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Date July 20, 1989

AN ALGEBRAIC APPROACH TO MULTIVARIATE APPROXIMATION

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ABSTRACT

An algebraic method for the study of certain multivariate approximation techniques is developed and examined. This method, introduced by Gordon [8], employs the properties of a distributive lattice Ψ generated by a set of linear projection operators $\{P_m\}_{m=1}^{M}$. Ψ has a maximal element M and a minimal element L . The maximal projector M combines all the approximation properties of the entire collection $\{P_m\}$, while the minimal projector L has only those properties common to all the P_m , $m = 1, 2, \ldots, M$.

A comparison of the maximal and minimal projectors for several examples is explored. These examples serve to illustrate the applications of this algebraic approach to the study of certain approximation techniques, as well as to provide specific interpolatory meaning to the algebraic properties of Ψ . In particular, an example involving the use of cubic splines contrasts the maximal method of successive decomposition with the minimal tensor product technique. Under fairly mild assumptions, the method of successive decomposition is much more efficient (in terms of the number of data points required) than is the tensor product method.

The application of this algebraic approach to other multivariate techniques is easily seen. This approach provides an interesting and useful technique for the study of multivariate approximation.

TABLE OF CONTENTS

INTRODUCTION

The following thesis examines an algebraic approach which may be used to study certain types of multivariate approximation methods. These results were first developed by William J. Gordon [8] , and we shall follow his approach closely. Many of his ideas were presented without proofs, which have therefore been furnished, and revisions and corrections have been supplied when necessary.

The algebraic structure which is developed in Chapter 1 is that of a distributive lattice Ψ generated by a set of linear projection operators $\{P_m\}_{m=1}^{M}$. The elements of the lattice are partially ordered with a relation $\leq$. Under this relation the lattice will have a unique maximal element M and a unique minimal element L . Throughout the paper the emphasis will be upon comparing and contrasting the properties of M and L . Various properties of the lattice Ψ are developed algebraically in Chapter 1 and summarized in the main result of the paper, Theorem 1.5.1.

Specific examples are explored in Chapter 2, to show how the lattice theoretic approach may be used to study certain multivariate approximation techniques. Among the techniques examined are the multivariate analogues of Taylor series expansion, Lagrange polynomial interpolation, and Fourier expansion in a Hilbert space. Each of the five examples of Chapter 2 provides concrete meaning to the results of Chapter 1. In particular, we see what the algebraic relation $L \leq M$ means in terms of the multivariate approximation properties of these projectors.

Chapter 3 is concerned with the contrast of two multivariate techniques which employ cubic splines. The technique of successive decomposition, which is based upon a maximal projector M, is contrasted with the tensor product method, which corresponds to a minimal projector L. The results of Chapter 3 show that, under certain realistic assumptions, an approximation of specified accuracy to a function F may be achieved through successive decomposition with far fewer data points than required for the tensor product method.

The utility of the algebraic method developed in Chapter 1 is therefore seen in Chapters 2 and 3, and its application to the study of other multivariate approximation techniques may be easily realized.

CHAPTER 1

DEVELOPMENT OF THE ALGEBRAIC FRAMEWORK

1.1. INTRODUCTION

In this chapter we shall be concerned with the study of an algebraic framework which we may use in the study of multivariate approximation techniques. The framework which we shall develop is that of a distributive lattice which is generated by a finite set of M linear projection operators (projectors). The M projectors are defined over a linear space $\mathcal{F}$ and have as their respective ranges M subspaces of $\mathcal{F}$. The construction of this distributive lattice will depend upon (i) the commutativity of the projectors under ordinary operator multiplication, (ii) the proper choice of algebraic operations used to combine the projectors, and (iii) the proper partial ordering chosen to compare the projectors. Section 2 of this chapter contains the definitions necessary for us to begin. The lattice is constructed in Section 3, and its properties are studied in Section 4. Section 5 contains the main result of this paper, Theorem 1.5.1, which is stated in our context of multivariate approximation.

1.2. PRELIMINARY DEFINITIONS

We begin the task of building our algebraic framework with several definitions.

<u>1.2.1. Definition</u>. Let $\leq$ be a relation on a set S which satisfies the following three properties:

(i) $a \leq a$ for all $a \in S$ (Reflexive) ,

(ii) If $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitive) ,

(iii) If $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetric).

Then $\leq$ is called a partial ordering on S , and the pair $(S, \leq)$ is called a partially ordered set, or poset.

1.2.2. Definition. A lower bound of a set X of elements of a poset $(P, \leq)$ is an element a of P satisfying $a \leq x$ for all $x \in X$. A greatest lower bound (g.l.b) c of X is a lower bound of X with the property that $a \leq c$ for any other lower bound a of X .

Similarly, we may define an upper bound, and least upper bound (l.u.b.), of a set X in the obvious way.

By property (iii) of Definition 1.2.1, we see that both the l.u.b. and the g.l.b. of a set X are unique when they exist.

1.2.3. Definition. A lattice is a poset in which any two elements have a l.u.b. and a g.l.b.

The lattice which we shall study will be constructed using a set of linear projection operators (projectors). Therefore we have the following definitions.

1.2.4. Definition. A projector P is an idempotent linear transformation from a linear space F onto a subspace Φ .

1.2.5. Definition. Given an element F of a linear space $\mathcal{F}$ and a projector $P: \mathcal{F} \to \Phi$, the element $\pi = P[F]$ (the image of F under P) is called an approximation to F ; the element $F - \pi$ is called the remainder, or error of the approximation.

1.3. CONSTRUCTION OF A DISTRIBUTIVE LATTICE OF PROJECTORS

We now turn our attention to the following problem: Given a collection of M projectors $P_1, P_2, \ldots, P_M$ from the linear space F onto the subspaces $\Phi_1, \Phi_2, \ldots, \Phi_M$, respectively, how may we generate a lattice from this collection? The answer to this question lies in choosing the proper way to combine the projectors (i.e. the proper algebraic operations) and in choosing the proper partial ordering to place upon the set resulting from these combinations. It is with these ideas in mind that we proceed.

The operations of addition $(+)$ and multiplication of projectors will be defined in the usual way for linear operators. Therefore, our collection of projectors $\{P_m\}$ will implicitly satisfy the following associative and distributive laws:

$$P_k(P_\ell P_m) = (P_k P_\ell)P_m \ , \tag{1}$$

$$P_k(P_\ell + P_m) = P_k P_\ell + P_k P_m \ , \tag{2}$$

$$(P_k + P_\ell)P_m = P_k P_m + P_\ell P_m \ , \tag{3}$$

for $1 \leq k, \ell, m \leq M$.

For our construction to proceed, we must assume that the projectors in our set commute under multiplication, i.e.,

$$P_k P_m = P_m P_k \quad \text{for} \quad 1 \leq k, m \leq M \ . \tag{4}$$

Although the product of any two commutative projectors is also a projector, the sum of two commutative projectors P_k and P_ℓ is a projector if and only if $P_k P_\ell = 0$. For this reason we shall use the Boolean addition defined below.

1.3.1. Definition (Boolean Addition). The Boolean sum $\oplus$ of two projectors a and b is

$$a \oplus b = a + b - ab \quad .$$

The following lemma shows that the problem with ordinary addition mentioned above is not present when the new Boolean addition is used.

1.3.2. Lemma. The Boolean sum $P_k \oplus P_\ell$, and the product $P_k P_\ell$ of any two projectors from the collection $\{P_m\}$ are also projectors.

Proof. Clearly both $P_k \oplus P_\ell$ and $P_k P_\ell$ are linear operators, and a simple calculation shows that they are idempotent. ■

We also see that Boolean sums of projectors retain the commutativity properties of the original set $\{P_m\}$.

1.3.3. Lemma. If $a = P_k \oplus P_\ell$, $b = P_m \oplus P_n$, where $1 \le k, \ell, m, n \le M$, then $ab = ba$.

By using Lemmas 1.3.2 and 1.3.3 and mathematical induction we arrive at the following lemma, whose proof is omitted.

1.3.4. Lemma. If a and b are linear operators which may be constructed as combinations of the elements of the set $\{P_m\}$ by using the operations of $\oplus$ and $\cdot$ (i.e., "Boolean polynomials" in the M variables $P_1, \ldots, P_M$), then both a and b are projectors and $ab = ba$.

We shall denote the set of all Boolean polynomials in the projectors $\{P_m\}$ by Ψ. We have the following result concerning the set Ψ.

<u>1.3.5. Lemma</u>. The set Ψ is closed under the operations of Boolean addition and operator multiplication. The following properties hold for all $a, b, c \in \Psi$:

(i) $a \oplus a = a$, $aa = a$,

(ii) $a \oplus b = b \oplus a$, $ab = ba$,

(iii) $a(bc) = (ab)c$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(iv) $a(b \oplus c) = ab \oplus ac$, $a \oplus (bc) = (a \oplus b)(a \oplus c)$.

<u>Proof</u>. The set Ψ is obviously closed under Boolean addition and multiplication, as the Boolean sum and/or product of two Boolean polynomials is again a Boolean polynomial. By Lemma 1.3.4 we have that $aa = a$ and $ab = ba$ for all $a, b \in \Psi$. For the remaining properties note that:

(i) $a \oplus a = a + a - aa = a + a - a = a$,

(ii) $a \oplus b = a + b - ab = b + a - ba = b \oplus a$,

(iii) $a(bc) = (ab)c$, by property (1) of this section, and

$a \oplus (b \oplus c) = a + (b + c - bc) - a(b + c - bc) =$

$(a + b - ab) + c - (a + b - ab)c = (a \oplus b) \oplus c$,

(iv) $a(b \oplus c) = ab + ac - abc = ab \oplus ac$, and

$a \oplus bc = a + bc - abc = (a + b - ab)(a + c - ac) = (a \oplus b)(a \oplus c)$ ∎

It is the set Ψ which we wish to use as our lattice. That is, we wish to place a partial order $\leq$ on the set Ψ so that the poset $(\Psi, \leq)$ will be a lattice. For this purpose we make the following definition.

<u>1.3.6. Definition</u>. $a \leq b$ means $ab = a$.

<u>1.3.7. Lemma</u>. The relation $\leq$ is a partial order on the set Ψ.

<u>Proof</u>. The proof consists of verifying the three conditions of Definition 1.2.1.

(i) (Reflexive property). By Lemma 1.3.5, $aa = a$ for all $a \in \Psi$. Thus $a \leq a$ for all $a \in \Psi$.

(ii) (Transitive Property). If $a \leq b$ and $b \leq c$ then by Lemma 1.3.5(iii),

$$ac = (ab)c = a(bc) = ab = a .$$

Thus $a \leq c$.

(iii) (Anti-Symmetric). If $a \leq b$ and $b \leq a$, then $a = ab = ba = b$. ∎

We wish to use the partial ordering $\leq$ as an aid in helping us decide when one projector b is "better" than another projector a. That is, if $a \leq b$ we would like to be able to conclude that, for any $F \in \mathcal{F}$, $b[F]$ is at least as good an approximation to F as is $a[F]$. As we shall see in Theorem 1.4.3, if $a \leq b$, then $\mathrm{Range}(a) \subseteq \mathrm{Range}(b)$. Thus, the ordering $\leq$ has a natural interpretation in terms of the ranges of the projectors: "bigger" (in terms of $\leq$) projectors have "bigger" (in terms of set inclusion) ranges.

Another way to interpret the partial ordering $\leq$ is as follows. Let $a \leq b$ and let $F \in \mathcal{F}$. Then, if we approximate F first with b (giving the approximation $b[F]$) and then approximate $b[F]$ with a, we have the same information as if we had first approximated F with a (i.e. $ab[F] = a[F]$). Thus, any information contained in the approximation $a[F]$ is also contained in the approximation $b[F]$.

As a concrete example of this concept, consider the linear space $\mathbb{R}^3$, and let a and b be the transformations whose matrix representations (with respect to the standard basis) are given by

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e., a is the projection onto the x-axis, and b is the projection onto the $x - y$ plane.

It is easy to see that a and b are projectors, a and b commute, and $a \leq b$. Furthermore, it is also obvious that given any $F \in \mathbb{R}^3$, the approximation $b[F]$ contains any information available in the approximation $a[F]$. Thus b is at least as good a way to approximate F as is a, which is what we wished to conclude from the relationship $a \leq b$.

Under the partial order $\leq$ we have the following result.

<u>1.3.8. Lemma</u>. If a and b are projectors in Ψ, then

(i) $\text{l.u.b.}\ \{a, b\} = a \oplus b$

(ii) $\text{g.l.b.}\ \{a, b\} = ab$.

<u>Proof</u>. Let $a, b \in \Psi$.

(i) By Lemma 1.3.5 (iv),

$$a(a \oplus b) = a \oplus ab = a + ab - ab = a,$$

$$b(a \oplus b) = ab \oplus b = ab + b - ab = b.$$

Thus $a \oplus b$ is an upper bound for $\{a, b\}$. If c is another upper bound for $\{a, b\}$, then

$$(a \oplus b)c = ac \oplus bc = a \oplus b \ .$$

Thus $a \oplus b$ is the least upper bound for $\{a, b\}$.

(ii) By commutativity of a and b we have

$$(ab)a = ab = (ab)b \ ,$$

so ab is a lower bound for $\{a, b\}$. If c is any other lower bound for $\{a, b\}$, then $(ab)c = a(bc) = ac = c$, so that $c \leq ab$. Therefore, g.l.b.$\{a, b\} = ab$. ∎

We see then that $(\Psi, \leq)$ is a poset in which any two elements have both a l.u.b. and a g.l.b.. By Definition 1.2.3 we have the following theorem.

<u>1.3.9. Theorem</u>. The poset $(\Psi, \leq)$ is a lattice.

In fact, by Lemma 1.3.5 we have that the ordered quadruple $(\Psi, \leq, \oplus, \cdot)$ is a lattice in which the algebraic operations of $\oplus$ and $\cdot$ are each distributive with respect to the other. That is, $(\Psi, \leq, \oplus, \cdot)$ is a <u>distributive lattice</u>. For brevity we shall refer to the distributive lattice as Ψ .

1.4. PROPERTIES OF THE LATTICE

In the sense of Definition 1.2.5, given any $F \in \mathcal{F}$, any projector $a \in \Psi$ provides an approximation to F. We wish to use our partial ordering $\leq$ to identify two particular projectors for study: the <u>algebraically maximal projector M</u> and the <u>algebraically minimal projector L</u>. By maximal (minimal) we mean that $a \leq M$ $(L \leq a)$ for all $a \in \Psi$. We note that by property (iii) of Definition 1.2.1. both M and L will be unique if they exist. The existence of M and L is the topic of the next lemma.

<u>1.4.1. Lemma</u>. (i) The maximal projector $M \in \Psi$ is given by

$$M = \text{l.u.b.}\{P_m\} = P_1 \oplus P_2 \oplus \dots \oplus P_M .$$

(ii) The minimal projector $L \in \Psi$ is given by

$$L = \text{g.l.b.}\{P_m\} = P_1 P_2 \dots P_M .$$

<u>Proof</u>. (i) The fact that $\text{l.u.b.}\{P_m\} = P_1 \oplus P_2 \oplus \dots \oplus P_M$ follows from Lemma 1.3.8. and a simple induction argument. It remains to see that this projector is the maximal element M of the lattice Ψ. The proof is an induction proof, where the induction is done on the number of projectors in the generating set.

For the case of a lattice generated by two projectors, P_1 and P_2, the result is easily verified by using the lattice $\Psi = \{P_1, P_2, P_1P_2, P_1 \oplus P_2\}$. For the general case, we shall denote by Ψ_{M-1} the lattice generated by the projectors $P_1, \ldots, P_{M-1}$. By the induction hypothesis, the maximal element of Ψ_{M-1} is $\mathcal{M}_{M-1} = P_1 \oplus \cdots \oplus P_{M-1}$.

Let $a \in \Psi$. Then $a = b \oplus P_M c \oplus P_M$, where each of b and c is either 0 or an element of Ψ_{M-1}. Thus,

$$a\mathcal{M} = (b \oplus P_M c \oplus P_M)(\mathcal{M}_{M-1} \oplus P_M)$$

$$= (b \oplus b\ P_M) \oplus (P_M c \oplus P_M c) \oplus (P_M\ \mathcal{M}_{M-1} \oplus P_M)$$

$$= b \oplus P_M c \oplus P_M = a\ .$$

Therefore $a \leq \mathcal{M}$ for all $a \in \Psi$, i.e. $\mathcal{M}$ is the maximal element of the lattice Ψ.

(ii) As in part (i), the fact that g.l.b.$\{P_m\} = P_1 P_2 \ldots P_M$ follows from Lemma 1.3.8. and a simple induction argument. The proof that $\mathcal{L}$ is the minimal element of Ψ follows the same induction argument used in the proof of (i). ■

The above result leads us directly to the following lemma.

<u>1.4.2. Lemma</u>. The maximal projector $\mathcal{M}$ has the property that $P_m\ \mathcal{M} = P_m$ for $1 \leq m \leq M$; the minimal projector $\mathcal{L}$ has the property that $P_1 P_2 \ldots P_M\ \mathcal{L} = P_1 P_2 \ldots P_M$.

As mentioned in the remarks following Lemma 1.3.7, we may interpret Lemma 1.4.2. as follows: the maximal projector $\mathcal{M}$ provides us with all the approximation information available from the entire set $\{P_m\}$, whereas the minimal projector $\mathcal{L}$ has only those properties common to all the projectors in the set $\{P_m\}$. A further interpretation of the interpolation properties of projectors is given in the theorem below.

<u>1.4.3. Theorem</u>. Let Φ_m be the range of P_m , $1 \le m \le M$. Under the correspondences $P_m \to \Phi_m$, $\cdot$ (multiplication) $\to \cap, \oplus \to$ span of union, and $\le \to \subseteq$ (set inclusion), the lattice Ψ is homomorphic to the distributive lattice generated by the ranges Φ_m , $m = 1, 2, \ldots, M$.

<u>Proof</u>. Given a projector $a \in \Psi$, we shall denote the range of a by $a(\mathcal{F})$.

It is clear that $(P_m \oplus P_n)(\mathcal{F}) = (P_m + P_n - P_m P_n)(\mathcal{F})$ $= \text{span}(P_n(\mathcal{F}) \cup P_m(\mathcal{F}))$, and $P_m P_n(\mathcal{F}) = P_m(\mathcal{F}) \cap P_n(\mathcal{F})$. If $P_m \le P_n$, then $P_m(\mathcal{F}) = P_m P_n(\mathcal{F}) = P_m(\mathcal{F}) \cap P_n(\mathcal{F})$. Thus $P_m(\mathcal{F}) \subseteq P_n(\mathcal{F})$. Therefore the correspondence described above does indeed define a homomorphism between the two lattices. ∎

Given an element $F \in \mathcal{F}$ and a projector $a \in \Psi$, the element $\pi = a[F]$ provides us with an approximation to F , and this approximation has a remainder (error) $F - \pi$. For the rest of this section we shall explore the relationships between the projectors in Ψ and their associated remainder operators. As we shall see, the collection of remainder operators will also form a lattice Ψ' which is isomorphic to the lattice Ψ . Both Ψ and Ψ' may be viewed as sublattices of a Boolean algebra Ω which contains all possible Boolean combinations of the projectors P_m and the associated remainder operators. We shall be particularly interested in two decompositions of the identity operator I which we shall denote by the <u>maximal decomposition</u> and the <u>minimal decomposition</u>. We begin with a study of the properties of the identity operator I and the null operator 0 on the linear space $\mathcal{F}$.

Clearly both I and 0 are projectors on F, and for any projector a the following three properties are satisfied:

(i) $0 \leq a \leq I$,

(ii) $a0 = 0a = 0$, $aI = Ia = a$,

(iii) $0 \oplus a = a$, $I \oplus a = I$.

By employing the identity operator we may study the remainders associated with the projectors in Ψ as operators themselves. Hence we have the following definition.

<u>1.4.4. Definition</u>. If a is a projector, the remainder operator a' associated with a is defined to be the complement of a :

$$a' \equiv I - a$$

where I is the identity operator on the space $\mathcal{F}$.

<u>1.4.5. Lemma</u>. If a is a projector from $\mathcal{F}$ onto a subspace Φ , then a' is a projector from $\mathcal{F}$ onto the subspace $\Phi' = \{F - aF | F \in \mathcal{F}\}$.

<u>Proof</u>. a' is clearly a linear operator, and it is easy to see that a' is idempotent: $a'a' = (I - a)(I - a) = I - a = a'$. Furthermore, the range of a' is Φ' , which is easily seen to be a subspace of $\mathcal{F}$. ■

<u>1.4.6. Lemma</u>. Projectors and their complements satisfy the following rules of composition:

(i) Complementation: $aa' = a'a = 0$ and $a \oplus a' = I$.

(ii) Dualization: $(a \oplus b)' = a'b'$, $(ab)' = a' \oplus b'$.

(iii) Involution: $(a')' = a$.

Proof. (i) $aa' = a(I - a) = a - a = 0 = (I - a)a = a'a$,

$a \oplus a' = a + a' - aa' = a + (I - a) = I$,

(ii) $(a \oplus b)' = I - (a + b - ab) = (I - a)(I - b) = a'b'$,

$a' \oplus b' = (I - a) + (I - b) - (I - a)(I - b) = I - ab$

$= (ab)'$, and

(iii) $(a')' = I - a' = I - (I - a) = a$ ■

We now wish to expand our study from the lattice Ψ to a larger algebraic structure which will contain not only Ψ but also the remainder operators corresponding to projectors in Ψ , as well as the null operator 0 and the identity operator I . We note that the lattice Ψ does not include the complements of projectors, and thus does not in general include the identity operator or the null operator. However, if we allow the unary operation of complementation in addition to the binary operations of $\oplus$ and $\cdot$, the resulting structure Ω is a Boolean algebra, which we define below.

1.4.7. Definition. A Boolean algebra is a distributive lattice Ω with an identity I , which is closed under the operation of complementation.

The lattice Ψ generated by the projectors $\{P_m\}$ has thus been enlarged to a Boolean algebra Ω which contains not only Ψ and the identity and null operators but also all possible Boolean combinations of the projectors P_m and the complements P_m' . We shall hereafter refer to the complements P_m' as the remainder R_m associated with P_m. The remainders also form a distributive lattice Ψ' which we shall call the <u>dual</u> of the lattice Ψ.

<u>1.4.8. Lemma</u>. The set Ψ' of all Boolean polynomials in the variables R_m, $m = 1,2,\ldots, M$ which are formed using the operations of multiplication and Boolean addition is a distributive lattice. The lattices Ψ and Ψ' are anti-isomorphic under the correspondence $a \in \psi \to a' \in \psi'$ (by "anti-isomorphic" we mean the correspondences $a \to a'$, $\cdot \to \oplus$ and $\oplus \to \cdot$ yield an isomorphism between Ψ and Ψ').

<u>Proof</u>. The proof that Ψ' is a distributive lattice is identical to the proof of Lemma 1.3.9. and therefore will not be given.

The correspondence $a \to a'$ is clearly a bijection, and by Lemma 1.4.6,

$$(a \oplus b)' = a'b'$$

$$(ab)' = a' \oplus b' .$$

Thus the two lattices are anti-isomorphic. ■

We now examine the relationship between the lattices Ψ and Ψ'. We note that for any projector $a \in \Psi$ we may associate a unique <u>dual projector</u> $\hat{a} \in \Psi'$, which is formed by replacing all of the P_m in the Boolean expression for a by their corresponding remainders R_m. We emphasize that the dual projector $\hat{a}$ is in general <u>not</u> the same as the remainder a'. For example, if $a = P_1 \oplus P_2$ then $\hat{a} = R_1 \oplus R_2$, while $a' = R_1R_2$.

We wish to explore the relationship between the maximal and minimal elements of the two lattices Ψ and Ψ' and to use these results to derive two decompositions of the identity operator for later use. The following lemma is the dual result of Lemma 1.4.1. The proof is identical to the proof of Lemma 1.4.1. and is therefore omitted. Lemma 1.4.10. reveals the relationship between the maximal and minimal elements of Ψ and Ψ' .

<u>1.4.9. Lemma</u>. The maximal element $\hat{M}$ and the minimal element $\hat{L}$ of the lattice Ψ' are given by

(i) $\hat{M} = \text{l.u.b.}\{R_m\} = R_1 \oplus R_2 \oplus \ldots \oplus R_M$

(ii) $\hat{L} = \text{g.l.b.}\{R_m\} = R_1 R_2 \ldots R_M$.

<u>1.4.10. Lemma</u>. The relationships between the maximal and minimal projectors of the lattices Ψ and Ψ' are given by

(i) $M = \text{l.u.b.}\{P_m\} = (\text{g.l.b.}\{R_m\})' = (\hat{L})'$

(ii) $L = \text{g.l.b.}\{P_m\} = (\text{l.u.b.}\{R_m\})' = (\hat{M})'$.

<u>Proof</u>. The results follow from Lemma 1.4.9. and parts (ii) and (iii) of Lemma 1.4.6.

(i) $$(\hat{L})' = (\text{g.l.b.}\{R_m\})' = \left(\prod_{m=1}^{M} R_m\right)' = R_1' \oplus R_2' \oplus \ldots \oplus R_M'$$

$$= P_1 \oplus P_2 \oplus \ldots \oplus P_M = \text{l.u.b.}\{P_m\} = M$$

(ii) $$(\hat{M})' = (\text{l.u.b.}\{R_m\})' = (R_1 \oplus R_2 \oplus \ldots \oplus R_M)' = \prod_{m=1}^{M} R_m'$$

$$= \prod_{m=1}^{M} P_m = \text{g.l.b.}\{P_m\} = L \quad \blacksquare$$

If a is any projector in the Boolean algebra Ω, then by Lemma 1.4.6.(i) the identity I may be decomposed as $I = a \oplus a'$. Since a and a' are mutually orthogonal, this decomposition may be expressed as $I = a \oplus a' = a + a'$. From Lemma 1.4.10 we have the following result, which gives us the decompositions of the identity, based upon the maximal and minimal elements of Ψ.

<u>1.4.11. Theorem</u>. With respect to the commutative set of projectors $\{P_m\}$ the identity operator has the following two decompositions:

(i) <u>Maximal Decomposition:</u> $I = M \oplus M' = M + M'$

$$= (P_1 \oplus P_2 \oplus \ldots \oplus P_M) + (R_1 R_2 \ldots R_M),$$

(ii) <u>Minimal Decomposition</u>: $I = L \oplus L' = L + L'$

$$= (P_1 P_2 \ldots P_M) + (R_1 \oplus R_2 \oplus \ldots \oplus R_M) .$$

1.5. THE LATTICE Ψ IN APPROXIMATION THEORY

We now summarize the results of Chapter 1 in the following theorem, which is the main result of this paper. The theorem is stated in the context of multivariate approximation and will be examined further in the examples of Chapters 2 and 3. The content of the theorem is a condensation of material presented in Sections 1.3 and 1.4. The proof is therefore not given.

<u>1.5.1. Theorem</u>. Let $\{P_m\}_{m=1}^{M}$ be a set of commutative projectors defined on the linear space $\mathcal{F}$ of functions of the N independent variables $x_1, x_2, \ldots, x_N$. Denote the range of P_m by Φ_m. If the collection $\{P_m\}$ satisfies (1)-(4) of Section 1.3, then any $F \in \mathcal{F}$ can be expanded as

$$F = a[F] + a'[F]$$

where a is any approximation operator in the lattice Ψ and a' is its complement in Ψ' . Among all approximations $a[F]$ to F the algebraically best approximation is the function $\pi = M[F]$ where $M = P_1 \oplus P_2 \oplus \ldots \oplus P_M$. The maximal approximation π is an element of the linear space span $(\Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_M)$ and satisfies

$$P_m[\pi] = P_m[F] \qquad \text{for} \quad m = 1, 2, \ldots, M .$$

The algebraically minimal approximation to F is the function $\pi' = L[F]$ where $L = P_1 P_2 \ldots P_M$. The minimal approximation π' is an element of $\Phi_1 \cap \Phi_2 \cap \ldots \cap \Phi_M$ and satisfies

$$P_1 P_2 \ldots P_M[\pi'] = P_1 P_2 \ldots P_M[F] .$$

The errors (remainders) associated with the maximal and minimal approximations are, respectively,

$$F - \pi = R_1 R_2 \ldots R_M[F]$$

and

$$F - \pi' = (R_1 \oplus R_2 \oplus \ldots \oplus R_M)[F] .$$

CHAPTER 2

EXAMPLES

2.1. INTRODUCTION

In this chapter we shall examine some ways in which the algebraic structure developed in Chapter 1 may be employed to study the problem of approximation of multivariate functions. Five examples will be considered in this chapter. The first three examples deal with the extension of a univariate approximation technique to a function of several variables. The univariate techniques employed are the familiar methods of Taylor series expansion, Lagrange polynomial interpolation, and orthogonal (Fourier) expansion in a Hilbert space. The last two examples are concerned with the interpolation of a function on the boundary of a geometric figure (the surface of a cylinder and perimeter of a triangle, respectively). These examples reveal some ways in which the results of Chapter 1 may be used, as well as offering a better understanding of these results. In particular, the contrast between the maximal and minimal projectors defined in Chapter 1 is evident in all five examples. The main results of this chapter are stated as corollaries to Theorem 1.5.1.

2.2. TAYLOR-TYPE EXPANSIONS

Our first example deals with the approximation of an N-variate function by means of a Taylor series type expansion. Our linear space F will be any linear space of functions F of the N independent variables $x_1, x_2, \ldots, x_N$ for which the two differentiability conditions stated below hold. The first condition is

$$\frac{\partial^{\nu}}{\partial x_{\ell}^{\nu}}\left[\frac{\partial^{\mu}}{\partial x_m^{\mu}}\ F|_{x_m=0}\right]\Big|_{x_\ell=0} = \frac{\partial^{\mu}}{\partial x_m^{\mu}}\left[\frac{\partial^{\nu}}{\partial x_{\ell}^{\nu}}\ F|_{x_\ell=0}\right]\Big|_{x_m=0}$$

$$(0 \le \mu \le K_m\ ,\ 0 \le \nu \le K_\ell) \qquad (5)$$

for $1 \le m, \ell \le N$. Here the operator

$$\frac{\partial^{\mu}}{\partial x_m^{\mu}}\ |_{x_m=0}$$

is the linear differential operator which takes the μ^{th} partial derivative with respect to x_m of a function $F(x_1,\ldots, x_N)$ and evaluates it on the hyperplane $x_m = 0$. We shall also use the notation

$$\overset{(0,\ \ldots,\ \mu,\ \ldots,\ 0\ )}{F}(x_1,\ldots,\ 0,\ \ldots,\ x_N) \equiv \frac{\partial^{\mu}}{\partial x_m^{\mu}}\ F(x_1,\ldots,\ x_N)|_{x_m=0}\ .$$

The second condition is

$$\frac{\partial^{K_1+1}}{\partial x_1^{K_1+1}}\frac{\partial^{K_2+1}}{\partial x_2^{K_2+1}}\cdots\frac{\partial^{K_N+1}}{\partial x_N^{K_N+1}}\ F(x_1,\ x_2,\ldots,\ x_N)$$

$$\equiv \overset{(K_1+1,\ K_2+1,\ldots,\ K_N+1)}{F}(x_1,\ x_2,\ldots,\ x_N) \qquad (6)$$

exists and is continuous. Here $K_1, K_2,\ldots, K_N$ is a given set of positive integers.

We shall use the phrase "any suitable F" to mean any function satisfying (5) and (6) .

We shall take the number of projectors to be equal to the number of variables N, and shall define the projectors P_m to be

$$P_m = \sum_{\mu=0}^{K_m} \frac{x_m^\mu}{\mu!} \frac{\partial^\mu}{\partial x_m}\Big|_{x_m=0} \qquad (1 \leq m \leq N) . \tag{7}$$

As before, we shall denote the image of F under P_m by π_m (i.e. $\pi_m = P_m[F]$). We note that the functions π_m may be interpreted as being the $K_m + 1$ term truncated parametric Taylor series expansion of $F(x_1, \ldots, x_N)$, where the expansion is in the m^{th} variable and the parameters are the remaining $N-1$ variables.

We now explore the properties of the projectors defined in (7) and the lattice Ψ which they generate. Our first task is to show that the operators P_m are actually projectors.

<u>2.2.1. Lemma</u>. The operator P_m defined by (7) is a projector.

<u>Proof</u>. P_m is clearly a linear operator. We notice first that for $F \in \mathcal{F}$ and $0 \leq \mu \leq K_m$, the μ^{th} partial derivatives with respect to x_m of π_m and of F coincide on the hyperplane $x_m = 0$, i.e.

$$\pi_m^{(0,\ldots,\,\mu,\,\ldots,\,0)}(x_1, \ldots, 0, \ldots, x_N) = F^{(0,\ldots,\,\mu,\,\ldots,\,0)}(x_1, \ldots, 0, \ldots, x_N) \qquad (0 \leq \mu \leq K_m). \tag{8}$$

To see (8) we simply note that

$$\pi_m^{(0,\ldots,\mu,\ldots,0)}(x_1,\ldots,0,\ldots,x_N)$$

$$= \sum_{i=0}^{K_m} \frac{\partial^\mu}{\partial x_m^\mu}\left[\frac{x_m^i}{i!} F^{(0,\ldots,i,\ldots,0)}(x_1,\ldots,0,\ldots,x_N)\right]\Big|_{x_m=0}$$

$$= F^{(0,\ldots,\mu,\ldots,0)}(x_1,\ldots,0,\ldots,x_N)$$

Thus, for suitable F ,

$$P_m(P_m[F]) = P_m[\pi_m] = \sum_{i=0}^{K_m} \frac{x_m^i}{i!} \frac{\partial^i}{x_m^i} \pi_m\Big|_{x_m=0}$$

$$= \sum_{i=0}^{K_m} \frac{x_m^i}{i!} \frac{\partial^i}{\partial x_m^i} F\Big|_{x_m=0} = P_m[F]$$

Therefore, P_m is idempotent, and hence P_m is a projector. ■

The main results of this section are stated in the following two corollaries to Theorem 1.5.1. Gordon [8] credits Stancu [14] with being the first to obtain these results.

2.2.2. Corollary. If F is a space of N-variate functions which satisfy (5) and (6) then the set of projectors $\{P_m\}_{m=1}^N$ defined by (7) generates a distributive lattice Ψ. The maximal projector $M = P_1 \oplus P_2 \oplus \ldots \oplus P_N$ has the properties that the approximation $\pi = M[F]$ agrees with F and its normal (pure partial) derivatives on the N hyperplanes $x_m = 0$, $(m = 1, 2,\ldots, N)$:

$$\pi^{(0,\dots,\,\mu_m,\dots,\,0)}(x_1,\dots,\,0,\dots,\,x_N) = F^{(0,\dots,\,\mu_m,\,\dots,\,0)}(x_1,\dots,\,0,\,\dots,\,x_N)$$

$$(0 \le \mu_m \le K_m \,,\; 1 \le m \le N)\,. \tag{9}$$

The function $\pi' = L[F]$, where $L = P_1P_2 \dots P_N$ is the minimal projector, agrees with F and its mixed partial derivatives at the point $(x_1,\dots,\,x_N) = (0,\dots,\,0)$:

$$\pi'^{(\mu_1,\dots,\,\mu_N)}(0,\dots,\,0) = F^{(\mu_1,\dots,\,\mu_N)}(0,\dots,\,0) \qquad (0 \le \mu_m \le K_m) \tag{10}$$

The remainder operator $R_m = I - P_m$ is the parametric integral operator

$$R_m[F] = \int_0^{x_m} t_m(x_m;\, \xi_m)\, \frac{\partial^{K_m+1}}{\partial \xi_m^{K_m+1}} F(x_1,\dots,\,\xi_m,\,\dots,\,x_N)\, d\xi_m \,, \tag{11}$$

where t_m is the Taylor kernel $t_m(x_m;\, \xi_m) = \dfrac{(x_m - \xi_m)^{K_m}}{k_m!}$.

Proof. The fact that the set of projectors $\{P_m\}_{m=1}^N$ generates a distributive lattice Ψ with maximal element M and minimal element L follows directly from Theorem 1.5.1 and the commutativity condition (5) .

To see that (9) holds, let m be given, $1 \le m \le N$. For each μ_m , $0 \le \mu_m \le K_m$, the operator $\dfrac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}\Big|_{x_m=0}$ commutes with each projector P_ℓ , $1 \le \ell \le N$, for by (5) ,

$$\frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} [P_\ell(F)]|_{x_m=0} = \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} [\sum_{\nu=0}^{K_\ell} \frac{x_\ell^\nu}{\nu!} \frac{\partial^\nu}{\partial x_\ell^\nu} F|_{x_\ell=0}]|_{x_m=0}$$

$$= \sum_{\nu=0}^{K_\ell} \frac{x_\ell^\nu}{\nu!} \frac{\partial^\nu}{\partial x_\ell^\nu} [\frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} F|_{x_m=0}]|_{x_\ell=0} = P_\ell[\frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} F|_{x_m=0}] .$$

Now,

$$\pi = M[F] = (P_1 \oplus P_2 \oplus \ldots \oplus P_N)[F]$$

$$= P_m[F] + (P_1 \oplus \ldots \oplus P_{m-1} \oplus P_{m+1} \oplus \ldots \oplus P_N)[F]$$

$$- (P_1 \oplus \ldots \oplus P_{m-1} \oplus P_{m+1} \oplus \ldots \oplus P_N)\, P_m[F] .$$

Therefore,

$$\frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} \pi|_{x_m=0} = \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} P_m[F]|_{x_m=0}$$

$$+ (P_1 \oplus \ldots \oplus P_{m-1} \oplus P_{m+1} \oplus \ldots \oplus P_N) \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} [F]|_{x_m=0}$$

$$- (P_1 \oplus \ldots \oplus P_{m-1} \oplus P_{m+1} \oplus \ldots \oplus P_N) \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} P_m[F]|_{x_m=0} .$$

By property (8), the above expression reduces to $\frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}} F|_{x_m=0}$ which proves (9) .

To see (10), we note that for each m, $1 \le m \le N$, and each μ_m, $0 \le \mu_m \le K_m$,

$$\pi'^{(\mu_1,\ldots,\mu_N)}(0,\ldots,0) = \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_N}}{\partial x_N^{\mu_N}} (P_1 P_2 \cdots P_N[F])|_{x_N=0}) \cdots |_{x_1=0})$$

$$= \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_{N-1}}}{\partial x_{N-1}^{\mu_{N-1}}} (P_1 \cdots P_{N-1}(\frac{\partial^{\mu_N}}{\partial x_N^{\mu_N}} P_N[F]|_{x_N=0}) \cdots |_{x_1=0})$$

The result (10) then follows by repeated application of (8).

To prove (11) we note that for each m, $1 \le m \le N$, the approximation $\pi_m = P_m[F]$ is the truncated Taylor series expansion of the function F where F is viewed as a function of x_m and the remaining $N - 1$ variables are treated as parameters. Using condition (6), we obtain (11) directly from Taylor's theorem for univariate functions. ∎

<u>2.2.3. Corollary</u>. The errors $F - \pi$ and $F - \pi'$ associated with the maximal and minimal projectors M and L are, respectively,

$$R(x_1,\ldots,x_N) = R_1 R_2 \cdots R_N[F]$$

$$= \int_0^{x_1} \cdots \int_0^{x_N} t_1 \cdots t_N \, F^{(K_1+1,\ldots,K_N+1)}(\xi_1,\ldots,\xi_N) \, d\xi_N \cdots d\xi_1 \quad (12)$$

and

$$R'(x_1,\ldots,x_N) = (R_1 \oplus R_2 \oplus \cdots \oplus R_N)[F]$$

$$= \Big(\sum_{m_1=1}^{N} R_{m_1} - \sum_{m_1=1}^{N} \sum_{m_2=m_1+1}^{N} R_{m_1} R_{m_2}$$

$$+ \ldots + (-1)^{N-2} \sum_{m_1=1}^{N} \sum_{m_2=m_1+1}^{N} \ldots \sum_{m_{N-1}=(m_{N-2})+1}^{N} R_{m_1} \ldots R_{m_{N-1}}$$

$$+ (-1)^{N} R_1 \ldots R_N \Big)[F] \,. \tag{13}$$

Proof. The relationships $R(x_1,\ldots, x_N) = (R_1 R_2 \ldots R_N)[F]$ and $R'(x_1,\ldots, x_N) = (R_1 \oplus R_2 \oplus \ldots \oplus R_N)[F]$ follow directly from Theorem 1.4.11.

To prove the rest of (12), we note that by (6) Liebnitz's rule for differentiation under the integral sign is applicable. Expanding $R_1R_2 \ldots R_N[F]$ by Corollary 2.2.2, we have

$$R(x_1,\ldots, x_N) = \int_0^{x_1} t_1(x_1;\, \xi_1)\, \frac{\partial^{K_1+1}}{\partial \xi_1^{K_1+1}} \int_0^{x_2} t_2(x_2;\, \xi_2)\, \frac{\partial^{K_2+1}}{\partial \xi_2^{K_2+1}}$$

$$\int_0^{x_3} \ldots \int_0^{x_N} K_N(x_N;\, \xi_N)\, \frac{\partial^{K_N+1}}{\partial \xi_N^{K_N+1}} F(\xi_1,\ldots, \xi_N)\, d\xi_N \ldots d\xi_1 \,.$$

Repeated applications of Liebnitz's rule yields the result (12).

To prove the rest of (13) we shall use induction on N, the number of projectors. For $N = 2$, (13) is easily verified, as $R_1 \oplus R_2 = R_1 + R_2 - R_1R_2 = \sum_{m_1=1}^{2} R_{m_1} - \sum_{m_1=1}^{2} \sum_{m_2=m_1+1}^{2} R_{m_1} R_{m_2}$. Assuming that the expansion holds for the sum $R_1 \oplus R_2 \oplus \ldots \oplus R_{N-1}$ yields

$$R_1 \oplus R_2 \oplus \ldots \oplus R_N = \sum_{m_1=1}^{N-1} R_{m_1} - \Big(\sum_{m_1=1}^{N-1} \sum_{m_2=m_1+1}^{N-1} R_{m_1} R_{m_2} + \ldots$$

$$+ (-1)^K \sum_{m_1=1}^{N-1} \ldots \sum_{m_{K+1}=(m_K)+1}^{N-1} R_{m_1} R_{m_2} \ldots R_{m_{K+1}} + \ldots + (-1)^{N-2} R_1 \ldots R_{N-1} \Big)$$

$$+ R_N - \Big(\sum_{m_1=1}^{N-1} R_{m_1} R_N - \sum_{m_1=1}^{N-1} \sum_{m_2=m_1+1}^{N-1} R_{m_1} R_{m_2} R_N + \ldots +$$

$$(-1)^{K-1} \sum_{m_1=1}^{N-1} \sum_{m_2=m_1+1}^{N-1} \ldots \sum_{m_K=(m_{K-1})+1}^{N-1} R_{m_1} R_2 \ldots R_{m_K} R_N$$

$$+ \ldots + (-1)^{N-2} R_1 \ldots R_{N-1} R_N \Big) \quad .$$

The first and last terms of (13) are apparent. The remaining terms result from subtracting corresponding terms from inside the two sets of parentheses:

$$(-1)^K \sum_{m_1=1}^{N-1} \sum_{m_2=m_1+1}^{N-1} \ldots \sum_{m_{K+1}=(m_K)+1}^{N-1} R_{m_1} R_{m_2} \ldots R_{m_{K+1}}$$

$$- (-1)^{K-1} \sum_{m_1=1}^{N-1} \sum_{m_2=m_1+1}^{N-1} \ldots \sum_{m_K=(m_{K-1})+1}^{N-1} R_{m_1} R_{m_2} \ldots R_{m_K} R_N$$

$$= (-1)^K \sum_{m_1=1}^{N-1} \sum_{m_2=m_1+1}^{N-1} \ldots \sum_{m_{K+1}=(m_K)+1}^{N} R_{m_1} R_{m_2} \ldots R_{m_{K+1}} \quad .$$

We notice that the upper indices of summation in the above expression may be increased to N , which yields (13) . ■

We now investigate a simple numerical example to illustrate the ideas of Corollary 2.2.2. and Corollary 2.2.3. Consider the bivariate function $f(x, y) = e^{x+y}$. Using the projectors P_1 and P_2 defined by (7) with $K_1 = K_2 = 2$ yields the following four approximations obtained from the lattice $\psi = \{P_1, P_2, P_1P_2, P_1 \oplus P_2\}$:

$$\pi_1 = P_1[F] = F(0, y) + x \frac{\partial F}{\partial x}(0, y) + \frac{x^2}{2} \frac{\partial^2 F}{\partial x^2}(0, y)$$

$$= e^y + xe^y + \frac{x^2}{2} e^y$$

$$\pi_2 = P_2[F] = F(x, 0) + y \frac{\partial F}{\partial y}(x, 0) + \frac{y^2}{2} \frac{\partial^2 F}{\partial y^2}(x, 0)$$

$$= e^x + ye^x + \frac{y^2}{2} e^x$$

$$\pi' = L[F] = F(0, 0) + y \frac{\partial F}{\partial y}(0, 0) + \frac{y^2}{2} \frac{\partial^2 F}{\partial y^2}(0, 0) + x \frac{\partial F}{\partial x}(0, 0)$$

$$+ xy \frac{\partial^2 F}{\partial x \partial y}(0, 0) + \frac{xy^2}{2} \frac{\partial^3 F}{\partial x \partial y^2}(0, 0) + \frac{x^2}{2} \frac{\partial^2 F}{\partial x^2}(0, 0)$$

$$+ \frac{x^2 y}{2} \frac{\partial^3 F}{\partial x^2 \partial y}(0, 0) + \frac{x^2 y^2}{4} \frac{\partial^4 F}{\partial x^2 \partial y^2}(0, 0)$$

$$= 1 + y + \frac{y^2}{2} + x + xy + \frac{xy^2}{2} + \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^2 y^2}{4}$$

$$\pi = M[F] = P_1[F] + P_2[F] - P_1P_2[F] =$$

$$e^y + xe^y + \frac{x^2}{2} e^y + e^x + ye^x + \frac{y^2}{2} e^x$$

$$- \left(1 + y + \frac{y^2}{2} + x + xy + \frac{xy^2}{2} + \frac{x^2}{2} + \frac{x^2 y}{2} + \frac{x^2 y^2}{4}\right) .$$

We see that π_1 (π_2) is the three term parametric Taylor series expansion of $F(x, y)$ as a function of x (resp. y) where the variable y (resp. x) is treated as a parameter.

A simple calculation shows that (9) and (10) hold.

The remainder operators R_1, R_2 and R_1R_2 are

$$R_1[F] = \int_0^x \frac{(x - \xi_1)^2}{2} e^{\xi_1 + y} \, d\xi_1 \ ,$$

$$R_2[F] = \int_0^y \frac{(y - \xi_2)^2}{2} e^{x + \xi_2} \, d\xi_2 \ ,$$

and

$$R_1R_2 = \int_0^x \int_0^y \frac{(x - \xi_1)^2}{2} \frac{(y - \xi_2)^2}{2} e^{\xi_1 + \xi_2} \, d\xi_1 \, d\xi_2 \ ,$$

and a simple integration reveals $I - M = R_1R_2$ and $I - L = R_1 \oplus R_2$.

A comparison of the order of magnitude estimates for the errors $F - \pi$ and $F - \pi'$ reveals some specific interpolatory meaning of our algebraic ordering $\leq$.

From (12), we see that

$$|R(x_1, \ldots, x_N)| \leq M \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_N} t_1 t_2 \cdots t_n \, d\xi_N d\xi_2 \cdots d\xi_1 \ ,$$

where $M = \sup\limits_{\prod_{i=1}^{N}[0, x_i]} F^{(K_1+1, \ldots, K_N+1)}(\xi_1, \ldots, \xi_N)$. By (6), $M < \infty$. Since each

t_m depends only on ξ_m, the iterated integral above is actually the product of N single integrals, each of which can be easily evaluated to obtain the estimate

$$R(x_1, \ldots, x_N) = O(x_1^{K_1+1} \; x_2^{K_2+1} \; \ldots \; x_N^{K_N+1}) \; .$$

Similarly, one can show

$$R'(x_1, \ldots, x_N) = O(x_1^{K_1+1}) + O(x_2^{K_2+1}) + \ldots + O(x_N^{K_N+1}) \; .$$

Thus, we see that the function π converges to F when $x_m \to 0$ for any value of m, but to deduce that the function π' converges to F we need $x_m \to 0$ for all $m = 1,2,\ldots, N$. The function π is therefore a good approximation to F near any of the hyperplanes $x_m = 0$, whereas π' is a good approximation to F only in a small neighborhood of the origin. From Chapter 1 we concluded that the relationship $L \leq M$ implies that the set of interpolation properties of L is a subset of the set of interpolation properties of M. This conclusion is seen clearly in this example, as interpolation on the hyperplanes $x_m = 0$ implies interpolation at the origin.

The function π' is the truncated Taylor series expansion of the N-variate function F. Another standard truncation of the Taylor expansion of F is the function

$$\pi'' = \Sigma^* \frac{x_1^{\mu_1} \; x_2^{\mu_2} \; \ldots \; x_N^{\mu_N}}{\mu_1! \; \mu_2! \; \ldots \; \mu_N!} \; F^{(\mu_1, \ldots, \mu_N)}(0, \ldots, 0) \quad ,$$

where Σ^* denotes summation over all values $\mu_1, \ldots, \mu_N$ with $\mu_1 + \mu_2 + \ldots + \mu_N \le K_1 + K_2 + \ldots K_N$. The function π'' is developed by Stancu [14] . The approach used by Stancu has some similarities to the algebraic results of Chapter 1. We note, however, that because the projector which yields π'' cannot be factored into the projectors defined by (7) , π'' will not be obtainable by any of the elements in our lattice Ψ .

2.2.4. Proposition. The function π is the unique solution to the partial differential equation

$$\frac{\partial^{K_1+K_2+\ldots+K_N+N}}{\partial x_1^{K_1+1} \partial x_2^{K_2+1} \ldots \partial x_N^{K_N+1}} \pi(x_1, x_2, \ldots, x_N) = 0 \tag{14}$$

subject to the boundary conditions (9) , and π' is the unique solution to the system of N partial differential equations

$$\frac{\partial^{K_m+1}}{\partial x_m^{K_m+1}} \pi'(x_1, \ldots, x_N) = 0 \qquad (1 \le m \le N) \tag{15}$$

subject to the conditions (10) .

Proof. To see that π is a solution of (14) , we recall from (12) that

$$F - \pi = R = \int_0^{x_1} \ldots \int_0^{x_N} t_1 \ldots t_N \, F^{(K_1+1, \ldots, K_N+1)}(\xi_1, \ldots, \xi_N) \, d\xi_N \ldots d\xi_1 .$$

Thus we will show (14) by showing

$$R(x_1, \ldots, x_N)^{(K_1+1, \ldots, K_N+1)} = F(x_1, \ldots, x_N)^{(K_1+1, \ldots, K_N+1)} . \tag{16}$$

By condition (6) we may apply Liebnitz's rule for differentiation under the integral sign to show (14) . Doing so K_1 times yields

$$\frac{\partial^{K_1}}{\partial x_1^{K_1}} R(x_1, \ldots, x_N)$$

$$= \int_0^{x_1} \cdots \int_0^{x_N} t_2 \cdots t_N \; F(\xi_1, \ldots, \xi_N)^{(K_1+1, \ldots, K_N+1)} \; d\xi_N \ldots d\xi_1 ,$$

since $\frac{\partial^{K_1}}{\partial x_1^{K_1}} t_1(x_1; \xi_1) = \frac{\partial^{K_1}}{\partial x_1^{K_1}} [\frac{(x - \xi_1)^{K_1}}{K_1!}] = 1$, and the other functions in the integrand are independent of x_1 . At this stage R depends upon x_1 only through the upper limit of integration in the outermost integral. Thus, by (6) ,

$$\frac{\partial^{K_1+1}}{\partial x_1^{K_1+1}} R(x_1, \ldots, x_N)$$

$$= \int_0^{x_2} \cdots \int_0^{x_N} t_2 \cdots t_N \; F(x_1, \xi_2, \ldots, \xi_N)^{(K_1+1, K_2+1, \ldots, K_N+1)} \; d\xi_N \cdots d\xi_2$$

Continuing this procedure for the other variables yields (16) , from which (14) follows.

That π is the unique solution of (14) subject to the conditions (9) follows from a standard integration argument.

Next, we turn our attention to the minimal approximation $\pi' = L[F] = P_1P_2 \ldots P_N[F]$. From the definition of the projector P_m in (7) we see that the dependence of π' upon X_m comes only through terms X_m^μ $(0 \leq \mu \leq K_m)$ and not through F and its partial derivatives, which are evaluated at the origin. Thus (15) clearly holds. As above, the fact that π' is the unique solution to (15) subject to the conditions (10) follows from a standard integration argument. ∎

We see therefore that Taylor-type expansions of a suitable function F may be studied using the algebraic framework developed in Chapter 1. The contrast between the maximal projector M and the minimal projector L exemplifies the patterns predicted by the relationship $L \leq M$. It is interesting to note that one of the two standard Taylor series representations of F (π') corresponds to our minimal projector.

2.3. MULTIVARIATE LAGRANGE POLYNOMIAL INTERPOLATION

As a second example of the way in which the ideas of Chapter 1 may be applied to yield standard techniques in approximation theory we now investigate multi-dimensional Lagrange polynomial interpolation. The following corollary to Theorem 1.5.1. is stated for the case of a bivariate function $F = F(x, y)$, and reveals the contrast between the maximal projector M and the minimal projector L for this case.

2.3.1. Corollary. Let $F(x, y) \in C([a, b] \times [c, d])$ and let $\Delta_1\colon a = x_1 < x_2 < \ldots < x_{k_1} = b$ and $\Delta_2\colon c = y_1 < y_2 < \ldots < y_{k_2} = d$ be partitions of the intervals $[a, b]$ and $[c, d]$, respectively. Let

$\{\phi_i^1(x)\}_{i=1}^{K_1}$ and $\{\phi_j^2(y)\}_{j=1}^{K_2}$ be the Lagrange interpolation polynomials for Δ_1 and Δ_2, respectively:

$$\phi_i^1(x) = \prod_{\substack{K=1 \\ K \neq i}}^{K_1} \left(\frac{x - x_K}{x_i - x_K}\right) \qquad (i = 1, 2, \ldots, K_1),$$

$$\phi_j^2(y) = \prod_{\substack{\ell=1 \\ \ell \neq j}}^{K_2} \left(\frac{y - y_\ell}{y_j - y_\ell}\right) \qquad (j = 1, 2, \ldots, K_2). \tag{17}$$

Define the projectors P_1 and P_2 by

$$P_1[F] = \sum_{i=1}^{K_1} \phi_i^1(x)\, F(x_i, y) \tag{18}$$

$$P_2[F] = \sum_{j=1}^{K_2} \phi_j^2(y)\, F(x, y_j). \tag{19}$$

The maximal approximation $\pi = M[F] = (P_1 + P_2 - P_1P_2)[F]$ agrees with the function F along the $K_1 + K_2$ mesh lines of the Cartesian product partition $\Delta_1 \times \Delta_2$ of $[a, b] \times [c, d]$:

$$\pi(x_i, y) = F(x_i, y) \qquad (i = 1, 2, \ldots, K_1),$$

$$\pi(x, y_j) = F(x, y_j) \qquad (j = 1, 2, \ldots, K_2). \tag{20}$$

The minimal approximation $\pi' = L[F] = P_1P_2[F]$ agrees with F at the $K_1 \cdot K_2$ mesh points (x_i, y_j):

$$\pi'(x_i, y_j) = F(x_i, y_j) \qquad (21)$$

$$(i = 1, 2, \ldots, K_1;\ j = 1, 2, \ldots, K_2) .$$

Proof. From (17) we note that

$$\phi_i^1(x_k) = \delta_{ik} \qquad (1 \leq i, k \leq K_1)$$

$$\phi_j^2(y_\ell) = \delta_{j\ell} \qquad (1 \leq j, \ell \leq K_2) . \qquad (22)$$

Using (18) and (19) we have

$$\pi = M[F] = \sum_{i=1}^{K_1} \phi_i^1(x)\, F(x_i, y) + \sum_{j=1}^{K_2} \phi_j^2(y)\, F(x, y_j)$$

$$- \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} \phi_i^1(x)\, \phi_j^2(y)\, F(x_i, y_j) .$$

Thus, by (22) ,

$$\pi(x_k, y) = F(x_k, y) \qquad (1 \leq k \leq K_1) ,$$

and

$$\pi(x, y_j) = F(x, y_j) \qquad (1 \leq j \leq K_2) .$$

The minimal projection is given by

$$\pi'(x, y) = L[F(x, y)] = \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} \phi_i^1(x)\, \phi_j^2(y)\, F(x_i, y_j) ,$$

and (21) follows readily from (22) . ■

We note again that the set of interpolation properties possessed by π' is a subset of the set of interpolation properties possessed by π . It is also interesting to note that, as in the case of the Taylor-type expansions of Section 2.2, the most common multivariate Lagrange interpolation technique (tensor product) corresponds to our minimal projector L .

2.4. ORTHOGONAL EXPANSION IN HILBERT SPACE

As our final example which deals with the extension of a univariate technique to multivariate interpolation we consider the technique of orthogonal (Fourier) expansion in the Hilbert space $L^2(\Omega)$, where Ω is some closed and bounded subset of $\mathbb{R}^N$. For the example at hand we will consider the bivariate case and let Ω be the rectangle $[a, b] \times [c, d] \subset \mathbb{R}^2$.

Our discussion of projections in Hilbert space relies upon the following well-known result.

2.4.1. Lemma. Let $\{\phi_i(x)\}_{i=1}^{\infty}$ be a complete orthonormal set for $L^2([a, b])$ and let $\{\psi_j\}_{j=1}^{\infty}$ be a complete orthonormal set for $L^2([c, d])$. Then the set $\{\phi_i(x)\, \psi_j(y)\}_{i,j=1}^{\infty}$ is a complete orthonormal set for $L^2([a, b] \times [c, d])$.

2.4.2. Corollary. Let $F(x, y)$ be in the Hilbert space $F = L^2([a, b] \times [c, d])$. Let $\{\phi_i(x)\}_{i=1}^{K_1}$ and $\{\psi_j(y)\}_{j=1}^{K_2}$ be orthonormal sets of functions over $[a, b]$ and $[c, d]$, respectively:

$$\int_a^b \phi_i(x)\ \phi_k(x)\ dx = \delta_{ik} \qquad (i, k = 1, 2, \ldots, K_1)\ ,$$
$$\int_c^d \psi_j(y)\ \psi_\ell(y)\ dy = \delta_{j\ell} \qquad (j, \ell = 1, 2, \ldots, K_2)\ . \tag{23}$$

Let

$$\Phi = \{F = \sum_{i=1}^{K_1} A_i(y)\ \phi_i(x) | A_i(y) \in L^2([c, d])\}\ ,$$
$$\Psi = \{F = \sum_{i=1}^{K_2} B_j(x)\ \psi_j(y) | B_j(x) \in L^2([a, b])\} \tag{24}$$

and define P_1 and P_2 to be the orthogonal projections from $L^2([a, b] \times [c, d])$ onto Φ and Ψ respectively:

$$P_1[F] = \sum_{i=1}^{K_1} \phi_i(x)\ (\int_a^b F(\xi, y)\ \phi_i(\xi)\ d\xi)\ ,$$
$$P_2[F] = \sum_{j=1}^{K_2} \psi_j(y)(\int_c^d F(x, \eta)\ \psi_j(\eta)\ d\eta)\ . \tag{25}$$

Then, the maximal approximation $\pi = P_1[F] + P_2[F] - P_1P_2[F]$ satisfies

$$\int_a^b \pi(\xi, y)\ \phi_i(\xi)\ d\xi = \int_a^b F(\xi, y)\ \phi_i(\xi) d\xi \quad (i = 1, 2, \ldots, K_1), \tag{26a}$$

$$\int_c^d \pi(x,\ \eta)\ \psi_j(\eta)\ d\eta = \int_c^d F(x,\ \eta)\ \psi_j(\eta)\ d\eta \quad (j = 1,\ 2,\ldots,\ K_2), \quad (26b)$$

and, among all functions in the space $\operatorname{span}\{\Phi \cup \Psi\}$, it is the unique function which minimizes the $L^2([a,\ b] \times [c,\ d])$ norm of the remainder:

$$\|F - \pi\|^2 = \int_a^b \int_c^d |F(x,\ y) - \pi(x,\ y)|^2\ dx\ dy\ . \qquad (27)$$

The minimal approximation $\pi' = P_1P_2[F]$ satisfies

$$\int_a^b \int_c^d \pi'(\xi,\ \eta)\ \phi_i(\xi)\ \psi_j(\eta)\ d\xi d\eta = \int_a^b \int_c^d F(\xi,\ \eta)\ \phi_i(\xi)\ \psi_j(\eta)\ d\xi d\eta$$

$$(i = 1,\ 2,\ldots,\ K_1\ ;\ j = 1,\ 2,\ldots,\ K_2)\ , \quad (28)$$

and is the unique function f in $\Phi \cap \Psi$ which minimizes $\|F - f\|$.

Proof. We begin by verifying that P_1 and P_2 are projectors which commute. From (25) P_1 and P_2 are clearly linear. To see that P_1 is idempotent we note that by (23).

$$P_1^2[F] = \sum_{\ell=1}^{K_1} \phi_\ell(x)\ \left(\int_a^b \sum_{i=1}^{K_1} \phi_i(\xi) \int_a^b F(\Omega,\ y)\ \phi_i(\Omega)\ d\Omega\ \phi_\ell(\xi)\ d\xi\right)$$

$$= \sum_{\ell=1}^{K_1} \phi_\ell(x)\ \left(\sum_{i=1}^{K_2} \int_a^b \phi_i(\xi)\ \phi_\ell(\xi)\ d\xi\right)\left(\int_a^b F(\Omega,\ y)\ \phi_i(\Omega)\ d\Omega\right)$$

$$= \sum_{\ell=1}^{K_1} \phi_\ell(x) \int_a^b F(\Omega,\ y)\ \phi_i(\Omega)\ d\Omega = P_1[F]\ .$$

Similarly, one can see that P_2 is idempotent. The fact that P_1 and P_2 commute follows easily from interchanging the orders of summation and integration.

Therefore, P_1 and P_2 generate a lattice with maximal element $\mathcal{M} = P_1 \oplus P_2$ and minimal element $\mathcal{L} = P_1 P_2$.

The next step is to verify that P_1 and P_2 are onto Φ and Ψ , respectively. For this, we simply note that if $F = A_j(y)\ \phi_j(x) \in \Phi$, then

$$P_1[F] = \sum_{i=1}^{K_1} \phi_i(x) \int_a^b A_j(y)\ \phi_j(\xi)\ \phi_i(\xi)\ d\xi = A_j(y)\ \phi_j(x) \quad .$$

By the linearity of P_1 , every element in Φ has a preimage (itself) in $L^2([a, b] \times [c, d])$. Thus P_1 is a projection onto all of Φ , and similarly P_2 is a projection onto all of Ψ .

We also note that Φ and Ψ are closed subspaces of $L^2([a, b] \times [c, d])$. To see this, let

$$F_N = \sum_{j=1}^{K_1} A_j^{(N)}(y)\ \phi_j(x) \in \Phi$$

and suppose $F_N \to F$. Then, by orthonormality of the ϕ_j's , for each j , as $N \to \infty$ $A_j^{(N)}$ converges to a limit, say $A_j(y)$. Since each $A_j^{(N)} \in L^2([c, d])$, $A_j(y) \in L^2([c, d])$. Thus, $F = \sum_{j=1}^{K_1} A_j(y)\ \phi_j(x) \in \Phi$, i.e. Φ is closed. Similarly, Ψ is also closed.

Consider the maximal operator $\mathcal{M} = P_1 + P_2 - P_1 P_2$, and the subspace $M = \mathrm{span}\{\Phi \cup \Psi\}$. The range of $\mathcal{M}$ is obviously a subset of M , and

and in the manner used above to show P_1 is onto Φ we may show $\mathcal{M}$ is actually onto all of M .

We will now show that M is a closed subspace of $L^2([a, b] \times [c, d])$. Since $\{\phi_i(x)\}_{i=1}^{K_1}$ and $\{\psi_j(y)\}_{j=1}^{K_2}$ are orthonormal sets in $L^2([a, b])$ and $L^2([c, d])$, respectively, they may be extended to orthonormal bases for these spaces. Let $\{\phi_i(x)\}_{i=1}^{\infty}$ and $\{\psi_j(y)\}_{j=1}^{\infty}$ denote these bases. By Lemma 2.4.1., the set $\{\phi_i(x)\ \psi_j(y)\}_{i,j=1}^{\infty}$ is an orthonormal basis for $L^2([a, b] \times [c, d])$.

Any function F in M may be expressed as

$$F = \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} \alpha_{ij}\ \phi_i(x)\ \psi_j(y) + \sum_{i=1}^{K_1} \sum_{j=K_2+1}^{\infty} \alpha_{ij}\ \phi_i(x)\ \psi_j(y)$$

$$+ \sum_{i=K_1+1}^{\infty} \sum_{j=1}^{K_2} \alpha_{ij}\ \phi_i(x)\ \psi_j(y)\ ,$$

i.e. $F = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij}\ \phi_i(x)\ \psi_j(y)$, where $\alpha_{ij} = 0$ whenever $i > K_1$ and $j > K_2$.

Thus, if $\{F_N\}$ is a sequence in M and $F_N \to F$, then by Parseval's formula

$$||F_N - F||^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_{ij}^{(N)} - F_{ij}|^2$$

where the $\alpha_{ij}^{(N)}$'s and F_{ij}'s are the Fourier coefficients of the functions F_N and F, respectively, with respect to the basis $\{\phi_i(x)\ \psi_j(y)\}_{i,j=1}^{\infty}$. Therefore, as $N \to \infty$, $\alpha_{ij}^{(N)} \to F_{ij}$ for all pairs of indices i and j. In particular, this means $F_{ij} = 0$ whenever $i > K_1$ and $j > K_2$. Thus $F \in M$, i.e. M is closed.

We see therefore that $\mathcal{M}$ is the projection onto the closed subspace M of $L^2([a, b] \times [c, d])$. Thus, by the projection theorem for Hilbert space, among all functions in M, $\pi = \mathcal{M}[F]$ is the unique function f which minimizes the L^2 norm of the remainder $\|F - f\|$.

Similarly, $\mathcal{L} = P_1P_2$ is the projection onto the closed subspace $\Phi \cap \Psi$, and among all functions $f \in \Phi \cap \Psi$, $\pi' = \mathcal{L}[F]$ is the unique function f which minimizes the quantity $\|F - f\|$.

To complete the proof of this theorem it remains only to verify the relationships (26a), (26b), and (28). To see (26a), we note that

$$\int_a^b \pi(\xi, y)\, \phi_i(\xi)\, d\xi$$

$$= \int_a^b \sum_{m=1}^{K_1} \phi_m(\xi) \left(\int_a^b F(\xi, y)\, \phi_m(\xi)\, d\xi\right)\phi_i(\xi)\, d\xi$$

$$+ \int_a^b \left(\sum_{j=1}^{K_2} \psi_j(y) \int_c^d F(\xi, \eta)\, \psi_j(\eta)\, d\eta\right) \phi_i(\xi)\, d\xi$$

$$- \int_a^b \sum_{m=1}^{K_1} \phi_m(\xi)\left[\int_a^b \sum_{j=1}^{K_2} \psi_j(y)\left(\int_c^d F(\xi, \eta)\, \psi_j(\eta)d\eta\right)\phi_m(\xi)d\xi\right]\phi_i(\xi)d\xi.$$

By integrating and employing the orthonormality of the ϕ_m's one sees that (26a) holds. A similar argument yields (26b). The verification of (28) is also completed using the orthonormality condition (23). ■

As in our two previous examples, we see that the relationship $\mathcal{L} \leq \mathcal{M}$ has an interpolatory meaning. In this example, the set of Fourier coefficients shared by π' and F is a subset of those shared by π and F. From (28), π' and F share the coefficients F_{ij} (with respect to the basis $\{\phi_i(x)\ \psi_j(y)\}_{i,j=1}^{\infty}$) for $1 \leq i \leq K_1$ <u>and</u> $1 \leq j \leq K_2$. The functions π and F, however, share the Fourier coefficients F_{ij} whenever $1 \leq i \leq K_1$ <u>or</u> $1 \leq j \leq K_2$, which is easily verified from (26a) and (26b).

2.5. MULTIVARIATE INTERPOLATION OVER CYLINDRICAL AND TRIANGULAR DOMAINS

In this section we shall consider two examples in which we interpolate a function on the boundary of a geometric figure. These examples differ from the three previous examples in that the projectors used are intrinsically multivariate projectors instead of parametric extensions of univariate techniques. The problem of multivariate interpolation over arbitrary curved element domains has been studied by Gordon and Hall [10] via the algebraic structure developed in Chapter 1. As in the previous examples, our main objective is to contrast the maximal and minimal projectors of our lattice.

We begin with the problem of approximating a function on the vertical surface and endcaps of the unit right circular cylinder whose vertical axis is the z-axis.

Let $F = F(r, \theta, z)$ be a trivariate function which satisfies the following condition:

> F is continuous in its first partial derivatives with respect to r and θ, and at least piecewise continuous in its second partial derivatives with respect to r and θ. (29)

We wish to construct an operator which will interpolate (exactly) to F on the surfaces Σ, D_1, and D_2 defined by

$$\Sigma = \{(r, \theta, z) | r = 1 , 0 \le \theta \le 2\pi , 0 \le z \le 1\} ,$$

$$D_1 = \{(r, \theta, z) | 0 \le r \le 1 , 0 \le \theta \le 2\pi , z = 0\} ,$$

$$D_2 = \{(r, \theta, z) | 0 \le r \le 1 , 0 \le \theta \le 2\pi , z = 1\} .$$

We shall define the projectors P_1 and P_2 by

$$P_1[F] = (1 - z)\, F(r, \theta, 0) + z\, F(r, \theta, 1) \tag{30}$$

$$P_2[F] = \int_0^{2\pi} G(r, \theta, \theta')\, F(1, \theta', z)\, d\theta' \tag{31}$$

where the Green's function G is the Poisson kernel

$$G(r, \theta, \theta') = \frac{1}{2\pi} \left[\frac{1 - r^2}{1 - 2r \cos(\theta - \theta') + r^2} \right] . \tag{32}$$

We see that the function $\pi_1 = P_1[F]$ agrees with F on the planes $z = 0$ and $z = 1$ (in particular on the two endcaps D_1 and D_2) :

$$\pi_1(r, \theta, 0) = F(r, \theta, 0) \tag{33a}$$

$$\pi_1(r, \theta, 1) = F(r, \theta, 1) . \tag{33b}$$

Also, for all values of z, the function $\pi_2 = P_2[F]$ satisfies Laplace's equation $\nabla^2 \pi_2 = 0$ and the boundary conditions

$$\pi_2(1, \theta, z) = F(1, \theta, z) \qquad (0 \le \theta \le 2\pi , 0 \le z \le 1) . \tag{34}$$

2.5.1. Lemma. The operators P_1 and P_2 are projectors and $P_1P_2 = P_2P_1$.

Proof. P_1 and P_2 are clearly linear. It is easily shown that P_1 is idempotent. To see that P_2 is idempotent we note that the integrand in (31) involves values of F only on the surface Σ , and by (34), π_2 and F agree on Σ . Thus $P_2^2[F] = P_2[\pi_2] = P_2[F]$.

Since the variable z is merely a parameter in the integral which defines the operator P_2 , the two projectors do indeed commute:

$$P_1P_2[F] = P_2P_1[F] = (1 - z)\int_0^{2\pi} G(r, \theta, \theta')F(1, \theta', 0)\, d\theta' + z \int_0^{2\pi} G(r, \theta, \theta')\, F(1, \theta', 1)\, d\theta' \qquad \blacksquare$$

The above observations lead us to the following corollary of Theorem 1.5.1.

2.5.2. Corollary. Let M and L be the maximal and minimal elements of the lattice Ψ generated by the projectors P_1 and P_2 defined in (30) and (31) .

For any function satisfying (29) the function $\pi = M[F]$ coincides with F on the three bounding surfaces D_1, D_2, and Σ of the unit right circular cylinder, i.e., π satisfies all three of the conditions (33a), (33b), and (34) . Moreover, it is the unique function which satisfies these boundary conditions and satisfies the partial differential equation

$$\frac{\partial^2}{\partial z^2} \nabla^2[\pi] = 0 \ .$$

The function π' given by $\pi' = L[F]$ agrees with F only on the circumferences of the two disks D_1 and D_2 :

$$\pi'(1, \theta, 0) = F(1, \theta, 0) \qquad (0 \leq \theta \leq 2\pi) \tag{35}$$

$$\pi'(1, \theta, 1) = F(1, \theta, 1) \qquad (0 \leq \theta \leq 2\pi) \ . \tag{36}$$

Furthermore, π' satisfies the pair of partial differential equations $\nabla^2[\pi'] = 0$ and $\frac{\partial^2}{\partial z^2}[\pi'] = 0$. The remainders associated with the approximations π and π' are, respectively:

$$F - \pi = R_1 R_2[F] \ , \tag{37}$$

$$F - \pi' = R_1[F] + R_2[F] - R_1 R_2[F] \ , \tag{38}$$

where

$$R_1[F] = \int_0^1 K_1(z; z')\, F^{(0, 0, 2)}(r, \theta, z)\, dz' \tag{39}$$

$$R_2[F] = \int_0^{2\pi}\int_0^1 K_2(r, \theta; r', \theta')\, \nabla^2 F(r', \theta', z)\, dr' d\theta' \ .$$

Here, $K_1(z; z')$ and $K_2(r, \theta; r', \theta')$ are well-known Green's functions for the differential operators $\frac{\partial^2}{\partial z^2}$ and ∇^2 respectively, subject to homogeneous boundary conditions:

$$K(z, z') = \begin{cases} (z' - 1)z & z \leq z' \\ \\ (z - 1)z' & z > z' \end{cases} \tag{41}$$

$$K_2(r, \theta; r', \theta') = -\frac{1}{2\pi} \log r'$$

$$+ \frac{1}{4\pi} \log \left[\frac{(r \cos \theta - r' \cos \theta')^2 + (r \sin \theta - r' \sin \theta')^2}{(r \cos\theta - \cos \theta')^2 + (r \sin \theta - \sin \theta')^2} \right] \tag{42}$$

Proof. We note that

$$\pi = M[F] = (1 - z)\, F(r, \theta, 0) + z\, F(r, \theta, 1)$$

$$+ \int_0^{2\pi} G(r, \theta, \theta')\, F(1, \theta', z)\, d\theta'$$

$$+ (z - 1) \int_0^{2\pi} G(r, \theta, \theta')\, F(1, \theta', 0)\, d\theta'$$

$$- z \int_0^{2\pi} G(r, \theta, \theta')\, F(1, \theta', 1)\, d\theta' .$$

It is therefore easily seen that $\pi(r, \theta, 0) = F(r, \theta, 0)$, $\pi(r, \theta, 1) = F(r, \theta, 1)$ and $\pi(1, \theta, z) = F(1, \theta, z)$. Furthermore, since $\nabla^2 P_2[F] = 0$ we have $\nabla^2\pi = (1 - z)\, \nabla^2 F(r, \theta, 0) + z\, \nabla^2 F(r, \theta, 1)$. Thus the function π does satisfy the partial differential equation $\frac{\partial^2}{\partial z^2} \nabla^2[\pi] = 0$. The fact that π is the unique solution to this equation subject to (33a), (33b) and (34) follows from a standard integration argument.

It is also easy to verify (35) and (36), as well as the facts that $\nabla^2[\pi'] = 0$ and $\frac{\partial^2}{\partial z^2}[\pi'] = 0$.

Let us consider the remainder operators R_1 and R_2 associated with the projectors P_1 and P_2 , respectively:

$$R_1[F] = F - P_1[F] = F(r, \theta, z) - [(1 - z)F(r,\theta, 0) + zF(r,\theta,1)]$$

$$R_2[F] = F - P_2[F] = F(r, \theta, z) - \int_0^{2\pi} G(r, \theta, \theta') F(1, \theta', z) \, d\theta'.$$

We see that

$$\frac{\partial^2}{\partial z^2} R_1[F] = \frac{\partial^2}{\partial z^2} F \text{ , } R_1(r, \theta, 1) = R_1(r, \theta, 0) = 0 \tag{43}$$

$$(0 \leq r \leq 1 \text{ , } 0 \leq \theta \leq 2\pi)$$

$$\nabla^2 R_2[F] = \nabla^2 F \text{ , } R_2(1, \theta, z) = 0$$

$$(0 \leq \theta \leq 2\pi \text{ , } 0 \leq z \leq 1) \text{ .} \tag{44}$$

The functions $R_1[F]$ and $R_2[F]$ are therefore the solutions of the partial differential equations (43) and (44). The results (39)-(42) then follow from (43) and (44) (see, for example, Courant and Hilbert [6]) . ■

The example above helps to illustrate the fact that the maximal projector M combines all the approximation properties of the collection $\{P_m\}_{m=1}^N$, while the minimal projector L has only those properties which the projectors P_m have in common. The projector P_1 agrees with F on the two endcaps D_1 and D_2, and the projector P_2 agrees with F on the vertical surface Σ of the cylinder. The maximal projector combines these attributes and the maximal approximation π agrees with F on the entire surface of the cylinder. The minimal approximation π', however, agrees with F only on the circumferences of D_1 and D_2 (i.e. $(D_1 \cup D_2) \cap \Sigma$).

As our last example of this chapter we shall develop a method for interpolation of a continuous bivariate function $F(x, y)$ on the perimeter of the triangle T whose vertices are $(0, 0)$, $(1, 0)$ and $(1, 1)$. The purpose of this example is to further illustrate how the maximal projector M combines the properties of the P_m, $1 \leq m \leq N$, while the minimal projector L enjoys only those properties which the P_m hold in common. The three projectors P_1, P_2, and P_3 defined below are derived purely to provide exact approximations to F on the edges of T, and to commute with each other.

We define the following three projectors:

$$P_1[F] = \left(\frac{x - y}{x}\right) F(x, 0) \quad , \tag{45}$$

$$P_2[F] = \left(\frac{x - y}{1 - y}\right) F(1, y) \quad , \tag{46}$$

$$P_3[F] = \left(\frac{1 - x}{1 - y}\right) \left(\frac{y}{x}\right) F(x, x) \quad . \tag{47}$$

We note that $\pi_1 = P_1[F]$ and $\pi_3 = P_3[F]$ are undefined along the line $x = 0$, and that $\pi_2 = P_2[F]$ and π_3 are undefined along the line $y = 1$. However, since our area of interest is the perimeter of T, the only singularities which concern us are at the points $(0, 0)$ and $(1, 1)$. We shall merely exclude these points and proceed.

It is easy to see that

$$\pi_1(x, y) = \begin{cases} F(x, 0) & (y = 0), \\ (1 - y)\, F(1, 0) & (x = 1), \\ 0 & (x = y), \end{cases} \tag{48}$$

$$\pi_2(x, y) = \begin{cases} x\, F(1, 0) & (y = 0), \\ F(1, y) & (x = 1), \\ 0 & (x = y), \end{cases} \tag{49}$$

$$\pi_3(x, y) = \begin{cases} 0 & (y = 0), \\ 0 & (x = 1), \\ F(x, x) & (y = x). \end{cases} \tag{50}$$

A simple computation shows that P_1, P_2, and P_3 commute, and therefore these three projectors generate a lattice with maximal element M and minimal element L given by

$$M = P_1 + P_2 + P_3 - P_1P_2 - P_1P_3 - P_2P_3 + P_1P_2P_3$$

and

$$L = P_1 \, P_2 \, P_3 = 0 \; .$$

The maximal and minimal approximations to F are, respectively:

$$\pi(x, y) = M[F(x, y)] = \left(\frac{x - y}{x}\right) F(x, 0) + \left(\frac{x - y}{1 - y}\right) F(1, y)$$

$$+ \left(\frac{1 - x}{1 - y}\right) \left(\frac{y}{x}\right) F(x, x) - (x - y) F(1, 0) \; ,$$

$$\pi' = L[F] = 0 \quad .$$

It is easy to see that π agrees with F on the perimeter of T (excluding $(0, 0)$ and $(1, 1)$), and thus combines the approximation properties of P_1, P_2, and P_3. The function π', however, agrees with F only where $F = 0$. This result is expected, as P_1, P_2, and P_3 have no interpolation properties in common.

The examples of this chapter help to illustrate the ideas of Chapter 1 as well as to reveal the ways in which some common approximation techniques may be studied by a lattice theoretic approach. In Chapter 3 we shall examine another example, using cubic splines, to further illustrate the ideas of Chapter 1.

CHAPTER 3

AN EXAMPLE USING CUBIC SPLINES

3.1. INTRODUCTION

As our final example we shall investigate the applications of our algebraic structure to the problem of interpolating a multivariate function by the use of cubic splines. In the Taylor series example of Section 2.2, we saw how an N-variate function F may be approximated by a combination of functions of $N - 1$ and fewer variables. In this example we shall apply this technique recursively (using cubic splines) to obtain an approximation to $F(x_1, \ldots, x_N)$ based only upon a finite sample of function values at specified data points. This technique of recursive approximation is known as the <u>method of successive decomposition</u>.

In the lattice which we shall construct by using cubic spline projectors, the method of successive decomposition will be based upon our maximal projector M, whereas the standard technique of tensor product approximation will correspond to our minimal projector L. A contrast of these two methods will reveal that, under some mild assumptions on the continuity of F and its partial derivatives, the number of data points required to achieve an approximation of predetermined accuracy is considerably less for the method of successive decomposition than for the tensor product method.

The method of successive decomposition will be developed for the bivariate case in Section 3.2, and Section 3.3 will examine this method using cubic splines for the bivariate function $F = F(x, y)$.

In Section 3.4 we shall investigate briefly the method for N-variate functions with $N \geq 2$.

3.2. THE METHOD OF SUCCESSIVE DECOMPOSITION FOR THE BIVARIATE CASE

In the Lagrange polynomial example of Section 2.3 we saw how a bivariate function $F = F(x, y)$ may be approximated by a combination of univariate functions $F(x_i, y)$ and $F(x, y_j)$ $(1 \leq i \leq K_1 ;\ 1 \leq j \leq K_2)$ where the points (x_i, y_j) are mesh-points in a two-dimensional subdivision of $[0, 1] \times [0, 1]$. The maximal projector $M = P_1 \oplus P_2$ performed the task of fitting a smooth surface $\pi(x, y)$ through the network of curves $F(x_i, y)$ and $F(x, y_j)$ $(1 \leq i \leq K_1 ;\ 1 \leq j \leq K_2)$. This technique, known as blending, has been investigated by Gordon [7], [9], and Gordon and Hall [10] . The function $\pi(x, y)$ is therefore called a blending function. For the bivariate case, the error associated with the blending function is $F - \pi = R_1R_2[F]$, where $R_j = I - P_j$ $(j = 1, 2)$.

In actual applications, however, we are generally not presented with a network of curves through which to pass a surface, but rather with a finite collection of sample values extracted from the function F . It will therefore be necessary to approximate the functions $F(x_i, y)$ and $F(x, y_j)$ by some finite parameter univariate scheme. We shall denote these approximations by $g_i(y)$ and $h_j(x)$, respectively.

If the errors in these approximations may be uniformly bounded in norm, say

$$||F(x_i, y) - g_i(y)|| \leq \varepsilon$$

and

$$||F(x, y_j) - h_j(x)|| \leq \varepsilon \qquad (1 \leq i \leq K_1 \ ; \ 1 \leq j \leq K_2)$$

then, for small ε, the error made by replacing the univariate curves by their approximations will be less than the error resulting from the initial blending function approximation $(||R_1R_2 F||)$. Therefore, the error involved in using successive decomposition will be of the same order of magnitude as the error involved in blending the plane curves $F(x_i, y)$ and $F(x, y_j)$. Also, since the error will be at least as great as $||R_1R_2 F||$, it is unnecessary to approximate the functions $F(x_i, y)$ and $F(x, y_j)$ to any higher accuracy than is required by the initial blending function approximation. The two levels of approximation should therefore be consistent in terms of the accuracy required.

For the bivariate case we shall have two levels of decomposition, each of which uses two projectors. We shall denote these projectors by P_1^1, P_2^1, P_1^2, and P_2^2, where superscripts denote the level of decomposition.

<u>3.2.1. Lemma</u>. Let $F(x, y)$ be a bivariate function for which the projectors P_1^1 and P_2^1 commute. Let P_1^2 and P_2^2 be two projectors such that $P_1^2P_2^1[F] = P_2^1P_1^2[F]$ and $P_2^2P_1^1[F] = P_1^1P_2^2[F]$. Then, with $R_j^i = I - P_j^i$ $(1 \leq i, j \leq 2)$, the bivariate identity operator I has the decomposition

$$I = P_2^2 P_1^1 + P_1^2 P_2^1 - P_1^1 P_2^1 + (R_2^2 P_1^1 + R_1^2 P_2^1) + R_1^1 R_2^1 \ . \tag{51}$$

<u>Proof</u>. Since P_1^1 and P_2^1 commute, from the results of Chapter 1 we have

$$I = P_1^1 \oplus P_2^1 + R_1^1 R_2^1 = P_1^1 + P_2^1 - P_1^1 P_2^1 + R_1^1 R_2^1 \ . \tag{52}$$

Furthermore, since also $I = P_1^2 + R_1^2 = P_2^2 + R_2^2$, we may write (52) as

$$I = (P_2^2 + R_2^2)\, P_1^1 + (P_1^2 + R_1^2)\, P_2^1 - P_1^1 P_2^1 + R_1^1 R_2^1 \ ,$$

from which (51) follows immediately. ∎

<u>3.2.2. Lemma</u>. If $F(x, y)$ belongs to the normed linear space F , then with $A = P_2^2 P_1^1 + P_1^2 P_2^1 - P_1^1 P_2^1$ and the usual definition of operator norm, the error operator

$$I - A = P_1^1 R_2^2 + P_2^1 R_1^2 + R_1^1 R_2^2$$

satisfies

$$\|I - A\| \le \|P_1^1\| \ \|R_2^2\| + \|P_2^1\| \, \|R_1^2\| + \|R_1^2\| \, \|R_2^1\| \ . \tag{53}$$

<u>Proof.</u> The formula for $I - A$ follows from (51) ; the error bound (53) follows from a simple application of the triangle inequality and the fact that $\|AB\| \le \|A\| \, \|B\|$ for all projectors A and B for which AB is defined. ∎

3.2.3. Remark. If the linear space F is actually a Hilbert space, and if P and Q are orthogonal projectors, then the algebraic ordering $P \leq Q$ is equivalent to the ordering in norm given by

$$\|Px\| \leq \|Qx\| \qquad (x \in F) . \tag{54}$$

If F is a Hilbert space and if the P_j^i $(1 \leq i, j \leq 2)$ are orthogonal projectors, then

$$\|(I - A)[F]\| \leq \|R_2^2 F\| + \|R_2^1 F\| + \|R_1^1 R_2^1[F]\| . \tag{55}$$

3.3. SUCCESSIVE DECOMPOSITION VS. TENSOR PRODUCTS IN CUBIC SPLINE INTERPOLATION (BIVARIATE CASE)

We shall now focus our attention on the contrast between successive decomposition and the method of tensor products for the case when the projectors used will be based upon cubic splines. We shall deal with the bivariate case and, for simplicity, we shall use projectors which are symmetric in x and y. That is, the projectors P_1^1 and P_2^1 (respectively, P_1^2 and P_2^2) used in the first (respectively, second) level of decomposition are actually the same projector with the roles of x and y interchanged. The details of the general bivariate case are easily inferred.

3.3.1. Definition. Let $C^{M,N}([0, 1] \times [0, 1])$ denote the class of bivariate functions $F(x, y)$ defined on $[0, 1] \times [0, 1]$ such that

$F^{(i,j)}(x, y) \equiv \dfrac{\partial^{i+j}}{\partial x^i \, \partial y^j} F(x, y)$ exists and is continuous for $0 \le i \le M$ and $0 \le j \le N$.

The normed linear space F which we shall consider will be the space $C^{4,4}([0, 1] \times [0, 1])$ equipped with the sup (uniform) norm:

$$\|F(x, y)\| = \sup_{(x,y)\in[0,1]\times[0,1]} |F(x, y)| \; .$$

We shall employ the set of cubic splines $\{\phi_i(\xi)\}_{i=1}^{M_1}$ with joints at the M_1 points of the uniform partition Δ_1 of $[0, 1]$ given by

$$\Delta_1 = \{0, h_1, \ldots, (M_1 - 1)h_1\},$$

where $h_1 = 1/(M_1 - 1)$. The cubic splines $\{\phi_i(\xi)\}_{i=1}^{M_1}$ shall satisfy

$$\phi_i(\xi_k) = \delta_{ik} \qquad \text{for } \xi_k \in \Delta_1 \qquad \text{(cardinal splines)} \tag{56}$$

$$\phi_i''(0) = \phi_i''(1) = 0 \qquad \text{(natural boundary conditions).} \tag{57}$$

The projectors P_1^1 and P_2^1 used at the first level of decomposition are

$$P_1^1[F] = \sum_{i=1}^{M_1} \phi_i(x) \, F(x_i, y) \qquad (x_i \in \Delta_1) \; , \tag{58}$$

$$P_2^1[F] = \sum_{j=1}^{M_1} \phi_j(y) \, F(x, y_j) \qquad (y_j \in \Delta_1) \; . \tag{59}$$

Although the operators P_1^1 and P_2^1 (and the other cubic spline operators used in this chapter) are idempotent linear operators, they are not really projectors in the sense of Definition 1.2.4, since their ranges are not subspaces of $C^{4,4}([0, 1] \times [0, 1])$. For example, the range of P_1^1 is a subspace of $C^{2,4}([0, 1] \times [0, 1])$. However, since this fact does not affect the results of this chapter, we shall refer to these cubic spline operators as projectors for consistency.

We note that each of P_1^1 and P_2^1 interpolates the function $F(x, y)$ as a function of one variable, with the other variable being held as a parameter. Therefore, the standard error bounds for the univariate case (Ahlberg, Nielson and Walsh [1]) yield the following results:

$$\|R_1^1[F]\| \leq \alpha\|F^{(4,0)}\| (h_1)^4 \quad , \tag{60}$$

$$\|R_2^1[F]\| \leq \alpha\|F^{(0,4)}\|(h_1)^4 \quad , \tag{61}$$

for some constant α independent of F .

The maximal projector of the lattice generated by P_1^1 and P_2^1 , $M = P_1^1 \oplus P_2^1$, yields the blending function $S(x, y) = M[F]$ described in Section 3.2. We shall therefore call the function $S(x, y)$ the <u>spline-blended interpolant</u> of F .

<u>3.3.2. Lemma</u>. The spline-blended interpolant $S(x, y)$ given by

$$S(x, y) = (P_1^1 \oplus P_2^1)[F] = \sum_{i=1}^{M_1} \phi_i(x)\, F(x_i, y) + \sum_{j=1}^{M_1} \phi_j(y)\, F(x, y_j) \tag{62}$$

$$- \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} \phi_i(x)\ \phi_j(y)\ F(x_i,\ y_j)$$

agrees with F along the $2M_1$ lines $x = x_i$ and $y = y_j$ $(1 \leq i,j \leq M_1)$ of $[0, 1] \times [0, 1]$.

The error $F - S = R_1R_2[F]$ satisfies

$$||R_1^1R_2^1[F]|| \leq A||F^{(4,4)}||\ (h_1)^8 \tag{63}$$

for some constant A independent of F .

Proof. The interpolation results follow directly from condition (56) :

$$S(x_k,\ y) = \sum_{i=1}^{M_1} \phi_i(x_k)\ F(x_i,\ y) + \sum_{j=1}^{M_1} \phi_j(y)\ F(x_k,\ y_j)$$

$$- \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} \phi_i(x_k)\phi_j(y)\ F(x_k,\ y_j)$$

$$= F(x_k,\ y) + \sum_{j=1}^{M_1} \phi_j(y)\ F(x_k,\ y_j) - \sum_{j=1}^{M_1} \phi_j(y)\ F(x_k,\ y_j)$$

$$= F(x_k,\ y)\ , \qquad (1 \leq k \leq M_1)\ .$$

Similarly, $S(x, y_k) = F(x, y_k)$, $(1 \leq k \leq M_1)$.

The error result (63) is easily derived from the standard error results for univariate spline interpolation. In the univariate case, the error in spline interpolation to an $F \in C^4([0, 1])$ is given by

$$e(x) = f(x) - \sum_{i=1}^{M_1} \phi_i(x)\, F(x_i) = \int_0^1 G(x, s)\, F^{(iv)}(s)\, ds ,$$

where $G(x, s)$ is the appropriate Green's function (cf. Birkhoff and deBoor [3]) .

The actions of the remainder operators R_1^1 and R_1^2 upon $F(x, y)$ are therefore given by

$$R_1^1[F] = \int_0^1 G(x, s)\, F^{(4,0)}(s, y) ds$$

and

$$R_2^1[F] = \int_0^1 G(y, t)\, F^{(0,4)}(x, t)\, dt .$$

Thus, for $(x, y) \in [0, 1] \times [0, 1]$,

$$R_1^1 R_2^1[F(x, y)] = \int_0^1\int_0^1 G(x, s)\, G(y, t)\, F^{(4,4)}(s, t)\, ds\, dt .$$

The error result (63) may now be shown by employing the mean-value theorem and properties of the kernel function G (see Gordon [7] and [9]) . ∎

The significance of Lemma 3.3.2 is that S converges to F with an error which is $\mathcal{O}(h_1)^8$, and thus, for small h_1 , S is a very good approximation to F . From the obvious extension of the proof above, we see that in the case of N variables the maximal projector $\mathcal{M} = P_1^1 \oplus \ldots \oplus P_N^1$ will have an error satisfying

$$\|R_1^1 R_2^1 \ldots R_N^1[F]\| \le c\|F^{(4,4,\ldots 4)}\| \, (h_1)^{4N} ,$$

where c is independent of F .

The maximal projector $S(x, y)$ does provide a good approximation to F , but the problem with S arises from the fact that we do not, in general, know the functions $F(x_i, y)$ and $F(x, y_j)$ $(1 \le i,j \le M_1)$. We therefore proceed to the second level of decomposition and approximate the univariate functions $F(x_i, y)$ and $F(x, y_j)$ by cubic spline projectors.

For this second level, we shall define the M_2 point uniform partition of $[0, 1]$, Δ_2, by

$$\Delta_2 = \{0, h_2, \ldots, (M_2 - 1)h_2\}$$

where $h_2 = 1/(M_2 - 1)$. We shall use the set $\{\psi_k\}$ of cubic splines satisfying

$$\psi_k(\xi_\ell) = \delta_{k\ell} \qquad (\xi_\ell \in \Delta_2) , \tag{64}$$

and the natural boundary conditions (57) with ϕ replaced by ψ .

The projectors P_1^2 and P_2^2 will be given by

$$P_1^2[F] = \sum_{k=1}^{M_2} \psi_k(x) \, F(x_k, y) \qquad (x_k \in \Delta_2) , \tag{65}$$

$$P_2^2[F] = \sum_{\ell=1}^{M_2} \psi_\ell(y) \, F(x, y_\ell) \qquad (y_\ell \in \Delta_2) , \tag{66}$$

and they have remainders R_1^2 and R_2^2 satisfying

$$\|R_1^2[F]\| \leq \mathcal{B}\|F^{(4,0)}\|(h_2)^4 \;, \tag{67}$$

$$\|R_2^2[F]\| \leq \mathcal{B}\|F^{(4,0)}\|(h_2)^4 \;, \tag{68}$$

for some constant $\mathcal{B}$ independent of F .

Using the decomposition scheme outlined in Lemma 3.2.1 and Lemma 3.2.2, we derive the following approximation to F , which uses only numerical data.

<u>3.3.3. Lemma</u>. The function S^* given by

$$S^*(x, y) = \sum_{i=1}^{M_1} \phi_i(x) \sum_{k=1}^{M_2} \psi_k(y)\, F(x_i, y_k) \tag{69}$$

$$+ \sum_{j=1}^{M_1} \phi_j(y) \sum_{\ell=1}^{M_2} \psi_\ell(x)\, F(x_\ell, y_j)$$

$$- \sum_{i=1}^{M_1} \sum_{j=1}^{M_1} \phi_i(x)\, \phi_j(y)\, F(x_i, y_j)$$

approximates $F(x, y)$ with an error satisfying

$$\|F - S^*\| \leq k[(h_2)^4 + (h_1)^8] \;, \tag{70}$$

where

$$k = \max\{\mathcal{B}(\|P_1^1\|\,\|F^{(0,4)}\| + \|P_2^1\|\,\|F^{(0,4)}\|) \;,\;\; A\|F^{(4,4)}\|\}.$$

Proof. From Lemma 3.2.2 we have

$$F - S^* = P_1^1\, R_2^2[F] + P_2^1 R_1^2[F] + R_1^1 R_2^1[F] \ . \tag{71}$$

Therefore,

$$\|F - S^*\| \leq \|P_1^1\|\,\|R_2^2[F]\| + \|P_2^1\|\,\|R_1^2[F]\| + \|R_1^1 R_2^1[F]\| \ ,$$

and (70) follows from (67), (68) , and (63) . ■

3.3.4 Remark. We notice that the formula

$$S^* = (P_1^1\, P_2^2 + P_1^2 P_2^1 - P_1^1 P_2^1)[F]$$

is actually a linear combination of three tensor product approximations to F . The function $P_1^1 P_2^2[F]$ is the tensor product approximation over the mesh $\Delta_1 \times \Delta_2$, and $P_1^2\, P_2^1$ and $P_1^1 P_2^1$ correspond to the meshes $\Delta_2 \times \Delta_1$ and $\Delta_1 \times \Delta_1$, respectively. This combination can be achieved by subtracting (72c) from the sum of (72a) and (72b) below:

$$I = P_2^1 P_1^1 + R_2^2 + R_1^1 - R_2^2 R_1^1 \tag{72a}$$

$$I = P_1^2 P_2^1 + R_1^2 + R_2^1 - R_1^2 R_2^1 \tag{72b}$$

$$I = P_1^1 P_2^1 + R_1^1 + R_2^1 - R_1^1 R_2^1 \ . \tag{72c}$$

This approach yields the remainder (71) in the form

$$F - S^* = R_1^2 + R_2^2 - R_1^2R_2^1 - R_2^2R_1^1 + R_1^1R_2^1 .$$

We note that terms involving R_1^1 and R_2^1 alone do not appear. If, as the case will be, $h_2 << h_1$, then these are the dominant error terms in (72a), (72b), and (72c). It is because of this cancellation that S^* is a much better approximation to F than is any of $P_2^2P_1^1$, $P_1^2P_2^1$, or $P_1^1P_2^1$.

We now consider the tensor product approximation to F over the cartesian product mesh $\tilde{\Delta} \times \tilde{\Delta}$, where $\tilde{\Delta} = \{0, \tilde{h}, \ldots, (\tilde{M} - 1)\tilde{h}\}$ with $\tilde{h} = 1/(\tilde{M} - 1)$. We shall use the cubic splines $\{\tilde{\phi}_i(\xi)\}_{i=1}^{\tilde{M}}$ which satisfy

$$\tilde{\phi}_i(\xi_k) = \delta_{ik} \qquad (\xi_k \in \tilde{\Delta}) \tag{73}$$

and the natural boundary conditions (57) with ϕ replaced by $\tilde{\phi}$.

The projectors $\tilde{P}_1$ and $\tilde{P}_2$ are given by

$$\tilde{P}_1 = \sum_{i=1}^{\tilde{M}} \tilde{\phi}_i(x)\, F(x_i, y) \qquad (x_i \in \tilde{\Delta}) , \tag{74}$$

$$\tilde{P}_2 = \sum_{j=1}^{\tilde{M}} \tilde{\phi}_j(y)\, F(x, y_j) \qquad (y_j \in \tilde{\Delta}) , \tag{75}$$

and the errors $\tilde{R}_1$ and $\tilde{R}_2$ satisfy

$$||\tilde{R}_1[F]|| \le \tilde{\alpha}||F^{(4,0)}||\, (\tilde{h})^4 , \tag{76}$$

$$||\tilde{R}_2[F]|| \le \tilde{\alpha}||F^{(0,4)}||\, (\tilde{h})^4 , \tag{77}$$

$$\|\tilde{R}_1\tilde{R}_2[F]\| \leq \tilde{A}\|F^{(4,4)}\|(\tilde{h})^8 , \tag{78}$$

with $\tilde{\alpha}$ and $\tilde{A}$ independent of F .

The tensor product (bicubic spline) approximation $T(x, y)$ to F is given by

$$T(x, y) = \tilde{P}_1\tilde{P}_2[F] = \sum_{\mu=1}^{\tilde{M}} \sum_{\nu=1}^{\tilde{M}} \tilde{\phi}_\mu(x)\, \tilde{\phi}_\nu(y)\, F(x_\mu, y_\nu) . \tag{79}$$

(It is interesting to note that if we choose $\Delta_1 = \tilde{\Delta}$, $P_1^2 = P_1^1$ and $P_2^2 = P_2^1$ above then the expression (69) for S^* reduces to (79)) .

<u>3.3.5. Lemma</u>. The bicubic spline (79) agrees with F at the $\tilde{M}^2$ mesh points of $\tilde{\Delta} \times \tilde{\Delta}$ and approximates F with an error satisfying

$$\|F - T\| \leq \tilde{k}[(\tilde{h})^4 + (\tilde{h})^8] , \tag{80}$$

where

$$\tilde{k} = \max\{\tilde{\alpha}(\|F^{(4,0)}\| + \|F^{(0,4)}\|) ,\ \tilde{A}\|F^{(4,4)}\|\} .$$

<u>Proof.</u> From (73), we see that for $1 \leq j, k \leq \tilde{M}$

$$T(x_k, y_j) = \sum_{\mu=1}^{\tilde{M}} \sum_{\nu=1}^{\tilde{M}} \tilde{\phi}_\mu(x_k)\, \tilde{\phi}_\nu(y_j)\, F(x_\mu, y_\nu) = F(x_k, y_j) .$$

The error associated with $T(x, y) = \tilde{P}_1\tilde{P}_2[F]$ is

$$F - T = \tilde{R}_1[F] + \tilde{R}_2[F] - \tilde{R}_1\tilde{R}_2[F] .$$

Thus, (80) follows from the triangle inequality and (76) - (78) . ∎

For $\tilde{h} << 1$, $(\tilde{h})^4 + (\tilde{h})^8 \approx (\tilde{h})^4$. Thus, we may replace (80) by the (approximate) error bound

$$||F - T|| \leq \tilde{k}(\tilde{h})^4 \ . \tag{82}$$

We shall now examine the number of points required to approximate F by the functions T and S^* , respectively, to the accuracy of 2ε .

From (70), we see that we may achieve $||F - S^*|| \leq 2\varepsilon$ by choosing

$$h_1 = (\varepsilon/k)^{1/8} \ , \tag{83}$$

and

$$h_2 = (\varepsilon/k)^{1/4} \ . \tag{84}$$

From (83) and (84) we conclude that we should have

$$(h_2) = (h_1)^2 \ . \tag{85}$$

The partition Δ_2 should therefore have a much smaller mesh size than Δ_1 . It is not necessary that Δ_2 be a refinement of Δ_1 , however, and generally this will not be the case. Thus, the number of distinct points, n_D , at which we must know the value of F to construct S^* is given by

$$n_D = M_1M_2 + M_2M_1 + M_1^2 - 4M_1 \ . \tag{86}$$

Since (85) implies $M_1 << M_2$ we have the approximate point count

$$n_D \approx 2\ M_1M_2 \ . \tag{87}$$

If Δ_2 is actually a refinement of Δ_1, then $n_D = 2\ M_1M_2 - M_1^2$, and (87) remains valid for $M_1 << M_2$. Since $M_1 = (h_1)^{-1} + 1$ and $M_2 = (h_2)^{-1} + 1$, we see that (87) may be written as

$$n_D \approx 2(\frac{1 + h_1}{h_1})\ (\frac{1 + h_2}{h_2}) \approx 2(h_1h_2)^{-1} \ , \tag{88}$$

which, with the relations (83) and (84) yield

$$n_D \approx 2(h_1)^{-3} = 2(k/\varepsilon)^{3/8} \ . \tag{89}$$

To approximate F to the same 2ε accuracy using T, we see from (82) that we should choose the mesh size $\tilde{h}$ such that

$$\tilde{h} = (2\varepsilon/\tilde{k})^{1/4} \ . \tag{90}$$

The number of mesh-points required would then be

$$n_T = \tilde{M}^2 = (\frac{1 + \tilde{h}}{\tilde{h}})^2 \ . \tag{91}$$

For $\tilde{h}$ small (91) yields the approximation

$$n_T \approx (\tilde{h})^{-2} = \frac{1}{\sqrt{2}} (\tilde{k}/\varepsilon)^{1/2} . \tag{92}$$

One way to compute the relative efficiency of methods (69) and (79) is to consider the ratio

$$\rho = \frac{n_D}{n_T} = 2\sqrt{2} \left(\frac{k^{3/8}}{\tilde{k}^{1/2}}\right) \varepsilon^{1/8} . \tag{93}$$

In actual approximation we would not, in general, have any information regarding k and $\tilde{k}$. If, however, one may assume that k and $\tilde{k}$ are of approximately the same magnitude, then calculation of the quantity (93) reveals the superiority of method (69) over method (79). Tables listing the quantities n_D, n_T, and ρ for various values of ε, k, and $\tilde{k}$ are included in the appendix.

3.4. SUCCESSIVE DECOMPOSITION VS. TENSOR PRODUCTS IN CUBIC SPLINE INTERPOLATION

We shall briefly examine the application of the techniques of Section 3.3 to functions of three or more variables.

For the trivariate case we shall approximate a function $F(x, y, z) \in C^{4,4,4}(\prod_{i=1}^{3} [0, 1])$, where $C^{4,4,4}(\prod_{i=1}^{3} [0, 1])$ is the obvious three-dimensional analogue of $C^{4,4}([0, 1] \times [0, 1])$ defined in Section 3.3. We shall employ three levels of decomposition, and, as before, we shall let superscripts denote the level of decomposition.

Let Δ_1, Δ_2, and Δ_3 denote the M_1, M_2, and M_3 point uniform partitions of $[0, 1]$. For each partition Δ_k let $\{\phi_j^k\}_{j=1}^{M_k}$ denote the set of natural cubic splines which have joints at the points of Δ_k and satisfy

$$\phi_j^k(\xi_\ell^k) = \delta_{j\ell} \qquad (\xi_\ell^k \in \Delta_k) \ .$$

At each level j our projectors $\{P_i^j\}_{i=1}^3$ will be the three-dimensional analogues of the projectors (58) and (59) of Section 3.3. That is, P_i^j will interpolate the function F as a function of the i^{th} variable over the mesh Δ_j .

A simple, but tedious, calculation shows that the function S^* which corresponds to our method of successive decomposition approximates F with an error satisfying

$$||F - S^*|| \leq k[(h_3)^4 + (h_2)^8 + (h_1)^{12}] \tag{94}$$

where $h_j = 1/(M_j - 1)$ for $j = 1, 2, 3$.

We may approximate F to an accuracy of 3ε by setting

$$h_1 = (\varepsilon/k)^{1/12} \ , \tag{95}$$

and

$$h_3 = (h_2)^2 = (h_1)^3 \ . \tag{96}$$

To a good approximation, the number of distinct points required to achieve this accuracy is

$$n_D \approx \left(\frac{3}{h_1}\right)\left(\frac{2}{h_2}\right)\left(\frac{1}{h_3}\right) = 6(h_1)^{-11/2} = 6(K/\varepsilon)^{11/24} \tag{97}$$

To achieve the same 3ε accuracy with the tensor product approach on the mesh $\tilde{\Delta} \times \tilde{\Delta} \times \tilde{\Delta}$, where $\tilde{\Delta}$ has a uniform spacing of $\tilde{h}$ would require the number of points

$$n_T \approx \left(\frac{1}{3}\right)^{3/4} \left(\frac{\tilde{K}}{\varepsilon}\right)^{3/4} . \tag{98}$$

Thus, in this case the ratio $\rho = n_D/n_T$ is given by

$$\rho \approx 13.7\left(\frac{k^{11/24}}{\tilde{k}^{3/4}}\right) \varepsilon^{7/24} . \tag{99}$$

For the case of N variables a similar argument gives the approximate point counts required for an error of $N\varepsilon$

$$n_D \approx N!(k/\varepsilon)^{1/4+1/8+\ldots+1/4N} , \tag{100}$$

$$n_T \approx (\tilde{k}/N\varepsilon)^{N/4} \tag{101}$$

and the ratio $\rho = n_D/n_T$

$$\rho \approx c\ e^{1/4[N-1-1/2-1/3-\ldots-1/N]} , \tag{102}$$

where C depends on k, $\tilde{k}$, and N.

The estimates (100)-(102) are valid for small N, but because of the assumptions made in the derivation of (100), the estimates (100) and

(102) may not be asymptotically valid as $N \to \infty$. Also, for extremely large N , the method of successive decomposition may become quite cumbersome, and other techniques may be preferred in this case.

CHAPTER 4

CONCLUSIONS

The algebraic results developed in Chapter 1 provide one with a framework which may be used for the study of certain multivariate approximation methods. Since any set of commutative projectors $\{P_m\}_{m=1}^{M}$ generates a lattice Ψ, many multivariate techniques which use projectors may fit nicely into this framework. In particular, the multivariate analogues of three classical techniques, Taylor series expansion, Lagrange polynomial interpolation, and Fourier expansion in a Hilbert space, may be studied by our lattice theoretic approach. The superiority of the maximal projector over the minimal projector was seen in Chapter 2, and in Chapter 3 the method of successive decomposition, which is based on a maximal projector, was contrasted with the minimal method of tensor products. The utility of the lattice theoretic approach was evident in these examples, and one can see that this algebraic technique provides an interesting and useful tool for the study of certain multivariate approximation techniques.

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BIBLIOGRAPHY

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APPENDIX

APPENDIX

The following tables (A1)-(A3) give the approximate numbers of points required to approximate a bivariate function to an accuracy of 2ε by the method of successive decomposition (n_D) and the method of tensor products (n_T). The constants k and $\tilde{k}$ are defined in Lemmas 3.3.3 and 3.3.5. Table A1 shows that, under the assumption that k and $\tilde{k}$ are of the same magnitude, the method of successive decomposition is much more efficient (in terms of the number of points required) than is the method of tensor products.

TABLE A1

Number of points required for successive decomposition and tensor product methods with $k = \tilde{k}$ (Bivariate function).

ε	$k = \tilde{k}$	n_D	n_T	$\rho = n_D/n_T$
10^{-6}	10^{-2}	6.30×10^{1}	7.07×10^{1}	0.887
10^{-6}	10^{-1}	1.50×10^{2}	2.24×10^{2}	0.670
10^{-6}	10^{0}	3.56×10^{2}	7.07×10^{2}	0.504
10^{-6}	10^{1}	8.43×10^{2}	2.24×10^{3}	0.377
10^{-6}	10^{2}	2.00×10^{3}	7.07×10^{3}	0.283
10^{-8}	10^{-2}	3.56×10^{2}	7.07×10^{2}	0.504
10^{-8}	10^{-1}	8.43×10^{2}	2.24×10^{3}	0.377
10^{-8}	10^{0}	2.00×10^{3}	7.07×10^{4}	0.283
10^{-8}	10^{1}	4.74×10^{3}	2.24×10^{4}	0.212
10^{-8}	10^{2}	1.12×10^{4}	7.07×10^{4}	0.160
10^{-10}	10^{-2}	2.00×10^{3}	7.07×10^{3}	0.283
10^{-10}	10^{-1}	4.74×10^{3}	2.24×10^{4}	0.212
10^{-10}	10^{0}	1.12×10^{4}	7.07×10^{4}	0.160
10^{-10}	10^{1}	2.67×10^{4}	2.24×10^{5}	0.119
10^{-10}	10^{2}	6.32×10^{4}	7.07×10^{5}	0.089
10^{-12}	10^{-2}	1.12×10^{4}	7.07×10^{4}	0.160
10^{-12}	10^{-1}	2.67×10^{4}	2.23×10^{5}	0.119
10^{-12}	10^{0}	6.32×10^{4}	7.07×10^{5}	0.089
10^{-12}	10^{1}	1.50×10^{5}	2.24×10^{6}	0.067
10^{-12}	10^{2}	3.56×10^{5}	7.07×10^{6}	0.050

TABLE A2

Number of points required for successive decomposition and tensor product methods with $k > \hat{k} = 1$ (Bivariate function)

ε	k	n_D	n_T	$\rho = n_D/n_T$
10^{-6}	10^0	3.56×10^2	7.07×10^2	0.503
10^{-6}	10^1	8.44×10^2	7.07×10^2	1.193
10^{-6}	10^2	2.00×10^3	7.07×10^2	2.828
10^{-6}	10^3	4.74×10^3	7.07×10^2	6.707
10^{-6}	10^4	1.12×10^4	7.07×10^2	15.905
10^{-8}	10^0	2.00×10^3	7.07×10^3	0.283
10^{-8}	10^1	4.74×10^3	7.07×10^3	0.671
10^{-8}	10^2	1.12×10^4	7.07×10^3	1.591
10^{-8}	10^3	2.67×10^4	7.07×10^3	3.772
10^{-8}	10^4	6.32×10^4	7.07×10^3	8.944
10^{-10}	10^0	1.12×10^4	7.07×10^4	0.160
10^{-10}	10^1	2.67×10^4	7.07×10^4	0.377
10^{-10}	10^2	6.32×10^4	7.07×10^4	0.894
10^{-10}	10^3	1.50×10^5	7.07×10^4	2.121
10^{-10}	10^4	3.56×10^5	7.07×10^4	5.030
10^{-12}	10^0	6.33×10^4	7.07×10^5	0.089
10^{-12}	10^1	1.50×10^5	7.07×10^5	0.212
10^{-12}	10^2	3.56×10^5	7.07×10^5	0.503
10^{-12}	10^3	8.43×10^5	7.07×10^5	1.193
10^{-12}	10^4	2.00×10^6	7.07×10^5	2.828

TABLE A3

Number of points required for successive decomposition and tensor product methods with $k = 1 < \tilde{k}$ (Bivariate function).

ε	$\tilde{k}$	n_D	n_T	$\rho = n_D/n_T$
10^{-6}	10^{0}	3.56×10^{2}	7.07×10^{2}	0.503
10^{-6}	10^{1}	3.56×10^{2}	2.24×10^{3}	0.160
10^{-6}	10^{2}	3.56×10^{2}	7.07×10^{3}	0.050
10^{-6}	10^{3}	3.56×10^{2}	2.24×10^{4}	0.016
10^{-6}	10^{4}	3.56×10^{2}	7.07×10^{4}	0.005
10^{-8}	10^{0}	2.00×10^{3}	7.07×10^{3}	0.283
10^{-8}	10^{1}	2.00×10^{3}	2.24×10^{4}	0.089
10^{-8}	10^{2}	2.00×10^{3}	7.07×10^{4}	0.028
10^{-8}	10^{3}	2.00×10^{3}	2.24×10^{5}	0.009
10^{-8}	10^{4}	2.00×10^{3}	7.07×10^{5}	0.003
10^{-10}	10^{0}	1.13×10^{4}	7.07×10^{4}	0.160
10^{-10}	10^{1}	1.13×10^{4}	2.24×10^{5}	0.050
10^{-10}	10^{2}	1.13×10^{4}	7.07×10^{5}	0.016
10^{-10}	10^{3}	1.13×10^{4}	2.24×10^{6}	0.005
10^{-10}	10^{4}	1.13×10^{4}	7.07×10^{6}	0.002
10^{-12}	10^{0}	6.33×10^{4}	7.07×10^{5}	0.089
10^{-12}	10^{1}	6.33×10^{4}	2.24×10^{6}	0.028
10^{-12}	10^{2}	6.33×10^{4}	7.07×10^{6}	0.009
10^{-12}	10^{3}	6.33×10^{4}	2.24×10^{7}	0.003
10^{-12}	10^{4}	6.33×10^{4}	7.07×10^{7}	0.001

Table A4 gives approximate point counts required to approximate a trivariate function to an accuracy of 3ε, with $\varepsilon = 10^{-8}$. Calculations are shown for $k = \tilde{k}$, $k > \tilde{k}$, and $k < \tilde{k}$.

TABLE A4

Number of points required for successive decomposition and tensor product methods for various values of k and $\tilde{k}$, with $\varepsilon = 10^{-8}$.

k	$\tilde{k}$	n_D	n_T	$\rho = n_D/n_T$
10^{-2}	10^{-2}	3.37×10^3	1.39×10^4	0.243
10^{-1}	10^{-1}	9.69×10^3	7.80×10^4	0.124
10^0	10^0	2.78×10^4	4.39×10^5	0.063
10^1	10^1	8.00×10^4	2.47×10^6	0.032
10^2	10^2	2.30×10^5	1.39×10^7	0.017
1	10^0	2.78×10^4	4.39×10^5	0.063
1	10^1	2.78×10^4	2.47×10^6	0.011
1	10^2	2.78×10^4	1.39×10^7	0.002
1	10^3	2.78×10^4	7.80×10^7	0.0003
1	10^4	2.78×10^4	4.39×10^8	0.00006
1	1	2.78×10^4	4.39×10^5	0.063
10	1	8.00×10^4	4.39×10^5	0.018
10^2	1	2.30×10^5	4.39×10^5	0.052
10^3	1	6.60×10^5	4.39×10^5	1.505
10^4	1	1.90×10^6	4.39×10^5	4.328

VITA

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