal{G}'_y(I - Fx\frac{\mathbf{1}^T}{\mathbf{1}^T Fx}))$.

Let $P = I - Fx\frac{\mathbf{1}^T}{\mathbf{1}^T Fx}$. Then

$$Py = (I - Fx\frac{\mathbf{1}^T}{\mathbf{1}^T Fx})\frac{Fx}{(\mathbf{1}^T Fx)^2} = \frac{Fx}{(\mathbf{1}^T Fx)^2} - \frac{Fx\mathbf{1}^T Fx}{(\mathbf{1}^T Fx)^3} = 0.$$

Hence 0 is an eigenvalue of P with corresponding eigenvector y. In [52] Vose proved that $d\mathcal{M}_y y = 2\mathcal{M}(y)$. So we also have

$$d\mathcal{G}'_y y = Fd\mathcal{M}_y y = F2\mathcal{M}(y) = 2F\mathcal{M}(y) = 2\mathcal{G}'(y) = 2y.$$

Hence 2 is an eigenvalue of $d\mathcal{G}'_y$ with corresponding eigenvector y. Therefore $Py = 0$, $d\mathcal{G}'_y y = 2y$, and by Lemma 8.2, $(d\mathcal{G}'_y)^T : y^\perp \to y^\perp$.

Next we show that for $z \in y^\perp$, $P^T z = z$. Note that

$$y^T z = 0 \Rightarrow \frac{x^T F^T}{(1^T Fx)^2} z = 0 \Rightarrow x^T F^T z = 0.$$

It follows that $P^T z = (I - \frac{1x^T F^T}{1^T Fx})z = z - \frac{1x^T F^T z}{1^T Fx} = z - 0 = z$. Therefore $P^T|_{y^\perp} = I$. Since $d\mathcal{G}'_y Py = 0y$, Theorem 6.12 in [52] implies that

$$\text{spec}(d\mathcal{G}'_y P) = \text{spec}((d\mathcal{G}'_y P)^T|_{y^\perp}) \cup \{0\} = \text{spec}(P^T(d\mathcal{G}'_y)^T|_{y^\perp}) \cup \{0\}.$$

But $P^T(d\mathcal{G}'_y)^T|_{y^\perp} = (d\mathcal{G}'_y)^T|_{y^\perp}$, since $P^T z = z \ \forall z \in y^\perp$ and $(d\mathcal{G}'_y)^T : y^\perp \to y^\perp$. Thus $\text{spec}(d\mathcal{G}'_y P) = \text{spec}((d\mathcal{G}'_y)^T|_{y^\perp}) \cup \{0\}$.

Since $d\mathcal{G}'_y y = 2y$, Theorem 6.12 in [52] tells us that

$$\text{spec}(d\mathcal{G}'_y) = \text{spec}((d\mathcal{G}'_y)^T|_{y^\perp}) \cup \{2\},$$

and so $\text{spec}((d\mathcal{G}'_y)^T|_{y^\perp}) = \text{spec}(d\mathcal{G}'_y) \setminus \{2\}$. Thus

$$\text{spec}(d\mathcal{G}'_y P) = \text{spec}(d\mathcal{G}'_y) \setminus \{2\} \cup \{0\}$$

or equivalently, $\text{spec}(d\mathcal{G}_x) = \text{spec}(d\mathcal{G}'_y) \setminus \{2\} \cup \{0\}$. ■

Lemma 8.4 *Consider a heuristic $\mathcal{G} : \Lambda \to \Lambda$ such that the entries of $\mathcal{G}$'s fitness vector are positive. For every fixed point x of $\mathcal{G}$, the corresponding fixed point $y = \frac{Fx}{(1^T Fx)^2}$ of $\mathcal{G}'$ is in S.*

Proof Note that if a fixed point y of $\mathcal{G}'$ is in S^c, then either $y_i = 0$ or $\mathcal{M}(y)_i = 0$ for some i. Consider the i^{th} fixed point equation of $\mathcal{G}'$:

$$y_i - f_i \mathcal{M}(y)_i = 0$$

Clearly if $\mathcal{M}(y)_i = 0$, $y_i = 0$ also. So a fixed point y of $\mathcal{G}'$ is in S^c if and only if $y_i = 0$ for some i.

Because mutation is positive, Theorem 4.6 of [52] (see appendix) tells us that $x_i = \mathcal{G}(x)_i > 0$ for all i. Hence all components of Fx are positive. It follows that all entries in y are positive as well. Therefore fixed point y of $\mathcal{G}'$ is in S. ■

Theorem 8.5 *For all fitness vectors in a dense open set in* $(\Re^+)^n$, *the map* $\mathcal{G} = \mathcal{M}\mathcal{F}$ *has only hyperbolic fixed points in* Λ.

Proof of Theorem 8.5 We have already shown that for all fitness vectors in the dense open set D, all fixed points of $\mathcal{G}'$ which lie in S are hyperbolic. Let f be a fitness vector in $D \cap (\Re^+)^n$, and $F = \text{diag}(f)$. We must prove that $\mathcal{G} = \mathcal{M}\mathcal{F}$ has hyperbolic fixed points.

Let $x \in \Lambda$ be an arbitrary fixed point of $\mathcal{G} = \mathcal{M}\mathcal{F}$. We want to show that x is hyperbolic. We have showed that $\mathcal{G}' = F\mathcal{M}$ has only hyperbolic fixed points in S. Let $y = \frac{Fx}{(1^T Fx)^2}$, the fixed point of $\mathcal{G}' = F\mathcal{M}$ that corresponds to x. By Lemma 8.4, $y \in S$. We know from Lemma 8.3 that

$$\text{spec}(d\mathcal{G}_x) = \text{spec}(d\mathcal{G}'_y) \setminus \{2\} \cup \{0\}.$$

Since y is a hyperbolic fixed point of $\mathcal{G}'$, none of the eigenvalues in $\text{spec}(d\mathcal{G}'_y)$ have modulus one. It follows that no element of $\text{spec}(d\mathcal{G}_x)$ has modulus one either. Hence x is a hyperbolic fixed point of $\mathcal{G}$. ■

Chapter 9

Future Work

The results in this report apply to genetic algorithms which use proportional selection. Our most immediate goal is to extend these results to other types of selection schemes, particularly ranking and tournament selection. In tournament and ranking selection, the functions which define the probability of selecting each string are not continuous in the fitness values. Therefore generic hyperbolicity results about these selection schemes will probably claim hyperbolicity for generic values of some parameter **other** than fitness. Because tournament and ranking selection are more complicated than proportional selection, we expect the extension of generic hyperbolicity to these schemes to be non-trivial.

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Appendix

A.1 Some Results about Continuous Functions

The first result is a prerequisite of Theorem A.2.

Theorem A.1 *If a polynomial in n real-valued variables is zero on a nonempty open set, then the polynomial is identically zero.*

Note: The following proof is essentially the proof of Proposition 5 in Chapter 1 of [11] (see appendix).

Proof The proof will be by induction on n.

basis case: $(n = 1)$ If p is a polynomial in a single variable, then either $p \equiv 0$ or p has at most $\deg(p)$ zeros. So if p is zero on a nonempty open set, it follows that p is identically zero.

Assume that the claim holds for $n - 1$.

Let $p(x_1, x_2, ..., x_n)$ be a polynomial in n variables, and assume that p is zero on the open box $\mathcal{O} = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$, where $a_i < b_i$ for all i. We want to show that $p \equiv 0$.

We can write p in the following way:

$p = \sum_{i=0}^{N} g_i(x_1, ..., x_{n-1}) x_n^i$, where g_i is a polynomial in $x_1, ..., x_{n-1}$.

Choose arbitrary $(m_1, ..., m_{n-1}) \in (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_{n-1}, b_{n-1})$.

Note the polynomial $p(m_1, ..., m_{n-1}, x_n)$ is a polynomial in one variable. Since $p(m_1, ..., m_{n-1}, x_n)$ is zero on the open set (a_n, b_n), it follows from the basis case that p is zero for all $x_n \in \Re$. Therefore the coefficients of $p(m_1, ..., m_{n-1}, x_n)$ are zero. So $g_i(m_1, ..., m_{n-1}) = 0$ for all i. Recall that m_i was chosen arbitrarily from (a_i, b_i) for $i = 1, ..., n-1$. So the polynomials g_i are polynomials in $n-1$ variables, and each g_i is zero on the open set $(a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1})$. It follows from the induction hypothesis that g_i is identically zero. Therefore $p(x_1, x_2, ..., x_n) = \sum_{i=0}^{N} g_i(x_1, ..., x_{n-1}) x_n^i = \sum_{i=0}^{N} 0 x_n^i = 0$. So p is identically zero, and we have proved the claim. ■

Theorem A.2 *A polynomial $p : \Re^n \to \Re$ is either identically zero or is nonzero on a dense open set.*

Proof Clearly $p^{-1}(\{0\})$ is closed since p is continuous and $\{0\}$ is closed. So $p^{-1}(\{0\})^c$ is open. Assume that p is not identically zero. We will show that $p^{-1}(\{0\})^c$ is dense. Consider an arbitrary open box $\mathcal{O} = (a_1, b_1) \times \cdots \times (a_n, b_n)$. We want to show that there exists $x \in \mathcal{O}$ such that $p(x) \neq 0$. Suppose that no such x exists by way of contradiction. Then p is zero on the open set $\mathcal{O}$, and therefore it follows from the previous theorem that p is identically zero. But we assumed that p was not identically zero, so we have reached a contradiction. ■

Theorem A.3 *Let* $p(x) = \sum_{i=0}^{n} a_i x^i$ *be an* n^{th} *degree polynomial with real coefficients, and let* Y *be the set of roots of* p. *Given* $\epsilon > 0$, *any* n^{th} *degree polynomial which is sufficiently close to* p *in terms of coefficients will have roots in* $\mathcal{B}_\epsilon(Y)$.

Proof Let $M > 0$ be a constant such that $|a_i| < M$ for $0 \leq i \leq n-1$. Note that for sufficiently large x, say $|x| > N$,

$$|\frac{a_n}{2}x^n| > (n-1)M\,|x|^{n-1}\,.$$

Then by the triangle inequality, no n^{th} degree polynomial $\sum b_i x^i$ such that $|b_n| \geq |\frac{a_n}{2}|$ and $|b_i| < M$ for $0 \leq i \leq n-1$ has roots outside a closed disc $\mathcal{L}$ of radius N. So all n^{th} degree polynomials which are sufficiently close to p have all their roots in $\mathcal{L}$.

Suppose the claim is false. Let $p_1, p_2, \ldots$ be a sequence of polynomials which all have a root outside of $\mathcal{B}_\epsilon(Y)$, but such that $p_i \to p$, and p_i is closer to p than p_{i-1} (in terms of coefficients). Note that $p_n \to p$ uniformly on $\mathcal{L}$ (by compactness).

Suppose that r_i is a root of p_i such that $r_i \notin \mathcal{B}_\epsilon(Y)$. Then the roots $\{r_i\}$ will have a convergent subsequence in $\mathcal{L}$. For simplicity of notation, assume that $r_i \to r$.

Since $p_n \rightarrow p$ uniformly on $\mathcal{L}$, there exists T such that $| p(x) - p_n(x) | < \epsilon$ for all $n \geq T$ and $x \in \mathcal{L}$. So in particular, $| p(r_n) | = | p(r_n) - p_n(r_n) | < \epsilon$ for all $n \geq T$. Therefore $p(r_n) \rightarrow 0$. However, since $r_n \rightarrow r$, $p(r_n) \rightarrow p(r)$, and hence $p(r) = 0$. But this is a contradiction, and so the claim holds. ■

Theorem A.4 *Let $p \in \mathbb{C}[x_1, \ldots, x_n]$, and suppose that $p \neq 0$ on a dense open set in $\mathbb{C}^n$. Then p is nonzero on a dense open set of $\Re^n$ as well.*

Proof Let $p = \Sigma c_i x_1^{s_1} \cdots x_n^{s_n}$, $c_i \in \mathbb{C}$. Suppose by way of contradiction that $p = 0$ on a nonempty open set $\mathcal{O}$ in $\Re^n$. Note that $p = \Sigma c_i x_1^{s_1} \cdots x_n^{s_n} = \Sigma(a_i + ib_i) x_1^{s_1} \cdots x_n^{s_n}$ for some $a_i, b_i \in \Re$; so $p = \Sigma a_i x_1^{s_1} \cdots x_n^{s_n} + i\Sigma b_i x_1^{s_1} \cdots x_n^{s_n}$. Setting $g = \Sigma a_i x_1^{s_1} \cdots x_n^{s_n}$ and $h = \Sigma b_i x_1^{s_1} \cdots x_n^{s_n}$, we have $p = g + ih$, where g and h are real-valued on $\mathcal{O}$. Therefore $g \equiv 0$ and $h \equiv 0$, that is, $a_i = b_i = 0$ for all i. Hence $c_i = 0$ for all i, and it follows that $p = 0$. Therefore p is nonzero on a dense open set in $\Re^n$. ■

A.2 Algebra

A.2.1 Definitions and Helpful Facts

We will need the following definitions:

Let k be a field, and let $x_1, \ldots, x_n$ be indeterminates over k. Let $J \subset k[x_1, \ldots, x_n]$ be an ideal. Define $V(J)$ to be the set

$$V(J) = \{(a_1, ..., a_n) \in k^n : \forall f \in J. f(a_1, ..., a_n) = 0\}.$$

We call $V(J)$ the *variety* defined by J.

For a set $S \subset k^n$, the set $I(S)$ is defined by

$$I(S) = \{f \in k[x_1, ..., x_n] : \forall a \in S. f(a) = 0\}.$$

The *radical* of J, denoted $\sqrt{J}$, is the set defined by

$$\sqrt{J} = \{f \in k[x_1, \ldots, x_n] : \exists m \geq 1. f^m \in J\}.$$

Alternatively, we can define $\sqrt{J}$ by $\sqrt{J} = \cap P$ where the intersection is taken over all prime ideals which contain J.

Given a subset $I \subset k[x_1, \ldots, x_n]$, the corresponding *affine algebraic set* is defined to be

$$Z(I) = \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

Let K be a field. If $I_0, \ldots, I_r$ are prime ideals in $K[x_1, \ldots, x_n]$ such that $I_j \subset I_{j+1}$ and $I_j \neq I_{j+1}$ for all $j = 0, \ldots, r$, then the collection $I_0, \ldots, I_r$ of primes is called a *chain of prime ideals* of *length* r. The *height* of a prime ideal $I \in K[x_1, \ldots, x_n]$, denoted by height$(I)$, is the maximal length of a chain of prime ideals $I_0, \ldots, I_r$ with $I_r = I$. The height of a non-prime ideal I, denoted height(I), is the minimum height of a prime ideal containing I. The *dimension* (or *Krull dimension*) of a ring R, denoted by $\dim(R)$, is the supremum of the lengths of chains of prime ideals in R. If I is a proper ideal of R, the *dimension*, $\dim(I)$, of I is defined to be $\dim(R/I)$.

A ring R is *Artinian* if, for every chain $I_1 \supset I_2 \supset \cdots$ of ideals of R, there exists an integer m such that $I_j = I_m$ for all $j \geq m$.

A ring R is *Noetherian* if, for every chain $I_1 \subset I_2 \subset \cdots$ of ideals of R, there exists an integer m such that $I_j = I_m$ for all $j \geq m$.

Let S be a commutative ring with identity 1_S and R a subring of S containing 1_S. Then S is called an *extension ring* of R.

Let S be an extension ring of R and $s \in S$. If there exists a monic polynomial $f(x) \in R[x]$ such that $f(s) = 0$, then s is *integral* over

R. If every element of S is integral over R, then S is an *integral extension* of R.

Lemma A.5 *Let $f : R \to S$ be an onto homomorphism with kernel K. There is a 1-1 correspondence between the set of all prime ideals in R that contain K and the set of all prime ideals in S.*

Proof Let I be a prime ideal of R such that $K \subset I$. We will show that $f(I)$ is a prime ideal in S. Suppose that $a, b \in S$ and $ab \in f(I)$, and assume that $a \notin f(I)$. Let $a' \in f^{-1}(a)$ and $b' \in f^{-1}(b)$. Note that $f(a'b') = f(a')f(b') = ab \in f(I)$. Thus $a'b' \in I$ since $K \subset I$, and therefore $b' \in I$ since I is prime. It follows that $b = f(b') \in f(I)$. So $f(I)$ is prime.

Next we assume that J is a prime ideal in S, and show that $f^{-1}(J)$ is a prime ideal of R which contains K. Assume that $u, v \in R$, and $uv \in f^{-1}(J)$. Suppose that $u \notin f^{-1}(J)$; therefore $f(u) \notin J$. Note that $f(u)f(v) = f(uv) \in J$, and therefore $f(v) \in J$ since J is prime. It follows that $v \in f^{-1}(J)$. So $f^{-1}(J)$ is prime in R. Since $0 \in J$, $K = f^{-1}(0) \subset f^{-1}(J)$. ■

Lemma A.6 *Given ring $R = k[x_1, \ldots, x_n]$ and ideal J of R such*

that $|V(J)| < \infty$, *all prime ideals between* J *and* R *are maximal.*

Proof Let $V(J) = \{z^1, ..., z^r\}$. Note that

$\sqrt{J} = I \circ V(J)$ by Hilbert's Nullstellensatz (see appendix)

$= I(\{z^1, ..., z^r\})$

$= \{p \in k[x_1, \ldots, x_n] : p(z^1) = 0, ..., p(z^r) = 0\}.$

Note that if $p \in < x - z^t >$, for some $t \in \{1, \ldots, r\}$, then $p = \sum s_j(x_1 - z_1^t)...(x_n - z_n^t)$ for some $s_j \in R$. So $p(z^t) = 0$. Hence if $p \in < x - z^1 > \cap \cdots \cap < x - z^r >$, then $p \in \sqrt{J}$. Therefore $< x - z^1 > \cap \cdots \cap < x - z^r > \subset \sqrt{J}$.

Since $\sqrt{J} = \bigcap Q_i$, where the intersection is taken over all prime ideals containing J (see [11]), there are no prime ideals between J and $\sqrt{J}$.

From Proposition 5.9 and Proposition 4.10 in [11] (see appendix), we know that $< x - z^i >$ is maximal for all i, and thus prime. Let $P_i = < x - z^i >$, and note that $P_1 P_2 ... P_r \subset P_1 \cap ... \cap P_r \subset \sqrt{J}$.

Suppose that P' is a prime ideal containing J. Then P' contains $\sqrt{J}$. So $P_1 P_2 ... P_r \subset P'$, and since P' is prime, $P_i \subset P'$ for some i. But P_i is maximal, and hence $P' = P_i$. Therefore every prime ideal

containing J is maximal. ■

Lemma A.7 *If k is an algebraically closed field, $S = k[x_1, .., x_n]$ and $0 \neq f \in S$ then dim S = dim $S[1/f]$.*

Proof Theorem A of section 8.2 in [18] (see appendix) says that dim S is the length of all maximal chains of prime ideals in S. Let $(a_1, \ldots, a_n) \in k^n$ such that $f(a_1, \ldots, a_n) \neq 0$. Therefore $f \notin (x_1 - a_1, \ldots, x_n - a_n)$, and consequently, $f^i \notin (x_1 - a_1, \ldots, x_n - a_n)$ for any integer $i \geq 1$ since Proposition 5.9 and Proposition 4.10 of [11] (see appendix) tell us that $(x_1 - a_1, \ldots, x_n - a_n)$ is prime. By Proposition 5.9 of [11] $(x_1 - a_1, \ldots, x_n - a_n)$ is also a maximal ideal. The chain

$$(0) \subset (x_1 - a_1) \subset (x_1 - a_1, x_2 - a_2) \subset \cdots \subset (x_1 - a_1, \ldots, x_n - a_n)$$

is a maximal chain of prime ideals in S. Since there is a 1-1 correspondence between the set of prime ideals in S which miss $\{f, f^2, \ldots, f^t, ...\}$ and the prime ideals of $S[1/f]$ (see Theorem 3.4.10 of [31] in appendix), it follows that dim S = dim $S[1/f]$. ■

Lemma A.8 *Let k be a field, and consider a polynomial ring $k[t_1, \ldots, t_l, x_1, \ldots, x_n]$ such that $l \geq n \geq 1$. Let $I = (f_1, \ldots, f_n)$ where $f_i = x_i - t_i Q_i(x_1, \ldots, x_n)$ for $1 \leq i \leq n$, and let P be a prime*

ideal which contains I. Then height$(P) \geq n$.

Proof This result and a proof were provided by S. Mulay; the proof which follows is a slightly altered version of Mulay's proof. The proof is by induction on n.

Base case ($n = 1$): We have $I = (x_1 - t_1 Q_1) \subset k[t_1, \ldots, t_l, x_1]$. Since $I \neq 0$, I is a principal ideal. The Principal Ideal Theorem (see appendix) says that height$(P) \geq 1$.

Assume the result is true for $m \leq n-1$. Let $g = Q_1 \cdots Q_n$.

Case a: Assume $g \in P$. So there exists Q_i such that $Q_i \in P$, since P is prime. Therefore $x_i \in P$. Let $\bar{P} = P/(x_i)$. Then $\bar{P} \subset k[t_1, \ldots, t_l, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Note that by Theorem 3.2.13 in [31] $\bar{P}$ is a prime ideal since $(x_i) \subset P$. Let

$$\bar{f}_j = x_j - t_j Q_j(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

Clearly $(\bar{f}_1, \ldots, \bar{f}_{i-1}, \bar{f}_{i+1}, \ldots, \bar{f}_n) \subset \bar{P}$. By the Induction Hypothesis, height$(\bar{P}) \geq n-1$. Therefore by Proposition 7.64 in [7]

$$\begin{aligned}
&\dim(k[t_1, \ldots, t_l, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]/\bar{P}) \\
&\quad = (l + n - 1) - \text{height}(\bar{P}) \\
&\quad \leq (l + n - 1) - (n - 1) = l.
\end{aligned}$$

Define

$$\pi : k[t_1, \ldots, t_l, x_1, \ldots, x_n] \to k[t_1, \ldots, t_l, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]/\bar{P})$$

by $\pi(d) = d \mid_{x_i=0} + \bar{P}$. Since $\bar{P} = P/(x_i)$, $\mathrm{Ker}(\pi) = P$. Note that for $d_1, d_2 \in k[t_1, \ldots, t_l, x_1, \ldots, x_n]$,

$$\begin{aligned}
\pi(d_1 d_2) &= d_1 d_1 \mid_{x_i=0} + \bar{P} \\
&= d_1 \mid_{x_i=0} d_2 \mid_{x_i=0} + \bar{P} \\
&= (d_1 \mid_{x_i=0} + \bar{P})(d_2 \mid_{x_i=0} + \bar{P}) \\
&= \pi(d_1)\pi(d_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi(d_1 + d_2) &= (d_1 + d_2) \mid_{x_i=0} \\
&= d_1 \mid_{x_i=0} + d_2 \mid_{x_i=0} + \bar{P} \\
&= d_1 \mid_{x_i=0} + \bar{P} + d_2 \mid_{x_i=0} + \bar{P} \\
&= \pi(d_1) + \pi(d_2).
\end{aligned}$$

So π is a homomorphism. Clearly π is onto. By applying the First Isomorphism Theorem to π, we get

$$k[t_1, \ldots, t_l, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]/\bar{P} \cong k[t_1, \ldots, t_l, x_1, \ldots, x_n]/P.$$

Therefore $(l + n) - \mathrm{height}(P) \leq l$, and so $\mathrm{height}(P) \geq n$ by Proposition 7.64 in [7].

Case b: Assume $g \notin P$. Then $g \neq 0$, and therefore none of the Q_i

are zero.

Let $T = \{1, g, g^2, \ldots\}$, the multiplicative set generated by powers of g, and let

$$A = T^{-1}k[t_1, \ldots, t_l, x_1, \ldots, x_n]$$
$$= k[t_1, \ldots, t_l, x_1, \ldots, x_n, \tfrac{1}{g}]$$
$$= k[x_1, \ldots, x_n, \tfrac{1}{g}][t_1, \ldots, t_l]$$
$$= B[t_1, \ldots, t_l] \text{ where } B = k[x_1, \ldots, x_n, \tfrac{1}{g}].$$

Note that $t_1, \ldots, t_l$ are indeterminates over B. Clearly B is an integral domain.

Note that by Theorem 3.4.8 and Theorem 3.4.10 in [31] (see appendix), $T^{-1}P \neq A$ and $T^{-1}P$ is a prime ideal in A. Since $I \subset P$, it follows that $T^{-1}I \subset T^{-1}P$. Note that $T^{-1}I = (t_1 - \frac{x_1}{Q_1}, \ldots, t_n - \frac{x_n}{Q_n})$, and define $\beta_i = \frac{x_i}{Q_i}$. Then $T^{-1}P$ is a prime ideal in $B[t_1, \ldots, t_l]$ containing $(t_1 - \beta_1, \ldots, t_n - \beta_n)$ where $\beta_i \in B$. Since $(t_1 - \beta_1, \ldots, t_n - \beta_n)$ is a prime ideal of height n in $B[t_1, \ldots, t_l]$, height$(T^{-1}P) \geq n$ and therefore height$(P) \geq n$ by Theorem 3.4.10 in [31]. ■

Lemma A.9 *Let k be an algebraically closed field, and let $S = k[t_1, ..., t_m]$ and $D = S[x_1, ..., x_n]$. Let J be a proper ideal of D such that $J \cap S = 0$, and assume that J is generated by the*

polynomials $g_1(t,x),...,g_n(t,x)$. *Suppose that for* $t = b$, *the system* J *has a positive number of solutions in* x. *Define* K *to be the prime ideal of* D *generated by* $t_1 - b_1,...,t_m - b_m$. *Then* $J + K$ *is a proper ideal of* D.

Proof We must show that $D \setminus (J + K)$ is nonempty. Choose $l_i \in k$ such that $l_i \neq b_i$ for $i = 1,...,m$, and choose d such that $x = d$ is a solution of J for $t = b$. Clearly $\prod(t_i - l_i) \in D$. Suppose by way of contradiction that $\prod(t_i - l_i) \in J + K$. Then $\prod(t_i - l_i) = j + k$ for some $j \in J, k \in K$. Therefore $\prod(t_i - l_i) = \sum r_i g_i(t,x) + \sum s_i(t_i - b_i)$, for some $r_i \in k$, $s_i \in S$. When we evaluate $\prod(t_i - l_i)$ at (b,d), we get $\prod(b_i - l_i) = j\,|_{(b,d)} + \sum s_i(b_i - b_i) = 0 + 0 = 0$ since (b,d) is a solution of J. But this contradicts our choice of l. Therefore $\prod(t_i - l_i) \notin J + K$, and hence $J + K \neq D$. It follows that $J + K$ is a proper ideal. ■

Lemma A.10 *Let* k *be an algebraically closed field, and let* $x_1,\ldots,x_n$ *be indeterminates over* k. *If* $p \in k[x_1,\ldots,x_n]$, $r \in k^n$ *and* $p(r) = 0$, *then* $p \in < x - r >$.

Proof First note that $V(< x-r >) \subset \{r\}$, since if $c \in V(< x-r >)$, $c_i = r_i$ for all i. Since $p(r) = 0$, $p \in I(\{r\}) \subset I(V(< x - r >))$ by Lemma 8.7.1 in [31] (see appendix). By Hilbert's Nullstellensatz

and the primality of $< x - r >$,

$$I(V(< x - r >)) = \sqrt{< x - r >} =< x - r > .$$

So $p \in < x - r >$. ■

A.2.2 Proof of Mulay's Theorem

The statement and proof outline of the following claim are from S. Mulay:

Mulay's Theorem *Let k be an algebraically closed field, and let $t_1, ..., t_m, x_1, ..., x_n$ be indeterminates over k. Define polynomial rings S and D by $S = k[t_1, ..., t_m]$ and $D = S[x_1, ..., x_n]$. Let J be a proper ideal of D such that $J \cap S = 0$, and so that J is generated by n polynomials $g_1(t, x), ..., g_n(t, x)$, where $t = (t_1, ..., t_m)$, and $x = (x_1, ..., x_n)$. Assume also that for each minimal prime P of J, height$(P) \geq n$. If there exists $b \in k^m$ such that for $t = b$ the system J has a finite, positive number of solutions in x, then there exists a dense open set C of values for t such that corresponding to each m-tuple in C, there are only finitely many solutions in x of the system J. Additionally, if P is a minimal prime of J and $P \cap S = 0$, there exists $0 \neq p \in S$ such that $V(P)$ is finite for all values of t such that $p(t) \neq 0$.*

Proof Let K be the ideal of D which is generated by

$t_1 - b_1, ..., t_m - b_m$, where $b = (b_1, ..., b_m)$.

By hypothesis we know that $V(J\mid_{t=b}) = \{x \in k^n \mid \forall j \in J.j(b,x) = 0\}$ is finite and non-empty; let $V(J\mid_{t=b}) = \{z^1, ..., z^r\}$.

Note that

$V(J/K) = \{x \mid \forall j(t,x) \in J.j + K = K\} = \{x \mid \forall j \in J.j(t,x) \in K\}$.

Also note that

$j(t,x) \in K$

$\Leftrightarrow j(t,x) = \Sigma_{i=1}^m A_i(t_i - b_i)$, for some $A_i \in D$

$\Leftrightarrow j(b,x) = 0$.

Therefore

$V(J/K) = \{x \mid \forall j \in J.j(b,x) = 0\} = \{z^1, \ldots, z^r\}$.

Step 1: We first want to show that J/K is zero dimensional, or equivalently that $(D/K)/(J/K)$ is a zero dimensional ring. So we must show that all primes of $(D/K)/(J/K)$ are maximal, or equivalently, that all primes between J/K and D/K are maximal.

Consider the homomorphism $\pi : D \to k[x_1, ..., x_n]$ given by

$\pi(f) = f \mid_{t=b}$. First note that π is onto. We will show that $\mathrm{Ker}(\pi) = K$:

Suppose $g \in \mathrm{Ker}(\pi)$. Then $g \mid_{t=b} = 0$. So by Lemma A.10 $g = \sum B_i(t_i - b_i)^{l_i}$ for positive integers l_i, and $B_i \in D$. Hence $g \in K$, since K is generated by $t_1 - b_1, ..., t_m - b_m$. Now suppose that $g \in K$. Then $g = \sum B_i(t_i - b_i)$ for some $B_i \in D$. Hence $g \mid_{t=b} = \sum B_i(b_i - b_i) = 0$. So $g \in \mathrm{Ker}(\pi)$.

Since $\mathrm{Ker}(\pi) = K$, it follows from the First Isomorphism Theorem (see appendix) that D/K is isomorphic to $k[x_1, \ldots, x_n]$.

The problem of showing that all primes between D/K and J/K are maximal was generalized in Lemma A.6. The lemma tells us that J/K is zero dimensional, or equivalently that all primes between D/K and J/K are maximal.

Step 2: Next we prove that $\mathrm{height}(J + K) = m + n$.

Note that if $\pi : D \to D/K$ is the canonical homomorphism, then $\pi(J + K) = J/K$. Hence it follows from Theorem 3.2.13 in [31] (see appendix) that there is a 1-1 correspondence of

the ideals between D/K and J/K, and the ideals between D and $J+K$.

Since by Step 1 every prime between D/K and J/K is maximal, so is every prime between D and $J+K$.

By definition, height$(J+K) = \min\{\text{height}(Q_j)\}$ where the Q_j's are the minimal primes containing $J+K$. Since Q_j is maximal for all j, it follows that $\dim(Q_j) = 0$. By Proposition 7.64 of [7] (see appendix), height$(Q_j) \equiv m + n - \dim(Q_j)$. Hence height$(Q_j) = m+n$ for all j. Therefore

$$\text{height}(J+K) = m+n.$$

Step 3: Now we show that height$(J) = n$.

Recall that J has n generators. By the Principal Ideal Theorem, height(Q) $\leq n$. By hypothesis, height$(Q) \geq n$; therefore height$(Q) = n$.

By definition, height$(J) = \min\{\text{height}(Q_j)\}$ where the Q_j's are the minimal primes containing J. Since we showed in the previous step that every minimal prime of J has height n, it follows that height$(J) = n$.

Step 4: Let $\sqrt{J} = P_1 \cap ... \cap P_r$ be the minimal prime decomposition of $\sqrt{J}$. Recall that $J \cap S = 0$.

We want to show that $\sqrt{J} \cap S = 0$:

Note that $\sqrt{J} = \{x \mid \exists k > 0. x^k \in J\}$. Suppose by way of contradiction that $0 \neq x \in \sqrt{J} \cap S$. By definition of $\sqrt{J}$ there exists k such that $x^k \in J$. Since $x \in S$, $x^k \in S$, because S is closed under multiplication. Hence $x^k \in J \cap S$, contradicting that $J \cap S = 0$.

Step 5: Next we will show that for some j, $P_j \cap S = 0$:

Since $\sqrt{J} \cap S = 0$ (Step 4) we know that $(\bigcap_{j=1}^{r} P_j) \cap S = 0$. Now suppose by way of contradiction that there exists $0 \neq p_j \in P_j \cap S$ for all j. Since P_j is an ideal, $p_1 p_2 ... p_r \in P_j$ for all j. Hence $p_1 p_2 ... p_r \in \bigcap P_j$. Since $p_j \in S$ for all $j, p_1 ... p_r \in S$, and so $p_1 ... p_r \in (\bigcap P_j) \cap S$. But this is a contradiction, since $\sqrt{J} \cap S = 0$.

Step 6: Let P be a minimal prime of J such that $P \cap S = 0$. Now we will prove that since D is finitely generated over S, $R = D/P$ is finitely generated over S/P.

Let $\phi : D \to D/P$ be the canonical homomorphism (i.e., $\phi(d) = d+P$). We will show that $D/P = S/P[\phi(x_1), ..., \phi(x_n)]$. Let $q + P \in D/P$. Since D is finitely generated over S, with generators $x_1, ..., x_n$, $q = \sum s_i x_1^{m_{1i}} ... x_n^{m_{ni}}$ for natural numbers m_{ji} and $s_i \in S$. Let $x_1^{m_{1i}} ... x_n^{m_{ni}}$ be called q_i. So $q = \sum s_i q_i$. Consider that

$$\sum (s_i + P)\{(x_1 + P)^{m_{1i}}(x_2 + P)^{m_{2i}} \cdots (x_n + P)^{m_{ni}}\}$$

$$= \sum (s_i + P)(q_i + P) = \sum s_i q_i + P = q + P.$$

Therefore

$$D/P = S/P[x_1 + P, ..., x_n + P] = S/P[\phi(x_1), ..., \phi(x_n)].$$

Hence D/P is finitely generated over S/P.

Step 7: Next we show that S/P is isomorphic to S:

Recall that $P \cap S = 0$. Define $\pi : S \to S/P$ by $\pi(s) = s + P$. Note that

$$\pi(s_1) = \pi(s_2) \Rightarrow \pi(s_1) - \pi(s_2) = 0 \Rightarrow \pi(s_1 - s_2) = 0$$

$$\Rightarrow s_1 - s_2 \in \text{Ker } \pi = P.$$

Since $s_1, s_2 \in S$, it follows that $s_1 - s_2 \in S$. Therefore $s_1 - s_2 \in P \cap S$, and hence $s_1 - s_2 = 0$. So $s_1 = s_2$. Therefore π is an isomorphism.

Step 8: We now show that for $0 \neq p \in S$, $R[1/p]$ is integral over $S[1/p]$. (Recall that $R = D/P$).

The Normalization Theorem (see appendix) says that there exist $z_1, \ldots, z_t \in R$ and $0 \neq p \in S$ such that $z_1, \ldots, z_t$ are algebraically independent over S, and $R[1/p]$ is integral over $S[z_1, \ldots, z_t, 1/p]$.

Theorem 48 in [34] (see appendix) implies that

$$\begin{aligned}\dim R[1/p] &= \dim S[z_1, ..., z_t, 1/p] \\ &= \dim S[1/p][z_1, \ldots, z_t] \\ &= t + \dim S[1/p] \text{ by Corollary 10.13 in [18].}\end{aligned}$$

Lemma A.7 tells us that $\dim S[1/p] = \dim S = m$, and $\dim R[1/p] = \dim R$. By Proposition 7.64 in [7],

height$(P) = m+n-\dim(D/P)$, and by Step 3 height$(P) = n$. Therefore dim $R = m$. So $m = t + m$, and therefore $t = 0$. Hence $R[1/p]$ is integral over $S[1/p]$.

Step 9: Now we show that if N is a maximal ideal of S not containing p, then R/NR is zero dimensional.

Since N is maximal in S, N is also a prime ideal of S by Proposition 4.10 in [11]. Consider the homomorphism $\phi_S : S \to S[\frac{1}{p}]$ given by $\phi_S(s) = \frac{sp}{p}$. By Lemma A.5 $\phi(N) = N[\frac{1}{p}]$ is prime in $S[\frac{1}{p}]$. Since there is a 1-1 correspondence of prime ideals in S which are disjoint from $\{p, p^2, p^3, \ldots\}$ and prime ideals in $S[\frac{1}{p}]$, $N[\frac{1}{p}]$ is also maximal. Recall from Step 8 that $R[\frac{1}{p}]$ is integral over $S[\frac{1}{p}]$. Therefore the Lying Over Theorem (see appendix) says that $NR[\frac{1}{p}]$ is a prime ideal in $R[\frac{1}{p}]$. By Theorem 8.5.12 in [31] (see appendix), $NR[\frac{1}{p}]$ is maximal in $R[\frac{1}{p}]$. Let $\phi_R : R \to R[\frac{1}{p}]$ be given by $\phi_R(r) = \frac{rp}{p}$. By Theorem 3.4.10 in [31], $\phi_R^{-1}(NR[\frac{1}{p}]) = NR$ is prime as well as maximal in R. Therefore the chain of primes between R and NR consisting only of NR is of maximal length. It follows that dim $(R/NR) = 0$.

Step 10: Let F be the ideal in D generated by $t_1 - b_1', ..., t_m - b_m'$, where $(b_1', b_2', ..., b_m') \in k^m$ and $p(b_1', b_2', ..., b_m') \neq 0$.

We will show that $p \notin F$. So suppose by way of contradiction that $p \in F$. Then $p = \sum a_i(t_1 - b_1')^{n_{1i}} ... (t_m - b_m')^{n_{mi}}$, for $a_i \in S$, and consequently $p(b_1', b_2', ..., b_m') = 0$. But this is a contradiction, and so $p \notin F$.

Step 11: Next we show that $V(P)$ is finite on a dense open set. Let $N = F \cap S$ and $K' = NR$. Since dim $S = m$ and N is generated by $t_1 - b_1', ..., t_m - b_m'$, N is a maximal ideal of S by Proposition 5.9 in [11]. It follows that N is also prime by Proposition 4.10 in [11].

We have already shown in step 9 that NR is a maximal ideal of R; therefore all prime ideals in R containing $t_1 - b_1', ..., t_m - b_m'$ are maximal. By Corollary 9.1 of [18] (see appendix), R/K' is Artinian. Corollary 2.15 of [18] (see appendix) says that $(D/P)/K'$ is Artinian if and only if $V(P/K')$ is finite. (We can apply Corollary 2.15 by setting $X = V(P/K')$; then $A(X) = D/P/K' = R/K'$. Note that since P is prime, P/K' is prime as well. Hence Hilbert's Nullstellensatz says

$I(V(P/K')) = \sqrt{P/K'} = P/K'$.) Hence V(P) is finite for $t_1 = b'_1, ..., t_m = b'_m$. Since the vector $(b'_1, b'_2, ..., b'_m)$ was chosen arbitrarily from $\{t : p(t) \neq 0\}$, it follows that V(P) is finite for all t such that $p(t) \neq 0$.

The zero set of any multi-variate nonzero polynomial is closed and nowhere dense. Since $p \neq 0$, we know that V(P) is finite on a dense open set.

Step 12: Next we complete the proof by showing that $V(J)$ is finite on a dense open set of parameters in k^m. We can apply the argument in Step 11 to all $P = P_j$ such that $P_j \cap S = 0$. Suppose for some j $P_j \cap S \neq 0$. Choose $0 \neq f_j \in P_j \cap S$. Since $f_j \neq 0$ on $(f_j^{-1}(0))^c$ and $f_j \in P_j$, $V(P_j) = \emptyset$ on the dense open set $(f_j^{-1}(0))^c$. So we have shown then that for every j, $V(P_j)$ is finite on a dense open set of parameters t. Because $V(J) = V(P_1) \cup ... \cup V(P_r)$, it follows that $V(J)$ is finite on a dense open set of parameters in k^m. Therefore the claim is true. ■

Claim *For a dense open set of fitness vectors $f \in \Re^n$, the number*

of solutions in x to the system

$$< x_0 - f_0\mathcal{M}(x)_0, x_1 - f_1\mathcal{M}(x)_1, \ldots, x_{n-1} - f_{n-1}\mathcal{M}(x)_{n-1} >$$

is bounded above by some positive integer k.

Proof We will first show that the hypotheses of Mulay's Theorem hold for $S = \mathbb{C}[f_0, \ldots, f_{n-1}]$, $D = S[x_0, \ldots, x_{n-1}]$, and

$$J =< x_0 - f_0\mathcal{M}(x)_0, x_1 - f_1\mathcal{M}(x)_1, \ldots, x_{n-1} - f_{n-1}\mathcal{M}(x)_{n-1} > .$$

For our purposes, the algebraically closed field k will be the field of complex numbers, and $m = n$.

We must verify the following:

1. $J \cap S = 0$.
2. There exists $\beta \in \mathbb{C}^n$ such that for $f = \beta$, the system J has a positive, finite number of solutions over $\mathbb{C}$.
3. For each minimal prime P of J, height$(P) \geq n$.

Note first that by Lemma A.8, height$(P) \geq n$ for every minimal prime P of J.

First we show that $J \cap S = 0$. Let $g \in J \cap S$. Since $g \in J$, $g = \sum A_i g_i$, where $A_i \in D$, and $g_i = x_i - f_i \sum_{u,v} x_{i+u} x_{i+v} M_{u,v}$. Because $g \in S$,

$g = \sum c_j h_j$ where $c_j \in \mathbb{C}$, and $h_j \in S = \mathbb{C}[f_0, ..., f_{n-1}]$. Since $\sum c_j h_j$ is independent of x,

$$g = \sum c_j h_j = \sum c_j h_j \mid_{x=\mathbf{0}} = \sum A_i g_i \mid_{x=\mathbf{0}}$$
$$= \sum A_i (x_i - f_i \sum_{u,v} x_{i+u} x_{i+v} M_{u,v}) \mid_{x=\mathbf{0}} = \sum A_i(0) = 0.$$

Therefore $g = 0$. It follows that $J \cap S = 0$.

Now we show that there exists $\beta \in \mathbb{C}^n$ such that for $f = \beta$, the system J has a positive and finite number of solutions over $\mathbb{C}$. Consider $f = \mathbf{0}$. The system J becomes $x_1 = 0, x_2 = 0, ..., x_n = 0$. So there is only one solution, $x = \mathbf{0}$.

So the hypotheses of the claim hold for

$$< x_0 - f_0 \sum_{u,v} x_{0+u} x_{0+v} M_{u,v}, ..., x_{n-1} - f_{n-1} \sum_{u,v} x_{(n-1)+u} x_{(n-1)+v} M_{u,v} > .$$

It remains to show that $V(J)$ is finite on a dense open set of parameters f in $\Re^n$.

Let $P_1, \ldots, P_r$ be the minimal primes of J. Consider $P = P_j$ such that $P_j \cap S = 0$. By Mulay's Theorem, $V(P)$ is finite on a dense open set in $\mathbb{C}^n$, namely, the set of all f such that, for some $0 \neq p \in S$, $p(f) \neq 0$. It follows from Theorem A.4 that $V(P_j)$ is finite for all $f \in R_j$, for some dense open set $R_j \subset \Re^n$.

Now consider P_j such that $P_j \cap S \neq 0$. Choose $0 \neq f \in P_j \cap S$. Since $f \in \mathbb{C}[f_0, ..., f_{n-1}]$ is not equivalently zero, Theorem A.4 tells us that f is nonzero on a dense open set $R_j \subset \Re^n$. Therefore $V(P_j) = \emptyset$ for f in dense open set R_j. Since $V(J) = V(P_1) \cup \cdots \cup V(P_r)$, it follows that $V(J)$ is finite when f is chosen from a dense open subset of $\Re^n$.

■

A.3 Useful Theorems

In this section, we state some well-known results that are referenced in this report. Proofs of these theorems can be found in the sources listed in the bibliography.

First Isomorphism Theorem [31]

If $f : R \to S$ is a homomorphism of rings, then f induces an isomorphism of rings $R/\text{Ker}f \cong \text{Im}f$.

Inverse Function Theorem [48]

Suppose that $f : \Re^n \to \Re^n$ is continuously differentiable in an open set containing a, and $\det(df_a) \neq 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that

$f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable and for all $y \in W$ satisfies

$$d(f^{-1})_y = [df_{f^{-1}(y)}]^{-1}.$$

Symmetric Function Theorem [31]

Suppose K is a field, $K(x_1, ..., x_n)$ is the field of rational functions in indeterminates $x_1, \ldots, x_n$ over K, E is the subfield of all symmetric rational functions in $K(x_1, ..., x_n)$. Let $f_1, ..., f_n$ be the elementary symmetric functions in $K(x_1, ..., x_n)$, that is, for $i = 1, \ldots, n$,

$$f_i = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} x_{j_2} \cdots x_{j_i}.$$

Then $E = K(f_1, ..., f_n)$.

Sard's Theorem [48]

Let $g : A \to \Re^n$ be continuously differentiable, where $A \subset \Re^n$ is open, and let $B = \{x \in A : \det(dg_x) = 0\}$. Then $g(B)$ has measure 0.

Brouwer's Fixed Point Theorem

Let $g : \Lambda \to \Lambda$ be a continuous function. Then g has a fixed point in Λ.

Normalization Theorem [43]

Assume that a ring R is generated by elements $b_1, ..., b_n$ over an integral domain I. Assume furthermore that no element $a \neq 0$ of I is a zero divisor in R. Then there are elements $z_1, ..., z_t$ of I which are algebraically independent over I and an element $a \neq 0$ of I such that $R[a^{-1}]$ is integral over $I[a^{-1}, z_1, ..., z_t]$.

Hilbert's Nullstellensatz [11]

Let k be an algebraically closed field. If J is an ideal in $k[x_1, ..., x_n]$, then $I(V(J)) = \sqrt{J}$.

Lying Over Theorem [31]

Let S be an integral extension ring of R and P a prime ideal of R. Then there exists a prime ideal Q in S which lies over P (that is, $Q \cap R = P$).

Principal Ideal Theorem [18]

Let R be a Noetherian ring. If $x_1, \ldots, x_c \in R$, and P is minimal among primes of R containing $x_1, \ldots, x_c$, then height$(P) \leq c$.

Mulay's Theorem [41]

Let k be an algebraically closed field, and let $t_1, ..., t_m, x_1, ..., x_n$

be indeterminates over k. Define polynomial rings S and D by $S = k[t_1, ..., t_m]$ and $D = S[x_1, ..., x_n]$. Let J be a proper ideal of D such that $J \cap S = 0$, and so that J is generated by n polynomials $g_1(t, x), ..., g_n(t, x)$, where $t = (t_1, ..., t_m)$, and $x = (x_1, ..., x_n)$. Assume also that for each minimal prime P of J, height$(P) \geq n$. If there exists $b \in k^m$ such that for $t = b$ the system J has a finite, positive number of solutions in x, then there exists a dense open set C of values for t such that corresponding to each m-tuple in C, there are only finitely many solutions in x of the system J. Additionally, if P is a minimal prime of J and $P \cap S = 0$, there exists $0 \neq p \in S$ such that $V(P)$ is finite for all values of t such that $p(t) \neq 0$.

Proposition 5.9 in [11]

If k is a field and $x_1, \ldots, x_n$ are indeterminates over k, then an ideal $I \subset k[x_1, \ldots, x_n]$ of the form

$$I = (x_1 - a_1, \ldots, x_n - a_n),$$

where $a_1, \ldots, a_n \in k$, is maximal.

Theorem A of section 8.2 in [18]

If R is an affine domain over a field k, then dim R is the common length of all maximal chains of prime ideals of R.

Theorem 3.4.10 in [31]

Let S be a multiplicative subset of a commutative ring R with identity. Then there is a 1-1 correspondence between the set $\mathcal{U}$ of prime ideals of R which are disjoint from S and the set $\mathcal{V}$ of prime ideals of $S^{-1}R$, given by $P \mapsto S^{-1}P$.

Theorem 8.5.12 in [31]

Let S be an integral extension ring of R and let Q be a prime ideal in S which lies over a prime ideal P in R. Then Q is maximal in S if and only if P is maximal in R.

Corollary 2.15 in [18]

Let X be an affine algebraic set over a field k, and let $\vec{x} = (x_1, \ldots, x_n)$ be a vector of indeterminates over k. The following are equivalent:

- X is finite.
- $A(X) = k[\vec{x}]/I(X)$ is a finite dimensional vector space over k whose dimension is the number of points in X.
- $A(X)$ is Artinian.

Corollary 9.1 in [18]

If R is Noetherian, then $\dim R = 0$ if and only if R is Artinian. An affine algebraic set has dimension 0 if and only if it is a finite set.

Theorem 3.2.13 in [31]

If I is an ideal of ring R, there exists a 1-1 correspondence between the set of all ideals of R which contain I and the set of all ideals of R/I, given by $J \rightarrow J/I$.

Proposition 7.54 in [7]

An ideal I of ring R is zero-dimensional if and only if all prime ideals of R/I are maximal.

Corollary 10.13 in [18]

If S is a Noetherian domain and $S[x]$ is the polynomial ring in one variable over S, then dim $S[x] =$ dim $S + 1$.

Theorem 48 in [34]

If R is integral over S, then dim $R =$ dim S.

Theorem 13.1 in [18]

If k is a field then dim $k[x_1, \ldots, x_r] = r$.

Proposition 7.64 in [7]

Let I be a prime ideal of $K[X_1, \ldots, X_n]$. Then height$(I) = n -$dim I.

Proposition 1.5 in [11]

Let k be an infinite field, and let $f \in k[x_1, \ldots, x_n]$. Then $f = 0$ in $k[x_1, \ldots, x_n]$ if and only if $f : k^n \to k$ is the zero function.

Lemma 8.7.1 in [31]

Let K be a field, F be an algebraically closed extension field of K, and $X, Y \subset F^n$. Then $X \subset Y \Rightarrow I(Y) \subset I(X)$.

Proposition 4.10 in [11]

If k is any field, every maximal ideal in $k[x_1, \ldots, x_n]$ is prime.

Theorem 4.6 in [52]

If mutation is positive, then $\mathcal{G}(x)$ is positive for every $x \in \Lambda$.

Theorem 6.12 in [52]

Suppose $Ax = \lambda x$. Then

$$\text{spec}(A) = \text{spec}(A^T \mid_{x^\perp}) \cup \{\lambda\}.$$

Theorem 6.13 in [52]

Considered as a map $\mathcal{M} : \Re^N \to \Re^N$,

$$\text{spec}(d\mathcal{M}_x) = 2(\mathbf{1}^T x)\text{spec}(M^*).$$

Theorem 3.4.8 in [31]

Let S be a multiplicative subset of a commutative ring R with identity and let I be an ideal of R. Then $S^{-1}I = S^{-1}R$ if and only if $S \cap I \neq \emptyset$.

VITA

Mary Eberlein was born in Mobile, Alabama, on November 27, 1966. After graduating from Davidson High School in June of 1984, she enrolled at the University of South Alabama. She graduated from South Alabama with a B.S. in mathematics in 1988, and then enrolled in the graduate program in mathematics at the University of Illinois. She received a M.S. in mathematics from Illinois in 1990, at which time she moved to the University of Tennessee to pursue a Ph.D. in Computer Science. Mary Eberlein was awarded the Doctor of Philosophy degree in Computer Science from the University of Tennessee in August of 1996.