The Conway Polynomial and Amphicheiral Knots

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To my parents, M.T. Weerasena and Kusuma Gunarathna,
who dedicated their lives to make my life better...

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“ Whatever living creatures there be,
Without exception, weak or strong,
Long, huge or middle-sized,
Or short, minute or bulky,

Whether visible or invisible,
And those living far or near,
The born and those seeking birth,
May all beings be happy!

Let none deceive or decry
His fellow anywhere;
Let none wish others harm
In resentment or in hate.

Just as with her own life
A mother shields from hurt
Her own son, her only child,
Let all-embracing thoughts
For all beings be yours.”
- Gautama Buddha

Abstract

The Conant's conjecture [7] which has foundation on the Conway polynomial and Vassiliev invariants is the main theme of this research. The Conant's conjecture claim that the Conway polynomial of amphicheiral knots split over integer modulo 4 space. We prove Conant's conjecture for amphicheiral knots coming from braid closure in certain way. We give several counter examples to a conjecture of A. Stoimenow[32] regarding the leading coefficient of the Conway polynomial. We also construct integer bases for chord diagrams up to order 7 and up to order 6 for Vassiliev invariants. Finally we develop a method to extract integer valued Vassiliev invariants from coefficients of the Jones polynomial.

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Chapter 1

Introduction

A knot is a smooth embedding of S^1 into $\mathbb{R}^3$. At very beginning of the knot theory, the enumeration of all possible knots for a given crossing number became a central problem. Tait and Little were the first ones to produce knot tables up to 10 crossings. These knot tables needed to check for completeness and non redundancy. It became obvious that completeness or generating all possible knots for given crossing number is tedious work. However non redundancy or in other words determining whether given two knots are topologically equivalent(ambient isotopic) became an attractive problem and draws many mathematicians into the field of knot theory. It has been proven that Reidemeister moves and planer isotopy, can be used to prove equivalence of two knots [22] . However, finding an appropriate sequence of Reidemeister moves for two equivalent knots is not an easy task. The number of Reidemeister moves needed for this task is discouraging. Thus use of this approach is not feasible in most cases. This called for finding other topological invariants which are invariant under Reidemeiseter moves. One successful invariant used to distinguish knots comes from considering the complement of knots. It has been proved that two knots neglecting mirror images are equivalent if and only if the complements of the two knots are homeomorphic [14]. With mirror image the map needed to preserve the orientation. But proving complements are homeomorphic may not be an easy task, so other

methods were developed. Most of these methods work only in one direction. That is they can show that two knots are not equivalent but cannot show they are equivalent. For example, linking number, crossing number, bridge number, unknotting number are some of earliest methods developed to detect two knots are not equivalent

Polynomial invariants came in to the world of knot theory with the introduction of the Alexander polynomial[22]. The fact that they were easy to use and easy to compute made them very attractive to knot theorists. Most of these invariants are one or two variables Laurent polynomials defined over the ring $\mathbb{Z}$. Using skein relations and normalizing the Alexander polynomial, a new polynomial invariant named the Conway polynomial was introduced in 1970 [9]. Then in the 1980's several other polynomial invariants came in to play. Namely, the Jones, Kauffman and HOMFLYPT polynomials. By using specific substitutions into HOMFLYPT polynomial, one can get Jones, Conway and the Alexander polynomial. Thus one can think of it as a generalized version of those polynomials. However these polynomial invariant are not complete. That means, by using only these polynomial invariant, one cannot distinguish all knots. This is particularly true when one try to use these polynomials to distinguish knots and their mirror images. For example, the first two introduced polynomials, Alexander and Conway both cannot distinguish knots from its mirror image.

Mirror image of knots are a fascinating subject of knot theory. A knot is called amphicheiral if it coincides with its mirror image. More formerly, we can say there is an orientation reversing map from $S^3 \to S^3$ which fixes the knot setwise. With orientation of knots, we can distinguish this amphicheirality as positive and negative depending on whether the map preserves or reverses the orientation of the knot. If the map used in this process is an involution, we call the resulting amphicheirality "strong". For almost 100 years, knot theorists believed, minimal crossing number of an amphicheiral knot is an even [33]. However, with availability of computer technology, M. Thistlethwaite in 1990 shown that this is false. Now it has been proven that for each odd natural n number greater than 14 there exists an amphicheiral knots

of crossing number n [33]. Contrary to common belief, one can use the Conway and Alexander polynomials to detect chirality among knots in to certain extent. Our first chapter discusses two conjecture proposed by J. Conant and A. Stoimenow. The first conjecture has its foundation in Vassiliev knot invariants, another branch of knot theory. However after careful reformulation of the conjecture it can be seen that this conjecture closely resembles to a theorem proved by A. Kawauchi and R. Hartley [17]. Their theorem proved that for negative and strong positive amphicheiral knots, the Conway polynomial splits. That is, the Conway polynomial of those two types of amphicheiral knots can be written as $f(z)f(-z)$ for some integer polynomial f. However this is not true for the remaining category, nonstrong positive amphicheiral knots and counter examples have been found [12]. J. Conant's conjecture claims that instead of considering split over ring $\mathbb{Z}$, if we consider splitting over $\mathbb{Z}_4$ instead, then in fact we can write the Conway polynomial of any amphicheiral knot as $f(z)f(-z)$ for some $f \in \mathbb{Z}_4[z]$, including the nonstrong positive amphicheiral category [7]. In general to prove the conjecture for all amphicheiral knots, one needs only to prove it for hyperbolic knots. The argument will follows the argument used by R. Hartley to generalize his and Kawauchi's theorem about strong positive amphicheiral knots to positive amphicheiral knots[15].

Now, the symmetry groups for hyperbolic knots are classified: they are either dihedral or cyclic[20, 26, 30]. Because the knot is amphicheiral, they have to contain an orientation-reversing element. If there is an orientation-reversing element of order 2, that means if the map is involution, so we know the knot is strong amphicheiral and A. Kawauchi and R. Hartley proved the Conway polynomial of the knot splits as $f(z)f(-z)$. The presentation of dihedral group is $\{x, y | x^n = y^2 = (xy)^2 = 1\}$. Thus one can see that, dihedral group possess an orientation-reversing element of order 2, so we are left with cyclic groups Z_{2n} with orientation reversing generator h. If n is odd, then h^n is an orientation-reversing involution, so we are done in that case. Hence we may assume n is even. All knots with these cyclic symmetry groups can be constructed by taking a tangle T, considering the concatenation $(T.m(T))^n$

and closing it up via the standard closure. Here $m(T)$ is the mirror image tangle with all crossings reversed. By considering this, In chapter 2, we proved J. Conant's conjecture is true for all amphicheiral knots coming from braid formation $(ww^*)^{2^n.m}$, where w^* is the braid w with crossing reversed. However, there are amphicheiral knots which cannot be obtained using this formation[16]. For general tangle T, the problem becomes much harder due to lack of methods for calculating the Conway polynomial of a tangle $(T.m(T))^n$.

The leading coefficient of the Conway polynomial can be used to detect chirality among knots to some extent. In 1997, K. Murasugi and J. Prztycki proved [29] that if the leading coefficient of the Conway polynomial of an alternating knot is prime, then it is not amphicheiral. Another implication of A. Kawauchi and R. Hartley's theorem discussed above is that except for the non strong positive amphicheiral category, the leading coefficient of the Conway polynomial of an amphicheiral knot is square. These results lead to the conjecture that the leading coefficient of amphicheiral knots is square or at least not prime. A. Stoimenow further strengthens this conjecture to claim that the leading coefficient of the Conway polynomial is square for certain other amphichiral knot classes such as the classes of alternating amphicheiral knots. In chapter 2, we give several counter examples to this conjecture by constructing nonstrong positive amphicheiral knots.

As we mentioned above, J. Conant's conjecture has its roots in Vassiliev invariants. When the circle S^1 is immersed in to the 3-sphere S^3 with n double points and no other singular points, we get a singular knot. When we have a knot invariant taking values on an abelian group, we can extend it to the singular knot category by introducing Vassiliev skein relation. The process may be called as desingularization. If a knot invariant extended to singular knots through the Vassiliev skein relation, vanishes on all singular knots with more than n double points, we call the invariant a Vassiliev invariant and the order of the invariant is $\leq n$. A Vassiliev invariant, v, is called *primitive* if it is additive under connected sum. That is $v(K_1 \# K_2) = v(K_1) + v(K_2)$. The conjecture we mentioned above that the Conway polynomial for amphicheiral

knot splits in ring $\mathbb{Z}_4[z]$ can be rephrased : for amphicheiral knots, $pc_{4n} \equiv 0 \mod 2$ where pc_{4n} denote specific type of primitive Vassiliev invariants coming from the Conway polynomial(see chapter 3).

One way to tackle the conjecture we mentioned earlier is going through this Vassiliev invariant formulation and using brute force calculations in low orders. Singular knots can be represented by chord diagrams, which consists of a circle and chords. The pre image of each double point in a singular knot is connected by a chord. Thus a chord diagram of n chords can be thought of as a singular knot with n double points without considering isotopies and normal crossing. All non singular knots represent by the empty chord diagram. A beautiful theorem in knot theory states that the value of a Vassiliev invariant of order n over singular knots with n double points depends only on its chord diagram. The fundamental theorem of Vassiliev invariants states that there exists a surjective map from the set of order $\leq n$ Vassiliev invariants to the quotient space of the vector space(where coefficients come from the field with characteristic 0) freely generated by chord diagrams of order n by the subspace spanned by $4T$ and $1T$ relation. The kernel of this map is set of order $n-1$ Vassiliev invariants [6]. Lets go back to our task, that is evaluating the value of Vassiliev invariant pc_{4n} on any amphicheiral knot. First we use the Vassiliev skein relation in backward direction. That is using Vassiliev skein relation backward, we write the knot as formal sum of several nonsingular and singular knots. Since chord diagrams encompass singularities, we can represent this formal sum as formal sum of chord diagrams and calculate the value of Vassiliev invariant pc_{4n} over these chord diagrams. The process would be easy, if we could find basis chord diagrams and basis Vassiliev invariants upto order $4n$. This motivation led us to search an integer and rational bases for chord diagrams up to order 7. Some of these were unknown before our work. We also calculated integer bases on Vassiliev invariants up to order 6. The work was done earlier by T. Kanenobu over the rationals. For order 7, T. Kanenobu claimed that there exists a Vassiliev invariant of order 7, which is not determined by either Kauffman or HOMFLYPT polynomials. Thus in order to build a basis for

Vassiliev invariants of order 7, one has to come up with invariants which are not related to those two polynomials. We have summarized our results in chapter 3. The effort to calculate bases for chord diagrams evolved in to computerized Mathematica program which is flexible for higher order calculations and future modifications.

The pc_{4n} conjecture lead to the speculation, whether one can find similar amphicheirality criteria using coefficients of Jones polynomial. Jones polynomials were found by V. F. R. Jones in 1984. Unlike the Alexander and the Conway polynomial, the Jones polynomial is able to detect the chirality up to a certain extent. Still, for some knots, the polynomial cannot detect chirality. One of the main unsolved problem in knot theory is whether the Jones polynomial is able to distinguish non trivial knots from the trivial one. The Alexander and the Conway polynomials both fail to distinguish nontrival knots from trivial ones. It has been proved that each coefficient of the Jones polynomial is the point wise limit of a sequence of Vassiliev invariants [25]. By substituting $\exp(h)$ in Jones polynomial and expanding as a power series one can get a series whose coefficients are rational valued Vassiliev invariants. These Vassiliev invariants can be converted to integer Vassiliev invariants by multyplying a suitable constant. Another motivation for this extraction of integer valued Vassiliev invariants comes from chapter 3 . In order to calculate integer bases one has to come up with a set of integer valued Vassiliev invariants. These two motivations led us to extract integer valued Vassiliev invariants from the Jones polynomial. These Vassiliev invariants seemed to satisfies multiple congruence relations. Chapter 4 is devoted for this task.

Chapter 2

The Conway Polynomial and Amphicheirality

2.1 Introduction

Detecting chirality is one of the main questions in knot theory. To this end, various methods have been developed in the past. The polynomial invariants like the Jones, HOMFLYPT and Kauffman polynomials are the most prominent methods among them. In 2006, James Conant found that for every natural number n, a certain polynomial in the coefficients of the Conway polynomial is a primitive integer-valued degree n Vassiliev invariant, which he named pc_n. He conjectured that $pc_{4n} \mod 2$ vanishes on all amphicheiral knots. Another way to formulate this conjecture is: if K is an amphicheiral knot then there is a polynomial F such that $C(z)C(iz)C(z^2) = F^2$ Where $F \in \mathbb{Z}_4[z^2]$, $C(z)$ is the Conway polynomial of knot K and $i = \sqrt{-1}$. Using theorems of Kawauchi and Hartley on amphicheiral knots, it can be shown that this is true for all negative and strong positive amphicheiral knots. However the conjecture still remains unsolved for positive amphicheiral knots which are not strong.

A singular knot is a map $S^1 \to \mathbb{R}^3$ that fails to be an embedding. We shall consider the simplest singularities, namely *double points*.

Definition 2.1.1. *[6] Let f be a smooth map of a one-dimensional manifold to $\mathbb{R}^3$. A point $p \in im(f) \subset \mathbb{R}^3$ is a double point if $f^{-1}(p)$ consists of two points t_1 and t_2 and the two tangent vectors $f'(t_1)$, $f'(t_2)$ are linearly independent.*

Any knot invariant with values in an abelian group can be extended to knots with finitely many double points by means of the *Vassiliev skein relation* shown in 2.1.

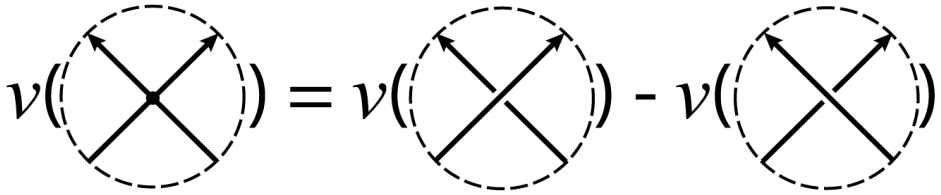


Figure 2.1: Vassilliev skein relation

Definition 2.1.2. *[35] A knot invariant is said to be a Vassiliev invariant of order n if it is vanishes on all singular knots with more than n double points but does not vanish on singular knots with n double points. A Vassiliev knot invariant is said to be primitive if it is additive under connected sum.*

Definition 2.1.3. *[9] The Conway polynomial $C(t)$ is an invariant of oriented links taking values in the ring $\mathbb{Z}[t]$ and characterized by the two properties:*

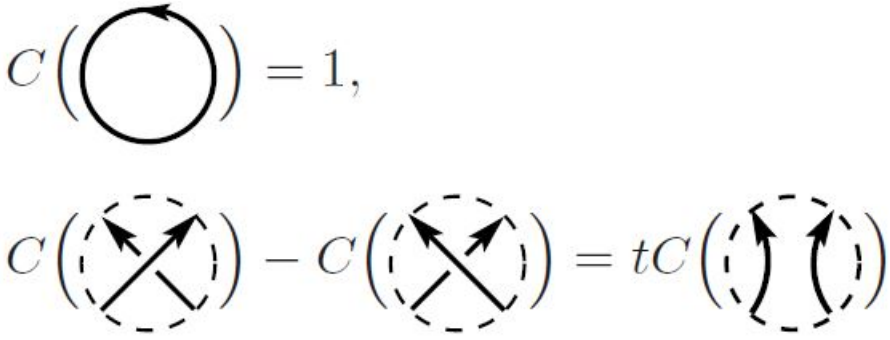


Figure 2.2: Conway skein relation

Let the Conway polynomial of a knot K be written as $C_K(z) = 1+\Sigma_{i=1}^{n} c_{2i} z^{2i}$ where $c_{2i} \in \mathbb{Z}$. It is an easy exercise to prove that the coefficients of the Conway polynomial, c_{2i}, are Vassiliev invariants. However they are not additive, or *primitive*, invariants. In 2006, J. Conant [7] obtained primitive integer valued Vassiliev invariants from the coefficients of the Conway polynomial. He introduced a discrete logarithm function as follows. Define:

$$log_{\mathbb{Z}} : 1 + x \cdot \mathbb{Z}[[x]] \to x \cdot \mathbb{Z}[[x]] \text{ and the inverse function,}$$
$$exp_{\mathbb{Z}} : x \cdot \mathbb{Z}[[x]] \to 1 + x \cdot \mathbb{Z}[[x]] \text{ as follows.}$$
$$log_{\mathbb{Z}}(1 + \Sigma_{i=0}^{\infty} a_i x^i) = \Sigma_{i=1}^{\infty} b_i x^i \text{ where,}$$
$$exp_{\mathbb{Z}}(\Sigma_{i=1}^{\infty} b_i x^i) = \prod_{i=1}^{\infty} (1 + (-x)^i)^{b_i}$$

We can apply the $log_{\mathbb{Z}}$ function to the Conway polynomial $C(z)$. We get an additive invariant of knots taking values in $\mathbb{Z}[[z^2]]$, and let us denote the coefficients by pc_{2i}. For example,

1. $pc_2 = -c_2$
2. $pc_4 = c_4 - \frac{1}{2}(c_2^2 + c_2)$
3. $pc_8 = c_8 + c_2(c_6 + c_4) + \frac{1}{2}(c_4^2 - c_4) + \frac{1}{4}(c_2^4 + c_2^2 + 2c_2)$

It has been proved that pc_{2i} are degree $2i$ Vassiliev invariants [8]. We use these pc_{2i} invariants to detect amphicheirality of a knot in this chapter.

Definition 2.1.4. *[17] A knot $K \subset S^3$ is said to be an amphicheiral knot if there exists diffeomorphism, $\phi : S^3 \to S^3$ such that*

- *ϕ reverses the orientation of S^3*
- *$\phi(K) = K$ as a set*

When a knot K has an orientation, amphicheirality can be divided into two categories.

- Positive amphicheiral knots (ϕ preserves the orientation of K).
- Negative amphicheiral knots (ϕ reverses the orientation of K).

If K is an amphicheiral knot and $\phi^2 = id$, then we call K a strong amphicheiral knot. It is worthwhile to note that a knot can be positive and negative amphicheiral at the same time [5]. The following Figure from [5] depicts two projections of the same 10 crossing knot, which shows that this knot is both positive and negative amphicheiral.

Figure 2.3: A positive and negative amphicheiral knot

The main conjecture of this chapter is follows:

Conjecture 2.1.1. *[7]If K is a amphicheiral knot then $pc_{4i}(K) \equiv 0 \mod 2$ for all i.*

2.2 Background

Before we prove the main theorem 2.3.2, we need to reformulate conjecture 2.1.1 and prove the equivalence of reformulated versions. The reformulated version of the conjecture is easier to grasp.

Theorem 2.2.1. *Let $C(z)$ denote the Conway polynomial of a knot K. Then the following are equivalent.*

1. $pc_{4i} \equiv 0 \mod 2$
2. $C(z)C(iz)C(z^2)$ *is a perfect square in* $\mathbb{Z}_4[z^2]$
3. $C(z)C(iz) \equiv C(z^2) \mod 4$
4. $C(z)$ *splits in* $\mathbb{Z}_4[z]$. *I.e.*$C(z) \equiv f(z)f(-z) \mod 4$ *for some* $f \in \mathbb{Z}[z]$

Proof. In [7], J. Conant proved equivalence of (1) and (2). We can easily see that (3) implies (2).

To prove (2) implies (3), suppose $C(z)C(iz)C(z^2)$ is a perfect square in $\mathbb{Z}_4[z^2]$. The Conway polynomial can be written as, $C(z) = 1 + \Sigma a_{2i+1}z^{2(2i+1)} + \Sigma a_{2i}z^{4i}$. Let $y = z^2$. Then, $C(y) = 1 + f_o(y) + f_e(y)$ where f_o denotes odd power terms and f_e denotes even power terms. Similarly we denote $C(iz)$ using the notation $C(y)^*$. Now, $C(iz) = C(y)^* = 1 - f_o(y) + f_e(y)$. By assumption, $C(z)C(iz)C(z^2) = C(y)C(y)^*C(y^2) = F^2$ for some $F \in \mathbb{Z}_4[y]$. In $\mathbb{Z}_2$; $\bar{C}(y)\bar{C}(y)^*\bar{C}(y^2) = \bar{F}^2 \in \mathbb{Z}_2[y]$. Note that in $\mathbb{Z}_2$; $\bar{C}(y) = \bar{C}(y)^*$. So, $\bar{C}^2(y)\bar{C}(y^2) = \bar{F}^2 \in \mathbb{Z}_2[y]$. This implies, $\bar{C}^4(y) = \bar{F}^2(y) \to \bar{C}^2(y) = \bar{F}(y)$. So we can see, $F = C^2(y)+2g(y) \in \mathbb{Z}_4(y)$, for some $g \in \mathbb{Z}_4[y]$. Thus $F^2 = C^4(y) \in \mathbb{Z}_4[y]$. Since $F^2 = C^4(y) = C(y)C(y)^*C(y^2)$, we conclude that (since $C(y)$is a non-zero divisor), $C(y)^*C(y^2) = C^3(y) \in \mathbb{Z}_4[y]$. Now observe that, $(C(y)^*)^2 = C^2(y)$. Therefore, we can conclude that $C(y)^*C(y^2) = (C(y)^*)^2C(y)$ which implies $C(y^2) = C(y)C(y)^* \in \mathbb{Z}_4[y]$. This implies that $C(z)C(iz) \equiv C(z^2) \mod 4$.

To prove (3) implies (4), substitute z for z^2 in (3), then $C(z) \equiv f(z)f(-z) \mod 4$ for some integer polynomial f.

To prove (4) implies (3) we use following argument. Suppose $C(z) \equiv f(z)f(-z)$ for some $f \in \mathbb{Z}[z]$. Let $\alpha(z), \beta(z), \gamma(z), \delta(z) \in \mathbb{Z}[z^4]$. Then a general integer polynomial $f(z)$ can be written $f(z) = \alpha(z) + \beta(z)z + \gamma(z)z^2 + \delta(z)z^3$.

$$
\begin{aligned}
&f(z)f(-z)f(iz)f(-iz) = \\
&= (\alpha+\beta z+\gamma z^2+\delta z^3)(\alpha-\beta z+\gamma z^2-\delta z^3)(\alpha+i\beta z-\gamma z^2-i\delta z^3)(\alpha-i\beta z- \\
&\gamma z^2+\delta z^3) \\
&= [(\alpha+\gamma z^2)^2-(\beta z+\delta z^3)^2][(\alpha-\gamma z^2)^2-(i\beta z-\delta z^3)^2] \\
&= [(\alpha+\gamma z^2)^2-(\beta z+\delta z^3)^2][(\alpha-\gamma z^2)^2+(\beta z-\delta z^3)^2] \\
&\equiv [(\alpha+\gamma z^2)^2-(\beta z+\delta z^3)^2][(\alpha+\gamma z^2)^2+(\beta z+\delta z^3)^2] \\
&= (\alpha+\gamma z^2)^4-(\beta z+\delta z^3)^4 \\
&= F(z)^4-G(z)^4
\end{aligned}
$$

Here we have used the fact $(A+B)^2 \equiv (A-B)^2 \mod 4$. On the other hand

$$
\begin{aligned}
f(z^2)f(-z^2) = (\alpha(z^2)+\beta(z^2)z^2+\gamma(z^2)z^4+\delta(z^2)z^6)(\alpha(z^2)-\beta(z^2)z^2+\gamma(z^2)z^4- \\
\delta(z^2)z^6) \\
= (\alpha(z^2)+\gamma(z^2)z^4)^2-(\beta(z^2)z^2+\delta(z^2)z^6)^2 \\
= F(z^2)^2-G(z^2)^2
\end{aligned}
$$

So, to show that $f(z)f(-z)f(iz)f(-iz) \equiv f(z^2)f(-z^2) \mod 4$. It suffices to show that $F(z^2)^2 \equiv F(z)^4 \mod 4$ for any integer polynomial $F(z)$. This can easily be proven by induction using the fact that $(A+B)^4 \equiv A^4+2A^2B^2+B^4 \mod 4$. Thus we proved that if $C(z) = f(z)f(-z) \mod 4$ then,

$$
\begin{aligned}
C(z)C(iz) &\equiv f(z)f(-z)f(iz)f(-iz) \mod 4 \\
&\equiv f(z^2)f(-z^2) \mod 4 \\
&\equiv C(z^2) \mod 4
\end{aligned}
$$

□

Surprisingly, the reformulated version of Conjecture 2.1.1 is very close to a theorem proved by A. Kawauchi and R.I. Hartley :

Theorem 2.2.2. *[15, 17, 23]Let $C(z)$ be the Conway polynomial of a knot K. If K is a $(-)$ amphicheiral knot then $C(z) = f(z)f(-z)$ for some $f \in \mathbb{Z}[z]$. If K is a strong $(+)$ amphicheiral knot then $C(z) = f(z^2)^2$ for some $f \in \mathbb{Z}[z]$.*

This proves that Conjecture 2.1.1 is true for all $(-)$ and strong $(+)$ amphicheiral knots. From Hartley and Kawauchi's theorem, it is natural to formulate the following question.

Conjecture 2.2.1. *For any amphicheiral knot K, $C_K(z) = \phi(z)\phi(-z)$ for some $\phi \in \mathbb{Z}[z]$.*

If this is true then Conjecture 2.1.1 becomes trivial. However a counterexample for Conjecture 2.2.1 was found in 2011 by N. Ermotti, C. V. Q. Hongler, C. Weber[12] The following Figure from [12] shows the counter example knot. For the knot in Figure 2.4, the Conway polynomial is $C(z) = (4z^8 + 16z^6 + 12z^4 - 16z^2 + 1)(1 + z)(1 - z)(2z^4 - 1)^2 \neq f(z)f(-z)$ for any $f(z) \in \mathbb{Z}[z]$. Thus it is indeed a counter example for Conjecture 2.2.1. However note that $C(z) \equiv (1 + z)(1 - z)(2z^4 - 1)^2 \equiv \phi(z)\phi(-z) \mod 4$. According to N. Ermotti [12], this is an alternating amphicheiral knot. We found a nonalternating knot as a counter example to Conjecture 2.2.1. Let $w = \sigma_1^2\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2\sigma_4\sigma_3^{-1}\sigma_4^2$ be the 5-strand braid (Figure 2.5), then the Conway polynomial of closure of ww^*ww^* is: $C_K(z) = (1+z^2)^2(1+3z^2)^2(1-11z^2+33z^4+8z^6)$ where w^* is the braid w with all crossings reversed. Thus it is a counter example to the Conjecture 2.2.1 because braids of this form are amphicheiral(see section 2.3). The knot is nonalternating due to the fact that the absolute value of the leading coefficient of the Conway polynomial of an alternating amphicheiral knot is square [32]. Still, this is not a counter example to conjecture 2.1.1 because, $C_K(z) \equiv (1 - z + z^2)(1 + z + z^2)(1 + z^4)^2 \mod 4$.

Now observe that pc_{4i} invariants are primitive Vassiliev invariants. Thus to prove Conjecture 2.1.1 one can focus only on prime amphicheiral knots. Due to theorem

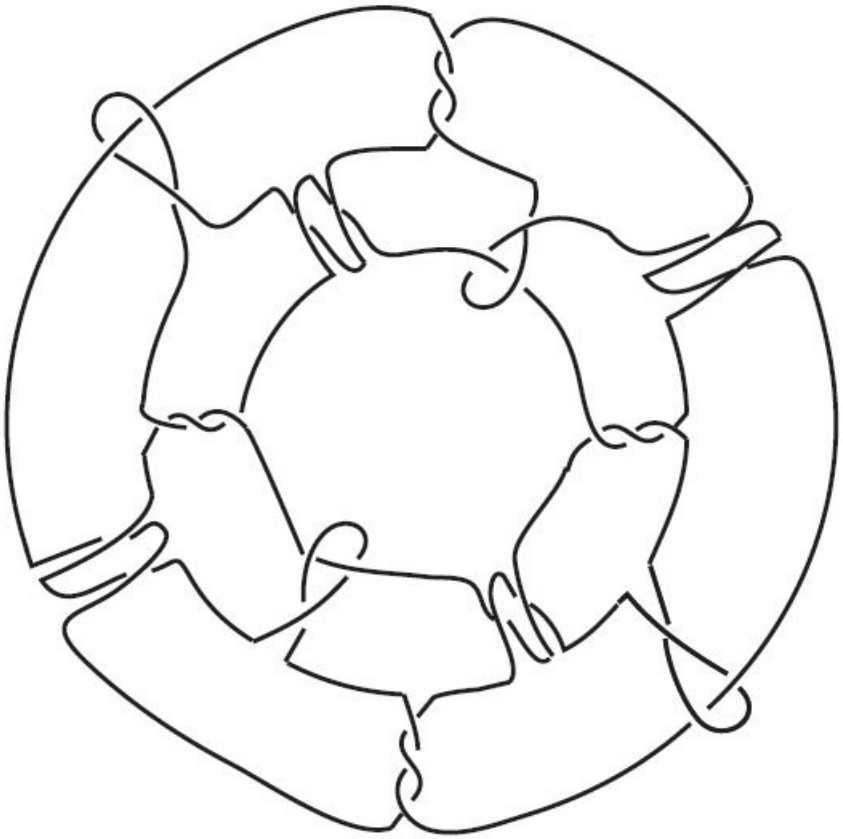


Figure 2.4: Ermotti etal. Knot

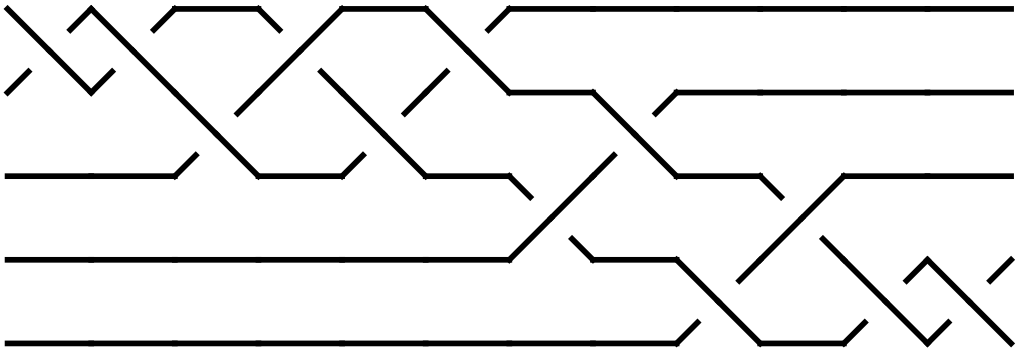


Figure 2.5: Braid w

2.2.2, one needs to consider only prime, positive (non-strong) amphicheiral knots. Also due to the following theorem one can focus only on prime periodically positive amphicheiral knots.

Definition 2.2.1. *[22]A knot $K \subset \mathbb{R}^3$ is a periodic knot of period n if there is a periodic map f of $(\mathbb{R}^3, K)$ which satisfies following conditions.*

1. *f is a $2\pi/n$-rotation about a line F in $\mathbb{R}^3$*

2. *$F \cap K = \emptyset$*

Theorem 2.2.3. *If $C_K(z) = \phi(z)\phi(-z) \mod 4$ for periodically positive amphicheiral knots then $C_K(z) = \phi(z)\phi(-z) \mod 4$ for all positive amphicheiral knots.*

Proof. Follows by theorem 3.1 of [15]. □

So the question is whether $C(z) \equiv f(z)f(-z) \mod 4$ for periodically positive (nonstrong) amphicheiral knots. If, yes then we have solved Conjecture 2.1.1 affirmatively.

2.3 The Alexander polynomial and the Burau representation

Since we are focusing on periodically positive amphicheiral knots, we need a method to build these knots systematically. Let w be a $(2n+1)$-braid, and w^* be the mirror image of the braid w (which means all crossings are reversed). Then the closure ww^* is a strong positive amphicheiral knot and the closure ww^*ww^* is a periodically positive amphicheiral knot.

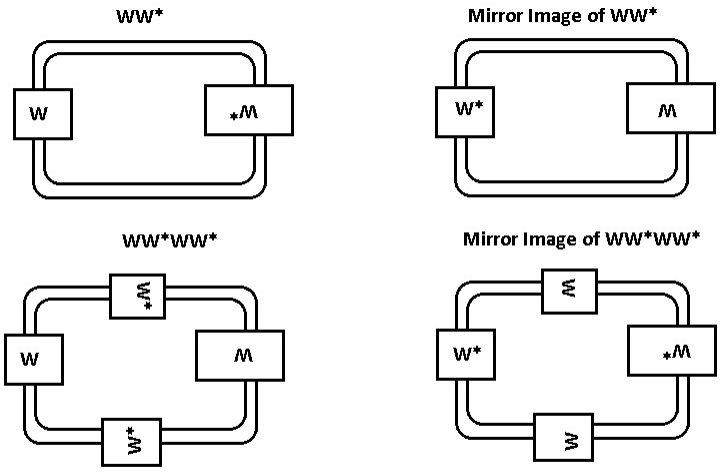


Figure 2.6: Braids ww^* and ww^*ww^*

Definition 2.3.1. *[4] Let σ_i denote the standard generator of the braid group B_n with $n \geq 3$. Then the reduced Burau representation of B_n, for $n \geq 3$ is given by*

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array}\right),$$

$$\sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array}\right), \quad 2 \leq i \leq n-2,$$

$$\sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & t & -t \end{array}\right),$$

Here I_k denotes the $k \times k$ identity matrix. For $n = 2$ it maps $\sigma_1 \mapsto (-t)$.

Theorem 2.3.1. *If a knot K is a closure of a braid Ψ in B_n then the Alexander polynomial of the knot K is given by*

$$\Delta_K(t) = \frac{\det(I - \beta(\Psi))}{1 + t + t^2 + ...t^{n-1}} \text{ where } \beta(\Psi) \text{ is Burau representation of } \Psi$$

Before we prove the main theorem of this section, we need a few results.

Lemma 2.3.1. *For any braid w, the closure of w and the closure of the reverse of w, $Re(w)$, are ambient isotopic. E.g: If $w = \sigma_1\sigma_3^2\sigma_2$ then $Re(w) = \sigma_2\sigma_3^2\sigma_1$.*

Proof. Let $w = \sigma_{k_1}.....\sigma_{k_n}$, where $k_i > 0$ be a $m-$strand braid. After 180 rotation it should be: $\sigma_{m-k_n}.....\sigma_{m-k_1}$ So when we take a reflection across the plane where w is in, we get $\sigma_{-(m-k_n)}.....\sigma_{-(m-k_1)}$. Now we take reflection through a plane perpendicular to the plane where w is in, $\sigma_{k_n}.....\sigma_{k_1} = Re(w)$. Here we denoted σ_i^{-1} by σ_{-i}. This proves that closure of w and $Re(w)$ are ambient isotopic. □

Lemma 2.3.2. *For any braid w, the closure of ww^* and $(ww^*)^{-1}$ are isotopic. Here w^* denotes w with all crossings reversed.*

Proof. Let $w = \sigma_{k_1}.....\sigma_{k_n}$. Then $ww^* = \sigma_{k_1}.....\sigma_{k_n}\sigma_{k_1}^{-1}.....\sigma_{k_n}^{-1}$. Then $Re(ww^*) = \sigma_{k_n}^{-1}.....\sigma_{k_1}^{-1}\sigma_{k_n}.....\sigma_{k_1}$. Now using Markov moves we can show that $Re(ww^*) = (ww^*)^{-1} = \sigma_{k_n}.....\sigma_{k_1}\sigma_{k_n}^{-1}.....\sigma_{k_1}^{-1}$. So the closure of ww^* and $(ww^*)^{-1}$ represent the same knot. □

Lemma 2.3.3. *Let A be an $n \times n$ matrix with $\det(A) = 1$. Suppose A and A^{-1} have the same characteristic polynomial, χ_A. Then the coefficient of λ^k equals the coefficient of λ^{n-k}. That means $\chi_A = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_2\lambda^2 + a_1\lambda + 1$*

Proof. Let $\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_2\lambda^2 + a_1\lambda + a_0$ be the characteristic polynomial of A. Then $\chi_{A^{-1}}(\lambda) = \lambda^n + \frac{1}{a_0}(a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-2}\lambda^2 + a_{n-1}\lambda + 1)$. However, $\det(A) = 1 \to a_0 = 1$. Since $\chi_A = \chi_{A^{-1}}$ we get the above result. □

Now we give the definition of Lucas Polynomial, which will use in the Theorem 2.3.2.

Definition 2.3.2. *The Lucas polynomials are defined recursively by,* $L_0 = 2, L_1 = z, L_n = zL_{n-1} + L_{n-2}$ *for all* $n \geq 2$.

Theorem 2.3.2. *If* K *is a positive amphicheiral knot coming from the closure of the braid* $(ww^*)^{2^m}$*, where* w *is a* $(2n+1)-$ *braid and* w^* *is the mirror image of the braid* w*, then the Conway polynomial of* K *satisfies conjecture 2.1.1. i.e*

$$C_K(z) \equiv f(z)f(-z) \mod 4 \text{ for some } f \in \mathbb{Z}[z]$$

Proof. Consider the characteristic polynomial of ww^* : $\det(\lambda I - \beta(ww*))$ where w is a $2n+1$-braid, w^* is w with all crossings reversed and $\beta(ww^*)$ represents the Burau representation of the braid ww^*. First we claim that the characteristic polynomials of ww^* and $(ww^*)^{-1}$ are the same. This is true from Lemma 2.3.1 and 2.3.2 and the observation that $\det(\beta(ww^*)) = 1$.

Consider the polynomial $\chi_{\beta(ww^*)}(\lambda)$. When $\lambda = 1$, $\chi_{\beta(ww^*)}(1) = 1 + a_1(t) + v + a_{2n-1}(t) + 1$. Since the characteristic polynomials of $\beta(ww^*)$ and $\beta((ww^*)^{-1})$ are the same, using Lemma 2.3.3 we conclude that, $\chi_{\beta(ww^*)}(1) = 2(1 + a_1(t) + \cdots + a_n(t))$. Similarly, $\chi_{\beta((ww^*)}(-1) = 2(1 - a_1(t) + a_2(t) + \cdots + (-1)^n a_n(t))$. Thus $\chi_{\beta(ww^*)}(1) + (-1)^n \chi_{\beta(ww^*)}(-1) = \det(I - \beta(ww^*)) + (-1)^n \det(I + \beta(ww^*)) \equiv 0 \mod 4$. In this proof we use this fact that $a_i(t)$, when converted using the substitution $\sqrt{t} - \frac{1}{\sqrt{t}} = z$ yields integer polynomials.

Now observe $(1+t+\cdots t^{2n}) = ((t^n + \frac{1}{t^n}) + \cdots + (t+1/t)+1) \equiv (-1)^n(1+L_1(z)+\cdots + L_n(z))(1-L_1(z)+\cdots+(-1)^n L_n(z))) \mod 4$. Here $z = \sqrt{t} - 1/\sqrt{t}$ and $L_n(z)$ is the n^{th} Lucas Polynomial. Also we used the identity [36], $L_n L_m = L_{n+m} + (-1)^m L_{n-m}$ to get above result.

The closure of ww^* yield a strong positive amphicheiral knot. Thus from Theorem 2.2.2, we conclude that the Alexander polynomial of the knot closure of ww^* is

$$\frac{\det(I - \beta(ww^*))}{1 + t + \cdots + t^{2n}} = f^2(z) \text{ for some} f(z) \in \mathbb{Z}[z]$$

.

From above we have, $\det(I-\beta(ww^*))+(-1)^n\det(I+\beta(ww^*)) \equiv 0 \mod 4$. Thus we conclude

$$\det(I+\beta(ww^*)) \equiv f^2(z)(1+L_1(z)+\cdots+L_n(z))(1-L_1(z)+\cdots+(-1)^nL_n(z))) \mod 4$$

Now consider the closure of braid ww^*ww^*. We can see that

$$\begin{aligned}\frac{\det(I-\beta(ww^*ww^*))}{1+t+\cdots+t^{2n}} &= \frac{\det(I-\beta(ww^*))\det(I+\beta(ww^*))}{1+t+\cdots+t^{2n}} \\ &\equiv f^4(z)(1+L_1(z)+\cdots+L_n(z))(1-L_1(z)+\cdots+(-1)^nL_n(z)) \\ &\equiv g(z)g(-z) \mod 4 \\ \text{where } g(z) &= f^2(z)(1+L_1(z)+\cdots+L_n(z))\end{aligned}$$

Here we use the fact [36], $L_{2n+1}(-z) = -L_{2n+1}(z)$. For $(ww^*)^{2^m}, m \geq 3$ observe that above method is still valid, and we have following induction hypothesis : $\frac{\det(I-\beta((ww^*)^{2^{m-1}}))}{1+t+\cdots+t^{2n}} \equiv g(z)g(-z) \mod 4$ for some $g \in \mathbb{Z}[z]$. Hence the theorem follows. □

For any positive amphicheiral knot coming from the closure of $(ww^*)^{2^m.k}$ with k being odd can be rearranged to look like $(ww^*)^{2^n}$ for some n. Thus Theorem 2.3.2, essentially proves the conjecture for all amphicheiral knots coming from braid closure $(ww^*)^{2^m.k}$, with $m, k \in \mathbb{Z}$.

There is a beautiful relationship between the Conway polynomial and Fibonacci polynomial. In the following lemma, this relationship is explored. Before we start the lemma, we need the following definition.

Definition 2.3.3. *[18] The two variable generalized Fibonacci polynomial is defined recursively as follows:*
$U_n(x,y) = xU_{n-1}(x,y) + yU_{n-2}(x,y)$, $U_1(x,y) = 1, U_0(x,y) = 0$

The one variable Fibonacci polynomial is defined by,
$F_1(z) = 1, F_2(z) = z, F_n(z) = zF_{n-1}(z) + F_{n-2}(z)$.

Corollary 2.3.1. *Let* $\Omega_n(t) = \Sigma_{i=0}^{n-1}(-t)^i$. *When* n *is odd,* $\frac{\Omega_n^2}{t^{n-1}} = F_n^2(z)$, *where* F_n *denotes the* n^{th} *Fibonacci polynomial and* $z = \sqrt{t} - \frac{1}{\sqrt{t}}$.

Proof. We need to prove $t^{-(n-1)/2}\Omega_n(t) = F_n(t)$, when n is odd. First we claim that $U_n(t-1,t) = (-1)^{n-1}\Sigma_{i=0}^{n-1}(-t)^i$. This can be proved using induction. In particular when n is odd, $U_n(t-1,t) = \Omega_n(t)$. Next we use following identity [18]. $U_n(x,y) = y^{(n-1)/2}F_n(\frac{x}{\sqrt{y}})$. So we have $U_n(t-1,t) = t^{(n-1)/2}F_n(\sqrt{t} - \frac{1}{\sqrt{t}}) = t^{(n-1)/2}F_n(z)$. Letting n be odd, we get our result.

□

Lemma 2.3.4. *Let* K *be the closure of the 3-braid* ww^*ww^*, *where* $w = \sigma_1^n\sigma_2^m, w^* = \sigma_1^{-n}\sigma_2^{-m}$ *with* n, m *are odd integers. Then*

$$C_K(z) = F_n^2(z)F_m^2(z)(4 - F_n^2(z)F_m^2(z)(z^2+3))$$

where $F_n(z)$ *is n-th Fibonacci polynomial.*

Proof. Consider the Alexander polynomial, $\Delta_K(t)$, of the closure of ww^*ww^*. Then using the Burau representation,

$$\begin{aligned}\Delta_K(t) &= \frac{\det(I - \beta(ww^*ww^*))}{1+t+t^2} \\ &= \frac{\det(I-\beta(ww^*))\det(I+\beta(ww^*))}{1+t+t^2}\end{aligned}$$

Let matrices A, B denote the Burau representation of σ_1 and σ_2 respectively. That means,

$$A = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

Since $w = \sigma_1^n \sigma_2^m$, this implies,

$$\begin{aligned}\Delta_K(t) &= \frac{\det(I - A^n B^m A^{-n} B^{-m} A^n B^m A^{-n} B^{-m})}{1+t+t^2} \\ &= \frac{\det(B^m A^n - A^n B^m)\det(B^m A^n + A^n B^m)}{(1+t+t^2)\det(A^n B^m)^2}\end{aligned}$$

Now observe that,

$$A^n = \begin{pmatrix} (-t)^n & \Omega_n(t) \\ 0 & 1 \end{pmatrix}, \quad B^m = \begin{pmatrix} 1 & 0 \\ t\Omega_m(t) & (-t)^m \end{pmatrix}$$

$$A^n B^m = \begin{pmatrix} (-t)^n + t\Omega_m\Omega_n & (-t)^m\Omega_n \\ t\Omega_m & (-t)^m \end{pmatrix}, \quad B^m A^n = \begin{pmatrix} (-t)^n & \Omega_n \\ t(-t)^n\Omega_m & t\Omega_m\Omega_n \end{pmatrix}$$

here $\Omega_n = \Sigma_{i=0}^{n-1}(-t)^i$. Also note that $\det(A^n B^m) = t^{n+m}$. Now, tedious calculations shows that,

$$\det(B^m A^n - A^n B^m) = \Omega_n\Omega_m t(1 + t^m + t^n + t^{m+n} - \Omega_m\Omega_n t) \text{ and}$$
$$\det(B^m A^n + A^n B^m) = 4(-t)^{m+n} - \Omega_n\Omega_m t(1 + t^m + t^n + t^{m+n} - \Omega_m\Omega_n t)$$

Now using the identity, $\frac{1+t^m+t^n+t^{m+n}-\Omega_m\Omega_n t}{1+t+t^2} = \Omega_m\Omega_n$,

$$\Delta_K(t) = \frac{\Omega_m\Omega_n t(4t^{m+n} - \Omega_m^2\Omega_n^2 t(1+t+t^2))}{t^{2(m+n)}}$$

Now using the Corollary 2.3.1, we have $\frac{\Omega_n^2}{t^{n-1}} = F_n^2(z)$ and $\frac{\Omega_m^2}{t^{m-1}} = F_n^2(z)$. Substituting these identities in above equation, we get

$$\Delta_K(t) = \frac{F_n^2 F_m^2}{t}(4 - F_n^2 F_m^2(t + \frac{1}{t} + 1))$$

Now observing $t + \frac{1}{t} + 1 = z^2 + 3$ and the Alexander polynomial is defined up to multiplication by $t^{\pm 1}$, we conclude that the Conway polynomial of the closure of braid ww^*ww^* is given by $C(z) = F_n^2 F_m^2(4 - F_n^2 F_m^2(z^2 + 3))$. □

In above n, m need to be odd, in order to get a knot from the closure of ww^*ww^*. Otherwise, we would get a link with more than one component. The beauty of the Lemma 2.3.4 is, it enable us to write the Conway polynomial of amphicheiral knot, $\sigma_1^n\sigma_2^m\sigma_1^{-n}\sigma_2^{-m}\sigma_1^n\sigma_2^m\sigma_1^{-n}\sigma_2^{-m}$ without using the definition of the Conway polynomial(avoiding the process of resolving crossings). The only parameters we need are n and m. This theorem suggests that there may be a general relationship between the Conway polynomial of amphicheiral knots and the Fibonacci polynomial. Also note that under $\mod 4$,

$$C(z) \equiv F_n^4 F_m^4 (z-1)(z+1) \mod 4$$

which imply that Conjecture 2.1.1 is true for these specific types of braids.

2.4 The leading coefficient of the Conway polynomial and amphicheiral knots

In this section we explore two conjectures proposed by A. Stoimenow regarding the leading coefficient of the Conway polynomial of an amphicheiral knot. The conjectures are based on the following theorem.

Theorem 2.4.1. *[32]Let K be an amphicheiral knot. Then the leading coefficient of Alexander(and Conway) polynomial of the knot K, $\Delta_K(z)$ is a square(up to sign) and the sign of the leading coefficient of the Alexander polynomial $\Delta_K(z)$ is $(-1)^{\max \deg \Delta_K(z)}$ if any of the following hold.*

1. *K is an alternating knot,*

2. *K is strongly amphicheiral or negative amphicheiral,*

3. *K is any knot with at most 16 crossings or*

4. *K is a fibered homogeneous knot.*

This theorem provides strong background for the following conjectures.

Conjecture 2.4.1. *[32]The leading coefficient of the Alexander(Conway) polynomial of an amphicheiral knot is a square.*

Conjecture 2.4.2. *[32]The sign of the leading coefficient of the Alexander polynomial, Δ_K of an amphicheiral knot K is $(-1)^{max\ deg\Delta_K}$.*

It is easy to see that any counter example to Conjecture 2.4.1 is again a counter example to the Kawauchi conjecture 2.2.1. More formally we can say,

Fact 2.4.1. *If the leading coefficient of the polynomial $G(z) \in \mathbb{Z}[z]$ is not square then $G(z) \neq F(z)F(-z)$ for any $F(z) \in \mathbb{Z}[z]$.*

Now we give several counterexamples to the conjectures we mentioned at the beginning.

Counterexample 2.4.1. *Let $w = \sigma_1^2\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2\sigma_4\sigma_3^{-1}\sigma_4^2$, which is a 5-strand braid. Then the Conway polynomial of the closure of a ww^*ww^* is $1 - 3z^2 - 33z^4 + 54z^6 + 535z^8 + 869z^{10} + 489z^{12} + 72z^{14}$. The knot ww^*ww^* is shown in the Figure 2.7. It was obtained using Knotscape[19].*

Theorem 6 in [29] states that if K is an alternating knot and the leading coefficient of the Alexander polynomial is prime, then K is non-amphicheiral. With this theorem, one may think that leading coefficient of the Alexander polynomial of amphicheiral knots which does not satisfy above conjectures must be composite. However the leading coefficient of the Alexander polynomial of the following amphicheiral knot is prime and serves as a counterexample to three Conjectures 2.2.1, 2.4.1 and 2.4.2.

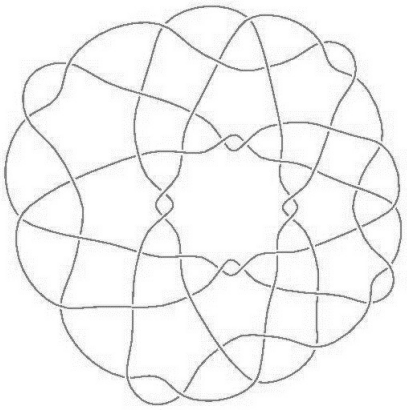


Figure 2.7: Amphicheiral knot with the leading coefficient 72

Counterexample 2.4.2. *Let* $w = \sigma_1^2\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_4\sigma_3\sigma_2^2\sigma_3^2\sigma_4$ *be the 5-strand braid. Then the Conway polynomial of the closure of* ww^*ww^* *is* $1 + 5z^2 + 39z^4 + 246z^6 + 657z^8 + 743z^{10} + 301z^{12} - 78z^{14} - 105z^{16} - 31z^{18} - 3z^{20}$. *The knot is positive amphicheiral but not strongly amphicheiral.*

After seeing these counter examples, one may wish to see a prime non-alternating knot which does not satisfy all of the above three conjectures. Upon our request, M. Thistlethwaite [34] kindly provided us the list of 19-crossings prime positive amphicheiral knots. In that list, the following knot does not satisfy any of the above three conjectures and it is a non-alternating positive(not strong) amphicheiral knot.

Counterexample 2.4.3. *Let* K *be the knot with the following Dowker-Thistlethwaite code.*

$$6, -12, 32, -18, -26, 16, -4, -22, 34, -38, 30, -14, 20, 36, -10, 24, 2, 28, -8$$

Then K *is a prime non-alternating and positive non strong amphichiral knot with the Conway polynomial,* $1 + 3z^2 + 8z^4$. *The knot is depicted in the Figure 2.8. It was obtained using Knotscape[19].Observe that* $1 + 3z^2 + 8z^4 \equiv (1 - z)(1 + z) \mod 4$. *Thus it is not a counter example to Conjecture 2.1.1*

We finish this section with following realization problem.

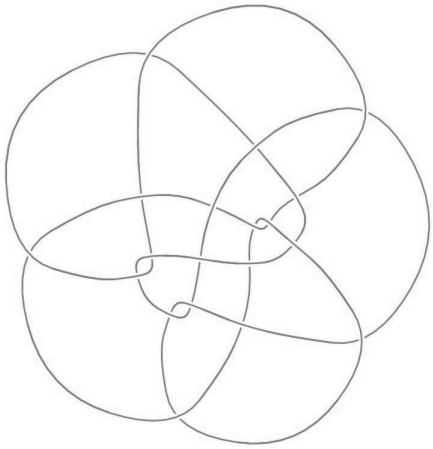


Figure 2.8: Non alternating prime amphicheiral knot with the leading coefficient 8

Question 2.4.1. *Let n be an arbitrary integer. Is it possible to have an amphicheiral knot K with n as a leading coefficient of its Conway polynomial?*

If n is a square, then we have a positive answer due to E. Flapan . Namely she has shown if $C(z) = F(z)^2$ and $F(0) = 1$ for some integer polynomial $F(z)$, then there exists a strong positive amphicheiral knot which has $C(z)$ as its Conway polynomial [13]. The question will be more interesting when the absolute value of n is prime.

Question 2.4.2. *For a given prime p, is there an amphicheiral knot such that the absolute value of the leading coefficient of the Conway polynomial is p?*

More ambitious question: If answer for above question is yes, can we choose the amphicheiral knot to be prime? That means we have prime amphicheiral knots with prime numbers as leading coefficient of their Conway polynomials. It is clear from the Theorem 2.4.1, if it exists then they are non alternating positive non strong amphicheiral knots.

We found several amphicheiral knots which have odd prime numbers as leading coefficients. The table 2.1 and the Figure 2.9 summarize our finding. Figures were obtained using Knotscape[19] and the Conway polynomial calculated using the Mathematica package KnotTheory[3].

Table 2.1: Leading coefficients of the Conway polynomial for knot ww^*ww^*

Braid word w	Leading coefficient of the Conway polynomial of ww^*ww^*	Number of strands
$\sigma_1^2\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_4\sigma_3\sigma_2^2\sigma_3^2\sigma_4$	-3	5
$\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_4^{-1}$	5	5
$\sigma_1\sigma_2\sigma_3^{-1}\sigma_4\sigma_4\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_4\sigma_3^{-1}$	-7	5
$\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3\sigma_2\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2$	-11	5
$\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3\sigma_3\sigma_3\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3^{-1}\sigma_2\sigma_2\sigma_3\sigma_4^{-1}\sigma_3$	13	5
$\sigma_1\sigma_2^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}$	17	5

The table suggest that using 5- strand braid words w, we may be able to get all prime numbers (absolute value) as leading coefficient . The table also direct us to the question, whether we can have ± 2 as leading coefficient of positive non strong amphicheiral knots. This is quite interesting because if exists, then it serves as a counterexample to Conjecture 2.1.1. This is because, if ± 2 is the leading coefficient of the Conway polynomial then $C(z)C(iz) - C(z^2) \ncong 0 \mod 4$.

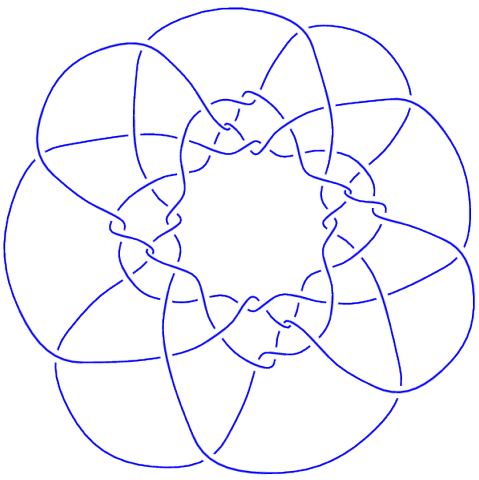
For periodic knots we have the following result from [10]

Theorem 2.4.2. *[10] If K is of period p^r and the leading coefficient of the Conway polynomial Δ_K is divisible by p, then it is divisible by p^r.*

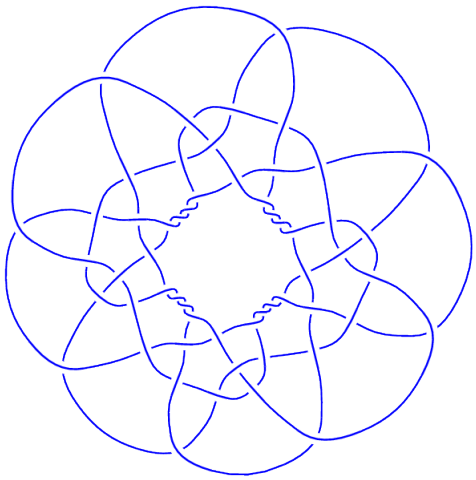
Now for periodically positive non strong amphicheiral knots, period is 2^r with $r \geq 4$.Thus for periodically positive non strong amphicheiral knots we conclude that the leading coefficient of the Conway polynomial cannot be 2. If there exist an amphicheiral knot with the leading coefficient of the Conway polynomial is 2, it must be a non periodic,non hyperbolic, non alternating, non strong positive amphicheiral knot. The non hyperbolic condition comes from the following proposition.

Proposition 2.4.1. *([22], Proposition 10.4.3) If K is a hyperbolic amphicheiral knot then it is a periodic amphicheiral knot.*

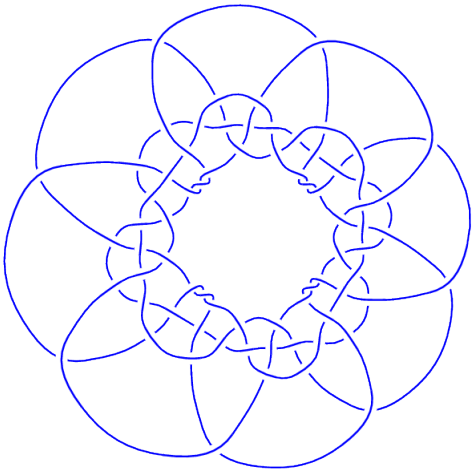
It is known that there are amphicheiral knots which are not periodically amphicheiral[16].We leave the question 2.4.2 as future research question.



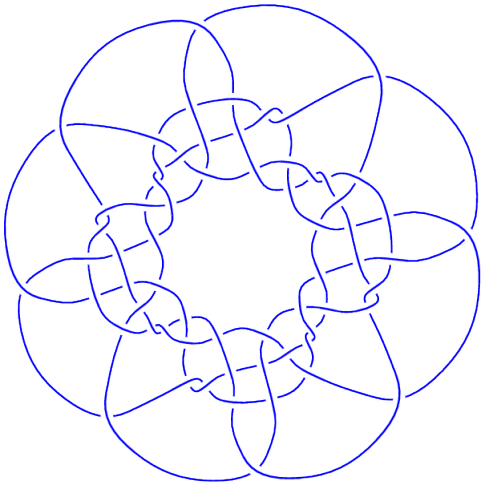
Leading coefficient = -3



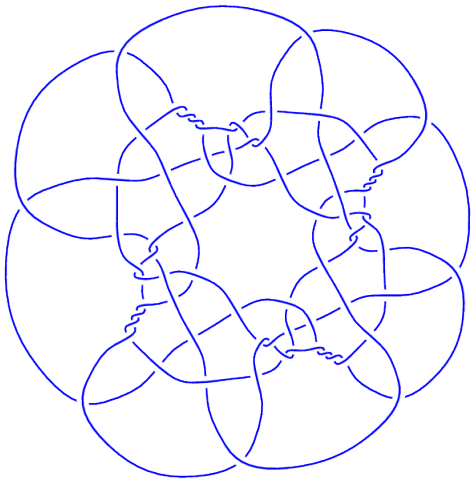
Leading coefficient = 5



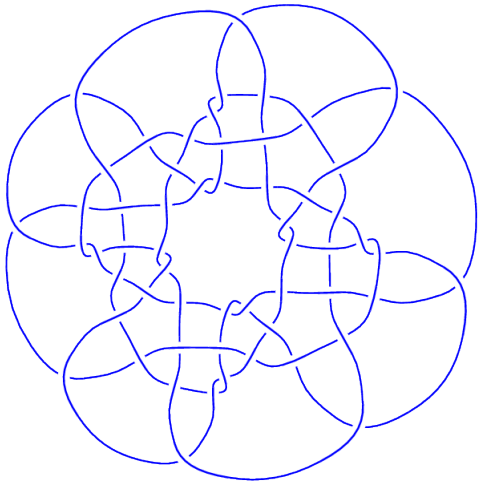
Leading coefficient = -7



Leading coefficient = -11



Leading coefficient = 13



Leading coefficient = 17

Figure 2.9: Amphicheiral knots where the leading coefficient of the Conway polynomial is prime

Chapter 3

Integer Vassiliev Knot Invariants

3.1 Introduction

We give a basis for the space of integer valued Vassiliev invariants of order less than six over $\mathbb{Z}$ and a basis for the chord diagrams of order less than six over $\mathbb{Z}$. For order seven, let W_n denote the integer valued Vassiliev invariants of order less than n. We calculate a basis for $W_7/W_6 \mod 2$ and $W_7/W_6 \mod p$ where p is any odd prime. We also give a rational basis for order seven. As a result we prove J. Conant's conjecture 2.1.1 regarding primitive Vassiliev invariants pc_{2n} for $n = 3$ and 4.

The aim of this chapter is to find a basis for Vassiliev invariants of order less than seven over $\mathbb{Z}$. In the past there were several attempts made to calculate the rational basis of Vassiliev knot invariants. A rational basis for Vassiliev knot invariants of order less than six was computed by T. Kanenobu [21] in 2001 and in the same year T. Stanford calculated the basis for Vassiliev invariants of order less than six with some added conditions. T. Stanford's [31] basis almost works as a basis over $\mathbb{Z}$ for invariants of order less than six. The main obstacle for computing a basis is the size of the relation matrix which come from 1T and 4T relations. For order seven, the relation matrix size is 3218×3218. For order eight this size increases to 42335×42335.

The dimension of the basis of Vassiliev invariants up to order 9 was computed by Dror Bar-Natan [1] in 1995 and up to twelve by J. Kneissler [24] in 1997.

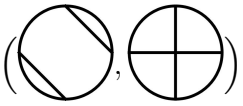
Since Dror Bar-Natan's program computes basis over the rationals and since it is highly inaccessible to modification, we wrote a separate Mathematica program to calculate a basis of chord diagram of order n over integers and rational. As a result we obtained a different chord diagram basis over the rationals compare to Dror Bar-Natan's basis of chord diagrams up to order nine posted on his webpage. The Mathematica program is more readable and has functions which may be useful for future computations.

3.2 Background

Definition 3.2.1. *[6] A chord diagram of order n is an oriented circle with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. The set of all chord diagrams of order n will be denoted by $\boldsymbol{A}_n$*

For example,

$$\mathbf{A}_2 = \Big(\quad , \quad \Big) \tag{3.1}$$



Chord diagrams can be used to code certain information about singular knots with n double points. Gauss codes can be used to differentiate the code diagrams. For example in above, the Gauss code for the diagrams are 1122 and 1212.

Definition 3.2.2. *[6] The chord diagram $\sigma(K) \in \boldsymbol{A}_n$ of a singular knot with n double points is obtained by marking on the parametrizing circle n pairs of points whose images are the n double points of the knot.*

The following figure from [6] shows this relationship.

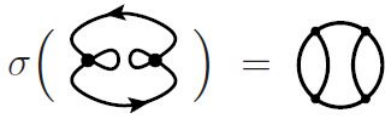


Figure 3.1: Chord diagrams and singular knots

The following proposition shows an important connection between chord diagrams of order n and the value of Vassiliev invariants of order n over singular knots with n double points.

Proposition 3.2.1. *[35] The value of a Vassiliev invariant v of order $\leq n$ on a knot K with n double points depends only on the chord diagram of K: $\sigma(K_1) = \sigma(K_2) \implies v(K_1) = v(K_2)$*

Even though, there are an infinite number of knots, by using the chord diagram representation of singular knots, we can calculate value of a Vassiliev knot invariant, v, of order n over any given knot. This process is called building an *actuality table*. Suppose we have a knot K. Then using Vassiliev skein relation 2.1, we can write the knot K as a formal sum of a non singular knot and a singular knot with one double point. Then we can write the knot with one double point as formal sum of a singular knot with one double point and several knots with two double points. By continuing this process, we can obtain a representation of the knot K as sum of basic knots with at most n double points and several knots with $n+1$ double points. Now v is of order n, thus it vanishes on all singular knots with more than n double points. Therefore, $v(K)$ can be calculate using the values of v on basic knots with at most n double point. Now the value of v on singular knots with exactly n double points depends only on its chord diagram from proposition 3.2.1. However for singular knots with less than n double points, the value of v depend on not only the chord digram but also on underlying singular knot.

Let $\mathbf{V}_n$ denote the set of Vassiliev invariants of order $\leq n$ over a ring $\mathbf{R}$. Let $\mathbf{RA}_n$ denote the $\mathbf{R}-$module of $\mathbf{R}$ valued functions on the set $\mathbf{A}_n$. Define a function $f : \mathbf{V}_n :\to \mathbf{RA}_n$ as follows. $f(v) = (f(v))$ where $(f(v))(D) = v(K)$ and K is

an arbitrary knot with $\sigma(K) = D$. Since the value of any Vassiliev invariant of order $\leq n-1$ over any singular knot with n double points is zero we conclude that $\mathbf{ker}f = \mathbf{V}_{n-1}$. Thus we have an injective map $\bar{f} : \mathbf{V}_n/\mathbf{V}_{n-1} \to \mathbf{RA}_n$. Before we describe the image of $\bar{f}$, we need the following two definitions.

Definition 3.2.3. *[6] A function $f \in \boldsymbol{RA}_n$ is said to satisfy the 4T relation if the alternating sum of the values of f is zero on the following quadruples of diagrams. (The figure is obtained from [6])*

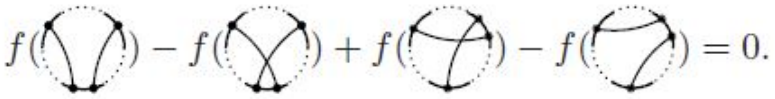


Figure 3.2: 4T Relation

Definition 3.2.4. *[6] A function $f \in \boldsymbol{RA}_n$ is said to satisfy the 1T relation if it vanishes on all chord diagrams with isolated chords. An isolated chord is a chord that does not intersect any other chord of the diagram.*

Theorem 3.2.1. *For $\boldsymbol{R} = \mathbf{C}$, the map $\bar{f}$ identifies $\boldsymbol{V}_n/\boldsymbol{V}_{n-1}$ with the subspace $\frac{\boldsymbol{RA}_n}{<4T,1T>}$. In other words,*

$$\frac{\boldsymbol{V}_n}{\boldsymbol{V}_{n-1}} \cong \frac{\boldsymbol{RA}_n}{< 4T, 1T >}$$

Two important assertions of this theorem is that any Vassiliev invariant of order n satisfies $4T$ and $1T$ relations. Above, we mentioned that in order to build actuality tables for Vassiliev invariants of order n, we need all possible chord diagrams up to order n. But from theorem 3.2.1, to describe the set $\mathbf{V}_n/\mathbf{V}_{n-1}$ we need only the basis diagram set for the vector space generated by the set $\mathbf{A}_n$ modulo the subspace spanned by all 4T and 1T relators. Next for $\mathbf{V}_{n-1}/\mathbf{V}_{n-2}$ we need only the basis diagram set for the space $\mathbf{A}_{n-1}$ modulo the subspace spanned by $4T$ and $1T$ relations. We can continue this operation until we have all basis chord diagrams up to order n. This process reduce the number of chord diagrams needed in actuality tables. Next we need to choose a set of Vassiliev invariants to present in actuality tables. For given

order, there are an infinite number of Vassiliev invariants. This make it impossible to make actuality tables for all Vassiliev invariants of given order. We can avoid this problem by choosing a basis for $\mathbf{V}_n$. In this chapter we focus mainly on integer valued Vassiliev invariants. To find a basis for integer valued Vassiliev invariants, we need set of basis chord diagrams over $\mathbb{Z}$ and we need a set of integer valued Vassiliev invariants.

Let V_n be the $\mathbb{Q}$ vector space of Vassiliev knot invariants of order $\leq n$. There does not seem to be a canonical way to choose a basis for V_n. Ted Stanford [31] mentioned some of the desirable properties that such a basis $B_n = \{ b_1, b_2, ...b_{dim(V_n)}\}$ might have:

1. B_n should consist of $\mathbb{Z}$ valued invariants.
2. B_n should be a basis over $\mathbb{Z}$ for the integer valued invariants of order $\leq n$.
3. For each $i \leq n$, the set $\{ b_1, b_2, ...b_{dim(V_i)}\}$ should be a basis for V_i.
4. If w is any even order basis element, then $w(m(K)) = w(K)$ for any knot K, where $m(K)$ denote the mirror image of K. If w is an odd order basis element then $w(m(K)) = -w(K)$ for any knot K.

We use following theorem extensively in our calculations.

Theorem 3.2.2. *Let V_n denote the space of Vassiliev invariants of order less than or equal to n over $\mathbb{Q}$.*

1. *[1] Let c_n denote the n-th coefficient of Conway polynomial. Then $c_n(.) \in V_n$*
2. *[21] Let $p(t, y)$ denote the HOMFLYPT polynomial. Let $p_{2n}^{(m)}(, 1)$ denote the m th derivative of coefficient of y^{2n} evaluated at 1. Then $p_{2n}^{(m)} \in V_{2n+m}$.*
3. *[21] Let $f(t, y)$ denote the Kauffman polynomial. Let $f_n^{(m)}(, \sqrt{-1})$ denote the m th derivative of coefficient of y^n evaluated at $\sqrt{-1}$. Then $(\sqrt{-1})^{n+m} f_n^{(m)}(, \sqrt{-1})$ $\in V_{n+m}$*

4. *[1] Let $J(t)$ denote the Jones polynomial. Substitute $t = e^h$ and then expand the resulting polynomial as a formal power series. The $n-th$ coefficient of this polynomial, j_n is an element of V_n*

5. *Let $V(t)$ denote the logarithm of Jones polynomial. Expand the resulting polynomial as a formal power series. The n th coefficient of this polynomial, v_n is an element of V_n*

6. *[27] Let $cj_{m,n}$ denote the n th coefficient of m coloured Jones polynomial. Then $cj_{m,n} \in V_n$.*

7. *[6] If $w_n \in V_n$ and $w_m \in V_m$ then $(w_n * w_m) \in V_{n+m}$.*

The invariants above are rational valued invariants. By multiplying these invariants with suitable minimal constant λ we can make them integer valued invariants. For example we can make j_4 an integer valued invariant by multiplying it by 4 and for j_5 it is 2. However there is no known formula to calculate these constants λ in general. In chapter 3, we calculate these constants for j_n.

Let W_n denote the set of integer valued Vassiliev invariants of order $\leq n$. First we show why T. Kanenobu [21] basis is not a basis over $\mathbb{Z}$. Second we give a basis for W_6 over $\mathbb{Z}$ which satisfies T. Stanford's basis conditions except the last condition. Third, we give basis for $(W_7/W_6) \mod 2$ and $(W_7/W_6) \mod p$ where p is any odd prime. Fourth, we give basis for V_7 over $\mathbb{Q}$. This basis is not a basis over $\mathbb{Z}$ but still we present it because as far as we know it was unknown until our work.

Before we continue, we are going to show how to calculate integer basis for chord diagrams of order 4. The process is computerized for higher orders and only the results are given in subsequent sections.

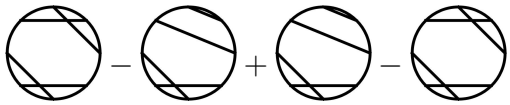
3.3 Calculating integer basis for chord diagrams of order 4

There are 18 distinct diagrams of order 4. After removing diagrams with isolated chords(1T relation) we are left with 7 diagrams. Those diagrams using Gauss codes are,

Table 3.1: Chord diagrams of order 4 without isolated chords

Name	$CD_{(4,1)}$	$CD_{(4,2)}$	$CD_{(4,3)}$	$CD_{(4,4)}$	$CD_{(4,5)}$	$CD_{(4,6)}$	$CD_{(4,7)}$
Gauss Code	12123434	12132434	12134234	12134243	12314234	12314324	12341234

Let's choose the first chord diagram $CD_{(4,1)}$. We can apply 4T relation to chord 1 and 2. We get the following relation.



(3.2)

This relation does not contribute to the relation matrix since everything cancels. We can continue this process for next two chords in $CD_{(4,1)}$. For example we can choose chords 2 and 3 and write $4T$ relations. By continuing this process we get a relation matrix. For order 4, the relation matrix is as follows.

$$\begin{pmatrix}
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1
\end{pmatrix}$$

Here column represents the chord diagrams $CD_{(4,1)} \cdots CD_{(4,7)}$ respectively. Now to find integer basis elements, we can calculate hermite normal form of this matrix. This gives

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 3 & -2 & -1 \\
0 & 1 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Thus we conclude that the diagrams $CD_{(4,5)}, CD_{(4,6)}, CD_{(4,7)}$ forms an integer basis for the vector space $\frac{\mathbf{A}_4}{<4T,1T>}$.

Hermite normal form is a great tool to find an integer basis as long as it is in row reduced echelon form. However we do not get this form all the time. In the event its not look like row reduced echelon form we have to use another approach to find basis. In that case we choose arbitrary chord diagram set as possible basis set and check whether that set form an integer basis. We can add a row for each suspected chord diagram and calculate the hermite normal form. If the rank of the new matrix is the number of columns, then we know that the list we have forms a spanning set. If the dimension of the list is equal to the dimension of the rational basis, then we conclude our list form a basis over $\mathbb{Z}$. For example suppose we want to check whether $\{CD_{(4,1)}, CD_{(4,2)}, CD_{(4,3)}\}$ forms an integer basis. After adding four new rows to above relation matrix, new relation matrix look like

$$\begin{pmatrix}
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

When we calculate the hermite normal form of this matrix, we do not get the identity matrix in the upper 7×7 matrix. This confirms that the set $\{CD_{(4,1)}, CD_{(4,2)}, CD_{(4,3)}\}$ does not form an integer basis for $\frac{\mathbf{A}_4}{<4T,1T>}$.

3.4 T. Kanenobu basis

We checked whether T. Kanenobu's basis [21] is a basis over $\mathbb{Z}$. His bases for chord diagrams of order up to six using Gauss codes are given by:

Table 3.2: T. Kanenobu's basis for chord diagrams of order less than six.

CD2	CD3	CD4	CD5	CD6
$M^2 = [1212]$	$M^3_1 = [121323]$	$M^4_1 = [12134243]$ $M^4_2 = [12314324]$ $M^4_3 = [12132434]$	$M^5_1 = [1213452543]$ $M^5_2 = [1231453254]$ $M^5_3 = [1213453254]$ $M^5_4 = [1213425453]$	$M^6_1 = [121345626543]$ $M^6_2 = [123145632654]$ $M^6_3 = [123415643265]$ $M^6_4 = [121345632654]$ $M^6_5 = [121345265643]$ $M^6_6 = [121342564365]$ $M^6_7 = [121345265463]$ $M^6_8 = [121342564653]$ $M^6_9 = [121342564635]$

Let A be the relation matrix generated by chord diagrams of order 5 using $4T$ and $1T$ relations. Now if the set $\{M^5_1 \cdot\cdot, M^5_4\}$ form a basis over $\mathbb{Z}$ then it should be a basis over $\mathbb{Z}_p$ for any prime p. If we add four rows to the relation matrix A such that each row looks like $(0,0,0,0,\cdots,1,\cdots,0)$ where 1 occurs at the places relating to M^5_i then the rank of the new matrix must be the number of columns in A.

We calculated the rank of $\tilde{A}_{\mod 2}$, where $\tilde{A}$ is the hermite normal form of the matrix A with added rows. The rank of new matrix is 35. But According to Bar-Natan[1] rank must be 36. Thus T. Kanenobu's basis does not form a basis over $\mathbb{Z}$. Similar calculations were done for a basis of order 6 chord diagrams. Modulo 2 the rank we got is 298 and modulo 3 the rank we got is 299. But According to Bar-Natan [1] there are 300 chord diagrams of order 6 without isolated chords. So we conclude that T. Kanenobu's basis is not a basis over $\mathbb{Z}$ for chord diagrams of order six.

3.5 Basis for chord diagrams of order up to seven over $\mathbb{Z}$

This is a crucial part if we want to write a given Vassiliev invariant as a linear combination of a basis of integer valued Vassiliev invariants. However even without the Vassiliev invariant connection, the fact that it is unknown until now is the main motivation behind our work in this section. Dror Bar-Natan[2] mentioned that his basis is basis over $\mathbb{Z}_p$ for primes p such that $p \leq 97$. However it is not confirmed whether these are basis over $\mathbb{Z}_p$ for all primes p. Here we confirmed Dror Bar-Natan's basis up to order six is a basis over $\mathbb{Z}_p$ for all primes p. We also give our own basis which we are going to use in our calculations.

First we build base for chord diagrams of order less than six over $\mathbb{Z}$. To this end we have to heavily rely on the Mathematica program in appendix A. For order three and four we can easily get a basis for chord diagrams over $\mathbb{Z}$ even without the Mathematica program. For orders five and six we used Mathematica program to build the relation matrix. Then we calculated a common basis for $A_{\text{ mod } 2}$ and $A_{\text{ mod } 3}$ where A is the relation matrix generated by chord diagrams of order five or six. After that we computed the hermite normal form of $\tilde{A}$, where again $\tilde{A}$ is the matrix obtained from the relation matrix A by adding extra rows to it, as explained in the previous section. If the rank of the new matrix $\tilde{A}$ is full(i.e equal to the number of columns in $\tilde{A}$), and the number of elements in the set is equal to the dimension of rational basis, we conclude that it is in fact an integer basis. We did same for order 6 chord diagrams. To make sure the program doesn't have an error we check whether the rank of $\tilde{A}_{\text{ mod } p}$ is equal to the number of columns in that matrix for the first $10,000$ primes. This was also checked in a separate program written by J. Conant. By doing that we got the following basis set.

For Bar-Natan [2] basis, we computed the hermite normal form of the relation matrix $\tilde{A}$ and we got full column rank. This confirm Bar-Natan basis is a basis over $\mathbb{Z}$. Here is his basis :

Table 3.3: Basis chord diagrams of order up to six over $\mathbb{Z}$

CD2	CD3	CD4	CD5	CD6
$CD_{2,1} = [1212]$	$CD_{3,1} = [121323]$	$CD_{4,5} = [12314234]$ $CD_{4,6} = [12314324]$ $CD_{4,7} = [12341234]$	$CD_{5,28} = [1231435425]$ $CD_{5,29} = [1231452345]$ $CD_{5,33} = [1231453254]$ $CD_{5,34} = [1234152345]$	$CD_{6,281} = [123415263546]$ $CD_{6,283} = [123415264356]$ $CD_{6,284} = [123415623456]$ $CD_{6,287} = [123415632456]$ $CD_{6,294} = [123415632465]$ $CD_{6,295} = [123415643265]$ $CD_{6,296} = [123451623456]$ $CD_{6,298} = [123451624356]$ $CD_{6,300} = [123456123456]$

Table 3.4: Dror Bar-Natan's basis chord diagrams of order up to six over $\mathbb{Z}$

CD2	CD3	CD4	CD5	CD6
$CD_{2,1} = [1212]$	$CD_{3,1} = [121323]$	$CD_{4,1} = [12123434]$ $CD_{4,2} = [12132434]$ $CD_{4,5} = [12314234]$	$CD_{5,1} = [1212343545]$ $CD_{5,3} = [1213243545]$ $CD_{5,9} = [1213425345]$ $CD_{5,27} = [1231425345]$	$CD_{6,1} = [121323454656]$ $CD_{6,31} = [121234354656]$ $CD_{6,32} = [121324354656]$ $CD_{6,34} = [121234345656]$ $CD_{6,47} = [121234536456]$ $CD_{6,87} = [121342536456]$ $CD_{6,88} = [121324536456]$ $CD_{6,180} = [123142536456]$ $CD_{6,195} = [123142563456]$

For order seven we got one basis over $\mathbb{Z}_p$, when p is odd prime and another basis when p equal to 2. The odd prime basis was tested for first 1000 odd primes and confirmed it is a basis over those primes. Both bases (over $\mathbb{Z}_2$, over $\mathbb{Z}_p$, p is odd prime) serve as bases over $\mathbb{Q}$. Also we checked Bar-Natan basis for order seven and it is not a basis over $\mathbb{Z}_2$.

Table 3.5: Basis chord diagrams of order seven over $\mathbb{Z}_2$ and $\mathbb{Z}_p$

$CD7 \mod 2$	$CD7 \mod p$
$CD_{7,3180} = [12345162753647]$	$CD_{7,3180} = [12345162753647]$
$CD_{7,3182} = [12345163752647]$	$CD_{7,3182} = [12345163752647]$
$CD_{7,3183} = [12345162734657]$	$CD_{7,3183} = [12345162734657]$
$CD_{7,3184} = [12345162743657]$	$CD_{7,3184} = [12345162743657]$
$CD_{7,3188} = [12345162743567]$	$CD_{7,3188} = [12345162743567]$
$CD_{7,3189} = [12345162745367]$	$CD_{7,3189} = [12345162745367]$
$CD_{7,3190} = [12345162754367]$	$CD_{7,3190} = [12345162754367]$
$CD_{7,3196} = [12345167324567]$	$CD_{7,3196} = [12345167324567]$
$CD_{7,3199} = [12345167245376]$	$CD_{7,3199} = [12345167245376]$
$CD_{7,3204} = [12341567234765]$	$CD_{7,3204} = [12341567234765]$
$CD_{7,3207} = [12341567432765]$	$CD_{7,3207} = [12341567432765]$
$CD_{7,3211} = [12345167324576]$	$CD_{7,3211} = [12345167324576]$
$CD_{7,3214} = [12345617234567]$	$CD_{7,3214} = [12345617234567]$
$CD_{7,3217} = [12345617243657]$	$CD_{7,3218} = [12345671234567]$

There are many different bases for chord diagrams of order seven over $\mathbb{Z}_2$ and $\mathbb{Z}_p$. However above we tried to choose a basis such that the difference between the two categories is minimal and both are bases over $\mathbb{Q}$. In fact the only difference in the above two categories is the last element.

3.6 Basis for Vassiliev invariants of order up to six over $\mathbb{Z}$

Now we have bases for $\frac{\mathbf{A}_n}{<4T,1T>}$ over $\mathbb{Z}$ up to order 6. Next we need to find a set of integer valued Vassiliev invariants which forms a basis up to order 6. We are going to stick with Ted Stanfords desirable conditions for a basis which we mentioned in the background section.

According to Bar-Natan[1] neglecting the constant map, $dim(V_6/V_1) = 19$. Therefore any Vassiliev invariant v of order less than six on a knot K can be written as

$$v(K) = v(U) + p_1 v(CD_{2,1}) + p_2 v(CD_{3,1}) + \cdots p_{19} v(CD_{6,300})$$

Here the set $\{CD_{2,1}, CD_{3,1}, \cdots, CD_{6,300}\}$ is the basis chord diagrams over $\mathbb{Z}$ from the table 3.3. Thus, $p_1, p_2, \cdots, p_{19} \in \mathbb{Z}$. It should be notice that value of $v(CD_{i,j})$ depend on the singular knot K in the embedding $\sigma(K) \to CD_{i,j}$ when the order of $CD_{i,j}$ is less than the order of Vassiliev invariant v. The embedding we used for each $CD_{i,j}$ is given in Figure 3.12 . Next we have to find nineteen linearly independent integer Vassiliev invariants which make a basis over $\mathbb{Z}$. It is evident that we have to evaluate Vassiliev invariants over all the above chord diagrams.

First we evaluate T. Kanenobu's[21] invariants of order up to six over our basis chord diagrams in Figure 3.12 and 3.12. 3.3 using the Mathematica program in the appendix A. By doing that we got the table 3.6. When we take the determinant of the matrix(from table 3.6), it is non-zero, which confirms that T. Kanenobu's invariants form a basis over $\mathbb{Q}$. However when we calculate the rank of that matrix over $\mathbb{Z}_2$ we got 7. Since $dim(V_6/V_1) = 19$ we conclude that T. Kanenobu's basis is not a basis over $\mathbb{Z}$.

Table 3.6: T. Kanenobu basis for Vasislive invariants of order less than six over $\mathbb{Q}$

$Invariant$	c_2	$\frac{p_0^{(3)}}{12}$	c_2^2	c_4	$\frac{p_0^{(4)}}{24}$	$\frac{(c_2 p_0^{(3)})}{24}$	$\frac{p_0^{(5)}}{120}$	$p_4^{(1)}$	$\frac{f_4^{(1)}}{\sqrt{-1}}$	c_2^3	$(c_2 * c_4)$	$\frac{(c_2 * p_0^{(4)})}{24}$	$(\frac{p_0^{(3)}}{24})^2$	$\frac{p_0^{(6)}}{6!}$	$\frac{p_4^{(2)}}{2}$	c_6	$\frac{f_4^{(2)}}{2}$	$f_5^{(1)}$
$CD_{2,1}$	1	3	1	0	−25	3	44	0	0	1	0	−25	9	−70	0	0	0	0
$CD_{3,1}$	0	2	2	0	−20	4	38	0	0	0	0	−30	10	−63	0	0	0	0
$CD_{4,5}$	0	0	4	0	−32	−2	32	0	2	6	0	−56	24	−64	0	0	9	0
$CD_{4,6}$	0	0	4	0	−16	4	32	0	0	0	0	−36	12	−56	0	0	0	0
$CD_{4,7}$	0	0	6	1	−48	26	160	−4	−2	24	3	−342	134	−392	10	0	11	0
$CD_{5,28}$	0	0	0	0	0	−10	32	8	12	6	2	52	−4	16	8	1	62	10
$CD_{5,29}$	0	0	0	0	0	−20	−64	4	2	−12	−2	184	−72	160	−10	0	17	2
$CD_{5,33}$	0	0	0	0	0	−12	−32	0	2	0	0	8	0	48	0	0	5	−2
$CD_{5,34}$	0	0	0	0	0	30	96	−2	−2	18	4	−300	116	−272	7	0	−9	−2
$CD_{6,281}$	0	0	0	0	0	0	0	0	0	36	8	−240	112	−64	16	0	20	−2
$CD_{6,283}$	0	0	0	0	0	0	0	0	0	36	4	−272	128	−64	8	0	8	0
$CD_{6,284}$	0	0	0	0	0	0	0	0	0	54	7	−432	192	−192	12	1	36	4
$CD_{6,287}$	0	0	0	0	0	0	0	0	0	48	4	−352	160	−128	8	0	28	2
$CD_{6,294}$	0	0	0	0	0	0	0	0	0	36	0	−272	120	−128	0	0	32	−2
$CD_{6,295}$	0	0	0	0	0	0	0	0	0	36	0	−144	72	−64	0	0	20	2
$CD_{6,296}$	0	0	0	0	0	0	0	0	0	72	8	−576	256	−256	16	0	24	−2
$CD_{6,298}$	0	0	0	0	0	0	0	0	0	60	4	−464	208	−192	4	0	24	−4
$CD_{6,300}$	0	0	0	0	0	0	0	0	0	90	15	−720	320	−320	36	1	52	2

Table 3.7: Basis Vasislive invariants of order up to six over $\mathbb{Z}$

$Invariants$	$w21$	$w31$	$w41$	$w42$	$w43$	$w51$	$w52$	$w53$	$w54$	$w61$	$w62$	$w63$	$w64$	$w65$	$w66$	$w67$	$w68$	$w69$
$CD_{2,1}$	1	−1	0	0	0	0	0	−1	0	0	0	0	0	0	0	1	8	−3
$CD_{3,1}$	0	−1	0	1	0	0	0	−1	0	0	0	−1	0	0	0	1	5	−2
$CD_{4,5}$	0	0	0	1	−1	0	−1	3	−1	0	0	−1	0	2	0	4	−60	11
$CD_{4,6}$	0	0	0	2	−1	0	0	0	0	0	0	−2	0	0	0	1	−10	2
$CD_{4,7}$	0	0	1	−1	−1	−2	1	−10	9	0	0	−3	−1	6	3	12	−139	19
$CD_{5,28}$	0	0	0	0	0	4	−6	5	1	1	5	−6	9	11	10	−5	403	−115
$CD_{5,29}$	0	0	0	0	0	2	−1	10	−6	0	1	0	1	0	4	−5	117	−29
$CD_{5,33}$	0	0	0	0	0	0	−1	6	−3	0	−1	−4	0	2	1	3	−64	13
$CD_{5,34}$	0	0	0	0	0	−1	1	−15	9	0	−1	−4	0	−4	−3	7	54	−13
$CD_{6,281}$	0	0	0	0	0	0	0	0	0	0	−1	−3	4	7	5	9	−5	−16
$CD_{6,283}$	0	0	0	0	0	0	0	0	0	0	0	0	2	−7	2	13	2	−11
$CD_{6,284}$	0	0	0	0	0	0	0	0	0	1	2	−4	3	6	9	20	−4	−30
$CD_{6,287}$	0	0	0	0	0	0	0	0	0	0	1	4	2	−3	7	18	−2	−24
$CD_{6,294}$	0	0	0	0	0	0	0	0	0	0	−1	10	0	0	8	17	−5	−27
$CD_{6,295}$	0	0	0	0	0	0	0	0	0	0	1	18	0	−2	5	9	−8	−18
$CD_{6,296}$	0	0	0	0	0	0	0	0	0	0	−1	−2	4	0	6	28	−2	−32
$CD_{6,298}$	0	0	0	0	0	0	0	0	0	0	−2	6	1	−7	6	25	0	−28
$CD_{6,300}$	0	0	0	0	0	0	0	0	0	1	1	−15	9	22	13	30	−11	−48

Next give our bases:

Proposition 3.6.1. *Let W_n denote the integer valued Vassiliev invariants of order $\leq n$ and let V_n denote the rational valued Vassiliev invariants of order $\leq n$. Then,*

1. *The Vassiliev invariants $\{w_{2,1}, w_{3,1}, \cdots, w_{6,9}\}$ form a basis for W_6 over $\mathbb{Z}$, where the invariants $w_{i,j}$ are given in the table 3.8. The value of $w_{i,j}$ over basis chord diagrams is given in the table 3.7. Moreover for each $2 \leq i \leq 6$, the set $\{w_{2,1}, w_{3,1}, ...w_{i,dim(W_i)}\}$ forms a basis for W_i over $\mathbb{Z}$.*

2. *The Vassiliev invariants $c_2, \frac{j_3}{6}, c_4, \frac{f_3^{(1)}}{2}, pc_4, \frac{p_4^{(1)}}{2}, \frac{f_4^{(1)}}{2\sqrt{-1}}, \frac{(c_2 * j_3)}{6}, \frac{p_2^{(3)}}{24}, c_6, \frac{f_5^{(1)}}{2}, \frac{(c_2 * f_3^{(1)})}{2}, \frac{p_4^{(2)}}{2}, \frac{f_3^{(3)}}{6}, \frac{f_4^{(2)}}{2}, \frac{(c_2 * p_2^{(2)})}{2}, \frac{c_2 * (pc_4 - (j_3/6))}{2}, \frac{p_2^{(4)}}{24}$ are integer valued and form a basis for V_6 over $\mathbb{Z}_p$ for all primes $p \geq 5$ and over $\mathbb{Q}$. Moreover for each $2 \leq i \leq 6$, the set $\{c_2, \frac{j_3}{6}, ...b_{dim(V_i)}\}$ form a basis for V_i over $\mathbb{Z}_p$ for $p \geq 5$ and over $\mathbb{Q}$. Values of invariants over basis chord diagrams are given in the table 3.9.*

3. *The Vassiliev invariants $c_2, \frac{j_3}{6}, c_4, \frac{f_3^{(1)}}{2}, pc_4, \frac{p_4^{(1)}}{2}, \frac{f_4^{(1)}}{2\sqrt{-1}}, \frac{(c_2 * j_3)}{6}, \frac{cj_{2,5}}{16}, c_6, (c_2 * c_4), \frac{p_0^{(6)}}{720}, \frac{p_2^{(4)}}{24}, \frac{f_1^{(5)}}{120}, \frac{f_2^{(4)}}{24}, \frac{f_3^{(3)}}{6}, \frac{(c_2 * p_0^{(4)})}{24}, pc_6$ are integer valued and form a basis for V_6 over $\mathbb{Z}_p$ for all primes p such that $p \neq 2$ or $p \neq 7$. This invariant set is also forms a basis for V_6 over $\mathbb{Q}$. Values of invariants over basis chord diagrams are given in the table 3.10*

The Hermite normal form of table 3.8 is the identity. Thus, it is a basis for W_6 over $\mathbb{Z}$. The main reason behind second and third parts of above proposition is complexity of invariants $w_{i,j}$. We also noticed that the complexity of $w_{i,j}$ arise because it forms a basis over $\mathbb{Z}_2$.

Table 3.8: Basis for Vassiliev invariants up to order six over $\mathbb{Z}$

order 2	$w_{2,1} := c_2$
order 3	$w_{3,1} := (j_3/6)$
order 4	$w_{4,1} := c_4$
	$w_{4,2} := (f_3^{(1)})/2$
	$w_{4,3} := (pc_4 - w_{3,1})/2$
order 5	$w_{5,1} := p_4^{(1)}/2$
	$w_{5,2} := f_4^{(1)}/2\sqrt{-1}$
	$w_{5,3} := w_{2,1} * w_{3,1}$
	$w_{5,4} := 1/18(-w_{2,1} - 12w_{5,3} + 52w_{4,1} - w_{4,2} - 16w_{5,2} + 10w_{3,1} + (p_2^{(3)}/24) - 2w_{5,1} - 3w_{4,3})$
order 6	$w_{6,1} := c_6$
	$w_{6,2} := f_5^{(1)}/2$
	$w_{6,3} := w_{2,1} * w_{4,2}$
	$w_{6,4} := ((p_4^{(2)}/2) + 7w_{5,1})/4$
	$w_{6,5} := ((f_3^{(3)}/6) + w_{4,2})/8$
	$w_{6,6} := ((f_4^{(2)}/2) + 56(w_{3,1}) + 264w_{4,1} - 20w_{5,1} - 71w_{5,2} - 56w_{5,3} - 88w_{5,4})/4$
	$w_{6,7} := ((c_2 * p_2^{(2)})/2 + 4w_{3,1} + 24w_{4,1} - 4w_{5,1} - 8w_{5,2} - 5w_{5,3} - 8w_{5,4})/4$
	$w_{6,8} := (12180w_{6,1} - 1264w_{6,2} + 666w_{6,3} + 11312w_{6,4} - 1224w_{6,5} + 1176w_{6,6} - 630w_{6,7} + 2502(w_{2,1} * w_{4,3}) + 463(p_2^{(4)}/24)$ $+589w_{2,1} - 3408w_{3,1} - 16598w_{4,1} + 607w_{4,2} + 1225w_{4,3} + 1290w_{5,1} + 5196w_{5,2} + 3954w_{5,3} + 6144w_{5,4})/216$
	$w_{6,9} := (-12552w_{6,1} + 1184w_{6,2} - 1044w_{6,3} - 11728w_{6,4} + 720w_{6,5} - 2064w_{6,6} - 180w_{6,7} - 2556(w_{2,1} * w_{4,3}) - 473(p_2^{(4)}/24)$ $+1471w_{2,1} - 16566w_{3,1} - 79886w_{4,1} + 1291w_{4,2} + 2953w_{4,3} + 6384w_{5,1} + 24024w_{5,2} + 18084w_{5,3} + 28212w_{5,4})/864$

Table 3.9: Basis for Vasisliev invariants of order up to six over $\mathbb{Q}$ and $\mathbb{Z}_p$, $p \geq 5$

$Invariants$	c_2	$j_3/6$	c_4	$f_3^{(1)}/2$	pc_4	$p_4^{(1)}/2$	$f_4^{(1)}/2I$	$c2*(j_3/6)$	$(p_2^{(3)}/24$	c_6	$f_5^{(1)}/2$	$c_2*(f_3^{(1)}/2$	$p_4^{(2)}/2$	$f_3^3/6$	$f_4^{(2)}/2$	$c2*(p_2^2/2)$	$w21)*(w43)$	$(p_2^4/24$
$CD_{2,1}$	1	-1	0	0	-1	0	0	-1	-1	0	0	0	0	0	0	3	0	5
$CD_{3,1}$	0	-1	0	1	-1	0	0	-1	-1	0	0	-1	0	-1	0	3	0	5
$CD_{4,5}$	0	0	0	1	-2	0	-1	3	0	0	0	-1	0	15	9	15	-4	6
$CD_{4,6}$	0	0	0	2	-2	0	0	0	-1	0	0	-2	0	-2	0	4	-1	5
$CD_{4,7}$	0	0	1	-1	-2	-2	1	-10	-2	0	0	-3	10	49	11	46	-1	-6
$CD_{5,28}$	0	0	0	0	0	4	-6	5	-10	1	5	-6	8	88	62	-19	-2	-28
$CD_{5,29}$	0	0	0	0	0	2	-1	10	0	0	1	0	-10	0	17	-18	-3	32
$CD_{5,33}$	0	0	0	0	0	0	-1	6	2	0	-1	-4	0	16	5	10	-3	-4
$CD_{5,34}$	0	0	0	0	0	-1	1	-15	-4	0	-1	-4	7	-32	-9	29	5	8
$CD_{6,281}$	0	0	0	0	0	0	0	0	0	0	-1	-3	16	56	20	36	-6	-48
$CD_{6,283}$	0	0	0	0	0	0	0	0	0	0	0	0	8	-56	8	52	-7	-16
$CD_{6,284}$	0	0	0	0	0	0	0	0	0	1	2	-4	12	48	36	80	-10	-16
$CD_{6,287}$	0	0	0	0	0	0	0	0	0	0	1	4	8	-24	28	72	-10	0
$CD_{6,294}$	0	0	0	0	0	0	0	0	0	0	-1	10	0	0	32	68	-9	32
$CD_{6,295}$	0	0	0	0	0	0	0	0	0	0	1	18	0	-16	20	36	-9	16
$CD_{6,296}$	0	0	0	0	0	0	0	0	0	0	-1	-2	16	0	24	112	-14	0
$CD_{6,298}$	0	0	0	0	0	0	0	0	0	0	-2	6	4	-56	24	100	-13	32
$CD_{6,300}$	0	0	0	0	0	0	0	0	0	1	1	-15	36	176	52	120	-15	-80

Table 3.10: Basis for Vasisliev invariants of order up to six over $\mathbb{Q}$ and $\mathbb{Z}_p$, $p \neq 5, p \neq 7$

$Invariants$	c_2	$j_3/6$	c_4	$f_3^{(1)}/2$	pc_4	$p_4^{(1)}/2$	$f_4^{(1)}/2I$	$c_2*(j_3/6)$	$cj_{2,5}/16$	c_6	c_2*c_4	$p_0^{(6)}/720$	$p_2^{(4)}/24$	$f_1^{(5)}/120$	$f_2^{(4)}/24$	$f_3^{(3)}/6$	$c_2*(p_0^{(4)}/24)$	pc_6
$CD_{2,1}$	1	-1	0	0	-1	0	0	-1	11	0	0	-70	5	105	30	0	-25	0
$CD_{3,1}$	0	-1	0	1	-1	0	0	-1	11	0	0	-63	5	104	25	-1	-30	0
$CD_{4,5}$	0	0	0	1	-2	0	-1	3	-36	0	0	-64	6	142	52	15	-56	-2
$CD_{4,6}$	0	0	0	2	-2	0	0	0	0	0	0	-56	5	104	21	-2	-36	0
$CD_{4,7}$	0	0	1	-1	-2	-2	1	-10	100	0	3	-392	-6	1078	449	49	-342	-5
$CD_{5,28}$	0	0	0	0	0	4	-6	5	-40	1	4	16	-28	-128	-36	88	52	1
$CD_{5,29}$	0	0	0	0	0	2	-1	10	-100	0	-2	160	32	-560	-260	0	184	2
$CD_{5,33}$	0	0	0	0	0	0	-1	6	-76	0	0	48	-4	-64	8	16	8	0
$CD_{5,34}$	0	0	0	0	0	-1	1	-15	172	0	4	-272	8	832	240	-32	-300	-2
$CD_{6,281}$	0	0	0	0	0	0	0	0	0	0	6	-64	-48	512	256	56	-240	-6
$CD_{6,283}$	0	0	0	0	0	0	0	0	0	0	4	-64	-16	480	80	-56	-272	-8
$CD_{6,284}$	0	0	0	0	0	0	0	0	0	1	7	-192	-16	1024	448	48	-432	-12
$CD_{6,287}$	0	0	0	0	0	0	0	0	0	0	4	-128	0	704	176	-24	-352	-12
$CD_{6,294}$	0	0	0	0	0	0	0	0	0	0	0	-128	32	544	96	0	-272	-12
$CD_{6,295}$	0	0	0	0	0	0	0	0	0	0	0	-64	16	288	32	-16	-144	-12
$CD_{6,296}$	0	0	0	0	0	0	0	0	0	0	8	-256	0	1312	480	0	-576	-16
$CD_{6,298}$	0	0	0	0	0	0	0	0	0	0	4	-192	32	928	176	-56	-464	-16
$CD_{6,300}$	0	0	0	0	0	0	0	0	0	1	15	-320	-80	1824	992	176	-720	-16

3.7 The conjecture regarding primitive Conway Vassiliev invariants

Conjecture 3.7.1. *[7] There exists integer valued Vassiliev invariants, v_{4n-1}, of order $4n-1$ such that*

$$pc_{4n} \equiv v_{4n-1} \mod 2$$

Furthermore v_{4n-1} can be chosen as odd invariant. That is,if mK denote the mirror image of the knot, $v_{4n-1}(mK) = -v_{4n-1}(K)$.

The first few primitive Conway Vassiliev invariants are defined as follows,

$$pc_2 = -c_2 \tag{3.3}$$

$$pc_4 = c_4 - \frac{c_2 + c_2^2}{2} \tag{3.4}$$

$$pc_6 = -c_6 + c_2c_4 + \frac{c_2 - c_2^3}{3} \tag{3.5}$$

$$pc_8 = c_8 + c_2(c_2c_4 - c_6) + \frac{(c_4 - c_4^2)}{2} - \frac{(c_2^4 + c_2^2 + 2c_2)}{4} \tag{3.6}$$

Matt Dawson[11] explicitly calculated Vassiliev invariants, which are congruent to $pc_4 \mod 2$ and $pc_6 \mod 2$. His results can be stated as follows.

Theorem 3.7.1. *[11]*

$$pc_4 = \frac{1}{6}j_3 \mod 2 \tag{3.7}$$

$$pc_6 = -\frac{1}{8}j_3 + 4 * j_4 - \frac{1}{2}j_5 + k_5^{(2)} \mod 2 \text{ on all knots up to 11 crossings} \tag{3.8}$$

where j_3 is the third coefficient of power series expansion of the logarithm of the Jones polynomial, j_4, j_5 are the fourth and fifth coefficient of the same polynomial. k_5^2 is obtained as follows. Take the logarithm of the Kauffman polynomial. Then take the power series expansion of this polynomial. Suppose $K[a, z]$ is the power series expansion of the logarithm of kauffman polynomial. Make the substitution $a =$

$-ze^p - \sqrt{-1}$ *Once this substitution made, the coefficient of* z^5 *is of the form*

$$2\sqrt{-1}k_5^{(1)}e^p + \sqrt{-1}k_5^{(2)}e^{2p} + 2\sqrt{-1}k_5^{(3)}e^{3p} + 2\sqrt{-1}k_5^{(4)}e^{4p} + \sqrt{-1}k_5^{(5)}e^{5p}$$

and $k_5^{(2)}$ *is the above mentioned invariant*

We will prove that Matt Dawson's invariant is congruent to pc_6 mod 2 for all knots. We evaluated Dawson invariant and $j_3/6$ over basis chord diagrams and we got table 3.11. The table proves the validity of Dawson's assertion for all knots.

Table 3.11: M. Dawson's invariants of order three and five

Invariants	pc_2	pc_4	pc_6	$j_3/6$	Dawson invariant
$CD_{2,1}$	-1	-1	0	-1	-30
$CD_{3,1}$	0	-1	0	-1	-44
$CD_{4,5}$	0	-2	-2	0	-224
$CD_{4,6}$	0	-2	0	0	-122
$CD_{4,7}$	0	-2	-5	0	-91
$CD_{5,28}$	0	0	1	0	-105
$CD_{5,29}$	0	0	2	0	-172
$CD_{5,33}$	0	0	0	0	-82
$CD_{5,34}$	0	0	-2	0	226
$CD_{6,281}$	0	0	-6	0	0
$CD_{6,283}$	0	0	-8	0	0
$CD_{6,284}$	0	0	-12	0	0
$CD_{6,287}$	0	0	-12	0	0
$CD_{6,294}$	0	0	-12	0	0
$CD_{6,295}$	0	0	-12	0	0
$CD_{6,296}$	0	0	-16	0	0
$CD_{6,298}$	0	0	-16	0	0
$CD_{6,300}$	0	0	-16	0	0

Here we give another invariant which is congruent to pc_6. However neither this invariant nor Matt Dawson's invariant satisfies the odd invariant condition.

Proposition 3.7.1.

Let $w_{2,1}$, $w_{5,1}$ *and* $w_{5,3}$ *be the integer Vassiliev invariants defined in the table 3.8. Then,*

$$pc_6 \equiv w_{2,1} + w_{4,2} + w_{5,1} + w_{5,3} \mod 2$$

Table 3.12: Basis chord diagrams and their singular knot representation

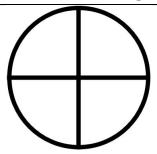
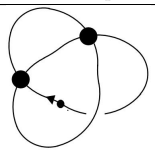
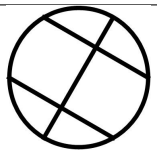
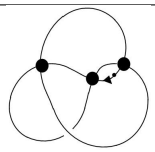
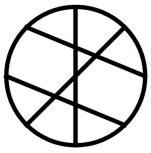
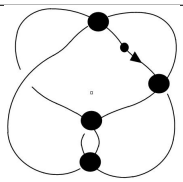
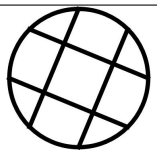
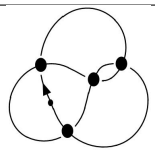
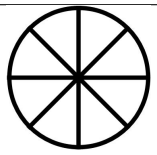
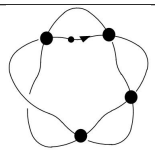
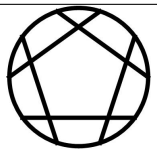
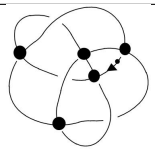
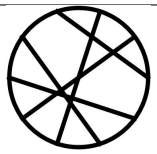
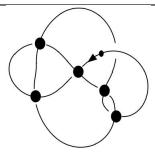
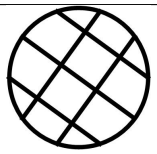
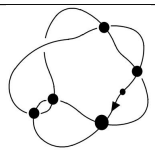
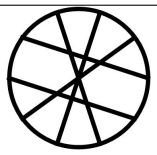
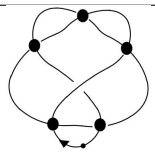
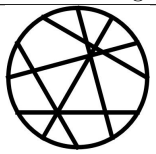
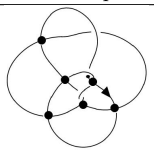
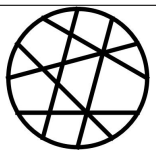
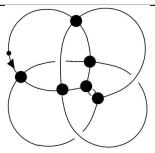
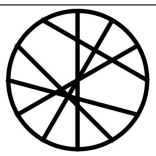
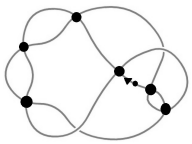
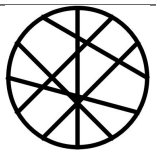
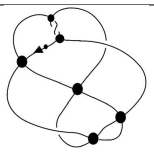
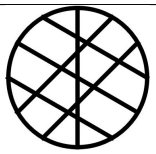
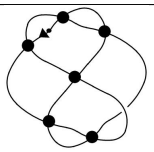
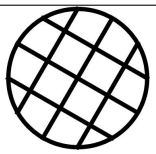
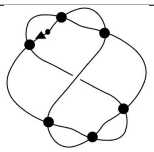
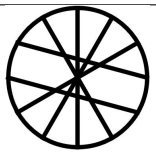
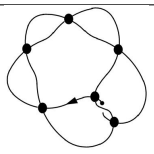
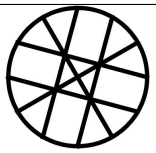
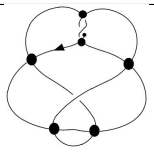
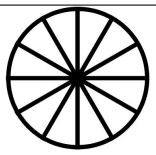
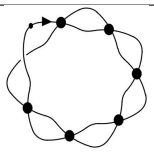
Index	GaussCode	Gauss Code Diagram	Singular Knot Representation
$CD_{2,1}$	1212		
$CD_{3,1}$	121323		
$CD_{4,5}$	12314234		
$CD_{4,6}$	12314324		
$CD_{4,7}$	12341234		
$CD_{5,28}$	1231435425		
$CD_{5,29}$	1231452345		
$CD_{5,33}$	1231453254		
$CD_{5,34}$	1234152345		

Table 3.12: Basis chord diagrams and their singular knot representation cont..

Index	Gauss Code	Gauss Code Diagram	Singular Knot Representation
$CD_{6,281}$	123415263546		
$CD_{6,283}$	123415264356		
$CD_{6,284}$	123415623456		
$CD_{6,287}$	123415632456		
$CD_{6,294}$	123415632465		
$CD_{6,295}$	123415643265		
$CD_{6,296}$	123451623456		
$CD_{6,298}$	123451624356		
$CD_{6,300}$	123456123456		

Chapter 4

The Coefficients of The Jones Polynomial

4.1 Introduction

It has been known that coefficients of certain knot polynomials are Vassiliev invariants. For example the n^{th} coefficient of the Conway polynomial is a Vassiliev invariant of order n [1]. Vassiliev invariants are an important tool to distinguish knots. It has been conjectured that the set of all rational valued Vassiliev invariants are complete. If this is true, then we can use rational valued Vassiliev invariants to approximate any other rational valued knot invariant. When we expand the Jones polynomial evaluated at e^x as a power series, the coefficients become rational valued Vassiliev invariants. Using this fact, Kofman and Rong [25] proved that each coefficient of the Jones polynomial of a knot is the limit of a sequence of Vassiliev invariants. However these coefficients are not additive under connected sum, or in other words are not primitive Vassiliev invariants. Since the Jones polynomial, $J_k(t)$, is multiplicative under connected sum, we can take the logarithm of $J_k(e^x)$ and expand it as a power series. However this turns integer coefficients of Jones polynomial, $J_k(t)$, into rational valued ones. Whenever we have rational valued Vassiliev invariants we

can construct an integer valued invariant by multiplying by a suitable "large" integer. However if we do that resulting invariants tend to have common factors. In this chapter we calculate minimal multiplying factor, λ, needed for the coefficient of the logarithm of the Jones polynomial, $\log(J_k(e^x))$ to become integer valued. By doing that we obtain a set of primitive integer-valued Vassiliev invariants.

4.2 Background

Definition 4.2.1. *The Jones polynomial $J_K(t)$ is defined as the polynomial satisfying the following* skein relation

$$t^{-1}J_K\Big(\times\Big) - tJ_K\Big(\times\Big) = \big(t^{1/2} - t^{-1/2}\big)J_K\Big(\frown\frown\Big). \tag{4.1}$$

$$J_K(unknot) = 1 \tag{4.2}$$

Let the Jones polynomial of a knot K be $J_K(t)$ where ,

$$\begin{aligned} J_k(t) &= \Sigma_{i=-m}^{n} c_i t^i \\ &= c_{-m}t^{-m} + c_{-m+1}t^{-m+1} + ... + c_{-1}t^{-1} + c_0 + c_1 t^1 + ... + c_n t^n \quad \text{where } c_i \in \mathbb{Z} \end{aligned}$$

Let $t = e^x$ then $J_K(e^x) = \Sigma_{k=0}^{\infty} a_k x^k$ is the Jones polynomial of knot K expanded as a power series in x. It is known that $a_0 = 1, a_1 = 0$ and for $k > 1$, $a_k = \frac{1}{k!}\Sigma_{i=-m, i\neq 0}^{n} c_i i^k$ is a rational valued Vassiliev invariant. We can convert a_k to be integer valued by multiplying by a suitable constant λ_k. For example, we can choose $\lambda_k = k!$ and then $\lambda_k a_k$ is an integer valued Vassiliev invariant. In this chapter we calculate minimal λ_k such that $\lambda_k a_k$ is an integer valued Vassiliev invariant. First we define what we meant by "minimal" λ_k. By "minimal" λ_k, we mean the unique positive constant λ_k which makes the values of $\lambda_k a_k$ integers and makes it so that not all values of $\lambda_k a_k$ have a common factor. The formal definition as follows:

Definition 4.2.2. *Let $\mathcal{K}$ denote the set of equivalence classes of knots. Let,*

$$q_k = lcm\{q_i | a_k(K_i) = \frac{p_i}{q_i}, \gcd(p_i, q_i) = 1, K_i \in \mathcal{K}\}$$
$$p_k = \gcd\{p_i | a_k(K_i) = \frac{p_i}{q_i}, \gcd(p_i, q_i) = 1, K_i \in \mathcal{K}\} \quad \textit{then,}$$
$$\lambda_k = \frac{q_k}{p_k}$$

The existence of q_k and p_k over infinite set $\mathcal{K}$ may come in to the question at this point. However it is due to the fact that a_k are Vassiliev invariants and the value of Vassiliev invariant over any given knot is a integer linear combination of its value on only finite number of singular knots.

It is clear from the definition $\lambda_k a_k \in \mathbb{Z}$, $\gcd(p_k, q_k) = 1$ and $k!$ is an upper bound of λ_k. One way to calculate λ_k is actuality tables. Suppose v_n is a Vassiliev invariant of order $\leq n$. To construct actuality tables one must choose a singular knot for every basis chord diagram of order $\leq n$. Then one must calculate the value of a given invariant over all those basis knot diagrams. This can be done easily if n is relatively small. For example, from actuality tables, we can see that $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = \frac{1}{6}$. However this method is not feasible when $k \geq 4$. So, from this point onward we are interested about a_k, and λ_k, only when $k \geq 4$. The following lemma gives more insight to the above definition.

Lemma 4.2.1. *Suppose there exists λ'_k and two knots $K_1, K_2 \in \mathcal{K}$ such that,*

1. $\lambda'_k a_k \in \mathbb{Z}$
2. $\gcd(\lambda'_k a_k(K_1), \lambda'_k a_k(K_2)) = 1$

Then λ'_k is minimal.

Proof. Let $\lambda_k = \frac{q_k}{p_k}$ and $\lambda'_k = \frac{q'_k}{p'_k}$, where $\gcd(p_k, q_k) = 1, \gcd(p'_k, q'_k) = 1$. Then from the definition of λ_k, it is clear that $q_k | q'_k$ and $p'_k | p_k$. We can easily prove that, $\lambda'_k = r\lambda_k$ for some $r \in \mathbb{Z}$. Since $\lambda_k a_k(K_1), \lambda_k a_k(K_2) \in \mathbb{Z}$, we conclude $\gcd(\lambda'_k a_k(K_1), \lambda'_k a_k(K_2)) = r = 1$. Hence $\lambda'_k = \lambda_k$. □

The following lemma lists some known results about coefficients of Jones polynomial. We use this lemma extensively in subsequent theorems.

Lemma 4.2.2. *Let Jones polynomial of knot K be $J_K(t) = \Sigma_{i=-m}^{n} c_i t^i$. Then,*

1. $J_K(1) = \Sigma_{i=-m}^{n} c_i = 1$
2. $J_K(-1) = \Sigma_{i=-m}^{n} (-1)^i c_i \equiv 1$ *or* $5 \mod 8$
3. $J_K(e^{2\pi i/3}) = 1$
4. $J_K(\sqrt{-1}) = (-1)^{c_2(K)}$ *where $c_2(K)$ is the second coefficient of the Conway polynomial of knot K.*
5. $J_K(1) - J_K(-1) = \Sigma_{i=-m}^{n} (c_i - (-1)^i c_i) \equiv 0 \mod 4$ *which implies* $\Sigma_{i=-m,2\nmid i}^{n} c_i \equiv 0 \mod 2$
6. $\Sigma_{i=-m,3|i}^{n} c_i = \cdots + c_{-6} + c_{-3} + c_0 + c_3 + c_6 + \cdots = 1$
7. $\Sigma_{i=-m,3\nmid i}^{n} (-1)^{(i-1) \mod 3} c_i = \cdots - c_{-4} + c_{-2} - c_{-1} + c_1 - c_2 + c_4 - \cdots = 0$
8. $\Sigma_{i=-m,3\nmid i}^{n} c_i = \cdots + c_{-5} + c_{-4} + c_{-2} + c_{-1} + c_1 + c_2 + c_4 + c_5 + \cdots = 0$
9. $\Sigma_{i=-m,2\nmid i}^{n} (-1)^{(i-1)/2} c_i = \cdots - c_{-5} + c_{-3} - c_{-1} + c_1 - c_3 + c_5 \cdot\cdot = 0$

Proof. We only prove (6), (7), (8) and (9). Others are standard results. Now from (3), we have, $J_K(e^{(2\pi i/3)}) = \Sigma_{i=-m,3|i}^{n} c_i - \frac{1}{2}\Sigma_{i=-m,3\nmid i}^{n} c_i + \frac{i\sqrt{3}}{2}\Sigma_{i=-m,3\nmid i}^{n} (-1)^{(i-1) \mod 3} c_i = 1$. This imply, $\Sigma_{i=-m,3|i}^{n} c_i - \frac{1}{2}\Sigma_{i=-m,3\nmid i}^{n} c_i = 1$ and $\Sigma_{i=-m,3\nmid i}^{n} (-1)^{(i-1) \mod 3} c_i = 0$. Using (1), we can rewrite, $\Sigma_{i=-m,3|i}^{n} c_i - \frac{1}{2}\Sigma_{i=-m,3\nmid i}^{n} c_i = \Sigma_{i=-m,3|i}^{n} c_i - \frac{1}{2}(1 - \Sigma_{i=-m,3|i}^{n} c_i) = 1$. Thus we have $\Sigma_{i=-m,3|i}^{n} c_i = 1$. Substituting this result in (1), we get $\Sigma_{i=-m,3\nmid i}^{n} c_i = 0$. Similarly, $J_K(\sqrt{-1}) = \sqrt{-1}(\Sigma_{i=-m,2\nmid i}^{n} (-1)^{(i-1)/2} c_i) + (c_0 + \Sigma_{i=-m,2|i,i\neq 0}^{n} (-1)^{(i-1)/2} c_i)$ $= (-1)^{c_2(K)}$. This implies the result of (9). □

Now consider the k^{th} coefficient, a_k of the power series expansion of the Jones polynomial of a knot K where $J_K(t) = \Sigma_{i=-m}^{n} c_i t^i$. Then it is clear that when $k > 0$, $a_k = \frac{1}{k!}\Sigma_{i=-m,i\neq 0}^{n} c_i i^k$. The following lemma gives some basic properties of a_k.

Lemma 4.2.3.

1. $(2k)!a_{2k} \equiv 0 \mod 2$, *where* $k \geq 1$.
2. *Let* n *be any non negative integer and* $k \geq 2$, *If* $k \equiv 0 \mod 3^n$ *then* $(k)!a_k \equiv 0 \mod 3^{n+1}$.
3. $(2k+1)!a_{2k+1} \equiv 0 \mod 4$, *where* $k \geq 1$.

Proof.

1. First observe that, $(2k)!a_{2k} \mod 2 \equiv \sum_{i=-m,2\nmid i}^{n} c_i \mod 2$. Now from lemma 4.2.2(5) we conclude that $(2k)!a_{2k} \equiv 0 \mod 2$.
2. If $k \equiv 0 \mod 3^n$ then,

$$i^{2k} \equiv \begin{cases} 0 \mod 3^{n+1}, & \text{if } i \equiv 0 \mod 3 \\ 1 \mod 3^{n+1}, & \text{otherwise} \end{cases}$$

   Hence, $(2k)!a_{2k} \equiv \sum_{i=-m,3\nmid i}^{n} c_i \mod 3^{n+1}$. Now from lemma 4.2.2(8), we conclude that $(2k)!a_{2k} \equiv 0 \mod 3^{n+1}$.

   Similarly,

$$i^{2k+1} \equiv \begin{cases} 1 \mod 3^{n+1}, & \text{if } i \equiv 1 \mod 3 \\ (-1) \mod 3^{n+1}, & \text{if } i \equiv (-1) \mod 3 \\ 0 \mod 3^{n+1}, & \text{otherwise} \end{cases}$$

   Thus, $(2k+1)!a_{2k+1} \equiv \Sigma_{i=-m,3\nmid i}^{n}(-1)^{(i-1) \mod 3} c_i \mod 3^{n+1}$. Therefore, from lemma 4.2.2(7) we conclude that, $(2k+1)!a_{2k+1} \equiv 0 \mod 3^{n+1}$.

3. Since $2k+1 \geq 3$,

$$i^{2k+1} \equiv \begin{cases} 1 \mod 4, & \text{if } i \equiv 1 \mod 4 \\ (-1) \mod 4, & \text{if } i \equiv (-1) \mod 4 \\ 0 \mod 4, & \text{otherwise} \end{cases}$$

Therefore, $(2k+1)!a_{2k+1} \equiv \Sigma_{i=-m,2\nmid i}^{n}(-1)^{(i-1)/2}c_i \mod 4$. Thus from lemma 4.2.2(9) we conclude that $(2k+1)!a_{2k+1} \equiv 0 \mod 4$.

□

4.3 Main Theorem

Now we are ready to prove the main result of this chapter. In the following theorem we give a formula for λ_{2k} when $k \geq 1$.

Theorem 4.3.1. *Let $J_K(e^x) = \sum_{i=0}^{\infty} a_k x^k$ be the Jones polynomial of knot K expanded as a power series in x. When $k \geq 1$, let $\lambda_{2k} = \frac{(2k)!}{2.3^{n+1}}$ where $3^n | k$ but $3^{n+1} \nmid k$ and n is a non negative integer. Then,*

1. *$v_{2k} = \lambda_{2k} a_{2k}(K)$ is an integer valued Vassiliev invariant .*

2. *λ_{2k} is minimal.*

Proof. From lemma 4.2.3, we have $2.3^{n+1} | (2k)!a_{2k}$. Therefore, $v_{2k} = \lambda_{2k} a_{2k} \in \mathbb{Z}$. Next we prove λ_{2k} is minimal. We claim $gcd(v_{2k}(3_1), v_{2k}(4_1)) = 1$ for any given k. First observe that $J_{3_1}(t) = -t^{-4} + t^{-3} + t^{-1}$ and $J_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$. Let $3^n | k, 3^{n+1} \nmid k$ then,

$$\begin{aligned} v_{2k}(3_1) &= \left[\frac{(2k)!}{2.3^{n+1}}\right]\left[\frac{-(-4)^{2k} + (-3)^{2k} + (-1)^{2k}}{(2k)!}\right] \\ &= \frac{-16^k + 9^k + 1}{2.3^{n+1}} \end{aligned}$$

Similarly,

$$\begin{aligned} v_{2k}(4_1) &= \left[\frac{(2k)!}{2.3^{n+1}}\right]\left[\frac{(-2)^{2k}-(-1)^{2k}-(1)^{2k}+(2)^{2k}}{(2k)!}\right] \\ &= \frac{2(4^k-1)}{2.3^{n+1}} \end{aligned}$$

We prove,$\gcd(-16^k+9^k+1, 2(4^k-1)) = 2.3^{n+1}$. From above we have $2.3^{n+1}|(2k)!a_{2k}$. Therefore $2.3^{n+1}|\gcd(-16^k+9^k+1, 2(4^k-1))$. Now observe that, $2^2 \nmid 2(4^k-1)$ thus $2^2 \nmid \gcd(-16^k+9^k+1, 2(4^k-1))$. Next observe that if, $3^{n+2}|\gcd(-16^k+9^k+1, 2(4^k-1))$ then $3^{n+2}|2(4^k-1)$. This imply $3^{n+2}|4^k-1$. But $k = 3^n.q$ where $3 \nmid q$. Thus we have $3^{n+2}|(3+1)^{3^n.q}-1$. This give us $3^{n+2}|3^{n+1}$, hence a contradiction.

Let P be a prime such that $P \neq 2, 3$ and $P|\gcd(-16^k+9^k+1, 2(4^k-1))$. This imply, $P|4^k-1$ and $P|-16^k+9^k+1$. Now observe that, $-16^k+1 = -[4^k-1]-[4^k(4^k-1)]$. Therefore, $P|-16^k+1$. Thus $P|9^k$. This give contradiction, since $P \neq 2, 3$. Hence we conclude, $\gcd(-16^k+9^k+1, 2(4^k-1)) = 2.3^{n+1}$. Thus $\gcd(v_{2k}(3_1), v_{2k}(4_1)) = 1$ for any given k. So, from Lemma 4.2.1, λ_{2k} is minimal for any k.

□

By induction, we can show easily $\lambda_{2k} \in \mathbb{Z}$ when $k \geq 2$. Thus when $k \geq 2$,

$$\begin{aligned} &\text{lcm}\{q_i|a_{2k}(K_i) = \frac{p_i}{q_i}, \gcd(p_i, q_i) = 1, K_i \in \mathcal{K}\} = \frac{(2k)!}{2.3^{n+1}} \quad \text{where } 3^n|k,\ 3^{n+1} \nmid k \\ &\gcd\{p_i|a_{2k}(K_i) = \frac{p_i}{q_i}, \gcd(p_i, q_i) = 1, K_i \in \mathcal{K}\} = 1 \end{aligned}$$

This is nice fact to know given that family of knots is infinite.

Now we should focus on finding a general formula for λ_{2k+1}. However it seems that this is not as easy as the λ_{2k} case. First observe that from Lemma 4.2.3, $\lambda_{2k+1} \leq \frac{(2k+1)!}{12}$. Computational results shows that this is indeed the least upper bound for some k. For example, $\lambda_{35} = 35!/12$, $\lambda_{95} = 95!/12$, $\lambda_{119} = 119!/12$, $\lambda_{143} = 143!/12$.

This is because, when $k = 35, 95, 119, 143$ we can show $\text{lcm}\{\, q_k(3_1), q_k(5_1), q_k(9_1)\} = \frac{k!}{12}$ and $\gcd\{\, p_k(3_1), p_k(5_1), p_k(9_1)\} = 1$. The seeming randomness of when $\lambda_{2k+1} = (2k+1)!/12$ to hampers giving a general formula for λ_{2k+1}.

Let $\mathbb{Z}_{(2)}$ denote the localization of $\mathbb{Z}$ at (2), where (2) is the prime ideal generate by $2 \in \mathbb{Z}$. Then we have following interesting lemma.

Lemma 4.3.1. *When $k \geq 1$,*

1. $2^{k-1}k!a_{2k} \in \mathbb{Z}_{(2)}$
2. $2^{k-2}k!a_{2k+1} \in \mathbb{Z}_{(2)}$

Proof. Using mathematical induction, we can show that $\frac{2^k k!}{(2k)!} \in \mathbb{Z}_{(2)}$. Now $2^{k-1}k!a_{2k} = \frac{2^{k-1}k!}{(2k)!}(2k)!a_{2k}$. Since we proved $(2k)!a_{2k} \equiv 0 \mod 2$ in Lemma 4.2.3, the first part of the lemma follows. Similarly $2^{k-2}k!a_{2k+1} = \frac{2^{k-2}k!}{(2k+1)(2k)!}(2k+1)!a_{2k+1}$ and since we proved $(2k+1)!a_{2k+1} \equiv 0 \mod 4$ in Lemma 4.2.3, second part of the lemma follows. □

The following lemma shows an interesting characteristic of $\lambda_k a_k \mod 2$.

Lemma 4.3.2. *Let $v_k = \lambda_k a_k$. Then $v_{2k} \equiv v_{2k+2} \mod 2$ for all $k \geq 1$.*

Proof. First observe that $v_{2k} = \frac{(2k)!}{2.3^{n+1}} \Sigma_{i=-m, i\neq 0}^{n'} \frac{c_i i^{2k}}{(2k)!} \in \mathbb{Z}$. Thus $v_{2k}3^{n+1} = \frac{1}{2}\Sigma_{i=-m,i\neq 0}^{n'} c_i i^{2k}$ is integer valued and we conclude, $v_{2k} \equiv \frac{1}{2}\Sigma_{i=-m,i\neq 0}^{n'} c_i i^{2k} \mod 2$. Now $v_{2k+2} - v_{2k} \equiv \Sigma_{i=-m,i\neq 0}^{n'} c_i(\frac{i^{2k+2}-i^{2k}}{2}) \mod 2$. It is easy to prove that $i^{2k+2} - i^{2k} \equiv 0 \mod 4$. Hence $v_{2k+2} \equiv v_{2k} \mod 2$. □

If the following conjecture is true, then we can extend the lemma 4.3.2 to odd case using a similar type proof.

Conjecture 4.3.1. *When $k \geq 1$, $(2k+1)!a_{2k+1} \not\equiv 0 \mod 8$*

Now, we give λ_{2k} values and conjectured λ_{2k+1} values.

Table 4.1: Values of λ_{2k} and conjectured λ_{2k+1} up to order 15

$\lambda_2 = \frac{1}{3} = \frac{2!}{2\times 3}$	$\lambda_3 = \frac{1}{6} = \frac{3!}{2^2\times 3^2}$
$\lambda_4 = 4 = \frac{4!}{2\times 3}$	$\lambda_5 = 2 = \frac{5!}{2^2\times 3\times 5}$
$\lambda_6 = 40 = \frac{6!}{2\times 3^2}$	$\lambda_7 = 60 = \frac{7!}{2^2\times 3\times 7}$
$\lambda_8 = 6720 = \frac{8!}{2\times 3}$	$\lambda_9 = 672 = \frac{9!}{2^2\times 3^3\times 5}$
$\lambda_{10} = 604800 = \frac{10!}{2\times 3}$	$\lambda_{11} = 302400 = \frac{11!}{2^2\times 3\times 11}$
$\lambda_{12} = 26611200 = \frac{12!}{2\times 3^2}$	$\lambda_{13} = 1140480 = \frac{13!}{2^2\times 3\times 5\times 7\times 13}$
$\lambda_{14} = 14529715200 = \frac{14!}{2\times 3}$	$\lambda_{15} = 36324288000 = \frac{15!}{2^2\times 3^2}$

These integer valued Vassiliev invariants seemed to satisfies multiple congruence relations.

Proposition 4.3.1.

Let Jones polynomial of knot K be $J_K(t) = \Sigma_{i=-m}^{n} c_i t^i$. Assume, v_7, v_9, v_{11} and v_{15} are integer valued and $\lambda_7, \lambda_9, \lambda_{11}$ and λ_{15} as in the above table. Then,

1. $v_4 \equiv v_8 \mod 5$
2. $v_4 \equiv v_{10} \mod 7$
3. $v_6 \equiv v_{12} \mod 7$
4. $v_3 \equiv v_{15} \mod 13$
5. $v_3 \equiv v_9 \mod 7$
6. $v_7 \equiv v_{11} \mod 24$

Proof. First observe that,

$$\frac{\lambda_4}{4!} = \frac{\lambda_8}{8!}, \frac{\lambda_4}{4!} = \frac{\lambda_{10}}{10!}, \frac{\lambda_6}{6!} = \frac{\lambda_{12}}{12!}, \frac{\lambda_3}{3!} = \frac{\lambda_{15}}{15!}$$

Now $v_4 - v_8 = \frac{1}{6}\sum_{i=-m, i\neq 0}^{n} c_i(i^4 - i^8)$. Since we proved v_{2k} is an integer,

$$6(v_4 - v_8) \mod 5 \equiv (v_4 - v_8) \mod 5 \equiv \sum_{i=-m, i\neq 0}^{n} c_i i^4(1 - i^4) \mod 5.$$

But $i^4(1-i^4) \equiv 0 \mod 5$. Hence the result follows. Similar proofs can be given for (2) and(3).

For (4),

$$v_{15} - v_3 = \frac{1}{2^2 3^2} \sum_{i=-m, i\neq 0}^{n} c_i(i^{15} - i^3) = \frac{1}{2^2 3^2} \sum_{i=-m, i\neq 0}^{n} c_i i^3(i^{12} - 1).$$

But $i^{12} - 1 \equiv 0 \mod 13, i^3(i^{12} - 1) \equiv 0 \mod 4, i^3(i^{12} - 1) \equiv 0 \mod 9$. Hence result follows.

Assume, v_9, v_7 and v_{11} are integer valued invariants. From actuality tables we know that v_3 is an integer valued.

For(5) First observe that $2^2 3^2 v_3 \equiv v_3 \mod 7$ and $2^2 3^3 5 v_9 \equiv v_9 \mod 7$. Now $(v_9 - v_3) \equiv \sum_{i=-m, i\neq 0}^{n} c_i(i^9 - i^3) \mod 7$. But it is easy to prove that $i^9 - i^3 \equiv 0 \mod 7$. Hence result follows.

For(6), Similar proof can be given. □

A careful reader may have noticed that the above invariants, $v_k = \lambda_k a_k$, are integer valued but not primitive ones. Let $J_K(e^x) = 1 + \Sigma_{i=1}^{\infty} a_i x^i$. Then $\log(J_k(e^x)) = \Sigma_{i=2}^{\infty} w_i x^i$ where,

$$|w_i| = \sum_{\Sigma k_j n_j = i} \left(\frac{1}{n_1 + \cdots + n_m} \right) \binom{n_1 + n_2 + \cdots + n_m}{n_1, n_2, \cdots, n_m} a_{k_1}^{n_1} \cdots a_{k_m}^{n_m}$$

We conjecture that $\lambda_{2k} w_{2k}$ is a primitive integer valued Vassiliev invariant for all $k > 1$ and λ_{2k} is "minimal". Before we end this chapter, we state following two lemmas showing two characteristics of λ_{2k}, which we calculated above .

Lemma 4.3.3. *λ_{2i} is divisible by λ_j for all $j \leq 2i$*

Proof. Suppose $j = 2k$ and assume, $\lambda_{2i} = \frac{(2i)!}{2.3^{n+1}}, 3^n | i, 3^{n+1} \nmid i$ and $\lambda_{2k} = \frac{(2i)!}{2.3^{m+1}}, 3^m | k, 3^{m+1} \nmid k$ where $m, n \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned} \frac{\lambda_{2i}}{\lambda_{2k}} &= \frac{(2i)!3^{m+1}}{(2k)!3^{n+1}} \\ &= \frac{(2i).q}{3^{n-m}} \text{ for some } q \in \mathbb{Z} \end{aligned}$$

If $n \leq m$, then it is clear that,$\lambda_{2k} | \lambda_{2i}$. Suppose $n > m$. Then,

$$\frac{\lambda_{2i}}{\lambda_{2k}} = \frac{(2.3^n.t_1).q}{3^{n-m}} \qquad \text{where } i = 3^n.t_1$$

Thus, it is clear that $\lambda_{2k} | \lambda_{2i}$. Now suppose, $j = 2k + 1$. Then, $\lambda_{2k+1} = \frac{(2k+1)!}{2.3^{m+1}.r}$ where $3^m | 2k + 1, 3^{m+1} \nmid 2k + 1$, $m \in \mathbb{Z}_{\geq 0}$and $r \in \mathbb{Z}$.

$$\begin{aligned} \frac{\lambda_{2i}}{\lambda_{2k+1}} &= \frac{(2i)!2.3^{m+1}r}{(2k+1)!3^{n+1}} \\ &= \frac{(2i)q_2}{3^{n-m}} \text{ for some } q_2 \in \mathbb{Z} \end{aligned}$$

Now using same argument as above, we can conclude $\lambda_{2k+1} | \lambda_{2i}$. Thus $\lambda_j | \lambda_{2i}$ for all $j \leq 2i$ □

Lemma 4.3.4. *$\prod_{\Sigma_j k_j = 2i} \lambda_{k_j} | \lambda_{2i}$ where $2 \leq k_j \leq 2i - 2$.*

Proof. From theorem 4.3.1, we have $\lambda_{2i} = \frac{(2i)!}{2.3^{n+1}}$ where $3^n | i$ but $3^{n+1} \nmid i$ and $n \in \mathbb{Z}_{\geq 0}$. Now

$$\frac{\lambda_{2i}}{\prod_{\Sigma_j k_j = 2i} \lambda_{k_j}} = \left(\frac{(2i)!}{\prod_{\Sigma_j k_j = 2i} (k_j)!} \right) \left(\frac{q_1}{2.3^{n+1}} \right) \text{for some } q_1 \in \mathbb{N}$$

Observe that $\frac{(2i)!}{\prod_{\Sigma_j k_j = 2i} (k_j)!}$ denote multinomial coefficients, which is a generalization of binomial coefficients and thus an integer. Since $\lambda_{2i-2} = \frac{(2i-2)!}{2.3^{m+1}}$ for some $m \in \mathbb{Z}_{\geq 0}$ we can see $q_1 = 2.3^{m+1} q_2$ for some $q_2 \in \mathbb{N}, m \in \mathbb{Z}_{\geq 0}$. Thus we have to consider only

the term 3^{n+1} in the denominator of the $\left(\frac{q_1}{2.3^{n+1}}\right)$ in order to claim $\frac{\lambda_{2i}}{\prod_{\Sigma_j k_j=2i}\lambda_{k_j}}$ is an integer. If $n=0$, then since the minimum m can take is 0, we are done. Otherwise we know that $3|i$. Suppose $\frac{i}{3}\neq 2$. Then consider the $\lambda_2\lambda_{i/3}=\left(\frac{(2)!}{2\times 3}\right)\left(\frac{(i/3)!}{2^2\times 3^n\times q_3}\right)$ for some $q_3\in\mathbb{N}$. Thus we can see that by dividing λ_{2i} by $\lambda_2\lambda_{i/3}$ we have $q_1=(2\times 3)(2^2\times 3^n\times q_4)$ for some $q_4\in\mathbb{N}$. Therefore we cancel, 3^{n+1} term in the denominator of $\left(\frac{q_1}{2.3^{n+1}}\right)$. Now suppose $i/3=2$. This implies $2i=12$. In that case $\lambda_{12}=\frac{12!}{2.3^2}$. Now using $\lambda_6=\frac{6!}{2\times 3^2}$ we can cancel the 2.3^2 term in the denominator of the λ_{12}. Hence result follows $\square$

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Appendix

Appendix A

Mathematica Program

A Mathematica [37] program created to calculate basis chord diagrams and invariants shown in below. The program is developed as two packages. First package named as '**CDPackage**' is used to calculate the integer and rational basis for chord diagrams up to order 7. Second package named as '**InvariantCal**' used to calculate Vassiliev invariants over various basis diagrams found by **CDPackage** program. Both packages can be downloaded from University of Tennessee Trace site(see under additional files).

A.1 CDPackage Program

The program can be loaded as follows. User must set the directory to where CDPackage is resides.

Listing A.1: Loading the package

```
1 SetDirectory["C:\\Users\\vmanathu\\Desktop\\PhD_Thesis"]
2 << CDPackage`
```

The program contain 9 useful functions which user can use and several other internal functions. We list only user friendly functions. Other internal functions can be used, but user may have to do some his/her own modifications. For each function, type **?Functionname** to get description a about that function. For example,

Listing A.2: Loading the package

```
? InitializeProg
```

returns description about InitializeProg function.

- InitializeProg[kk]: The program generate all required files need to calculate bases up to that order kk. For experiments, run this program up to order 7. To generate files above order 7 require sufficient computer memory and may take some time. User do not need to run this function every time they load the package but only the first time they use this program. After the first time run, the generated file can be used in future sessions.

- Getrationalbasiselement[kk]: This function generate rational basis for given order.

- Moduloprimebasisfinder[$kk, moduloprime$]: This function generate modulo prime basis for given order.

- CheckWhethergivenListformIntegerBasis[$kk, possiblebasisset$]: This function check whether given list form a spanning set over $\mathbb{Z}$. User must decide whether it is a basis by comparing the size of spanning set and the dimension of rational basis. If they are equal, user can conclude the list entered indeed form a basis.

- CheckWhethergivenListformModuloPrimeBasis[$kk, primenum, possiblebasisset$]: This function check whether given list form modulo prime spanning set for that given order.

- DrawGaussCode[$gausscode$]: This will draw given gauss code and output it as jpeg file.

- Index related functions: These three functions give detail about indexing system(naming convention) used by the program for chord diagrams without 1T relations.

– GetIndexgivenCD[$givencdlist, kk$]: This function gives index used by the program for diagrams without 1T relations.

– GetCDgivenIndex[$givenindex, kk$]: Given the index, this gives the chord diagrams.

– FindStandardRepofgivenCD[$givencd, kk$]: This give the standard representation used by the program for similar set of CD diagrams. For example 112233 and 223311 represent same chord diagram and this program use 112233 to represent all diagrams similar to 112233.

Listing A.3: Running program functions

```
InitializeProg [6];
Getrationalbasiselement [5];
Moduloprimebasisfinder[5, 19];
CheckWhethergivenListformIntegerBasis[5, {"1234512345",''1234521354"}];
CheckWhethergivenListformModuloPrimeBasis[3, 7, {"123123"}];
GetIndexgivenCD[{"123123", "121323"}, 3];
GetCDgivenIndex[{1, 2, 3}, 5];
FindStandardRepofgivenCD["231231", 3]
DrawGaussCode["123123"];
```

A.2 InvariantCalc Package

The program calculate bases of Vassiliev invariants up to order six over chord diagrams defined in the table 3.3. The main program is "MainProg[$kk\_$]". User can use five type of options.

1. "Z": To get the basis invariants in the table 3.8
2. "Q1" To get the basis invariants in the table 3.9
3. "Q2" To get the basis invariants in the table 3.10

4. “TK” To get the T. Kanenobu invariants in the table 3.6

5. “Dawson” Calculate Dawson invariant in the table 3.11

User need “KnotTheory” package developed by Dror Bar-Natan to run this program. It can be downloaded from [3]. After downloaded it in to current directory, use following code to load package KnotTheory and InvariantCalc Package.

Listing A.4: Loading InvariantCalc Package

```
1 SetDirectory["C:\\Users\\vmanathu\\Desktop\\PhD_Thesis"]
2 << KnotTheory`
3 Get["InvariantCalc.m"]
```

After that user can use MainProg function. This program contain many useful functions to calculate Vassiliev invariants coming from polynomial invariants. Mathematica expert user can defined his/her own function easily and include it in the InvariantCalc program. This program is very flexible for expansion.

Vita

Vajira Asanka Manathunga was born in Maho, Sri Lanka as the eldest son of M. T. Weerasena and Kusuma Gunarathna. After completing his secondary education at Royal College in Colombo, Sri Lanka he entered as a undergraduate in to the University of Colombo in 2001. He earned his Bachelor of Science in Mathematics in 2005 and entered to Middle Tennessee State University in 2008 as a graduate student. He completed Master degree in 2010 and began doctoral studies at the University of Tennessee, Knoxville. He graduated in May 2016.