To the Graduate Council:

I am submitting herewith a dissertation written by Stephen L. Clark entitled "Some Qualitative Properties of the Spectral Density Function for Hamiltonian Systems." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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SOME QUALITATIVE PROPERTIES OF THE SPECTRAL DENSITY FUNCTION FOR HAMILTONIAN SYSTEMS

A Dissertation
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Degree
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Stephen L. Clark

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DEDICATION

To my parents, Dorla J. and David S. Clark, and my friend, Nongluk Tunyavanich for their love and support, and to the memory of my grandparents, Evelyn F. and Milford J. Wagner.

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ABSTRACT

For Hamiltonian systems with one singular endpoint of limit point type, the problem is considered of determining when the continuous spectrum is absolutely continuous or continuously differentiable.

Following a method of Atkinson for $(-py')' + qy = \lambda wy$, a sequence of regular nonself-adjoint boundary value problems is examined and their Titchmarsh-Weyl $M(\lambda)$-functions derived. The Pick-Nevanlinna representation of each $M(\lambda)$-function produces a density function that is $C^1(\mathbb{R})$, and whose derivative is related to an energy-like functional for the Hamiltonian system. A sequence of densities $\{\rho_n(\lambda)\}$ associated with these regular nonself-adjoint operators is shown to converge to a density $\rho(\mu)$ associated with the $M(\lambda)$-function for the singular self-adjoint operator. The energy-like functionals are used to establish the absolute continuity and continuous differentiability of $\rho(\mu)$.

Examples are considered of a general second order system, and of a Dirac system, in which the potentials are of long range, short range, and oscillatory type. Conditions are stated in terms of these potentials such that the continuous spectrum is C^1. Examples are also considered of a coupled second order system and of a Dirac system in which the potentials include terms which are smooth and unbounded. Conditions are stated such that the continuous spectrum is C^1, and is all of $\mathbb{R}$.

In the last chapter, following a technique used by Hinton and Shaw in their consideration of the equation $y^{(4)} + q(x)y = \lambda y$, first order asymptotics are developed for the $M(\lambda)$-function associated

with a second order system with coupling terms present and with minimal restrictions on these terms.

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INTRODUCTION

Consider the following equation:

$$Ly = -(p(x)\ y')' + q(x)y = \lambda y \quad , \qquad 0 \leq x < \infty \tag{0.1}$$

where $p(x)$ is absolutely continuous, $q(x)$ is continuous, and $p(x) > 0$ on the half line. To construct an operator on the Hilbert space $L^2(0, \infty)$ using Ly , one is quite naturally led to consider square integrable solutions of (0.1) . The construction of such solutions involves the introduction of the Titchmarsh-Weyl $M(\lambda)$ function ([8, p. 226], [24]) .

Introduce separated boundary conditions that define a self-adjoint operator on a finite interval $0 \leq x \leq b$:

$$\sin\alpha\, y(0, \lambda) + \cos\alpha\, p(0)\, y'(0, \lambda) = 0 \tag{0.2}$$

$$\sin\beta\, y(b, \lambda) + \cos\beta\, p(b)\, y'(b, \lambda) = 0 \tag{0.3}$$

where $0 \leq \alpha, \beta < \pi$, and a fundamental pair of solutions $\phi(x, \lambda)$, $\theta(x, \lambda)$ such that

$$\begin{bmatrix} \theta(0, \lambda) & \phi(0, \lambda) \\ p(0)\ \theta'(0, \lambda) & p(0)\ \phi'(0, \lambda) \end{bmatrix} = \begin{bmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix} . \tag{0.4}$$

We define a function $M_b(\lambda) = M_b(\lambda, b, \alpha, \beta)$ where

$$M_b(\lambda) = - \frac{\sin\beta\ \theta(b, \lambda) + \cos\beta\ \ p(b)\ \theta'(b, \lambda)}{\sin\beta\ \phi(b, \lambda) + \cos\beta\ \ p(b)\ \phi'(b, \lambda)} \quad . \qquad (0.5)$$

This function $M_b(\lambda)$ is meromorphic as a function of λ with its poles corresponding to the eigenvalues of the regular boundary value problem on the finite interval $0 \leq x \leq b$ described by (0.1)-(0.3) .

For fixed α , $\operatorname{Im} M_b(\lambda) > 0$ when $\operatorname{Im} \lambda > 0$. For each λ , $M_b(\lambda)$ traces a circle in the half plane as β varies from 0 to π . A similar statement holds for λ with $\operatorname{Im} \lambda < 0$. As b tends to infinity, these circles nest one inside the next, and for this second order equation tend either to a circle or a point; thus leading (0.1) to the dichotomy of being either limit point or limit circle at infinity.

If (0.1) is limit point, then the function $M_b(\lambda)$ converges in each half plane to an analytic function. We shall confine our attention to the upper half plane, $\operatorname{Im} \lambda > 0$. For these λ the limiting function will be denoted by $M_\infty(\lambda)$.

Weyl [27] showed for $\alpha = \pi/2$ in (0.2) that the function

$$\chi(x, \lambda) = \theta(x, \lambda) + \phi(x, \lambda)\ M_\infty(\lambda) \qquad (0.6)$$

is in $L^2(0, \infty)$. In fact, when (0.1) is limit point, there is exactly 1 independent $L^2(0, \infty)$ solution.

Consider now the singular boundary value problem on the half line $0 \leq x < \infty$ defined by (0.1)-(0.2) . Let D be the set of functions $f(x)$ such that $f(x) \in L^2(0, \infty)$; $f(x)$ satisfies (0.2); and $L(f) \in L^2(0, \infty)$. When (0.1) is limit point, then the operator defined by $L(f)$ with domain D is a self adjoint operator on the Hilbert space $L^2(0, \infty)$.

Now, for the limit point case, both $M_\infty(\lambda)$ and $M_b(\lambda)$ are functions of Pick-Nevanlinna type. A function $M(\lambda)$ is of Pick-Nevanlinna type if it is analytic for λ in the upper half plane, and if in this half plane, $\operatorname{Im} M(\lambda) \geq 0$. Such functions $M(\lambda)$ have a representation in terms of a real valued nondecreasing function $\rho(\lambda)$ defined on $\mathbb{R}$ where

$$M(\lambda) = c + id + \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu^2}{1 + \mu^2}\right) d\rho(\mu) \quad , \tag{0.7}$$

where c and d are real constants, and $d \geq 0$ [1] . From this it follows that

$$\operatorname{Im} M(\lambda) = d + \operatorname{Im} \lambda \int_{-\infty}^{\infty} \frac{1}{|\mu - \lambda|^2} d\rho(\mu) \tag{0.8}$$

and inversely that

$$\rho(d) - \rho(c) = \pi^{-1} \lim_{\varepsilon \to 0} \int_{c+i\varepsilon}^{d+i\varepsilon} \operatorname{Im} M(\lambda)\, d\lambda \tag{0.9}$$

at the continuity points c and d of $\rho(\lambda)$ [8] .

The spectrum of the self adjoint operator defined in the limit point case is the support of the measure $d\rho(\mu)$ [8] . As mentioned above, $\rho(\mu)$ is nondecreasing. It can also be shown that the eigenvalues (if any) of the self adjoint operator correspond to jump discontinuities of $\rho(\mu)$.

In what follows we shall be concerned with the following Hamilton system:

$$Jy' = [\lambda\ A(x) + B(x)]y \quad , \quad a \le x < \infty \tag{0.10}$$

where J is skew hermitian and nonsingular; $A(x)$ and $B(x)$ are $2n \times 2n$ complex hermitian matrices; λ is complex parameter and $y(x)$ is a $2n \times 1$ vector function. In addition $A(x)$, $B(x)$ are locally integrable on $[a, \infty)$; $A(x) \ge 0$ (nonnegative definite) and for any nontrivial solution of (0.10) it is assumed that

$$\int_a^b y^*(x)\ A(x)\ y(x)\ dx > 0$$

for all $b > 0$, where y^* is the complex conjugate of the vector y . The equation (0.1) is a special case of this system. A general discussion of Hamiltonian systems can be found in [2, p. 252] . In [13], [14], and [15], Hinton and Shaw have extended the notion of a Titchmarsh-Weyl $M(\lambda)$- function to general Hamiltonian systems.

In [14] they consider the singular boundary value problem on the half line $[a, \infty)$ formed from (0.10) and the left hand boundary condition

$$y(a) = \begin{bmatrix} 0 & \alpha_2^* \\ & \\ 0 & -\alpha_1^* \end{bmatrix} v \ , \quad \text{or} \quad [\alpha_1, \alpha_2]\ y(a) = 0 \ , \tag{0.11}$$

where v is some $2n \times 1$ vector and where α_1, α_2 are $n \times n$ matrices such that

$$\operatorname{rank}[\alpha_1, \alpha_2] = n$$

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n \tag{0.12}$$

$$\alpha_1\alpha_2^* = \alpha_2\alpha_1^*$$

In the future this singular boundary value problem (0.10)-(0.12) will be denoted by (S) .

As in the scalar second order case, they are interested in the existence of square integrable solutions to (0.10) with a view toward the construction of self adjoint operators. By square integrable solution, we mean a solution $y(x, \lambda)$ of (0.10) which satisfies

$$\int_a^\infty y^*(x, \lambda)\, A(x)\, y(x, \lambda)\, dx < \infty . \tag{0.13}$$

From a fundamental solution of (0.10)

$$Y(x, \lambda) = \begin{bmatrix} \theta(x, \lambda) & \phi(x, \lambda) \\ \hat{\theta}(x, \lambda) & \hat{\phi}(x, \lambda) \end{bmatrix} , \tag{0.14}$$

where $\theta, \hat{\theta}, \phi, \hat{\phi}$ are $n \times n$ blocks and where the columns of $\begin{bmatrix} \phi \\ \hat{\phi} \end{bmatrix}$ satisfy the boundary condition (0.11) , they define an $n \times n$ matrix valued function $M_b(\lambda)$ in a manner analogous to (0.5) . The important thing is that they are able to develop an analog, in this higher dimension case, of the analytic and geometric aspects of $M_b(\lambda)$

and $M_\infty(\lambda)$ for the scalar second order equation (0.1) . These matters are discussed in more detail in Chapter 1.

In particular, they are able to identify a limit point case [14] for (0.10), as $b \to \infty$, which corresponds to the existence of exactly n independent square integrable solutions of (0.10) . In this case one is able to show that, as $b \to \infty$, $M_b(\lambda)$ converges elementwise, for $\operatorname{Im} \lambda \neq 0$, to a limiting matrix $M_\infty(\lambda)$. The entries of both $M_b(\lambda)$ and $M_\infty(\lambda)$ for $\operatorname{Im} \lambda \neq 0$ are analytic. $M_b(\lambda)$ possesses singularities on $\mathbb{R}$ corresponding to the eigenvalues of a regular boundary value problem on a finite interval $[a, b]$.

They are able to show, in analogy to the second order case, that when (0.10) is limit point then the matrix of solutions

$$\chi(x, \lambda) = \theta(x, \lambda) + \phi(x, \lambda)\, M(\lambda) \tag{0.15}$$

provides for n independent square integrable solutions of (0.10) ([13], [14]) .

They observe for $\operatorname{Im} \lambda > 0$ that $\operatorname{Im} M(\lambda) = \frac{1}{2i}\,[M(\lambda) - M^*(\lambda)] \geq 0$. This also follows as a result of Lemma 1.3 of the first chapter. In the appendix we observe that there is an analog, for analytic matrices, of the Pick-Nevanlinna representation of functions which are analytic in the upper half plane and whose imaginary parts in this half plane are nonnegative. The result is that there is a nondecreasing matrix valued function $\rho(\mu)$, defined on all of $\mathbb{R}$, that in the analog of $\rho(\mu)$ for the scalar second order case in (0.7)-(0.9).

Information about $\rho(\lambda)$ gives information about $M(\lambda)$ by means of matrix analogs of (0.8)-(0.9) . Information about the limiting nature of $M_\infty(\lambda)$ as λ tends to $\mathbb{R}$ through the upper half plane can thus be derived.

In [16] Hinton and Shaw consider (0.10) where $A(x)$ is invertible or of the form $\begin{bmatrix} A_1(x) & 0 \\ 0 & 0 \end{bmatrix}$ where $A_1(x)$ is invertible. For simplicity, they take $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. This still encompasses many cases of interest. The equation (0.1) is still a special case as is

$$Ly = -(p(x)y') + q(x)y = \lambda\, w(x)y \qquad 0 \leq x < \infty \tag{0.16}$$

where $w(x)$ is locally integrable and $w(x) > 0$.

When this system satisfies the limit point condition, they show that it is possible to construct a self adjoint operator T on the Hilbert space $L_A^2(a, \infty)$ from the singular boundary value problem (S) . By $L_A^2(a, \infty)$ we mean the collection of $2n \times 1$ vector functions $f(x)$ which satisfy (0.13) . More than this, they extend the results of Chaudhuri and Everitt [7] and show that there exists a connection between the spectrum of this self adjoint operator and the behavior of $M_\infty(\lambda)$ near $\mathbb{R}$.

They define the point spectrum, $P(T)$, to be the set of isolated eigenvalues of T . Where $\sigma(T)$ is the spectrum, they define the essential spectrum $E(T)$ to be $E(T) = \sigma(T) - P(T)$; the point continuous spectrum $PC(T)$ to be the set of eigenvalues embedded in $E(T)$; and the continuous spectrum $C(T)$ to be $C(T) = E(T) - PC(T)$. Among the things they show is that eigenvalues correspond to simple poles of $M_\infty(\lambda)$. This, by means of the analogs of (0.7) and (0.9), produces a corresponding jump discontinuity in $\rho(\lambda)$. Similarly, they show that $\lambda_0 \in C(T)$ if and only if $M(\lambda)$ is not analytic at λ_0 and $\lim_{\nu\to 0} \nu M_\infty(\lambda + i\nu) = 0$. Such a λ_0 can be shown to be a point of increase for the nondecreasing function $\rho(\mu)$. Thus for these operators, information about the nature of $M_\infty(\lambda)$ for λ near $\mathbb{R}$, or of $\rho(\lambda)$ for $\lambda \in \mathbb{R}$, can be translated into information about the spectrum.

Two problems of interest which we address are the following. First, locate the essential spectrum $E(T)$, and second, find conditions under which $\rho(\lambda)|_{E(T)}$ is absolutely continuous. This latter condition is important for the construction of certain operators which arise in scattering theory.

Much of what has been done in terms of finding information about the spectrum has involved asymptotic procedures. Frequently, the asymptotics of the solutions of an equation are derived and, through them, asymptotic information about $M(\lambda)$. Hinton and Shaw have used this technique in [17] and [18] to derive conditions sufficient to guarantee that the spectrum for certain self adjoint operators is absolutely continuous.

These techniques are certainly nontrivial. The difficulty in application comes in the necessity for asymptotic information about solutions. For general systems like (0.10) this may be difficult and in many cases perhaps impossible to come by. Even for perturbations of constant coefficient equations there is no general theory to cover the case of multiple eigenvalues of the constant coefficient equations.

Recently, Atkinson [3] has shown for the limit point case of (0.16) that, for λ restricted to an interval $(\Lambda_1, \Lambda_2) \subset \mathbb{R}$, certain limited asymptotic behavior of $\phi(x, \lambda)$ is sufficient to guarantee that $\rho(\lambda)$ is continuously differentiable with positive derivative over (Λ_1, Λ_2). Specifically he shows that

Theorem: Assume for (0.16) that over (a, b)

1. $p^{-1}(x)$, $q(x)$, $w(x)$ are real and locally integrable over (a, b) ;
2. $w(x) \geq 0$, $w > 0$ in some right hand neighborhood of a (or for sufficiently large $x < 0$ if $a = -\infty$) ;
3. (0.16) is limit circle at $x = a$;
4. (0.16) is limit point at $x = b$;
5. u, v are real solutions of (0.16) for $\lambda = 0$, such that $p(u'v - uv') = 1$;
6. $y(x, \lambda)$ is a solution of (0.16) such that $p(yv' - y'v) \to 0$, $p(yu' - y'u) \to 1$ as $x \to a$;
7. For some λ interval (Λ_1, Λ_2) and x-interval (x_0, b) there are positive functions $K(x, \lambda)$ and $\sigma(\lambda)$ such that as $x \to b$,

$$K(x, \lambda)\, y^2(x, \lambda) + K^{-1}(x, \lambda)\, p^2(x)\, y'^2(x, \lambda) \to \sigma(\lambda) \qquad (0.17)$$

8. $\sigma(\lambda) \in C(\Lambda_1, \Lambda_2)$ and on every compact subinterval of (Λ_1, Λ_2), $\ln K(x, \lambda)$ is continuous in λ , uniformly so as $x \to b$,

then $\rho(\lambda)$ is continuously differentiable in (Λ_1, Λ_2) , and in this interval $\rho'(\lambda) = \{\pi\, \sigma(\lambda)\}^{-1}$.

If $a > -\infty$, and a is a regular point for (0.16) , then the requirements (5) and (6) may be replaced by boundary conditions of the usual kind for y and py' at $x = a$.

Atkinson points out that by being able to hypothesize limit pointness rather than having to specify conditions guaranteeing limit pointness of (0.16), one gains some freedom of choice among the various asymptotic techniques available for second order equations.

By requiring asymptotic information for an energy-like functional (0.17), rather than for the solutions $y(x, \lambda)$ themselves, one seems to gain some additional freedom. This would be helpful in an extension of this procedure to systems of the form (0.10).

It should be noted that this result is stated for (0.16) with 2 singular endpoints a and b . In what follows we shall be interested in that case where $x = a$ is regular and $b = \infty$.

In what follows, we shall consider Hamiltonian systems of the form (0.10) defined on the half line $[a, \infty)$, where $x = a$ is assumed to be regular and J is taken to be $\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. We will be interested in those systems which satisfy a limit point condition at infinity.

In the Appendix, as mentioned earlier, an analog of the Pick-Nevanlinna representation (0.7) is presented for matrix valued functions which are analytic in the upper half complex plane and whose imaginary parts are nonnegative definite in this half plane. This affords a representation for the $M(\lambda)$ function of the system under consideration. And in particular, there is an associated nondecreasing matrix valued function $\rho(\lambda)$. It is this function, $\rho(\lambda)$, that is our immediate interest in Chapters 1, 2, and 3 .

In Chapter 1, for the Hamiltonian system just described, two results are developed for this half line problem that are similar to the just stated result of Atkinson. In Theorem 1.1, sufficient conditions are given on a functional analogous to (0.17) which guarantee that $\rho(\lambda)$ is absolutely continuous in an interval (Λ_1, Λ_2), in the sense that all entries of $\rho(\lambda)$ are absolutely continuous on (Λ_1, Λ_2). In Theorem 1.2 an analog of Atkinson's result is stated in which conditions are given on a functional like (0.17) sufficient to guarantee that $\rho'(\lambda) = \pi^{-1}\,\alpha(\lambda)$ a.e. on (Λ_1, Λ_2) where $\alpha(\lambda)$ is a positive matrix valued function given at the outset.

It should be noted that the system under consideratation requires only that $A(x)$ be nonnegative definite and not of the specific form that Hinton and Shaw considered in [16] .

The following two chapters consider applications of the results of Chapter 1 to examples of (0.10) . In particular we see that a technique used by Atkinson and more elementary than the asymptotic technique of Hinton and Shaw [17], [18] is useful in deriving conditions which guarantee the existence of absolutely continuous spectrum.

In the last chapter we derive first order asymptotics for the $M(\lambda)$ function for a specific example of a Hamiltonian system (0.10) with coupling terms present and minimal restrictions on these terms. The technique used here has also been used by Hinton and Shaw [17] in their treatment of the scalar fourth order equation $y^{(4)} + q(x)y = \lambda y$ for $0 \leq x < \infty$.

Aside from the intrinsic interest in generating asymptotics for $M(\lambda)$ functions, Atkinson [4] has shown for the scalar second order operator that better asymptotic estimates of the $M(\lambda)$-function can lead to approximations of the spectral density. Viewed in this context, the fourth chapter is perhaps a first step down a different road whose goal is a description of the spectral density for systems.

In the Appendix, A^*, where A is a matrix, is defined. Also defined are the following terms for matrix valued functions: nonnegative definite; nondecreasing; absolutely continuous; bounded variation; analytic; real and imaginary parts. In addition, presented in brief is the analog of the Pick-Nevanlinna representation theorem for analytic functions [1, p. 7] .

CHAPTER 1

ABSOLUTE CONTINUITY AND CONTINUOUS DIFFERENTIABILITY OF $\rho(\lambda)$ FOR A HAMILTONIAN SYSTEM

Consider the system of ordinary differential equations

$$Jy'(x) = [\lambda A(x) + B(x)]y(x) \ , \qquad a \le x < \infty \ . \tag{1.1}$$

Here $y(x)$ is a $2n \times 1$ vector, λ is a complex scalar, J is the matrix $\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity, and $A(x)$ and $B(x)$ are $2n \times 2n$ hermitian matrices with entries locally integrable on the given interval. It is further assumed that $A(x)$ is non-negative definite, denoted $A(x) \ge 0$, and that for any nontrivial solution vector $y(x)$ of (1.1)

$$\int_a^b y^*(x)\ A(x)\ y(x)\ dx > 0 \quad , \tag{1.2}$$

for all $b > a$, where y^* is the conjugate transpose of y .

First, we consider a regular boundary value problem examined by Atkinson in [2] . On the finite interval $[a, b]$, the equation in (1.1) is taken together with the separated boundary conditions

$$y(a) = Fv \ , \ y(b) = Gv \ . \tag{1.3}$$

Here v is some $2n \times 1$ vector while F and G are square matrices. Further, it is assumed that

$$F^* JF = G^* JG \tag{1.4}$$

and that

$$Fv = Gv = 0 \qquad \text{implies} \quad v = 0 . \tag{1.5}$$

The symmetry condition (1.4) implies that the eigenvalues are real.

In [13], [14] and [15] Hinton and Shaw, for a regular boundary value problem of type (1.1)-(1.5), develop the notion of a Titchmarsh-Weyl $M(\lambda)$ - function. It is an analog of the scalar-valued $M(\lambda)$-function whose analytic properties as a function of λ were first examined by Titchmarsh [22]. Geometric properties of $M(\lambda)$, a function dependent upon the boundary conditions and endpoints, were first noted by Weyl [27]. A discussion of these matters for the scalar second order case can be found in Chapter 9 of Coddington and Levinson [8] .

In [14] boundary conditions of type (1.3) are considered where specifically,

$$F = \begin{bmatrix} 0 & \alpha_2^* \\ 0 & -\alpha_1^* \end{bmatrix} \quad , \quad G = \begin{bmatrix} \beta_2^* & 0 \\ -\beta_1^* & 0 \end{bmatrix} \tag{1.6}$$

given the $n \times n$ matrices α_i and β_i such that $\text{rank}[\alpha_1, \alpha_2] = \text{rank}[\beta_1, \beta_2] = n$, and where it is also assumed that

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n \ ; \quad \alpha_1\alpha_2^* = \alpha_2\alpha_1^* \ ; \tag{1.7}$$

$$\beta_1\beta_1^* + \beta_2\beta_2^* = I_n \ ; \quad \beta_1\beta_2^* = \beta_2\beta_1^* \ . \tag{1.8}$$

Note: In the future the regular self-adjoint boundary value problem defined by (1.1)-(1.8) will be referred to as (R) .

$Y(x, \lambda)$ is a $2n \times 2n$ fundamental matrix of solutions of (1.1) . This matrix is taken to be partitioned into $n \times n$ blocks

$$Y(x, \lambda) = \begin{bmatrix} \theta(x, \lambda) & \phi(x, \lambda) \\ \hat{\theta}(x, \lambda) & \hat{\phi}(x, \lambda) \end{bmatrix} \tag{1.9}$$

where

$$Y(a, \lambda) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix} . \tag{1.10}$$

For the boundary value problem (R) , Hinton and Shaw define the function

$$\begin{aligned} M(\lambda, b) &= M(\lambda, b, \beta_1, \beta_2, \alpha_1, \alpha_2) \\ &= -[\beta_1\phi(b, \lambda) + \beta_2\, \hat{\phi}(b, \lambda)]^{-1}[\beta_1\theta(b, \lambda) + \beta_2\hat{\theta}(b, \lambda)] \end{aligned} \tag{1.11}$$

where $\beta_1\phi(b, \lambda) + \beta_2\hat{\phi}(b, \lambda)$ is singular precisely for the eigenvalues of (R) . It is this $n \times n$ matrix-valued function $M(\lambda, b)$ which is called the Titchmarsh-Weyl $M(\lambda)$, or just M-function.

Note that when the M-function does exist, it is a matrix whose entries are meromorphic functions of λ since $\theta, \hat{\theta}, \phi, \hat{\phi}$ are all $n \times n$ matrices whose entries are analytic functions of λ . One can show that the singularities of the entries of $M(\lambda, b)$ occur precisely at eigenvalues of (R) [14] .

Definition: We define the real and imaginary parts of a matrix A as follows:

$$\operatorname{Re} A = \frac{1}{2}[A + A^*] \; ; \; \operatorname{Im} A = \frac{1}{2i}[A - A^*] \; .$$

We note two things about this definition. First, in general, these matrices do not consist of the real and imaginary parts of the entries of A . Second, the entries are not, in general, real valued.

We shall now consider a regular boundary value problem related to (R) .

Note: In the future, (B) will refer to the boundary value problem on $[a, b]$ defined by the equation in (1.1) and conditions (1.2), (1.3), (1.5)-(1.7), where $\beta_1 = I_n$, $\beta_2 = \beta^*$, and β is an $n \times n$ matrix.

For boundary value problem (B) , $G = \begin{bmatrix} \beta & 0 \\ -I_n & 0 \end{bmatrix}$ and because of this,

$$G^* JG = \begin{bmatrix} -\beta_1\beta_2^* + \beta_2\beta_1^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\beta + \beta^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2i \text{ Im } \beta & 0 \\ 0 & 0 \end{bmatrix} .$$

Now, as α_1 and α_2 satisfy (1.7) we note also that $F^* JF = 0$. As a consequence, if $\text{Im } \beta \neq 0$ then (1.4) will no longer hold since $0 = F^* JF \neq G^* JG$. Furthermore, (1.8) will not hold, since

$$\beta_1\beta_1^* + \beta_2\beta_2^* = I_n + \beta^*\beta \neq I_n ,$$

and

$$\beta_1\beta_2^* - \beta_2\beta_1^* = 2i \text{ Im } \beta \neq 0 .$$

For boundary value problems of this type, we would like to be able to define and examine a function analogous to the M-function defined for (R) . We would like to let

$$M(\lambda, b, \beta) = -[\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)]^{-1} [\theta(b, \lambda) + \beta\hat{\theta}(b, \lambda)] .$$

However, we need to know the nature of the singularities of the matrix $\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)$. For (R) , the demonstration that the singularities of $\beta_1\phi(b, \lambda) + \beta_2\hat{\phi}(b, \lambda)$ coincide with the eigenvalues of (R) relies upon that part of (1.8) which states that $\beta_1\beta_2^* - \beta_2\beta_1^* = 0$. As we have seen, this will not be the case when $\operatorname{Im} \beta \neq 0$; thus we show

<u>Lemma 1.1</u>: The $n \times n$ matrix $\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)$ is singular if and only if λ is an eigenvalue of (B) .

<u>Proof</u>: Let λ be an eigenvalue of (B) and $y(x, \lambda) = Y(x, \lambda) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ a corresponding eigenfunction. To be an eigenfunction means that there is a vector $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ such that,

$$y(a,\lambda) = Y(a, \lambda) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1^* v_1 - \alpha_2^* v_2 \\ \alpha_2^* v_1 + \alpha_1^* v_2 \end{bmatrix} ,$$

and by (1.3) that

$$y(a, \lambda) = Fw = \begin{bmatrix} \alpha_2^* w_2 \\ -\alpha_1^* w_2 \end{bmatrix} .$$

Thus

$$(\alpha_1\alpha_1^* + \alpha_2\alpha_2^*)v_1 + (\alpha_2\alpha_1^* - \alpha_1\alpha_2^*)v_2 = (\alpha_1\alpha_2^* - \alpha_2\alpha_1^*)w_2 ,$$

and so by (1.7)

$$I_n\, v_1 + 0\, v_2 = 0\, w_2 .$$

Consequently,

$$y(x, \lambda) = Y(x, \lambda)\begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \begin{bmatrix} \phi(x, \lambda) \\ \hat{\phi}(x, \lambda) \end{bmatrix} v_2 .$$

By (1.3) it now follows that

$$y(b, \lambda) = \begin{bmatrix} \phi(b, \lambda) \\ \hat{\phi}(b, \lambda) \end{bmatrix} v_2$$

$$= G\,w = \begin{bmatrix} \beta\, w_1 \\ -w_1 \end{bmatrix}$$

and that

$$[\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)]v_2 = \beta w_1 - \beta w_1 = 0 .$$

Again, $v_2 \neq 0$ since $y(x, \lambda)$ is nontrivial; thus we see that $\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)$ is singular.

Suppose that $\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)$ is singular; that is, there is an $n \times 1$ vector $w \neq 0$ such that $[\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)]w = 0$

and hence that $\phi(b, \lambda)w = -\beta\hat{\phi}(b, \lambda)w$. It follows that

$y(x, \lambda) = \begin{bmatrix} \phi(x, \lambda) \\ \hat{\phi}(x, \lambda) \end{bmatrix} w$ will be an eigenfunction of (B) because,

$$y(a, \lambda) = \begin{bmatrix} \phi(a, \lambda) \\ \hat{\phi}(a, \lambda) \end{bmatrix} w = \begin{bmatrix} -\alpha_2^* \\ \alpha_1^* \end{bmatrix} w$$

$$= \begin{bmatrix} 0 & \alpha_2^* \\ 0 & -\alpha_1^* \end{bmatrix} \begin{bmatrix} \cdot \\ -w \end{bmatrix}$$

while

$$y(b, \lambda) = \begin{bmatrix} \phi(b, \lambda) \\ \hat{\phi}(b, \lambda) \end{bmatrix} w = \begin{bmatrix} -\beta\hat{\phi}(b, \lambda) \\ \hat{\phi}(b, \lambda) \end{bmatrix} w$$

$$= \begin{bmatrix} \beta & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} -\hat{\phi}(b, \lambda)\, w \\ \cdot \end{bmatrix} .$$

Thus, $y(x, \lambda) = \begin{bmatrix} \phi(x, \lambda) \\ \hat{\phi}(x, \lambda) \end{bmatrix} w$ satisfies the boundary conditions $y(a, \lambda) = Fv$ and $y(b, \lambda) = Gv$ when

$$v = \begin{bmatrix} -\hat{\phi}(b, \lambda)w \\ -w \end{bmatrix} .$$

■

With the exception of the eigenvalues of (B) the matrix $\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)$ is nonsingular; hence with the exception of these values we may define

$$M(\lambda, b) = M(\lambda, b, \beta) = M(\lambda, b, \beta, \alpha_1, \alpha_2)$$

$$= -[\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)]^{-1} [\theta(b, \lambda) + \beta\hat{\theta}(b, \lambda)] . \quad (1.12)$$

As before, $M(\lambda, b)$ is an $n \times n$ matrix-valued function whose entries are meromorphic functions in terms of λ . The set of singularities of these entries is contained in the set of eigenvalues of (B) .

In what follows, we shall want to consider the boundary value problem (B) and the associated M-function where $\text{Im}\, \beta \geq 0$.

<u>Lemma 1.2</u>: If $\text{Im}\, \beta \geq 0$, and λ is an eigenvalue for the boundary value problem (B) , then $\text{Im}\, \lambda \leq 0$.

<u>Proof</u>: One can show that if $y(x)$ and $z(x)$ are solutions of (1.1) then

$$[y^*(x)\, Jz(x)]' = (\lambda - \bar{\lambda})\, y^*(x)\, A(x)\, z(x) . \quad (1.13)$$

Suppose that $y(x)$ is an eigenfunction of (B) . Then by (1.13),

$$y^*(b)\, Jy(b) - y^*(a)\, Jy(a)$$

$$= 2i\ \text{Im}\, \lambda \int_a^b y^*(x)\, A(x)\, y(x)\, dx ; \quad (1.14)$$

By (1.3) there is a $2n \times 1$ vector v such that

$$v^* G^*\, JGv - v^*\, F^*\, JFv = 2i\ \text{Im}\, \lambda \int_a^b y^*(x)\, A(x) y(x)\, dx .$$

It has been noted that $F^* JF = 0$ and that $G^* JG = \begin{bmatrix} -2i \, \mathrm{Im}\beta & 0 \\ 0 & 0 \end{bmatrix}$ so that if $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ where v_i is an $n \times 1$ vector; then,

$$[v_1^*, v_2^*] \begin{bmatrix} -2i \, \mathrm{Im}\beta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2i \, \mathrm{Im}\, \lambda \int_a^b y^*(x) \, A(x) y(x) \, dx \, .$$

Consequently,

$$-v_1^* \, \mathrm{Im}\, \beta \, v_1 = \mathrm{Im}\, \lambda \int_a^b y^*(x) \, A(x) \, y(x) \, dx \, .$$

Neither v_1 nor v_2 can be 0 ; otherwise, either $y(a)$ or $y(b) = 0$ with the result that $y(x) \equiv 0$. Since (1.2) holds for all nontrivial solutions of (1.1) and because $\mathrm{Im}\, \beta \geq 0$, it follows that $\mathrm{Im}\, \lambda \leq 0$. Note also that if $\mathrm{Im}\, \beta > 0$ then it follows that $\mathrm{Im}\, \lambda < 0$. ■

We now examine the imaginary part of those matrix-valued functions $M(\lambda, b, \beta)$ for (B) when $\mathrm{Im}\, \beta \geq 0$.

Lemma 1.3: For the boundary value problem (B) in which $\mathrm{Im}\, \beta \geq 0$, the imaginary part of $M(\lambda, b, \beta)$ is nonnegative definite for those values of λ where $\mathrm{Im}\, \lambda \geq 0$ and λ is not an eigenvalue.

Proof: If $X(x, \lambda)$ is a $2n \times k$ matrix of solutions of (1.1), then (1.14) can be extended to read

$$X^*(b, \lambda)\, JX(b, \lambda) - X^*(a, \lambda)\, JX(a, \lambda)$$

$$= 2i \operatorname{Im} \lambda \int_a^b X^*(x, \lambda)\, A(x)\, X(x, \lambda)\, dx\,, \tag{1.15}$$

where the last integral is the matrix composed of the integrals of the entries of $X^*(x, \lambda)\, A(x)\, X(x, \lambda)$. By (1.2) the matrix $\int_a^b X^* A X$ is positive definite for each $X(x, \lambda)$ with nontrivial columns.

Now let M be an arbitrary $n \times n$ matrix and let

$$X(x, \lambda) = Y(x, \lambda) \begin{bmatrix} I \\ M \end{bmatrix} = \begin{bmatrix} \theta(x, \lambda) + \phi(x, \lambda)\, M \\ \hat{\theta}(x, \lambda) + \hat{\phi}(x, \lambda)\, M \end{bmatrix}.$$

For each $X(x, \lambda)$,

$$X^*(a, \lambda)\, JX(a, \lambda) = \begin{bmatrix} I \\ M \end{bmatrix}^* Y^*(a, \lambda)\, J\, Y(a, \lambda) \begin{bmatrix} I \\ M \end{bmatrix}$$

$$= [I, M^*] J \begin{bmatrix} I \\ M \end{bmatrix} = M^* - M$$

$$= -2i \operatorname{Im} M\,.$$

If $M = M(\lambda, b, \beta) = -[\phi(b, \lambda) + \beta\hat{\phi}(b, \lambda)]^{-1}\, [\theta(b, \lambda) + \beta\hat{\theta}(b, \lambda)]$ then

$$\theta(b, \lambda) + \phi(b, \lambda)\, M = -\beta[\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda) M]\,,$$

and we find that

$$X^*(b, \lambda)\ J\ X(b, \lambda) =$$

$$= \begin{bmatrix} \theta(b, \lambda) + \phi(b, \lambda)\ M \\ \hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M \end{bmatrix}^* J \begin{bmatrix} \theta(b, \lambda) + \phi(b, \lambda)\ M \\ \hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M \\ \beta(\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M) \end{bmatrix}^* \begin{bmatrix} -\beta(\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M) \\ \hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda) \end{bmatrix}$$

$$= [\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M]^*\ (\beta^* - \beta)\ [\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M]$$

$$= -2i[\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M]^*\ (\mathrm{Im}\ \beta)\ [\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda) M]\ .$$

Thus when $M = M(\lambda, b, \beta)$, (1.15) can be written as

$$\mathrm{Im}\ M = [\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda) M]^*\ (\mathrm{Im}\ \beta)\ [\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)\ M] + \mathrm{Im}\ \lambda \int_a^b X^*(x, \lambda)\ A(x)\ X(x, \lambda)\ dx \qquad (1.16)$$

and the result follows. ■

From (1.16), if $\mathrm{Im}\ \lambda \geq 0$, then $\mathrm{Im}\ M(\lambda, b, \beta) > 0$, that is positive definite, when either $\mathrm{Im}\ \lambda > 0$, or $\mathrm{Im}\ \beta > 0$.

<u>Lemma 1.4</u>: If $\lambda \in \mathbb{R}$ and $\mathrm{Im}\ \beta > 0$ then $\mathrm{Im}\ M(x, b, \beta)$ is

$$\{\phi^*(b, \lambda)(\mathrm{Im}\ \beta)^{-1}\ \phi(b, \lambda) + \hat{\phi}^*(b, \lambda)\ \beta^*(\mathrm{Im}\ \beta)^{-1}\ \beta\hat{\phi}(b, \lambda) - 4\ \mathrm{Im}\ [\phi^*(b, \lambda)\ (\beta - \beta^*)^{-1}\ \beta\hat{\phi}(b, \lambda)]\}^{-1}\ . \qquad (1.17)$$

Proof: When $\lambda \in \mathbb{R}$ (1.15) becomes

$$X^*(x, \lambda)\ J\ X(x, \lambda) - X^*(a, \lambda)\ J\ X(a, \lambda) = 0$$

and if $X(x, \lambda) = Y(x, \lambda)$ then

$$Y^*(x, \lambda)\ J\ Y(x, \lambda) - J = 0$$

or

$$\begin{bmatrix} \hat{\theta}^*\theta - \theta^*\hat{\theta} & \hat{\theta}^*\phi - \theta^*\hat{\phi} \\ \hat{\phi}^*\theta - \phi^*\hat{\theta} & \hat{\phi}^*\phi - \phi^*\hat{\phi} \end{bmatrix} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

with the resulting identities:

$$\hat{\theta}^*(x, \lambda)\ \theta(x, \lambda) - \theta^*(x, \lambda)\ \hat{\theta}(x, \lambda) =$$
$$\hat{\phi}^*(x, \lambda)\ \phi(x, \lambda) - \phi^*(x, \lambda)\ \hat{\phi}(x, \lambda) = 0$$

$$\hat{\phi}^*(x, \lambda)\ \theta(x, \lambda) - \phi^*(x, \lambda)\ \hat{\theta}(x, \lambda) =$$
$$\theta^*(x, \lambda)\ \hat{\phi}(x, \lambda) - \hat{\theta}^*(x, \lambda)\ \phi(x, \lambda) = I_n\ . \tag{1.18}$$

Now since $-JY^*(x, \lambda)\ JY(x, \lambda) = I_{2n}$, we have that $-JY(x, \lambda)\ JY^*(x, \lambda) = I_{2n}$ with the resulting identities,

$$\phi(x, \lambda)\, \theta^*(x, \lambda) - \theta(x, \lambda)\, \phi^*(x, \lambda)$$

$$= \hat{\phi}(x, \lambda)\, \hat{\theta}^*(x, \lambda) - \hat{\theta}(x, \lambda)\, \hat{\phi}^*(x, \lambda) = 0$$

$$\hat{\phi}(x, \lambda)\, \theta^*(x, \lambda) - \hat{\theta}(x, \lambda)\, \phi^*(x, \lambda)$$

$$= \theta(x, \lambda)\, \hat{\phi}^*(x, \lambda) - \phi(x, \lambda)\, \hat{\theta}^*(x, \lambda) = I_n\,. \qquad (1.19)$$

For simplicity we drop the notation of dependence upon b and λ in what follows. M is now the $M(\lambda)$- function.

$$M = M(\lambda, b, \beta) = -[\phi + \beta\hat{\phi}]^{-1}\,[\theta + \beta\hat{\theta}]\,.$$

$$M - M^* = -[\phi + \beta\hat{\phi}]^{-1}\,[\theta + \beta\hat{\theta}] + [\theta^* + \hat{\theta}^*\beta^*][\phi^* + \hat{\phi}^*\beta^*]$$

$$= -[\phi + \beta\hat{\phi}]^{-1}\{[\theta + \beta\hat{\theta}][\phi^* + \hat{\phi}^*\beta^*]$$

$$- [\phi + \beta\hat{\phi}][\theta^* + \hat{\theta}^*\beta^*]\}\,[\phi^* + \hat{\phi}^*\beta^*]^{-1}$$

$$= -[\phi + \beta\hat{\phi}]^{-1}\{\theta\phi^* - \theta\hat{\phi}^*\beta^* + \beta\hat{\theta}\phi^* + \beta\hat{\theta}\hat{\phi}^*\beta^* - \phi\theta^* - \phi\hat{\theta}^*\beta^*$$

$$- \beta\hat{\phi}\theta^* - \beta\hat{\phi}\hat{\theta}^*\beta^*\}\,[\phi^* + \hat{\phi}^*\beta^*]^{-1}$$

$$= -[\phi + \beta\hat{\phi}]^{-1}\{(\theta\phi^* - \phi\theta^*) + \beta(\hat{\theta}\phi^* - \hat{\phi}\theta^*)$$

$$+ (\theta\hat{\phi}^* - \phi\hat{\theta}^*)\beta^* + \beta(\hat{\theta}\hat{\phi}^* - \hat{\phi}\hat{\theta}^*)\beta^*\}[\phi^* + \hat{\phi}^*\beta^*]^{-1}\,.$$

By the identities in (1.19) the last expression becomes

$$M - M^* = -[\phi + \beta\hat{\phi}]^{-1}\,\{0 - \beta + \beta^* + 0\}\,[\phi^* + \hat{\phi}^*\beta^*]^{-1}$$

$$= \{[\phi^* + \hat{\phi}^*\beta^*](\beta - \beta^*)^{-1}\,[\phi + \beta\hat{\phi}]\}^{-1}$$

$$= \{\phi^*(\beta - \beta^*)^{-1}\phi + \hat{\phi}^*\beta^*(\beta - \beta^*)^{-1}\beta\hat{\phi} + 2i\ \mathrm{Im}\ \phi^*(\beta - \beta^*)^{-1}\beta\hat{\phi}\}^{-1}$$

and from this the conclusion follows upon division by $2i$. ∎

As a special case of this last result, let K be an $n \times n$ positive definite matrix, and let $\beta = i\,K$. Then $\operatorname{Im}\beta = \frac{1}{2i}[\beta - \beta^*] = \frac{1}{2}[K + K^*] = K$. The last equality follows because a positive definite matrix is hermitian. In this case, (1.16) becomes

$$\operatorname{Im} M(\lambda, b, \beta) = \{\phi^*(b, \lambda)\, K^{-1}\phi(b, \lambda) + \hat{\phi}^*(b, \lambda)\, K\hat{\phi}(b, \lambda)$$
$$- 2 \operatorname{Im} \phi^*(b, \lambda)\, \hat{\phi}(b, \lambda)\}^{-1} .$$

By the identities (1.18), we find that

$$2i \operatorname{Im} \phi^*\hat{\phi} = \phi^*\hat{\phi} - \hat{\phi}^*\phi = 0$$

and hence, for $\beta = iK$, $K > 0$ and $\lambda \in \mathbb{R}$, that

$$\operatorname{Im} M(\lambda, b, \beta) = \{\phi^*(b, \lambda)\, K^{-1}\, \phi(b, \lambda) +$$
$$+ \hat{\phi}^*(b, \lambda)\, K\hat{\phi}(b, \lambda)\}^{-1} . \tag{1.20}$$

Consider $M(\lambda, b, \beta)$ of (1.12) as a function of b . Since $A(x)$ and $B(x)$ in (1.1) are locally integrable for $x \geq a$, the fundamental solution (1.9) exists for all $x \geq a$; and thus by Lemma 1.1, $M(\lambda, b, \beta)$ exists, except when λ is an eigenvalue, for all $b > a$. As a result of Lemma 1.2, the eigenvalues of (B) are in the lower half complex plane when $\operatorname{Im}\beta \geq 0$. Thus $M(\lambda, b, \beta)$ is defined for all $b > a$ when $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\beta \geq 0$ Now within the proof of Lemma 1.3 it was shown when $\operatorname{Im}\beta \geq 0$, $M = M(\lambda, b, \beta)$, and $X(x, \lambda) = Y(x, \lambda) \begin{bmatrix} I \\ M \end{bmatrix}$, that

$$X^*(b, \lambda)(-i\ J)\ X(b, \lambda) =$$

$$= -2[\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)M]^* (\mathrm{Im}\ \beta)[\hat{\theta}(b, \lambda) + \hat{\phi}(b, \lambda)M] \ . \quad (1.21)$$

It is necessary to show when λ , with $\mathrm{Im}\ \lambda > 0$, is restricted to a compact set, that the entries of $M(\lambda, b, \beta)$ in (1.12) are bounded for $b > a$. Furthermore, when the system (1.1) is of a particular type we want to show that as b tends to infinity, $M(\lambda, b, \beta)$ converges, for λ with $\mathrm{Im}\ \lambda > 0$, to an analytic matrix valued function of λ . To do this, we need to use the geometric properties of matrix circles as developed in [6], [13], [14], [23] .

In [14], a function E is defined upon the set of $n \times n$ complex matrices M where $X(b, \lambda) = Y(b, \lambda) \begin{bmatrix} I \\ M \end{bmatrix}$ and

$$E(M) = \begin{cases} X^*(b, \lambda)\ (-iJ)\ X(b, \lambda) & , \quad \mathrm{Im}\ \lambda > 0 \\ \\ X^*(b, \lambda)\ (iJ)\ X(b, \lambda) & , \quad \mathrm{Im}\ \lambda < 0\ . \end{cases} \quad (1.22)$$

Now for (B) with $\mathrm{Im}\, \beta > 0$ and $\mathrm{Im}\ \lambda > 0$ we can see by (1.21) that

$$E(M(\lambda, b, \beta)) < 0\ .$$

Matrices which satisfy the inequality $E(M) \leq 0$ are examined in [6], [14], and [23] .

In connection with the function E is defined the matrix

$$\begin{bmatrix} \mathscr{A} & \mathscr{B}^* \\ \mathscr{B} & \mathscr{D} \end{bmatrix} = \begin{cases} -i\, Y^*(b, \lambda)\, J\, Y(b, \lambda) \ , & \operatorname{Im} \lambda > 0 \\ i\, Y^*(b, \lambda)\, J\, Y(b, \lambda) \ , & \operatorname{Im} \lambda < 0 \ . \end{cases} \tag{1.23}$$

where $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$ are $n \times n$ matrices. From this definition we see that

$$E(M) = [I, M^*] \begin{bmatrix} \mathscr{A} & \mathscr{B}^* \\ \mathscr{B} & \mathscr{D} \end{bmatrix} \begin{bmatrix} I \\ M \end{bmatrix} .$$

It was proved in [14] that

$$\begin{aligned} E(M) &= M^* \mathscr{D} M + M^* \mathscr{B} + \mathscr{B}^* M + \mathscr{A} \\ &= (M - C)^* R_1^{-2} (M - C) - R_2^2 \end{aligned} \tag{1.24}$$

where

$$\begin{aligned} C &= C(b, \lambda) = -\mathscr{D}^{-1} \mathscr{B} \\ R_1 &= R_1(b, \lambda) = \mathscr{D}^{-1/2} \\ R_2 &= R_2(b, \lambda) = [\mathscr{B}^* \mathscr{D}^{-1} \mathscr{B} - \mathscr{A}]^{-1/2} \ , \end{aligned} \tag{1.25}$$

and further noted that

$$\mathscr{D} = 2|\mathrm{Im}\ \lambda| \int_a^b \begin{bmatrix} \phi \\ \hat{\phi} \end{bmatrix}^* A(x) \begin{bmatrix} \phi \\ \hat{\phi} \end{bmatrix} dx \ ;$$

that

$$\mathscr{B}^*(b, \lambda)\mathscr{D}^{-1}(b, \lambda)\mathscr{B}(b, \lambda) - \mathscr{A}(b, \lambda) = \mathscr{D}^{-1}(b, \bar{\lambda}) \ ;$$

and as a consequence that

$$R_1(b, \lambda) = R_2(b, \bar{\lambda}) \ .$$

Notice that the matrices in (1.23) and hence in (1.25) are independent of the boundary condition at $x = b$. There is dependence upon the boundary condition at $x = a$, however, because $Y(x, \lambda)$ depends upon this condition.

It is shown in [14] that an $n \times n$ matrix M satisfies $E(M) = 0$ if and only if

$$M = -[\beta_1 \ \phi(b, \lambda) + \beta_2\hat{\phi}(b, \lambda)]^{-1} \ [\beta_1\theta(b, \lambda) + \beta_2\hat{\theta}(b, \lambda)] \ .$$

That is, M is the $M(\lambda)$ function (1.11) for some $[\beta_1, \beta_2]$ associated with a boundary value problem of type (R) on $[a, b]$.

By (1.24), the equation $E(M) = 0$ is equivalent to

$$[R_1^{-1}(M - C)\ R_2^{-1}]^* \ [R_1^{-1}(M - C)\ R_2^{-1}] = I_n \quad . \qquad (1.26)$$

This is equivalent to saying that $R_1^{-1}(M - C)\ R_2^{-1}$ is a unitary matrix. As a consequence, those M satisfying the equation $E(M) = 0$ can be identified with the set of unitary matrices by means of the equation

$$M = C + R_1 UR_2 , \tag{1.27}$$

where U is some unitary matrix. And so by this same equation, there is an identification between the M-functions associated with the boundary value problems (R) and the set of unitary matrices. The locus of these matrices is referred to as the "matrix circle".

Now for fixed b and λ where $\operatorname{Im} \lambda \neq 0$, elements of the set

$$S(b, \lambda) = \{M: E(M) \leq 0\}$$

can, by (1.24), be identified with matrices V such that $0 \leq V^* V \leq I_n$. This identification is described by $M = C + R_1 V R_2$. Furthermore, when $\operatorname{Im} \lambda \neq 0$, the matrices $C(b, \lambda)$ and $R_i(b, \lambda)$ are constant. The elements of the matrices V , where $0 \leq V \leq I_n$, are bounded as elements of $\mathbb{C}^{n^2}$ since

$$\sum_{k=1}^{n} |V_{ki}|^2 = [V^* V]_{ii} \leq 1$$

for all values of i . Thus for fixed b and λ , the elements of $S(b, \lambda)$ form a compact set in $\mathbb{C}^{n^2}$.

If we fix b and allow λ to vary with $\operatorname{Im} \lambda \neq 0$, we note that R_i and C depend continuously upon λ . As a result, the set $S(b, \lambda)$ depends continuously upon λ . Thus for a compact set Ω in the upper half of $\mathbb{C}$, $\bigcup_{\lambda \in \Omega} S(b, \lambda)$ is a bounded set in $\mathbb{C}$. Thus $M(\lambda, b, \beta)$, defined by (1.12), has entries bounded uniformly for $\lambda \in \Omega$.

Recalling (1.15) and (1.22), we see that

$$E(M) = -i(\mathrm{sgn}(\mathrm{Im}\ \lambda))\ X^*(b, \lambda)\ J\ X(b, \lambda)$$

$$= -i\ \mathrm{sgn}(\mathrm{Im}\ \lambda)\ X^*(a, \lambda)\ J\ X(a, \lambda) + 2|\mathrm{Im}\ \lambda| \int_a^b X^*\ AX\ .$$

From this, it can be observed that if $E(M) \leq 0$ when b is the upper limit of integration, then $E(M) \leq 0$ for any b_1, such that $a \leq b_1 \leq b$. And so we see that

$$S(b, \lambda) \subseteq S(b_1, \lambda)$$

for all $b \geq b_1$. That is, for fixed λ, $\mathrm{Im}\ \lambda \neq 0$, the sets $S(b, \lambda)$ are "nested" as b increases.

From the relations for $\mathscr{D}$, R_1, and R_2 in (1.25), we see that as b increases, $\mathscr{D}$ increases, and hence that $R_1(b, \lambda)$ and $R_2(b, \lambda)$ decrease. It is pointed out in [14] that there is a sequence $\{b_n\}$, $b_n \to \infty$, such that R_1 and R_2 decrease to non-negative definite limits. Further, it can be shown that $C(b, \lambda)$ also has a limit [6], [23].

Let

$$N(\lambda) = \dim\{y(x, \lambda) \text{ satisfying (1.1) } \mid \int_a^\infty y^*\ A(x)\ y\ dx < \infty\};$$

then $N(\lambda)$ is constant in $\mathrm{Im}\ \lambda > 0$ and in $\mathrm{Im}\ \lambda < 0$ [21]. When $N(i) = N(-i) = n$, we say that (1.1) is limit point. In [14] it is shown that in the limit point case, both $R_1(b, \lambda)$ and

$R_2(b, \lambda)$ tend to 0 as b tends to ∞. Thus for those systems (1.1) which are limit point, the sets $S(b, \lambda)$, for fixed λ with $\operatorname{Im} \lambda \neq 0$, collapse to the limiting value of $C(b, \lambda)$ as b tends to ∞. This limiting matrix we denote by $M_\infty(\lambda)$.

One can now see that, for the boundary value problem (B) defined for those β where $\operatorname{Im} \beta \geq 0$ and for those λ such that $\operatorname{Im} \lambda > 0$, the M-function $M(\lambda, b, \beta)$ satisfies $E(M(\lambda, b, \beta)) < 0$ and hence that $M(\lambda, b, \beta)$ is an element of $S(b, \lambda)$. Our previous comments show that $M(\lambda, b, \beta)$ is forced to converge to $M_\infty(\lambda)$ by the collapse of $S(b, \lambda)$ when (1.1) is limit point.

In the limit point case, as b tends to infinity, the entries of $M(\lambda, b, \beta)$ will converge pointwise to the entries of $M_\infty(\lambda)$ for $\lambda \in \Omega$. By a standard theorem in complex function theory, we may conclude that the entries of $M_\infty(\lambda)$ are analytic functions for $\operatorname{Im} \lambda > 0$. Furthermore, by the remark following Lemma 1.3, we note $\operatorname{Im} M(\lambda, b, \beta) > 0$ when $\operatorname{Im} \beta > 0$ and hence that $\operatorname{Im} M_\infty(\lambda) \geq 0$ for $\operatorname{Im} \lambda > 0$. Consequently, both $M(\lambda, b, \beta)$ and $M_\infty(\lambda)$ are matrix valued functions of Pick-Nevanlinna type.

By Theorem A.4 and equation (A.8) of the Appendix there is a unique nondecreasing matrix valued measure $\rho(\mu, b, \beta)$ as well as hermitian matrices $A(b, \beta)$, $B(b, \beta)$ where $B(b, \beta) \geq 0$ such that for λ, with $\operatorname{Im} \lambda > 0$,

$$M(\lambda, b, \beta) = A(b, \beta) + \lambda B(b, \beta) + \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2}\right) d\rho(\mu, b, \beta) . \tag{1.28}$$

The last integral is defined to be

$$\lim_{c,d\to\infty} \int_{-c}^{d} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2}\right) d\rho(\mu, b, \beta) .$$

As A, B are hermitian and $\rho(\mu)$ is nondecreasing it follows that

$$\operatorname{Im} M(\lambda, b, \beta) = (\operatorname{Im} \lambda)\, B(b, \beta) + \int_{-\infty}^{\infty} \frac{\operatorname{Im} \lambda}{|\mu - \lambda|^2}\, d\rho(\mu, b, \beta) . \tag{1.29}$$

It has already been noted that when $\operatorname{Im} \beta > 0$, $M(\lambda, b, \beta)$ has analytic entries with singularities in the lower open half of $\mathbb{C}$. Thus in this case, $M(\lambda, b, \beta)$ is analytic for $\lambda \in \mathbb{R}$. By an extension to the case of matrices of a standard argument involving the Stieltjes Inversion formula [8 , p. 239] , we can show that for such β ,

$$\rho(\zeta, b, \beta) - \rho(\mu, b, \beta) = \pi^{-1} \int_{\mu}^{\zeta} \operatorname{Im} M(\xi, b, \beta)\, d\xi .$$

Since $M(\xi, b, \beta)$ is continuous for all $\xi \in \mathbb{R}$, it follows that

$$\rho'(\mu, b, \beta) = \pi^{-1} \operatorname{Im} M(\mu, b, \beta) . \tag{1.30}$$

If we take $\lambda = i$, then (1.29) becomes

$$\operatorname{Im} M(i, b, \beta) = B(b, \beta) + \int_{-\infty}^{\infty} \frac{1}{1 + \mu^2}\, d\rho(\mu, b, \beta) . \tag{1.31}$$

Because $B(b, \beta) \geq 0$ it follows that

$$0 \leq \int_{-\infty}^{\mu} \frac{1}{1+\xi^2} \, d\rho(\xi, b, \beta) \leq \operatorname{Im} M(i, b, \beta) . \tag{1.32}$$

We have noted that $S(b, i)$ is a bounded set, and that as $b \to \infty$ these sets nest. It has also been noted that when $\operatorname{Im} \beta > 0$, $M(i, b, \beta) \in S(b, i)$. Thus the entries of $M(i, b, \beta)$ possess a uniform bound as $b \to \infty$.

Lemma 1.5: Let $\{b_n, \beta_n\}$ be a sequence where $b_n \in \mathbb{R}$, and $b_n \to \infty$ as $n \to \infty$, and where β_n is an $n \times n$ matrix such that $\operatorname{Im} \beta_n > 0$. Let $\rho_n(\mu) = \rho(\mu, b_n, \beta_n)$ be the measure defined in (1.28) and equation (A.8) of the Appendix. Given that the system (1.1) is limit point, there are constant $n \times n$ hermitian matrices $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B} \geq 0$, a nondecreasing matrix valued function $\rho(\mu)$, and a subsequence $\{\rho_{n_k}(\mu)\}$ such that $\rho_{n_k}(\mu)$ converges pointwise on $\mathbb{R}$ to $\rho(\mu)$; and for λ with $\operatorname{Im} \lambda > 0$,

$$\begin{aligned} M_\infty(\lambda) &= \lim_{n_k \to \infty} M(\lambda, b_{n_k}, \beta_{n_k}) \\ &= \mathcal{A} + \lambda \mathcal{B} + \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1+\mu^2}\right) d\rho(\mu) . \end{aligned} \tag{1.33}$$

Proof. In the remark following (1.31) it was noted that the nesting property of the sets $S(b, \lambda)$ implied that there was a bound for the elements of the matrices $M(i, b_n, \beta_n)$ and hence of $\operatorname{Im} M(i, b_n, \beta_n)$. By (1.28) we note that the elements of the matrices $A(b_n, \beta_n)$ and $B(b_n, \beta_n)$ must also be bounded. Thus from the sequence $\{b_n, \beta_n\}$ there is a subsequence $\{b_{n_k}, \beta_{n_k}\}$ and a pair of hermitian matrices A and B where $B \geq 0$ such that

$$\lim_{n_k \to \infty} A(b_{n_k}, \beta_{n_k}) = A$$

and

$$\lim_{n_k \to \infty} B(b_{n_k}, \beta_{n_k}) = B .$$

From this point, we work with the subsequence which without loss of generality we denote by $\{b_n, \beta_n\}$.

Let $\rho(\mu, b_n, \beta_n)$ be the unique nondecreasing matrix valued function described in (1.28) . Let

$$\Omega(\mu, b_n, \beta_n) = \int_{-\infty}^{\mu} \frac{1}{1 + \xi^2} \, d\rho(\xi, b_n, \beta_n) ; \tag{1.34}$$

$\Omega(\mu, b_n, \beta_n)$ is thus the unique nondecreasing matrix valued function associated with the representation of $M(\lambda, b_n, \beta_n)$ as described in Theorem A.4 and equation (A.7) of the appendix.

This collection $\{\Omega(\mu, b_n, \beta_n)\}$ is a family of left continuous, nondecreasing, matrix valued function defined on all of $\mathbb{R}$ and at $\pm\infty$ as well. Specifically,

$$\Omega(-\infty, b_n, \beta_n) = 0$$

and

$$\Omega(\infty, b_n, \beta_n) = \lim_{\mu\to\infty} \Omega(\mu, b_n, \beta_n)$$

$$= -B(b_n, \beta_n) + \operatorname{Im} M(i, b_n, \beta_n) ;$$

the last equality following from (1.31) and (1.34). The inequality (1.32) shows that this family of nondecreasing functions is uniformly bounded, and thus of uniform bounded variation.

By Lemma A.2 of the Appendix we know that there is a nonnegative definite, nondecreasing matrix valued function $\Omega(\mu)$ and a subsequence $\{b_{n_k}, \beta_{n_k}\}$ of the subsequence $\{b_n, \beta_n\}$ such that $\Omega(\mu, b_{n_k}, \beta_{n_k})$ converges pointwise on $\mathbb{R}$, as well as at $\pm\infty$ to $\Omega(\mu)$. This function $\Omega(\mu)$ will share the same bound on its variation as on the family $\{\Omega(\mu, b_n, \beta_n)\}$.

From this point on, we shall work with this latest subsequence. Again without loss of generality, we denote the subsequence by $\{\Omega(\mu, b_n, \beta_n)\}$.

We next define $\rho(\mu)$ up to a constant by

$$d\Omega(\mu) = \frac{1}{1 + \mu^2}\, d\rho(\mu) , \tag{1.35}$$

where $\Omega(\mu)$ is the limit, just described, of the subsequence $\{\Omega(\mu, b_n, \beta_n)\}$. We thus have the relation

$$\int_{-\infty}^{\infty} \frac{1 + \mu\lambda}{\mu - \lambda}\, d\Omega(\mu, b_n, \beta_n) = \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2}\right) d\rho(\mu, b_n, \beta_n)$$

where the improper integrals are defined as in (1.28).

To what does $\int_{-\infty}^{\infty} \frac{1 + \mu\lambda}{\mu - \lambda} d\Omega(\mu, b_n, \beta_n)$ converge?

To answer this, first note that the function $\frac{1 + \mu\lambda}{\mu - \lambda}$ is continuous and bounded on $\mathbb{R}$ since we are assuming that $\operatorname{Im} \lambda > 0$. Further note that as $\mu \to \pm\infty$ this function tends to λ as a limit. Now since $\frac{1 + \mu\lambda}{\mu - \lambda}$ is bounded and $\Omega(\lambda)$ is of bounded variation, the improper integrals

$$\int_{-\infty}^{\infty} \frac{1 + \mu\lambda}{\mu - \lambda} d\Omega(\mu) = \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2}\right) d\rho(\mu)$$

exist and are defined as those in (1.28). This observation together with our earlier observations about the family $\{\Omega(\mu, b_n, \beta_n)\}$ shows that the hypotheses of Theorem A.2 of the Appendix are satisfied. As a consequence, we have that for the final subsequence chosen,

$$M(\lambda, b_n, \beta_n) = A(b_n, \beta_n) + \lambda B(b_n, \beta_n)$$

$$+ \int_{-\infty}^{\infty} \frac{1 + \mu\lambda}{\mu - \lambda} d\Omega(\mu, b_n, \beta_n)$$

converges, as $n \to \infty$, to

$$A + \lambda(B + \alpha + \beta) + \int_{-\infty}^{\infty} \frac{1 + \mu\lambda}{\mu - \lambda} d\Omega(\mu) \qquad (1.36)$$

where

$$\alpha = \Omega(\infty) - \Omega(\infty^-) \qquad , \beta = \Omega(-\infty^+) - \Omega(-\infty) \quad ,$$

$\Omega(\infty^-) = \lim_{\mu \to \infty} \Omega(\mu)$, and $\Omega(-\infty^+) = \lim_{\mu \to \infty} \Omega(\mu)$. Note that as $\Omega(\mu)$ is nondecreasing, α and β are nonnegative definite. And so from our definition of $\rho(\mu)$, (1.36) is equivalent to

$$A + \lambda(B + \alpha + \beta) + \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 - \mu^2}\right) d\rho(\mu) \ . \tag{1.37}$$

At the same time, we know that because of the limit point hypothesis, $M(\lambda, b_n, \beta_n)$ converges $M_\infty(\lambda)$. Thus (1.37), with $\mathcal{A} = A$ and $\mathcal{B} = B + \alpha + \beta$, is a Pick-Nevanlinna representation for $M_\infty(\lambda)$. ∎

By Theorem A.4 and equation (A.8) of the appendix, while $\Omega(\mu)$ and $\rho(\mu)$ might not be the unique measures described in that theorem nevertheless the measures are essentially unique in that all such measures have, on $\mathbb{R}$, the same sets of points of continuity, and all such measures differ by at most a constant matrix on the set of points of continuity. We shall use the fact that $\rho(\mu)$ and $\Omega(\mu)$ are the pointwise limits of the subsequences $\{\rho(\mu, b_n, \beta_n)\}$ and $\{\Omega(\mu, b_n, \beta_n)\}$.

<u>Definition</u>: Let $\{K_n(\xi)\}$ be a family of $n \times n$ matrix valued functions all of which are positive definite for ξ (Λ_1, Λ_2) ; let $\phi(x, \xi)$ and $\hat{\phi}(x, \xi)$ be the $n \times n$ blocks of the fundamental matrix defined in (1.9) . Then define

$$J_n(\mu, \xi) = J(x_n, \mu, \xi) = \phi^*(x_n, \mu)\, K_n^{-1}(\xi)\, \phi(x_n, \mu)$$
$$+ \hat{\phi}^*(x_n, \mu)\, K_n(\xi)\, \hat{\phi}(x_n, \mu)$$

Theorem 1.1: Assume that system (1.1) is limit point. Let $\{K_n(\mu)\}$ be a family of $n \times n$ matrices which are positive definite on (Λ_1, Λ_2) . Let $\{x_n\}$ be a sequence such that $x_n \to \infty$ as $n \to \infty$.

Suppose that for each $\xi \in (\Lambda_1, \Lambda_2)$ there is an interval Δ centered at ξ and positive constants B_1, B_2, B_3 such that for $\mu \in \Delta$,

$$J(x_n, \mu, \mu) > B_1 I \tag{1.38}$$

and

$$B_2 I < K_n^{-1/2}(\mu)\, K_n(\xi)\, K_n^{-1/2}(\mu) < B_3 I . \tag{1.39}$$

One can show that the measure $\rho(\mu)$ in the Pick-Nevannlina representation of $M_\infty(\lambda)$ will be absolutely continuous in (Λ_1, Λ_2) .

Proof: Let $\xi \in (\Lambda_1, \Lambda_2)$. By our hypotheses there is a $\Delta \subset (\Lambda_1, \Lambda_2)$ centered at ξ , and positive constants B_1, B_2, B_3 such that (1.38) and (1.39) hold. It is shown in Coppel ([9 , p. 41] that if A and B are matrices such that $A > B > 0$, then $B^{-1} > A^{-1} > 0$. With this it follows from (1.39) that

$$B_2^{-1} I > K_n^{1/2}(\mu)\, K_n^{-1}(\xi)\, K_n^{1/2}(\mu) > B_3^{-1} I .$$

And so, there is a constant $B > 0$ such that for $\mu \in \Delta$

$$K_n^{1/2}(\mu)\ K_n^{-1}(\xi)\ K_n^{1/2}(\mu) > B\ I$$

and

$$K_n^{-1/2}(\mu)\ K_n(\xi)\ K_n^{1/2}(\mu) > B\ I\ .$$

Now through this we see that

$$\phi^*(x_n,\ \mu)\ K_n^{-1}(\xi)\ \phi(x_n,\ \mu) =$$

$$= \phi^*(x_n\ \mu)\ K_n^{-1/2}(\mu)[K_n^{1/2}(\mu)\ K_n^{-1}(\xi)\ K_n^{1/2}(\mu)]K_n^{-1/2}(\mu)\ \phi(x_n,\ \mu)$$

$$\geq B\phi^*(x_n,\ \mu)\ K_n^{-1}(\mu)\ \phi(x_n,\ \mu) \quad .$$

By similar reasoning,

$$\hat{\phi}^*(x_n,\ \mu)\ K(\xi)\ \hat{\phi}(x_n,\ \mu) \geq B\hat{\phi}^*(x_n,\ \mu)\ K_n(\mu)\ \hat{\phi}(x_n,\ \mu)\ .$$

Thus for $\mu \in \Delta$,

$$J_n(\mu,\ \xi) \geq B\ J_n(\mu,\ \mu)\ .$$

By our hypotheses,

$$J_n(\mu,\ \xi) \geq B\ B_1 I \qquad \text{for}\ \ \mu \in \Delta\ .$$

As a result,

$$J_n^{-1}(\mu, \xi) \leq C\, I \qquad \text{where} \quad C = (B\, B_1)^{-1} \tag{1.40}$$

for $\mu \in \Delta$.

Consider now the boundary value problem (B) with $\beta_n = i\, K_n(\xi)$. Then $\operatorname{Im} \beta_n = K_n(\xi) > 0$, and by (1.30) with $\rho_n(\mu) = \rho(\mu, x_n, \beta_n)$ we see that

$$\rho_n'(\mu) = \pi^{-1} \operatorname{Im} M(\mu, x_n, \beta_n) \ ;$$

and by (1.20) that

$$\rho_n'(\mu) = \pi^{-1}\, J_n^{-1}(\mu, \xi) \ .$$

By (1.40) this implies that for $\mu \in \Delta$,

$$\rho_n'(\mu) \leq \pi^{-1} C\, I \ .$$

So for $\mu_2, \mu_1 \in \Delta$ with $\mu_2 \geq \mu_1$, we see that

$$0 \leq \rho_n(\mu_2) - \rho_n(\mu_1) \leq \pi^{-1}\, C(\mu_2 - \mu_1) I \ . \tag{1.41}$$

It was observed in Lemma 1.5 that a subsequence of $\{\rho_n(\mu)\}$ can be found converging to $\rho(\mu)$ on $\mathbb{R}$. As a consequence of (1.41)

$$0 \leq \rho(\mu_2) - \rho(\mu_1) < \pi^{-1} C(\mu_2 - \mu_1) I .$$

Thus $\rho(\mu)$ is a nondecreasing matrix valued function satisfying a Lipschitz condition on Δ and is thus absolutely continuous on Δ and hence on (Λ_1, Λ_2) . ■

Lemma 1.6: Let $\{K_n(\mu)\}$ be a family of $n \times n$ positive definite matrix valued functions. Let $\{K_n^{-1}(\mu)\}$ be equicontinuous on (Λ_1, Λ_2) and assume that for each $\xi \in (\Lambda_1, \Lambda_2)$ there is a positive constant B such that $K_n^{-1}(\xi) > B \cdot I$. Then given $\varepsilon > 0$, and $\xi \in (\Lambda_1, \Lambda_2)$, there is an interval Δ centered at ξ , such that for $\mu \in \Delta$

$$(1 - \varepsilon/B)I < K_n^{-1/2}(\mu)\, K_n(\xi)\, K_n^{-1/2}(\mu) < (1 + \varepsilon/B)\, I . \quad (1.42)$$

Proof: Given ξ and B , choose $\varepsilon > 0$ such that $1 > \max\{\varepsilon, \frac{\varepsilon}{B}\} > 0$. By equicontinuity there is an interval Δ centered at ξ such that

$$\|K_n^{-1}(\mu) - K_n^{-1}(\xi)\| < \varepsilon$$

for $\mu \in \Delta$, where $\|\cdot\|$ is the operator norm. And so for $\mu \in \Delta$,

$$-\varepsilon I \leq K_n^{-1}(\mu) - K_n^{-1}(\xi) \leq \varepsilon I \quad ,$$

and hence

$$-\varepsilon I + K_n^{-1}(\xi) < K^{-1}(\mu) < \varepsilon\, I + K_n^{-1}(\xi) \quad .$$

As a consequence,

$$-\varepsilon K_n(\xi) + I < K_n^{1/2}(\xi)\, K_n^{-1}(\mu)\, K_n^{1/2}(\xi) < \varepsilon\, K_n(\xi) + I\ .$$

Now from the assumption that $K_n^{-1}(\xi) > B\, I$ it follows that $0 < K_n(\xi) < B^{-1}\, I$, and for $\mu \in \Delta$ that

$$0 < (1 - \varepsilon B^{-1})I < K_n^{1/2}(\xi)\, K_n^{-1}(\mu)\, K_n^{1/2}(\xi) < (1 + \varepsilon B^{-1})\, I. \quad (1.43)$$

Let $c_1(\varepsilon) = 1 - \varepsilon B^{-1}$ and $c_2(\varepsilon) = 1 + \varepsilon B^{-1}$. Now, it follows from (1.43) that

$$c_1(\varepsilon)\, K_1^{-1}(\xi) < K_n^{-1}(\mu) < c_2(\varepsilon)\, K_n^{-1}(\xi)\ .$$

From the left hand inequality it follows that

$$K_n^{1/2}(\mu)\, K_n^{-1}(\xi)\, K_n^{1/2}(\mu) < c_1^{-1}(\varepsilon)\, I$$

and from the right hand inequality that

$$c_2^{-1}(\varepsilon)\, I < K_n^{1/2}(\mu)\, K_n^{-1}(\xi)\, K_n^{1/2}(\mu)\ .$$

From these last two inequalities it follows that for $\mu \in \Delta$

$$c_2(\varepsilon)\, I > K_n^{-1/2}(\mu)\, K_n(\xi)\, K_n^{-1/2}(\mu) > c_1(\varepsilon)\, I\ . \qquad \blacksquare$$

Theorem 1.2: Assume that the system (1.1) is limit point. Let $\{K_n(\mu)\}$ be a family of positive definite $n \times n$ matrix valued functions satisfying the hypotheses of Lemma 1.6 . Suppose $\alpha(\mu)$ is an $n \times n$ positive definite matrix valued function whsoe entries are locally integrable on (Λ_1, Λ_2) and $\{x_n\}$ is a sequence such that $x_n \to \infty$ as $n \to \infty$ and such that $J_n(\mu, \mu)$ converges to $\alpha(\mu)$ pointwise a.e. on (Λ_1, Λ_2) . If for each $\xi \in (\Lambda_1, \Lambda_2)$ there is a positive constant B and an interval $\bar{\Delta}$ centered at ξ such that $J_n(\mu, \mu) > B\, I$ for $\mu \in \bar{\Delta}$, then the measure $\rho(\mu)$ in the Pick-Nevanlinna representation of $M_\infty(\lambda)$ will be absolutely continuous on (Λ_1, Λ_2) and $\pi'(\mu) = \pi^{-1}\, \alpha(\mu)$ a.e. .

Proof: Let $\xi \in (\Lambda_1, \Lambda_2)$. By Lemma 1.6 there is an interval Δ centered at ξ with $\Delta \subset \bar{\Delta}$, and constants $B_1 = B$, $B_2 = c_1(\varepsilon)$, $B_3 = c_2(\varepsilon)$ such that the hypotheses of Theorem 1.1 hold. As a result, we see that $\rho(\mu)$ is absolutely continuous on (Λ_1, Λ_2) ; thus $\rho'(\mu)$ exists a.e. on (Λ_1, Λ_2) .

As in Theorem 1.1 we let $\beta_n = i\, K_n(\xi)$ and $\rho_n(\mu) = \rho(\mu, x_n, \beta_n)$. And as noted there, as a result of Lemma 1.5 there is a subsequence of $\{\rho_n(\mu)\}$ converging to $\rho(\mu)$ on $\mathbb{R}$. Now let $[\xi, \xi + \delta] \subset \Delta$; then

$$\lim_{n\to\infty} \int_{\xi}^{\xi+\delta} (\mu^2 + 1)^{-1}\, d\rho_n(\mu) = \int_{\xi}^{\xi+\delta} (\mu^2 + 1)^{-1}\, d\rho(\mu) .$$

Let

$$E_n = \int_{\xi}^{\xi+\delta} (\mu^2 + 1)^{-1} \, d\rho_n(\mu) - \int_{\xi}^{\xi+\delta} (\mu^2 + 1)^{-1} \, d\rho(\mu) \, . \tag{1.44}$$

Then $\lim_{n\to\infty} E_n = 0$. Next note that

$$\begin{aligned} 0 &< \int_{\xi}^{\xi+\delta} [(\xi^2 + 1)^{-1} - (\mu^2 + 1)] \, d\rho_n(\mu) \\ &= \int_{\xi}^{\xi+\delta} \frac{\mu^2 - \xi^2}{(\mu^2 + 1)(\xi^2 + 1)} \, d\rho_n(\mu) \\ &\leq \int_{\xi}^{\xi+\delta} (\mu^2 - \xi^2) \, d\rho_n(\mu) \\ &\leq [(\xi + \delta)^2 - \xi^2] \, [\rho_n(\xi + \delta) - \rho_n(\xi)] \, . \end{aligned}$$

In equation (1.41) of Theorem 1.1, it was shown that there is a positive constant C , independent of n , such that for $\mu_2, \mu_1 \in [\xi, \xi + \delta] \subset \Delta$, with $\mu_1 < \mu_2$,

$$\rho_n(\mu_2) - \rho_n(\mu_1) \leq \pi^{-1} \, C(\mu_2 - \mu_1) \, I \leq \pi^{-1} \, C\delta \, I$$

Thus

$$\begin{aligned} 0 &< \int_{\xi}^{\xi+\delta} [(\xi^2 + 1)^{-1} - (\mu^2 + 1)] \, d\rho_n(\mu) \\ &\leq [2\xi + \delta] \, \pi^{-1} \, C \, \delta^2 \, I = O(\delta^2) \, I \, . \end{aligned} \tag{1.45}$$

Again, this bound is independent of n . Define

$$B_n(\xi, \delta) = \int_{\xi}^{\xi+\delta} [(\xi^2 + 1)^{-1} - (\mu^2 + 1)^{-1}] \, d\rho_n(\mu) \geq 0 \ .$$

By (1.45),

$$0 \leq B_n(\xi, \delta) \leq O(\delta^2) \cdot I \tag{1.46}$$

where the order of magnitude is independent of n. Now by a similar argument, if we define

$$A(\xi, \delta) = \int_{\xi}^{\xi+\delta} [(\xi^2 + 1)^{-1} - (\mu^2 + 1)^{-1}] \, d\rho(\mu) \ .$$

one can show that

$$0 \leq A(\xi, \delta) \leq O(\delta^2) \cdot I \ . \tag{1.47}$$

Now from the definition of $B_n(\xi, \delta)$,

$$\begin{aligned}\int_{\xi}^{\xi+\delta} (\mu^2 + 1)^{-1} \, d\rho_n(\mu) &= (\xi^2 + 1)^{-1} \int_{\xi}^{\xi+\delta} d\rho_n(\mu) - B_n(\xi, \delta) \\ &= (\xi^2 + 1)^{-1} \int_{\xi}^{\xi+\delta} \rho_n'(\mu) \, d\mu - B_n(\xi, \delta) \\ &= (\xi^2 + 1)^{-1} \int_{\xi}^{\xi+\delta} \pi^{-1} \, J_n^{-1}(\mu, \xi) \, d\mu - B_n(\xi, \delta) \ . \end{aligned} \tag{1.48}$$

Similarly,

$$\int_{\xi}^{\xi+\delta} (\mu^2 + 1)^{-1} \, d\rho(\mu) = (\xi^2 + 1)^{-1} \int_{\xi}^{\xi+\delta} d\rho(\mu) - A(\xi, \delta)$$

$$= (\xi^2 + 1)^{-1} [\rho(\xi + \delta) - \rho(\xi)] - A(\xi, \delta) . \qquad (1.49)$$

Combining (1.44), (1.48), and (1.49), we see that

$$\rho(\xi + \delta) - \rho(\xi) - (\xi^2 + 1) A(\xi, \delta)$$

$$= \pi^{-1} \int_{\xi}^{\xi+\delta} J_n^{-1}(\mu, \xi) \, d\mu - (\xi^2 + 1) B_n(\xi, \mu) - (\xi^2 + 1) E_n \quad (1.50)$$

Consider now that

$$\int_{\xi}^{\xi+\delta} J_n^{-1}(\mu, \delta) \, d\mu =$$

$$\int_{\xi}^{\xi+\delta} J_n^{-1}(\mu, \mu) \, d\mu + \int_{\xi}^{\xi+\delta} [J_n^{-1}(\mu, \xi) - J_n^{-1}(\mu, \mu)] \, d\mu \quad (1.51)$$

Next we examine $J_n^{-1}(\mu, \xi) - J_n^{-1}(\mu, \mu)$.

In Lemma 1.6 it was shown that if Δ and hence $[\xi, \xi + \delta]$ are chosen to be sufficiently small, that there are constants $c_1(\varepsilon) < 1$ and $c_2(\varepsilon) > 1$ that can be made arbitrarily close to 1 such that

$$c_2(\varepsilon) I > K_n^{-1/2}(\mu) K_n(\xi) K_n^{-1/2}(\mu) > c_1(\varepsilon) I > 0$$

As a consequence,

$$\hat{\phi}^*(x_n, \mu)\ K_n(\xi)\ \hat{\phi}(x_n, \mu)$$

$$= \hat{\phi}^*(x_n, \mu)\ K_n^{1/2}(\mu)[K_n^{-1/2}(\mu)\ K_n(\xi)\ K_n^{-1/2}(\mu)]K_n^{1/2}(\mu)\ \hat{\phi}(x_n, \mu);$$

and hence that,

$$c_2(\varepsilon)\ \hat{\phi}^*(x_n, \mu)\ K_n(\mu)\ \hat{\phi}(\alpha_n, \mu)$$

$$\leq \hat{\phi}^*(x_n, \mu)\ K_n(\xi)\ \hat{\phi}(x_n, \mu)$$

$$\geq c_1(\varepsilon)\ \hat{\phi}^*(x_n, \mu)\ K_n(\mu)\ \phi(x_n, \mu)\ .$$

By a similar argument one can show that

$$c_1^{-1}(\varepsilon)\ \phi^*(x_n, \mu)\ K_n^{-1}(\mu)\ \phi(x_n, \mu)$$

$$\geq \phi^*(x_n, \mu)\ K_n^{-1}(\xi)\ \phi(x_n, \mu)$$

$$\geq c_2^{-1}(\varepsilon)\ \phi^*(x_n, \mu)\ K_n^{-1}(\mu)\ \phi(x_n, \mu)\ .$$

Hence for $\mu \in [\xi, \xi + \delta]$,

$$D_1^{-1}(\varepsilon)\ J_n(\mu, \mu) \geq J_n(\mu, \xi) \geq D_2^{-1}(\varepsilon)\ J_n(\mu, \mu)$$

where $D_1^{-1}(\varepsilon) = \max\{c_2(\varepsilon), c_1^{-1}(\varepsilon)\}$ and $D_2^{-1}(\varepsilon) = D_1(\varepsilon)$. And thus for $\mu \in [\xi, \xi + \delta]$

$$D_2(\varepsilon)\ J_n^{-1}(\mu, \mu) \geq J_n^{-1}(\mu, \xi) \geq D_1(\varepsilon)\ J_n^{-1}(\mu, \mu)\ ,$$

and

$$\{D_2(\varepsilon) - 1) \, J_n^{-1}(\mu, \mu) \geq J_n^{-1}(\mu, \xi) - J_n^{-1}(\mu, \mu)$$

$$\geq (D_1(\varepsilon) - 1) \, J_n^{-1}(\mu, \mu) \, .$$

And finally,

$$\|J_n^{-1}(\mu, \xi) - J_n^{-1}(\mu, \mu)\| \leq D_3(\varepsilon) \ \ \|J_n^{-1}(\mu, \mu)\|$$

where $D_3(\varepsilon) = \max\{|D_1(\varepsilon) - 1|, |D_2(\varepsilon) - 1|\}$. From this definition note that $D_3(\varepsilon)$ can be made arbitrarily small by choosing Δ small at the outset.

From (1.50) and (1.51) it follows that

$$\begin{aligned}
&\| [\rho(\xi + \delta) - \rho(\xi)] - \pi^{-1} \int_{\xi}^{\xi+\delta} \alpha(\mu) \, d\mu - (\xi^2 + 1) \, A(\xi, \delta)\| \\
&\quad = \|\pi^{-1} \int_{\xi}^{\xi+\delta} [J_n^{-1}(\mu, \mu) - \alpha(\mu)] \, d\mu \\
&\qquad + \pi^{-1} \int_{\xi}^{\xi+\delta} [J_n^{-1}(\mu, \xi) - J_n^{-1}(\mu, \mu)] \, d\mu \\
&\qquad - (\xi^2 + 1) \, B_n(\xi, \delta) - (\xi^2 + 1) \, E_n\| \\
&\quad \leq \pi^{-1} \int_{\xi}^{\xi+\delta} \|J_n^{-1}(\mu, \mu) - \alpha(\mu)\| \, d\mu \, + \\
&\qquad + \pi^{-1} \, D_3(\varepsilon) \int_{\xi}^{\xi+\delta} \|J_n^{-1}(\mu, \mu)\| \, d\mu \\
&\qquad + (\xi^2 + 1) \, \|B_n(\xi, \delta)\| \ + (\xi^2 + 1) \, \|E_n\| \ .
\end{aligned} \tag{1.52}$$

On Δ and hence on $[\xi, \xi + \delta]$, $\|J_n^{-1}(\mu, \mu)\| \leq B^{-1}$ and $\|B_n(\xi, \delta)\| \leq O(\delta^2)$ hence the last expression is less than or equal to

$$\pi^{-1} \int_{\xi}^{\xi+\delta} \|J_n^{-1}(\mu, \mu) - \alpha(\mu)\| \, d\mu + \pi^{-1} D_3(\varepsilon) B^{-1}\delta$$

$$+ O(\delta^2) + (\xi^2 + 1) \|E_n\| . \tag{1.53}$$

Over Δ , $J_n^{-1}(\mu, \mu)$ is bounded. And as $J_n^{-1}(\mu, \mu)$ converges to $\alpha(\mu)$ a.e., this means that $\alpha(\mu)$ is also bounded a.e. on Δ . By the dominated convergence theorem,

$$\lim_{n\to\infty} \pi^{-1} \int_{\xi}^{\xi+\delta} \|J_n^{-1}(\mu, \mu) - \alpha(\mu)\| \, d\mu = 0 .$$

Recall from the definition of E_n that

$$\lim_{n\to\infty} \|E_n\| = 0 .$$

By taking the limit as $n \to \infty$ of both sides of the inequality formed by (1.52) and (1.53) we see that

$$\|[\rho(\xi + \delta) - \rho(\xi)] - \pi^{-1} \int_{\xi}^{\xi+\delta} \alpha(\mu) \, d\mu - (\xi^2 + 1) A(\xi, \delta)\|$$

$$\leq \pi^{-1} D_3(\varepsilon) B^{-1}\delta + O(\delta^2) \tag{1.54}$$

Recall by (1.47) that $A(\xi, \delta)$ is $O(\delta^2)$. Dividing both sides of (1.54) by δ and taking the limsup as $\delta \downarrow 0$ of both sides yields

$$\limsup_{\delta \downarrow 0} \| \delta^{-1}[\rho(\xi + \delta) - \rho(\xi)] - \pi^{-1} \delta^{-1} \int_{\xi}^{\xi+\delta} \alpha(\mu)\, d\mu \|$$

$$\leq \pi^{-1} D_3(\varepsilon)$$

where $D_3(\varepsilon)$ can be made arbitrarily small by choosing Δ sufficiently small at the outset. Thus

$$\limsup_{\delta \downarrow 0} \| \delta^{-1} [\rho(\xi + \delta) - \rho(\xi)] - \pi \delta^{-1} \int_{\xi}^{\xi+\delta} \alpha(\mu)\, d\mu \| = 0 \qquad (1.55)$$

This does not say that $\rho(\mu)$ is differentiable at ξ ; only that the difference of the ratios approaches 0 as $\delta \to 0$. We do know that $\rho(\mu)$ is differentiable a.e. on (Λ_1, Λ_2) . And since $\alpha(\mu)$ is locally integrable on (Λ_1, Λ_2) we also know that

$$\lim_{\delta \downarrow 0} \pi^{-1} \delta^{-1} \int_{\xi}^{\xi+\delta} \alpha(\mu)\, d\mu = \pi^{-1} \alpha(\xi) \qquad (1.56)$$

for almost every $\xi \in (\Lambda_1, \Lambda_2)$. A similar argument exists for $\delta < 0$. Thus if ξ is an element of (Λ_1, Λ_2) for which (1.56) holds then (1.55) says that $\rho'(\xi) = \pi^{-1} \alpha(\xi)$. ∎

Note that if $\alpha(\mu)$ is continuous on (Λ_1, Λ_2) then $\rho'(\mu) = \pi^{-1} \alpha(\mu)$ for all $\mu \in (\Lambda_1, \Lambda_2)$; that is, $\rho(\mu)$ is $C^1(\Lambda_1, \Lambda_2)$.

CHAPTER 2

ABSOLUTE CONTINUITY OF $\rho(\lambda)$ FOR TWO EXAMPLES OF SECOND ORDER SYSTEMS

As an example of an application of Theorem 1.2 let us consider the following system:

$$J\begin{bmatrix} y \\ y' \end{bmatrix}' = \left[\lambda\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} v_1 + v_2 + v_3 & 0 \\ 0 & I \end{bmatrix}\right]\begin{bmatrix} y \\ y \end{bmatrix},$$

$$0 \leq x < \infty \tag{2.1}$$

or equivalently, with $V = v_1 + v_2 + v_3$,

$$y'' + (\lambda + V)y = 0 \quad , \quad 0 \leq x < \infty$$

where $y(x)$ is an $n \times 1$ vector, $v_i(x)$ are $n \times n$ matrices

$$v_i^* = v_i \ , \quad i = 1, 2, 3 \tag{2.2}$$

$$v_2 \text{ is locally integrable on } [0, \infty) \tag{2.3}$$

v_1 is locally absolutely continuous on $[0, \infty)$ and

$$\lim_{x\to\infty} v_1(x) = 0 \tag{2.4}$$

$$v_3(x),\ v_1'(x)\ ,\ \ V_2(x) = \int_x^\infty v_2(s)\,ds \ \text{ are all } L^1[0, \infty). \tag{2.5}$$

We shall show for the singular boundary value problem (S) formed from this system that the spectrum is C^1 for $0 < \lambda < \infty$; discrete for $\lambda < 0$; and that for the derivative of the spectral density, $\rho'(\lambda)$, $[\rho'(\lambda)]_{ii}$ is asymptotic, as $\lambda \to \infty$, to

$$[\lambda^{1/2} \Phi^*(0, \lambda) \Phi(0, \lambda) + \lambda^{-1/2} \Phi'^*(0, \lambda) \Phi'(0, \lambda)]_{ii}$$

where $\Phi(x, \lambda)$ and $\Phi'(x, \lambda)$ are blocks from the fundamental solution $Y(x, \lambda)$ described in Chapter 1.

The first thing we must show, in order to apply Theorem 1.2, is that this system is limit point at infinity. A geometric characterization of limit pointness was mentioned in Chapter 1. An equivalent characterization of this property, as shown by Hinton and Shaw in [14], is that there are precisely n independent $L^2(0, \infty)$ solutions of (2.1)-(2.5) on $[0, \infty)$.

For a general Hamiltonian system, by $L^2(0, \infty)$ solution we mean that

$$\int_0^\infty \begin{bmatrix} y(x, \lambda) \\ \hat{y}(x, \lambda) \end{bmatrix}^* A(x) \begin{bmatrix} y(x, \lambda) \\ \hat{y}(x, \lambda) \end{bmatrix} dx < \infty$$

where y and $\hat{y}$ are $n \times 1$ vectors and where $\begin{bmatrix} y(x, \lambda) \\ \hat{y}(x, \lambda) \end{bmatrix}$ is a solution of (1.1). In the above case this reduces to

$$\int_0^\infty |y|^2 \, dx < \infty ,$$

where $y(x, \lambda)$ is the $n \times 1$ vector in the solution $\begin{bmatrix} y \\ y' \end{bmatrix}$ of (2.1) .

As there are at least n such solutions, a standard means of showing limit pointness is to show that if $\begin{bmatrix} y \\ y' \end{bmatrix}$ is an $L^2(0, \infty)$ solution of the system (2.1)-(2.5) then $\int_0^\infty |y'|^2 < \infty$. This being the case, it would follow, by the Cauchy-Schwarz inequality, that the bilinear form

$$[y, z] = z'^*(x)\, y(x) - z^*(x)\, y'(x)$$

is a function of integrable square. However if there are more than n independent $L^2(0, \infty)$ then there is a pair $y(x)$ and $z(x)$ for which $[y, z] \equiv 1$, with a resulting contradiction.

Lemma 2.1: If $\begin{bmatrix} y \\ y' \end{bmatrix}$ is an $L^2(0, \infty)$ solution of (2.1)-(2.5), then $\int_0^\infty |y'|^2 < \infty$.

Proof: To show this, we shall make use of a result that appears in [11] which states:

"Let F be a nonnegative continuous function on $[a, \infty)$ and define $H(t) = \int_a^t (t - s)^n F(s)\, ds$. If as $t \to \infty$, $H(t) = O(t^n [H^{(n)}]^\alpha)$, where $\alpha = \frac{2n - 1}{2n}$ then $\int_a^t F(s)\, ds = O(1)$ as $t \to \infty$".

We note that, as $t \to \infty$, integration by parts yields

$$\int_0^t [y^*(s)\, y'(s)]' \, (1 - \tfrac{s}{t})\, ds = O(1) + \frac{1}{t}\int_0^t y^* y' . \qquad (2.6)$$

And so,

$$\left|\int_0^t [y^*(s)\ y'(s)]'\ (1 - \tfrac{s}{t})ds\right| \le$$

$$\le 0(1) + \frac{1}{t}\left[\int_0^t |y|^2\right]^{1/2} \left[\int_0^t |y'|^2\right]^{1/2}$$

From this we observe that as $t \to \infty$,

$$\int_0^t (y^*\ y')'\ (1 - \tfrac{s}{t})\ ds = 0\left(\left[\int_0^t |y'|^2\right]^{1/2}\right) . \tag{2.7}$$

Next, consider that

$$\int_0^t (y^*\ y')'\ (1 - \tfrac{s}{t})ds =$$

$$= \int_0^t |y'|^2\ (1 - \tfrac{s}{t})\ ds + \int_0^t y^*\ y''\ (1 - \tfrac{s}{t})ds$$

$$= \int_0^t |y'|^2\ (1 - \tfrac{s}{t})\ ds - \int_0^t [y^*(\lambda + V)y](1 - \tfrac{s}{t})ds .$$

That is, as $t \to \infty$,

$$\int_0^t |y'|^2\ (1 - \tfrac{s}{t})ds =$$

$$= \int_0^t (y^*\ y')'\ (1 - \tfrac{s}{t})\ ds + \int_0^t [y^*(\lambda + V)y](1 - \tfrac{s}{t})ds$$

$$= 0\ \left(\left[\int_0^t |y'|^2\right]^{1/2}\right) + \int_0^t [y^*(\lambda + V)y]\ (1 - \tfrac{s}{t})\ ds , \tag{2.8}$$

where we have used (2.7) in the last equality.

We next wish to show that, as $t \to \infty$,

$$\int_0^t [y^*(\lambda + V)y] (1 - \frac{s}{t}) ds = O([\int_0^t |y'|^2]^{1/2}) .$$

We begin by noting that because $y \in L^2[0, \infty)$,

$$\int_0^t [y^*(\lambda + V)y] (1 - \frac{s}{t}) ds =$$

$$= O(1) + \int_0^t [y^* Vy] (1 - \frac{s}{t}) ds \tag{2.9}$$

as $t \to \infty$.

In the following, the Landau notation, $O(\cdot)$, is given with respect to behavior as $t \to \infty$.

Now consider

$$\int_0^t y^* Vy(1 - \frac{s}{t}) ds =$$

$$= \int_0^t y^*(v_1 + v_2 + v_3) y(1 - \frac{s}{t}) ds .$$

First,

$$|\int_0^t y^* v_1 y(1 - \frac{s}{t})ds| \leq K \int_0^t |y|^2 ds = O(1)$$

since $(1 - \frac{s}{t})$ is bounded, and by assumption, $\lim_{x\to\infty} v_1(x) = 0$.

Second,

$$\int_0^t y^* v_2 y(1 - \frac{s}{t}) ds = -\int_0^t y^* V_2' y(1 - \frac{s}{t}) ds$$

by (2.5) . And

$$\int_0^t y^* V_2' y(1 - \frac{s}{t})\, ds =$$

$$y^* V_2 y(1 - \frac{s}{t})|_0^t - \int_0^t y'^* V_2 y(1 - \frac{s}{t})\, ds$$

$$- \int_0^t y^* V_2 y'(1 - \frac{s}{t})\, ds + \frac{1}{t}\int_0^t y^* V_2 y\, ds \ .$$

By (2.5), the operator norm $\|V_2(x)\|$ is bounded, and so,

$$|\frac{1}{t}\int_0^t y^* V_2 y\, ds| \le \frac{K}{t}\int_0^t |y|^2\, ds = O(1) \quad .$$

As a consequence,

$$|\int_0^t y'^* V_2 y\ (1 - \frac{s}{t})\, ds| \le K \int_0^t |y'|\ |y|\ (1 - \frac{s}{t})\, ds$$

$$\le K \int_0^t |y'|\ |y|\ ds = O([\int_0^t |y'|^2]^{1/2}) \quad ,$$

and thus,

$$|\int_0^t y^* v_2 y\ (1 - \frac{s}{t})\, ds| = O([\int_0^t |y'|^2]^{1/2}) \ .$$

Third,

$$|\int_0^t y^* v_3\, y(1 - \frac{s}{t}) ds| = O([\int_0^t |y'|^2]^{1/2})$$

by an argument analogous to the second point above where we first define $V_3(x) = \int_0^x v_3(s)\, ds$. By these three points we see that

$$\left|\int_0^t y^* V y(1 - \tfrac{s}{t})\, ds\right| = O([\int_0^t |y'|^2]^{1/2})$$

and by (2.8)-(2.9) that as $t \to \infty$,

$$\int_0^t |y'|^2 (1 - \tfrac{s}{t})\, ds = O([\int_0^t |y'|^2]^{1/2}) .$$

As a consequence, as $t \to \infty$,

$$\int_0^t |y'|^2 (t - s)\, ds = O(t[\int_0^t |y'|^2]^{1/2}) \qquad (2.10)$$

Thus by the result in [11], (2.10) implies that $y'(x, \infty) \in L^2[0, \infty)$. This, as already explained, leads to a contradiction if we assume that there are more than n independent integrable square solutions of (2.1)-(2.5) . Hence (2.1)-(2.5) is limit point at infinity. ∎

Theorem 2.1: For the singular boundary value problem (S), constructed from (2.1)-(2.5), $\rho(\lambda)$ is continuously differentiable for $0 < \lambda < \infty$.

Proof: Following Atkinson [3] , we define the functional

$$E(x, \lambda) = \Phi^*(x, \lambda)\, [\lambda^{1/2} I_n]\, \Phi(x, \lambda)$$
$$+ \Phi'^*(x, \lambda)\, [\lambda^{-1/2} I_n]\, \Phi'(x, \lambda) . \qquad (2.11)$$

Our goal is to show that $E(x, \lambda)$ converges to a continuous, positive definite matrix valued function as $x \to \infty$ for $\lambda > 0$. It was shown in Lemma 2.1 that the system (2.1)-(2.5) is limit point. If we can show the above convergence, then we may invoke Theorem 1.2 of Chapter 1 with $K_n(\lambda) = \lambda^{-1/2} I_n$. Because the limit function is continuous, it follows that $\rho'(\lambda) = \lim_{x\to\infty} [\pi\, E(x, \lambda)]^{-1}$ for all $\lambda > 0$.

Instead of $E(x, \lambda)$, we shall consider the following functionals:

$$E_0(x, \lambda) = \lambda\Phi^*(x, \lambda)\, \Phi(x, \lambda) + \Phi^{*\prime}(x, \lambda)\, \Phi'(x, \lambda) \qquad (2.12)$$

and

$$E_1(x, \lambda) = \Phi^*(x, \lambda)\, (\lambda\, I_n + v_1(x))\, \Phi(x, \lambda) +$$
$$[\Phi'(x, \lambda) - V_2(x)\Phi(x, \lambda)]^*\, [\Phi'(x, \lambda) - V_2(x)\Phi(x, \lambda)]. \qquad (2.13)$$

Should $E_0(x, \lambda)$ converge to a positive definite continuous matrix valued function as $x \to \infty$, then so too will $E(x, \lambda)$. It is less clear that should $E_1(x, \lambda)$ exhibits this same convergence, E_0 and hence E also converge to continuous matrix valued functions as $x \to \infty$. However, since $v_1(x) \to 0$ as $x \to \infty$, it follows that for x sufficiently large,

$$|u^*\, v_1(x)\, u| \leq \|v_1(x)\| < \varepsilon$$

where $|u| = 1$ and $\|\cdot\|$ is the matrix operator norm. Thus for x

sufficiently large,

$$0 < (\lambda - \varepsilon)I_n < \lambda\ I_n + v_1(x) < (\lambda + \varepsilon\ )I_n\ . \tag{2.14}$$

From this it is clear that the elements of $\Phi^*(x, \lambda)\ \Phi(x, \lambda)$, of $\Phi(x, \lambda)$, and hence of $\Phi'(x, \lambda)$ are all bounded, for fixed λ and for x sufficiently large, when $E_1(x, \lambda)$ has a limit as $x_1 \to \infty$. It follows that

$$\|E_1(x, \lambda) - E_0(x, \lambda)\| =$$

$$\|\Phi^* v_1\ \Phi + \Phi^*\ V_2\ \Phi' + \Phi'^*\ V_2\Phi\ + \Phi^*\ V_2^2\ \Phi\|$$

will converge to 0 as $x \to \infty$ for fixed $\lambda > 0$ because $\|v_1\|$ and $\|V_2\|$ both converge to 0 as $x \to \infty$.

We now examine $[E_1(x, \lambda)]_{ij}$. First we consider the case $i = j$. To do so, it is sufficient to consider $e_i\ E(x, \lambda)\ e_i$. However, in order to consider the case $i \neq j$, we examine $u^*\ E_1(x, \lambda)u$, where u is an arbitrary $n \times 1$ vector.

Now if $\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} \Phi\ u \\ \Phi' u \end{bmatrix}$, then

$$u^*\ E_1(x, \lambda)u = y^*(x, \lambda)(\lambda I + v_1(v))\ y(x, \lambda) +$$

$$|y'(x, \lambda) - V_2(x)\ y(x, \lambda)|^2\ . \tag{2.15}$$

Let

$$q(x, \lambda) = u^* E_1(x, \lambda) u .$$

We wish to examine the integrability of $\frac{q'(x, \lambda)}{q(x, \lambda)}$.

By differentiating $q(x, \lambda)$ with respect to x we find that

$$q'(x, \lambda) = y^*(2\lambda V_2 + v_1 V_2 + V_2 v_1 + v_3 V_2 + V_2 v_3 + v_1)y +$$

$$+ y^* (-v_3 + V_2^2)y' + y'^*(-v_3 + V_2^2)y - 2y'^* V_2 y'$$

$$= y^*(2\lambda V_2 + v_1 V_2 + V_2 v_1 + v_1')y - y^* v_3(y' - V_2 y)$$

$$-(y' - V_2 y)^* v_3 y + [y^* V_2^2 y' - y'^* V_2 y'] . \qquad (2.16)$$

Now observe that

$$y'^* V_2^2 y - y'^* V_2 y' = (V_2 y)^* V_2(y' - V_2 y) - y'^* V_2(y' - V_2 y)$$

$$- (V_2 y)^* V_2(y' - V_2 y)$$

and so,

$$y'^* V_2^2 y - y'^* V_2 y' =$$

$$= (y' - V_2 y)^* V_2(y' - V_2 y) + y^* V_2^2(y' - V_2 y) .$$

Similarly,

$$y^* V_2^2 y' - y'^* V_2 y' =$$

$$= (y' - V_2 y)^* V_2 (y' - V_2 y) + (y' - V_2 y)^* V_2^2 y .$$

Thus (2.16) becomes,

$$q'(x, \lambda) = y^*(2\lambda V_2 + v_1 V_2 + V_2 v_1 + v_1')y - y^* v_3 (y' - V_2 y)$$

$$- (y' - V_2 y)^* v_3 y + 2(y' - V_2 y)^* V_2 (y' - V_2 y) +$$

$$+ (y' - V_2 y)^* V_2^2 y + y^* V_2^2 (y' - V_2 y) .$$

By (2.14)-(2.15), we note that for $0 < \varepsilon < \lambda$ and for x sufficiently large

$$q(x, \lambda) > (\lambda - \varepsilon) |y|^2 + |y' - V_2 y|^2 > 0$$

and

$$\left|\frac{q'(x, \lambda)}{q(x, \lambda)}\right| \le \frac{|q'(x, \lambda)|}{(\lambda - \varepsilon) |y|^2 + |y' - V_2 y|^2} .$$

Now we see that

$$\frac{|y^*(2\lambda V_2 + v_1 V_2 + V_2 v_1 + v_1')y|}{q(x, \lambda)} \le$$

$$\frac{|y^*(2\lambda V_2 + v_1 V_1 + V_2 v_1 + v_1')y|}{(\lambda - \varepsilon) |y|^2 + |y' - V_2 y|^2} \le$$

$$\leq (\lambda - \varepsilon)^{-1} \{2\lambda \|V_2\| + \|v_1\| \|V_2\| + \|v_1'\|\}$$

where $\|\cdot\|$ is the operator norm for our matrices.

Similarly,

$$\frac{|2(y' - V_2y)^* V_2(y' - V_2y)|}{q(x, \lambda)} \leq 2 \|V_2\| \quad ;$$

and

$$\frac{|(y' - V_2y)^* V_2^2 y + y^* V_2^2(y' - V_2y)|}{q(x, \lambda)} \leq$$

$$\leq \frac{2|y| |y' - V_2y| \|V_2\|^2}{q(x, \lambda)} \quad .$$

Now

$$2|y| |y' - V_2y| \leq (\lambda - \varepsilon)^{1/2} |y|^2 + (\lambda - \varepsilon)^{-1/2} |y' - V_2y|^2 \ .$$

It follows that for $0 < \varepsilon < \lambda$ and x sufficiently large, that

$$\frac{2|y| |y' - V_2y| \|V_2\|^2}{q(x, \lambda)} \leq (\lambda - \varepsilon)^{-1/2} \|V_2\|^2 \ .$$

Lastly, note that

$$\frac{|y^* v_3(y' - V_2 y)|}{q(x, \lambda)} \leq \frac{|y|\ |y' - V_2 y|\ \|v_3\|}{q(x, \lambda)}$$

$$\leq \frac{1}{2} (\lambda - \varepsilon)^{-1/2}\ \|v_3\| .$$

We can now see that

$$\left|\frac{q'(x, \lambda)}{q(x, \lambda)}\right| \leq (\lambda - \varepsilon)^{-1} \{2\lambda \|V_2\| + \|v_1\| \|V_2\| + \|v_1'\|\} +$$

$$+ (\lambda - \varepsilon)^{-1/2} \{\|v_3\| + \|V_2\|^2\} + 2 \|V_2\| .$$

From the hypothesis, we note that for sufficiently large values of x , $\|V_2\|$ and $\|v_1\|$ are bounded; also note that $\|V_2\|$, $\|v_3\|$ and $\|v_1'\|$ are $L(0, \infty)$. It follows for $\lambda > \varepsilon$ that $\int_0^\infty \frac{q'(x, \lambda)}{q(x, \lambda)}\, dx$ exists. As ε is arbitrarily chosen, this integral exists for $\lambda > 0$. Thus $\lim\limits_{x\to\infty} \ln[q(x, \lambda)]$ exists and hence $\lim\limits_{x\to\infty} q(x, \lambda)$ exists and is positive for $\lambda > 0$.

Recalling our argument to this point, we chose $\varepsilon > 0$, then observed that, for x sufficiently large and for $\lambda > \varepsilon$, the above inequality for $\frac{q'}{q}$ followed. From this inequality we see that

$$\lim_{x\to\infty} \int_x^\infty \frac{q'(s, \lambda)}{q(s, \lambda)}\, ds = 0$$

uniformly for $\lambda \geq \delta > \varepsilon$. For this same interval, it follows that $\frac{q(x, \lambda)}{q(0, \lambda)}$ converges uniformly to $\frac{q(\infty, \lambda)}{q(0, \lambda)}$ as $x \to \infty$. Now from the definition of $q(x, \lambda)$ and (2.12), we see that the continuous dependence of $y(x, \lambda)$ and $y'(x, \lambda)$ on λ implies the same continuous dependence upon λ of $q(x, \lambda)$. As $q(0, \lambda)$ is continuous and positive as a function of λ , we see that the ratio $\frac{q(x, \lambda)}{q(0, \lambda)}$ is continuous as a function of λ and converges as $x \to \infty$ uniformly for $\lambda > 0$ in a compact set, to $\frac{q(\infty, \lambda)}{q(0, \lambda)}$. The ratio $\frac{q(\infty, \lambda)}{q(0, \lambda)}$ is thus a positive continuous function of λ for $\lambda > 0$. And as a consequence, $q(\infty, \lambda)$ is also a positive continuous function for $\lambda > 0$.

If in (2.12) we take u to be the unit $n \times 1$ vector e_i then, by what we have just shown, $[E_1(x, \lambda)]_{ii}$ converges as $x \to \infty$ to a positive continuous function for $\lambda > 0$. This leaves us with the question of $[E_1(x, \lambda)]_{jk}$ when $j \neq k$. But in the definition of $q(x, \lambda)$ we notice that q is the quadratic form constructed from the $n \times n$ hermitian matrix $E_1(x, \lambda)$. By the polarization identity

$$
\begin{aligned}
[E_1(x, \lambda)]_{jk} &= e_j^* E_1(x, \lambda) e_k \\
&= \tfrac{1}{4}\{(e_j + e_k)^* E_1(x, \lambda)(e_j + e_k) - (e_j - e_k)^* E_1(x, \lambda)(e_j - e_k) \\
&\qquad + i(e_j + ie_k)^* E_1(x, \lambda)(e_j + ie_k) \\
&\qquad - i(e_j - ie_k)^* E_1(x, \lambda)(e_j - ie_k)\}
\end{aligned}
$$

and the preceeding discussion shows that $[E_1(x, \lambda)]_{jk}$ converges as $x \to \infty$ to a continuous function for $\lambda > 0$.

We have now shown that $\lim_{x\to\infty} E_1(x, \lambda)$ exists and is a matrix of continuous functions, for $\lambda > 0$. When we showed that for all

$\lambda > 0$, $\lim_{x\to\infty} u^* E_1(x, \lambda) u > 0$ for every $n \times 1$ vector u, we also demonstrated that $\lim_{x\to\infty} E_1(x, \lambda)$ is positive definite for $\lambda > 0$.

When $\Phi(x, \lambda)$ and $\Phi'(x, \lambda)$ are chosen to be $n \times n$ blocks of the fundamental matrix $Y(x, \lambda)$ described in (1.9), then what we have shown, together with Theorem 1.2, gives us the conclusion we seek. ■

And so for the singular self adjoint operator described in the introduction formed from the system (2.1)-(2.5) the spectrum is not only absolutely continuous but continuously differentiable in the interval $(0, \infty)$. We next consider the nature of the spectrum in the interval $(-\infty, 0)$.

Lemma 2.2: The spectrum is discrete in the interval $(-\infty, 0)$ for the self adjoint operator described in the introduction and based upon the system described in (2.1)-(2.5).

Proof: Let $Ly = -y'' + (v_1 + v_2 + v_3)y$ where $v_i(x)$ is defined in (2.2)-(2.5) and where $y(x, \lambda)$ is an $n \times 1$ vector.

As shown in [12] it is sufficient that there be a number $b > 0$ for every $0 < \varepsilon < 1$, such that for every $y(x, \lambda)$, with compact support in (b, ∞) where y and y' are absolutely continuous and $y'' \in L^2(b, \infty)$, the following holds:

$$\int_b^\infty y^* Ly < -\varepsilon \int_b^\infty |y|^2 .$$

The last statement is equivalent to

$$-\int_b^\infty y^* y'' + \varepsilon\int_b^\infty |y|^2 + \int_b^\infty y^*(v_1 + v_2 + v_3)y > 0 \ . \qquad (2.17)$$

First, since y has compact support in (b, ∞)

$$-\int_b^\infty y^* y'' = \int_b^\infty |y'|^2 = \|y'\|^2 \ .$$

Next, note that as $\lim_{x\to\infty} v_1(x) = 0$ that given ε_1 there is a b such that for $x > b$, $\|v_1(x)\| < \varepsilon$, where $\|v_1\|$ here stands for the matrix operator norm. So,

$$\left|\int_b^\infty y^* v_1 y\right| \leq \int_b^\infty |y| \ \|v_1\| \ |y|$$

$$\leq \varepsilon_1 \|y\|^2 \ .$$

Also we find that

$$\int_b^\infty y^* v_2 y = -\int_b^\infty [y^* V_2 y]' + \int_b^\infty y'^* V_2 y + \int_b^\infty y^* V_2 y' \ .$$

By the compactness of support for y,

$$\int_b^\infty [y^* V_2 y]' = 0$$

and so,

$$\left|\int_b^\infty y^* v_2 y\right| \leq 2\left|\int_b^\infty y'^* V_2 y\right|$$

$$\leq 2 \int_b^\infty |y'| \ \|V_2\| \ |y| \ .$$

Now since $V_2(x) = \int_x^\infty v_2 \, dx$ exists, it follows that $\lim_{x\to\infty} V_2(x) = 0$ and that for k sufficiently large, $\|V_2(x)\| < \varepsilon_2$ for a given $\varepsilon_2 > 0$. Thus,

$$|\int_b^\infty y^* v_2 y| \leq 2 \varepsilon_2 \int_b^\infty |y'| \; |y|$$

$$\leq 2\varepsilon_2[\|y'\|^2 + \|y\|^2] .$$

Since $v_3 \in L\,(b, \infty)$, if we define

$$V_3(x) = -\int_x^\infty v_3 ,$$

it follows that $\lim_{x\to\infty} V_3(x) = 0$ and that given $\varepsilon_3 > 0$ then for b sufficiently large

$$|\int_b^\infty y^* v_3 y| \leq 2 \varepsilon_3 \int_b^\infty |y'| \; |y|$$

$$\leq 2\varepsilon_3[\|y'\|^2 + \|y\|^2] .$$

From these observations we note that given a choice for ε_1, ε_2, ε_3, there is a b sufficiently large, that independently of our choice of $y(x, \lambda)$ satisfying our initial requirements,

$$-\int_b^\infty y^* y'' + \varepsilon \int_b^\infty |y|^2 + \int_b^\infty y^*(v_1 + v_2 + v_3)y \geq$$

$$\geq \|y'\|^2 + \varepsilon \|y\|^2 - \varepsilon_1 \|y\|^2 - 2[\varepsilon_2 + \varepsilon_3]\{ \|y'\|^2 + \|y\|^2$$

$$\geq \|y'\|^2 \{1 - 2(\varepsilon_2 + \varepsilon_3)\} + \|y\|^2 \{\varepsilon - \varepsilon_1 - 2(\varepsilon_2 + \varepsilon_3)\} .$$

So if we choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon/8$ then

$$1 - 2(\varepsilon_2 + \varepsilon_3) > 0$$

and

$$\varepsilon - \varepsilon_1 - 2(\varepsilon_2 + \varepsilon_3) > 0 \ .$$

Thus (2.17) holds, and as we wished

$$\int_b^\infty y^* Ly > -\varepsilon \int_b^\infty |y|^2$$

for those $y(x, \lambda)$ with compact support in (b, ∞) and for which y, y' are absolutely continuous and $y'' \in L^2(0, \infty)$. ■

We next show, analogous to the case $n = 1$, that for the dominant terms of $\rho'(\lambda)$ we can find some information concerning the asymptotic nature of these terms as $\lambda \to \infty$.

Lemma 2.3: For the singular boundary value problem (S) based upon (2.1)-(2.5)

$$\begin{aligned} [\rho'(\lambda)]_{ii} &= \\ &= [\lambda^{1/2}\Phi^*(0, \lambda)\Phi(0, \lambda) \\ &\quad + \lambda^{-1/2}\ \Phi'^*(0, \lambda)\ \Phi'(0, \lambda)]_{ii}\,(1 + o(1)) \end{aligned} \tag{2.18}$$

as $\lambda \to \infty$.

Proof: Let $E(x, \lambda)$, $E_0(x, \lambda)$ and $E_1(x, \lambda)$ be defined as in (2.11)-(2.13) . And as before, let $E(\infty, \lambda)$, $E_0(\infty, \lambda)$, and $E_1(\infty, \lambda)$ denote the limits of $E(x, \lambda)$ and $E_i(x, \lambda)$ as $x \to \infty$.

By Theorem 2 we know that $\rho'(\lambda) = E(\infty, \lambda) = \lambda^{1/2} E_0(\infty, \lambda)$. We wish to show that $\dfrac{[\rho'(\lambda)]_{ii}}{[E(0, \lambda)]_{ii}} \to 1$ as $\lambda \to \infty$. Since

$$\frac{[\rho'(\lambda)]_{ii}}{[E(0, \lambda)]_{ii}} = \frac{[E(\infty, \lambda)]_{ii}}{[E(0, \lambda)]_{ii}} = \frac{\lambda^{1/2}[E_0(\infty, \lambda)]_{ii}}{\lambda^{1/2}[E_0(0, \lambda)]_{ii}} \tag{2.19}$$

we consider whether

$$\frac{[E_0(\infty, \lambda)]_{ii}}{[E_0(0, \lambda)]_{ii}} \to 1 \quad \text{as} \quad \lambda \to \infty .$$

To do this we show that in general

$$\frac{u^* E_0(\infty, \lambda)\, u}{u^* E_0(0, \lambda)\, u} \to 1 \quad \text{as} \quad \lambda \to \infty$$

for any $n \times 1$ complex vector u .

We proceed as Atkinson in the case $n = 1$ to consider that

$$\frac{u^* E_0(\infty, \lambda)\, u}{u^* E_0(0, \lambda)\, u} =$$

$$= \frac{u^* E_0(x, \lambda)\, u}{u^* E_0(0, \lambda)\, u} \; \frac{u^* E_1(x, \lambda)\, u}{u^* E_0(x, \lambda)\, u} \; \frac{u^* E_0(\infty, \lambda)\, u}{u^* E_1(x, \lambda)\, u} \; . \tag{2.20}$$

Note that it was shown in Theorem 2 that

$$u^* E_0(\infty, \lambda)u = u^* E_1(\infty, \lambda) u ,$$

so the last factor in the above product is

$$\frac{u^* E_1(\infty, \lambda) u}{u^* E_1(x, \lambda) u} . \tag{2.21}$$

In Theorem 2 we saw that

$$\int_x^\infty \frac{[u^* E_1(s, \lambda) u]'}{[u^* E_1(s, \lambda) u]} ds$$

converged to 0 uniformly for $\lambda \geq \delta > 0$ as $x \to \infty$. Thus the ratio in (2.21) converges uniformly to 1 for $\lambda \geq \delta > 1$ as $x \to \infty$.

Next let $y = y(x, \lambda) = \Phi(x, \lambda) u$. Then we note that

$$\begin{aligned}[u^* E_0(x, \lambda) u]' &= \lambda y'^* y + \lambda y' y^* - y^*(\lambda + v_1 + v_2 + v_3)y' \\ &\qquad - y'^*(\lambda + v_1 + v_2 + v_3)y \\ &= -y^*(v_1 + v_2 + v_3)y' - y'^*(v_1 + v_2 + v_3)y .\end{aligned}$$

From this we observe that

$$\left|\frac{[u^* E_0(x, \lambda) u]'}{[u^* E_0(x, \lambda) u]}\right| \le \frac{2|y| |y'|(\|v_1\| + \|v_2\| + \|v_3\|)}{\lambda|y|^2 + |y'|^2}$$

$$\le \lambda^{-1/2}(\|v_1\| + \|v_2\| + \|v_3\|)$$

where $\|\cdot\|$ stands for the operator norm for matrices.

And thus,

$$\ln \left|\frac{u^* E_0(x, \lambda) u}{u^* E_0(0, \lambda) u}\right| \le \int_0^x \left|\frac{[u^* E_0(s, \lambda) u]'}{[u^* E_0(s, \lambda) u]}\right| ds$$

$$\le \lambda^{-1/2} \int_0^x \|v_1(s)\| + \|v_2(s)\| + \|v_3(s)\| \; ds \; .$$

For fixed x , we see that

$$\frac{u^* E_0(x, \lambda) u}{u^* E_0(0, \lambda) u} \to 1 \quad \text{as} \quad \lambda \to \infty \; . \tag{2.22}$$

This leaves the ratio $\dfrac{u^* E_1(x, \lambda) u}{u^* E_0(x, \lambda) u}$ to consider. As in Atkinson, we introduce the intermediate functional

$$E_2(x, \lambda) = \lambda\Phi^*(x, \lambda)\, \Phi(x, \lambda) +$$

$$(\Phi'(x, \lambda) - V_2(x)\, \Phi(x, \lambda))^*(\Phi'(x, \lambda) - V_2(x)\, \Phi(x, \lambda)) \; .$$

In Theorem 1, for fixed λ , the convergence, as $x \to \infty$, of $E_1(x, \lambda)$ gave the boundedness of $\|\Phi(x, \lambda)\|$ and of $\|\Phi'(x, \lambda)\|$. At that time it was argued that $\|E_1(x, \lambda) - E_0(x, \lambda)\| \to 0$, for fixed λ , as $x \to \infty$. Similarly,

$$||E_2(x, \lambda) - E_1(x, \lambda)|| \to 0 \quad ,$$

for fixed λ , as $x \to \infty$.

With the introduction of $E_2(x, \lambda)$ we now consider

$$\frac{u^* E_1(x, \lambda)u}{u^* E_0(x, \lambda)u} = \frac{u^* E_2(x, \lambda)u}{u^* E_0(x, \lambda)u} \cdot \frac{u^* E_1(x, \lambda)u}{u^* E_2(x, \lambda)u} \ . \tag{2.23}$$

Note that,

$$\frac{u^* E_2(x, \lambda)\, u}{u^* E_0(x, \lambda)\, u} = \frac{\lambda|y|^2 + |y' - V_2 y|^2}{\lambda|y|^2 + |y'|^2}$$

$$= 1 - \frac{2y'^* V_2 y + y^* V_2^2 y}{\lambda|y|^2 + |y'|^2} \quad .$$

Note also that

$$\frac{2|y'^* V_2 y|}{\lambda|y|^2 + |y'|^2} \le \lambda^{-1/2} \; ||V_2|| \left[\frac{2|y| \; |y'|}{\lambda^{1/2} \; |y|^2 + \lambda^{-1/2} \; |y'|^2}\right]$$

$$\le \lambda^{-1/2} \; ||V_2|| \ .$$

Similarly,

$$\frac{|y^* V_2^2 y|}{\lambda|y|^2 + |y'|^2} \le \lambda^{-1} \; ||V_2||^2 \quad .$$

Thus for fixed x ,

$$\lim \frac{u^* E_2(x, \lambda) u}{u^* E_0(x, \lambda) u} = 1 . \tag{2.24}$$

For the remaining ratio we note that

$$\frac{u^* E_1(x, \lambda) u}{u^* E_2(x, \lambda) u} = \frac{y^*(\lambda + v_1)y + |y' - V_2 y|^2}{|y|^2 + |y' - V_2|^2}$$

$$= 1 + \frac{y^* v_1 y}{\lambda|y|^2 + |y' - V_2 y|^2}$$

and that

$$\frac{|y^* v_1 y|}{\lambda|y|^2 + |y' - V_2 y|^2} \leq \lambda^{-1} \|v_1\| .$$

For fixed x , it follows that

$$\lim_{\lambda\to\infty} \frac{u^* E_1(x, \lambda) u}{u^* E_2(x, \lambda) u} = 1 . \tag{2.25}$$

Now we can make (2.20) arbitrarily close to 1 by fixing x so that for all $\lambda \geq \delta > 0$ the ratio in (2.21) is close to 1 , then choose λ sufficiently large so that the ratios in (2.23)-(2.25) are close to 1 . From this (2.18) follows . ∎

Before continuing it should be noted that the equation (2.1) for $v_1(x) = v_2(x) \equiv 0$ and $\lambda \neq 0$ may be studied by asymptotic methods. This has been done by S. Itatsu and H. Kaneta [20] on $(-\infty, \infty)$ (instead of $[a, \infty)$). However these methods fail when a v_1-type term is included because of the existence of repeated eigenvalues in the n-dimensional system of constant coefficients $y'' = \lambda y$.

Now consider the following system:

$$y'' + (\lambda I + V(x))y = 0 \ , \ 0 \leq x < \infty \ , \tag{2.26}$$

where $y(x)$ is a 2×1 vector function, and

$$V(x) = \begin{bmatrix} x^{\alpha} & x^{\gamma} \\ x^{\gamma} & x^{\beta} \end{bmatrix} \tag{2.27}$$

where $\alpha, \beta > 0$. We have a simple example of a coupled second order system involving potentials which are smooth and unbounded as $x \to \infty$.

For a system of the form

$$y'' + P(x)y = \lambda y \ , \ a \leq x < \infty$$

where $P(x)$ is an $n \times n$ real symmetric matrix of locally integrable functions, Lidskii [22] has shown that such a system is limit point at infinity if

$$P(x) \leq Kx^2 \cdot I_n$$

where K is a positive constant. Thus (2.26)-(2.27) will be limit point at infinity if α, β and $\gamma \leq 2$.

<u>Theorem 2.2</u>: For the system described in (2.26)-(2.27) we assume that $0 < \alpha, \beta \leq 2$. If $\gamma < \min\{\frac{\alpha + 3\beta}{4}, \frac{\beta + 3\alpha}{4}\}$, then $\rho'(\lambda)$ is continuous and positive definite for all $\lambda \in \mathbb{R}$.

<u>Proof</u>: Since $0 < \alpha, \beta \leq 2$ it follows from our previous observations that (2.26)-(2.27) is limit point at infinity.

Now the basic functional we consider is

$$E_0 = E_0(x, \lambda) = \phi^*(x, \lambda)\, A^{-2}(x)\, \phi(x, \lambda) + \phi'^*(x, \lambda)\, A^2(x)\, \phi'(x, \lambda) \qquad (2.28)$$

where

$$A(x) = \begin{bmatrix} x^{-\alpha/4} & 0 \\ 0 & x^{-\beta/4} \end{bmatrix} .$$

E_0 is a 2×2 matrix valued function. Note that as a function of λ , $A^{-2}(x)$ is equicontinuous for all $x > 0$. Note also that as $x \to \infty$, $A^{-2}(x)$ is bounded below. Let I be an arbitrarily chosen, closed subset of $\mathbb{R}$. If we can show, as $x \to \infty$, that $E_0(x, \lambda)$

converges uniformly for $\lambda \in I$ to a continuous positive definite matrix valued function $E_\infty(\lambda)$, then by Theorem 1.2 $\rho'(\lambda)$ will be continuous and positive for all $\lambda \in \mathbb{R}$.

The proof follows the same sequence of steps as in Theorem 2.1. We show for all 2×1 complex vectors u, that $u^* E_0 u$ converges, as $x \to \infty$, uniformly for $\lambda \in I$ to a positive continuous function of λ. As seen in the preceeding case, this is sufficient to establish the desired convergence of $E_0(x, \lambda)$. However, to do this we consider a perturbation $E = E(x, \lambda)$ of the functional $E_0(x, \lambda)$ chosen so that as $x \to \infty$, the ratio $\frac{u^* Eu}{u^* E_0 u}$ converges to 1 uniformly for $\lambda \in I$.

The perturbation we consider here is

$$E = E(x, \lambda) = (A\phi' - A'\phi)^*(A\phi' - A'\phi)$$
$$+ \phi^*(A^{-2} + \lambda A^2)\phi - \phi^* BA^2 \phi \qquad (2.29)$$

Let $y = y(x, \lambda) = \phi(x, \lambda)u$ where u is a 2×1 complex vector. Define $Q_0 = Q_0(x, \lambda)$, $Q = Q(x, \lambda)$ to be

$$\begin{aligned} Q_0 &= u^* E_0 u \\ &= y^* A^{-2} y + y'^* A^2 y \\ &= |A^{-1} y|^2 + |Ay'|^2 \end{aligned}$$

and

$$Q = u^* \; Eu$$

$$= (Ay' - A'y)^* \; (Ay' - A'y) + y^*(A^{-2} + \; A^2)y - y^* \; BA^2 \; y$$

$$= Q_0 - 2y'^* \; AA'y + y^*(A')^2 \; y + \lambda y^* \; A^2 y - y^* \; BA^2 y \qquad (2.30)$$

Thus,

$$\frac{Q}{Q_0} = 1 - 2\,\frac{y'^* \; AA'y}{Q_0} - 2\,\frac{y^*(A')^2 y}{Q_0} + \frac{\lambda y^* \; A^2 y}{Q_0} - \frac{y^* \; BA^2 y}{Q_0} \qquad (2.31)$$

Consider the last four terms of (2.31). Note that

$$y'^* \; AA'y = y'^* A(A'A) \; A^{-1} y$$

and thus

$$|y'^* AA'y| \leq |Ay'| \; |A^{-1}y| \; \|A'A\|$$

Now,

$$A'A = \begin{bmatrix} (\alpha/4)x^{-\alpha/2-1} & 0 \\ & \\ 0 & (-\beta/4)x^{-\beta/2-1} \end{bmatrix} \qquad (2.32)$$

and so $\|A'A\| \to 0$ as $x \to \infty$, independent of the value of λ in I . From this we see that

$$\left|\frac{2y'^* AA' y}{Q_0}\right| \le 2\left|\frac{Ay' \quad A^{-1}y}{Q_0}\right| \; ||A'A|| \le 2 \; ||A'A||$$

Thus as $x \to \infty$, the above ratio tends to 0 independently of $\lambda \in I$.

For the second term we note that $y^*(A')^2 y = y^* A^{-1}[A'A]^2 A^{-1}y$, and thus that

$$\left|\frac{y^*(A')^2 y}{Q_0}\right| \le \frac{|A^{-1}y|^2}{Q_0} \; ||A'A||^2 \le ||A'A||^2 .$$

As $x \to \infty$, this ratio also tends to 0 independent of $\lambda \in I$.

Next, we note that $y^* A^2 y = y^* A^{-1}(A^4)A^{-1}y$ and that

$$\left|\frac{\lambda y^* A^2 y}{Q_0}\right| \le |\lambda| \frac{|A^{-1} \; y|^2}{Q_0} \; ||A||^4 \le |\lambda| \; ||A||^4 .$$

Given our definition of A we see that $||A||$ tends to 0 as $x \to \infty$, and hence that, as $x \to \infty$, this ratio tends to 0 uniformly for λ in I.

For the last term in (2.31) note that $y^* BA^2 y = y^* A^{-1}(ABA^3)A^{-1}y$, and hence that

$$\frac{|y^* BA^2 y|}{Q} \le \frac{|A^{-1}y|^2}{Q} \; ||ABA^3|| \le ||ABA^3|| \quad .$$

A computation shows that

$$ABA^3 = x^{\gamma}\begin{bmatrix} 0 & x^{-(\frac{\alpha+3\beta}{4})} \\ x^{-(\frac{\beta+3\alpha}{4})} & 0 \end{bmatrix}$$

By our hypotheses both $\gamma - (\frac{\alpha + 3\beta}{4})$ and $\gamma - (\frac{\beta + 3\alpha}{4})$ are less than 0 and so $||ABA^3|| \to 0$ as $x \to \infty$. And with this last observation we see by (2.31) that as $x \to \infty$, $\frac{Q}{Q_0}$ converges uniformly for $\lambda \in I$ to 1.

Our next step is to compute Q'.

Let $Q_1 = y'^*A^2y'$, $Q_2 = -2y'^* AA'y$, $Q_3 = y^*(A')^2y$, $Q_4 = y^*(A^{-2} + A^2)y$, and $Q_5 = y^* BA^2 y$. Then by (2.30)

$$Q' = \sum_{i=1}^{5} Q_i'$$

First,

$$\begin{aligned} Q_1' &= 2y''^* A^2y' + 2y'^* AA'y' \\ &= -2y^*[\lambda I + A^{-4} + B]A^2y' + 2y'^* AA' y' \\ &= -2\lambda y^* A^2y' - 2y^* A^{-2}y' - 2y^* BA^2y' + 2y'^*AA'y' . \end{aligned}$$

For the next term,

$$Q_2' = -2y''^* AA'y - 2y'^*(AA')'y - 2y'^* AA'y'$$

$$= 2\lambda y^* AA'y + 2y^* A^{-3} A'y + 2y^* BAA'y$$

$$- 2y^*(A')^2y' - 2y^* AA''y' - 2y'^* AA'y' .$$

Continuing,

$$Q_3' = 2y^*(A')^2y' + 2y^* A'A'' y ;$$

and

$$Q_4' = 2y^*(A^{-2} + A^2)y' - 2y^* A^{-3} A'y + 2\lambda y^* AA'y .$$

And lastly,

$$Q_5' = 2y^* BA^2y' + y^* D_x[BA^2] y .$$

As a consequence,

$$Q' = 4\lambda y^* AA'y + 2y^* BAA'y - 2y^* AA''y'$$

$$+ 2y^* A'A''y + y^* D_x[BA^2]y . \tag{2.33}$$

Now we consider the integrability of $\left|\frac{Q'}{Q}\right|$. Note first that

$$y^* AA'y = y^* A^{-1}(A^3A')A^{-1} y .$$

From this it follows that

$$|y^* AA'y| \le |A^{-1}y|^2 \ \|A\|^2 \|AA'\| \ .$$

By (2.32) we see that $\|AA'\|$ is $L(1, \infty)$. The definition of A implies that $\|A\|^2 \to 0$ as $x \to \infty$. We have shown that as $x \to \infty$, $\frac{Q_0}{Q}$ converges to 1 uniformly for $\lambda \in I$. It follows for x sufficiently large that a constant K exists, independent of $\lambda \in I$, such that

$$\frac{|4\lambda y^* AA'y|}{Q} \le 4|\lambda| \ K \frac{|A^{-1}y|^2}{Q_0} \ \|A\|^2 \ \|AA'\|$$

$$\le 4|\lambda| \ K \|A\|^2 \|AA'\| \ . \tag{2.34}$$

Thus by (2.33) we see that the first term in the ratio $\left|\frac{Q'}{Q}\right|$ is bounded, uniformly for $\lambda \in I$, by a function in $L(x_0, \infty)$ where x_0 is sufficiently large.

Similar demonstrations can be made for the remaining terms in the ratio $\left|\frac{Q'}{Q}\right|$. For the second term first note that

$$y^* BAA'y = y^* A^{-1}[ABA^2A']A^{-1} y \ ,$$

and hence for x sufficiently large that there is a constant K for which

$$\frac{|2y^* BAA'y|}{Q} \le K \|ABA^2A'\| \ . \tag{2.35}$$

Now

$$ABA^2 A' = x^{\gamma}\begin{bmatrix} 0 & -\beta/4\ x^{-\delta-1} \\ -\alpha/4\ x^{-\zeta-1} & 0 \end{bmatrix}$$

where

$$\delta = \frac{\alpha + 3\beta}{4} \quad \text{and} \quad \zeta = \frac{\beta + 3\alpha}{4} \ . \tag{2.36}$$

We see by the hypotheses that $\|ABA^2 A'\|$ is integrable for $x > 0$, and hence that $\frac{|2y^* BAA'y|}{Q}$ is bounded, uniformly for $\lambda \in I$, by an $L(x_0, \infty)$ function where x_0 is sufficiently large.

For the next two terms we first note that since α, β are assumed to be positive, it follows that $\|AA''\|$ is integrable for $x > 0$. As in the previous cases it follows that for x sufficiently large there are constants K and C such that

$$\frac{|2y^* AA'' y|}{Q} \leq K\,\|AA''\| \tag{2.37}$$

and

$$\frac{|2y^* A' A''y|}{Q} \leq K\,\|AA'\|\,\|AA''\| \tag{2.38}$$

Thus the ratios in (2.37) and (2.38) are bounded by $L(x_0, \infty)$ functions for x_0 sufficiently large.

For the last term we begin by noting that

$$y^* D_x[BA^2]y = y^* A^{-1}(AD_x[BA^2]\ A)A^{-1}y\ .$$

Next note that

$$A\ D_x[BA^2]A = x^{\gamma} \begin{bmatrix} 0 & -\beta/2\ x^{-\delta-1} \\ & \\ -\alpha/2\ x^{-\zeta-1} & 0 \end{bmatrix}$$

where δ and ζ are defined in (2.36) . Again, by our hypotheses we see that $\|A\ D_x[BA^2]A\|$ is integrable for $x > 0$. Thus, for x sufficiently large there is a constant K such that

$$\frac{|y^*\ D_x[BA^2]\ y|}{Q} \leq K\|A\ D_x[BA^2]\ A\|\ . \qquad (2.39)$$

The inequalities (2.34)-(2.35) and (2.37)-(2.39) imply that for x sufficiently large the ratio $|\frac{Q'}{Q}|$ is bounded, uniformly for $\lambda \in I$, by an integrable function of the variable x . As in Theorem 2.1 and as noted at the outset of this proof, this last observation is sufficient to conclude that as $x \to \infty$, $E_0(x, \infty)$ converges, uniformly for λ restricted to compact subsets of $\mathbb{R}$, to a continuous positive definite matrix valued function. As a consequence of Theorem 1.2, $\rho'(\lambda)$ is continuous and positive definite for $\lambda \in \mathbb{R}$. ■

Note that when $0 < \alpha = \beta \leq 2$ then the restriction on γ in Theorem 2.2 reduces to $\gamma < \alpha$.

CHAPTER 3

ABSOLUTE CONTINUITY OF $\rho(\lambda)$ FOR TWO EXAMPLES OF DIRAC SYSTEMS

Consider the following Dirac System

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} p & \lambda + c + p_2 \\ -\lambda + c + p_1 & -p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$0 \le x < \infty ; \qquad (3.1)$$

or as written in Hamiltonian form

$$J\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \left[\lambda I + \begin{pmatrix} -c - p_1 & p \\ p & c + p_2 \end{pmatrix} \right] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad 0 \le x < \infty ,$$

where $y_1(x, \lambda)$ and $y_2(x, \lambda)$ are scalar valued functions; λ is a complex parameter; $p(x)$, $p_1(x)$ and $p_2(x)$ are real valued and locally integrable.

A specific example of such a system on $0 < x < \infty$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} k/x & \lambda + z/x + c \\ -\lambda - z/x + c & -k/x \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (3.2)$$

where c and z are constants and k is an integer. For appropriate choices of these constants this is the radial wave equation in quantum mechanics for a particle in a field of potential $v(x) = z/x$. It was shown by Titchmarsh [25] that the spectrum of (3.2) is discrete in $(-c, c)$ and continuous outside of this interval.

In [18], Hinton and Shaw extend the work of Weidmann [26] on (3.1) . They consider a system in which

$$\begin{aligned}
&p_i(x) = p_{i1}(x) + p_{i2}(x) + p_{i3}(x) \quad ; \quad i = 1, 2 \\
&p(x) = \Delta_1(x) + \Delta_3(x) \\
&p_{i1}(x), \Delta_1(x) \to 0 \quad \text{as} \quad x \to \infty , \quad i = 1, 2 \\
&p_{i1}'(x) , \Delta_1'(x) , p_{i3}(x) , \Delta_3(x) \in L(0, \infty) , \ i = 1, 2 \\
&P_{k2}(x) = \int_x^\infty p_{k2}(s)\, ds \quad \text{exists conditionally and} \\
&P_{k2}(x) \in L(x_0, \infty) \qquad k = 1, 2 .
\end{aligned} \tag{3.3}$$

The functions p_{i3} , $\Delta_3(x)$ are called short range potentials while $p_{i1}(x)$, $\Delta_1(x)$ are called long range potentials. The functions $p_{i2}(x)$ are called oscillatory potentials and have as their model, functions like $x^\alpha \sin(x^\beta)$.

Consider (3.1) together with the boundary condition

$$\sin\beta\, y_1(0) + \cos\beta\, y_2(0) = 0 , \ \beta \in [0, \pi) .$$

Let $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ and let $L^2[0, \infty)$ be the set

$$\{y | \int_0^\infty (|y_1|^2 + |y_2|^2) < \infty\}.$$

The system (3.1) has only 1 independent solution in $L^2(0, \infty)$ for $\mathrm{Im}\,\lambda \neq 0$; that is, it is limit point at infinity. Now let $T(y)$ be defined by

$$T(y) = Jy' - \begin{bmatrix} -c - p_1 & p \\ p & c + p_2 \end{bmatrix} y \tag{3.4}$$

and let the domain of T be the set $\mathscr{D} \subset L^2[0, \infty)$ which consists of those functions $y(x)$ which satisfy the above boundary condition, are locally absolutely continuous and such that $T(y) \in L^2[0, \infty)$. The operator so defined is self adjoint with domain and range in $L^2[0, \infty)$ ([13], [15], [16]) .

Hinton and Shaw have shown in [18] that the assumptions in (3.3) together with the absolutely integrability on $(0, \infty)$ of

$$\int_x^\infty |P_{k2}(s)\ p_{j1}'(s)|ds\ , \quad \int_x^\infty |P_{k2}(s)\ \Delta_1'(s)|\ ds\ ,$$

$$p_{k2}(x)\ P_{j2}(x)\ ,\ p_{m2}(x) \int_x^\infty |P_{k2}(s)\ p_{j1}'(s)|ds\ , \tag{3.5}$$

$$p_{m2}(x) \int_x^\infty |P_{k2}(s)\ \Delta_1'(s)|\ ds\ ,$$

where $j, k, m = 1, 2$, are sufficient to guarantee that the spectrum of the operator $T(y)$ is continuously differentiable in the complement of $[-c, c]$ and discrete in $(-c, c)$. Thus one of the things they show is that, in the complement of $[-c, c]$, the density function $\rho(\mu)$ is continuously differentiable with positive derivative.

They show that the hypotheses of (3.3) and (3.5) are sufficient to derive asymptotics for the solutions of (3.1) . These asymptotics are then used to obtain information about the $M(\lambda)$ function and through it, information about the density function $\rho(\lambda)$ and hence the spectrum. However, as a consequence of the results of Chapter 1, it can be shown that the additional hypotheses in (3.5) can be reduced in number.

Theorem 3.1: Let $T(y)$ be the self adjoint operator defined by (3.4) and having the set H as its domain. If $p(x)$, $p_1(x)$, $p_2(x)$ satisfy the hypotheses given in (3.3) and if in addition we assume that both $p_{12}(x)\, P_{22}(x)$ and $p_{22}(x)\, P_{12}(x) \in L\,(x_0, \infty)$, then the spectrum of T will be continuously differentiable in the complement of $[-c, c]$. That is, $\rho(\mu)$ is continuously differentiable with positive derivative in the complement of $[-c, c]$.

Proof: As in Theorem 2.1 we shall want to find a functional $E_0(x, \lambda)$ and examine its behavior as $x \to \infty$. In this case $E_0(x, \lambda)$ will be scalar valued rather than matrix valued; and as in the case of Theorem 2.1 we shall want to show that there is some functional, asymptotic to $E_0(x, \infty)$ as $x \to \infty$ which possesses a logarithmic derivative that is integrable as a function of x. But what shall we choose for a basic functional $E(x, \lambda)$?

If we consider (3.1) with $p = p_1 = p_2 \equiv 0$ and set

$$E(x, \lambda) = k^{-1}\, y_1^2(x, \lambda) + k y_2^2(x, \lambda) \quad , \quad k = k(\lambda) > 0 \ ,$$

which for (3.1) is the basic form of $J(x, \lambda, \lambda)$ in Theorem 1.2, then differentiating with respect to x we find

$$E'(x, \lambda) = 2y_1y_2\{\frac{(\lambda + c)}{k} - k(\lambda - c)\} \quad .$$

If

$$\frac{\lambda + c}{k} + k(\lambda - c) = 0$$

then $E'(x, \lambda) \equiv 0$ and hence $E(x, \lambda)$ is constant as a function of x as $x \to \infty$ and we observe that $E(0, \lambda) = \lim_{x\to\infty} E(x, \lambda)$. Now if $k(\lambda)$ is positive, continuous and solves this equation, then for those values of λ in the domain of $k(\lambda)$, $E(0, \lambda)$ will be positive and continuous, and $\rho'(\lambda) = \{\pi\ E(0, \lambda)\}^{-1}$ by Theorem 1.2 .

For λ in the complement of $[-c, c]$, we see that

$$k(\lambda) = \sqrt{\frac{\lambda + c}{\lambda - c}}$$

is just such a solution.

We shall show that the functional

$$E_0(x, \lambda) = (\frac{\lambda - c}{\lambda + c})^{1/2}\ y_1^2(x, \lambda) + (\frac{\lambda + c}{\lambda - c})^{1/2}\ y_2^2(x, \lambda) \qquad (3.6)$$

converges uniformly for λ in a compact subset of the complement of $[-c, c]$ (i.e. $[-c, c]'$) , as $x \to \infty$, to a positive continuous function. Note that $E_0(x, \lambda)$ is of the form $J(x, \lambda, \lambda)$, from Chapter 1, where $K_n^{-1}(\lambda) = (\frac{\lambda - c}{\lambda + c})^{1/2}$. As $K_n^{-1}(\lambda)$ is independent of n , equicontinuity follows trivially, as does the boundedness below of $K_n^{-1}(\lambda)$ for any fixed λ in $[-c, c]$. Thus the hypotheses of Theorem 1.2 are satisfied and the uniform convergence described above is sufficient to conclude that $\rho(\mu)$ is C^1 in $[-c, c]'$.

Define the following functional,

$$E(x, \lambda) = (\lambda - c - p_{11})(y_1 + P_{22}y_2)^2$$
$$+ (\lambda + c + p_{21})(y_2 + P_{12}\, y_1)^2$$
$$+ 2\Delta_1(y_1 + P_{22}y_2)(y_2 + P_{12}\, y_1) \quad . \tag{3.7}$$

We shall show that for compact subsets of $\lambda > c$ this function converges uniformly as $x \to \infty$, to $f(\lambda)$ a positive continuous function of λ . For $\lambda < -c$ we would consider instead $-E(x, \lambda)$. So without loss of generality we limit our discussion from this point on to $\lambda > c$. It follows that uniform convergence of $E(x, \lambda)$ upon compact subsets of $\lambda > c$ to $f(\lambda)$ as $x \to \infty$ implies that $E_0(x, \lambda)$ will converge uniformly on compact subsets of $[-c, c]'$, to a positive continuous function $E_\infty(\lambda)$.

To see this for $\lambda > c$, and hence for $\lambda < -c$ as well, let

$$z_1 = (\lambda - c - p_{11})^{1/2}(y_1 + P_{22}\, y_2)$$

$$z_2 = (\lambda + c + p_{21})^{1/2}(y_2 + P_{12}\, y_1)$$

where we assume x sufficiently large so that the square roots are defined and real. Then, we see that

$$E(x, \lambda) = z_1^2 + z_2^2 + 2\Delta_1(\lambda - c - p_{11})^{-1/2}(\lambda + c + p_{21})^{-1/2} z_1 z_2 .$$

Note that as $x \to \infty$, $2\Delta_1(\lambda - c - p_{11})^{-1/2}(\lambda + c + p_{21})^{-1/2}$ converges to 0 uniformly on compact subsets of $\lambda > c$. Choose $\varepsilon > 0$ to be an arbitrary small number. One can show that for x sufficiently large,

$$(1 - \varepsilon)(z_1^2 + z_2^2) \leq E(x, \lambda) \leq (1 + \varepsilon)(z_1^2 + z_2^2) \tag{3.8}$$

when λ is restricted for a compact subset of $\lambda > c$. As a consequence, we see that

$$\begin{aligned} z_1^2 + z_2^2 = {} & (\lambda - c - p_{11})(y_1 - P_{22}\, y_2)^2 \\ & + (\lambda + c + p_{21})(y_2 - P_{12}\, y_1)^2 \end{aligned}$$

also converges to $f(\lambda)$ uniformly on compact subsets of $\lambda > c$.

Next note that,

$$\begin{aligned} z_1^2 + z_2^2 = {} & (\lambda - c)y_1^2 + (\lambda + c)y_2^2 \\ & + [(\lambda - c - p_{11})P_{22}^2 - p_{11}]y_1^2 \\ & + [(\lambda + c + p_{21})\, P_{12}^2 - p_{21}]\, y_2^2 \\ & + 2[(\lambda - c - p_{11})P_{22} + (\lambda + c + p_{21})P_{12}]y_1 y_2 \, . \end{aligned}$$

Let $w_1 = (\lambda - c)^{1/2} y_1$ and $w_2 = (\lambda + c)^{1/2} y_2$; then

$$z_1^2 + z_2^2 = [1 + \delta_1(x, \lambda)]w_1^2 + [1 + \delta_2(x, \lambda)]w_2^2$$
$$+ \delta_3(x, \lambda)\, w_1 w_2$$

where $\delta_i(x, \lambda)$, $i = 1, 2, 3$ converges to 0 uniformly on compact subsets of $\lambda > c$. One can then show for an arbitrary small number $\varepsilon > 0$, that when x is sufficiently large,

$$(1 - \varepsilon)(w_1^2 + w_2^2) < z_1^2 + z_2^2 < (1 + \varepsilon)(w_1^2 + w_2^2) \,. \tag{3.9}$$

As a result, $w_1^2 + w_2^2$ converges uniformly on compact subsets of $\lambda > c$ to $f(\lambda)$. We need now only note that

$$E_0(x, \lambda) = (\lambda - c)^{-1/2}(\lambda + c)^{-1/2}(w_1^2 + w_2^2)$$

to conclude that as $x \to \infty$, $E_0(x, \lambda)$ converges uniformly on compact subsets of $\lambda > c$ to $(\lambda - c)^{-1/2}(\lambda + c)^{-1/2} f(\lambda)$. A similar argument can be constructed for $\lambda < -c$.

Note also that the inequalities (3.8) and (3. 9) together imply that the ratios $\dfrac{E(x, \lambda)}{z_1^2 + z_2^2}$, and $\dfrac{z_1^2 + z_2^2}{w_1^2 + w_2^2}$ converge to 1 uniformly on compact subsets of $\lambda > c$ as $x \to \infty$ without regard to whether or not $E(x, \lambda)$ converges to $f(\lambda)$. As a consequence we have the uniformity of convergence to 1 of $\dfrac{E(x, \lambda)}{w_1^2 + w_2^2}$.

Define the following functions

$$Q_1(x, \lambda) = (\lambda - c - p_{11})(y_1 + P_{22}\, y_2)^2$$

$$Q_2(x, \lambda) = (\lambda + c + p_{21})(y_2 + P_{12}\, y_1)^2$$

$$Q_3(x, \lambda) = 2\Delta_1(y_1 + P_{22}\, y_2)(y_2 + P_{12}\, y_1) ;$$

then by (3. 7)

$$E(x, \lambda) = Q_1(x, \lambda) + Q_2(x, \lambda) + Q_3(x, \lambda) . \tag{3.10}$$

We shall want to consider $\frac{E'(x, \lambda)}{E(x, \lambda)}$; so we next consider separately $Q_i'(x, \lambda)$.

First, differentiating Q_1 with respect to x ,

$$\begin{aligned} Q_1'(x, \lambda) &= -p_{11}'(y_1 + P_{22}\, y_2)^2 \\ &\quad + 2(\lambda - c - p_{11})(y_1 + P_{22}\, y_2)(y_1' + P_{22}'\, y_2 + P_{22}\, y_2') . \end{aligned} \tag{3.11}$$

Now using (3.1), part of the last term in (3.11) can be written as

$$y_1' + P_{22}'\, y_2 + P_{22}\, y_2' = \mathcal{B}_{11}\, y_1 + \mathcal{B}_{12}\, y_2 \quad , \tag{3.12}$$

where

$$\mathcal{B}_{11} = p + P_{22}(-\lambda + c + p_1)$$

$$\mathcal{B}_{12} = \lambda + c + p_2 - p_{22} - pP_{22} \quad .$$

We see that

$$\mathscr{B}_{11} = \Delta_1 + \Delta_3 + P_{22}(-\lambda + c + p_{11}) + P_{22}\, p_{13} + P_{22}\, p_{12}\ .$$

By (3.3) and the additional hypothesis that $P_{22}(x)\, p_{12}(x) \in L(x_0, \infty)$ we see that

$$\mathscr{B}_{11} = \mathscr{B}_{11}(x, \lambda) = \Delta_1(x) + V_{11}(x, \lambda) \tag{3.13}$$

where $V_{11}(x, \lambda)$ is continuous as a function of λ and as a function of x is $L(x_0, \infty)$. Also note that

$$\mathscr{B}_{12} = \lambda + c + p_{21} + p_{23} - \Delta_1\, P_{22} - \Delta_3\, P_{22}\ .$$

By (3.3) and specifically noting that $P_{22}(x)$ and $\Delta_1(x) \to 0$ as $x \to \infty$ we see that

$$\mathscr{B}_{12} = \mathscr{B}_{12}(x, \lambda) = \lambda + c + p_{21} + V_{12}(x) \tag{3.14}$$

where $V_{12} \in L(x_0, \infty)$. By applying (3.13) and (3.14) we find that

$$\begin{aligned}
&2(\lambda - c - p_{11})(y_1 + P_{22}\, y_2)(y_1' + P_{22}'\, y_2 + P_{21}\, y_2') = \\
&\quad = 2(\lambda - c - p_{11})(y_1 + P_{22}\, y_2)(\mathscr{B}_{11}\, y_1 + \mathscr{B}_{12}\, y_2) \\
&\quad = 2(\lambda - c - p_{11})[(\Delta_1 + V_{11})y_1^2 + P_{22}(\lambda + c + p_{21} + V_{12})y_2^2 \\
&\qquad\qquad + (P_{22}(\Delta_1 + V_{11}) + \lambda + c + p_{21} + V_{12})y_1 y_2] \\
&\quad = 2(\lambda - c - p_{11})\Delta_1\, y_1^2 + 2(\lambda - c - p_{11})(\lambda + c + p_{21})y_1 y_2 \\
&\qquad\qquad + C_1(x, \lambda)\, y_1^2 + C_2(x, \lambda)\, y_2^2 + C_3(x, \lambda)\, y_1 y_2
\end{aligned}$$

where $C_1(x, \lambda)$, $C_2(x, \lambda)$, and $C_3(x, \lambda)$ are continuous as functions of λ , and $L(x, \infty)$ as a function of α by our hypotheses.

From this we now see that (3.11) becomes

$$
\begin{aligned}
Q_1'(x, \lambda) &= \\
&= -p_{11}'(y_1 + P_{22}\, y_2)^2 + C_1(x, \lambda)\, y_1^2 \\
&\qquad + C_2(x, \lambda)\, y_2^2 + C_3(x, \lambda)\, y_1 y_2 + 2\Delta_1(\lambda - c - p_{11}) y_1^2 \\
&\qquad + 2(\lambda + c + p_{21})(\lambda - c - p_{11}) y_1 y_2 \,. \qquad (3.15)
\end{aligned}
$$

Now differentiating Q_2 we see that,

$$
\begin{aligned}
Q_2'(x, \lambda) &= p_{21}'(y_2 + P_{12}\, y_1)^2 + \\
&\quad + 2(\lambda + c + p_{21})(y_2 + P_{12} y_1)(y_2' + P_{12}'\, y_1 + P_{12}\, y_1') \,. \qquad (3.16)
\end{aligned}
$$

As before we define $\mathscr{B}_{21}$ and $\mathscr{B}_{22}$ by

$$
y_2' + P_{12}'\, y_1 + P_{12}\, y_1' = \mathscr{B}_{21}\, y_1 + \mathscr{B}_{22}\, y_2 \quad , \qquad (3.17)
$$

where

$$
\begin{aligned}
\mathscr{B}_{21} = \mathscr{B}_{21}(x, \lambda) &= -\lambda + c + p_{11} + P_{12}(x)\, \Delta_1(x) + P_{12}(x)\Delta_3(x) \\
&= -\lambda + c + p_{11} + V_{21}(x) \ , \qquad (3.18)
\end{aligned}
$$

where $V_{21}(x) \in L(x_0, \infty)$, where

$$\mathcal{B}_{22} = \mathcal{B}_{22}(x, \lambda) = -\Delta_1 - \Delta_3 + (\lambda - c + p_{21})P_{12} + p_{23}P_{12} + p_{22}P_{12}$$

$$= -\Delta_1(x) + V_{22}(x, \lambda) \quad , \tag{3.19}$$

and where $V_{22}(x, \lambda)$ is a continuous function of λ and with respect to x is $L(x_0, \infty)$. As a result,

$$2(\lambda + c + p_{21})(y_2 + P_{12}\, y_1)(y_2' + P_{12}\, y_1 + P_{12}\, y_1')$$

$$= 2(\lambda + c + p_{21})(y_2 + P_{12}\, y_1)(\mathcal{B}_{21}\, y_1 + \mathcal{B}_{22}\, y_2)$$

$$= -2\Delta_1(\lambda + c + p_{21})y_2^2 + 2(\lambda + c + p_{21})(-\lambda + c + p_{11})y_1y_2$$

$$+ D_1(x, \lambda)\, y_1^2 + D_2(x, \lambda)\, y_2^2 + D_3(x, \lambda)\, y_1y_2$$

where the $D_i(x, \lambda)$ are continuous as functions of λ and $L(x_0, \infty)$ as functions of x . As a result (3.16) becomes

$$Q_2'(x, \lambda) =$$

$$= p_{21}'(y_2 + P_{21}y_1)^2 + D_1(x, \lambda)y_1^2 + D_2(x, \lambda)\, y_2^2$$

$$+ D_3(x, \lambda)\, y_1y_2 - 2(\lambda + c + p_{21})(\lambda - c - p_{11})y_1y_2$$

$$- 2\Delta_1(\lambda + c + p_{21})y_2^2 \quad . \tag{3.20}$$

Now consider the derivative with respect to x of $Q_3(x, \lambda)$:

$$Q_3'(x, \lambda) = 2\Delta_1'(y_1 + P_{22}y_2)(y_2 + P_{12}y_1)$$

$$+ 2\Delta_1(y_1 + P_{22}y)'(y_2 + P_{12}y_1)$$

$$+ 2\Delta_1(y_1 + P_{22}y)(y_2 + P_{12}y_1)' \ . \tag{3.21}$$

By (3.12)-(3.14)

$$2\Delta_1(y_1 + P_{22}y_2)'(y_2 + P_{12}y_1) =$$

$$= 2\Delta_1(\mathcal{B}_{11}y_1 + \mathcal{B}_{12}y_2)(y_2 + P_{12}y_1)$$

$$= 2\Delta_1^2 \, y_1y_2 + 2\Delta_1(\lambda + c + p_{21})y_2^2$$

$$+ F_1(x, \lambda) \, y_1^2 + F_2(x, \lambda) \, y_2^2 + F_3(x, \lambda) \, y_1y_2 \tag{3.22}$$

where the $F_i(x, \lambda)$ are continuous with respect to λ and $L(x_0, \infty)$ with respect to x . Similarly by (3.17)-(3.19)

$$2\Delta_1(y_1 + P_{22}y_2)(y_2 + P_{12}y_1)'$$

$$= 2\Delta_1(y_1 + P_{22}y_2)(\mathcal{B}_{21} \, y_1 + \mathcal{B}_{22} \, y_2)$$

$$= -2\Delta_1(\lambda - c - p_{11})y_1^2 - 2\Delta_1^2 \, y_1y_2$$

$$+ G_1(x, \lambda) \, y_1^2 + G_2(x, \lambda) \, y_2^2 + G_3(x, \lambda) \, y_1y_2 \tag{3.23}$$

again where the $G_i(x, \lambda)$ are continuous with respect to λ and $L\,(x_0, \infty)$ with respect to x .

Therefore, by (3.21)-(3.23),

$$
\begin{aligned}
Q_3'(x, \lambda) = \\
&= 2\Delta_1'(y_1 + P_{22}y_2)(y_2 + P_{12}\, y_1) \\
&\quad - 2\Delta_1(\lambda - c - p_{11})y_1^2 + 2\Delta_1(\lambda + c + p_{21})y_2^2 \\
&\quad + (F_1 + G_1)y_1^2 + (F_2 + G_2)y_2^2 + (F_3 + G_3)y_1y_2 \qquad (3.24)
\end{aligned}
$$

Now by (3.15), (3.20), and (3.24)

$$
\begin{aligned}
E'(x, \lambda) &= Q_1' + Q_2' + Q_3' \\
&= p_{11}'(y_1 + P_{22}y_2)^2 + p_{21}'(y_2 + P_{12}y_1)^2 \\
&\quad + 2\Delta_1'(y_1 + P_{22}\, y_2)(y_2 + P_{12}\, y_1) \\
&\quad + H_1(x, \lambda)\, y_1^2 + H_2(x, \lambda)\, y_2^2 + H_3(x, \lambda)\, y_1y_2 \quad (3.25)
\end{aligned}
$$

where the $H_i(x, \lambda) = F_i(x, \lambda) + G_i(x, \lambda) + D_i(x, \lambda) + C_i(x, \lambda)$. Thus the $H_i(x, \lambda)$ are continuous as functions of λ and $L(x_0, \infty)$ with respect to x .

Next we wish to show that $\left|\frac{E'(x, \lambda)}{E(x, \lambda)}\right|$ is bounded by a function that is $L(x_0, \infty)$ for x_0 sufficiently large when λ is restricted to a compact subset of $\lambda > c$.

First note that for x sufficiently large and λ restricted to a compact subset of $\lambda > c$

$$\left|\frac{p'_{11}(y_1 + p_{22}y_2)^2}{E(x, \lambda)}\right| \leq C_1 |p'_{11}(x)| \quad ,$$

similarly that

$$\left|\frac{p'_{21}(y_2 + p_{12}\, y'_1)^2}{E(x, \lambda)}\right| \leq C_2 |p'_{21}(x)| \quad ,$$

and that

$$\left|\frac{2\Delta'_1(y_1 + p_{22}y_2)(y_2 + p_{12}y_1)}{E(x, \lambda)}\right| \leq C_3 |\Delta'_1(x)| \quad .$$

Thus the first three terms in the ratio $\left|\frac{E'(x, \lambda)}{E(x, \lambda)}\right|$ are bounded independent of λ in a compact subset of $\lambda > c$, where x is sufficiently large, by absolutely integrable functions.

It was shown earlier when $w_1 = (\lambda - c)^{1/2}\, y_1(x, \lambda)$, $w_2 = (\lambda + c)^{1/2}\, y_2(x, \lambda)$ the ratio $\frac{E(x, \lambda)}{w_1^2 + w_2^2}$ converges to 1 uniformly on compact subsets of $\lambda > c$ as $x \to \infty$. As a result note that for x sufficiently large, and λ restricted to a compact set

$$\left|\frac{H_1(x, \lambda)}{E(x, \lambda)}\right| y_1^2 = \left|\frac{H_1(x, \lambda)}{\lambda - c}\right| \cdot \left|\frac{w_1^2 + w_2^2}{E(x, \lambda)}\right| \cdot \frac{(\lambda - c)\, y_1^2}{w_1^2 + w_2^2}$$

$$\leq C_4\, |H_1(x, \lambda)| \quad .$$

By similar reasoning, for x sufficiently large and λ so restricted

$$\left|\frac{H_2(x, \lambda)}{E(x, \lambda)}\right| y_2^2 \leq C_5|H_2(x, \lambda)|$$

and

$$\left|\frac{H_3(x, \lambda)}{E(x, \lambda)} y_1y_2\right| \leq C_6|H_3(x, \lambda)| \quad .$$

Now in fact each $H_i(x, \lambda)$ is a polynomial in terms of λ with coefficients that are absolutely integrable functions of x. Thus, when λ is restricted to a compact set

$$|H_i(x, \lambda)| \leq T_i(x)$$

where $T_i(x) \in L(x_0, \infty)$ for x_0 sufficiently large. With this final remark we now see for λ restricted to a compact set and x_0 sufficiently large that $\int_{x_0}^{\infty} \frac{E'(x, \lambda)}{E(x, \lambda)}\, dx$ exists and that as in the proof of Theorem 2.1, $\int_{x_0}^{x} \left|\frac{E'(s, \lambda)}{E(s, \lambda)}\right| ds$ converges uniformly for λ in a compact set of $\lambda > c$ as $x \to \infty$; and as in that previous result, this is sufficient to conclude the uniform convergence, on compact subsets of $\lambda > c$, of $E(x, \lambda)$ to a positive continuous function $E_\infty(\lambda)$. ■

As another example, consider the Dirac system of the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} p(x) & \lambda + q_1(x) \\ -\lambda - q_2(x) & -p(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad 0 \leq x < \infty \qquad (3.26)$$

We now wish to consider the possibility that the potentials involve terms which grow without bounds.

As before λ is a complex parameter; the functions $p(x)$, $q_1(x)$, $q_2(x)$ are locally integrable. We further take for $k = 1, 2$ $q_k(x) = q_{k1}(x) + q_{k3}(x)$; where $q_{k1}(x)$ have the same sign and each tends to infinity as x tends to infinity. We shall without loss of generality assume now that $q_{k1}(x) > 0$.

Our goal, as before, is to stipulate conditions on the functions $p(x)$, $q_1(x)$, $q_2(x)$ sufficient to guarantee that the density function $\rho(\mu)$ from Chapter 1 is such that $\rho'(\mu) > 0$ for all μ in $\mathbb{R}$. In other words, we wish to find conditions which guarantee that the self adjoint operator formed from (3.26) in the same manner as one which was formed from the system (3.1) earlier, has a continuously differentiable spectrum that is all of $\mathbb{R}$.

We take, analogous to the first example of this chapter,

$$E_0(x, \lambda) = \sqrt{\frac{\lambda + q_{11}}{\lambda + q_{21}}}\, y_1^2 + \sqrt{\frac{\lambda + q_{21}}{\lambda + q_{11}}}\, y_2^2$$

to be the functional whose convergence on $\mathbb{R}$ we wish to examine. Note that

$$E_0(x, \lambda) =$$

$$\left[\left(\frac{q_{11} + \lambda}{q_{11}}\right)\left(\frac{q_{21} + \lambda}{q_{21}}\right)\right]^{-1/2}$$

$$\left[(1 + \lambda q_{11}^{-1})\sqrt{\frac{q_{11}}{q_{21}}}\, y_1^2 + (1 + \lambda q_{21}^{-1})\sqrt{\frac{q_{21}}{q_{11}}}\, y_2^2\right]$$

where the factor $\left[\left(\frac{q_{11}+\lambda}{q_{11}}\right)\left(\frac{q_{21}+\lambda}{q_{21}}\right)\right]$, as well as the terms λq_n^{-1} and λq_{21}^{-1} converge as $x \to \infty$ uniformly for λ in a compact subset of $\mathbb{R}$.

One can show by an argument similar to that used in the previous example of this section that $E_0(x, \lambda)$ as well as

$$E_1(x, \lambda) = (1 + \lambda q_{11}^{-1})\sqrt{\frac{q_{11}}{q_{21}}}\; y_1^2 + (1 + \lambda q_{21}^{-1})\sqrt{\frac{q_{21}}{q_{11}}}\; y_2^2 \tag{3.27}$$

and

$$E = E(x, \lambda) = \sqrt{\frac{q_{11}}{q_{21}}}\, y_1^2 + \sqrt{\frac{q_{21}}{q_{11}}}\, y_2^2 \tag{3.28}$$

all form ratios that converge to 1 as $x \to \infty$ uniformly for λ restricted to compact subsets of $\mathbb{R}$.

For this example we shall take $E(x, \lambda)$ to be the basic functional whose convergence we examine. We wish to give conditions which guarantee that $E(x, \lambda)$ converges as $x \to \infty$ uniformly on compact subsets of $\mathbb{R}$ to a positive continuous function of λ.

There is an advantage, in this instance, to working with $E(x, \lambda)$. As noted in Chapter 1 $E(x_n, \lambda) = J(x_n, \lambda, \lambda)$ where

$$J(x_n, \lambda, \mu) = \phi^*(x_n, \lambda)\, K_n^{-1}(\mu)\, \hat{\phi}(x_n, \lambda)$$
$$+ \hat{\phi}^*(x_n, \lambda)\; K_n(\mu)\, \hat{\phi}(x_n, \lambda)\,.$$

For this example, $\phi(x_n, \lambda) = y_1(x_n, \lambda)$, $\hat{\phi}(x_n, \lambda) = y_2(x_n, \lambda)$

and $K_n^{-1}(\mu) = \left[\frac{q_{11}}{q_{21}}\right]^{1/2} (x_n)$. It was shown in (1.30) and (1.20) that

$$\rho_n'(\lambda) = \pi^{-1} J^{-1}(x_n, \lambda, \mu) = \pi^{-1} J^{-1}(x_n, \lambda, \lambda) .$$

Thus by showing that $E(x_n, \lambda) = J(x_n, \lambda, \lambda)$ converges as $x_n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ to a positive continuous function $E_\infty(\lambda)$ we shall have shown directly that $\rho_n'(\lambda)$ converges uniformly on this compact set to $\pi^{-1} E_\infty^{-1}(\lambda)$. Now it was shown in Chapter 1 that a subsequence $\rho_{n_k}(\lambda)$ converges on $\mathbb{R}$, and because the system is limit point at infinity this subsequence converges to $\rho(\lambda)$ on $\mathbb{R}$. By a standard theorem in analysis the uniform convergence of $\rho_{n_k}'(\lambda)$ to $\pi^{-1} E_\infty^{-1}(\lambda)$ implies that $\rho'(\lambda) = \pi^{-1} E_\infty^{-1}(\lambda)$ on each compact subset of $\mathbb{R}$. We thus need not be concerned, as we were in Theorems 1.1 and 1.2, with whether or not $K_n^{-1}(\lambda)$ is bounded away from 0 for n sufficiently large, and hence with stipulating at the outset whether $q_{11}(x)$ or $q_{21}(x)$ is dominant as $x \to \infty$.

<u>Theorem 3.2</u>: For the system (3.26) let $q_{11}(x)$ and $q_{21}(x)$ be differentiable, of the same sign, and let both tend to infinity as $x \to \infty$. Let $q_{k3}(k)$, and $p(x)$ be locally integrable and let $p(x)$ be smooth in the sense that $D[(q_{11}q_{21})^{-1/2} p]$ exists. Let $\eta(x) = \dfrac{q_{11}(x)}{q_{21}(x)}$. If q_{k1}^{-1} , $\eta' \eta^{-1/2} q_{11}^{-1}$, and $(q_{11} q_{21})^{-1/2} p$ are terms which tend to 0 as $x \to \infty$ and whose derivatives are $L(x_0, \infty)$

and if $\left|\frac{q_{k3}}{q_{k1}}\right| \; (q_{21}\; q_{11})^{1/2} \in L(x_0, \infty)$, then $\rho'(\lambda)$ is continuous and positive on $\mathbb{R}$.

<u>Proof</u>: Note that E defined in (3.28) can be written as

$$E = \eta^{1/2}\, y_1^2 + \eta^{-1/2}\, y_2^2 \,.$$

As in previous examples we shall show that the logarithmic derivative of E is $L(x_0, \infty)$ and that $\int_{x_0}^{x} \left|\frac{E'(s, \lambda)}{E(s, \lambda)}\right| ds$ converges as, $x \to \infty$, uniformly for λ in a compact subset of $\mathbb{R}$.

By (3.26) we find that

$$\begin{aligned} E' = {} & 1/2\; \eta'\; \eta^{-1/2}\; y_1^2 - 1/2\; \eta'\; \eta^{-3/2}\; y_2^2 \\ & + 2p\; \eta^{1/2}\; y_1^2 - 2p\; \eta^{-1/2}\; y_2^2 \\ & + [2\lambda\; \eta^{1/2} - 2\lambda\eta^{-1/2}] y_1 y_2 \\ & + [2\eta^{1/2}\; q_{23} - 2\eta^{-1/2}\; q_{13}] y_1 y_2 \end{aligned} \tag{3.29}$$

In this example we define several auxillary functionals

$$\begin{aligned} Q_1 &= Q_1(x, \lambda) = 1/2\; \eta'\; \eta^{-1/2}\; q_{11}^{-1}\; y_1 y_2 \\ Q_2 &= Q_2(x, \lambda) = 2\eta^{1/2}\; p q_{11}^{-1}\; y_1 y_2 = 2[(q_{11} q_{21})^{-1/2}\; p]\; y_1 y_2 \\ Q_3 &= Q_3(x, \lambda) = \lambda q_{11}^{-1}\; \eta^{1/2}\; y_1^2 \\ Q_4 &= Q_4(x, \lambda) = \lambda q_{21}^{-1}\; \eta^{-1/2}\; y_2^2 \end{aligned} \tag{3.30}$$

Note that the coefficient functions for y^2, y_2^2 and $y_1\, y_2$ are taken by hypothesis to tend to 0 as $x \to \infty$.

For each of these functions one can show by (3.26) that

$$\begin{aligned} Q_1' = \tfrac{1}{2}\, D[\eta'\ \eta^{-1/2}\ q_{11}^{-1}]y_1y_2 \\ + \tfrac{1}{2}\lambda\,[\eta'\ \eta^{-1/2}\ q_{11}^{-1}]y_2^2 - \tfrac{1}{2}\lambda\,[\eta'\ \eta^{-1/2}\ q_{11}^{-1}]\ y_1^2 \\ + \tfrac{1}{2}[\eta'\eta^{-3/2}]\ y_2^2 - \tfrac{1}{2}[\eta'\ \eta^{-1/2}]\ y_1^2 \\ + \tfrac{1}{2}[\eta'\ \eta^{-1/2}\ q_{11}^{-1}\ q_{23}]y_2^2 - \tfrac{1}{2}[\eta'\ \eta^{-1/2}\ q_{11}^{-1}\ q_{13}]y_1^2 \end{aligned} \tag{3.31}$$

$$\begin{aligned} Q_2' = 2D[\eta^{1/2}\ pq_{11}^{-1}]y_1y_2 \\ + 2\lambda[\eta^{1/2}\ pq_{11}^{-1}]\ y_2^2 - 2\lambda[\eta^{1/2}\ pq_{11}^{-1}]y_1^2 \\ + 2[p\ \eta^{-1/2}]y_2^2 - 2[p\eta^{1/2}]y_1^2 \\ + 2[\eta^{1/2}\ pq_{11}^{-1}\ q_{23}]y_2^2 - 2[\eta^{1/2}\ pq_{11}^{-1}\ q_{13}]y_1^2 \end{aligned} \tag{3.32}$$

$$\begin{aligned} (Q_3 + Q_4)' = \lambda D[q_{11}^{-1}]\eta^{1/2}\ y_1^2 + \lambda D[q_{21}^{-1}]\ \eta^{-1/2}\ y_2^2 \\ + \tfrac{1}{2}\ \lambda[q_{11}^{-1}\ \eta'\ \eta^{-1/2}]y_1^2 - \tfrac{1}{2}\ \lambda[q_{11}^{-1}\ \eta'\ \eta^{-1}]\ {}^{2}_{2} \\ + 2\lambda[pq_{11}^{-1}\eta^{1/2}]\ y_1^2 - 2\lambda[pq_{11}^{-1}\ \eta^{1/2}]\ y_2^2 \\ + [2\lambda\eta^{-1/2} - 2\lambda\ \eta^{1/2}]\ y_1y_2 \\ + [2\lambda\ \eta^{1/2}\ q_{11}q_{23} - 2\lambda\ \eta^{-1/2}\ q_{21}^{-1}\ q_{13}]\ y_1y_2 \end{aligned} \tag{3.33}$$

By adding equations (3.29)-(3.33) we find that

$$[E + Q_1 + Q_2 + Q_3 + Q_4]'$$

$$= \{\tfrac{1}{2} D[\eta' \eta^{-1/2} q_{11}^{-1}] + 2D[\eta^{1/2} pq_{11}^{-1}]\} y_1y_2$$

$$+ \lambda D[q_{11}^{-1}] \eta^{1/2} y_1^2 + \lambda D[q_{21}^{-1}] \eta^{-1/2} y_2^2$$

$$+ 2[\eta^{1/2} q_{23}]y_1y_2 - 2[\eta^{-1/2} q_{13}]y_1y_2$$

$$+ \tfrac{1}{2}[\eta' q_{11}^{-1} q_{23}] \eta^{-1/2} y_2^2 - \tfrac{1}{2}[\eta' \eta^{-1} q_{11}^{-1} q_{13}]\eta^{1/2} y_1^2$$

$$+ 2[\eta pq_{11}^{-1} q_{23}]\eta^{-1/2} y_2^2 - 2[pq_{11}^{-1} q_{13}] \eta^{1/2} y_1^2$$

$$+ 2\lambda[\eta^{1/2} q_{11} q_{23}]y_1y_2 - 2\lambda[\eta^{-1/2} q_{21}^{-1} q_{13}] y_1y_2 \qquad (3.34)$$

Now let $\Delta = (q_{11} q_{21})^{1/2}$; then note the following:

$$\eta^{1/2} q_{23} = \Delta(\frac{q_{23}}{q_{11}}) \quad \text{and} \quad \eta^{-1/2} q_{13} = \Delta \frac{q_{13}}{q_{11}} \qquad (3.35)$$

are integrable by hypothesis;

$$\eta' q_{11}^{-1} q_{23} = [\eta' \eta^{-1/2} q_{11}^{-1}] \Delta \frac{q_{23}}{q_{21}}$$

and (3.36)

$$\eta' \eta^{-1} q_{11}^{-1} q_{13} = [\eta' \eta^{-1/2} q_{11}^{-1}] \Delta \frac{q_{13}}{q_{11}}$$

are integrable because $\eta' \eta^{-1/2} q_n^{-1} \to 0$ as $x \to \infty$ and $\Delta \dfrac{q_{k3}}{q_{k1}}$

are integrable;

$$\eta\, pq_{11}^{-1}\, q_{23} = [(q_{11}\, q_{21})^{-1/2}\, p]\, \Delta \frac{q_{23}}{q_{21}}$$

and (3.37)

$$pq_{11}^{-1}\, q_{13} = [(q_{11}\, q_{21})^{-1/2}\, p]\, \Delta \frac{q_{13}}{q_{11}}$$

are integrable since $(q_{11}\, q_{21})^{-1/2}\, p \to 0$ as $x \to \infty$;

$$\eta^{1/2}\, q_{11}\, q_{23} = [\frac{1}{q_{11}}]\, \Delta \frac{q_{23}}{q_{21}}$$

and (3.38)

$$\eta^{-1/2}\, q_{21}\, q_{13} = [\frac{1}{q_{21}}]\, \Delta \frac{q_{13}}{q_{11}}$$

are also integrable. Thus we see that each coefficient in (3.34) is, as a function of x , $L(x_0, \infty)$.

Let $E_2 = E + Q_1 + Q_2 + Q_3 + Q_4$ we note that

$$\begin{aligned} E_2 &= (1 + \lambda\, q_{11}^{-1})\, \eta^{1/2}\, y_1^2 + (1 + \lambda\, q_{21}^{-1})\, \eta^{-1/2}\, y_2^2 \\ &\quad + (\tfrac{1}{2}\, \eta'\, \eta^{-1/2}\, q_{11}^{-1} + 2\eta^{1/2}\, pq_{11}^{-1}) y_1 y_2 \\ &= E_1(x, \lambda) + (\tfrac{1}{2}\, \eta'\, \eta^{-1/2}\, q_{11}^{-1} + 2\eta^{1/2}\, pq_{11}^{-1})\, y_1 y_1 \end{aligned} \qquad (3.39)$$

where by our hypotheses $\frac{1}{2}\, \eta'\, \eta^{-1/2}\, q_{11}^{-1}$ and $2\eta^{1/2}\, pq_{11}^{-1}$ converge to 0 as $x \to \infty$. And as argued earlier, we can show that the ratio of E_2 to E_1 converges to 1 as $x \to \infty$ uniformly for λ restricted to a compact set in $\mathbb{R}$. The same is true then for $E_2(x, \lambda)$ and $E(x, \lambda)$. We may use this together with (3.34)-(3.38) to conclude that for x sufficiently large,

$$\left| \frac{E_2'(x, \lambda)}{E_2(x, \lambda)} \right| \leq C_1 |D[\eta'\eta^{-1/2} q_{11}^{-1}]|$$

$$+ \lambda C_2\{D[q_{11}^{-1}] + D[q_{21}^{-1}]\}$$

$$+ C_3(1 + \lambda) |(q_{11} q_{21})^{1/2} \frac{q_{13}}{q_{11}}|$$

$$+ C_4(1 + \lambda) |(q_{11} q_{21})^{1/2} \frac{q_{23}}{q_{21}}| .$$

As a consequence, $\int_{x_0}^{\infty} \left| \frac{E_2'(x, \lambda)}{E_2(x, \lambda)} \right| dx$ exists. Thus as in previous examples $E_2(x, \lambda)$ will converge as $x \to \infty$, uniformly on compact subsets to a positive continuous function; hence the same is true for $E(x, \lambda)$. As a consequence $\rho'(\mu)$ is continuous and positive for all μ in $\mathbb{R}$. ■

Erdelyi [10] considered the case where in

$$P(x) = -k/x , \quad q_1(x) = V(x) + c ,$$

$q_2(x) = V(x) - c$ where k and c are constants, and where $V(x) = V_1(x) + V_2(x)$; $V_2(x) \in L(x_0, \infty)$, $V_1(x)$ is differentiable and $|V_1(x)| \to \infty$ as $x \to \infty$. For this case he showed that the continuous spectrum is all of $\mathbb{R}$. It should be noted that this result also follows from Theorem 3.2.

A careful examination of the hypotheses of the result by Hinton and Shaw [18, Theorem 2] shows that they are identical to those of

Theorem 3.2 for the particular case considered. In their result, Hinton and Shaw allow the presence of weight functions $\alpha_1(x)$ and $\alpha_2(x)$ not identically equal to 1, and for the decomposition of the ratios $\frac{\alpha_k(x)}{q_{k1}(x)}$ and $\frac{q_{k3}(x)}{q_{k1}(x)}$, for $k = 1, 2$, into long range, short range, and oscillatory potentials of the kind described in Theorem 3.1. Thus although Theorem 3.1 is not an extension of the result of Hinton and Shaw, its proof is of a more elementary nature. There is also the hope of a simplification of the hypotheses, in this more general case, as occurred in Theorem 3.1.

Finally it should be noted that in Theorem 3.1 a more general functional could have been chosen for (3.7) with the result that somewhat weaker hypotheses are possible. For example one can choose

$$\begin{aligned} E(x, \lambda) = {} & (\lambda - c - p_{11})(y_1 + V_1 y_1 + V_2 y_2)^2 \\ & + (\lambda + c + p_{21})(y_2 + \overline{V}_1 y_1 + \overline{V}_2 y_2)^2 \\ & + 2\Delta_1 (y_1 + V_1 y_1 + V_2 y_2)(y_2 + \overline{V}_1 y_1 + \overline{V}_2 y_2) , \end{aligned}$$

with

$$V_1 = \int_x^\infty p_{12} \int_s^\infty p_{22}$$

$$V_2 = \int_x^\infty p_{22}[1 + \int_s^\infty p_{12} \int_t^\infty p_{22}]$$

$$\overline{V}_1 = \int_x^\infty p_{22} \int_s^\infty p_{12}$$

$$\overline{V}_2 = \int_x^\infty p_{12}[1 + \int_s^\infty p_{22} \int_t^\infty p_{12}] \quad .$$

If in addition to (3.3) we require the integrability of the V_i and $\overline{V}_i$, then the conclusion of Theorem 3.1 follows.

CHAPTER 4

SECTORIAL ASYMPTOTICS OF $M(\lambda)$ FOR A COUPLED SECOND ORDER SYSTEM

We shall now consider the following system:

$$Jz'(x) = \left(\begin{bmatrix} \lambda I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} v(x) & 0 \\ 0 & I \end{bmatrix} \right) z(x) \quad , \quad 0 \leq x < \infty \quad (4.1)$$

where $z = \begin{bmatrix} y \\ -y' \end{bmatrix}$, and y is a 2×1 vector. We also take $v(x) = \begin{bmatrix} p(x) & r(x) \\ r(x) & q(x) \end{bmatrix}$ where p, q, and r are real valued and locally integrable on $[0, \infty)$. Note that (4.1) is equivalent to $y'' + v(x)y = \lambda y$.

Specifically, we wish to develop the asymptotic form of the Titchmarsh-Weyl $M(\lambda)$ function associated with (4.1) as λ tends to infinity in sectors omitting the real axis. The approach is that of Hinton and Shaw [19], in their treatment of the scalar four order equation

$$y^{(4)} + q(x)\, y = \lambda y \qquad 0 \leq x < \infty .$$

For system (4.1) we shall consider a regular boundary value problem (R) as described in Chapter 1. We make specific choices $\alpha_1 = \beta_1 = I$, $\alpha_2 = \beta_2 = 0$ in (1.8) with the result that

$$M(\lambda, b) = -\Phi^{-1}(b, \lambda)\, \Theta(b, \lambda) , \tag{4.2}$$

where $0 \leq b < \infty$ and $\text{Im}\, \lambda \neq 0$.

In Chapter 1, results proven in [14] were summarized concerning the development of the concept of a matrix disk associated with these $M(\lambda)$ functions. At that time it was stated that $M(\lambda, b)$ satisfies

$$M(\lambda, b) = C(b, \lambda) + R_1(b, \lambda)\, U(b, \lambda)\, R_2(b, \lambda)$$

where $U(b, \lambda)$ is a unitary matrix and where $C(b, \lambda)$, $R_i(b, \lambda)$ are defined by (1.24) .

The critical idea was that the sets

$$S(b, \lambda) = \{M | M = C(b, \lambda) + R_1(b, \lambda)\, VR_2(b, \lambda)\ ,\ \ V^*V \leq I\},$$

for $0 < b < \infty$ and $\operatorname{Im} \lambda \neq 0$, nest as b tends to infinity; that is, $S(b_1, \lambda) \subset S(b_2, \lambda)$ for $b_2 < b_1$.

In this discussion we take $M(\lambda)$ to be the limit of a sequence of terms of the form (4.2) . In [14] it was proven that all such sequences have a common limit when the system is limit point. We do not assume that (4.1) is limit point. However, as we shall see, all limits of sequences of the form (4.2) have common first order asymptotics.

We make use of the following observation from Chapter 1: As $M(\lambda)$ is an element of each $S(b, \lambda)$, it must have a form

$$M(\lambda) = C(b, \lambda) + R_1(b, \lambda)\, V(b, \lambda)\, R_2(b, \lambda)\ .$$

And so,

$$M(\lambda) - M(b, \lambda) = R_1(b, \lambda)[V(b, \lambda) - U(b, \lambda)]\, R_2(b, \lambda) \,. \quad (4.3)$$

Thus from this,

$$\|M(\lambda) - M(b, \lambda)\|$$
$$\leq \|R_1(b, \lambda)\| \, \|V(b, \lambda) - U(b, \lambda)\| \, \|R_2(b, \lambda)\| \,, \quad (4.4)$$

where $\|\cdot\|$ is a matrix norm. Given the nature of U and V, $\|V(b, \lambda) - U(b, \lambda)\| \leq K$ for some constant K independent of b and λ ; hence,

$$\|M(\lambda) - M(b, \lambda)\| \leq K\|R_1(b, \lambda)\| \; \|R_2(b, \lambda)\| \quad (4.5)$$

It was noted in Chapter 1 that $R_1(b, \lambda) = R_2(b, \overline{\lambda})$. It can be shown that when the system is real then $R_2(b, \overline{\lambda}) = \overline{R_2(b, \lambda)}$, thus (4.5) becomes

$$\|M(\lambda) - M(b, \lambda)\| \leq K\|R_1(b, \lambda)\|^2 \quad (4.6)$$

for $0 < b < \infty$ and $\operatorname{Im} \lambda \neq 0$.

Our goal is to consider the inequality

$$\|M(\lambda) - M_0(\lambda)\|$$
$$\leq \|M(\lambda) - M(b, \lambda)\| + \|M(b, \lambda) - M_0(\lambda)\| \,, \quad (4.7)$$

where b is appropriately chosen and where $M_0(\lambda)$ is the $M(\lambda)$

function for the system (4.1) where $v(x) \equiv 0$. We shall show for an appropriate choice of b as a function of λ, that as λ tends to infinity through a sector in $\mathbb{C}$ which excludes $\mathbb{R}$, $\|M(\lambda) - M(b, \lambda)\|$ and $\|M(b, \lambda) - M_0(\lambda)\|$ are $o(1)$ and hence that $M(\lambda) = M_0(\lambda) + o(1)$.

We shall restrict our attention to those values of λ in $\mathbb{C}$ such that

$$\lambda = \Delta^2 , \quad \Delta = \sigma + i\tau$$

$$0 < \varepsilon_0 < \tau/\sigma < K < 1 , \sigma > 2 . \tag{4.8}$$

This describes a region in $\mathbb{C}$ such that $\pi - \varepsilon_1 > \arg \lambda > \varepsilon_2$ and $\pi > \varepsilon_1, \varepsilon_2 > 0$. It is a sector which excludes the real line. Let us also define for λ satisfying (4.8),

$$X = 3/2 \, \sigma^{-1} \ln \sigma .$$

Note that as defined $X > 0$, and that as $\sigma \to \infty$, $X \to 0$. For λ restricted to this sector it is equivalent to say either that $\lambda \to \infty$ or $\sigma \to \infty$ because

$$\sigma(1 + \varepsilon_0^2)^{1/2} < |\Delta| < \sigma(1 + K^2)^{1/2} . \tag{4.9}$$

Theorem 4: Let

$$Y(x, \lambda) = \begin{bmatrix} \theta(x, \lambda) & \Phi(x, \lambda) \\ \theta'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix} \qquad (4.10)$$

denote the fundamental solution of (4.1) where $\theta(x, \lambda)$ and $\Phi(x, \lambda)$ 2×2 blocks and where $Y(0, \lambda) = I$. For $\operatorname{Im} \lambda \neq 0$ let $M(\lambda)$ be the limit as $b_n \to \infty$ of some sequence $\{M(\lambda, b_n)\}$ where

$$M(\lambda, \; b) = -\Phi^{-1}(\lambda, b)\, \theta(b, \lambda) \; .$$

As $\lambda \to \infty$ in the sector described by (4.8) , $M(\lambda) = M_0(\lambda) + o(1)$, where

$$M_0(\lambda) = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}, \text{ and } \Delta \text{ is described in (4.8) .}$$

Proof: For the system (4.1) with $v(x) \equiv 0$ one finds that the characteristic equation is $(r^2 - \lambda)^2 = 0$; the eigenvalues are $\pm\Delta$ where each is of order two .

The fundamental solution of (4.1) with $v(x) = 0$ which is equal to the identity when $x = 0$ is

$$Y_0(x, \lambda) = \begin{bmatrix} \cosh(\Delta x) & 0 & -\Delta^{-1} \sinh(\Delta x) & 0 \\ 0 & \cosh(\Delta x) & 0 & -\Delta^{-1}\sinh(\Delta x) \\ -\Delta\sinh(\Delta x) & 0 & \cosh(\Delta x) & 0 \\ 0 & -\Delta\sinh(\Delta x) & 0 & \cosh(\Delta x) \end{bmatrix}$$

If we partition $Y_0(x, \lambda)$ into 2×2 submatrices and take

$$\theta_0(x, \lambda) = \begin{bmatrix} \cosh(\Delta x) & 0 \\ 0 & \cosh(\Delta x) \end{bmatrix}$$

and

$$\Phi_0(x, \lambda) = \begin{bmatrix} -\Delta^{-1} \sinh(\Delta x) & 0 \\ 0 & -\Delta^{-1}\sinh(\Delta x) \end{bmatrix} ,$$

then

$$-\Phi_0^{-1} \theta_0 = \begin{bmatrix} \Delta \coth(\Delta x) & 0 \\ 0 & \Delta \coth(\Delta x) \end{bmatrix}$$

and we observe that for $\mathrm{Im}(\lambda) \neq 0$

$$\lim_{x\to\infty} -\Phi_0^{-1}(x, \lambda)\, \theta_0(x, \lambda) = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$$

Thus we define

$$M_0(\lambda) = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} . \tag{4.11}$$

Let

$$\theta(x, \lambda) = \{\theta_{ij}(x, \lambda)\} = \{\theta_{ij}\}$$

$$\Phi(x, \lambda) = \{\phi_{ij}(x, \lambda)\} = \{\phi_{ij}\}$$

for $i,j = 1, 2$ be the 2×2 blocks of the fundamental matrix defined by (4.10) . Now let

$$\begin{aligned} M(x , \lambda) &= -\Phi^{-1}(x, \lambda)\ \theta(x, \lambda) \\ &= -\det[\Phi(x, \lambda)]^{-1}\ [\mathrm{adj}(\Phi(x, \lambda)]\ \theta(x, \lambda) \\ &= -\det[\Phi(x, \lambda)]^{-1}\ \{a_{ij}\} \end{aligned} \tag{4.12}$$

where $i,j = 1, 2$.

One can show

$$a_{11} = \begin{vmatrix} \theta_{11} & \phi_{12} \\ \theta_{21} & \phi_{22} \end{vmatrix} , \quad a_{12} = \begin{vmatrix} \theta_{12} & \phi_{12} \\ \theta_{22} & \phi_{22} \end{vmatrix}$$

$$(4.13)$$

$$a_{21} = - \begin{vmatrix} \theta_{11} & \phi_{11} \\ \theta_{21} & \phi_{21} \end{vmatrix} , \quad a_{22} = - \begin{vmatrix} \theta_{12} & \phi_{11} \\ \theta_{22} & \phi_{21} \end{vmatrix}$$

and

$$\det \Phi = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \\ \phi_{21} & \phi_{22} \end{vmatrix} \quad . \qquad ((4.13)\text{ con't})$$

We now wish to develop asymptotic expressions for the determinants in (4.13), as λ tends to infinity in the sector described by (4.8) .

We begin by first noticing that each determinant is of the form $\begin{vmatrix} y_1 & w_1 \\ y_2 & w_2 \end{vmatrix}$ where the vectors

$$y(x, \lambda) = \begin{bmatrix} y_1 \\ y_2 \\ -y_1' \\ -y_3' \end{bmatrix} \quad , \quad \text{and} \quad w(x, \lambda) = \begin{bmatrix} w_1 \\ w_2 \\ -w_1' \\ -w_2' \end{bmatrix}$$

are columns in the matrix $Y(x, \lambda)$.

Let $y(x, \lambda)$ and $w(x, \lambda)$ be solutions of (4.1) . Define

$$\Gamma_1 = \Gamma_1(x, \lambda) = \begin{vmatrix} y_1 & w_1 \\ y_2 & w_2 \end{vmatrix}$$

$$\Gamma_2 = \Gamma_2(x, \lambda) = \begin{vmatrix} y_1' & w_1' \\ y_2' & w_2' \end{vmatrix} \qquad (4.14)$$

$$\Gamma_3 = \Gamma_3(x, \lambda) = \begin{vmatrix} y_1 & w_1 \\ y_2' & w_2' \end{vmatrix}$$

$$\Gamma_4 = \Gamma_4(x, \lambda) = \begin{vmatrix} y_1' & w_1' \\ y_2 & w_2 \end{vmatrix}$$

$$\Gamma_5 = \Gamma_5(x, \lambda) = \begin{vmatrix} y_2 & w_2 \\ y_2' & w_2' \end{vmatrix} + \begin{vmatrix} y_1' & w_1' \\ y_1 & w_1 \end{vmatrix} \qquad ((4.14) \text{ con't})$$

and Γ to be the 5×1 vector $\Gamma(x, \lambda) = (\Gamma_i)$. With this definition one can show that Γ satisfies

$$\Gamma' = \left[\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & \lambda & \lambda & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p & -q & -r \\ -q & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 \\ 2r & 0 & 0 & 0 & 0 \end{bmatrix}\right]\Gamma \qquad (4.15)$$

or $\Gamma' = (B(\lambda) + Q(x))\Gamma$.

The characteristic equation for the system $Z' = B(\lambda)\, Z$ is $r^3(r^2 - 4\lambda) = 0$; thus the eigenvalues are 0, $\pm 2\Delta$. The fundamental solution of the homogeneous system is

$$Z = Z(x) = Z(x, \lambda) = \qquad (4.16)$$

(con't)

$$\frac{1}{2}\cdot\begin{bmatrix} 1+\cosh 2\Delta x & \Delta^{-2}(-1+\cosh 2\Delta x) & \Delta^{-1}\sinh 2\Delta x & \Delta^{-1}\sinh 2\Delta x & 0 \\ \Delta^{2}(-1+\cosh 2\Delta x) & 1+\cosh 2\Delta x & \Delta\sinh 2\Delta x & \Delta\sinh 2\Delta x & 0 \\ \Delta\sinh 2\Delta x & \Delta^{-1}\sinh 2\Delta x & 1+\cosh 2\Delta x & -1+\cosh 2\Delta x & 0 \\ \Delta\sinh 2\Delta x & \Delta^{-1}\sinh 2\Delta x & -1+\cosh 2\Delta x & 1+\cosh 2\Delta x & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (4.16)$$

where $Z(0, \lambda) = I$.

Let $\eta = \eta(x) = \eta(x, \lambda)$ be the fundamental solution of (4.15) such that $\eta(0, \lambda) = I$. We note the following: By (4.13) and (4.14),

$$\begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} = \eta_{12}$$

since this corresponds to $\begin{vmatrix} y_1 & w_1 \\ y_2 & w_2 \end{vmatrix}$ where $y_1'(0, \lambda) = w_2'(0, \lambda) = -1$

$0 = y_i(0, \lambda) = w_i(0, \lambda)$ for $i = 1, 2$ and $0 = y_2'(0, \lambda) = w_1'(0, \lambda)$.

And so, with

$$\Gamma_1 = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} \quad (4.17)$$

we find that

$$\Gamma(0, \lambda) = (\Gamma_i(0, \lambda)) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Thus not only is $\Gamma(x, \lambda)$ with Γ_1 defined by (4.17) a solution of (4.15) but in fact is the second column of the fundamental solution η of (4.15) .

By a similar analysis we find that

$$-\eta_{13} = \begin{vmatrix} \theta_{11} & \phi_{12} \\ \theta_{21} & \theta_{22} \end{vmatrix} \quad , -\eta_{14} = -\begin{vmatrix} \theta_{12} & \phi_{11} \\ \theta_{22} & \phi_{21} \end{vmatrix}$$

$$-\eta_{15} = \begin{vmatrix} \theta_{12} & \phi_{12} \\ \theta_{22} & \phi_{22} \end{vmatrix} = -\begin{vmatrix} \theta_{11} & \phi_{11} \\ \theta_{21} & \phi_{21} \end{vmatrix} \tag{4.18}$$

As a result of (4.12), (4.13) , and (4.18), we have that

$$M(x,\lambda) = -\Phi^{-1}(x, \lambda)\ \theta(x, \lambda) = \eta_{12}^{-1} \begin{bmatrix} \eta_{13} & \eta_{15} \\ \eta_{15} & \eta_{14} \end{bmatrix} \tag{4.19}$$

Next, we wish to compute bounds for the η_{ij} . We shall do so by using the variation of parameters formulation of η_{ij} and Gronwall's inequality.

By variation of parameters, we find that

$$\eta(x) = Z(x) + \int_0^x Z(x - s)\ Q(s)\ \eta(s)\ ds\ . \tag{4.20}$$

Thus

$$\eta_{12} = \frac{1}{2}\Delta^{-2}(-1 + \cosh 2\Delta x) -$$

$$+ \frac{1}{2}\int_0^x \{\Delta^{-1}\eta_{12}(s)[p(s) + q(s)]\sinh 2\Delta(x - s)$$

$$+ \Delta^{-2}[p\eta_{32} + q\eta_{42} + r\eta_{52}](-1 + \cosh 2\Delta(x - s))\}ds \quad (4.21)$$

By (4.9) there are constants C_{11} and C_{12} such that

$$|\eta_{12}| \le C_{11}\sigma^{-2}e^{2\sigma x} + C_{12}\int_0^x T(s)\, e^{2\sigma(x-s)}$$

$$\cdot \{\sigma^{-1}|\eta_{12}| + \sigma^{-2}(|\eta_{32}| + |\eta_{42}| + |\eta_{52}|)\}\, ds\ , \quad (4.22)$$

where $T(s) = |p(s)| + |q(s)| + |r(s)|$.

Similarly, we find for $k = 3, 4$ that

$$|\eta_{k2}| \le C_{k1}\sigma^{-1}e^{2\sigma x} +$$

$$C_{k2}\int_0^x T(s)\, e^{2\sigma(x-s)}\{|\eta_{12}| + \sigma^{-1}(|\eta_{32}| + |\eta_{42}| + |\eta_{52}|)\}ds;$$

and for $k = 5$ that,

$$\eta_{52} = \int_0^x 2r(s)\,\eta_{12}(s)\, ds$$

and

$$|n_{52}| \leq 2 \int_0^x T(s)\, |n_{12}|\, ds$$

$$\leq 2 \int_0^x T(s)\{|n_{12}| + \sigma^{-1}(|n_{32}| + |n_{42}| + |n_{52}|)\, ds \ .$$

As a consequence,

$$|n_{12}| + \sigma^{-1}(|n_{32}| + |n_{42}| + |n_{52}|) \leq C_1\, \sigma^{-2}\, e^{2\sigma x} \tag{4.23}$$

$$+ C_2 \int_0^x R(s)\, e^{2\sigma(x-s)}\{|n_{12}| + \sigma^{-1}(|n_{32}| + |n_{42}| + |n_{52}|)\}ds$$

where

$$R(s) = R(s, \sigma, x) = (3 + 2\, e^{2\sigma(s-x)})\, T(s) \ .$$

Recall, for $X = 3/2\, \sigma^{-1} \ln \sigma$ and $\sigma \geq 2$, that X is a bounded variable as $\sigma \to \infty$. For (4.23), we assume that

$$0 \leq s < x \leq X \ . \tag{4.24}$$

On this interval, Gronwall's inequality implies that

$$|n_{12}| + \sigma^{-1}(|n_{32}| + |n_{42}| + |n_{52}|) \leq L\sigma^{-2}\, e^{2\sigma x} \ , \tag{4.25}$$

where, as $\sigma \to \infty$, L is a constant not depending upon x for all $x \in [0, X]$ because X remains bounded, and in fact tends to 0 as $\sigma \to \infty$.

By (4.21), (4.24) and (4.9) we see that

$$|\eta_{12} - Z_{12}| \le$$

$$\le C\sigma^{-1} e^{2\sigma x} \int_0^x T(s)\, e^{-2\sigma s}$$

$$\cdot \{|\eta_{12}| + \sigma^{-1}(|\eta_{32}| + |\eta_{42}| + |\eta_{52}|)\}\, ds \ ,$$

and by (4.25), that

$$\le C_1\, \sigma^{-3}\, e^{2\sigma x} \int_0^x T(s)\, ds \ , \qquad 0 \le s < x \le X \ .$$

As a result,

$$\eta_{12} = Z_{12} + \sigma^{-3}\, e^{2\sigma x}\, R_{12}(x, \lambda) \tag{4.26}$$

where $|R_{12}(X, \lambda)| \to 0$ as $\lambda \to \infty$. By a similar analysis one can show that

$$\eta_{1k} = Z_{1k} + \sigma^{-2}\, e^{2\sigma x}\, R_{1k}(x, \lambda) \quad , \quad k = 3, 4, 5 \tag{4.27}$$

where $|R_{1k}(X, \lambda)| \to 0$ as $\lambda \to \infty$.

The description of $M(x, \lambda)$ given in (4.12) and (4.19) involves the ratios $\dfrac{\eta_{1k}}{\eta_{12}}$, $k = 3, 4, 5$. Now by (4.26) and (4.27) ,

$$\frac{\eta_{1k}}{\eta_{12}} = \frac{Z_{1k} + \sigma^{-2}\, e^{2\sigma x}\, R_{1k}(x, \lambda)}{Z_{12} + \sigma^{-3}\, e^{2\sigma x}\, R_{12}(x, \lambda)} \tag{4.28}$$

for $k = 3, 4, 5$. For cases $k = 3, 4$, where $x = X$

$$\left| \frac{\sigma^{-2} e^{2\sigma X}}{Z_{1k}(X, \lambda)} \right| \leq C\sigma^{-1}$$

and so,

$$\frac{\eta_{ik}(X)}{\eta_{12}(X)} = \frac{Z_{1k}}{Z_{12}} \quad \frac{1 + \delta_{1k}(X, \lambda)}{1 + \delta_{12}(X, \lambda)}$$

where $\sigma\, \delta_{1k}(X, \lambda) \to 0$ as $\lambda \to \infty$ or

$$\frac{\eta_{1k}(X)}{\eta_{12}(X)} = \frac{Z_{ik}}{Z_{12}} (1 + \zeta_{1k}(X, \lambda)) \tag{4.29}$$

again, where $\sigma\zeta_{1k}(X, \lambda) \to 0$ as $\lambda \to \infty$. On the other hand,

$$\frac{\eta_{15}}{\eta_{12}} = \frac{\sigma^{-2} e^{2\sigma x} R_{15}(x, \lambda)}{Z_{12} + \sigma^{-3} e^{2\sigma x} R_{12}(x, \lambda)}$$

Thus

$$\frac{\eta_{15}(X)}{\eta_{12}(X)} = \left[\frac{\sigma^{-2} e^{2\sigma X}}{Z_{12}} \quad R_{15}(X, \lambda) \right] \quad (1 + \zeta_{15}(X, \lambda))$$

where $\sigma\zeta_{15}(X, \lambda) \to 0$ as $\lambda \to \infty$, and

$$\frac{\sigma^{-2} e^{2\sigma X}}{Z_{12}(X, \lambda)} R_{15}(X, \lambda) = o(1)$$

as $\lambda \to \infty$. As a result,

$$\frac{\eta_{15}(X)}{\eta_{12}(X)} = o(1) .$$

Next note with the constraints imposed by (4.8) that for $k = 3, 4$

$$\left| \frac{Z_{1k}(X)}{Z_{12}(X)} - \Delta \right| \leq C\sigma \left| \frac{\sinh 2\Delta X - \cosh 2\Delta X + 1}{\cosh 2\Delta X - 1} \right|$$

$$\leq C\sigma \left| \frac{(e^{2\Delta X} - e^{-2\Delta X}) - (e^{2\Delta X} + e^{-2\Delta X}) + 2}{e^{2\Delta X} + e^{-2\Delta X} - 2} \right|$$

$$\leq C\sigma \left| \frac{e^{-2\Delta X} - e^{-4\Delta X}}{1 + e^{-4\Delta X} - 2e^{-2\Delta X}} \right|$$

and because $X = \frac{3}{2} \sigma^{-1} \ln \sigma$,

$$\leq C\sigma(\sigma^{-3} + e^{-6}) .$$

As a result,

$$\frac{Z_{1k}(X)}{Z_{12}(X)} = \Delta + o(1) .$$

By (4.29), for $k = 3, 4$,

$$\frac{\eta_{1k}(X)}{\eta_{12}(X)} = \Delta + o(1) ,$$

as $\lambda \to \infty$. And by (4.19)

$$M(X, \lambda) = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} + o(1)$$

or by (4.11)

$$M(X, \lambda) = M_0(\lambda) + o(1) . \tag{4.30}$$

If in inequality (4.7) we choose $b = X$ subject to restrictions (4.8) then

$$||M(\lambda) - M_0(\lambda)|| \leq ||M(\lambda) - M(X, \lambda)|| + o(1) , \tag{4.31}$$

and (4.6) becomes

$$||M(\lambda) - M(X, \lambda)|| \leq K ||R_1(X, \lambda)||^2 . \tag{4.32}$$

Recalling the definitions in (1.23) and (1.25) we see that $R_1^2(X, \lambda) = \mathscr{D}^{-1}(X, \lambda)$, and that

$$\begin{aligned} \mathscr{D}(X) &= \mathscr{D}(X, \lambda) \\ &= -i\,[\Phi'^{*}\Phi - \Phi^{*}\Phi'] \\ &= -i\,[\Phi'^{*}(X, \lambda)\,\Phi(X, \lambda) - \Phi^{*}(X, \lambda)\,\Phi'(X, \lambda)] . \end{aligned}$$

Thus,

$$\begin{aligned} -iR_1^2(X) &= -i\ R_1^2(X, \lambda) \\ &= [\Phi^{*\prime}\Phi - \Phi^*\Phi']^{-1} \\ &= -\Phi^{-1}[\Phi'\Phi^{-1} - (\Phi'\Phi^{-1})^*]^{-1}\ \Phi^{*-1} \quad . \end{aligned} \tag{4.33}$$

As mentioned in Chapter 1, it is proven in [13] that, for $\mathrm{Im}\ \lambda \neq 0$, $\Phi^{-1}(X, \lambda)$ exists. We seek to show that $\|R_1(X, \lambda)\| \to 0$ as $\lambda \to \infty$ by demonstrating that $\|\Phi^{-1}(X, \lambda)\| \to 0$ and that $\|[\Phi'\Phi^{-1} - (\Phi'\Phi^{-1})^*]^{-1}(X, \lambda)\| \to 0$ as $\lambda \to \infty$ subject to (4.8). This, together with (4.31) and (4.32) will complete the demonstration that

$$M(\lambda) = M_0(\lambda) + o(1) \ .$$

By (4.13) and (4.17), we have seen that $\eta_{12} = \det \Phi$; thus

$$\Phi^{-1} = \Phi^{-1}(X, \lambda) = \frac{1}{\eta_{12}} \begin{bmatrix} \phi_{22} & -\phi_{12} \\ -\phi_{21} & \phi_{11} \end{bmatrix}$$

To show that $|\Phi^{-1}(X, \lambda)| \to 0$ as $\lambda \to \infty$ we need to examine $\dfrac{\phi_{ij}}{\eta_{12}}$, $i,j = 1, 2$.

Given the fundamental solution $Y_0(X, \lambda)$,

$$\Phi(x, \lambda) = Y_0(x, \lambda) + \int_0^x Y_0(x - s, \lambda)\ Q(s)\ \Phi(s, \lambda)\ ds$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & -r & 0 & 0 \\ -r & -q & 0 & 0 \end{bmatrix}$$

As a consequence, for $i \neq j$ and $i,j = 1, 2$

$$\phi_{jj} = \phi_{jj}(x, \lambda) =$$

$$= -\Delta^{-1} \sinh \Delta x + \int_0^x \Delta^{-1} \sinh(x - s)[p\phi_{jj} + q\phi_{ij}]\, ds$$

$$\phi_{ij} = \phi_{ij}(x, \lambda) = \int_0^x \Delta^{-1} \sinh(x - s)\, [r\phi_{jj} + q\phi_{ij}]\, ds\ .$$

From this we see that

$$|\phi_{jj}| + |\phi_{ij}| \leq$$

$$\leq C_1\, \sigma^{-1}\, e^{\sigma x} + \int_0^x \sigma^{-1}\, e^{\sigma(x-s)}\, T(s)\, (|\phi_{jj}| + |\phi_{ij}|)\, ds$$

where $T(s) = |p(s)| + |q(s)| + |r(s)|$; thus by Gronwall we see that

$$|\phi_{jj}| + |\phi_{ij}| \leq \sigma^{-1}\, e^{\sigma x}\, D_i(x, \lambda)$$

for x $[0, X]$, and where for λ subject to (4.8) $|D_i\ (X, \lambda)| \to 0$ as $\lambda \to \infty$. And,

$$\left|\frac{\phi_{ij}(X)}{\eta_{12}(X)}\right| \le \frac{\sigma^{-1}\ e^{\sigma X}|D_i(X,\ \lambda)|}{|Z_{12} + \sigma^{-3}\ e^{2\sigma X}\ R_1(X,\ \lambda)|}$$

$$\le C\ \frac{\sigma^{-1}\ e^{\sigma X}|D_i(X,\ \lambda)|}{\sigma^{-2}\ e^{2\sigma X}|1 + \delta(X,\ \lambda)|}$$

$$\le C\sigma^{-1/2}\ \frac{D_i(X,\ \lambda)}{|1 + \delta(X,\ \lambda)|}\ .$$

where $\delta(X,\ \lambda) \to 0$ as $\lambda \to \infty$ and since $X = 3/2\ \frac{\ln \sigma}{\sigma}$.

From this we see that $\left|\frac{\phi_{ij}(X,\ \lambda)}{\eta_{12}(X,\ \lambda)}\right| \to 0$ and hence

$$\|\Phi^{-1}(X,\ \lambda)\| \to 0 \tag{4.34}$$

as $\lambda \to \infty$.

We now consider $[\Phi'\Phi^{-1} - (\Phi'\ \Phi^{-1})^*]^{-1}$.

As $-\Phi' = \begin{bmatrix} \phi'_{11} & \phi'_{12} \\ \phi'_{21} & \phi'_{22} \end{bmatrix}$ and $\Phi^{-1} = \frac{1}{\eta_{12}}\begin{bmatrix} \phi_{22} & -\phi_{12} \\ -\phi_{21} & \phi_{11} \end{bmatrix}$, one can show that

$$-\Phi'\ \Phi^{-1} = \frac{1}{\eta_{12}}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{4.35}$$

where $b_{11} = \begin{vmatrix} \phi'_{11} & \phi'_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix}$, $b_{12} = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi'_{11} & \phi'_{12} \end{vmatrix}$

$$b_{21} = \begin{vmatrix} \phi_{21}' & \phi_{22}' \\ \phi_{21} & \phi_{22} \end{vmatrix} , \quad b_{22} = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21}' & \phi_{22}' \end{vmatrix} .$$

We see that b_{11} is of the form $\begin{vmatrix} y_1' & w_1' \\ y_2 & w_2 \end{vmatrix}$ and hence of the form Γ_4 in (4.14). In this case, $y_1'(0) = w_2'(0) = -1$ and so we can see that

$$b_{11} = \eta_{42} .$$

Now b_{22} is like Γ_3 in (4.14) where again $y_1'(0) = w_2'(0) = -1$ so

$$b_{22} = \eta_{32} .$$

For b_{12} and b_{21} things are different. We see that b_{12} is of the form $\begin{vmatrix} y_1 & w_1 \\ y_1' & w_1' \end{vmatrix}$ while b_{21} is of the form $\begin{vmatrix} y_2' & w_2' \\ y_2 & w_2 \end{vmatrix}$. We observe then that $b_{12} + b_{21} = -\Gamma_5$ and that as a result,

$$b_{12} + b_{21} = -\eta_{52} . \tag{4.36}$$

Next, by differentiation, one can show that

$$\begin{vmatrix} y_1 & w_1 \\ y_1' & w_1' \end{vmatrix} + \begin{vmatrix} y_2 & w_2 \\ y_2' & w_2' \end{vmatrix} = \text{const.} = C .$$

For our case this would be

$$b_{12} - b_{21} = C \ .$$

And so by (4.36) and the equation above,

$$b_{12} = \frac{1}{2} [C - \eta_{52}]$$

$$b_{21} = - \frac{1}{2} [C + \eta_{52}] \ .$$

In considering (4.35), we would like descriptions of η_{32}, η_{42}, and η_{52} .

As mentioned in (4.20)

$$\eta_{k2} = \eta_{k2}(x, \lambda) = [Z(x) + \int_0^x Z(x - s) \, Q(s) \, \eta(s) \, ds]_{k2} \qquad (4.37)$$

where $Q(s)$ is defined in (4.15) . Thus,

$$\eta_{32} = \frac{\Delta^{-1}}{2} \sinh 2\Delta x - \frac{1}{2} \int_0^x [q(1 + \cosh 2\Delta(x - s)) + p(-1 + \cosh 2\Delta(x - s))]\eta_{12} + \Delta^{-1} \sinh 2\Delta(x - s) [p\eta_{32} + q\eta_{42} + r\eta_{52}] \, ds \ ,$$

and so

$$|\eta_{32} - \frac{\Delta^{-1}}{2} \sinh 2\Delta x| \leq$$

$$C_1 \int_0^x e^{2\sigma(x-s)}\, T(s)\, [|\eta_{12}| + \sigma^{-1}(|\eta_{32}| + |\eta_{42}| + |\eta_{52}|)]\, ds$$

again where $T(s) = |p(s)| + |q(s)| + |r(s)|$, and where $0 \leq s \leq x \leq X$. By (4.25) we then see that

$$|\eta_{32} - \frac{\Delta^{-1}}{2} \sinh 2\Delta x| \leq C_1\, \sigma^{-2}\, e^{2\sigma X} \int_0^X T(s)\, ds\ .$$

As a consequence,

$$\eta_{32}(x, \lambda) = \frac{\Delta^{-1} \sinh 2\Delta x}{2} + \sigma^{-2}\, e^{2\sigma X}\, R_{32}(x, \lambda)$$

for $x \in [0, X]$; where $R_{32}(X, \lambda) \to 0$ as $\lambda \to \infty$. If $x = X = \frac{3}{2}\, \sigma^{-1} \ln \sigma$,

$$\eta_{32}(X, \lambda) = \frac{\Delta^{-1} \sinh 2\Delta X}{2} + \sigma\, R_{32}(X, \lambda)\ . \tag{4.38}$$

A similar argument will show that

$$\eta_{42}(X) = \eta_{42}(X, \lambda) = \frac{\Delta^{-1} \sinh 2\Delta X}{2} + \sigma\, R_{42}(X, \lambda) \tag{4.39}$$

where $|R_{42}(X, \lambda)| \to 0$ as $\lambda \to \infty$.

Referring again to (4.37) we see that

$$\eta_{52} = \eta_{52}(x, \lambda) = \int_0^x 2r(s)\, \eta_{12}(s)\, ds\ .$$

By (4.25) it follows that

$$|\eta_{52}(X, \lambda)| \leq C \int_0^X |r(s)| \, \sigma^{-2} e^{2\sigma s} \, ds$$

$$\leq C \, \sigma^{-2} e^{2\sigma X} \int_0^X |r(s)| \, ds \; .$$

And so we note that

$$\eta_{52}(X) = \eta_{52}(X, \lambda) = \sigma \, R_{52}(X, \lambda) \tag{4.40}$$

where $|R_{52}(X, \lambda)| \to 0$ as $\lambda \to \infty$ in the sector described by (4.8) .

We note by (4.26) that

$$\eta_{12}(X) = \eta_{12}(X, \lambda) = \frac{\Delta^{-2}}{2} (1 + \cosh 2\Delta X) + R_{12}(X, \lambda) \; , \tag{4.41}$$

again where $|R_{12}(X, \lambda)| \to 0$ as $\lambda \to \infty$.

Considering again (4.35) we see that for $x = X$

$$\frac{b_{11}}{\eta_{12}(X)} = \frac{\eta_{42}(X)}{\eta_{12}(X)}$$

and by (4.39) and (4.41) we see that

$$\frac{\eta_{42}(X)}{\eta_{12}(X)} = \frac{\Delta^{-1} \sinh 2\Delta X + 2\sigma \, R_{42}(X, \lambda)}{\Delta^{-2} (-1 + \cosh 2\Delta X) + 2 \, R_{12}(X, \lambda)}$$

$$= \frac{\Delta^{-1}\ e^{2\Delta X}}{\Delta^{-2}\ e^{2\Delta X}} \left[\frac{1 - e^{-4\Delta X} + 4\sigma e^{-2\Delta X}\ R_{42}(X, \lambda)}{1 + e^{-4\Delta X} - 2e^{-2\Delta X} + 4e^{-2\Delta X}\ R_{12}(X, \lambda)} \right].$$

Note that $|\sigma e^{-2\Delta X}| = \sigma^{-2}$ since $X = \frac{3}{2}\sigma^{-1} \ln \sigma$, and so, in fact

$$\frac{\eta_{42}(X)}{\eta_{12}(X)} = \Delta \left[\frac{1 + \delta_{42}(X, \lambda)}{1 + \delta_{12}(X, \lambda)} \right]$$

where $|\delta_{42}(X, \lambda)|$, $|\delta_{12}(X, \lambda)| \to 0$ as $\lambda \to \infty$. And so,

$$\frac{b_{11}}{\eta_{12}(X)} = \Delta[1 + \Delta_{11}] \tag{4.42}$$

where $\Delta_{11} = \Delta_{11}(X, \lambda)$ and $|\Delta_{11}| \to 0$ as $\lambda \to \infty$.

By a similar argument using (4.38) and (4.41) one can show that

$$\frac{b_{22}}{\eta_{12}(X)} = \frac{\eta_{32}(X)}{\eta_{12}(X)} = \Delta(1 + \Delta_{22}) \tag{4.43}$$

where $|\Delta_{22}| = |\Delta_{22}(X, \lambda)| \to 0$ as $\lambda \to \infty$.

By using (4.40) and (4.41) one can show that

$$\frac{\eta_{52}(X)}{\eta_{12}(X)} = \frac{2\sigma\ R_{52}(X, \lambda)}{\Delta^{-2}(-1 + \cosh 2\Delta X) + R_{12}(X, \lambda)}$$

$$= \frac{4\sigma\ R_{52}(X, \lambda)}{\Delta^{-2}\ e^{2\Delta X}[1 + e^{-4\Delta X} - 2e^{-2\Delta X} + 4e^{-2\Delta X}\ R_{12}(X, \lambda)]}$$

$$= \frac{4\Delta^2 \sigma^{-2} e^{-2i\tau/\sigma} R_{52}(X, \lambda)}{1 + \delta_{12}(X, \lambda)}$$

$$= O(1)\ R_{52}(X, \lambda)\ .$$

Also note that

$$\eta_{12}^{-1}(X) = \frac{\Delta^2 e^{2\Delta X}}{1 + \delta_{12}(X, \lambda)}$$

and so $|\eta_{12}^{-1}(X)| = O(\sigma^{-1})$. From these last two observations it follows that

$$\frac{b_{12}}{\eta_{12}(X)} = \frac{1}{2}\left[\frac{C - \eta_{52}(X)}{\eta_{12}(X)}\right] = \Delta_{12}(X, \lambda) = \Delta_{12}$$

and that

$$\frac{b_{21}}{\eta_{12}(X)} = -\frac{1}{2}\left[\frac{C + \eta_{52}(X)}{\eta_{12}(X)}\right] = \Delta_{21}(X, \lambda) = \Delta_{21}$$

where $|\Delta_{12}|$ and $|\Delta_{21}| \to 0$ as $\lambda \to \infty$ in the sector described by (4.8) .

We now write (4.35) as

$$\Phi'(X, \lambda)\ \Phi^{-1}(X, \lambda) = \begin{bmatrix} \Delta(1 + \Delta_{11}) & \Delta_{12} \\ \Delta_{21} & \Delta(1 + \Delta_{22}) \end{bmatrix} \qquad (4.44)$$

From this we see that

$$-\det[\ '(X,\lambda)\ \Phi^{-1}(X,\lambda) - (\Phi'(X,\lambda)\ \Phi^{-1}(X,\lambda))^*]$$

$$= [\Delta - \overline{\Delta} + (\Delta\Delta_{11} - \overline{\Delta\Delta}_{11})][\Delta - \overline{\Delta} + (\Delta\Delta_{22} - \overline{\Delta\Delta}_{22})]$$

$$+ |\Delta_{12} - \Delta_{21}|^2$$

$$= -\sigma^2[2\sigma^{-1}\tau + \sigma^{-1}(\Delta\Delta_{11} - \overline{\Delta\Delta}_{11})][2\sigma^{-1}\tau + \sigma^{-1}(\Delta\Delta_{22} - \overline{\Delta\Delta}_{22})]$$

$$+ |\Delta_{12} - \Delta_{21}|^2 .$$

From (4.8) we recall that $2\frac{\tau}{\sigma} > 2\varepsilon_0 > 0$ while from (4.9) $|\sigma^{-1}\ \Delta| < (1 + k^2)^{1/2}$. Since for $i,j = 1, 2$, $|\Delta_{ij}| \to 0$ as $\lambda \to \infty$, it follows that for $|\lambda|$ sufficiently large,

$$|\det[\Phi'\Phi^{-1} - (\Phi'\Phi^{-1})^*]| > C^2 .$$

From this and (4.44) it follows that

$$||[\Phi'\Phi^{-1} - (\Phi'\ \Phi^{-1})^*]^{-1}|| = O(\sigma^{-1})$$

as $\lambda \to \infty$ in the sector (4.8), where $\Phi' = \Phi'(X,\lambda)$ and $\Phi = \Phi(X,\lambda)$. Together with (4.34) and (4.33) we see that $||R_1^2(X,\lambda)||$ and $||R_1(X,\lambda)||$ tend to 0 as $\lambda \to \infty$ in the sector described by (4.8) . This shows by (4.32) that

$$\|M(\lambda) - M(X, \lambda)| = o(1)$$

as $\lambda \to \infty$, and by (4.31) completes the demonstration that

$$\|M(\lambda) - M_0(\lambda)\| = o(1)$$

as $\lambda \to \infty$ in a sector.

It recently came to our attention that Atkinson [5] has succeeded in developing the first order asymptotics as $\lambda \to \infty$ in a sector excluding the real axis for matrix Sturm-Liouville equations. That is, for λ such that $\varepsilon \leq \arg \lambda \leq \pi - \varepsilon$ he has considered the system

$$Jy' = (\lambda A + B)y \quad , \qquad a \leq x < b$$

for which

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \quad ; \quad A = A(x) = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B = B(x) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where A_{11}, B_{ij} are $n \times n$ matrix valued functions with locally integrable entries. When applied to the specific case considered in Theorem 4 it seems to show that

$$M(\lambda) = M_0(\lambda) + o(\sqrt{\lambda}) \quad .$$

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APPENDIX

APPENDIX

Note: M^T represents the transpose of the matrix M, and M^* stands for the conjugate transpose of M.

Definition: An $n \times n$ matrix M is hermitian if $M^* = M$.

Note: An $n \times n$ matrix M is hermitian if and only if $u^* Mu$ is a real number for all complex vectors u.

Definitions: A square matrix M is said to be nonnegative definite when $u^* Mu \geq 0$ for all complex vectors u, and is denoted by $M \geq 0$.

For the square matrices A and B, $A \geq B$ means that $A - B \geq 0$.

A matrix valued function $A(t)$ is said to be nondecreasing on (a, b) if for $a < t_1 < t_2 < b$, $0 \leq A(t_2) - A(t_1)$. If in addition, $A(t)$ is hermitian for $t \in (a, b)$ then the diagonal elements $a_{ii}(t)$ are nonnegative and nondecreasing on (a, b).

For any matrix M, let the real and imaginary parts be defined as follows:

$$\text{Re } M = \frac{1}{2} [M + M^*] \quad \text{and} \quad \text{Im } M = \frac{1}{2i} [M - M^*] .$$

An immediate property of nonnegative definite matrices is that the diagonal elements are real and nonnegative. An important property of such matrices is that the diagonal elements dominate in the sense that

$$|a_{ij}|^2 \leq a_{ii}\, a_{jj} \quad . \tag{A.1}$$

The matrices Re M and Im M are hermitian and their entries are not necessarily real valued. From the above definition, M is hermitian if and only if Im M = 0 .

Definitions: By a matrix valued function $A(t)$ which is continuous, or absolutely continuous, or of bounded variation, we mean a matrix of complex-valued functions each of whose entries possess these properties. We interpret $\int_0^t f(t)\, dA(t)$ to mean the limit of Riemann sums of the form: $\Sigma f(\xi_i)[A(t_{i+1}) - A(t_i)]$. The integral, thus defined, is nothing more than an array of Reimann-Stieltjes integrals.

Note: In a straight forward manner, one can extend many of the standard theorems of integration to include integrals of this type. An account of some standard integration theorems together with a discussion of some extensions can be found in [2, pp. 416-435] .

Definition: A matrix valued function $M(z)$ is said to be analytic when its entries are analytic functions of z .

It is known, when $f(z)$ is analytic in $|z| < r$ and continuous in $|z| \leq r$, that

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{it} + z}{re^{it} - z} \operatorname{Re}[f(re^{it})]\, dt \tag{A.2}$$

for $|z| < r$.

A similar statement can be made for analytic matrices.

Lemma A.1: Let $M(z)$ be an $n \times n$ matrix with entries analytic for $|z| < r$ and continuous for $|z| \leq r$. For such $M(z)$,

$$M(z) = i\,\mathrm{Im}M(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{it} + z}{re^{it} - z}\,\mathrm{Re}M(re^{it})dt$$

for $|z| < r$.

Proof: Let $M(z) = A(z) + iB(z)$ where $A(z)$ and $B(z)$ have real-valued entries. Now by (A.2)

$$M(z) = i\beta + \frac{1}{2\pi}\int_0^{2\pi} K(t, z)\, A(re^{it})\, dt \tag{A.3}$$

where $\beta = B(0)$, $K(t, z) = \dfrac{re^{it} + z}{re^{it} - z}$, and $|z| < r$.

Since the entries of $iM(z)$ are also analytic for $|z| < r$ and continuous for $|z| \leq r$, we have

$$iM(z) = i\alpha - \frac{1}{2\pi}\int_0^{2\pi} K(t, z)B(re^{it})dt$$

where $\alpha = A(0)$ and $|z| < r$. Consequently

$$M(z) = \alpha + \frac{1}{2\pi}\int_0^{2\pi} K(t, z)(iB(re^{it}))dt \tag{A.4}$$

Now $M(z) = \dfrac{M(z) + M^T(z)}{2} + \dfrac{M(z) - M^T(z)}{2}$ and by (A.2),

$$\frac{M(z) + M^T(z)}{2} = i\left(\frac{\beta + \beta^T}{2}\right)$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} K(t, z)\left(\frac{A(re^{it}) + A^T(re^{it})}{2}\right)dt$$

and by (A.4) ,

$$\frac{M(z) - M^T(z)}{2} = \left(\frac{\alpha - \alpha^T}{2}\right)$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} K(t, z)\left[i\left(\frac{B(re^{it}) - B^T(re^{it})}{2}\right)\right]dt \ .$$

So

$$M(z) = i\left[\left(\frac{\beta + \beta^T}{2}\right) - i\left(\frac{\alpha - \alpha^T}{2}\right)\right]$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} K(t, z)\left[\frac{A(re^{it}) + A^T(re^{it})}{2} + i\left(\frac{B(re^{it}) - B^T(re^{it})}{2}\right)\right] dt \ .$$

Next note that,

$$\operatorname{Re} M(z) = \frac{M(z) + M^*(z)}{2} = \left(\frac{A + A^T}{2}\right) + i\left(\frac{B - B^T}{2}\right) \quad \text{and}$$

$$\operatorname{Im} M(z) = \frac{M(z) - M^*(z)}{2i} = \left(\frac{B + B^T}{2}\right) - i\left(\frac{A - A^T}{2}\right) \ .$$

As a consequence,

$$M(z) = i\,\mathrm{Im}\,M(0) + \frac{1}{2\pi}\int_0^{2\pi} K(t,\ z)\ \mathrm{Re}\ M(re^{it})dt \qquad (A.5)$$

for $|z| < r$. ■

Note that if $M = M^T$, and as a consequence, that $A = A^T$ and $B = B^T$, then (A.5) reduces to (A.3) .

If $\sigma_r(t) = \int_0^t \mathrm{Re}\ M(re^{is})\ ds$ then (A.5) becomes

$$M(z) = i\,\mathrm{Im}\ M(0) + \frac{1}{2\pi}\int_0^{2\pi} K(t,\ z)\ d\sigma_r(t)\ .$$

<u>Lemma A.2</u>: Let $\{A_\alpha(t)\}$ be a family of $n \times n$ nonnegative definite, nondecreasing matrix valued functions defined on $(a,\ b)$. If there is a nonnegative definite matrix C such that $0 \leq A_\alpha(t) \leq C$ for $t \in (a,\ b)$ and all α , then the elements of $A_\alpha(t)$ are collectively, uniformly bounded and of uniform bounded variation.

<u>Proof</u>: Let $A(t)$ be an arbitrarily chosen matrix from the family; then, first we note that $a_{ii}(t) \leq c_{ii}$. By (A.1) this implies that $|a_{jk}(t)| \leq \max_{i=1,\ldots,n} \{c_{ii}\}$ for $j,k = 1,\ldots,n$. Hence the entries are bounded independent of $A(t)$.

Note that for $A(t)$

$$\sum_k |a_{ij}(t_{k+1}) - a_{ij}(t_k)| \leq$$

$$\leq \sum_k([a_{ii}(t_{k+1}) - a_{ii}(t_k)]^{1/2}[a_{jj}(t_{k+1} - a_{jj}(t_k)]^{1/2})$$

$$\leq \left(\sum_k [a_{ii}(t_{k+1}) - a_{ii}(t_k)] \sum_k [a_{jj}(t_{k+1}) - a_{jj}(t_k)]\right)^{1/2}$$

$$\leq (c_{ii}\ c_{jj})^{1/2} \leq \max_{i=1,\ldots,n} \{c_{ii}\}$$

where $\sum_k (a_{ii}(t_{k+1}) - a_{ii}(t_k)) \leq c_{ii}$ because $a_{ii}(t)$ is nondecreasing and bounded by c_{ii} .

For such a family of matrix valued functions as described in Lemma A.2 one can, by a diagonal procedure, produce a subsequence $\{A_n(t)\}$ of the family $\{A_\alpha(t)\}$ which converges, on the rationals of (a, b) , to a nonnegative definite, nondecreasing function $B(t)$. On the irrationals of (a, b) we can define $B(t)$ by means of left continuity. $B(t)$ thus defined is nonnegative, nondecreasing and shares the same bound on the variation of its elements as $\{A_\alpha(t)\}$.

Since the elements $b_{ii}(t)$ are nondecreasing on (a, b) the union of the sets of discontinuity is countable. Again by (A.1) we see that set of discontinuity of any $b_{ij}(t)$ is contained in the above union. Thus the set of discontinuity for $B(t)$ is countable.

One can show that the subsequence $\{A_n(t)\}$ must, in fact, converge to $B(t)$ at its points of continuity. As the set of discontinuity of $B(t)$ is countable, a further subsequence $\{A_{n_k}(t)\}$ can be found converging on this set and with suitable redefinition we can have a nonnegative nondecreasing $B(t)$ of bounded variation to which $\{A_{n_k}(t)\}$ converges on (a, b) . ■

We state without proof the following matrix generalizations of Helly-Bray theorems appearing in [2, pp. 425-428] .

Theorem A.1: For a family $\{A_\alpha(t)\}$ of $n \times n$ nonnegative definite, nondecreasing matrix valued functions defined on a finite interval (a, b) and which satisfy $0 \leq A_\alpha(t) \leq C$ for a constant matrix C, there is a sequence $\{A_n(t)\}$ and a nonnegative definite nondecreasing function $B(t)$ of bounded variation to which $\{A_n(t)\}$ converges on (a, b), such that, for functions $f(t)$ continuous on $[a, b]$

$$\lim_{n\to\infty} \int_a^b f(t)\, dA_n(t) = \int_a^b f(t)\, dB(t) .$$

When the interval $(a, b) = (-\infty, \infty)$ then one has:

Theorem A.2: For a family of matrices $\{A_\alpha(t)\}$ satisfying the hypotheses of Theorem A.1 where in addition, $A_n(\infty) = \lim_{t\to\infty} A_n(t)$ and $A_n(-\infty) = \lim_{t\to-\infty} A_n(t)$, there is a sequence $\{A_n(t)\}$ and a nonnegative definite, nondecreasing function $B(t)$ of bounded variation to which $\{A_n(t)\}$ converges on $-\infty \leq t \leq \infty$, such that for a function $f(t)$, continuous on $\mathbb{R}$ and with finite limits $f(\infty) = \lim_{t\to\infty} f(t)$, $f(-\infty) = \lim_{t\to-\infty} f(t)$, the following holds:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, dA_n(t) = \int_{-\infty}^{\infty} f(t)\, dB(t) + f(\infty)\,[B(\infty) - B(\infty^-)]$$

$$+ f(-\infty)\,[B(-\infty^+) - B(-\infty)]$$

where $B(\infty^-) = \lim_{t\to\infty} B(t)$ and $B(-\infty^+) = \lim_{t\to-\infty} B(t)$.

Before considering the next result, note that if $f(t) \geq 0$ and $A(t)$ is nonnegative definite and of bounded variation for

$t \in [a, b]$, then $\int_c^d f(t)\, dA(t) \geq 0$, $a \leq c \leq d \leq b$. And further, note that $\int_a^t f(\xi)\, dA(\xi)$ is a nonnegative definite, nondecreasing matrix valued function for $t \in [a, b]$.

We next restate a standard representation theorem for functions which are analytic and have nonnegative real part in the circle $|z| < 1$ [1] .

Theorem A.3: A matrix-valued function $M(z)$, with entries which are finite in the circle $|z| < 1$, will admit the representation

$$M(z) = iB + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, dA(t)$$

where B is hermitian and $A(t)$ is a non-negative definite, non-decreasing matrix-valued function of bounded variation if and only if $M(z)$ has a nonnegative definite real part and is analytic in $|z| < 1$.

Proof: We modify a standard proof found in [1 , p. 5] .

Suppose that $M(z)$ does possess this representation. Then if $K(t, z) = \frac{e^{it} + z}{e^{it} - z}$, we have

$$\operatorname{Re} M(z) = \operatorname{Re}\left[iB + \int_0^{2\pi} K(t, z)\, dA(t)\right] = \operatorname{Re}\left[\int_0^{2\pi} K(t, z)\, dA(t)\right]$$

$$= \int_0^{2} \operatorname{Re}(K(t, z))\, dA(t) .$$

The last equality follows because $A(t)$ is nondecreasing so that $A(t) - A(s) = B(t, s)$ is hermitian, and so, if $\alpha \in \mathbb{C}$, then

$$\mathrm{Re}(\alpha B(t, s)) = \frac{1}{2} [\alpha^* B^*(t, s) + \alpha B(t, s)]$$

$$= \frac{1}{2} (\alpha^* + \alpha) B(t, s)$$

$$= (\mathrm{Re}\ \alpha)\ B(t, s)\ .$$

Now $\mathrm{Re}\{K(t, z)\} = \dfrac{1 - \zeta^2}{1 - 2\zeta\cos(t - \phi) + \zeta^2}$ where $\zeta = |z| < 1$, $\phi = \arg z$. As a result, $\mathrm{Re}\{K(t, z)\} \geq 0$, and as a consequence, $\int_0^{2\pi} \mathrm{Re}\{K(t, z)\}\ dA(t) \geq 0$.

Given the representation, it is clear that each entry of M is analytic.

Suppose on the other hand that $M(z)$ is matrix-valued, analytic with nonnegative definite real part for $|z| < 1$. By the initial Lemma, A.1,

$$M(z) = \imath\,\mathrm{Im}\ M(0) + \frac{1}{2\pi} \int_0^{2\pi} K(t, r, z)\ \mathrm{Re}\ M(re^{it})\ dt$$

for $|z| < r < 1$ where $K(t, r, z) = \dfrac{re^{it} + z}{re^{it} - z}$. By assumption $\mathrm{Re}\ M(re^{it}) \geq 0$, hence $A_R(t) = \int_0^t \mathrm{Re}\ M(re^{is})\ ds$ is nonnegative definite and nondecreasing as a function of t and $0 \leq A_r(t) \leq A_r(2\pi) = \mathrm{Re}\ M(0)$. The set of matrix-valued functions $\{A_r(t)\}$ for $0 < r < 1$ thus forms a family of such functions which are uniformly bounded and of uniform bounded variation. By Theorem A.1 there is a nonnegative, nondecreasing function of bounded variation $A(t)$ to which a subsequence $A_{r_n}(t)$ converges pointwise on $[0, 2\pi]$, as $r_n \to 1$, and for which

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} K(t, r_n, z)\, dA_{r_n}(t) = \frac{1}{2\pi}\int_0^{2\pi} K(t, 1, z)\, dA(t)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, dA(t) .$$

This limit exists because of the Helly theorem and because $K(t, r_n, z)$ converges to $K(t, 1, z)$ uniformly for $t \in [0, 2\pi]$ as $r_n \to 1$.

As a consequence,

$$M(z) = \lim_{n\to\infty} \{i\operatorname{Im} M(0) + \frac{1}{2\pi}\int_0^{2\pi} K(t, r_n, z)\, dA_{r_n}(t)\}$$

$$= i\operatorname{Im} M(0) + \frac{1}{2\pi}\int_0^{2\pi} K(t, 1, z)\, dA(t)$$

$$= i\operatorname{Im} M(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, dA(t)$$

for all $|z| < 1$.

In what sense is the matrix-valued function $A(t)$ unique? Within the circle $|z| < 1$, the function

$$K(t, 1, z) = -1 + \frac{2}{1 - z/e^{it}} = -1 + 2 \sum_{n=0}^{\infty} (z/e^{it})^n$$

$$= 1 + 2 \sum_{n=1}^{\infty} z^n e^{-int}$$

and so

$$M(z) = i \text{ Im } M(0) + \frac{1}{2\pi}\int_0^{2\pi} dA(t) + \sum_{n=1}^{\infty} [\frac{1}{\pi} \int_0^{2\pi} e^{-int} dA(t)]z^n .$$

Now,

$$\int_0^{2\pi} dA(t) = \lim_{n\to\infty} \int_0^{2\pi} dA_{r_n}(t)$$

$$= \lim_{n\to\infty} [A_{r_n}(2\pi) - A_{r_n}(0)]$$

$$= \lim_{n\to\infty} [\text{Re } M(0) - 0] = \text{Re } M(0) .$$

Consequently, if there is a second matrix-valued measure $B(t)$ such that $M(z) = i\text{Im } M(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dB(t)$, then by comparing the resulting Taylor expansions we note that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-int} dA(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} dB(t)$$

for $k = 0, 1, 2, 3, \ldots$.

Both $A(t)$ and $B(t)$ are nondecreasing, thus both $A(t) - A(s)$ and $B(t) - B(s)$ are hermitian and so,

$$[\frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} dA(t)]^* = [\frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} dB(t)]^*$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikt} dA(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikt} dB(t) .$$

Now the difference $A(t) - B(t) = C(t)$ is a matrix-valued function of bounded variation for which

$$\int_0^{2\pi} e^{ikt} dC(t) = 0 \quad \text{for} \quad k = 0, \pm 1 \quad \pm 2 \quad \ldots \quad .$$

The last statement we recall is in fact a statement about elements; namely,

$$\int_0^{2\pi} e^{ikt} dc_{jm}(t) = 0 \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots \quad . \tag{A.6}$$

This means that

$$\int_0^{2\pi} e^{ikt} dc_{jm}(t) = e^{ikt} c_{jm}(t) \Big|_0^{2\pi} - ik \int_0^{2\pi} e^{ikt} c_{jm}(t)dt$$

$$= \int_0^{2\pi} dc_{jm}(t) - ik \int_0^{2\pi} e^{ik} c_{jm}(t)dt$$

and hence by (A.6), that

$$0 = 0 - ik \int_0^{2\pi} e^{ikt} c_{jm}(t)dt \, .$$

so for $k \neq 0$, $\int_0^{2\pi} e^{ikt} c_{jm}(t)dt = 0$. Setting $d_{jm} = \int_0^{2\pi} c_{jm}(t)dt$ we note that

$$\int_0^{2\pi} [c_{jm}(t) - d_{jm}] e^{ikt} dt = 0 \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots$$

so that $c_{jm}(t) - d_{jm}$ has Fourier coefficients all of which are 0 .

The uniqueness of these coefficients implies that $c_{jm}(t) - d_{jm} = 0$ and hence $C(t) - D = 0$ at each point of continuity of $C(t)$ and as a consequence $A(t)$ and $B(t)$ differ by a constant matrix at their points of continuity.

Since $A(t) - A(s) \geq 0$ for $t > s$, the diagonal elements of this difference dominate the off-diagonal elements. As a consequence, a point at which all diagonal elements are continuous is a point where all elements of $A(t)$ are continuous.

In order to guarantee that $A(t)$ in the theorem is unique it is necessary to impose some normalization conditions. For example, if we require that $A(t)$ be left continuous in $(0, 2\pi)$, and if we define $\tilde{A}(t)$ such that $\tilde{A}(t) = A(t)$ for $0 < t < 2\pi$,

$$\hat{A}(2\pi) = \tilde{A}(2\pi - 0) = A(2\pi - 0) ,$$

$$\tilde{A}(+0) - \tilde{A}(0) = A(+0) - A(0) + A(2\pi) - A(2\pi - 0) ,$$

then for functions f where $f(0) = f(2\pi)$, $\int_0^{2\pi} f(t)\, dA(t) = \int_0^{2\pi} f(t)\, dA(t)$. If we define $\Omega(t) = A(t) - A(0)$ then $\Omega(0) = 0$,

$$\int_0^{2\pi} f(t)\, d\Omega\, (t) = \int_0^{2\pi} f(t)\, dA(t)$$

and $\Omega(t)$ is unique.

With the representation theorem for matrix valued functions one can, as is done in [1, p. 7] , by means of the transformations $z = i(\frac{1 + \zeta}{1 - \zeta})$ and $\phi(z) = if(\zeta)$, establish the following representation theorem:

Theorem A.4: A matrix valued function $M(z)$, which is finite for z such that $\operatorname{Im} z > 0$, has a representation

$$M(z) = A + zB + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \, d\Omega(t) , \tag{A.7}$$

where $B \geq 0$, A is hermitian, and $\Omega(t)$ is nonnegative definite, nondecreasing and of bounded variation, if and only if $M(z)$ is analytic and has nonnegative definite imaginary part for z with $\operatorname{Im} z > 0$.

If one defines, up to a constant matrix, the matrix valued function $\rho(t)$ by

$$d\Omega(t) = \frac{1}{1 + t^2} \, d\rho(t) ,$$

then $\rho(t)$ will also be nonnegative definite, and nondecreasing. It will share with $\Omega(t)$ the same set of points of continuity in $\mathbb{R}$. In terms of this measure $\rho(t)$,

$$\int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \, d\Omega(t) = \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) d\rho(t)$$

and the characterization of $M(z)$ can now be written as

$$M(z) = A + zB + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{1}{1 + t^2}\right) d\rho(t) . \tag{A.8}$$

VITA

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