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To the Graduate Council:

I am submitting herewith a thesis written by Robert James Smith entitled "Groups and Algebraicity in Complete Rank Rings". I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

J. A. Cooley
Major Professor

We have read this thesis and recommend its acceptance:

O. J. Harned, Jr.

Wallace Givens

Edward J. Harris

David T. King

Accepted for the Council:

Dean of the Graduate School

GROUPS AND ALGEBRAICITY IN COMPLETE RANK RINGS

A DISSERTATION

Submitted to
The Graduate Council
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy

---

by

Robert James Smith

December 1959

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CHAPTER I

INTRODUCTION

It is well known that many of the results in classical linear algebra have an unequivocal extension to the more general situation when the scalars are drawn from an arbitrary division ring K. There are, however, three distinct theories of determinants for matrices over a division ring. One of these, originated by Study, "applies only to very particular non-commutative fields and to matrices of special type" (Dieudonné) and will not concern us here. The remaining two, one due to Ore and the other due to Dieudonné, reflect together, if not separately, the basic properties of the classical determinant. The diversity is, as we will see, due to the fact that the ordinary determinants for square matrices play different roles; first, in connection with ideals in the matrix ring and then, if a matrix be non-singular, in connection with the group of invertible matrices.

The elegant Dieudonné theory deals with a homomorphism of the matrix group into the commutative quotient group of cosets of K modulo its commutator subgroup. When K is commutative, the images of non-singular matrices under the homomorphism become simply the values of the ordinary determinant. A refinement of the usual methods of row or column expansion of a determinant is the main apparatus of

this theory; it fails, however, to assign a determinantal value to singular matrices unless a zero is somewhat artificially adjoined to the class of determinantal values of non-singular matrices.

The Ore determinant, on the other hand, maps the matrix ring into a class of two elements, essentially the ideals in K, with singular matrices corresponding to the zero ideal and non-singular matrices to K. Although it fails to assign the classical determinant to a matrix over a commutative field K, it does maintain precisely some other standard properties.

We will examine possible extensions of the definitions of these two determinants to apply to a class of rings more general than matrix rings, namely, the complete rank rings. Such a ring is, among other things, irreducible, regular and complete in the topology of a certain metric associated with it.

In Chapter II we will give von Neumann's definition of a complete rank ring and state results of constant later application which are derived from the work of von Neumann and Ehrlich. We will also establish some basic lemmas not explicitly stated in their work, which we will use in the sequel.

The succeeding chapter examines the technique of the Dieudonné determinant as applied to elements in a continuous

ring, that is, a complete rank ring whose rank function has as its range the unit interval. In this case a trivial determinant emerges but a substantial characterization of the group of invertible ring elements has been found as an immediate consequence of the author's results.

Chapter IV analyses the concept of the determinant due to Ore. We are able to obtain a slight generalization of Ore's result for matrices over a division ring and obtain as well an analogous result for continuous rings. We will indicate how the definition of determinant for matrices over a division ring can be extended to a limited class of matrices with polynomial elements, will define a characteristic equation for a λ-matrix and will show, for a few cases, that a square matrix, with elements in an arbitrary division ring, satisfies its characteristic equation.

J. von Neumann published in a paper "Continuous rings and their arithmetics" (see Bibliography) a sequence of results on the algebraicity over the center of the ring for an arbitrary element of a complete rank ring. These results "can be used to build up a theory of proper (eigen-) values and of elementary divisors in the ring" (von Neumann). The proofs of the theorems, however, have not been published heretofore and we give them in Chapter V. Surprisingly, in view of what happens for matrix rings over a commutative field there is no apparent connection between the results of

Chapters IV and V for matrix rings over a general division ring. Perhaps the difference lies in the fact that in Chapter IV we essentially view the matrices in relation to non-unilateral equations, a more natural, but more difficult, viewpoint than the unilateral one of Chapter V. We do not pursue the elementary divisor theory in the present work but do obtain a complete picture of the algebraicity to be expected for an arbitrary element of the ring.

CHAPTER II

COMPLETE RANK RINGS

The literature dealing with the general properties of complete rank rings consists primarily of five papers and a three volume publication by J. von Neumann, two publications by G. Ehrlich and three papers by K. D. Fryer and I. Halperin.

Material drawn from some of these sources will be used continually in the remainder of this work and we will give an informal account of the pertinent material, listing relevant publications forthwith so as to have available the suitable abbreviations indicated below.

Publications by J. von Neumann

Papers in the Proceedings of the National Academy of Sciences

"Continuous geometry," 22:92-100, 1936 ("C. G.");

"On regular rings," 22:707-13, 1936 ("R. R.");

"Algebraic theory of continuous geometries," 23: 19-22, 1937 ("A. T.");

"Continuous rings and their arithmetics," 23:341-9, 1937 ("C. R. A.");

Continuous Geometry Vol. II. (Planographed lecture notes, Ann Arbor: Edwards Brothers, 1937) ("N II").

A paper by G. Ehrlich in The Transactions of the American Mathematical Society

"Characterization of a continuous geometry within the unit group," 83, 2:397-416, 1956 ("E").

A paper by K. D. Fryer and I. Halperin in Acta Scientiarum Mathematicarum

"The von Neumann coordinatization theorem for complemented modular lattices," 17, 3-4:203-49, 1956 ("F, H").

Although we are principally in a certain class of rings, namely, the complete rank rings, and these can be defined without reference to the geometrical context in which they arise, it appears more natural to approach the subject by way of the lattice of principal right ideals in the general such ring; moreover, we will use the properties of the lattices.

Axioms of Continuous Geometry [C. G., 94-6]

We consider a class L consisting of at least two distinct elements, the elements being denoted by $a, b, \ldots$; there is defined among the elements of L a relation $\leq$ with the following properties

1. Under the relation, L is a complete, complemented, modular lattice. (Axioms I, II, IV, V).

2. L is irreducible, that is, the sole elements in L with unique complements are $0=\bigcap(L)$, $1=\bigcup(L)$, or equivalently, L is no direct sum. (Axiom VI).

3. We introduce a special limit notion in L as follows:

Let Ω be some infinite aleph and S a sequence of $\mathfrak{a}_\alpha \in$ L where α runs over all ordinals $\alpha < \Omega$. Define

If $\alpha < \beta < \Omega$ implies $\mathfrak{a}_\alpha \leqq \mathfrak{a}_\beta$, then

$$\lim_{\alpha \to \Omega}{}^{*}(\mathfrak{a}_\alpha) = \cup(S).$$

If $\alpha < \beta < \Omega$ implies $\mathfrak{a}_\alpha \geqq \mathfrak{a}_\beta$, then

$$\lim_{\alpha \to \Omega}{}^{*}(\mathfrak{a}_\alpha) = \cap(S).$$

$\lim^{*}_{\alpha \to \Omega}(\mathfrak{a}_\alpha)$ is otherwise not defined.

Lattice join and meet are continuous in L, in other words,

If $\alpha < \beta < \Omega$ implies $\mathfrak{a}_\alpha \geqq \mathfrak{a}_\beta$, then

$$\lim_{\alpha \to \Omega}{}^{*}(\mathfrak{a}_\alpha \cup \mathfrak{t}) = (\lim_{\alpha \to \Omega}{}^{*} \mathfrak{a}_\alpha) \cup \mathfrak{t} .$$

If $\alpha < \beta < \Omega$ implies $\mathfrak{a}_\alpha \leqq \mathfrak{a}_\beta$, then

$$\lim_{\alpha \to \Omega}{}^{*}(\mathfrak{a}_\alpha \cap \mathfrak{t}) = (\lim_{\alpha \to \Omega}{}^{*} \mathfrak{a}_\alpha) \cap \mathfrak{t} .$$

(Axiom III).

The axioms I-VI are invariant under dualization, that is, under reversal of partial order, with subsequent interchanges of lattice join and meet and of 0 and 1.

Dimension Function [C. G., 96-9]

It is possible to define uniquely a numerical dimension function D on the elements of L by the following conditions

A) $D(\mathfrak{a})$ is defined for all $\mathfrak{a} \in$ L, its values being real numbers in the unit interval.

B) $D(0) = 0$, $D(1) = 1$.

C) $D(\mathfrak{a} \cup \mathfrak{t}) + D(\mathfrak{a} \cap \mathfrak{t}) = D(\mathfrak{a}) + D(\mathfrak{t})$.

Von Neumann proves that the range $\underset{\sim}{D}$ of D is one of $\underset{\sim}{D}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ for some integer n and the set of all real numbers $\geqq 0, \leqq 1$ $(\underset{\sim}{D}_\infty)$. Moreover, if $\underset{\sim}{D} = \underset{\sim}{D}_n$, then L is the lattice of subspaces of an irreducible projective geometry whose ordinary projective dimension is n-1. If $\underset{\sim}{D} = \underset{\sim}{D}_\infty$, the lattice is referred to as a continuous geometry.

Regular rings [R. R., 708-12; F, H, 208]

An element e of a ring $\mathcal{R}$ is said to be idempotent if $e^2 = e$. Then also, 1-e is idempotent.

A non-empty set $\mathcal{G}$ of elements a, b, ... is called a semi-group if an associative multiplication is defined in $\mathcal{G}$. $\mathcal{G}$ is called a regular semi-group and the multiplication is called regular multiplication if for each $a \in \mathcal{G}$, $aba = a$ for at least one $b \in \mathcal{G}$. $\mathcal{G}$ is regular if and only if for each a there exists an idempotent $e \in \mathcal{G}$ and an element $b \in \mathcal{G}$ such that $ea = a$, $ab = e$ (if $aba = a$, then choose $e = ab$). Also, $\mathcal{G}$ is regular if and only if for each a there exists an idempotent e such that $ae = a$, $ba = e$, for some $b \in \mathcal{G}$.

We infer that $\mathcal{G}$ is regular if and only if each left coset $(a)_\ell$ contains a and is identical with $(e)_\ell$ for some idempotent $e \in \mathcal{G}$ and if and only if each right coset $(a)_r$ contains a and is identical with $(e)_r$ for some idempotent $e \in \mathcal{G}$.

A ring $\mathcal{R}$, with multiplication associative by definition, is called a regular ring if its multiplication is

regular, that is, for each a, $aba = a$ for some $b \in \mathfrak{K}$; equivalently, one of the two conditions

there exists an idempotent $e \in \mathfrak{K}$ such that $(e)_\ell = (a)_\ell$ for each $a \in \mathfrak{K}$

there exists an idempotent $e \in \mathfrak{K}$ such that $(e)_r = (a)_r$ for each $a \in \mathfrak{K}$

holds. These definitions and properties do not depend on the existence of a unit in $\mathfrak{K}$; however, we assume as of now, that $\mathfrak{K}$ is a regular ring with unit. Of course, the cosets $(a)_\ell$, $(a)_r$ are simply the principal left and right ideals generated by a in $\mathfrak{K}$.

We denote by $\mathfrak{a}^\ell$ ($\mathfrak{a}^r$) the left (right) annihilator of the right (left) ideal $\mathfrak{a}$ in $\mathfrak{K}$. If $e^2 = e$ and $\mathfrak{a} = (e)_r$ ($(e)_\ell$), then $\mathfrak{a}^\ell = (e)_r^\ell = (1-e)_\ell$ ($\mathfrak{a}^r = (e)_\ell^r = (1-e)_r$). If $R_{\mathfrak{K}}$ denotes the class of all principal right ideals and $L_{\mathfrak{K}}$ the class of all principal left ideals of the regular ring $\mathfrak{K}$ and if, in addition, we impose the condition that $\mathfrak{K}$ is irreducible, that is, its center is a field, [R. R., Theorem 5] then

$R_{\mathfrak{K}}$ and $L_{\mathfrak{K}}$ are complemented, modular and irreducible lattices and

$R_{\mathfrak{K}}$ and $L_{\mathfrak{K}}$ are anti-isomorphic under the mappings $\mathfrak{a} \in R_{\mathfrak{K}} \to \mathfrak{a}^\ell \in L_{\mathfrak{K}}$ and $\mathfrak{a} \in L_{\mathfrak{K}} \to \mathfrak{a}^r \in R_{\mathfrak{K}}$.

<u>The Coordinatization Theorem [M II, F, H]</u>

The lattices L as defined by von Neumann's axioms

are isomorphic to the lattice of principal right ideals of an irreducible regular ring $\mathfrak{R}$; and also to the principal left ideal lattice of a suitable irreducible regular ring. Moreover, $\mathfrak{R}$ is uniquely determined up to isomorphism by the lattice when $\underset{\sim}{D} = \underset{\sim}{D}_\infty$ and when $\underset{\sim}{D} = \underset{\sim}{D}_n$ for $n \geqq 3$, provided L is not a non-Desarguesian projective plane.

If L is a projective geometry of ordinary dimension n-1, then $\mathfrak{R}$, called a discrete ring, is the matrix ring $M_n(K)$ of order n over a division ring K. If $\underset{\sim}{D} = \underset{\sim}{D}_\infty$, $\mathfrak{R}$ is described as a continuous ring.

The Geometrisation Theorem [C. R. A., 343]

Suppose that $\mathfrak{R}$ is a regular ring with unit and that it is irreducible, that is, its center is a field.

$\mathfrak{R}$ is a rank ring if a numerical function R(a) can be defined for all $a \in \mathfrak{R}$ which possesses the properties

(1) Always $0 \leqq R(a) \leqq 1$

(2) $R(a) = 0$ if and only if $a = 0$

(3) $R(1) = 1$

(4) $R(ab) \leqq R(a)$, $R(ab) \leqq R(b)$

(5) For $e^2 = e$, $f^2 = f$, $ef = 0 = fe$ we have $R(e + f) = R(e) + R(f)$.

These requirements themselves imply

(6) $R(a) = 1$ if and only if a^{-1} exists in $\mathfrak{R}$

(7) $R(a) = R(b)$ if and only if $a = ubv$ where u^{-1}, v^{-1} exist in $\mathfrak{R}$

(8) $R(a + b) \leqq R(a) + R(b)$

We can then infer that R(a-b), called the rank-distance, is a metric in $\mathfrak{R}$, that is,

(i) $R(a-b) = 0$ for $a = b$

> 0 for $a \neq b$

(ii) $R(a-b) = R(b-a)$

(iii) $R(a-c) \leqq R(a-b) + R(b-c)$

and moreover, addition and multiplication in $\mathfrak{R}$ satisfy conditions

(iv) $R((a + b)-(c + d)) \leqq R(a-c) + R(b-d)$

(v) $R(ab-cd) \leqq R(a-c) + R(b-d)$.

A ring $\mathfrak{R}$ satisfying (1)-(5) and the subsequent conditions (6)-(8) and (i)-(v) is termed a complete rank ring if it is complete in the topology of the rank-distance. If the range of the rank function is the set $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ for some integer n, then $\mathfrak{R}$ is a discrete ring; and if the range of R is the unit interval, then $\mathfrak{R}$ is a continuous ring.

The two classes of complete rank rings are indeed the only ones possible [C. R. A., 345]. If $\mathfrak{R}$ is a complete rank ring, then the principal right ideal lattice $R_{\mathfrak{R}}$ (and the principal left ideal lattice $L_{\mathfrak{R}}$) fulfil the Axioms I-VI; conversely, if the principal right ideal lattice of a regular and irreducible ring $\mathfrak{R}$ satisfies the Axioms I-VI, then $\mathfrak{R}$ is a complete rank ring.

The relation between rank and dimension function is

as might be anticipated; namely, for any $a \in \mathfrak{R}$

$$R(a) = D((a)_r) = D((a)_\ell).$$

Preliminary Lemmas

In the remainder of this chapter we will discuss and in some cases derive some rather more special results for complete rank rings $\mathfrak{R}$ and their associated principal right (and left) ideal lattices. We will also introduce some groups in $\mathfrak{R}$ which will be basic in the next chapter. Throughout this and the next chapter $\mathfrak{R}$ will denote a complete rank ring. Unless otherwise indicated, we will assume also that $\mathfrak{R}$ does not have characteristic two; and if the discrete case applies, then $n > 2$.

The Rank-Distance Topology

We recall that if $\mathfrak{R}$ is a discrete ring, then it is equivalent to the ring of all $n \times n$ matrices over a division ring, for some integer n; in this case the range of R is the set $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ and a fundamental sequence is constant after a finite number of terms.

In the general case, if $\{a_1, a_2, \ldots\}$ is a sequence in $\mathfrak{R}$, then the existence of a limit $a \in \mathfrak{R}$, that is, an $a \in \mathfrak{R}$ such that

$$\lim_{i \to \infty} R(a_i - a) = 0$$

is equivalent to

$$\lim_{i,j\to\infty} R(a_i - a_j) = 0$$

2.1 Lemma. Continuity of Multiplication. Let $\{a_i\}$, $\{b_j\}$ (i, j = 1, 2, ...) be sequences in $\mathfrak{R}$ converging to a, b respectively. Then $\lim_{i\to\infty} a_i b_i = ab$.

Proof: $R(a_i b_i - ab) \leqq R(a_i - a) + R(b_i - b)$ by (v)

$\to 0$ as $i \to \infty$.

2.2. Lemma. Let $e_i \in \mathfrak{R}$ be a sequence of idempotents converging to e. Then e is idempotent.

Proof: By Lemma 2.1, $e^2 = (\lim_{i\to\infty} e_i)(\lim_{i\to\infty} e_i) = \lim_{i\to\infty} e_i e_i$ $\lim_{i\to\infty} e_i = e$.

2.3. Definition. We denote by $\mathfrak{G}$ the group of non-singular (invertible) ring elements, that is, $a \in \mathfrak{G} \subset \mathfrak{R}$ if and only if $R(a) = 1$.

2.4. Lemma. Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$. Then the closure of $\mathfrak{H}$ in the rank-distance topology is also a subgroup.

Proof; Let $\{t_i;\ t_i \in \mathfrak{H},\ i = 1, 2, \ldots\}$ be a convergent sequence. Then $\lim_{i,j\to\infty} R(t_i - t_j) = 0$ implies

$$\begin{aligned}\lim_{i,j\to\infty} R(t_i^{-1} - t_j^{-1}) &= \lim_{i,j\to\infty} R\{t_j^{-1}(t_j - t_i)t_i^{-1}\} \\ &= \lim_{i,j\to\infty} (t_i - t_j) \qquad \text{by (7)} \\ &= 0.\end{aligned}$$

Hence $\lim_{i\to\infty} t_i^{-1}$ exists in $\mathfrak{R}$.

By the continuity of multiplication $(\lim_{i\to\infty} t_i)(\lim_{i\to\infty} t_i^{-1}) = 1$ so that $\lim_{i\to\infty} t_i \in \mathfrak{G}$ and its inverse in $\mathfrak{G}$ is $\lim_{i\to\infty} t_i^{-1}$.

Moreover, as $\mathcal{G}$ is a group, $\lim_{i\to\infty} t_i^{-1}$ belongs to the closure of $\mathcal{G}$.

Let $\{t_i\}$, $\{u_j\}$ $(i, j = 1, 2, \ldots)$ be sequences in $\mathcal{G}$ converging in the rank-distance topology. Then

$$(\lim_{i\to\infty} t_i)(\lim_{i\to\infty} u_i) = \lim_{i\to\infty} t_i u_i$$

is also in the closure of $\mathcal{G}$.

Elements of Class 2

2.5. Definition. $t \in \mathfrak{R}$ is of class 2 (written henceforth as $t \in \mathfrak{C}^2$) if $t = 1 + r$ where $r \neq 0$, $r^2 = 0$.

2.6. Lemma. The inverse of $t \in \mathfrak{C}^2$ with $t = 1 + r$ is 1-r, also of class 2.

Proof: $(1 + r)(1-r) = 1-r^2 = 1$ and $(-r)(-r) = r^2 = 0$.

We are justified, then in speaking of the group generated by the class 2 elements. Because of Lemma 2.4 we have

2.7. Lemma. The closure $\mathfrak{R}^+$ of the group generated by the elements of class 2 is a group.

2.8. Lemma. Let $t \in \mathfrak{C}^2$, $s \in \mathfrak{G}$. Then $sts^{-1} \in \mathfrak{C}^2$.

Proof: Let $t = 1 + r$, $r^2 = 0$; then $sts^{-1} = 1 + srs^{-1}$ and $(srs^{-1})(srs^{-1}) = sr^2s^{-1} = 0$.

2.9. Corollary. Let $t \in \mathfrak{C}^2$, $s \in \mathfrak{G}$. Then $st = t_1 s$ for some $t_1 \in \mathfrak{C}^2$.

Proof: Let $sts^{-1} = t_1$. Then $st = t_1 s$ where $t_1 \in \mathfrak{C}^2$.

2.10. Corollary. Let $t \in \mathfrak{R}^+$, $s \in \mathfrak{G}$. Then $st = t_1 s$

for some $t_1 \in \mathfrak{R}^+$.

Proof: Let $t \in \mathfrak{R}^+$, $s \in \mathfrak{C}$. Then $sts^{-1} \in \mathfrak{R}^+$. For, if $t = \lim_{i\to\infty} t_i$ where $\{t_i;\ i = 1, 2, \ldots\}$ is a sequence in the group generated by the class 2 elements and which converges in $\mathfrak{R}$, then $sts^{-1} = s(\lim_{i\to\infty} t_i)s^{-1} = \lim_{i\to\infty} (st_i s^{-1})$ which belongs to $\mathfrak{R}^+$ since $t_i = t_i^{(1)} t_i^{(2)} \ldots$, where $t_i^{(k)} \in C^2$, implies $st_i s^{-1} = (st_i^{(1)} s^{-1})(st_i^{(2)} s^{-1})\ldots$ is a product of class 2 elements, by Lemma 2.8.

The equation $sts^{-1} = t_1$, t_1 in $\mathfrak{R}^+$, implies $st = t_1 s$.

2.11. **Definition.** We write $u \cong s$ for non-singular $u, s \in \mathfrak{R}$ when $u = ts$ for some $t \in \mathfrak{R}^+$.

2.12. **Corollary.** The relation $\cong$ is an equivalence relation.

Proof: Clearly $u \cong u$ as $u = 1.u$ and $1 \in \mathfrak{R}^+ \subset \mathfrak{R}$.

Also $u = ts$ with $t \in \mathfrak{R}^+$ implies $s = t^{-1}u$ and $t^{-1} \in \mathfrak{R}^+$ by Lemma 2.7, that is, $u \cong s$ implies $s \cong u$.

Finally, $u = t_1 s$, $s = t_2 v$ with $t_1, t_2 \in \mathfrak{R}^+$ imply $u = t_1 t_2 v$ or $u \cong s$, $s \cong v$ imply $u \cong v$.

2.13. **Lemma (Givens).** $t \in C^2$ if and only if there exists $u \in \mathfrak{C}$ such that $utu^{-1} = t^2$.

Proof: This result is proved in [E, 405].

The Subrings $\mathfrak{R}(e)$, $e^2 = e$

Let e $(e \neq 0)$ be an idempotent in $\mathfrak{R}$. Then the subring $\mathfrak{R}(e)$ consisting of all quantities exe, $x \in \mathfrak{R}$, is

again a regular ring with unit e. More strongly, $\mathfrak{R}(e)$ is a complete rank ring whose lattice of principal right ideals satisfies the Axioms I-VI. This lattice is isomorphic to the lattice interval $[(0)_r, (e)_r]$. [N II, 146].

With slight modifications, then, equations (1)-(8) and (i)-(v) apply to the ring $\mathfrak{R}(e)$, $e \neq 0$. Also, we may normalize the rank R(exe) for arbitrary $exe \in \mathfrak{R}(e)$ to obtain a rank function R^* such that $R^*(exe) = R(exe)/R(e)$; R^*, defined for all members of $\mathfrak{R}(e)$, then satisfies the conditions (1)-(8) and (i)-(v) precisely.

2.14. Lemma. Let $(1-e)t(1-e)$ belong to the closure of the group generated by the elements of class 2 of $\mathfrak{R}(1-e)$. Then $e + (1-e)t(1-e) \in \mathfrak{R}^+$.

Proof: If $(1-e)t(1-e) = 1-e + (1-e)r(1-e)$, with $((1-e)r(1-e))^2 = 0$, then $e + (1-e)t(1-e) = 1 + (1-e)r(1-e) \in \mathfrak{C}^2 \subset \mathfrak{R}^+$.

Again if $(1-e)t(1-e)$ is a finite product of class 2 elements in $\mathfrak{R}(1-e)$, that is,

$$(1-e)t(1-e) = ((1-e) + (1-e)r_1(1-e))((1-e) + (1-e)r_2(1-e))\ldots,$$

then

$$e+(1-e)t(1-e)=(e+(1-e)+(1-e)r_1(1-e))(e+(1-e)+(1-e)r_2(1-e))\ldots$$
$$=(1+(1-e)r_1(1-e))(1+(1-e)r_2(1-e))\ldots$$

Now let $\{(1-e)t_i(1-e);\ i = 1, 2, \ldots\}$ be a sequence of finite products of class 2 elements in $\mathfrak{R}(1-e)$ converging to $(1-e)t(1-e)$ in the topology of rank-distance in $\mathfrak{R}(1-e)$ (using the rank function R^* with $R^*(1-e) = 1$, $R^*(0) = 0$).

We have

$$\lim_{i,j\to\infty} R((e + (1-e)t_i(1-e))-(e + (1-e)t_j(1-e)))$$
$$= \lim_{i,j\to\infty} R((1-e)t_i(1-e)-(1-e)t_j(1-e))$$
$$= \lim_{i,j\to\infty} R(1-e)R^*((1-e)t_i(1-e)-(1-e)t_j(1-e))$$
$$= 0.$$

Hence $\{ e + (1-e)t_i(1-e);\ i = 1, 2, \ldots\}$ forms a converging sequence of finite products of class 2 elements in $\mathfrak{R}$. In general, the closure of the group generated by the class 2 elements in $\mathfrak{R}(1-e)$ consists of quantities $(1-e)t(1-e)$ such that $e + (1-e)t(1-e) \in \mathfrak{R}^*$.

2.15. Lemma. Let $a \in \mathfrak{R}$, $(a)_r \leqq (e)_r$, $e^2 = e \in \mathfrak{R}$. Then $(a)_r = (f)_r$ where $f = efe$ and $f^2 = f$. [N II, Lemmas 15.5, 15.7].

2.16. Lemma. Let e, f be idempotents in $\mathfrak{R}$ such that $R(e) = R(f)$. Then there exists $t \in \mathfrak{C}$ such that $tet^{-1} = f$ [E, Lemma 9].

Matrix Bases

2.17. Lemma. If $e_1, e_2, \ldots, e_n$ are independent idempotents in $\mathfrak{R}$, that is,

$$e_i e_j = 0,\ i \neq j$$
$$= e_i,\ i = j$$

then $(e_{i_1})_r \cup (e_{i_2})_r \cup \ldots \cup (e_{i_t})_r = (e_{i_1} + e_{i_2} + \ldots + e_{i_t})_r$ [N II, Lemma 3.1].

2.18. Lemma. The principal right ideals $\mathfrak{a}_i$

$(i = 1, 2, \ldots, n)$ are independent if and only if there exist n independent idempotents $e_1, e_2, \ldots, e_n \in \mathfrak{R}$ such that $\mathfrak{a}_i = (e_i)_r$. Also $\mathfrak{a}_1 \cup \mathfrak{a}_2 \cup \ldots \cup \mathfrak{a}_n = (1)_r$ if and only if $e_1 + e_2 + \ldots + e_n = 1$ [N II, Lemma 3.2].

2.19. Definition. The principal right ideals $\mathfrak{a}_i$ $(i = 1, 2, \ldots, n)$ in $R_{\mathfrak{R}}$ form a homogeneous basis if the $\mathfrak{a}_i$ are independent, $D(\mathfrak{a}_i)$ is the same for all i and $\mathfrak{a}_1 \cup \mathfrak{a}_2 \cup \ldots \cup \mathfrak{a}_n = (1)_r$.

2.20. Lemma. $\mathfrak{a}_i \in R_{\mathfrak{R}}$ form a homogeneous basis if and only if there exist $s_{ij} \in \mathfrak{R}$ $(i, j = 1, 2, \ldots, n)$ such that $\mathfrak{a}_i = (s_{ii})_r$, and the s_{ij} are matrix units; that is,

$$s_{ij}\, s_{kh} = s_{ih} \quad \text{if } j = k$$
$$= 0 \quad \text{if } j \neq k$$

and the s_{ii} are idempotents of the same rank and with sum one [N II, Lemma 3.6].

2.21. Lemma. Let e_1, e_2 be independent idempotents and $R(e_1) = R(e_2)$. Then there exist s_{12}, s_{21} such that $e_1 s_{12} = s_{12} e_2 = s_{12}$, $e_2 s_{21} = s_{21} e_1 = s_{21}$, $e_1 s_{21} = 0 = s_{12} e_1$, $e_2 s_{12} = 0 = s_{21} e_2$.

Proof: In the complete rank ring $\mathfrak{R}(e_1 + e_2)$ the principal right ideals generated by e_1, e_2 form a homogeneous basis; by Lemma 2.20 there exist $s_{11} = e_{11}$, $s_{22} = e_{22}$, s_{12}, s_{21} in $\mathfrak{R}(e_1 + e_2) \subset \mathfrak{R}$ such that $(e_{11})_r = (e_1)_r$, $(e_{22})_r = (e_2)_r$, $e_{11}^2 = e_{11}$, $e_{22}^2 = e_{22}$, $e_{11} e_{22} = 0 = e_{22} e_{11}$, $e_{11} + e_{22} = e_1 + e_2$, $e_{11} s_{12} = s_{12} e_{22} = s_{12}$, $e_{22} s_{21} = s_{21} e_{11} = s_{21}$, $e_{11} s_{21} = 0 = s_{12} e_{11}$, $e_{22} s_{12} = 0 = s_{21} e_{22}$.

But $e_{11} + e_{22} = e_1 + e_2$ and $e_{22} e_2 = e_2$ imply $e_{11} e_2 = e_1 e_2 = 0$ and further as $e_{11} e_1 = e_1$, then $e_{11} = e_1$. Similarly $e_{22} = e_2$ and the result follows.

CHAPTER III

GROUPS IN A COMPLETE RANK RING

The Dieudonné Theory [1]

Let $M_n^x(K)$ be the multiplicative group of non-singular nxn matrices A, B, ... with elements in a division ring K. If K^x is the multiplicative group of K and C is the commutator subgroup of K^x, let θ be the homomorphism $K^x \longrightarrow K^x/C$. The Dieudonné determinant $\Delta_n(A)$ is defined inductively as follows:

If $A = (\lambda)$, a 1x1 matrix, write $\Delta_1(\lambda) = \theta(\lambda)$.

If $A = (\xi_{ij})$ in $M_n^x(K)$ has ξ_{i1} a non-zero element of the first column, suppose A_i is obtained by adding $-\xi_{j1}(\xi_{i1})^{-1}$.(i-th row of A) to the j-th row (all $j \neq i$). We define

$$\Delta_n(A) = \theta((-1)^{i+1}\xi_{i1})\ \Delta_{n-1}(A_i')$$

where A_i' is obtained by deleting the first column and i-th row from A_i.

In the case where K is commutative C is a one element group and the Dieudonné determinant obviously coincides with the ordinary determinant. In the general case, the Dieudonné original paper [1, 33] implies that although the determinantal values will be in the quotient group K^x/C rather than in K, the coset obtained is independent of the particular non-zero element ξ_{i1} chosen in the first column and subsequent similar alternatives in A_i' etc.

The addition of a left multiple of a row of the matrix

A to another row is, of course, the equivalent of left multiplying A by a transvection, that is, a matrix of the form $I_n + uv$ where u is a column vector and v a row vector and $vu = 0$. The subgroup B_n of $M_n^x(K)$, generated by the transvections, is the commutator subgroup C_n of $M_n^x(K)$ except for the cases (i) $n = 2$, $K = GF(2)$ and (ii) $n = 1$. Writing $B_1 = C$ and omitting the exceptional case (i) we can obtain [1, Theorem 1] that K^x/C, $M_n^x(K)/C_n$, $M_n^x(K)/B_n$, $(n = 1, 2, \ldots)$ are isomorphic. Indeed, with no exceptions, $M_n^x(K)/B_n$ is isomorphic to K^x/C.

This suggests that the cosets $M_n^x(K)/C_n$ will serve just as well for determinantal values as the cosets in K^x noting, of course, that by the definition, we are restricting ourselves to non-singular matrices only and can expect complete parallelism with the classical theory for theorems involving the multiplication of determinants only. The best results obtained [Givens, 5] in weakening the stipulations involve the adjunction of a zero to the coset determinantal values, that is, say $\Delta_n(X) = 0 \in K$ if $X \in M_n(K)$ has no inverse; we can then show that if X, Y, Z are matrices, singular or not, of the same order and with identical elements except in the i-th row (or column) where they have row (or column) vectors u_1, u_2, $u_1 + u_2$ respectively, then

$$\Delta_n(Z) \subset \Delta_n(X) + \Delta_n(Y)$$

where the right hand side (a union of cosets) is the class

of all sums of an element of $\Delta_n(X)$ and one of $\Delta_n(Y)$. We will later use as our determinantal values the cosets in $M_n^x(K)$; there will be, under these circumatances, no results obtained for determinants which are not purely multiplicative.

The Subgroup $\mathfrak{R}^+$

In the remainder of this chapter $\mathfrak{R}$ denotes a fixed complete rank ring, of characteristic not 2.

3.1. Lemma. Let e be any idempotent of rank $\frac{1}{2}$ and let s be non-singular and otherwise arbitrary in $\mathfrak{R}$. Then for some $t \in \mathfrak{R}$,

$$s \cong e + (1-e)t(1-e)$$

Proof: The existence of idempotents of rank $\frac{1}{2}$ is assured in continuous rings, that is, when the range of R is the unit interval. In the discrete case the result has no meaning if the order of the matrices is odd.

Now suppose that the principal left ideal $((1-e)se)_\ell = (g_1)_\ell$ where $g_1 = eg_1e$, $g_1^2 = g_1$ (Lemma 2.16). By the Peirce decomposition, s is the sum of the quantities in the blocks of

$$\begin{bmatrix} g_1sg_1 & g_1s(e-g_1) & \\ & & es(1-e) \\ (e-g_1)sg_1 & (e-g_1)s(e-g_1) & \\ (1-e)sg_1 & (1-e)s(e-g_1) & (1-e)s(1-e) \end{bmatrix}$$

where a matrix notation is used for clarity and to permit the comparison of later processes with standard matrix ones; we will simply equate such a partitioned array to the sum of its members. We have

$$g_i = y_i(1-e)se = y_i(1-e)seg_i = y_i(1-e)sg_i$$

for some $y_i \in \mathfrak{R}$ so that

$$\{1 + g_i(g_i - g_i s g_i)y_i(1-e)\}s$$

$$= \begin{bmatrix} g_i & g_i s(e-g_i) & g_i s^*(1-e) \\ (e-g_i)sg_i & (e-g_i)s(e-g_i) & (e-g_i)s(1-e) \\ (1-e)sg_i & 0 & (1-e)s(1-e) \end{bmatrix}$$

for some $s^* \in \mathfrak{R}$ since

$$g_i s g_i + (g_i - g_i s g_i)y_i(1-e)sg_i = g_i s g_i + g_i - g_i s g_i = g_i$$

and

$$(1-e)s(e-g_i) = (1-e)se - (1-e)sg_i = (1-e)seg_i - (1-e)sg_i = 0.$$

Moreover, as $(1-e)g_i = 0 = g_i(1-e)$ the first factor on the left side is of class 2 as are $1-(1-e)sg_i$, $1-(e-g_i)sg_i$, $1-g_i s(e-g_i)$ and $1-g_i s^*(1-e)$. Multiplying on the left by $(1-(1-e)sg_i)(1-(e-g_i)sg_i)$ and on the right by $((1-g_i)s(e-g_i))$ $\cdot(1-g_i s^*(1-e))$ gives

$$t_1 s = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & (e-g_1)s(e-g_1) & (e-g_1)s_1(1-e) \\ 0 & (1-e)s_1(e-g_1) & (1-e)s_1(1-e) \end{bmatrix} = s_1$$

for some $s_1 \in \mathfrak{R}$ and some $t_1 \in \mathfrak{R}^\dagger$ by Corollary 2.10.

Define g_{n+1}, s_{n+1}, t_{n+1} for n = 1, 2, ... as follows:

Let $((1-e)s_n(e-g_1-\ldots-g_n))_\ell = (g_{n+1})_\ell$ where $g_{n+1}^2 = g_{n+1}$ and $(e-g_1-\ldots-g_n)g_{n+1}(e-g_1-\ldots-g_n) = g_{n+1}$. We have, similarly to the above, the existence of a $t_{n+1} \in \mathfrak{R}^\dagger$ and an $s_{n+1} \in \mathfrak{R}$ such that $t_{n+1} s$

$$= \begin{bmatrix} g_1 & & & \\ & \ddots & 0 & 0 \\ & & g_n & \\ 0 & & g_{n+1}(e-g_1-\ldots-g_{n+1})s_{n+1}(e-g_1-\ldots-g_{n+1}) & (e-g_1-\ldots-g_{n+1})s_{n+1}(1-e) \\ 0 & & (1-e)s_{n+1}(e-g_1-\ldots-g_{n+1}) & (1-e)s_{n+1}(1-e) \end{bmatrix}$$

$= s_{n+1}$.

Now $g_1, \ldots, g_n$ are independent idempotents and so

$$\tfrac{1}{2} \geq R(g_1 + \ldots + g_n) = R(g_1) + \ldots + R(g_n) = \sum_{m=1}^{n} R((1-e)s_m(e-g_1-\ldots-g_m))$$

so that $\lim_{i\to\infty} R((1-e)s_i(e-g_1-\ldots-g_i) = 0$ and in turn

$$\lim_{i\to\infty} (1-e)s_i(e-g_1-\ldots-g_i) = 0. \tag{3.1}$$

In addition, $\lim_{n,p\to\infty} R(g_{n+1} + \ldots + g_{n+p})$

$= \lim_{n,p\to\infty} R(g_{n+1}) + \ldots + R(g_{n+p})$

$= 0.$

Hence $\lim_{n\to\infty} (g_1 + \ldots + g_n) = g$, say, exists in $\mathfrak{R}$; also by Lemmas 2.1, 2.2 we have $g = ege$ and g is idempotent.

In order to prove that $\lim_{n\to\infty} t_n$ exists in $\mathfrak{R}$ and so belongs to $\mathfrak{R}^\dagger$ we note that

$$(3.2)\qquad (1-(1-e)s_n g_{n+1})(1-(e-g_1-\ldots-g_{n+1})s_n g_{n+1})$$
$$\cdot(1 + g_{n+1}(g_{n+1} - g_{n+1} s_n g_{n+1})y_{n+1}(1-e))t_n s$$
$$\cdot(1-g_{n+1} s_n(e-g_1-\ldots-g_{n+1}))(1-g_{n+1} s_n^*(1-e))$$
$$= t_{n+1} s$$

where $s_n^* \in \mathfrak{R}$ and y_{n+1} is defined by the condition $g_{n+1} = y_{n+1}(1-e)s_n e$. The last two factors on the left side of (3.2) may be transferred after a similarity transformation to the left of $t_n s$, by Corollary 2.9, giving

$$(1 + \Phi(g_{n+1}))t_n s = t_{n+1} s$$

where $\Phi(g_{n+1})$ is an expression involving no more than $2^5 - 1 = 31$ terms, each containing g_{n+1} as a factor and so of rank $\leqq R(g_{n+1})$. Hence $t_{n+1} - t_n = \Phi(g_{n+1})t_n$ and

$$R(t_{n+1} - t_n) \leqq R\Phi(g_{n+1}) \leqq 31 R(g_{n+1}),$$
$$R(t_{n+p} - t_n) \leqq \sum_{i=1}^{p} R(t_{n+i} - t_{n+i-1}) \quad \text{[Chapter II, (iii)]}$$
$$\leqq 31 \sum_{i=1}^{p} R(g_{n+1})$$
$$\to 0 \text{ as } n, p \to \infty.$$

We conclude that

$$\lim_{n\to\infty}(1-g_1-\ldots-g_n)s_n(1-g_1-\ldots-g_n) = \lim_{n\to\infty}(t_n s-(g_1 + \ldots + g_n))$$

exists in $\mathfrak{R}$. It equals $(1-g)t(1-g)$ for some $t \in \mathfrak{R}$. Moreover, $(1-e)t(e-g) = 0$ by (3.1). Then

$$(3.3) \qquad s \cong \begin{bmatrix} g & 0 & 0 \\ 0 & (e-g)t(e-g) & (e-g)t(1-e) \\ 0 & 0 & (1-e)t(1-e) \end{bmatrix}$$

where $R((e-g)t(e-g) \leq \frac{1}{2}$ and, since the right member has an inverse in $\mathfrak{R}$, $R((e-g)t(e-g)) = R(e-g)$ and so $(e-g)t(e-g)$ has an inverse in the subring $\mathfrak{R}(e-g)$.

By Lemma 2.21, if $(1-e)h(1-e) = h$ is an idempotent of rank equal to $R(e-g)$, then $e-g$, h define quantities $x, y \in \mathfrak{R}$ such that

$$xh = (e-g)x = x$$
$$hy = y(e-g) = y$$
$$xy = e-g, \quad yx = h.$$

We have that $1 + x$, $1 + y \in \mathfrak{C}^2$ since $x^2 = xh(e-g)x = 0$, $y^2 = y(e-g)hy = 0$ and so $(1 + x)(1-y)(1 + x)$

$$= (1 + x-y-xy)(1 + x)$$
$$= 1 + x + x + x^2-y-yx-xy-xyx$$
$$= 1 + 2x-y-h-(e-g)-x$$
$$= 1-(e-g)-h + x-y$$

belongs to $\mathfrak{R}^\dagger$. Multiplying the right member of (3.3) by $(1 + x)(1-y)(1 + x)$ gives

$$(3.4) \qquad s \cong \begin{bmatrix} g & 0 & 0 \\ 0 & 0 & (e-g)t^*(1-e) \\ 0 & -y(e-g)t(e-g) & (1-e)t^*(1-e) \end{bmatrix}$$

for some $t^* \in \mathfrak{R}$ since

$$\begin{aligned} (1-(e-g)-h + x-y)g &= g-hg + xg-yg \\ &= g-xhg-y(e-g)g \\ &= g \qquad\qquad (g = ege) \end{aligned}$$

and $(1-(e-g)-h + x-y)(e-g)t(e-g)$

$$= -h(e-g)t(e-g) + x(e-g)t(e-g)-y(e-g)t(e-g)$$

$$= -y(e-g)t(e-g).$$

Since $R(-y(e-g)t(e-g)) = R(e-g)$, then

$$(-y(e-g)t(e-g))_\ell = (e-g)_\ell$$

and so $e-g = -zy(e-g)t(e-g)$ for some $z \in \mathfrak{R}$.

Multiplication of the right member of (3.4) on the left by the class 2 element $1-(e-g)z(1-e)$ gives

$$s \cong \begin{bmatrix} g & 0 & 0 \\ 0 & e-g & (e-g)t_1^*(1-e) \\ 0 & -y(e-g)t(e-g) & (1-e)t^*(1-e) \end{bmatrix}$$

with $t_1^* \in \mathfrak{R}$ since

$$(1-(e-g)z(1-e))(-y(e-g)t(e-g))$$

$$= -y(e-g)t(e-g) + (e-g)z(1-e)y(e-g)t(e-g)$$

$$= -y(e-g)t(e-g) + zy(e-g)t(e-g)$$

$$= -y(e-g)t(e-g) + e-g.$$

Multiplication on the left by $1 + (1-e)y(e-g)t(e-g)$ and on the right by $1-(e-g)t_i^*(1-e)$, both of class 2, gives, for some $t' \in \mathfrak{R}$ that

$$s \cong \begin{bmatrix} g & 0 & 0 \\ 0 & e-g & 0 \\ 0 & 0 & (1-e)t'(1-e) \end{bmatrix}$$

In other words

$$s \cong g + e-g + (1-e)t'(1-e) = e + (1-e)t'(1-e).$$

3.2. Lemma. In a continuous ring $\mathfrak{R}$, let $e^2 = e$, $R(e) < 1$ and s be non-singular. Then, for some $t \in \mathfrak{R}$,

$$s \cong e + (1-e)t(1-e).$$

Proof: If $R(e) < \frac{1}{2}$, a similar proof to that of Lemma 3.1 yields the result.

We may suppose then, that

$$\sum_{i=1}^{p-1} 2^{-i} \leqq R(e) < \sum_{i=1}^{p} 2^{-i}$$

for $p > 1$.

Let $e_1 = ee_1e$ be an idempotent of rank $\frac{1}{2}$. Then, by Lemma 3.1

$$t_1 s = e_1 + (1-e_1)s_1(1-e_1)$$

for some $t_1 \in \mathcal{R}^\dagger$ and $s_1 \in \mathcal{R}$. If $p > 2$, we let $e_2 = (e-e_1)e_2(e-e_1)$ be an idempotent of rank $\frac{1}{2}$; then e_2 has normalized rank $\frac{1}{2}$ in the continuous ring $\mathcal{R}(1-e_1)$ and $(1-e_1)s_1(1-e_1)$ is non-singular in this ring. Hence there exists t_2 in the group $\mathcal{R}^\dagger$ of $\mathcal{R}(1-e_1)$ such that

$$t_2(1-e_1)s_1(1-e_1) = e_2 + (1-e_1-e_2)s_2(1-e_1-e_2)$$

where $s_2 \in \mathcal{R}(1-e_1) \subset \mathcal{R}$. Then

$$(e_1 + t_2)(e_1 + (1-e_1)s_1(1-e_1)) = e_1 + e_2 + (1-e_1-e_2)s_2(1-e_1-e_2);$$

moreover, $e_1 + t_2 \in \mathcal{R}^\dagger$ by Lemma 2.14.

Proceeding in a similar fashion we have eventually for some s_{p-1} and independent idempotents $e_i = ee_ie$ $(i = 1, 2, \ldots, p-1)$ with $R(e_i) = 2^{-i}$

$$(3.5) \qquad s \cong e_1 + e_2 + \ldots + e_{p-1} + (1-e_1-e_2-\ldots-e_{p-1})s_{p-1}(1-e_1-e_2-\ldots-e_{p-1}).$$

Application of the first statement of the proof to the idempotent $e-e_1-e_2\ldots-e_{p-1}$ in the subring $\mathcal{R}(1-e_1-e_2-\ldots-e_{p-1})$ gives

$$t_p(1-e_1-e_2-\ldots-e_{p-1})s_{p-1}(1-e_1-e_2-\ldots-e_{p-1}) = e-e_1-e_2-\ldots-e_{p-1} + (1-e)s_p(1-e)$$

where $t_p \in \mathcal{R}(1-e_1-e_2-\ldots-e_{p-1})$, $e_1 + e_2 + \ldots + e_{p-1} + t_p \in \mathcal{R}^\dagger$ and $s_p \in \mathcal{R}$.

Multiplying the right side of (3.5) by $e_1 + e_2 + \ldots + e_{p-1} + t_p$ gives

$$s \cong e_1 + e_2 + \ldots + e_{p-1} + t_p(1-e_1-e_2-\ldots-e_{p-1})s_{p-1}(1-e_1-e_2-\ldots-e_{p-1})$$
$$= e_1 + e_2 + \ldots + e_{p-1} + e - e_1 - e_2 - \ldots - e_{p-1} + (1-e)s_p(1-e)$$
$$= e + (1-e)s_p(1-e).$$

3.3. Theorem (Amemiya). In a continuous ring $\mathfrak{R}^{\dagger} = \mathfrak{C}$.

Proof: Let $f_1, f_2, \ldots$ be idempotents such that $R(f_i) = 2^{-i}$ and $f_{i+1} \leq f_i f_{i+1} f_i$ $(i = 1, 2, \ldots)$. Write $e_i = 1-f_i$. By Lemma 3.2, if $s \in \mathfrak{C}$, there exist $t_i \in \mathfrak{R}^{\dagger}$ and $u_i \in \mathfrak{R}$ $(i = 1, 2, \ldots)$ such that

$$t_i s = e_i + (1-e_i)u_i(1-e_i).$$

Then $R(t_{i+j} - t_i) = R((t_{i+j} - t_i)s) \leqq R(e_{i+j} - e_i) + R((1-e_{i+j})u_{i+j}$
$\cdot(1-e_{i+j})) + R((1-e_i)u_i(1-e_i))$
$\longrightarrow 0$ as $i, j \to \infty$.

Hence $\lim_{i\to\infty} t_i$ exists, $(\lim_{i\to\infty} t_i)s = 1$ and $s = (\lim_{i\to\infty} t_i)^{-1} \in \mathfrak{R}^{\dagger}$.

3.4. Definition. We denote by $\mathfrak{R}$ the closure of the commutator subgroup of $\mathfrak{C}$.

3.5. Theorem. In a continuous ring $\mathfrak{R} = \mathfrak{R}^{\dagger}$.

Proof: By Lemma 2.13 the arbitrary $t \in \mathfrak{C}^{\dagger}$ satisfies $t^2 = utu^{-1}$ for some $u \in \mathfrak{C}$, that is,

$$t = utu^{-1}t^{-1}. \tag{3.6}$$

Therefore $\mathfrak{R}^{\dagger} \subset \mathfrak{R}$.

By Lemma 3.1, if $a_1, a_2 \in \mathfrak{C}$ and e is an idempotent such that $R(e) = \frac{1}{2}$, then $a_1 = b_1 d_1$, $a_2 = b_2 d_2$ where $b_1, b_2 \in \mathfrak{R}^{\dagger}$ and

$$d_1 = e + (1-e)d_1(1-e)$$
$$d_2 = e + (1-e)d_2(1-e).$$

The commutator $a_1 a_2 a_1^{-1} a_2^{-1}$ has the form $bd_1 d_2 d_1^{-1} d_2^{-1}$ with $b \in \mathcal{R}^+$ by Corollary 2.10. It is sufficient to show that $d_1 d_2 d_1^{-1} d_2^{-1} \in \mathcal{R}^+$ and we need only show that $d_1 d_2 = b^{(1)} d_2 d_1 b^{(2)}$ where $b^{(1)}$, $b^{(2)} \in \mathcal{R}^+$. Write $(1-e)d_1(1-e) = \lambda$, $(1-e)d_2(1-e) = \mu$.

Now e, $1-e$ define a matrix basis s_{ij} with $s_{11} = e$, $s_{22} = 1-e$, $s_{12} = es_{12} = s_{12}(1-e)$, $s_{21} = (1-e)s_{21} = s_{21}e$ by Lemma 2.21. Then

$$\begin{aligned}(1 + s_{12})(1-s_{21})(1 + s_{12}) &= (1 + s_{12} - s_{21} - e)(1 + s_{12}) \\ &= 1 + s_{12} + s_{12} - s_{21} - (1-e) - e - s_{12} \\ &= s_{12} - s_{21}\end{aligned}$$

and $(s_{12} - s_{21})^2 = (s_{12} - s_{21})(s_{12} - s_{21}) = -e - (1-e) = -1$ both belong to $\mathcal{R}^+$.

Noticing that λ, μ have inverses λ^{-1}, μ^{-1}, say, respectively, in $\mathcal{R}(1-e)$ we obtain

$$d_1 d_2 \cong (1 + es_{12}\lambda^{-1}(1-e)) \begin{bmatrix} e & 0 \\ 0 & \lambda\mu \end{bmatrix}$$

$$= \begin{bmatrix} e & s_{12}\mu \\ 0 & \lambda\mu \end{bmatrix}$$

$$\cong (1-(1-e)\lambda s_{21} e) \begin{bmatrix} e & s_{12}\mu \\ 0 & \lambda\mu \end{bmatrix}$$

$$= \begin{bmatrix} e & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix}$$

$$\cong (1 + e s_{12} \lambda^{-1}(1-e)) \begin{bmatrix} e & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix} \tag{3.7}$$

Multiplying the last quantity on the left by $-(s_{12} - s_{21})$ gives

$$d_1 d_2 \cong \begin{bmatrix} s_{12}\lambda s_{21} & 0 \\ 0 & \mu \end{bmatrix}$$

$$\cong \begin{bmatrix} s_{12}\lambda s_{21} & 0 \\ 0 & \mu \end{bmatrix} (s_{12} - s_{21}) \qquad \text{(Corollary 2.10)}$$

$$= \begin{bmatrix} 0 & s_{12}\lambda \\ -\mu s_{21} & 0 \end{bmatrix}.$$

This has a form similar to that of (3.7) but with λ, μ interchanged. As the relation $\cong$ is an equivalence relation, the steps leading to (3.7) can be reversed to replace the last quantity by

$$\begin{bmatrix} e & 0 \\ 0 & \mu\lambda \end{bmatrix} = d_2 d_1 .$$

3.6. Theorem. In a continuous ring $\mathcal{R} = \mathcal{E}$.

Proof: Theorems 3.3, 3.5.

3.7. Remark. When $\mathcal{R}$ is a matrix ring over a division ring (discrete ring), $\mathcal{R}$, $\mathcal{R}^+$ are respectively the commutator subgroup and the group generated by the elements of class 2, for in the discrete case a sequence convergent in the rank-distance metric is constant after a finite number of terms. Provided the order of the matrices exceeds two, as we assume, (3.6) holds [Ehrlich, 2, Theorem 2.12], and again $\mathcal{R}^+ \subset \mathcal{R}$; also $\mathcal{R}^+$ contains the group generated by the transvections, that is, by the matrices of the form $I_n + uv$ where u is a column vector, v a row vector and $vu = 0$. Moreover, with the exceptions mentioned before, the group generated by the transvections equals the commutator group. With our exceptions, then, Theorem 3.4 holds for discrete rings.

However, Theorem 3.6 does not hold in a discrete ring; the fact that it does hold for continuous rings prevents us from obtaining non-trivial determinantal values, as cosets of $\mathcal{E}$ modulo $\mathcal{R}$, for members of $\mathcal{E}$.

The identification of $\mathcal{R}$ and $\mathcal{R}^+$ for discrete rings has apparently been unnoticed previously. Using it, we are

able to obtain generalizations of some well known results in determinants; the restrictions on characteristic and order apply and the determinants are Dieudonné determinants, reducing to ordinary determinants if the relevant division ring is commutative. We note again that the following Sections 3.8-3.13 apply non-trivially only when Δ denotes the Dieudonné determinant of a non-singular square matrix over a division ring.

3.8. Theorem. Let c be non-singular and e any idempotent in $\mathcal{R}$. Then

$$\Delta(1-e+ec^{-1}e)\Delta(c) = \Delta(e+(1-e)c(1-e)).$$

Proof:
$$\begin{aligned}
&\Delta(1-e+ec^{-1}e)\Delta(c)\\
&\quad=\Delta\{(1+ec^{-1}(1-e))(1-e+ec^{-1}e)\}\Delta(c)\\
&\quad=\Delta\{1-e+ec^{-1}e+ec^{-1}(1-e)\}\Delta(c)\\
&\quad=\Delta\{(1-e)c+ec^{-1}c\}\\
&\quad=\Delta\{(1-e)c+e\}\\
&\quad=\Delta\{(1-(1-e)ce)((1-e)ce+(1-e)c(1-e)+e)\}\\
&\quad=\Delta\{(1-e)ce+(1-e)c(1-e)+e-(1-e)ce\}\\
&\quad=\Delta\{e+(1-e)c(1-e)\}.
\end{aligned}$$

3.9. Theorem (The Laplace development; compare Dieudonné 1, 37). Let $e^2=e$, $x\in\mathcal{R}$. If $R(exe)=R(e)$, then $\Delta(x)=\Delta(exe+(1-e))\Delta(e+(1-e)x(1-e)-(1-e)xe.eye.ex(1-e))$ where eye is the inverse of exe in $\mathcal{R}(e)$.

Proof: $\Delta(x) = \Delta\{(1-(1-e)xe.eye)x\}$

$=\Delta(exe + ex(1-e) + (1-e)xe + (1-e)x(1-e)-(1-e)xe.eye.exe$
$-(1-e)xe.eye.ex(1-e))$

$=\Delta(exe + ex(1-e) + (1-e)x(1-e)-(1-e)xe.eye.ex(1-e))$

$=\Delta\{(exe + ex(1-e) + (1-e)x(1-e)-(1-e)xe.eye.ex(1-e))$
$.(1-eye.ex(1-e))$

$=\Delta(exe + ex(1-e) + (1-e)x(1-e)-(1-e)xe.eye.ex(1-e)$
$-exe.eye.ex(1-e))$

$=\Delta(exe + (1-e)x(1-e)-(1-e)xe.eye.ex(1-e))$

$=\Delta(exe + (1-e))\,\Delta(e + (1-e)x(1-e)-(1-e)xe.eye.ex(1-e))$.

3.10. Theorem (Cramer's Rule). Let $ax = b$ be satisfied by $a, b, x \in \mathfrak{R}$. Then $\Delta(be + a(1-e))$ $= \Delta(a)\Delta(exe + (1-e))$ for any idempotent $e \in \mathfrak{R}$.

Proof: $ax = b$ implies $axe = be$ and so

$$\Delta(be + a(1-e)) = \Delta(axe + a(1-e))$$
$$= \Delta(a)\Delta(xe + (1-e))$$
$$= \Delta(a)\Delta\{(exe + (1-e)xe + (1-e))(1-(1-e)xe)\}$$
$$= \Delta(a)\Delta(exe + (1-e)).$$

3.11. Remark. The fact that Theorem 3.8 includes Cramer's Rule can be seen as follows.

The matrix equation $Ax = b$ with $A = (a_{ij})$, an $n \times n$ matrix and $x = \{x_1, \ldots, x_n\}$, $b = \{b_1, \ldots, b_n\}$, the components being in a division ring K, can be expressed

$$(a_{ij})\begin{pmatrix} x_1 & x_j \\ \cdot & \cdot \\ \cdots & \cdots \\ \cdot & \cdot \\ x_n & x_n \end{pmatrix} = \begin{pmatrix} b_1 & b_1 \\ \cdot & \cdot \\ \cdots & \cdots \\ \cdot & \cdot \\ b_n & b_n \end{pmatrix}$$

where each vector is replaced by a ring element with identical columns.

Taking e = diag(0,0,...,1,...) with 1 in the i-th place, Theorem 3.8 gives

$$\Delta\begin{pmatrix} a_{11} & b_1\, a_{i+1,1} & \\ \cdot & \cdot\ \cdot & \\ \cdot\ \cdots & \cdot\ \cdot & \cdots \\ \cdot & \cdot\ \cdot & \\ a_{1n} & b_n\, a_{i+1,n} & \end{pmatrix} = \Delta(A)\Delta\{\mathrm{diag}(1,\ldots,x_i,1,\ldots)\}$$

.

If C is the commutator subgroup of $K^\times$, the isomorphism of $M_n^\times(K)/C_n$ and $K^\times/C$ implies the preceding equation holds when we interpret Δ as the Dieudonné determinant (K non-commutative) or as the ordinary determinant (K commutative).

3.12. Remark. The matrix equation yA = b with $A = (a_{ij})$, an nxn matrix, and $y = (y_1, \ldots, y_n)$, $b = (b_1, \ldots, b_n)$ and subsequent treatment by the Dieudonné determinant to give a generalization of Cramer's Rule can be further generalized in a similar fashion to Theorem 3.10. We have

3.13. Theorem. Let ya = b be satisfied by a, b, y in the complete rank ring $\mathcal{R}$. Then

$$\Delta(eb + (1-e)a) = \Delta(a)\Delta(eye + (1-e)).$$

Proof: ya = b implies eya = eb and

$$\begin{aligned}\Delta(eb + (1-e)a) &= \Delta(ey + (1-e))\Delta(a) \\ &= \Delta(a)\Delta(eye + ey(1-e) + (1-e)) \\ &= \Delta(a)\Delta\{(1-ey(1-e))(eye + ey(1-e) + (1-e)\} \\ &= \Delta(a)\Delta(eye + (1-e)).\end{aligned}$$

CHAPTER IV

THE ORE DETERMINANT

The early paragraphs of this chapter consist of a rephrasing of the Ore theory of determinants, originally discussed in relation to the solution of simultaneous equations. Although the language of vector spaces could no doubt be introduced, there appears no direct advantage in doing so and, as we shall see, the theory tends more toward the ring theoretic viewpoint of matrix algebra rather than the vector space interpretations.

4.1. Definition. An integral domain S is said to have the common right multiple property if, for any two non-zero elements $a, b \in S$, there exist $m \neq 0$, $n \neq 0$ in S such that $an = bm$.

Among the rings satisfying Definition 4.1 are the division rings. Moreover, a ring with the common right multiple property can be imbedded in a right quotient (division) ring consisting of all formal quantities ab^{-1}, $b \neq 0$ in S, with the identification of ab^{-1}, $a_1 b_1^{-1}$ if and only if $bm_1 = b_1 m$ implies $am_1 = a_1 n$. In addition, if the formal quantities ab^{-1} with $a, b \neq 0$ belonging to an integral domain S form a division ring, then S has the common right multiple property. However, this does not preclude an integral domain without the common (right)

multiple property being a subring of a division ring [Ritt, 8], although there do exist non-commutative integral domains that cannot be imbedded in division rings.

4.2. Definition. Let $A = (a_{ij})$ be a 2x2 matrix with elements in an integral domain S with the common multiple property. Then suppose A_{22}, A_{21} are such that

$$a_{21} A_{22} + a_{22} A_{21} = 0$$

and

$$A_{22} \neq 0,\ A_{21} \neq 0 \text{ when } a_{21} \neq 0,\ a_{22} \neq 0$$
$$A_{22} \neq 0,\ A_{21} = 0 \text{ when } a_{21} = 0,\ a_{22} \neq 0$$
$$A_{22} = 0,\ A_{21} \neq 0 \text{ when } a_{21} \neq 0,\ a_{22} = 0$$
$$A_{22} = 0,\ A_{21} = 0 \text{ when } a_{21} = 0,\ a_{22} = 0.$$

We define the right-hand determinant $|A|$ of A to be

(4.1) $$a_{11} A_{22} + a_{12} A_{21}.$$

By a different choice of A_{22}, A_{21} we obtain different determinants; introducing the quotient division ring K, the different expressions (4.1) can each be obtained from a given one by right-hand multiplication by an element $k \neq 0$ in K. A right-hand determinant is therefore either zero or non-zero in K; the determinant, in other words, maps the matrix ring into the right ideals $(0)_r$, $(1)_r$ in K.

4.3. Definition. Suppose A'_{22}, A'_{12} are such that

$$A'_{22} a_{12} + A'_{12} a_{22} = 0$$

and

$A'_{22} \neq 0$, $A'_{12} \neq 0$ when $a_{12} \neq 0$, $a_{22} \neq 0$

$A'_{22} \neq 0$, $A'_{12} = 0$ when $a_{12} = 0$, $a_{22} \neq 0$

$A'_{22} = 0$, $A'_{12} \neq 0$ when $a_{12} \neq 0$, $a_{22} = 0$

$A'_{22} = 0$, $A'_{12} = 0$ when $a_{12} = 0$, $a_{22} = 0$.

We define the left-hand determinant $\|A|$ of A to be

$$A'_{22} a_{11} + A'_{12} a_{21}.$$

Again, a left-hand determinant is either zero or non-zero in K; the determinant maps the matrix ring into the left ideals $(0)_\ell$, $(1)_\ell$ in K.

For the general nxn matrix $A = (a_{ij})$, with elements in K and $n > 1$, we can define the right-hand and left-hand determinants similarly as follows.

4.4. Definition. Write $e_i = \mathrm{diag}(0,0,\ldots,1,0,\ldots)$ with 1 in the i-th place. In the case where $((1-e_1)A)^r_\ell$ has normalized dimension $\frac{1}{n}$, let $\{A_1, A_2, \ldots, A_n\}$ be a non-zero vector occurring as a column of a matrix of $((1-e_1)A)^r_\ell$. The Ore right-hand determinant $|A\|$ is defined as

$$a_{11} A_1 + a_{12} A_2 + \ldots + a_{1n} A_n.$$

If $D((1-e_1)A)^r_\ell > \frac{1}{n}$, we define $|A\| = 0$.

4.5. Definition. If $(A(1-e_1))^\ell_r$ has dimension $\frac{1}{n}$, let $(A'_1, A'_2, \ldots, A'_n)$ be a non-zero vector occurring in a row of a matrix of $(A(1-e_1))^\ell_r$. The Ore left-hand determinant $\|A|$ is defined as

$$A'_1 a_{11} + A'_2 a_{21} + \ldots + A'_n a_{n1}.$$

If $D(A(1-e_1))^\ell_r > \frac{1}{n}$, we define $\|A| = 0$.

In the nxn case also, the right-hand and left-hand determinants are either zero or non-zero. Any different possible choice of the $A_1, A_2, \ldots, (A_1', A_2', \ldots)$ will produce a right- (left-) hand determinant which is a right (left) multiple of a given one by some non-zero element in K.

The principal result of Ore is to the effect that a matrix is non-singular if and only if its right or its left determinant is non-zero. We will generalize this result in what follows; the result is stated and proved by non-inductive methods for more general cases, including that when the ring elements are drawn from a complete rank ring.

4.6. Lemma. Let a be non-singular in a complete rank ring $\mathfrak{R}$ and let $e \in \mathfrak{R}$, $e^2 = e$, $R(e) > 0$. If $((1-e)a)_\ell^\tau = (g)_r$, then $ag = eag \neq 0$.

Proof: We have $(1-e)ag = 0$ whence $ag = eag$ or $g = a^{-1}eag$. That is, $eag = 0$ implies $g = 0$ or $((1-e)a)_\ell^\tau = (0)_r$. Hence $R((1-e)a) = R(1-e) = 1$, a contradiction.

4.7. Corollary. Let A be a non-singular nxn matrix over a division ring K. Then $|A\| \neq 0$.

Proof: Put $e = e_1$. Then $(g)_r$ consists of all matrices whose columns right annihilate the last n-1 rows of A and g is of rank $\frac{1}{n}$. By the Lemma the row vector $e_1 Ag$ is not zero and any non-zero element in $e_1 Ag$ is a value of $|A\|$, by Definition 3.4.

4.8. Lemma. In a discrete ring $\mathfrak{R}$ (matrices of order n)

let $e^2 = e$, $(g)_r = ((1-e)a)_\ell^r$, $g^2 = g$, $R(g) = R(e) = \frac{1}{n}$ and $ag = eag \neq 0$. Then a is non-singular.

Proof: By Lemma 2.16 there exists $t \in \mathcal{F}$ such that $g = tet^{-1}$. Then $1-g = t(1-e)t^{-1}$. Also $(1-e)ag = 0$ and hence $(1-e)ate = 0$. That is,

$$at = eate + eat(1-e) + (1-e)at(1-e)$$

by the Peirce decomposition.

Since $eag \neq 0$ and $R(e) = \frac{1}{n}$,

$$R(eate) = R(eatet^{-1}) = R(eag) = \frac{1}{n}.$$

Let ese be the inverse of eate in $\mathcal{R}(e)$. Noting that $1-ese.eat(1-e)$ is of class 2 and so has an inverse in $\mathcal{R}$, we have that

$$\begin{aligned} R(a) &= R(at(1-ese.eat(1-e)) \\ &= R(eate + eat(1-e) + (1-e)at(1-e)-eat(1-e)) \\ &= R(eate + (1-e)at(1-e)). \end{aligned}$$

The result follows if $R((1-e)at(1-e)) = 1-\frac{1}{n}$. To show this we note that

$$(1-g)_\ell = (g)_r^\ell = ((1-e)a)_\ell^{r+\ell} = ((1-e)a)_\ell$$

and so $(1-e)a(1-g) = (1-e)a$. But this shows that $(1-e)at.(1-e)t^{-1} = (1-e)a$ or $(1-e)at(1-e) = (1-e)at$. Hence $R((1-e)at(1-e)) = R((1-e)at) = R((1-e)a) = 1-\frac{1}{n}$.

We remove the requirement that the generator g of $((1-e)a)_\ell^r$ be idempotent in the next Corollary.

4.9. Corollary. In a discrete ring $\mathcal{R}$ let $e^2 = e$, $(p)_r = ((1-e)a)_\ell^r$, $R(p) = R(e) = \frac{1}{n}$, $ap = eap \neq 0$. Then

a is non-singular.

Proof: Let $(g)_r = (p)_r$ where $g^2 = g$. Then $p = gu$ for some $u \in \mathcal{R}$. The condition $eap \neq 0$ implies $eagu \neq 0$, hence $eag \neq 0$. The Lemma can now be applied.

4.10. Lemma. Let e be an idempotent in the discrete ring $\mathcal{R}$ with $R(e) = \frac{1}{n}$. If $D((1-e)a)_\ell^r > \frac{1}{n}$, then a is singular.

Proof: $D((1-e)a)_\ell^r > \frac{1}{n}$ implies $R((1-e)a) < 1-\frac{1}{n}$ and if a is non-singular, then $R((1-e)a) = R(1-e) = 1-\frac{1}{n}$, a contradiction.

4.11. Corollary. Let $A = (a_{ij})$ be an nxn matrix over a division ring K and $|A| \neq 0$. Then A is non-singular.

Proof: If $|A| \neq 0$, in the terminology of Definition 4.4 we have that $((1-e_1)A)_\ell^r$ has dimension $\frac{1}{n}$ and the conditions of Lemma 4.10 do not apply.

As $0 \neq |A| = a_{11}A_1 + a_{12}A_2 + \ldots + a_{1n}A_n$ where $\{A_1, A_2, \ldots, A_n\}$ is a non-zero vector occurring in a column of a matrix of $((1-e_1)A)_\ell^r$, then $e_1Ag \neq 0$ if $(g)_r = ((1-e_1)A)_\ell^r$. The rank of e_1Ag is $\frac{1}{n}$. By Corollary 4.9, A is non-singular.

4.12. Theorem. In a complete rank ring $\mathcal{R}$ let $e \neq 0$ be idempotent and $(g)_r = ((1-e)a)_\ell^r$.

(i) If $R(g) = R(e) = R(eag)$, then a is non-singular.

(ii) If $R(g) = R(e) > R(eag)$, then a is singular.

(iii) If $R(g) > R(e)$, then a is singular.

Proof: We adapt the proofs of Lemmas 4.8, 4.10 to this more general situation.

(i), (ii). Suppose first that g is idempotent. Let $t \in \mathcal{C}$ have the property that $g = tet^{-1}$. We have again

$$(1-e)a(1-g) = (1-e)a$$

and so

$$(1-e)at(1-e) = (1-e)at.$$

Hence $R((1-e)at(1-e)) = R((1-e)at) = R((1-e)a)$. But, by the definition of g, $R((1-e)a) = R(1-g) = 1-R(g)$ and so by the assumption $R(g) = R(e)$,

$$R((1-e)at(1-e)) = 1-R(e) = R(1-e).$$

Let $(1-e)v(1-e)$ be the inverse of $(1-e)at(1-e)$ in $\mathcal{R}(1-e)$. The quantity $1-eat(1-e).(1-e)v(1-e)$ is of class 2, is thus non-singular and

$$\begin{aligned}&(1-eat(1-e).(1-e)v(1-e))at\\ &= (1-eat(1-e).(1-e)v(1-e))(eate + eat(1-e) + (1-e)at(1-e))\\ &= eate + (1-e)at(1-e).\end{aligned}$$

We conclude that

$$R(a) = R(at) = R(eate) + 1-R(e) = R(eag) + 1-R(e).$$

If g is not idempotent, let $(g)_r = (h)_r$ where $h^2 = h$. Then $g = hg$, $h = gw$ for some $w \in \mathcal{R}$. Hence

$$R(eah) \geq R(eahg) = R(eag) \geq R(eagw) = R(eah),$$

or $R(eag) = R(eah)$. Because of the first portion of the proof

$$R(a) = R(eah) + 1-R(e)$$

and the result then follows.

(iii). $R(g) > R(e)$ implies $R((1-e)a) < 1-R(e)$; if a is non-singular, then $R((1-e)a) = R(1-e) = 1-R(e)$, a contradiction.

4.13. Corollary. Let a be an arbitrary member of a complete rank ring $\mathfrak{R}$, $e \neq 0$ an idempotent in $\mathfrak{R}$ and $(g)_r = ((1-e)a)_\ell^r$. Then a necessary and sufficient condition that a be non-singular is that

$$R(e) = R(g) = R(eag).$$

There are analogues of the results 4.6-4.16 corresponding to the Ore left-hand determinant. The principal results are

4.14. Theorem. The $n \times n$ matrix A with elements in a division ring is non-singular if and only if $\|A| \neq 0$.

4.15. Corollary. $\|A| \neq 0$ if and only if $|A\| \neq 0$. $\|A| = 0$ if and only if $|A\| = 0$.

4.16. Theorem. In a complete rank ring $\mathfrak{R}$ let $e \neq 0$ be idempotent and $(h)_\ell = (a(1-e))_r^\ell$.

(i) If $R(h) = R(e) = R(hae)$, then a is non-singular.

(ii) If $R(h) = R(e) > R(hae)$, then a is singular.

(iii) If $R(h) > R(e)$, then a is singular.

Lemma 4.8, Corollary 4.9 and the corresponding results for left-hand determinants show that the particular n-1 rows in the case of the right-hand determinant and n-1 columns in the case of the left-hand determinant chosen to determine a vector to multiply into the remaining row or column in order to obtain $|A\|$, $\|A|$ is immaterial. Moreover, the matrices

effecting elementary operations are non-singular; hence we can interchange rows or columns of a given square matrix A, left multiply a row or right multiply a column of A by a non-zero quantity of K and add a left multiple of one row of A to another etc. and obtain a new matrix whose right-hand and left-hand determinants are zero or non-zero according to the value with A.

With a certain qualification, also, the Ore determinants have the property of splitting a row or column. We have the trivial

4.17. Corollary. In a complete rank ring, if $a = ea_1 + ea_2 + (1-e)a$ and $(g)_r = ((1-e)a)^*_\ell$, then $eag = ea_1 g + ea_2 g$. In particular, if $A = (a_{ij})$, with $a_{ij} = a_i^{(1)} + a_i^{(2)}$ (a_{ij}, $a_i^{(1)}$, $a_i^{(2)} \in K$ for i, j = 1, 2, ..., n) and the same non-zero $\{A_1, A_2, \ldots, A_n\}$, a column in a matrix of $((1-e_1)A)^*_\ell$ is used in all determinants, then $|A|$ is the sum of the right-hand determinants of

$$\begin{pmatrix} a_1^{(1)} & a_2^{(1)} & \cdots \\ a_{21} & \cdot & \cdots \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{pmatrix}, \begin{pmatrix} a_1^{(2)} & a_2^{(2)} & \cdots \\ a_{21} & \cdot & \cdots \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{pmatrix} .$$

4.13. Among the basic results of classical linear algebra which have so far had no extension to the case of

matrices over an arbitrary division ring is the Cayley-Hamilton Theorem. This states that the matrix equation $|\lambda I_n - A| I_n = 0$ is satisfied by the $n \times n$ matrix A, where the elements of A belong to a commutative field. We will be able to obtain fragmentary results in the direction of a possible extension. The determinant used will be the Ore determinant, and the results will be, in the main, purely formal, principally because there are apparently few known properties of the "polynomial" ring involved.

Let $K[\lambda]$ be the ring (with unity $1 \in K$) generated by the division ring K and the symbol λ. No law for the commutation of λ with the quantities of K is assumed, except that λ may be taken to commute with the quantities in the center of K.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{ij} \in K$. Then

$\lambda I - A = \begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix}$. The various Ore determinants of $\lambda I - A$ include in the cases

(i) $a_{12} \neq 0$, $a_{21} \neq 0$

(4.2) $$(\lambda - a_{11}) a_{21}^{-1} (\lambda - a_{22}) - a_{12}$$

(4.3) $$(\lambda - a_{22}) a_{12}^{-1} (\lambda - a_{11}) - a_{21}.$$

The expression (4.2) is obtained as a formal right-hand determinant as

$$-a_{21}\cdot a_{21}^{-1}(\lambda - a_{22}) + (\lambda - a_{23})\cdot 1 = 0$$

and as a left-hand determinant as

$$1\cdot(\lambda - a_{11}) + (\lambda - a_{11})a_{21}^{-1}(-a_{21}) = 0.$$

The expression (4.3) is obtained as a right-hand determinant by interchanging the roles of the rows and as a left-hand determinant by interchanging the roles of the columns.

Moreover, both (4.2) and (4.3) when equated to zero form matrix equations

(4.4) $$(\lambda - a_{11}I)a_{21}^{-1}(\lambda - a_{22}I) - a_{12}I = 0$$

(4.5) $$(\lambda - a_{22}I)a_{12}^{-1}(\lambda - a_{11}I) - a_{21}I = 0$$

which are satisfied by $\lambda = A$. For

$$(A - a_{11}I)a_{21}^{-1}(A - a_{22}I) - a_{12}I$$

$$= \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} - a_{11} \end{pmatrix} \begin{pmatrix} a_{21}^{-1}(a_{11} - a_{22}) & a_{21}^{-1}a_{12} \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} a_{12} & 0 \\ 0 & a_{12} \end{pmatrix}$$

$$= \begin{pmatrix} a_{12} & 0 \\ 0 & a_{12} \end{pmatrix} - \begin{pmatrix} a_{12} & 0 \\ 0 & a_{12} \end{pmatrix} = 0$$

and $(A - a_{22}I)a_{12}^{-1}(A - a_{11}I) - a_{21}I$

$$= \begin{pmatrix} a_{11} - a_{22} & a_{12} \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a_{12}^{-1}a_{21} & a_{12}^{-1}(a_{22} - a_{11}) \end{pmatrix} - \begin{pmatrix} a_{21} & 0 \\ 0 & a_{21} \end{pmatrix}$$

$$= 0.$$

We are justified then, it seems, in calling (4.2), (4.3) the characteristic functions of A, particularly as

(4.4), (4.5) become, when K is commutative and λ is assumed to commute with the members of K, the matrix version of the ordinary characteristic equation.

(ii) $a_{12} = 0,\ a_{21} \neq 0.$

$$(\lambda - a_{11})a_{21}^{-1}(\lambda - a_{22}).$$

This expression is again a formal right-hand and left-hand determinant. The matrix equation

$$(\lambda - a_{11}I)a_{21}^{-1}(\lambda - a_{22}I) = 0$$

is satisfied by $\lambda = A$ for

$$\begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} - a_{11} \end{pmatrix} \begin{pmatrix} a_{21}^{-1}(a_{11} - a_{22}) & 0 \\ 1 & 0 \end{pmatrix} = 0.$$

(iii) $a_{12} \neq 0,\ a_{21} = 0.$

$$(\lambda - a_{22})a_{12}^{-1}(\lambda - a_{11}).$$

This is a formal right-hand and left-hand determinant and the matrix equation

$$(\lambda - a_{22}I)a_{12}^{-1}(\lambda - a_{11}I) = 0$$

is satisfied by $\lambda = A$.

(iv) $a_{12} = 0,\ a_{21} = 0.$

$$(\lambda - a_{11})(\lambda - a_{22})$$

$$(\lambda - a_{22})(\lambda - a_{11}).$$

In this case,

$$(\lambda - a_{11}I)(\lambda - a_{22}I) = 0$$

$$(\lambda - a_{22}I)(\lambda - a_{11}I) = 0$$

are satisfied by $\lambda = A$.

In all cases, equations in λ are obtainable from $\det(\lambda I-A) = 0$, which in the commutative case are identical with the characteristic equations. Also, obviously, the elements of A need not have originally been drawn from a division ring; the following result holds under the weaker condition that K can be imbedded in a division ring.

4.19. Theorem. Let A be a 2×2 matrix with elements in a division ring K. Then there exists a formal Ore determinant of $\lambda I-A$ in $K[\lambda]$ such that A satisfies the matrix equation obtained by equating the determinant to zero (and replacement of scalar quantities by corresponding scalar matrices).

4.20. The procedures of the previous sections do not go quite so smoothly for general 3×3 matrices A. For example, to find in this situation a column vector which annihilates under matrix (right) multiplication two rows of $\lambda I-A$ we must, it seems, use formal inverses of the quantities of $K[\lambda]$. We will examine effectively all cases, however, and will find positive results in almost all. As the Ore determinant does not depend essentially on whether we examine the right- or left-hand determinant and, as it is often immaterial what two rows (columns) of $\lambda I-A$ we find an annihilating column (row) vector for, we can reduce the number of particular matrices to be examined.

4.21. In this paragraph we will use formal inverses

of quantities in $K[\lambda]$ and no attempt is made to justify this procedure rigorously. It is apparently a problem of some difficulty to determine in what measure $K[\lambda]$ can be imbedded in a division ring. Because of the unique importance of the Cayley-Hamilton Theorem in classical linear algebra, due to its initiation of the deeper results of that theory, the author feels that the present extensions, although fragmentary, are not without value.

Let $A = (a_{ij})$ be a 3X3 matrix with elements in a division ring K, and suppose that $a_{23} \neq 0$, $a_{32} \neq 0$. We will show by computation that the matrix product

$$(4.6)\quad \begin{pmatrix} -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ \left[(\lambda - a_{33})a_{23}^{-1}(\lambda - a_{22}) - a_{32}\right]^{-1}\left[(\lambda - a_{33})a_{23}^{-1}a_{21} + a_{31}\right] \\ \left[(\lambda - a_{22})a_{32}^{-1}(\lambda - a_{33}) - a_{23}\right]^{-1}\left[(\lambda - a_{22})a_{32}^{-1}a_{31} + a_{21}\right] \end{pmatrix}$$

is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. An Ore determinant of $\lambda I - A$ is then

$$(4.7)\quad \begin{aligned} \lambda - a_{11} - a_{12}\left[(\lambda - a_{33})a_{23}^{-1}(\lambda - a_{22}) - a_{32}\right]^{-1}\left[(\lambda - a_{33})a_{23}^{-1}a_{21} + a_{31}\right] \\ - a_{13}\left[(\lambda - a_{22})a_{32}^{-1}(\lambda - a_{33}) - a_{23}\right]^{-1}\left[(\lambda - a_{22})a_{32}^{-1}a_{31} + a_{21}\right]. \end{aligned}$$

On equating to zero, (4.7) simplifies to the ordinary characteristic equation if K is commutative and λ is assumed to commute with the members of K.

Returning to (4.6), the first element in the matrix

product is

$$-a_{21}+\left[(\lambda-a_{22})^{-1}\right]^{-1}\left[(\lambda-a_{33})a_{23}^{-1}(\lambda-a_{22})-a_{32}\right]^{-1}\left[(\lambda-a_{33})a_{23}^{-1}a_{21}+a_{31}\right]$$
$$-(a_{23}^{-1})^{-1}\left[(\lambda-a_{22})a_{32}^{-1}(\lambda-a_{33})-a_{23}\right]^{-1}\left[(\lambda-a_{22})a_{32}^{-1}a_{31}+a_{21}\right]$$
$$=-a_{21}+\left[(\lambda-a_{33})a_{23}^{-1}-a_{32}(\lambda-a_{22})^{-1}\right]^{-1}(\lambda-a_{33})a_{23}^{-1}a_{21}$$
$$+\left[(\lambda-a_{33})a_{23}^{-1}-a_{32}(\lambda-a_{22})^{-1}\right]^{-1}a_{31}$$
$$-\left[(\lambda-a_{22})a_{32}^{-1}(\lambda-a_{33})a_{23}^{-1}-1\right]^{-1}(\lambda-a_{22})a_{32}^{-1}a_{31}$$
$$-\left[(\lambda-a_{22})a_{32}^{-1}(\lambda-a_{33})a_{23}^{-1}-1\right]^{-1}a_{21}$$
$$=-a_{21}+\left[1-a_{23}(\lambda-a_{33})^{-1}a_{32}(\lambda-a_{22})^{-1}\right]^{-1}a_{21}$$
$$+\left[(\lambda-a_{33})a_{23}^{-1}-a_{32}(\lambda-a_{22})^{-1}\right]^{-1}a_{31}$$
$$-\left[(\lambda-a_{33})a_{23}^{-1}-a_{32}(\lambda-a_{22})^{-1}\right]^{-1}a_{31}$$
$$-\left[(\lambda-a_{22})a_{32}^{-1}(\lambda-a_{33})a_{23}^{-1}-1\right]^{-1}a_{21}$$
$$=\left[-1+\left\{1-a_{23}(\lambda-a_{33})^{-1}a_{32}(\lambda-a_{22})^{-1}\right\}^{-1}\right.$$
$$\left.+\left\{1-(\lambda-a_{22})a_{32}^{-1}(\lambda-a_{33})a_{23}^{-1}\right\}^{-1}\right]a_{21}$$
$$=\left[-1+(\lambda-a_{22})a_{32}^{-1}\left\{(\lambda-a_{22})a_{32}^{-1}-a_{23}(\lambda-a_{33})^{-1}\right\}^{-1}\right.$$
$$\left.+a_{23}(\lambda-a_{33})^{-1}\left\{a_{23}(\lambda-a_{33})^{-1}-(\lambda-a_{22})a_{32}^{-1}\right\}^{-1}\right]a_{21}$$
$$=\left[-1+\left\{(\lambda-a_{22})a_{32}^{-1}-a_{23}(\lambda-a_{33})^{-1}\right\}\left\{(\lambda-a_{22})a_{32}^{-1}-a_{23}(\lambda-a_{33})^{-1}\right\}^{-1}\right]a_{21}$$
$$=0.$$

The other element of the matrix product (4.6) is

$$-a_{31}-(a_{32}^{-1})^{-1}\left[(\lambda-a_{33})a_{23}^{-1}(\lambda-a_{22})-a_{32}\right]^{-1}\left[(\lambda-a_{33})a_{23}^{-1}a_{21}+a_{31}\right]$$
$$+\left[(\lambda-a_{33})^{-1}\right]^{-1}\left[(\lambda-a_{22})a_{32}^{-1}(\lambda-a_{33})-a_{23}\right]^{-1}\left[(\lambda-a_{22})a_{32}^{-1}a_{31}+a_{21}\right]$$
$$=-a_{31}-\left[(\lambda-a_{33})a_{23}^{-1}(\lambda-a_{22})a_{32}^{-1}-1\right]^{-1}(\lambda-a_{33})a_{23}^{-1}a_{21}$$
$$-\left[(\lambda-a_{33})a_{23}^{-1}(\lambda-a_{22})a_{32}^{-1}-1\right]^{-1}a_{31}$$
$$+\left[(\lambda-a_{22})a_{32}^{-1}-a_{23}(\lambda-a_{33})^{-1}\right]^{-1}(\lambda-a_{22})a_{32}^{-1}a_{31}$$
$$+\left[(\lambda-a_{22})a_{32}^{-1}-a_{23}(\lambda-a_{33})^{-1}\right]^{-1}a_{21}$$

$$= \left[-1 - a_{32}(\lambda - a_{22})^{-1} \left\{ (\lambda - a_{33}) a_{23}^{-1} - a_{32}(\lambda - a_{22})^{-1} \right\}^{-1} \right.$$
$$\left. + (\lambda - a_{33}) a_{23}^{-1} \left\{ (\lambda - a_{33}) a_{23}^{-1} - a_{32}(\lambda - a_{22})^{-1} \right\} \right] a_{31}$$
$$= \left[-1 + \left\{ (\lambda - a_{33}) a_{23}^{-1} - a_{32}(\lambda - a_{22})^{-1} \right\} \left\{ (\lambda - a_{33}) a_{23}^{-1} - a_{32}(\lambda - a_{22})^{-1} \right\}^{-1} \right] a_{31}$$
$$= 0.$$

If we make the further assumption that either a_{12} or a_{13} but not both are zero, say $a_{12} = 0$, we obtain on equating (4.7) to zero the expression

$$(\lambda - a_{22}) a_{32}^{-1} (\lambda - a_{33}) a_{13}^{-1} (\lambda - a_{11}) - a_{23} a_{13}^{-1} (\lambda - a_{11})$$
$$- (\lambda - a_{22}) a_{32}^{-1} a_{31} - a_{21} = 0.$$

The corresponding matrix equation

$$(4.8) \quad (\lambda - a_{22} I) a_{32}^{-1} (\lambda - a_{33} I) a_{13}^{-1} (\lambda - a_{11} I) - a_{23} a_{13}^{-1} (\lambda - a_{11} I)$$
$$- (\lambda - a_{22} I) a_{32}^{-1} a_{31} - a_{21} I = 0$$

is satisfied by

$$\lambda = A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with $a_{13} \neq 0$, $a_{32} \neq 0$ (we may drop the condition that $a_{23} \neq 0$). For, the left side of (4.8) becomes

$$\begin{pmatrix} a_{11} - a_{22} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} - a_{22} \end{pmatrix} a_{32}^{-1} \begin{pmatrix} a_{11} - a_{33} & 0 & a_{13} \\ a_{21} & a_{22} - a_{33} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

$$\cdot a_{13}^{-1}\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22}-a_{11} & a_{23} \\ a_{31} & a_{32} & a_{33}-a_{11} \end{pmatrix} - a_{23}\, a_{13}^{-1}\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22}-a_{11} & a_{23} \\ a_{31} & a_{32} & a_{33}-a_{11} \end{pmatrix}$$

$$- \begin{pmatrix} a_{11}-a_{22} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & a_{33}-a_{22} \end{pmatrix} a_{32}^{-1} a_{31} - a_{21} I$$

$$= \begin{pmatrix} (a_{11}-a_{22})a_{32}^{-1} & 0 & a_{13}\, a_{32}^{-1} \\ a_{21}\, a_{32}^{-1} & 0 & a_{23}\, a_{32}^{-1} \\ a_{31}\, a_{32}^{-1} & 1 & (a_{33}-a_{22})a_{32}^{-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} a_{31} & a_{32} & a_{11}-a_{33}+a_{33}-a_{11} \\ (a_{22}-a_{33})a_{13}^{-1} a_{21} + a_{23} a_{13}^{-1} a_{31} & (a_{22}-a_{33})a_{13}^{-1}(a_{22}-a_{11}) + a_{23} a_{13}^{-1} a_{32} & a_{21} + (a_{22}-a_{33})a_{13}^{-1} a_{23} + a_{23} a_{13}^{-1}(a_{33}-a_{11}) \\ a_{32}\, a_{13}^{-1} a_{21} & a_{32}\, a_{13}^{-1}(a_{22}-a_{11}) & a_{31} + a_{32}\, a_{13}^{-1} a_{23} \end{pmatrix}$$

$$
- \begin{pmatrix}
(a_{11}-a_{22})a_{32}^{-1}a_{31} + a_{21} & 0 & a_{23} + a_{13}a_{32}^{-1}a_{31} \\
a_{23}a_{13}^{-1}a_{21} + a_{21}a_{32}^{-1}a_{31} & a_{23}a_{13}^{-1}(a_{22}-a_{11}) + a_{21} & a_{23}a_{13}^{-1}a_{23} + a_{23}a_{32}^{-1}a_{31} \\
a_{23}a_{13}^{-1}a_{31} + a_{31}a_{32}^{-1}a_{31} & a_{23}a_{13}^{-1}a_{32} + a_{31} & a_{23}a_{13}^{-1}(a_{33}-a_{11}) + (a_{33}-a_{22})a_{32}^{-1}a_{31} + a_{21}
\end{pmatrix}
$$

$= 0$, after some further computation.

We obviously will be able to obtain corresponding matrix equations directly from the Ore determinant of $\lambda I - A$ and satisfied by the 3×3 matrix $A = (a_{ij})$ when one $a_{ij} = 0$, $i \neq j$ and $a_{ik} \neq 0$, $a_{\ell j} \neq 0$ $(k \neq i,\ k \neq j,\ \ell \neq j,\ \ell \neq i)$.

The other types of possible A, apart from the case where no off-diagonal element is zero, have typical representatives

(i) $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ (ii) $\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$

(iii) $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$.

Case (i). The λ-matrix is

$$\begin{pmatrix} \lambda - a_{11} & 0 & 0 \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{pmatrix}$$

Among the column vectors giving a matrix product of zero on multiplication on the right of the first two rows of $\lambda I - A$ is $\{0,\ \lambda - a_{11},\ a_{23}^{-1}(\lambda - a_{22})(\lambda - a_{11})\}$, provided $a_{23} \neq 0$. An Ore determinant is

$$\left[-a_{32} + (\lambda - a_{33})a_{23}^{-1}(\lambda - a_{22})\right](\lambda - a_{11}).$$

The corresponding matrix equation is

$$\left[(\lambda - a_{33}I)a_{23}^{-1}(\lambda - a_{22}I) - a_{32}I\right](\lambda - a_{11}I) = 0.$$

When λ is replaced by A, the equation is satisfied because the left side will be, after a little computation,

$$\begin{pmatrix} (a_{11} - a_{33})a_{23}^{-1}(a_{11} - a_{22}) - a_{32} & 0 & 0 \\ a_{21}a_{23}^{-1}(a_{11} - a_{22}) + (a_{22} - a_{11})a_{23}^{-1}a_{21} + a_{31} & 0 & 0 \\ a_{31}a_{23}^{-1}(a_{11} - a_{22}) + a_{32}a_{21}^{-1}a_{21} & 0 & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} - a_{11} & a_{23} \\ a_{31} & a_{32} & a_{33} - a_{11} \end{pmatrix} = 0.$$

If $a_{23} = 0$, the matrix becomes of type (iii).

Case (ii).

$$\lambda I - A = \begin{pmatrix} \lambda - a_{11} & 0 & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & 0 & \lambda - a_{33} \end{pmatrix}$$

This case is equivalent to (i), if we find a left-hand determinant by means of a row vector multiplying into the last two columns of $\lambda I - A$ to give a zero product; if $a_{13} \neq 0$, a left-hand determinant is

$$(\lambda - a_{22})\left[(\lambda - a_{33})a_{13}^{-1}(\lambda - a_{11}) - a_{31}\right].$$

Also, $(A - a_{22}I)\left[(A - a_{33}I)a_{13}^{-1}(A - a_{11}I) - a_{31}I\right]$

$$= \begin{pmatrix} a_{11} - a_{22} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} - a_{22} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & 0 & \\ (a_{22} - a_{33})a_{13}^{-1}a_{21} + a_{23}a_{13}^{-1}a_{21} & (a_{22} - a_{33})a_{13}^{-1}(a_{22} - a_{11}) - a_{31} & \\ 0 & 0 & \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ a_{21} + (a_{22} - a_{33})a_{13}^{-1}a_{23} + a_{23}a_{13}^{-1}a_{23} \\ 0 \end{pmatrix}$$

$$= 0.$$

Case (iii).

$$\lambda I - A = \begin{pmatrix} \lambda - a_{11} & 0 & 0 \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & 0 & \lambda - a_{33} \end{pmatrix}.$$

As in Case (i), if $a_{23} \neq 0$, we obtain the matrix equation $(\lambda - a_{33} I) a_{23}^{-1} (\lambda - a_{22} I)(\lambda - a_{11} I) = 0$. By the discussion of (i), $\lambda = A$ satisfies this. If $a_{23} = 0$, the matrix product of the first two rows and the column vector $\{0, 0, (\lambda - a_{22})(\lambda - a_{11})\}$, in this order is zero and an Ore determinant is

$$(\lambda - a_{33})(\lambda - a_{22})(\lambda - a_{11}).$$

The corresponding matrix equation is

$$(\lambda - a_{33} I)(\lambda - a_{22} I)(\lambda - a_{11} I) = 0$$

which is satisfied by $\lambda = A$.

We notice that in all cases, including the general one, the expression obtained for an Ore determinant of $\lambda I - A$ becomes, on equating to zero, the characteristic equation if K is commutative.

4.22. Theorem. Let $A = (a_{ij})$ be a 3X3 matrix over a division ring K. If some a_{ij} $(i \neq j)$ is zero, then there exists a matrix equation equivalent to the ordinary characteristic equation if K is commutative and satisfied by A. The matrix equation is obtained by equating some formal Ore determinant of $\lambda I - A$ to zero.

We note finally that a characteristic function as above, using the Ore determinant, is apparently a more successful concept than the "characteristic determinant" defined similarly using the Dieudonné determinant [1, 39].

CHAPTER V

ALGEBRAICITY IN COMPLETE RANK RINGS

In this chapter we give proofs of statements (13)-(20) of C. R. A. leading to a clear-cut picture of the algebraicity over the center of an arbitrary element of a complete rank ring $\mathfrak{R}$. Let Z be the center of $\mathfrak{R}$ and P be the set of all polynomials

$$p(x) \equiv x^{\ell} + a_1 x^{\ell-1} + \ldots + a_{\ell}. \quad (a_i \in Z,\ i = 1, 2, \ldots, \ell)$$

We are interested in how small $R(p(a))$ can be made; the $p(x) \in P$ such that $R(p(a)) < 1$ we call a-singular.

5.1. Definition. We denote by T the set of all a-singular polynomials from P that are irreducible with respect to the coefficient domain Z.

5.2. Theorem. T is enumerable.

Proof: Suppose $\{p_i(x);\ i \in I\}$ are the distinct a-singular monic polynomials which are irreducible with respect to Z.

We will first establish that $\{(p_i(a))_{\ell}^{r};\ i \in I\}\perp$; for this it will be sufficient to prove $\{(p_i(a))_{\ell}^{r};\ i \in I_0\}\perp$ for every finite subset $I_0 \subset I$ [von Neumann, 10, Theorem 2.3]. Clearly this holds for sets I_0 of cardinality one. Assume the result for all sets I_0 of cardinality n. Let $\{\alpha_m;\ m = 1, 2, \ldots, n+1\}$ be a subset of I. Then the distinct, monic and irreducible $p_{\alpha_m}(x)$ $(m = 1, 2, \ldots, n+1)$

are relatively prime in pairs and there exist polynomials $g(x)$, $h(x)$ with coefficients in Z such that

$$1 = g(x)p_{\alpha_{n+1}}(x) + h(x)p_{\alpha_1}(x)p_{\alpha_2}(x)\ldots p_{\alpha_n}(x)$$

whence

$$1 = g(a)p_{\alpha_{n+1}}(a) + h(a)p_{\alpha_1}(a)p_{\alpha_2}(a)\ldots p_{\alpha_n}(a).$$

Therefore

$$(1)_\ell = (p_{\alpha_{n+1}}(a))_\ell \cup (p_{\alpha_1}(a)p_{\alpha_2}(a)\ldots p_{\alpha_n}(a))_\ell$$
$$\leqq (p_{\alpha_{n+1}}(a))_\ell \cup [(p_{\alpha_1}(a))_\ell \cap (p_{\alpha_2}(a))_\ell \cap \ldots \cap (p_{\alpha_n}(a))_\ell]$$

since the polynomials $p_{\alpha_m}(a)$, with coefficients in Z, commute. Obviously equality holds and so

$$(0)_r = (1)^r_\ell = (p_{\alpha_{n+1}}(a))^r_\ell \wedge [(p_{\alpha_1}(a))^r_\ell \cup (p_{\alpha_2}(a))^r_\ell \cup \ldots \cup (p_{\alpha_n}(a))^r_\ell];$$

[N II, Lemma 2.3]

that is, $\{(p_{\alpha_m}(a))^r_\ell;\ m = 1, 2, \ldots, n+1\}\perp$.

The cardinality of $\{(p_i(a))^r_\ell;\ i \in I\}$ is therefore at most $\aleph_0$ [Maeda, 6, 119, Anmerkung 2.2]; and since the independence shows that distinct $p_i(x)$ $(i \in I)$ yield distinct $(p_i(a))^r_\ell$, we have the result.

5.3. <u>Theorem.</u> Let α be the g.l.b. of all $R(p(a))$, $p(x) \in P$. Let $\mathfrak{o}$ ($\mathfrak{o}'$) be the intersection of all $(p(a))_r$ ($(p(a))_\ell$). Then there exists a unique idempotent $e \in \mathfrak{R}$ such that

$$(e)_r = \mathfrak{o}, \quad (e)_\ell = \mathfrak{o}'$$

and $R(e) = D(\mathfrak{o}) = D(\mathfrak{o}') = \alpha$.

Proof: Obviously $D(\bigcap_{p\in P}(p(a)_r) \leqq \alpha$.

If there is no a-singular polynomial, then $\mathfrak{a} = (1)_r$, $\mathfrak{a}' = (1)_\ell$ and 1 is the unique idempotent with this property. An arbitrary $p(x) \in P$ with $R(p(a)) < 1$ can be factored uniquely, apart from the order of the factors, into irreducible factors; that is

$$p(x) = p_1(x)p_2(x)\ldots p_s(x)p_{s+1}(x)\ldots p_n(x)$$

where the $p_i(x)$ $(i = 1, 2, \ldots, n)$ are irreducible and the $p_i(x)$ $(i = 1, 2, \ldots, s \leqq n)$ are a-singular, the remaining factors not being a-singular. Then

$$(p(a))_r = (p_1(a)p_2(a)\ldots p_s(a))_r$$

so that the cardinality of the distinct $(p(a))_r$ is no greater than $\sum_{s=1}^{\infty} \aleph_0^s = \aleph_0$. Hence the distinct $(p(a))_r$ can be enumerated thus: $\{(p_m(a))_r ; m = 1, 2, \ldots\}$.

Write $p_1(x)p_2(x)\ldots p_m(x) = p^m(x)$. Then $p^m(x) \in P$. We have $\bigcap_{i=1}^{m} (p_m(a))_r \geqq (p^m(a))_r$ where the $(p^m(a))_r$ form a descending chain in the lattice and

$$D(\bigcap_{p\in P}(p(a))_r) = D(\bigcap_{m=1}^{\infty}(p_m(a))_r) \geqq D(\bigcap_{m=1}^{\infty}(p^m(a))_r) \geqq \alpha$$

so $D(\mathfrak{a}) = \alpha$. Similarly $D(\mathfrak{a}') = \alpha$.

Now $(p'(a))_r \geqq (p^2(a))_r \geqq \ldots$. We will first consider the case, as far as the existence of a suitable idempotent e is concerned, when this sequence of lattice elements terminates; that is, when for some positive integer n, $(p^n(a))_r = (p^{n+s}(a))_r$ $(s = 1, 2, \ldots)$. This case will be covered by the more general proof later but because of the special interest we will give the brief proof possible

here. If $(p^n(a))_r = (p^{n+s}(a))_r$ for some positive integer n and for all s = 1, 2, ..., then

$$\mathfrak{a} = (p^n(a))_r.$$

Let $(p^n(a))_r = (f)_r$, $f^2 = f$. Then $R(f) = R(p^n(a)) = R(p^{n+s}(a))$ and

$$\mathfrak{a}' = \bigcap_{p \in P} (p(a))_l = (p^n(a))_l = (p^{n+s}(a))_l. \quad (s=1, 2, \ldots)$$

Also $fp^n(a) = p^n(a)$ so

$$(p^n(a)f)_r \geqq (p^n(a)fp^n(a))_r = ([p^n(a)]^2)_r \geqq \mathfrak{a} = (p^n(a))_r.$$

But $(p^n(a))_r \geqq (p^n(a)f)_r$ so equality holds and there exists $x \in \mathfrak{R}$ such that

$$p^n(a)fx = p^n(a).$$

Then $e = f + fx(1-f)$ has the property that $\mathfrak{a} = (e)_r$, $\mathfrak{a}' = (e)_l$. For the idempotent e generates $(f)_r = \mathfrak{a}$ and $(f + fx(1-f))_l$ contains

$$p^n(a)f + p^n(a)fx(1-f) = p^n(a)f + p^n(a)(1-f) = p^n(a).$$

As $R(e) = R(p^n(a))$, we have $\mathfrak{a}' = (p^n(a))_l = (e)_l$.

The uniqueness of e follows here, with the same reasoning applied in the general case, because if $(e)_r = (g)_r$, $(e)_l = (g)_l$, $g^2 = g$, then $eg = g$, $eg = e$ and $e = g$.

If $\mathfrak{R}$ is a discrete ring, that is, a matrix ring over a division ring, and thus satisfying the descending chain condition for right ideals, we do have $\mathfrak{a} = (p^n(a))_r$ for some positive integer n and consequently the result. Simple proofs of later theorems for $\mathfrak{R}$, a discrete ring, can also be given.

Returning now to the general case, let $\bigcap_{p\in P}(p(a))_r = \bigcap_{m=1}^{\infty}(p^m(a))_r = (f)_r$ with $f^2 = f$. Then

$$f = p^m(a)u_m \qquad (u_m \in \mathfrak{R},\ m = 1, 2, \ldots)$$

and hence

$$f = fp^m(a)u_m$$

or $(fp^m(a))_r \geqq (f)_r$. But obviously $(f)_r \geqq (fp^m(a))_r$ so

$$(f)_r = (fp^m(a))_r. \qquad (m = 1, 2, \ldots).$$

We thus have $\alpha \subset R(f) = R(fp^m(a)).$ $(m = 1, 2, \ldots).$

By the definition of f,

$$\{fx;\ x \in \mathfrak{R}\} = \{v;\ v = p^1(a)v_1 = p^2(a)v_2 = \ldots \text{ for some } v_1, v_2, \ldots \in \mathfrak{R}\}$$

and so

$$\{p^m(a)fx;\ x \in \mathfrak{R}\} = \{u;\ u = p^m(a)v \text{ where } v = p^1(a)v_1 = p^2(a)v_2 = \ldots \text{ for some } v_1, v_2, \ldots \in \mathfrak{R}.$$

That is, u belongs to the right side if and only if u belongs to $\bigcap_{n=1}^{\infty}(p^m(a)p^n(a))_r \geqq (f)_r$; but obviously $R(f) \geqq R(p^m(a)f)$ so

$$(p^m(a)f)_r = (f)_r. \qquad (m = 1, 2, \ldots).$$

Hence for each $m = 1, 2, \ldots$ $fp^m(a)f = p^m(a)f$ has an inverse fq_mf, say, in $\mathfrak{R}(f)$.

Also $\{p_{m+1}(a)\ldots p_{m+n}(a)fx;\ x \in \mathfrak{R}\} = \{u;\ u = p_{m+1}(a)\ldots p_{m+n}(a)v \text{ where } v = p^1(a)v_1 = p^2(a)v_2 = \ldots \text{ for some } v_1, v_2, \ldots \in \mathfrak{R}.$ $(m, n = 1, 2, \ldots).$

Hence u belongs to the right side if and only if u belongs to $\bigcap_{t=1}^{\infty}(p_{m+1}(a)\ldots p_{m+n}(a)p^t(a))_r \geqq (f)_r$; so, as before, we have $(f)_r = (p_{m+1}(a)\ldots p_{m+n}(a)f)_r$ and

$fp_{m+1}(a)\ldots p_{m+n}(a)f = p_{m+1}(a)\ldots p_{m+n}(a)f$ has an inverse of $fq'_{m+n}f$, say, in $\mathfrak{R}(f)$.

Then $fp^{m+n}(a)f = fp_1(a)\ldots p_m(a)p_{m+1}(a)\ldots p_{m+n}(a)f$

$= fp_1(a)\ldots p_m(a)fp_{m+1}(a)\ldots p_{m+n}(a)f$

$= fp_1(a)\ldots p_m(a)f.fp_{m+1}(a)\ldots p_{m+n}(a)f$

and so $fq_{m+n}f = fq'_{m+n}f.fq_mf = fq_mf.fq'_{m+n}f$, the inverses in $\mathfrak{R}(f)$ commuting as the corresponding factors commute.

We examine the idempotents $f + fq_mfp^m(a)(1-f)$ $(m = 1, 2, \ldots)$. For each m,

$$(f + fq_mfp^m(a)(1-f))_+ = (f)_+ = \bigcap_{p\in P}(p(a))_+$$

and $(f + fq_mfp^m(a)(1-f))_\ell$ contains

$fp^m(a)f + fp^m(a)f.fq_mfp^m(a)(1-f)$

$= fp^m(a)f + fp^m(a)(1-f)$

$= fp^m(a)$.

Since $R(f + fq_mfp^m(a)(1-f)) = R(f) = R(fp^m(a))$, then

$$(f + fq_mfp^m(a)(1-f))_\ell = (fp^m(a))_\ell \leqq (p^m(a))_\ell \leqq (p^{m-1}(a))_\ell \leqq \ldots \leqq (p'(a))_\ell . \qquad (m = 1, 2, \ldots).$$

Moreover, the sequence of idempotents $f + fq_mfp^m(a)(1-f)$ approaches a limit as $m \to \infty$ since

$R(\{f + fq_mfp^m(a)(1-f)\} - \{f + fq_{m+n}fp^{m+n}(a)(1-f)\})$

$= R(fq_mfp^m(a)(1-f) - fq_{m+n}fp^{m+n}(a)(1-f))$

$= R(fq_mf\{f - fq'_{m+n}fp_{m+1}(a)\ldots p_{m+n}(a)\}p^m(a)(1-f))$

$= R(fq_mf\{f - fq'_{m+n}fp_{m+1}(a)\ldots p_{m+n}(a)f$

$- fq'_{m+n}fp_{m+1}(a)\ldots p_{m+n}(a)(1-f)\}p^m(a)(1-f))$

$= R(fq_mf\{f - f - fq'_{m+n}fp_{m+1}(a)\ldots p_{m+n}(a)(1-f)\}p^m(a)(1-f))$

$\leqq R((1-f)p^m(a)(1-f) \to 0$ when $m \to \infty$ since

$((1-f)p^m(a)(1-f))_r$ contains $(1-f)p^m(a)(1-f)p_{m+1}(a)(1-f)$

$= (1-f)\{p^{m+1}(a)-p^m(a)fp_{m+1}(a)\}(1-f)$

$= (1-f)\{p^{m+1}(a)-fp^m(a)fp_{m+1}(a)\}(1-f)$

$= (1-f)p^{m+1}(a)(1-f)$.

Hence $\{((1-f)p^m(a)(1-f))_r ; m = 1, 2, \ldots\}$ form a descending chain in the lattice and $(p^m(a))_r = (fp^m(a) + (1-f)p^m(a)(1-f))_r$ $= (fp^m(a))_r \cup ((1-f)p^m(a)(1-f))_r$ since $(1-f)p^m(a)f$ $= (1-f)fp^m(a)f = 0$ and $(fp^m(a) + (1-f)p^m(a)(1-f))_r$ contains $fp^m(a)f$ and so $fp^m(a)$ as $(fp^m(a))_r = (f)_r$ $= (fp^m(a)f)_r$; consequently $(fp^m(a) + (1-f)p^m(a)(1-f))_r$ contains $fp^m(a) + (1-f)p^m(a)(1-f)-fp^m(a) = (1-f)p^m(a)(1-f)$. The inclusion in the opposite direction is obvious.

Then $(p^m(a))_+ = (f)_r \cup ((1-f)p^m(a)(1-f))_r$ and

$$(f)_+ = \bigcap_{m=1}^{\infty} (p^m(a))_r = \bigcap_{m=1}^{\infty} [(f)_+ \cup ((1-f)p^m(a)(1-f))_+]$$

$$= (f)_r \cup [\bigcap_{m=1}^{\infty} ((1-f)p^m(a)(1-f))_r]$$

[Axiom III].

That is, $(f)_r \geqq \bigcap_{m=1}^{\infty} ((1-f)p^m(a)(1-f))_r$. As the principal right ideals are disjoint, apart from 0, we must have $\bigcap_{m=1}^{\infty} ((1-f)p^m(a)(1-f))_+ = (0)_r$ and so $\lim_{m\to\infty} R((1-f)p^m(a)(1-f)) = 0$.

We conclude with the observations necessary to show that if $\lim_{m\to\infty} (f + fq_m fp^m(a)(1-f)) = e$, then e satisfies the requirements of the theorem. For e is idempotent, being the limit of a sequence of idempotents; also e has the form $f + fx(1-f)$ for some $x \in \mathfrak{A}$ by the continuity of multiplication since, as we have shown, $\lim_{m\to\infty} (fq_m fp^m(a)(1-f))$ exists

and it equals $f(\lim_{m\to\infty}(fq_m fp^m(a)(1-f)))(1-f)$ whence $(e)_r$ $= (f + fx(1-f))_r = (f)_r = \bigcap_{p\in P}(p(a))_r$. To show that $(e)_\ell$ $= \bigcap_{p\in P}(p(a))_\ell$ we note first that if a principal left ideal in $\mathfrak{R}$ contains a convergent sequence, it contains the limit of the sequence. For, let $(g)_\ell$ $(g^2 = g)$ contain the convergent sequence u_i $(i = 1, 2, \ldots)$. Then $u_i = u_i g$ implies by the continuity of multiplication that $\lim_{i\to\infty} u_i$ $= (\lim_{i\to\infty} u_i)g$ or $\lim_{i\to\infty} u_i \in (g)_\ell$. As we showed above

$$(p^m(a))_\ell \geqq (fp^{m+n}(a))_\ell = (f + fq_{m+n}fp^{m+n}(a)(1-f))_\ell$$
$$(m, n = 1, 2, \ldots)$$

so for each $m = 1, 2, \ldots$ $(p^m(a))_\ell$ contains $e = \lim_{t\to\infty}(f + fq_t fp^t(a)(1-f))$. Hence $\bigcap_{p\in P}(p(a))_\ell = \bigcap_{m=1}^{\infty}(p^m(a))_\ell$ $\geqq (e)_\ell$. But as the dimensions are equal, equality holds.

5.4. Theorem. Let $T = \{ q_i(x);\ i = 1, 2, \ldots$ and $q_i(x) \in P$ and is irreducible and a-singular$\}$. Define α_i, σ_i, σ_i' $(i = 1, 2, \ldots)$ similarly to the α, σ, σ' of Theorem 5.3 but restricting $p(x)$ to the $(q_i(x))^t$ $(t = 1, 2, \ldots)$. Then there exists a unique idempotent $e_i \in \mathfrak{R}$ such that $(e_i)_r = \sigma_i$, $(e_i)_\ell = \sigma_i'$.

Also $R(e_i) = D(\sigma_i) = D(\sigma_i') = \alpha_i$.

Proof: The proof follows much the same lines as that of Theorem 5.3 and we will include only a summary of it.

Let $q_i(x)$ be an irreducible a-singular polynomial. Write $(q_i(a))^t = q^t$. Clearly $D(\bigcap_{t=1}^{\infty}(q^t)_r) = \alpha_i = D(\bigcap_{t=1}^{\infty}(q^t)_\ell)$ since $(q^t)_{r \text{ or } \ell}$ form a descending chain.

Now let $\bigcap_{t=1}^{\infty}(q^t)_r = (f)_r$ with $f^2 = f$. Then $f = q^n u_n$ and $(f)_r = (fq^n)_r$ or $\alpha_i = R(f) = R(fq^n)$.

$(u_n \in \mathfrak{R}\ ,\ n = 1, 2, \ldots)$

Also $q^m f \in \bigcap_{s=m+1}^{\infty}(q^s)_r = \bigcap_{s=1}^{\infty}(q^s)_r = (f)_r$ and $fq^m f = q^m f$; as in Theorem 5.3 we have $R(q^m f) = R(f)$ so that $fq^m f$ has an inverse in $\mathfrak{R}(f)$. In particular there exists a ring element $u = fuf$ such that

$$fuf.fqf = fqf.fuf = f$$

and since $(qf)^m = q^m f$, the inverse of $fq^m f$ in $\mathfrak{R}(f)$ is $(fuf)^m = fu^m f$.

Then $(f + fu^m fq^m(1-f))_r = (f)_r$ and $(f + fu^m fq^m(1-f))_\ell = (fq^m)_\ell$. The idempotents $f + fu^m fq^m(1-f)$ form a sequence converging in the rank metric since

$$\begin{aligned}
&R(fu^m fq^m(1-f) - fu^{m+n} fq^{m+n}(1-f)) \\
=&R(fu^m f(f-fu^n fq^n)q^m(1-f)) \\
=&R(fu^m f(f-fu^n fq^n f-fu^n fq^n(1-f))q^m(1-f)) \\
=&R(fu^m fu^n fq^n(1-f)q^m(1-f)) \\
\leqq& R((1-f)q^m(1-f)) \to 0 \text{ as } m,\ n \to \infty, \text{ as before.}
\end{aligned}$$

The remainder of the proof is similar to that for Theorem 5.3. We have $e_i = \lim_{m\to\infty}(f + fu^m fq^m(1-f))$.

We note the following results to be used in later theorems.

(i) $(1-e_i)q^m e_i = 0 = e_i q^m(1-e_i)$. $(m = 1, 2, \ldots)$

For $e_i q^m e_i = q^m e_i$ holds, as the above equality $fq^m f = q^m f$ requires only that $f^2 = f$, $(f)_r = \bigcap_{n=1}^{\infty}(q^n)_r$ and e_i

has these properties. Similarly $e_i q^m e_i = e_i q^m$ since $(e_i)_\ell = \bigcap_{n=1}^{\infty} (q^m)_\ell$.

(ii) $e_i q^m e_i = q^m e_i = e_i q^m$. $(1-e_i)q^m = q^m(1-e_i)$ as q^m commutes with e_i and consequently with $1-e_i$. Moreover $e_i q^m e_i$ has an inverse in $\mathfrak{R}(e_i)$.

(iii) $\lim_{m\to\infty} (1-e_i)q^m(1-e_i) = 0$.

This holds when e_i is replaced by an idempotent f, a generator of $\bigcap_{n=1}^{\infty}(q^n)_r$, so holds for $f = e_i$.

5.5. Theorem. (i) Let $\beta_i = 1-\alpha_i$, $f_i = 1-e_i$. Then $R(f_i) = \beta_i$.

(ii) $ef_i = f_i e = 0$, $f_i f_j = 0$ $(i \neq j)$ and $e + \lim_{n\to\infty} \sum_{i=1}^{n} f_i = 1$; $\alpha + \sum_{i=1}^{\infty} \beta_i = 1$ and the rings $\mathfrak{R}(e)$, $\mathfrak{R}(f_i)$ $(i = 1, 2, \ldots)$ are mutually orthogonal.

Proof: (i) $R(f_i) = R(1-e_i) = 1-R(e_i) = 1-\alpha_i = \beta_i$.

(ii) $(e_i)_r = \bigcap_{t=1}^{\infty} ((q_i(a))^t)_r \geqq \bigcap_{p\in P} (p(a))_+ = (e)_+$ so $e = e_i e$. Similarly we have $(e_i)_\ell \geqq (e)_\ell$ and $e = ee_i$. Then

$$ef_i = e(1-e_i) = 0, \quad f_i e = (1-e_i)e = 0.$$

Now let $q_i(x)$, $q_j(x)$ be distinct members of T and let e_i, e_j be defined as in Theorem 5.4. Then

$$1 = u_n(x)(q_i(x))^n + v_n(x)(q_j(x))^n \quad (n = 1, 2, \ldots)$$

for some polynomials $u_n(x)$, $v_n(x)$ with coefficients in Z; that is,

$$1 = u_n(a)(q_i(a))^n + v_n(a)(q_j(a))^n.$$

Hence

$$1-e_i = u_n(a)(q_i(a))^n(1-e_i) + v_n(a)(q_j(a))^n(1-e_i)$$
$$= u_n(a)(e_i + (1-e_i))(q_i(a))^n(1-e_i) + v_n(a)(q_j(a))^n(1-e_i)$$
$$= u_n(a)(1-e_i)(q_i(a))^n(1-e_i) + v_n(a)q_j(a)^n(1-e_i)$$

[Theorem 5.4, (i)]

and $(1-e_j)(1-e_i) = (1-e_j)u_n(a)(1-e_i)(q_i(a))^n(1-e_i)$
$$+(1-e_j)(q_j(a))^n v_n(a)(1-e_i)$$
$$=(1-e_j)u_n(a)(1-e_i)(q_i(a))^n(1-e_i)$$
$$+(1-e_j)(q_j(a))^n(1-e_j)v_n(a)(1-e_i).$$

As $R((1-e_j)u_n(a)(1-e_i)(q_i(a))^n(1-e_i)) \leqq R((1-e_i)\cdot(q_i(a))^n(1-e_i)) \to 0$ as $n \to \infty$ by Theorem 5.4 (iii) and similarly $R((1-e_j)(q_j(a))^n(1-e_j)v_n(a)(1-e_i)) \to 0$ as $n \to \infty$, we have, taking the limit,

$$f_j f_i = (1-e_j)(1-e_i) = 0. \qquad (i \neq j)$$

Then $f_i f_j = 0 = f_j f_i$ $(i \neq j)$ implies

$$1-e_i-e_j + e_i e_j = 1-e_i-e_j + e_j e_i$$

or $e_i e_j = e_j e_i$. Moreover $e_i e_j$ is idempotent and $(e_i e_j)_r = (e_i)_r \wedge (e_j)_r$, $(e_i e_j)_\ell = (e_i)_\ell \wedge (e_j)_\ell$. For, obviously $(e_i e_j)_r \leqq (e_i)_r \wedge (e_j)_r$ and $x \in (e_i)_r \wedge (e_j)_r$ implies $x = e_i x = e_j x = e_i e_j x$ or $x \in (e_i e_j)_r$. Similarly for left ideals. We have $(e_i e_j)_r = (e_i)_r \wedge (e_j)_r = \bigwedge_{m=1}^{\infty}((q_i(a))^m)_r \wedge \bigwedge_{n=1}^{\infty}((q_j(a))^n)_r$ so $e_i e_j = (q_j(a))^n w_n$ $(w_n \in \mathfrak{R}$, $n = 1, 2, \ldots)$. That is,

$$q_i(a)e_i e_j = q_i(a)(q_j(a))^n w_n = (q_j(a))^n q_i(a)w_n$$
$$(n = 1, 2, \ldots)$$

and $(q_i(a)e_ie_j)_r \leqq \bigcap_{n=1}^{\infty}((q_j(a))^n)_r = (e_j)_r$. Also $(q_i(a)e_ie_j)_r$ $= (e_iq_i(a)e_j)_r \leqq (e_i)_r$ so $(q_i(a)e_ie_j)_r \leqq (e_i)_r \cap (e_j)_r = (e_ie_j)_r$ and hence $e_ie_jq_i(a)e_ie_j = q_i(a)e_ie_j$.

Using left ideals we have similarly $e_ie_jq_i(a)e_ie_j$ $=e_ie_jq_i(a)$ and so $q_i(a)e_ie_j = e_ie_jq_i(a)$. Now

$$\begin{aligned}
e_jq_i(a) &= e_ie_jq_i(a) + (e_j-e_ie_j)q_i(a)\\
&= e_ie_jq_i(a) + e_j(1-e_i)q_i(a)\\
&= e_ie_jq_i(a) + e_j(1-e_i)q_i(a)e_j + e_j(1-e_i)q_i(a)(1-e_j)\\
&= e_ie_jq_i(a) + e_j(1-e_i)q_i(a)(1-e_i)e_j\\
&\qquad + e_j(1-e_i)q_i(a)(1-e_i)(1-e_j)\\
&\qquad\qquad [\text{Theorem 5.4 (ii)}]\\
&= e_ie_jq_i(a) + e_j(1-e_i)q_i(a)(1-e_i)e_j
\end{aligned}$$

since $(1-e_i)(1-e_j) = f_if_j = 0$.

Also

$$\begin{aligned}
q_i(a)e_j &= q_i(a)e_ie_j + q_i(a)(e_j-e_ie_j)\\
&= e_ie_jq_i(a) + e_jq_i(a)(1-e_i)e_j + (1-e_j)q_i(a)(1-e_i)e_j\\
&= e_ie_jq_i(a) + e_j(1-e_i)q_i(a)(1-e_i)e_j.
\end{aligned}$$

We conclude $e_iq_j(a) = q_j(a)e_i$ $(i, j = 1, 2, \ldots)$ and consequently $e_i(q_j(a))^m = (q_j(a))^me_i$ $(m = 1, 2, \ldots)$. Now

$$p^m(a) = (q_{i_1}(a))^{t_{i_1}}(q_{i_2}(a))^{t_{i_2}}\ldots(q_{i_n}(a))^{t_{i_n}}$$

for some $i_1, i_2, \ldots, i_n, t_{i_1}, t_{i_2}, \ldots, t_{i_n} = 1, 2, \ldots$, so

$$\begin{aligned}
(p^m(a))_r &\geqq ((q_{i_1}(a))^{t_{i_1}}(q_{i_2}(a))^{t_{i_2}}\ldots(q_{i_n}(a))^{t_{i_n}}e_{i_1}e_{i_2}\ldots e_{i_n})_r\\
&= (e_{i_1}e_{i_2}\ldots e_{i_n})_r
\end{aligned}$$

since the e_i and the $(q_j(a))^{t_j}$ commute and $(q_j(a))^{t_j}e_j$ has an inverse in $\mathcal{R}(e_j)$. Hence

$$(p^m(a))_r \geq (e_{i_1})_r \cap (e_{i_2})_r \cap \cdots \cap (e_{i_n})_r .$$

Consequently $(e)_r \geq \bigcap_{i=1}^{\infty}(e_i)_r$; but $(e)_r \leq \bigcap_{i=1}^{\infty}(e_i)_r$ since $e = e_i e$ so

$$(e)_r = \bigcap_{i=1}^{\infty}(e_i)_r .$$

Obviously $\bigcap_{i=1}^{\infty}(e_i)_r = \bigcap_{n=1}^{\infty}(e_1e_2\cdots e_n)_r$ and $\bigcap_{i=1}^{\infty}(e_i)_\ell = \bigcap_{n=1}^{\infty}(e_1e_2\cdots e_n)_\ell = (e)_\ell$; also, if $g_n = e_1e_2\cdots e_n$ $(n = 1, 2, \ldots)$, we have $g_mg_n = g_ng_m = g_n$ if $m < n$ so that the sequence of idempotents $g_1, g_2, \ldots$ converges in the rank metric [Maeda, 6, 155]. Moreover, if $\lim_{n\to\infty} g_n = \lim_{n\to\infty}(e_1e_2\cdots e_n) = g$, $g^2 = g$ and $(g)_r = (e)_r$, $(g)_\ell = (e)_\ell$ [Maeda, loc. cit.]. By the uniqueness established in Theorem 5.3 we have $g = e$. That is, $e = \lim_{n\to\infty}(e_1e_2\cdots e_n)$.

Consider now the sequence $h_n = 1-\sum_{i=1}^{n}f_i$ $(n = 1, 2, \ldots)$. These are idempotent and $h_nh_{n+1} = h_{n+1}h_n = h_{n+1}$. Again $\lim_{n\to\infty}h_n$ exists and so does $\lim_{n\to\infty}\sum_{i=1}^{n}f_i = 1-\lim_{n\to\infty}h_n$. We write $\lim_{n\to\infty}\sum_{i=1}^{n}f_i = \sum_{i=1}^{\infty}f_i$.

We prove $1-e_1e_2\cdots e_n = (1-e_1) + (1-e_2) + \cdots + (1-e_n)$. The result is obviously true for $n = 1$. Assuming for n, we have

$$\begin{aligned}
&(1-e_1) + (1-e_2) + \cdots + (1-e_n) + (1-e_{n+1}) \\
&= e_{n+1}\left[(1-e_1) + (1-e_2) + \cdots + (1-e_n) + (1-e_{n+1})\right] \\
&\quad + (1-e_{n+1})\left[(1-e_1) + (1-e_2) + \cdots + (1-e_n) + (1-e_{n+1})\right] \\
&= e_{n+1}\left[1-e_1e_2\cdots e_n\right] + (1-e_{n+1})
\end{aligned}$$

$= e_{n+1} - e_1 e_2 \dots e_n e_{n+1} + 1 - e_{n+1}$

$= 1 - e_1 e_2 \dots e_{n+1}$.

Hence $1 - e_1 e_2 \dots e_n = \sum_{i=1}^{n} f_i$. Taking limits we have $1 - \lim_{n\to\infty} e_1 e_2 \dots e_n = \sum_{i=1}^{\infty} f_i$ or $\sum_{i=1}^{\infty} f_i = 1 - e$. That is, $e + \sum_{i=1}^{\infty} f_i = 1$.

By the orthogonality of the idempotents $e, f_1, f_2, \dots$, $D(e + f_1 + f_2 + \dots + f_n)_r = R(e + f_1 + f_2 + \dots + f_n)$ $= R(e) + R(f_1) + \dots + R(f_n)$ and so

$$1 = D(e + \sum_{i=1}^{\infty} f_i)_r = D(\lim{}^{*}(e + f_1 + \dots + f_n)_r)$$

$$= \lim_{n\to\infty} R(e + f_1 + \dots + f_n) \qquad [\text{C. G. (15)(v)}]$$

$$= \lim_{n\to\infty} (R(e) + R(f_1) + \dots + R(f_n))$$

$$= \lim_{n\to\infty} (\alpha + \beta_1 + \dots + \beta_n)$$

or $\alpha + \sum_{i=1}^{\infty} \beta_i = 1$.

Finally $ef_i = f_i e = 0$, $f_i f_j = 0$ $(i \neq j)$ imply $u \in \mathfrak{R}(e)$, $v \in \mathfrak{R}(f_i)$, $w \in \mathfrak{R}(f_j)$ $(i \neq j)$ mean $uv = 0 = vu$, $vw = 0 = wv$.

5.6. Theorem. e and all f_i commute with every x which commutes with a, hence in particular with a; for every such x there exists a unique decomposition

$$x = x_e + x_{1-e} = x_e + \sum_{i=1}^{\infty} x_{f_i} \text{ where } x_e \in \mathfrak{R}(e),$$

$x_{1-e} \in \mathfrak{R}(1-e)$, $x_{f_i} \in \mathfrak{R}(f_i)$ and $\sum_{i=1}^{\infty} x_{f_i} = \lim_{n\to\infty} \sum_{i=1}^{n} x_{f_i}$.

Proof: Since x commutes with a, it commutes with each $(q_i(a))^t$ $(i, t = 1, 2, \dots)$. Also e_i and consequently $1-e_i$ commute with $(q_i(a))^t$ by Theorem 5.4 (ii).

Then $(1-e_i)xe_i.e_i(q_i(a))^t e_i = (1-e_i)x(q_i(a))^t e_i$ $= (1-e_i)(q_i(a))^t xe_i = (1-e_i)(q_i(a))^t(1-e_i).(1-e_i)xe_i$ $(t = 1, 2, \ldots)$. Let $e_i u_t e_i$ be the inverse of $e_i(q_i(a))^t e_i$ in $\mathcal{K}(e_i)$. Then

$$(1-e_i)xe_i = (1-e_i)(q_i(a))^t(1-e_i).(1-e_i)xe_i.e_i u_t e_i .$$

Taking the limit as $t \to \infty$ we have

$$(1-e_i)xe_i = 0. \qquad \text{[Theorem 5.4 (iii)]}$$

Similarly $e_i x(1-e_i) = 0$ so

$$x = e_i xe_i + (1-e_i)x(1-e_i)$$

commutes with $1-e_i = f_i$. That is, x commutes with $\sum_{i=1}^{\infty} f_i$ and in turn with $1-\sum_{i=1}^{\infty} f_i = e$.

As $xe = ex$ we have $ex(1-e) = 0 = (1-e)xe$ and $x = exe + (1-e)x(1-e)$; the uniqueness of this decomposition is obvious.

The idempotents $e, f_1, f_2, \ldots, f_n, 1-e-\sum_{i=1}^{n} f_n$ are independent and hence

$$x = exe + \sum_{i=1}^{n} f_i xf_i + (1-e-\sum_{i=1}^{n} f_i)x(1-e-\sum_{i=1}^{n} f_i).$$

[E, Lemma 12, 400]

Now $R((1-e-\sum_{i=1}^{n} f_i)x(1-e-\sum_{i=1}^{n} f_i)) \leqq R(1-e-\sum_{i=1}^{n} f_i) \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} (\sum_{i=1}^{n} f_i xf_i) = x-exe-\lim_{n\to\infty} ((1-e-\sum_{i=1}^{n} f_i)x(1-e-\sum_{i=1}^{n} f_i))$ exists. We write this limit as $\sum_{i=1}^{\infty} x_{f_i}$.

The decomposition is unique for, not considering the obviously unique exe, suppose we had

$$\lim_{n\to\infty} (\sum_{i=1}^{n} f_i xf_i) = \lim_{n\to\infty} (\sum_{i=1}^{n} f_i x'f_i).$$

Then, for any $j = 1, 2, \ldots$

$$\sum_{i=1}^{j-1} f_i x f_i + f_j x f_j + \lim_{n\to\infty} \left(\sum_{i=j+1}^{n} f_i x f_i\right)$$
$$= \sum_{i=1}^{j-1} f_i x' f_i + f_j x' f_j + \lim_{n\to\infty} \left(\sum_{i=j+1}^{n} f_i x' f_i\right).$$

Pre-multiplying by f_j and using the continuity of multiplication together with $f_i f_j = 0$ $(i \neq j)$ we have $f_j x f_j = f_j x' f_j$ $(j = 1, 2, \ldots)$.

5.7. Theorem. For every $p(x) \in P$ the $p(a_e)$, when formed in $\mathfrak{R}(e)$, has an inverse in $\mathfrak{R}(e)$.

Proof: Now $(p(a))_r$ is one of an enumerable set, $\{(p_m(a))_r;\ m = 1, 2, \ldots\}$ say, whether $p(a)$ is singular or not, so $(p(a))_r \geqq (p^m(a))_r$ for some $m = 1, 2, \ldots$, defining $p^m(x)$ along the lines of the corresponding quantity of Theorem 5.2. Hence $p^m(a) = p(a)u$ $(u \in \mathfrak{R})$ and $ep^m(a) = ep(a)u$. That is, $(ep(a))_r \geqq (ep^m(a))_r = (e)_r$. Clearly, equality holds. Since e commutes with a and thus with $p(a)$, we have $R(ep(a)e) = R(e)$ and $ep(a)e$ has an inverse in $\mathfrak{R}(e)$. But if

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_n \qquad (a_1, \ldots, a_n \in Z)$$

then $ep(a)e = (eae)^n + a_1 (eae)^{n-1} + \ldots + a_n e$.

5.8. Theorem. Form $p(a_{1-e})$ in $\mathfrak{R}(1-e)$. Then $R(p(a_{1-e}))$ can be made arbitrarily small by a suitable selection of p.

Proof: We have shown in Theorem 5.3 that

$$\lim_{m\to\infty} R((1-e)p^m(a)(1-e)) = 0.$$

But, as a commutes with e and thus with 1-e, we have $(1-e)p^m(a)(1-e) = p^m(a_{1-e})$; hence the result.

5.9. **Theorem.** Form $(q_i(a_{f_i}))^t$ $(t = 1, 2, \ldots)$ in $\mathfrak{K}(f_i)$. Then

$$\lim_{t\to\infty} R((q_i(a_{f_i}))^t) = 0.$$

Proof: We have shown in Theorem 5.4 that

$$\lim_{t\to\infty} R((1-e_i)(q_i(a))^t(1-e_i)) = 0$$

or $\lim_{t\to\infty} R(f_i(q_i(a))^t f_i) = 0$ so, since f_i commutes with a and thus $f_i a^s f_i = (f_i a f_i)^s$ $(s = 1, 2, \ldots)$, we have the result.

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