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General Relativistic Gravity in Core-Collapse Supernova Simulations

A Dissertation Presented for the
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James Nicholas Roberts II
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Acknowledgements

It has been said that we are the sum total of our experiences, each shaping the course of our life like stones guiding a river. While most experiences amount to pebbles on the shore, others can reshape the flow entirely. I would like to take this opportunity to acknowledge those people who have made lasting impacts on my life.

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Abstract

Core-collapse supernovae (CCSNe) are some of the most extreme and complex phenomena in the universe. The toolkit for high-order neutrino-radiation hydrodynamics (**thornado**) is being developed to simulate CCSNe, which will provide insight into the mechanisms underlying these events. The **thornado** framework is a collection of modules used to calculate the effects of gravity, hydrodynamics, neutrino transport, and nuclear physics through the Weaklib equation of state and opacity tables. This dissertation will present the development of the **Poseidon** code, which provides the general relativistic gravity solver for the **thornado** framework.

The **Poseidon** code solves for the general relativistic metric using the xCFC formulation of the Einstein equations. The first part of this dissertation will present the derivation of the xCFC metric equations. This will be done by first casting the Einstein equations in the 3+1 formalism and applying the conformally flat approximation (CFA). The CFA system of equations suffers from non-uniqueness properties, therefore a further approximation is made to arrive at the xCFC system of equations for the metric variables: ψ , the conformal factor; α , the lapse function; and β , the shift vector.

The xCFC equations involve two types of equations, non-linear scalar Laplacians, and linear vector Laplacians. **Poseidon** uses Galerkin projections with real spherical harmonics and radial finite element polynomials to discretize both types of equations. The Fixed-point iteration method with Anderson acceleration is used to solve the non-linear equations that govern the non-linear equations. We will also discuss the

coupling of `Poseidon` with `thornado`'s hydrodynamics and the incorporation of the adaptive mesh refinement framework, AMReX.

We also present results from `Poseidon` and the coupled `thornadoPoseidon` framework. These results include the evolution of realistic stellar progenitors through collapse, core bounce, and the subsequent outward shock propagation. These simulations are done using relativistic hydrodynamics and gravity, and the Weaklib tabulated equation of state table and reveal the importance of relativity on the simulation of CCSNe.

Table of Contents

1	Introduction	1
2	The 3+1 Formulation of General Relativity	8
2.1	Introduction	8
2.2	The Einstein Equations	9
2.3	Foliating Spacetime	9
2.4	Derivation of the 3+1 Metric	10
2.4.1	The Normal Unit Vector	10
2.4.2	Normal Evolution Vector	12
2.4.3	The Lie Derivative	15
2.4.4	Lapse Function	19
2.4.5	Defining the 3-Metric on Σ_t	20
2.4.6	Defining the Time Vector	21
2.4.7	Defining the 3+1 Metric	24
2.4.8	Projection Operators	26
2.5	3+1 Covariant Derivative	28
2.6	Gauss - Codazzi Relations	32
2.6.1	Gauss Relations	32
2.6.2	Codazzi Relations	38
2.6.3	Mixed Projection of the Ricci Tensor	40
2.7	Time Evolution of the Spatial Metric, γ	43

2.8	Decomposition of Einstein Equation	47
2.8.1	Decomposition of the Stress-Energy Tensor, T	48
2.8.2	Projection Perpendicular To Σ_t	49
2.8.3	Projection onto Σ_t	50
2.8.4	Mixed Projection	50
2.9	Summary	51
3	Approximating General Relativistic Gravity: The Conformally Flat Condition, CFC	53
3.1	Introduction	53
3.2	Defining the Conformal Metric	54
3.3	Conformal Transformations	56
3.3.1	Christoffel Symbol Transformation	56
3.3.2	Covariant Derivative Transformation	58
3.3.3	Ricci Conformal Transformations	59
3.4	The Conformal Factor Equation	62
3.5	The Lapse Function Equation	63
3.5.1	Extrinsic Curvature Transformation	65
3.6	The Shift Vector Equation	69
3.7	Summary	72
4	The eXtended Conformally Flat Condition, xCFC	74
4.1	Non-Uniqueness in the CFC Equations	74
4.2	Source Variable Scaling	75
4.3	Approximation of K_{ij}	75
4.4	Summary	80
5	Numerical Methods	82
5.1	Domain Decomposition	83
5.2	Quadrature	84

5.3	Conformal Factor and Lapse Function Method	85
5.3.1	Galerkin Projection	85
5.3.2	Fixed Point Method and Anderson Acceleration	93
5.3.3	Properties of the Conformal Factor and Lapse Function Linear Systems	94
5.4	X and Shift Vector Method	96
5.5	Hydrodynamics Coupling	104
5.6	AMReX Incorporation	106
6	Code Tests	108
6.1	Introduction	108
6.2	Hamiltonian Constraint Test	108
6.3	Homogenous Ellipsoid Test	110
6.4	Conformal Vector Laplacian Test	113
6.5	Yahil Self-Similar Collapse Test	118
6.6	Conclusion	124
7	General Relativistic Adiabatic Collapse in Spherical Symmetry	125
7.1	Collapse of a 15 $M_{\odot}$ Star	126
7.1.1	Validation of AMR Implementation	126
7.1.2	Resolution Study	128
7.2	Collapse of a 37 $M_{\odot}$ Star	133
7.3	Performace	133
8	Conclusion	138
8.1	Summary	138
8.2	My Contributions	139
8.3	Outlook	140
	Bibliography	142

List of Tables

6.1	Error norms for the Homogenous Ellipsoid problem run with 706 radial leaf elements on 4 levels of refinement and 256 theta elements at the finest resolution.	114
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List of Figures

2.1	The spacetime manifold, $\mathcal{M}$, foliated into a set of non-intersecting spacelike hypersurfaces, Σ_t	11
2.2	Diagrammatic representation of the 3+1 foliation showing the relationship between two adjacent space-like hypersurfaces. The four-vector $\mathbf{t}$ relates the evolution of the point x^i . The timelike component α measures the temporal distance between hypersurfaces with the lapse function α measuring the proper time. The shift vector, β , measures the shift in coordinates.	27
5.1	The hat function λ_n is the sum of the two linear Lagrange polynomials, $\ell_{i,2}$ and $\ell_{i+1,1}$	88
5.2	Nonzero structure of the matrix $S_{l_{nn'}}$ used in Poseidon for the case of $k = 2$ and $N_r = 4$. The diagonal lines denote the band structure of the matrix. The blue squares surrounding the crosses denote the overlapping block structure within the band, and the crosses denote individual, non-zero-valued, elements. The single overlapping element shared by the consecutive blocks comes from the shared interpolation node at element interfaces.	97

6.1	L^1 (left panel) and L^2 (right panel) errors between the analytic and numerical solution of the conformal factor for a uniform density sphere. The error norms are scaled by the number of degrees of freedom to obtain an average error per node. The dashed lines serve as a reference for the convergence rates of the numerical solutions.	111
6.2	Results from Poseidon using a homogenous ellipsoid as the source profile. This test demonstrates the code's ability to capture multi-dimensional details using different limits on the spherical harmonic expansion. The left column shows the Conformal Factor as calculated by Poseidon while the right column shows the relative error.	115
6.3	Results from Poseidon using a homogenous ellipsoid as the source profile. This test demonstrates the code's ability to capture multi-dimensional details using different limits on the spherical harmonic expansion. The left column shows the Conformal Factor as calculated by Poseidon while the right column shows the relative error.	116
6.4	Comparable results can be achieved with a refined mesh using far fewer elements than with a uniform mesh.	117
6.5	The L^1 (Left) and L^2 (Right) norms of the relative difference between low-resolution solutions and a very fine-resolution reference solution are represented by the solid lines. The dashed lines give the norms that one would expect based on the convergence properties of the finite element method. Different colors represent the degrees of finite element polynomials used to approximate the solution. In both cases, we see the expected convergence rate for this method.	119
6.6	Results from thornado-Poseidon showing the collapse of the Yahil progenitor using an ideal equation of state. The upper left and upper right panels show the fluid density and velocity respectively. The lower left panel shows the conformal factor, ψ , and lapse function, α , while the lower right presents the radial component of the shift vector. . . .	121

6.7	Here, we compare the simulated conformal factor and lapse function (solid lines) with the values predicted by the Schwarzschild metric in Isotropic coordinates (dashed lines). This comparison is only appropriate in the weak-field limit and while $\beta/c \ll 1$ as can be seen by the increasing difference between predicted and simulated results.	122
6.8	Anderson acceleration is used to reduce the number of fixed point iterations needed to converge to a solution. This figure shows the effects of using information from the previous M iterates to update the solution.	123
7.1	The number of radial elements during collapse and post-bounce. The domain is refined based on density values therefore as the star collapses and becomes more compact, more cells are refined to higher degrees. This leads to an increasing number of leaf elements during collapse which peak at bounce. In the post-bounce regime, there is an initial drop followed by long periods when the mesh is unchanging.	127
7.2	Adiabatic collapse of a $15 M_{\odot}$ star using a uniform grid and an AMR grid. (Top Left) Mass density in g cm^{-3} , (Top Right) fluid velocity normalized to the speed of light, (Upper Middle Left) temperature (K), (Upper Middle Right) entropy per baryon in k_B , (Lower Middle Left) conformal factor, (Lower Middle Right) electron fraction, (Lowest Left) lapse function, (Lowest Right) radial component of the shift vector are plotted as functions of radius in km. The solutions are compared during collapse when the central density is approximately $[10^{10}, 10^{11}, 10^{12}, 10^{13}, 10^{14}] \text{ g cm}^{-3}$.	129

7.3	Post-bounce evolution of a $15 M_{\odot}$ star using a uniform grid and an AMR grid. (Top Left) Mass density in g cm^{-3} , (Top Right) fluid velocity normalized to the speed of light, (Upper Middle Left) temperature (K), (Upper Middle Right) entropy per baryon in k_B , (Lower Middle Left) conformal factor, (Lower Middle Right) electron fraction, (Lowest Left) lapse function, (Lowest Right) radial component of the shift vector are plotted as functions of radius in km. The solutions are compared at intervals measured relative to core bounce.	130
7.4	Results from the adiabatic collapse of a $15 M_{\odot}$ star run with different maximum levels of refinement. The resolution of the simulations on the finest levels are 1km, 0.5km, and 0.25km given by the different line styles. The different colors represent different moments during collapse corresponding to the first instant the central density reaches a new decade. The panels show the same variables as in Figure 7.2	131
7.5	Post-bounce phase of the collapse of a $15 M_{\odot}$ star run with different maximum levels of refinement. The resolution of the simulations on the finest levels are 1 km, 0.5 km, and 0.25 km given by the different line styles. The different colors represent different moments during collapse corresponding to the first instant the central density reaches a new decade.	132
7.6	Post-bounce evolution of a 15 solar mass star and 37 solar mass star using an AMR grid. (Top Left) Mass density in grams per cubic centimeter, (Top Right) fluid velocity normalized to the speed of light, (Upper Middle Left) temperature, (Upper Middle Right) entropy, (Lower Middle Left) conformal factor, (Lower Middle Right) electron fraction, (Lowest Left) lapse function, (Lowest Right) radial component of the shift vector are plotted as functions of radius in kilometers. The solutions are compared at bounce and after 100 ms of post-bounce evolution.	134

7.7	Collapse of a 37 solar mass star using different order methods. The panels relate the same variables as in Figure 7.2. The solutions are plotted as each run reaches a new central density decade.	135
7.8	Comparisons of the density (panels A and B), Temperature, (panels C and D), and Entropy (panels E and F) during the pre-bounce collapse (left panels) and post-bounce shock propagation (right panels) for the adiabatic collapse of a 15 $M_{\odot}$ star. We see good agreement between the runs. Spacing out gravity solves without impacting the evolution of the star will help reduce computational costs.	137

Chapter 1

Introduction

The most advanced core-collapse supernova (CCSN) simulations rely on relativistic treatments of hydrodynamics, gravity, and neutrino transport. Each of these fields, in turn, relies on knowing the spacetime metric to properly account for the effects of gravity. Therefore, we present **Poseidon**, a relativistic gravity solver that calculates the spacetime metric subject to the extended conformally flat condition (xCFC).

During the lifetime of massive stars, multiple fusion cycles create layers of elemental ash until an iron core forms. For stars with zero-age main sequence (ZAMS) masses greater than approximately 8 solar masses, $M_{\odot}$, the iron core can grow until it reaches its effective Chandrasekhar mass, $\sim 1.4M_{\odot}$ (Baron and Cooperstein, 1990). At this point, the electrons have been squeezed close enough to the nucleus that electrons begin to be captured by protons and turned into neutrons. As the number of electrons begins to decrease so does the degeneracy pressure and collapse begins. Iron photodisintegrates into neutrons and helium as the core collapses, and the helium further disintegrates into neutrons and protons. This increased number of protons accelerates electron capture, further reducing degeneracy pressure and leading to a neutron-rich core called a proto-neutron star. The collapse and subsequent explosion release 10^{53} erg of gravitational energy in the form of neutrinos in what is called a core-collapse supernova (CCSN) (Branch and Wheeler, 2017).

Understanding the mechanics of CCSNe has been a subject of much research as they are related to many important phenomena. Many heavy elements are synthesized during a CCSN, potentially including some from rapid neutron capture (Tamamura et al., 2003; Martínez-Pinedo et al., 2014; Cowan et al., 2021). They are then ejected along with outer elemental layers, thereby promoting galactic chemical evolution. The material in the collapsed core can have densities exceeding $10^{14} \text{ g cm}^{-3}$ and fluid speeds up to 30% of the speed of light. At these extremes, aspherical movement can produce gravitational waves that should be detectable with current-generation detectors (e.g., LIGO, Virgo, and KAGRA) as well as with the next-generation detectors such as Cosmic Explorer and the Einstein Telescope.

While many aspects of the CCSN mechanism are known, a full understanding remains elusive (for reviews see Mezzacappa, 2001, 2005; Janka et al., 2012, 2016; Burrows, 2013; Hix et al., 2014; Müller et al., 2016; Couch, 2017). A key problem remaining concerns the re-energization of the stalled shock. When the collapsing core reaches nuclear matter densities, repulsive nucleon-nucleon interactions halt the collapse and cause the core to rebound sending a shockwave outward. It was originally thought that this shock would propagate immediately to the stellar surface, exploding the star. However, early computer simulations showed that the shock loses energy by photo-dissociating the in-falling matter and emitting copious amounts of neutrinos (Burrows and Lattimer, 1985). This energy loss causes the shock to stall before it can reach the stellar surface, and it is ultimately pushed back into the core. The failure of these simulations to explode prompted the search for a delayed explosion mechanism. Several methods for re-invigoration of the shock have been proposed, with the most widely accepted explanation being the delayed neutrino heating mechanism (Bethe and Wilson, 1985). During a CCSNe only 1% of the energy released is contained in the visible explosion, while the majority of the energy emitted from the core as neutrinos. The neutrino heating mechanism proposes that $\sim 1\%$ of that energy is transferred back into the shock, giving it the energy to resume its outward motion. Other methods include a magneto-rotational mechanism (Akiyama et al., 2003).

To better understand the dynamics of CCSNe, computer simulations have been developed to model the complex interactions of gravity, hydrodynamics, neutrino transport, and nuclear physics at relativistic extremes. Early simulations were able to use full general relativity in spherical symmetry (Colgate and Johnson, 1960; Colgate and White, 1966; van Riper, 1979; Bethe and Wilson, 1985; Mezzacappa and Bruenn, 1993c,a,b; Rampp and Janka, 2002; Tamamura et al., 2003; Thompson et al., 2003; Sumiyoshi et al., 2007; H  depohl et al., 2010; Boccioli et al., 2021; Gogilashvili et al., 2023) as the restriction to spherical symmetry simplified the Einstein equations. These simulations revealed that something was missing as the self-consistent models only produced explosions for very low-mass progenitors. Two-dimensional simulations gave the hydrodynamics freedom for convective flows within the star (Arnett et al., 1989; Herant et al., 1992; Burrows and Fryxell, 1992; Janka and Mueller, 1995; Mezzacappa et al., 1998). These flows allowed for the more efficient transfer of energy from near the core to the shock out beyond the neutrinosphere where neutrinos have larger mean free paths and interact less with matter. Further, it was discovered that the stalled accretion shock was sensitive to non-radial perturbations and could oscillate about the proto-neutron star (PNS) core (Blondin et al., 2003; Blondin and Mezzacappa, 2006). These standing accretion shock instabilities (SASI) have been shown to impact the post-bounce evolution of the CCSN explosion and PNS remnant (Blondin and Mezzacappa, 2007; Yoshida et al., 2007; Endeve et al., 2016; Kuroda et al., 2016a; Blondin et al., 2017; Walk et al., 2018; Vartanyan and Burrows, 2020; Dunham et al., 2023)

In the last decade and a half, computational technology has advanced to the point that 3-dimensional CCSNe simulations have become realizable. Beginning with the early simulations of Takiwaki et al. (2012); Hanke et al. (2013); Lentz et al. (2015), 3D models showed differences from the earlier 2D models including greater neutrino luminosities, stronger SASI amplitudes, and delayed shock revival. Further models have been presented (Summa et al., 2018; O’Connor and Couch, 2018; Nakamura et al., 2019; Powell and M  ller, 2020; Stockinger et al., 2020; Sandoval et al., 2021;

Aksenov and Chechetkin, 2023; Wang and Burrows, 2023), with some focusing on the role that magnetic fields play in the evolution of CCSN (Varma et al., 2023; Matsumoto et al., 2024).

The computational expense of 3D CCSN simulations is high, requiring several months to complete; therefore each of these codes has been developed to run on leadership-class supercomputers utilizing techniques to efficiently distribute work across CPUs and GPUs. Even so, approximations are made to keep runtimes feasible with the most prominent approximation being the degree to which general relativity is represented. One of the simplest approximations widely used modifies the monopole term of the Newtonian potential (Marek et al., 2006). More advanced methods include free-evolution schemes, such as the BSSNOK (Nakamura et al., 1987; Shibata and Nakamura, 1995; Baumgarte and Shapiro, 1998) and CZ4 (Bona et al., 2003) models, in which the hyperbolic evolution equations are solved and the constraint equations are used to ensure violations remain within some tolerance. In contrast, constrained models, including the Conformally Flat Condition (CFC) (Wilson et al., 1996) and the eXtended Conformally Flat Condition (xCFC) (Cordero-Carrión et al., 2009), solve both evolution and constraint equations. In general, constrained models are slower than free-evolution models due to the cost of solving elliptic rather than hyperbolic systems. The balance is that solving the full system gives constrained models long-term accuracy (Baiotti and Rezzolla, 2017) whereas free-evolution models will suffer from constraint violations.

Currently, several production codes are maintained and used to simulate 3D CCSNe including CHIMERA (Bruenn et al., 2020), CoCoNuT-VERTEX (Müller et al., 2010), FLASH-X (Fryxell et al., 2000), FORNAX (Skinner et al., 2019), Gmunu (Cheong et al., 2020), NADA-FLD (Rahman et al., 2019), ZELMANI (Ott et al., 2012), and the code in (Kuroda et al., 2016b). Of the ways to approximate GR, CHIMERA, FLASH-X, and FORNAX use the effective potential method of approximating GR. While NADA-FLD, and ZELMANI use free-evolution schemes, CoCoNuT-VERTEX and Gmunu use xCFC.

Just as there are different ways to approximate GR, there are multiple ways to numerically approximate the solution of GR metric equations. The xCFC formulation results in a set of eight nonlinear coupled elliptic equations and therefore requires a combination of linear and non-linear methods. CoCoNuT-VERTEX uses a fixed point iteration where on each iteration the metric equations are solved using a multipole expansion using spherical harmonic functions. In contrast, Gmunu solves the xCFC metric equations using a nonlinear cell-centered multigrid method.

In 1D, the use of GR has been shown to lead to a more compact PNS core and neutrinospheres from which most of the neutrinos are radiated (Bruenn et al., 2001). The increased number of high-energy neutrinos being radiated leads to more energy being deposited in the post-shock fluid, enhancing the neutrino heating mechanism. While in 2D, the inclusion of GR has been shown to lead to a successful explosion where the modified Newtonian potential failed (Müller et al., 2012), and affects the rate at which the SASI grows (Dunham et al., 2024). From these results, it is clear that the inclusion of GR is essential to the CCSN mechanism.

Here, we present the **Poseidon** code, a relativistic metric solver and part of the Toolkit for High Order Neutrino-Radiation Hydrodynamics (**thornado**) framework. **Poseidon** solves for the relativistic metric using the xCFC formulation which approximates the 3+1 decomposition of general relativity by placing restrictions on the spatial portion of the metric as well as how space is allowed to evolve with time. These approximations transform the Einstein system to a set of 8 non-linear coupled elliptic PDEs and restrict the metric to 5 degrees of freedom. The approximation is exact in spherical symmetry and has been shown to be accurate for slowly rotating neutron stars (Cordero-Carrión et al., 2009; Müller et al., 2010)

To discretize the xCFC metric system, **Poseidon** uses a mixed method consisting of an angular Galerkin projection and a radial finite element method. The angular projection uses a subset of the real spherical harmonics (Blanco et al., 1997; Atkinson and Han, 2012), to project the system onto the radial dimension. The spherical nature of CCSNe enables the angular decomposition to capture details with fewer degrees

of freedom while the relationship between the spherical harmonic functions and the primary differential operator simplifies discretization of the equations in spherical coordinates. The continuous Galerkin (CG) finite element method (FEM) ([Brenner and Scott, 1994](#); [Clason, 2017](#)) discretizes the system on the radial domain. The FEM method provides a polynomial representation of the solution with hp-adaptivity on a compact stencil and adapts well to complex meshes such as arise in grids determined using AMR. Once discretized, the non-linear equations are solved using fixed point iterations with Anderson acceleration until a convergence criterion is met.

By itself, **Poseidon** can only calculate the metric at an instant in time, therefore it was incorporated into the **thornado** framework*. Within this framework, **Poseidon** serves as the gravity solver, providing either the xCFC metric variables or the Newtonian potential. **thornado** is the first discontinuous Galerkin (DG) solver of the GR hydrodynamics equations under the xCFC formalism for CCSNe applications. Together the two modules enable the full GR simulation of the gravitational collapse of a realistic progenitor with a tabulated equation of state using AMR and finite-element/DG methods.

The future of the **thornado** framework includes the completion of the neutrino transport modules, which will allow for the first CCSN simulations with AMR and finite-element methods to be conducted. Additionally, the multi-dimensional capabilities of **thornado** are currently being developed, which will initially allow for axis-symmetric simulations to be conducted. In the longer term, the code will be used in three dimensions to explore CCSNe mechanics fully, such as the SASI, which have the potential to become much more dynamic given a third spatial dimension of freedom.

In this dissertation, we will first present a derivation of the 3+1 Einstein equations in Chapter 2 following that of [Gourgoulhon \(2012\)](#). The 3+1 equations will then be approximated using the CFC formalism in Chapter 3 using the methods outlined in [Baumgarte \(2010\)](#) and [Rezzolla and Zanotti \(2013\)](#), and that will be extended to the

*<https://github.com/endeve/thornado>

xCFC formalism in Chapter 4. In Chapter 5, we will outline the mixed method used in **Poseidon**[†] to discretize and solve the xCFC metric equations. To demonstrate the capabilities of the code, results from test problems in 1 and 2 dimensions will be presented in Chapter 6. Included in these results will be tests showing **Poseidon**’s compatibility with the **AMReX**[‡], a block-structured adaptive mesh refinement (AMR) framework with MPI parallelism designed for exascale computing. Also in Chapter 6, we will show results using the Yahil self-similar profile, a source profile with qualities similar to those found in CCSNe that is evolved using the coupled **thornado-Poseidon** framework with AMR.

To demonstrate the current extent of **thornado**’s abilities, results from the evolution of 15 $M_{\odot}$ and 37 $M_{\odot}$ progenitors through collapse, bounce, and shock propagation will be presented. These simulations are conducted with GR gravity and hydrodynamics and use the equation of state and neutrino opacities provided by the open-source library, Weaklib[§]. Further, they are run in parallel and with an adaptive mesh using **AMReX**. This represents a significant accomplishment and will provide a base to compare to future CCSNe simulations.

[†]<https://github.com/jrober50/Poseidon>

[‡]<https://github.com/AMReX-Astro>

[§]<https://www.github.com/starkiller-astro/weaklib>

Chapter 2

The 3+1 Formulation of General Relativity

2.1 Introduction

The focus of this dissertation is the creation of solver of the xCFC metric equations. The extended conformally flat condition (xCFC) is an approximation of the 3+1 formalism of general relativity. Therefore, we begin by deriving the constraint and evolution equations from the Einstein equations.

We will follow the method outlined in [Gourgoulhon \(2012\)](#). First, we will present the decomposition of the 4-dimensional spacetime manifold into a foliation of 3-dimensional hypersurfaces. Next, the 3+1 metric will be defined and then used to define the covariant derivative. Using these tools, the Einstein equations will be projected onto a hypersurface, perpendicular to the hypersurface, and using a mixed projection to reveal the constrain and evolution equations.

2.2 The Einstein Equations

General relativity relates the presence of matter and energy in a system with the curvature of spacetime around it and defines how that curvature affects the matter and energy in return. This relationship is governed by the Einstein equations,

$$8\pi T_{\alpha\beta} = {}^{(4)}R_{\alpha\beta} - \frac{1}{2} {}^{(4)}R g_{\alpha\beta}, \quad (2.1)$$

where $T_{\alpha\beta}$ is the stress-energy tensor, ${}^{(4)}R_{\alpha\beta}$ is the four-dimensional Ricci tensor, ${}^{(4)}R$ is the four-dimensional Ricci scalar, and $g_{\alpha\beta}$ is the four-metric. The metric quantifies the relationships between coordinates, while the Ricci tensor and scalar can be viewed as a measure of how much the metric differs from Euclidean geometry. The stress-energy tensor contains information about the matter and energy within the system.

2.3 Foliating Spacetime

If a spacetime manifold, $\mathcal{M}$, admits a spatial hypersurface, Σ , which is intersected by every inextensible time-like, or null curve once and only once, then we can call $\mathcal{M}$ globally hyperbolic, and Σ a Cauchy Surface. The topology of such a manifold is $\Sigma \times \mathbb{R}$, which leads to the idea of foliation. Foliation is the decomposition of the manifold into an infinite number of Cauchy surfaces connected along a smooth scalar field, τ . This scalar field would need to have a non-vanishing gradient, and be such that every hypersurface, Σ_t , was a level surface of τ , i.e.

$$\forall t \in \mathbb{R} \quad \exists \quad \Sigma_t := \{\mathbf{x} \in \mathcal{M}, \tau(\mathbf{x}) = t\}. \quad (2.2)$$

Each hypersurface is called a leaf or slice of the foliation, and due to the regular nature of τ , they are non-intersecting (See Figure 2.1. Since τ is continuous and all Σ_t are non-intersecting, we assume the union of all Σ_t constitutes the entire spacetime

manifold $\mathcal{M}$,

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t. \quad (2.3)$$

2.4 Derivation of the 3+1 Metric

2.4.1 The Normal Unit Vector

We have defined our system such that each hypersurface Σ_t is defined as a level surface of the scalar field, τ . Thus the gradient 1-form of τ , $\nabla\tau$, is by definition normal to Σ_t . If $\mathbf{g}$ is the metric on $\mathcal{M}$, then the metric dual to $\nabla\tau$ is given by $\nabla^\alpha\tau = g^{\alpha\beta}\nabla_\beta\tau$. This vector gradient $\vec{\nabla}\tau$ defines the unique ”+1” direction normal to the 3-dimensional hypersurface Σ_t that gives the 3+1 formalism its name.

We now define $\mathbf{n}$, a unit vector normal to the hypersurface, Σ_t . The uniqueness of $\vec{\nabla}\tau$ implies that all vectors normal to the hypersurface Σ_t must be collinear to $\vec{\nabla}\tau$ and of the form $\mathbf{v} = c\vec{\nabla}\tau$, where c is a scalar. To normalize, $\mathbf{n}$, we define the scalar coefficient as

$$\alpha := (-\vec{\nabla}\tau \cdot \vec{\nabla}\tau)^{-1/2} \quad (2.4)$$

such that $\mathbf{n}$ can be defined

$$n_\beta := -\alpha\nabla_\beta\tau = \{-\alpha, 0, 0, 0\} \quad (2.5)$$

where we have chosen the negative sign to make $\mathbf{n}$ future-directed with

$$\mathbf{n} \cdot \mathbf{n} = -1. \quad (2.6)$$

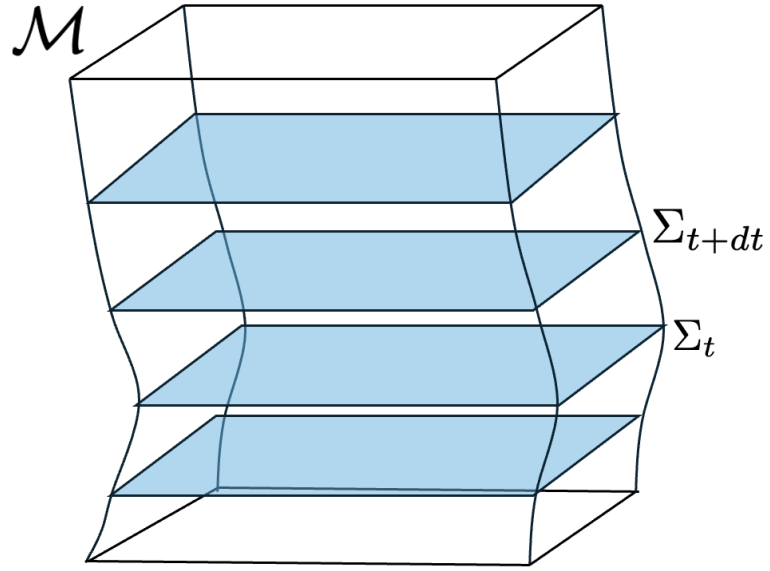


Figure 2.1: The spacetime manifold, $\mathcal{M}$, foliated into a set of non-intersecting spacelike hypersurfaces, Σ_t .

Lastly to define $\mathbf{n}$, the 1-form of $\mathbf{n}$, we can use the metric $\mathbf{g}$ on $\mathcal{M}$ to write,

$$\begin{aligned}
n^\beta &:= g^{\beta\alpha} n_\alpha \\
&= -\alpha g^{\beta\alpha} \nabla_\alpha \tau \\
&= -\alpha \nabla^\alpha \tau.
\end{aligned} \tag{2.7}$$

2.4.2 Normal Evolution Vector

To connect two adjacent hypersurfaces, we need an evolution vector that connects a point on Σ_t to the same point on $\Sigma_{t+\delta t}$. The normal unit vector, $\mathbf{n}$, does not satisfy this condition, as shown by the following example. If $\mathbf{x}_0$ is a point on Σ_t and $\mathbf{x}_1$ is the same point on $\Sigma_{t+\delta t}$ then

$$\tau(\mathbf{x}_1) - \tau(\mathbf{x}_0) = \delta t. \tag{2.8}$$

If $\mathbf{n}$ were the time evolution vector, we could relate $\mathbf{x}_1$ to $\mathbf{x}_0$ by

$$\mathbf{x}_1 = \mathbf{x}_0 + \delta t \mathbf{n} \tag{2.9}$$

where $\delta t \mathbf{n}$ is an infinitesimal displacement. Using this definition of $\mathbf{x}_1$ in τ

$$\begin{aligned}
\tau(\mathbf{x}_1) &= \tau(\mathbf{x}_0 + \delta t \mathbf{n}) \\
&= \tau(\mathbf{x}_0) + \delta t n^\beta \nabla_\beta \tau \\
&= \tau(\mathbf{x}_0) - \alpha \delta t \nabla^\beta \tau \nabla_\beta \tau \\
&= \tau(\mathbf{x}_0) + \alpha^{-1} \delta t
\end{aligned} \tag{2.10}$$

Clearly, $\mathbf{n}$ is not a proper time evolution operator but suggests the form of one that would satisfy the condition set in equation (2.8).

Let us define the time evolution vector $\mathbf{m}$ as

$$m_\beta := \alpha n_\beta \quad (2.11)$$

$$= -\alpha^2 \nabla_\beta \tau, \quad (2.12)$$

which has the property

$$\begin{aligned} \mathbf{m} \cdot \mathbf{m} &= m_\beta m^\beta \\ &= \alpha^2 n_\beta n^\beta \\ &= -\alpha^2 \end{aligned} \quad (2.13)$$

where we have defined the components of the 1-form of $\mathbf{m}$ by

$$\begin{aligned} m^\beta &:= g^{\beta\delta} m_\delta \\ &= \alpha g^{\beta\delta} n_\delta \\ &= \alpha n^\beta \\ &= -\alpha^2 \nabla^\beta \tau. \end{aligned} \quad (2.14)$$

Using $\mathbf{m}$ as the evolution vector,

$$\mathbf{x}_1 = \mathbf{x}_0 + \delta t \mathbf{m}, \quad (2.15)$$

we now attempt to time evolve $\mathbf{x}_0$ to $\mathbf{x}_1$,

$$\begin{aligned}
\tau(\mathbf{x}_1) &= \tau(\mathbf{x}_0 + \delta t \mathbf{m}) \\
&= \tau(\mathbf{x}_0) + \delta t m^\beta \nabla_\beta \tau \\
&= \tau(\mathbf{x}_0) + \alpha \delta t n^\beta \nabla_\beta \tau \\
&= \tau(\mathbf{x}_0) - \alpha^2 \delta t \nabla^\beta \tau \nabla_\beta \tau \\
&= \tau(\mathbf{x}_0) + \delta t.
\end{aligned} \tag{2.16}$$

Equation (2.8) is now satisfied, therefore $\mathbf{m}$ can function as a time evolution vector.

A close examination of equation (2.16) reveals an important property of $\mathbf{m}$;

$$\begin{aligned}
\mathbf{m} \nabla \tau &= m^\alpha \nabla_\alpha \tau \\
&= -\alpha^2 \nabla^\alpha \tau \nabla_\alpha \tau \\
&= 1.
\end{aligned} \tag{2.17}$$

The unity of the above relation gives $\mathbf{m}$ the ability to drag a point from Σ_t to its location on a neighboring hypersurface, $\Sigma_{t+\delta t}$. This action is reminiscent of the *Lie Derivative* (see below). The equation for the Lie derivative when applied to a scalar field such as ϕ and taken with respect to a general vector $\mathbf{v}$ is

$$\mathcal{L}_{\mathbf{v}} \phi := v^\alpha \nabla_\alpha \phi. \tag{2.18}$$

Comparing this with equation (2.17), we can now write

$$\mathbf{m} \nabla \tau = \mathcal{L}_{\mathbf{m}} \tau = 1. \tag{2.19}$$

This unitary nature of this Lie derivative means that when a point is Lie dragged along $\mathbf{m}$ it transitions one unit along τ . As τ defines the connection between hypersurfaces, this is equivalent to saying that the vector $\delta t \mathbf{m}$ drags the hypersurface Σ_t to the

neighboring hypersurface $\Sigma_{t+\delta t}$. Likewise, any time the Lie derivative of τ is unitary along some vector, $\mathbf{v}$, then that vector will drag points from a hypersurface to an adjacent hypersurface, a property we will use to help define a time coordinate.

2.4.3 The Lie Derivative

One of the key features of General Relativity is its coordinate-free formulation. This leaves one free to choose a system of coordinates and impose them on the problem. Instead of defining our manifold by a set of basis vectors, we can use a tensor field, $\mathbf{A}$, within the manifold as the defining structure. Because tensor fields are in general more complex than the familiar sets (Cartesian, Polar, etc) of basis vectors, we need to define a more general form of the derivative. As one can define the standard derivative of a tensor as a measure of how much that tensor field changes as it moves in the direction of a given basis, we therefore define the *Lie Derivative*, $\mathcal{L}_{\mathbf{A}}\mathbf{B}$, of a tensor field, $\mathbf{B}$, as a measure of how it changes as it moves along another tensor field, $\mathbf{A}$.

The Lie Derivative of a Vector Field

As it is easiest to visualize and understand in terms of vectors, we will begin by defining the Lie derivative of vector fields and then generalize it to tensor fields. To start, let us pick a point, m , and two different vector fields, $\mathbf{A}$ and $\mathbf{B}$, on the manifold, $\mathcal{M}$. Moving along $\mathbf{A}$ and $\mathbf{B}$ to define two new points m^* and m° by the infinitesimal translation equations,

$$m^* := m + \lambda \mathbf{A}(m) \tag{2.20}$$

and

$$m^\circ := m + \epsilon \mathbf{B}(m) \tag{2.21}$$

where λ and ϵ are infinitesimal scaling factors, or "distances" along $\mathbf{A}$ and $\mathbf{B}$ respectively.

If we further define another set of two points $m^{\star\circ}$ and $m^{\circ\star}$ which are related to $m^\star$ and m° by

$$\begin{aligned}
m^{\star\circ} &:= m^\star + \epsilon \mathbf{B}(m^\star) \\
&= m + \lambda \mathbf{A}(m) + \epsilon \mathbf{B}(m + \lambda \mathbf{A}(m)) \\
&= m + \lambda \mathbf{A}(m) + \epsilon \mathbf{B}(m) + \epsilon \lambda \mathbf{B}(\mathbf{A}(m))
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
m^{\circ\star} &:= m^\circ + \lambda \mathbf{A}(m^\circ) \\
&= m + \epsilon \mathbf{B}(m) + \lambda \mathbf{A}(m + \epsilon \mathbf{B}(m)) \\
&= m + \epsilon \mathbf{B}(m) + \lambda \mathbf{A}(m) + \epsilon \lambda \mathbf{A}(\mathbf{B}(m))
\end{aligned} \tag{2.23}$$

where we have treated displacement along $\mathbf{A}$ and $\mathbf{B}$ as linear due to the infinitesimal size of the displacements. We can think of point $m^{\star\circ}$ as the value of $\epsilon \mathbf{B}(m^\star)$, and $m^{\circ\star}$ is value of $\epsilon \mathbf{B}(m)$ after it has been dragged along $\mathbf{A}$ the same "distance", λ , that m is from $m^\star$.

Taking the difference between $m^{\circ\star}$ and $m^{\star\circ}$, we are left with

$$\begin{aligned}
m^{\circ\star} - m^{\star\circ} &= m + \epsilon \mathbf{B}(m) + \lambda \mathbf{A}(m) + \epsilon \lambda \mathbf{A}(\mathbf{B}(m)) \\
&\quad - (m + \lambda \mathbf{A}(m) + \epsilon \mathbf{B}(m) + \epsilon \lambda \mathbf{B}(\mathbf{A}(m))) \\
&= \epsilon \lambda [\mathbf{A}(\mathbf{B}(m)) - \mathbf{B}(\mathbf{A}(m))] \\
&= \epsilon \lambda [\mathbf{A}, \mathbf{B}](m).
\end{aligned} \tag{2.24}$$

The $[\mathbf{A}, \mathbf{B}](m)$ in the last line of equation (2.24) is just the commutator of the two vectors $\mathbf{A}$ and $\mathbf{B}$ at location m , which is also known by the name, *Lie Bracket*.

By our initial definition of the Lie derivative, we can now write

$$\begin{aligned}\mathcal{L}_{\lambda\mathbf{A}}(\epsilon\mathbf{B}) &= m^{\star\circ} - m^{\circ\star} \\ &= \epsilon\lambda[\mathbf{A}, \mathbf{B}](m).\end{aligned}\tag{2.25}$$

As m can be any point in the manifold, we will drop it from further definitions. We can set λ and ϵ equal to unity to define the Lie derivative between two unscaled vectors as just the commutator between them, i.e.

$$\mathcal{L}_{\mathbf{A}}\mathbf{B} = [\mathbf{A}, \mathbf{B}].\tag{2.26}$$

From equation (2.25), we can quickly deduce a few linear properties of the Lie derivative. If we set either λ or ϵ to unity and allow the other to be an arbitrary value we can show;

$$\begin{aligned}\mathcal{L}_{\lambda\mathbf{A}}(\mathbf{B}) &= \lambda[\mathbf{A}, \mathbf{B}] \\ &= \lambda\mathcal{L}_{\mathbf{A}}(\mathbf{B}),\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}\mathcal{L}_{\mathbf{A}}(\epsilon\mathbf{B}) &= \epsilon[\mathbf{A}, \mathbf{B}] \\ &= \epsilon\mathcal{L}_{\mathbf{A}}(\mathbf{B}).\end{aligned}\tag{2.28}$$

If we now choose a coordinate system, x^α , that is adapted to the vector $\mathbf{A}$ (i.e. $A^0 = 1$ and $A^\beta = 0$ for $\beta \neq 0$ when described by the system x^α) and place it on the manifold, $\mathcal{M}$. We can define the natural basis of the tangent vector field at any point as,

$$\partial_\alpha := \frac{\partial}{\partial x^\alpha} \quad \text{where } \alpha \in \{0, 1, 2, 3\}\tag{2.29}$$

Given this basis, a general vector, $\mathbf{v}$, would be defined in component form as

$$\mathbf{v} := v^\alpha \partial_\alpha \quad (2.30)$$

Using equation (2.30), we can write the components of the commutator of two vectors in the form

$$\begin{aligned} [\mathbf{A}, \mathbf{B}]^\alpha &= (\mathbf{AB})^\alpha - (\mathbf{BA})^\alpha \\ &= A^\beta \partial_\beta B^\alpha - B^\beta \partial_\beta A^\alpha. \end{aligned} \quad (2.31)$$

The covariant derivative of a general vector, v^α , on a manifold is given by

$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\beta\mu} v^\beta, \quad (2.32)$$

which can be rewritten as

$$\partial_\beta v^\alpha = \nabla_\beta v^\alpha - \Gamma^\alpha_{\beta\eta} v^\eta. \quad (2.33)$$

Rewriting the partial derivatives in equation (2.31) in the form of equation (2.33), we show

$$\begin{aligned} [\mathbf{A}, \mathbf{B}]^\alpha &= A^\beta \partial_\beta B^\alpha - B^\beta \partial_\beta A^\alpha \\ &= A^\beta (\nabla_\beta B^\alpha - \Gamma^\alpha_{\beta\eta} B^\eta) - B^\beta (\nabla_\beta A^\alpha - \Gamma^\alpha_{\beta\eta} A^\eta) \\ &= A^\beta \nabla_\beta B^\alpha - B^\beta \nabla_\beta A^\alpha - A^\beta B^\eta \Gamma^\alpha_{\beta\eta} + B^\beta A^\eta \Gamma^\alpha_{\beta\eta} \\ &= A^\beta \nabla_\beta B^\alpha - B^\beta \nabla_\beta A^\alpha - A^\beta B^\eta \Gamma^\alpha_{\beta\eta} + A^\beta B^\eta \Gamma^\alpha_{\eta\beta} \\ &= A^\beta \nabla_\beta B^\alpha - B^\beta \nabla_\beta A^\alpha, \end{aligned} \quad (2.34)$$

where we have used the symmetry of the Christoffel symbols to eliminate the third and fourth terms. This allows us to define the components of the Lie derivative in

terms of the covariant derivative which is associated with a metric, $\mathbf{g}$, as

$$(\mathcal{L}_{\mathbf{A}}\mathbf{B})^\alpha := A^\beta \nabla_\beta B^\alpha - B^\beta \nabla_\beta A^\alpha. \quad (2.35)$$

Lie Derivative of General Tensors

We can extend the Lie derivative to any tensor field by first defining its action on a scalar field, ϕ , by

$$\begin{aligned} \mathcal{L}_{\mathbf{A}}\phi &:= A^\alpha \partial_\alpha \phi \\ &= A^\alpha \nabla_\alpha \phi. \end{aligned} \quad (2.36)$$

The action of the Lie derivative on a general tensor field of rank (m,n) can be found by application of the Leibniz Rule to show,

$$\begin{aligned} \mathcal{L}_{\mathbf{A}} B^{\alpha_1, \dots, \alpha_m}_{\beta_1, \dots, \beta_n} &:= A^\mu \nabla_\mu B^{\alpha_1, \dots, \alpha_m}_{\beta_1, \dots, \beta_n} \\ &\quad - \sum_{i=1}^m B^{\alpha_1, \dots, \alpha_i-1, \xi, \alpha_i+1, \dots, \alpha_m}_{\beta_1, \dots, \beta_n} \nabla_\xi A^{\alpha_i} \\ &\quad + \sum_{i=1}^n B^{\alpha_1, \dots, \alpha_m}_{\beta_1, \dots, \beta_i-1, \xi, \beta_i+1, \dots, \beta_n} \nabla_{\beta_i} A^\xi. \end{aligned} \quad (2.37)$$

2.4.4 Lapse Function

Consider an Eulerian observer, an observer who is traveling with a unit timelike 4-velocity, $\mathbf{n}$, and two events, $\mathbf{x}_0$ and $\mathbf{x}_1$ separated by $\delta t \mathbf{m}$ as in equation (2.15). The proper time measured by the Eulerian observer would then be the metric length

between the $\mathbf{x}_0$ and $\mathbf{x}_1$.

$$\begin{aligned}
\delta\tau &= \sqrt{-\delta t \mathbf{m} \cdot \delta t \mathbf{m}} \\
&= \sqrt{-g_{\alpha\beta} \delta t m^\alpha \delta t m^\beta} \\
&= \sqrt{-m_\beta m^\beta} \delta t \\
&= \sqrt{\alpha^2} \delta t \\
&= \alpha \delta t.
\end{aligned} \tag{2.38}$$

From equation (2.38), we see that α relates the physical time seen by the Eulerian observer to the coordinate time of the system. Because of this, we call α the *Lapse Function*.

The lapse function can be defined in terms of the four-dimensional metric, $\mathbf{g}$, of the manifold, $\mathcal{M}$, by use of the definition, equation (2.5), and normalization, equation (2.6), of $\mathbf{n}$,

$$\begin{aligned}
-1 &= \mathbf{n} \cdot \mathbf{n} \\
&= g^{\alpha\beta} n_\beta n_\alpha \\
&= g^{00} n_0 n_0 \\
&= g^{00} \alpha^2.
\end{aligned} \tag{2.39}$$

Solving equation (2.39) for α , we can write

$$\alpha = \frac{1}{\sqrt{-g^{00}}}. \tag{2.40}$$

2.4.5 Defining the 3-Metric on Σ_t

For each spatial hypersurface, Σ_t , there exists a 3-metric, $\boldsymbol{\gamma}$, that defines the relations between the three spatial coordinates on the hypersurface. With the unit normal vector, $\mathbf{n}$, as defined in equation (2.5), we can relate the spatial 3-metric of Σ_t to the

4-metric, $\mathbf{g}$, of the manifold, $\mathcal{M}$, by the following equations:

$$\gamma_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta \quad (2.41)$$

and

$$\gamma^{\alpha\beta} := g^{\alpha\beta} + n^\alpha n^\beta. \quad (2.42)$$

where we have adopted the index convention that Greek letters such as $\mu, \nu \in \{0, 1, 2, 3\}$ and Latin letters such as $i, j \in \{1, 2, 3\}$. A convention we will continue to use throughout this paper. By this definition $\gamma_{\alpha 0} = \gamma_{0\beta} = 0$, and $\gamma_{ij} = g_{ij}$, but in general $\gamma^{ij} \neq g^{ij}$. Also by these definitions and conventions, we have the standard metric/inverse metric relationship on the spatial dimensions,

$$\gamma_{ij}\gamma^{ij} = \delta^i_j, \quad (2.43)$$

where δ^i_j is the Kronecker delta.

2.4.6 Defining the Time Vector

We now introduce a general spatial coordinate system, x^i , on each hypersurface, Σ_t . Given that this spatial system transitions smoothly from hypersurface to hypersurface, the four-dimensional coordinate system formed by $x^\alpha = (\nabla\tau, x^i)$, $= (\nabla\tau, x^1, x^2, x^3)$ is well-behaved (i.e. smooth, differentiable, and monotonic) on the manifold, $\mathcal{M}$. With this coordinate system established, we now pick a point within the manifold and examine its tangent vector space, $\mathcal{T}_m(\mathcal{M})$. There exists a basis, ∂_α , associated with the coordinates of the vector space called the natural basis of $\mathcal{T}_m(\mathcal{M})$

and is given by

$$\boldsymbol{\partial}_\alpha := \frac{\partial}{\partial x^\alpha} \quad (2.44)$$

The dual to the natural basis, $\boldsymbol{\partial}_\alpha$, of $\mathcal{T}_m(\mathcal{M})$ is the gradient 1-form, $\mathbf{d}x^\alpha$. In particular, the 1-form dual to the time vector, $\boldsymbol{\partial}_0$, is $\mathbf{d}t = \nabla\tau$. By definition the relationship between $\boldsymbol{\partial}_0$ and $\nabla\tau$ is

$$(\partial_0)^\alpha \nabla_\alpha \tau = 1 \quad (2.45)$$

or in terms of the Lie derivative,

$$\mathcal{L}_{\boldsymbol{\partial}_0} \tau = 1. \quad (2.46)$$

As we saw with the evolution vector, $\mathbf{m}$, the condition satisfied by equation (2.46) means $\boldsymbol{\partial}_0$ Lie drags a hypersurface, Σ_t , to an adjacent hypersurface, $\Sigma_{t+\delta t}$. While $\mathbf{m}$ is a strictly timelike vector, $\boldsymbol{\partial}_0$ generally has both temporal and spatial components. We, therefore, relate the two by the equation

$$\boldsymbol{\partial}_0 := \mathbf{m} + \boldsymbol{\beta}, \quad (2.47)$$

where $\boldsymbol{\beta}$ is a purely spatial vector called the *Shift Vector*. We can easily check that $\boldsymbol{\beta}$ is purely spatial, and hence tangent to the hypersurface, Σ_t , by the Lie derivative

of τ along β ,

$$\begin{aligned}
\mathcal{L}_\beta \tau &= \beta \nabla \tau \\
&= (\partial_0 - \mathbf{m}) \nabla \tau \\
&= \partial_0 \nabla \tau - \mathbf{m} \nabla \tau \\
&= 1 - 1 \\
&= 0.
\end{aligned} \tag{2.48}$$

Since β is tangent to the hypersurface, Σ_t , and $\mathbf{n}$ is orthogonal to Σ_t , we can write

$$\begin{aligned}
\beta \cdot \mathbf{m} &= \alpha(\beta \cdot \mathbf{n}) \\
&= \alpha \beta_\alpha n^\alpha \\
&= 0.
\end{aligned} \tag{2.49}$$

We can use equation (2.12) to rewrite the time vector in equation (2.47) as

$$\partial_0 := \alpha \mathbf{n} + \beta, \tag{2.50}$$

and use this time vector to form expressions for n^α . We begin by solving equation (2.50) for $\mathbf{n}$, and then express it using index notation and the natural basis given by equation (2.44) to find

$$n^\alpha \partial_\alpha = \frac{1}{\alpha} [(\partial_0)^\alpha \partial_\alpha - \beta^\alpha \partial_\alpha]. \tag{2.51}$$

As $(\partial_0)^\alpha \partial_\alpha = (\partial_0)^0 \partial_0 = 1$ and $\beta^\alpha = (0, \beta^i)$, we can rewrite equation (2.51) as

$$\begin{aligned}
n^0 &= \frac{1}{\alpha} [1 - 0] \\
&= \frac{1}{\alpha}
\end{aligned} \tag{2.52}$$

and

$$\begin{aligned} n^i &= \frac{1}{\alpha} [0 - \beta^i] \\ &= \frac{-\beta^i}{\alpha}. \end{aligned} \tag{2.53}$$

To summarize, we now define,

$$n_\alpha := (-\alpha, 0, 0, 0), \tag{2.54}$$

$$n^\alpha := \left(\frac{1}{\alpha}, \frac{-\beta^1}{\alpha}, \frac{-\beta^2}{\alpha}, \frac{-\beta^3}{\alpha} \right). \tag{2.55}$$

2.4.7 Defining the 3+1 Metric

The elements of the metric, $\mathbf{g}$, on the manifold, $\mathcal{M}$, can be expressed as

$$g_{\alpha\beta} = \partial_\alpha \cdot \partial_\beta, \tag{2.56}$$

but we will now derive more explicit forms of the elements of $\mathbf{g}$. Starting with g_{00} , by the definitions in equation (2.56) and equation (2.50), we can write

$$\begin{aligned} g_{00} &= \partial_0 \cdot \partial_0 \\ &= (\partial_0)^\mu (\partial_0)_\mu \\ &= (\alpha n^\mu + \beta^\mu)(\alpha n_\mu + \beta_\mu) \\ &= \alpha^2 n^\mu n_\mu + \alpha n^\mu \beta_\mu + \alpha \beta^\mu n_\mu + \beta^\mu \beta_\mu \\ &= -\alpha^2 + \beta^\mu \beta_\mu. \end{aligned} \tag{2.57}$$

We have used the orthogonality of $\mathbf{n}$ and β as well as the normalization of $\mathbf{n}$ in the last step in equation (2.57).

Next, we assess g_{i0}

$$\begin{aligned}
g_{i0} &= \boldsymbol{\partial}_i \cdot \boldsymbol{\partial}_0 \\
&= \boldsymbol{\partial}_i \cdot (\alpha \mathbf{n} + \boldsymbol{\beta}) \\
&= \alpha(\boldsymbol{\partial}_i \cdot \mathbf{n}) + \boldsymbol{\partial}_i \cdot \boldsymbol{\beta}
\end{aligned}$$

as $\mathbf{n}$ is by definition orthogonal to the spatial hypersurface, $\boldsymbol{\partial}_i \cdot \mathbf{n} = 0$, and by expressing $\boldsymbol{\beta}$ as $\boldsymbol{\beta} = \beta^\mu \partial_\mu$, we have $\boldsymbol{\partial}_i \cdot \boldsymbol{\beta} = \beta_i$, which simplifies the above equation for g_{i0} to

$$g_{i0} = \beta_i \tag{2.58}$$

likewise, by symmetry, we have

$$g_{0j} = \beta_j. \tag{2.59}$$

Lastly, by choice of foliation, we have

$$g_{ij} = \gamma_{ij}. \tag{2.60}$$

Putting equations (2.57) - (2.60) together, the full metric is given by

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{2.61}$$

and inverting we find

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix}. \tag{2.62}$$

With the metric made explicit, we can define the proper distance by the equation,

$$\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= (-\alpha^2 + \beta^i \beta_i) dt^2 + \gamma_{ij} (dx^i + \beta_i dt) (dx^j + \beta_j dt) \\
&= -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j.
\end{aligned} \tag{2.63}$$

Figure 2.2 shows how the metric variables relate the same point on two adjacent hypersurfaces.

2.4.8 Projection Operators

To aid in the decomposition of the Einstein Equations into our 3+1 spacetime, we will need the use of projection operators. To this end, we now define:

The *Spatial projection operator*

$$\begin{aligned}
\gamma^\mu{}_\nu &:= g^{\mu\xi} \gamma_{\xi\nu} = g^{\mu\xi} (g_{\xi\nu} + n_\xi n_\nu) \\
&= \delta^\mu{}_\nu + n^\mu n_\nu,
\end{aligned} \tag{2.64}$$

which projects tensors onto the hypersurface, Σ_t . And, the *Time projection operator*

$$N^\mu{}_\nu := -n^\mu n_\nu, \tag{2.65}$$

which projects perpendicular to the hypersurface and onto the time axis. As expected these two operators are orthogonal to each other.

$$\begin{aligned}
\gamma^\mu{}_\xi N^\xi{}_\nu &= -\delta^\mu{}_\xi n^\xi n_\nu - n^\mu n_\xi n^\xi n_\nu \\
&= -n^\mu n_\nu - (-n^\mu n_\nu) \\
&= 0
\end{aligned} \tag{2.66}$$

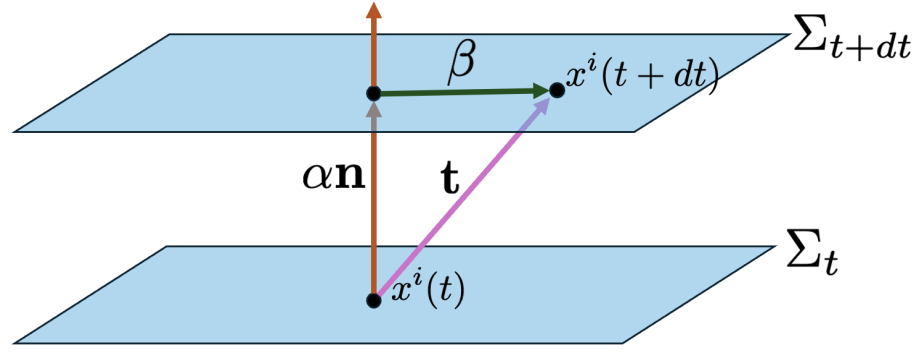


Figure 2.2: Diagrammatic representation of the 3+1 foliation showing the relationship between two adjacent space-like hypersurfaces. The four-vector $\mathbf{t}$ relates the evolution of the point x^i . The timelike component $\alpha \mathbf{n}$ measures the temporal distance between hypersurfaces with the lapse function α measuring the proper time. The shift vector, β , measures the shift in coordinates.

2.5 3+1 Covariant Derivative

With the metric established and projection operators defined, we turn our attention to defining the covariant derivative in the 3+1 formalism. Using Eq. (2.64), we project the standard four-dimensional covariant tensor, ∇_ν onto the hypersurface, Σ_t , to create the three-dimensional spatial covariant derivative,

$$D_\mu := \gamma^\nu_\mu \nabla_\nu. \quad (2.67)$$

When D_μ is applied to a general tensor, $\mathbf{v}$, all indices of the four-dimensional covariant derivative must be projected onto the hypersurface, Σ_t . For instance, If $\mathbf{v}$ is a general mixed tensor, v^α_β , its three-dimensional covariant derivative would be expressed

$$D_\mu v^\alpha_\beta = \gamma^\nu_\mu \gamma^\alpha_\delta \gamma^\eta_\beta \nabla_\nu v^\delta_\eta, \quad (2.68)$$

where $\nabla_\nu v^\delta_\mu$ is given by the equation

$$\nabla_\nu v^\delta_\mu = {}^{(4)}\partial_\nu v^\delta_\mu + {}^{(4)}\Gamma^\delta_{\nu\eta} v^\eta_\mu - {}^{(4)}\Gamma^\eta_{\nu\mu} v^\delta_\eta, \quad (2.69)$$

where ${}^{(4)}\Gamma^\eta_{\nu\mu}$ is the four-dimensional Christoffel symbol,

$${}^{(4)}\Gamma^\eta_{\nu\mu} := \frac{1}{2} \gamma^{\eta\alpha} ({}^{(4)}\partial_\nu \gamma_{\mu\alpha} + {}^{(4)}\partial_\mu \gamma_{\alpha\nu} - {}^{(4)}\partial_\alpha \gamma_{\nu\mu}) \quad (2.70)$$

We use ${}^{(4)}$ to denote the dimensionality. From these expressions, it is clear that we need to explore the projections of ${}^{(4)}\partial_i$ and ${}^{(4)}\Gamma^\mu_{\nu\eta}$.

As one might guess, the standard four-dimensional derivative ${}^{(4)}\partial_\mu$ becomes a three-dimensional spatial derivative, ${}^{(3)}\partial_\mu := \partial_i$, when projected onto Σ_t as can be

seen by

$$\begin{aligned}
\gamma^\alpha{}_\beta{}^{(4)}\partial_\alpha &= (\delta^\alpha{}_\beta + n^\alpha n_\beta)^{(4)}\partial_\alpha \\
&= {}^{(4)}\partial_\beta + n_\beta n^\alpha \partial_\alpha \\
&= (\partial_0 - \partial_0, \partial_1, \partial_2, \partial_3) \\
&= {}^{(3)}\partial_\beta \\
&= \partial_i.
\end{aligned} \tag{2.71}$$

On the other hand, the Christoffel symbol transforms like

$$\begin{aligned}
\gamma^\alpha{}_\mu \gamma^\nu{}_\beta \gamma^\epsilon{}_\delta {}^{(4)}\Gamma^\mu{}_{\nu\epsilon} &= \frac{1}{2} \gamma^\alpha{}_\mu \gamma^\nu{}_\beta \gamma^\epsilon{}_\delta g^{\mu\eta} (\partial_\nu g_{\epsilon\eta} + \partial_\epsilon g_{\eta\nu} - \partial_\eta g_{\nu\epsilon}) \\
&= \frac{1}{2} \gamma^\alpha{}_\mu g^{\mu\eta} (\gamma^\nu{}_\beta \gamma^\epsilon{}_\delta \partial_\nu g_{\epsilon\eta} + \gamma^\nu{}_\beta \gamma^\epsilon{}_\delta \partial_\epsilon g_{\eta\nu} - \gamma^\nu{}_\beta \gamma^\epsilon{}_\delta \partial_\eta g_{\nu\epsilon}).
\end{aligned} \tag{2.72}$$

Applying the spatial projection operator on the full four-dimensional metric will return the 3-metric of the hypersurface.

$$\begin{aligned}
\gamma^\alpha{}_\delta g^{\delta\beta} &= (\delta^\alpha{}_\delta + n^\alpha n_\delta) g^{\delta\beta} \\
&= g^{\alpha\beta} + n^\alpha g^{\delta\beta} n_\delta \\
&= g^{\alpha\beta} + n^\alpha n^\beta \\
&= \gamma^{\alpha\beta}.
\end{aligned} \tag{2.73}$$

Using equation (2.73) in equation (2.72), we can continue and write the projection of the Christoffel symbol as

$$\gamma^\alpha{}_\mu \gamma^\nu{}_\beta \gamma^\epsilon{}_\delta {}^{(4)}\Gamma^\mu{}_{\nu\epsilon} = \frac{1}{2} \gamma^{\alpha\eta} (\partial_\beta \gamma_{\delta\eta} + \partial_\delta \gamma_{\eta\beta} - \partial_\eta \gamma_{\beta\delta}). \tag{2.74}$$

From this, we see that when projected onto the spatial hypersurface Σ_t , the Christoffel symbol retains the same form but is associated with the 3-metric of the hypersurface.

Therefore we define

$${}^{(3)}\Gamma^\alpha_{\beta\delta} := \frac{1}{2}\gamma^{\alpha\eta} \left({}^{(3)}\partial_\beta\gamma_{\delta\eta} + {}^{(3)}\partial_\delta\gamma_{\eta\beta} - {}^{(3)}\partial_\eta\gamma_{\beta\delta} \right) \quad (2.75)$$

with ${}^{(3)}\Gamma^\alpha_{\beta\delta} = 0$ if any $\alpha, \beta, \delta = 0$.

Taking into consideration equations (2.71) and (2.75), we know that the three-dimensional covariant derivative D_i behaves identically to the four-dimensional covariant derivative ∇_μ except that it is limited to the spatial coordinates and thus can be written in general form as

$$D_\nu v^\delta{}_\mu = {}^{(3)}\partial_\nu v^\delta{}_\mu + {}^{(3)}\Gamma^\delta_{\nu\eta} v^\eta{}_\mu - {}^{(3)}\Gamma^\eta_{\nu\mu} v^\delta{}_\eta \quad (2.76)$$

where $v^\delta{}_\mu$ is a general tensor living in the 3-dimensional hypersurface. As D_ν , ${}^{(3)}\partial_\beta$, ${}^{(3)}\Gamma^\alpha_{\beta\delta}$ and $v^\delta{}_\mu$ all live purely on the 3-dimensional hypersurface, Σ_t , they can be indexed with Latin characters, and ${}^{(3)}$ will be dropped when using Latin indices. Using that notation, the 3-dimensional covariant derivative on a hypersurface can be written as

$$D_i v^j{}_k := \partial_i v^j{}_k + \Gamma^j_{ik} v^\eta{}_\eta - \Gamma^\eta_{ik} v^j{}_l \quad (2.77)$$

where

$$\Gamma^i_{jk} := \frac{1}{2}\gamma^{il} (\partial_j\gamma_{kl} + \partial_k\gamma_{lj} - \partial_l\gamma_{jk}) \quad (2.78)$$

is the Christoffel symbol defined on the hypersurface.

The three-dimensional covariant derivative retains the properties of its four-dimensional counterpart. Two such properties that we will make use of are its compatibility with the four-metric,

$$D_\mu g_{\alpha\beta} = \gamma^\nu{}_\mu \nabla_\nu g_{\alpha\beta} = 0 \quad (2.79)$$

and the three-metric,

$$\begin{aligned}
D_i \gamma_{jk} &= \partial_i \gamma_{jk} - \Gamma_{ij}^l \gamma_{lk} - \Gamma_{ik}^l \gamma_{jl} \\
&= \partial_i \gamma_{jk} - \frac{1}{2} \gamma^{lm} (\partial_i \gamma_{mj} + \partial_j \gamma_{im} - \partial_m \gamma_{ij}) \gamma_{lk} - \frac{1}{2} \gamma^{lm} (\partial_i \gamma_{mk} + \partial_k \gamma_{im} - \partial_m \gamma_{ik}) \gamma_{jl} \\
&= \partial_i \gamma_{jk} - \frac{1}{2} \delta^m_k (\partial_i \gamma_{mj} + \partial_j \gamma_{im} - \partial_m \gamma_{ij}) - \frac{1}{2} \delta^m_j (\partial_i \gamma_{mk} + \partial_k \gamma_{im} - \partial_m \gamma_{ik}) \\
&= \partial_i \gamma_{jk} - \frac{1}{2} (\partial_i \gamma_{kj} + \partial_j \gamma_{ik} - \partial_k \gamma_{ij}) - \frac{1}{2} (\partial_i \gamma_{jk} + \partial_k \gamma_{ij} - \partial_j \gamma_{ik}) \\
&= \partial_i \gamma_{jk} - \frac{1}{2} (2 \partial_i \gamma_{jk}) \\
&= 0.
\end{aligned} \tag{2.80}$$

Using equation (2.80), we can also see the inverse metric is also compatible with the three-dimensional covariant derivative,

$$\begin{aligned}
0 &= D_i (\delta^j_k) = D_i (g^{jl} g_{lk}) \\
&= g^{jl} D_i (g_{lk}) + g_{lk} D_i (g^{jl}) \\
&= 0 + g_{lk} D_i (g^{jl}).
\end{aligned} \tag{2.81}$$

As g_{lk} is in general non-zero, we can write

$$D_i (g^{jl}) = 0. \tag{2.82}$$

If we now take the three-dimensional covariant derivative of the spatial projection operator, equation (2.64), and use equations (2.80) and (2.82), we find

$$\begin{aligned}
D_i (\gamma^j_k) &= D_i (g^{jl} \gamma_{lk}) \\
&= g^{jl} D_i (\gamma_{lk}) + \gamma_{lk} D_i (g^{jl}) \\
&= g^{jl} 0 + \gamma_{lk} 0 \\
&= 0.
\end{aligned} \tag{2.83}$$

A consequence of equation (2.83) is the ability to commute the spatial projection tensor with the three-dimensional covariant derivative. If v_k^j is a general tensor then

$$\begin{aligned}
D_i(\gamma_m^n v_k^j) &= \gamma_m^n D_i(v_k^j) + v_k^j D_i(\gamma_m^n) \\
&= \gamma_m^n D_i(v_k^j) + v_k^j 0 \\
&= \gamma_m^n D_i(v_k^j).
\end{aligned} \tag{2.84}$$

Equation (2.84) also reveals that the spatial projection operator commutes with the four-dimensional covariant derivative due to the relation between D_i and ∇_i established in equation (2.71).

2.6 Gauss - Codazzi Relations

With a definition of the three-dimensional covariant derivative in hand, we now turn our attention to the Riemann Tensor and its relation to the hypersurface Σ_t .

2.6.1 Gauss Relations

Let us begin by defining the three-dimensional Riemann tensor of the hypersurface using the Ricci Identity on a vector field, $\mathbf{v}$, which is tangent to Σ_t

$$D_\alpha D_\beta v^\delta - D_\beta D_\alpha v^\delta = {}^{(3)}R^\delta_{\sigma\alpha\beta} v^\sigma \tag{2.85}$$

where we have reverted to the four-dimensional form of the three-dimensional covariant derivative, D_α given by equation (2.76). Relating this equation to the

four-dimensional covariant derivative ∇_α using equation (2.67), we find

$$\begin{aligned}
D_\alpha D_\beta v^\delta &= D_\alpha (D_\beta v^\delta) \\
&= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \nabla_\mu (D_\nu v^\eta) \\
&= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \nabla_\mu (\gamma^\rho_\nu \gamma^\eta_\sigma \nabla_\rho v^\sigma) \\
&= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta (\gamma^\eta_\sigma \nabla_\rho v^\sigma \nabla_\mu \gamma^\rho_\nu + \gamma^\rho_\nu \nabla_\rho v^\sigma \nabla_\mu \gamma^\eta_\sigma + \gamma^\rho_\nu \gamma^\eta_\sigma \nabla_\mu \nabla_\rho v^\sigma). \quad (2.86)
\end{aligned}$$

If we write

$$\begin{aligned}
\nabla_\mu \gamma^\rho_\nu &= \nabla_\mu (n^\rho n_\nu) \\
&= n_\nu \nabla_\mu n^\rho + n^\rho \nabla_\mu n_\nu, \quad (2.87)
\end{aligned}$$

equation (2.86) becomes

$$\begin{aligned}
D_\alpha D_\beta v^\delta &= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \left(\gamma^\eta_\sigma \nabla_\rho v^\sigma n_\nu \nabla_\mu n^\rho \right. \\
&\quad + \gamma^\eta_\sigma \nabla_\rho v^\sigma n^\rho \nabla_\mu n_\nu \\
&\quad + \gamma^\rho_\nu \nabla_\rho v^\sigma n_\sigma \nabla_\mu n^\eta \\
&\quad + \gamma^\rho_\nu \nabla_\rho v^\sigma n^\eta \nabla_\mu n_\sigma \\
&\quad \left. + \gamma^\rho_\nu \gamma^\eta_\sigma \nabla_\mu \nabla_\rho v^\sigma \right). \quad (2.88)
\end{aligned}$$

The first and fourth terms of equation (2.88) equal zero as $n_\nu \gamma^\nu_\beta = n^\eta \gamma^\delta_\eta = 0$.

Therefore,

$$\begin{aligned}
D_\alpha D_\beta v^\delta &= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \left(\gamma^\eta_\sigma \nabla_\rho v^\sigma n^\rho \nabla_\mu n_\nu \right. \\
&\quad + \gamma^\rho_\nu \nabla_\rho v^\sigma n_\sigma \nabla_\mu n^\eta \\
&\quad \left. + \gamma^\rho_\nu \gamma^\eta_\sigma \nabla_\mu \nabla_\rho v^\sigma \right). \quad (2.89)
\end{aligned}$$

We now take a moment to define the extrinsic curvature of a hypersurface, Σ_t . The extrinsic curvature can be defined as the projection of the gradient of the normal vector, $\mathbf{n}$, on its associated hypersurface and is a measure of how that hypersurface is curved in relation to the 4-dimensional spacetime manifold, $\mathcal{M}$. With this definition, the extrinsic curvature can be written mathematically as

$$K_{\alpha\beta} = -\gamma^\delta_\beta \nabla_\delta n_\alpha. \quad (2.90)$$

If we expand the projection operator in equation (2.90), we can write

$$\begin{aligned} K_{\alpha\beta} &= -\gamma^\delta_\beta \nabla_\delta n_\alpha \\ &= -(\delta^\delta_\beta - n^\delta n_\beta) \nabla_\delta n_\alpha \\ &= -\delta^\delta_\beta \nabla_\delta n_\alpha - n^\delta n_\beta \nabla_\delta n_\alpha \\ &= -\nabla_\beta n_\alpha - n_\beta n^\delta \nabla_\delta n_\alpha. \end{aligned} \quad (2.91)$$

The $n^\delta \nabla_\delta n_\alpha$ term in equation (2.91) can be expanded and rewritten as,

$$\begin{aligned} n^\delta \nabla_\delta n_\alpha &= n^\delta \nabla_\delta (-\alpha \nabla_\alpha t) \\ &= -n^\delta \alpha \nabla_\delta (\nabla_\alpha t) - n^\delta \nabla_\delta (\alpha) \nabla_\alpha t \\ &= -n^\delta \alpha \nabla_\alpha \nabla_\delta t - n^\delta \nabla_\alpha t \nabla_\delta \alpha \\ &= n^\delta \alpha \nabla_\alpha \left(\frac{n_\delta}{\alpha} \right) + n^\delta \left(\frac{n_\alpha}{\alpha} \right) \nabla_\delta \alpha \\ &= n^\delta \nabla_\alpha n_\delta + n^\delta n_\delta \alpha \nabla_\alpha \left(\frac{1}{\alpha} \right) + n^\delta \left(\frac{n_\alpha}{\alpha} \right) \nabla_\delta \alpha \\ &= n^\delta \nabla_\alpha n_\delta + \frac{1}{\alpha} \nabla_\alpha \alpha + \frac{n^\delta n_\alpha}{\alpha} \nabla_\delta \alpha \\ &= 0 + \frac{1}{\alpha} (\nabla_\alpha \alpha + n^\delta n_\alpha \nabla_\delta \alpha) \\ &= \frac{1}{\alpha} \gamma^\delta_\alpha \nabla_\delta \alpha. \end{aligned} \quad (2.92)$$

Putting in the result of equation (2.92) in equation (2.91), we can write an equation for the extrinsic curvature tensor in the form

$$\begin{aligned}
K_{\alpha\beta} &= -\nabla_\beta n_\alpha - n_\beta n^\delta \nabla_\delta n_\alpha \\
&= -\nabla_\beta n_\alpha - \frac{1}{\alpha} n_\alpha \gamma^\delta_\beta \nabla_\delta \alpha \\
&= -\nabla_\beta n_\alpha - \frac{1}{\alpha} n_\alpha D_\beta \alpha.
\end{aligned} \tag{2.93}$$

Solving for $\nabla_\beta n_\alpha$,

$$\nabla_\beta n_\alpha = -K_{\alpha\beta} - \frac{1}{\alpha} n_\alpha D_\beta \alpha, \tag{2.94}$$

we can use equation (2.94) to define the gradient of $\mathbf{m}$ as,

$$\begin{aligned}
\nabla_\beta m_\alpha &= \nabla_\beta (\alpha n_\alpha) \\
&= \alpha \nabla_\beta n_\alpha + n_\alpha \nabla_\beta \alpha \\
&= -\alpha K_{\alpha\beta} - n_\alpha D_\beta \alpha + n_\alpha \nabla_\beta \alpha.
\end{aligned} \tag{2.95}$$

From equation (2.93), one can see that $K_{\alpha\beta}$ is symmetric with regards to α and β . Lastly, by orthogonality we can use $n_\alpha v^\alpha = 0$ to write,

$$\begin{aligned}
\nabla_\beta (n_\alpha v^\alpha) &= n_\alpha \nabla_\beta v^\alpha + v^\alpha \nabla_\beta n_\alpha = 0 \\
&\Rightarrow n_\alpha \nabla_\beta v^\alpha = -v^\alpha \nabla_\beta n_\alpha.
\end{aligned} \tag{2.96}$$

Using equation (2.90), the three terms on the right-hand side of equation (2.89) can be rewritten as

$$\begin{aligned}
\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \gamma^\eta_\sigma n^\rho \nabla_\rho v^\sigma \nabla_\mu n_\nu &= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\sigma n^\rho \nabla_\rho v^\sigma \nabla_\mu n_\nu \\
&= -\gamma^\delta_\sigma n^\rho \nabla_\rho v^\sigma \gamma^\nu_\beta K_{\alpha\nu} \\
&= -\gamma^\delta_\sigma n^\rho \nabla_\rho v^\sigma K_{\alpha\beta}
\end{aligned} \tag{2.97}$$

and

$$\begin{aligned}
\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \gamma^\rho_\nu \nabla_\rho v^\sigma n_\sigma \nabla_\mu n^\eta &= \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta \nabla_\rho v^\sigma n_\sigma \nabla_\mu n^\eta \\
&= -\gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta v^\sigma \nabla_\rho n_\sigma \nabla_\mu n^\eta \\
&= -\gamma^\delta_\eta v^\sigma (\gamma^\rho_\beta \nabla_\rho n_\sigma) (\gamma^\mu_\alpha \nabla_\mu n^\eta) \\
&= -\gamma^\delta_\eta v^\sigma K_{\beta\sigma} K^\eta_\alpha \\
&= -K_{\beta\sigma} K^\delta_\alpha v^\sigma
\end{aligned} \tag{2.98}$$

and lastly,

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\delta_\eta \gamma^\rho_\nu \gamma^\eta_\sigma \nabla_\mu \nabla_\rho v^\sigma = \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\sigma \nabla_\mu \nabla_\rho v^\sigma. \tag{2.99}$$

Replacing the terms in equation (2.89) with equations (2.97) - (2.99) reveals

$$D_\alpha D_\beta v^\delta = -\gamma^\delta_\sigma n^\rho \nabla_\rho v^\sigma K_{\alpha\beta} - K^\delta_\alpha K_{\beta\sigma} v^\sigma + \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\sigma \nabla_\mu \nabla_\rho v^\sigma. \tag{2.100}$$

Likewise, by symmetry, we have

$$D_\beta D_\alpha v^\delta = -\gamma^\delta_\sigma n^\rho \nabla_\rho v^\sigma K_{\beta\alpha} - K^\delta_\beta K_{\alpha\sigma} v^\sigma + \gamma^\mu_\beta \gamma^\rho_\alpha \gamma^\delta_\sigma \nabla_\mu \nabla_\rho v^\sigma. \tag{2.101}$$

Substituting equations (2.100) and (2.101) into equation (2.85) and remembering the symmetry of $K_{\alpha\beta}$, we find

$$\begin{aligned}
{}^{(3)}R^\delta_{\sigma\alpha\beta} v^\sigma &= D_\alpha D_\beta v^\delta - D_\beta D_\alpha v^\delta \\
&= (K^\delta_\beta K_{\alpha\sigma} - K^\delta_\alpha K_{\beta\sigma}) v^\sigma + \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta (\nabla_\mu \nabla_\rho v^\eta - \nabla_\rho \nabla_\mu v^\eta) \\
&= (K^\delta_\beta K_{\alpha\sigma} - K^\delta_\alpha K_{\beta\sigma}) v^\sigma + \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta {}^{(4)}R^\eta_{\kappa\mu\rho} v^\kappa \\
&= (K^\delta_\beta K_{\alpha\sigma} - K^\delta_\alpha K_{\beta\sigma}) v^\sigma + \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} v^\sigma
\end{aligned} \tag{2.102}$$

where in the third line we used the four-dimensional Ricci identity to write $\nabla_\mu \nabla_\rho v^\eta - \nabla_\rho \nabla_\mu v^\eta = {}^{(4)}R^\eta_{\kappa\mu\rho} v^\kappa$ and in the last line we have used $\gamma^\kappa_\sigma v^\sigma = v^\kappa$. As $\mathbf{v}$ is a general vector tangent to Σ_t , we can write the general relation known as the *Gauss Relation*,

$$\gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} = {}^{(3)}R^\delta_{\sigma\alpha\beta} + K^\delta_\alpha K_{\beta\sigma} - K^\delta_\beta K_{\alpha\sigma}. \quad (2.103)$$

Further contracting equation (2.103) by applying γ^α_δ , we find

$$\begin{aligned} \gamma^\alpha_\delta \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\delta_\eta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} &= \gamma^\alpha_\delta {}^{(3)}R^\delta_{\sigma\alpha\beta} + \gamma^\alpha_\delta K^\delta_\alpha K_{\beta\sigma} - \gamma^\alpha_\delta K^\delta_\beta K_{\alpha\sigma} \\ \gamma^\alpha_\eta \gamma^\mu_\alpha \gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} &= {}^{(3)}R^\alpha_{\sigma\alpha\beta} + K^\alpha_\alpha K_{\beta\sigma} - K^\alpha_\beta K_{\alpha\sigma} \\ &= {}^{(3)}R_{\sigma\beta} + K K_{\beta\sigma} - K_{\sigma\alpha} K^\alpha_\beta \end{aligned} \quad (2.104)$$

where we define the extrinsic curvature scalar as $K = K^\alpha_\alpha$ and the Ricci Tensor as $R_{\sigma\beta} := R^\alpha_{\sigma\alpha\beta}$. If one combines the $\gamma^\alpha_\eta \gamma^\mu_\alpha = \gamma^\mu_\eta$ term on the left-hand side, equation (2.104) can be expanded using equation (2.64),

$$\begin{aligned} \gamma^\mu_\eta \gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} &= (\delta^\mu_\eta + n^\mu n_\eta) \gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} \\ &= \gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R_{\kappa\rho} + n^\mu n_\eta \gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R^\eta_{\kappa\mu\rho} \\ &= \gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R_{\kappa\rho} + n^\rho n^\kappa \gamma^\mu_\beta \gamma_{\eta\sigma} {}^{(4)}R^\eta_{\kappa\mu\rho}. \end{aligned} \quad (2.105)$$

Combining equations (2.104) and (2.105), we arrive at a second form of the Gauss Relation,

$$\gamma^\rho_\beta \gamma^\kappa_\sigma {}^{(4)}R_{\kappa\rho} + n^\rho n^\kappa \gamma^\mu_\beta \gamma_{\eta\sigma} {}^{(4)}R^\eta_{\kappa\mu\rho} = {}^{(3)}R_{\sigma\beta} + K K_{\beta\sigma} - K_{\sigma\alpha} K^\alpha_\beta. \quad (2.106)$$

Contracting the left-hand side of equation (2.106) along σ and β using $\gamma^{\sigma\beta}$ reveals

$$\begin{aligned}
\gamma^{\sigma\beta}\gamma^\mu_\eta\gamma^\rho_\beta\gamma^\kappa_\sigma{}^{(4)}R^\eta_{\kappa\mu\rho} &= \gamma^{\sigma\beta}(\gamma^\rho_\beta\gamma^\kappa_\sigma{}^{(4)}R_{\kappa\rho} + n^\rho n^\kappa\gamma^\mu_\beta\gamma_{\eta\sigma}{}^{(4)}R^\eta_{\kappa\mu\rho}) \\
&= {}^{(4)}R + \gamma^{\sigma\beta}\gamma^\kappa_\sigma n^\rho n_\beta R_{\kappa\rho} + n^\rho n^\kappa\gamma^{\mu\sigma}\gamma_{\eta\sigma}{}^{(4)}R^\eta_{\kappa\mu\rho} \\
&= {}^{(4)}R + 2n^\rho n^\kappa{}^{(4)}R_{\kappa\rho}.
\end{aligned} \tag{2.107}$$

This is called the *contracted Gauss relation*. If we contract the right-hand side of equation (2.104) along σ and β and set it equal to equation (2.107), we can write what is known as the *scalar Gauss relation*,

$${}^{(4)}R + 2n^\rho n^\kappa{}^{(4)}R_{\kappa\rho} = {}^{(3)}R + K^2 - K_{\alpha\sigma}K^{\alpha\sigma}. \tag{2.108}$$

2.6.2 Codazzi Relations

When considering the Gauss relations, we applied the Ricci Identity to a vector tangent to the hypersurface Σ_t . Now we wish to follow a similar process but with a vector normal to Σ_t . As $\mathbf{n}$ is by definition normal to the 3-dimensional hypersurface, we begin by applying the four-dimensional version of the Ricci Identity to it,

$$\nabla_\alpha\nabla_\beta n^\delta - \nabla_\beta\nabla_\alpha n^\delta = {}^{(4)}R^\delta_{\sigma\alpha\beta}n^\sigma. \tag{2.109}$$

The projection of equation (2.109) onto Σ_t ,

$$\gamma^\alpha_\mu\gamma^\beta_\nu\gamma^\rho_\delta{}^{(4)}R^\delta_{\sigma\alpha\beta}n^\sigma = \gamma^\alpha_\mu\gamma^\beta_\nu\gamma^\rho_\delta(\nabla_\alpha\nabla_\beta n^\delta - \nabla_\beta\nabla_\alpha n^\delta) \tag{2.110}$$

can be more compactly written as the terms on the right-hand side become

$$\begin{aligned}
\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\delta \nabla_\alpha \nabla_\beta n^\delta &= \gamma^\alpha_\mu \gamma^\rho_\delta \nabla_\alpha \gamma^\beta_\nu \nabla_\beta n^\delta \\
&= -\gamma^\alpha_\mu \gamma^\rho_\delta \nabla_\alpha K^\delta_\nu \\
&= -D_\mu \gamma^\rho_\delta K^\delta_\nu \\
&= -D_\mu K^\rho_\nu.
\end{aligned} \tag{2.111}$$

Using equation (2.111), we can now rewrite equation (2.110) as

$$\begin{aligned}
\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\delta {}^{(4)}R^\delta_{\sigma\alpha\beta} n^\sigma &= \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\delta (\nabla_\alpha \nabla_\beta n^\delta - \nabla_\beta \nabla_\alpha n^\delta) \\
&= D_\nu K^\rho_\mu - D_\mu K^\rho_\nu
\end{aligned} \tag{2.112}$$

which is known as the *Codazzi Relation*. If we project the Codazzi relation onto Σ_t by applying γ^μ_ρ , equation (2.112) becomes

$$\begin{aligned}
\gamma^\mu_\rho \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\delta {}^{(4)}R^\delta_{\sigma\alpha\beta} n^\sigma &= \gamma^\mu_\rho \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\delta (\nabla_\alpha \nabla_\beta n^\delta - \nabla_\beta \nabla_\alpha n^\delta) \\
&= \gamma^\mu_\rho (D_\nu K^\rho_\mu - D_\mu K^\rho_\nu) \\
&= D_\nu K^\rho_\rho - D_\rho K^\rho_\nu \\
&= D_\nu K - D_\rho K^\rho_\nu.
\end{aligned} \tag{2.113}$$

Examining the left-hand side of equation (2.113), we can express it in the form,

$$\begin{aligned}
\gamma^\mu_\rho \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\rho_\delta {}^{(4)}R^\delta_{\sigma\alpha\beta} n^\sigma &= \gamma^\beta_\nu \gamma^\alpha_\delta {}^{(4)}R^\delta_{\sigma\alpha\beta} n^\sigma \\
&= \gamma^\beta_\nu {}^{(4)}R_{\sigma\beta} n^\sigma
\end{aligned} \tag{2.114}$$

Using equation (2.114) for the left-hand side equation (2.113), we arrive at the *contracted Codazzi Relation*,

$$\gamma^\beta_\nu {}^{(4)}R_{\sigma\beta} n^\sigma = D_\nu K - D_\rho K^\rho_\nu \tag{2.115}$$

2.6.3 Mixed Projection of the Ricci Tensor

In equation (2.115), we completely projected the Ricci tensor onto the hypersurface Σ_t , and in the scalar Gauss relation of equation (2.108) we have a term, $n^\alpha n^{\beta(4)} R_{\alpha\beta}$ that represents the projection of the Ricci Tensor perpendicular to the hypersurface. To complete the set of projections, we now need a mixed projection of the Ricci tensor. We begin with the Ricci Identity as defined in equation (2.109), and apply a mixed projection operator, $\gamma_{\alpha\delta} \gamma^\eta_\beta n^\nu$, to both sides,

$$\begin{aligned}
\gamma_{\alpha\delta} \gamma^\eta_\beta n^\nu R^\delta_{\mu\eta\nu} n^\mu &= \gamma_{\alpha\delta} \gamma^\eta_\beta n^\nu (\nabla_\eta \nabla_\nu n^\delta - \nabla_\nu \nabla_\eta n^\delta) \\
&= \gamma_{\alpha\delta} \gamma^\eta_\beta n^\nu \left(\nabla_\eta \left(-K^\delta_\nu - n_\nu \frac{D^\delta \alpha}{\alpha} \right) - \nabla_\nu \left(-K^\delta_\eta - n_\eta \frac{D^\delta \alpha}{\alpha} \right) \right) \\
&= \gamma_{\alpha\delta} \gamma^\eta_\beta n^\nu \left(-\nabla_\eta K^\delta_\nu - \nabla_\eta \left(n_\nu \frac{D^\delta \alpha}{\alpha} \right) + \nabla_\nu K^\delta_\eta + \nabla_\nu \left(n_\eta \frac{D^\delta \alpha}{\alpha} \right) \right) \\
&= \gamma^\eta_\beta n^\nu \left(-\nabla_\eta K_{\alpha\nu} - \nabla_\eta \left(n_\nu \frac{D_\alpha \alpha}{\alpha} \right) + \nabla_\nu K_{\alpha\eta} + \nabla_\nu \left(n_\eta \frac{D_\alpha \alpha}{\alpha} \right) \right)
\end{aligned} \tag{2.116}$$

where we used the relation $\nabla_\nu n^\delta = -K^\delta_\eta + n_\eta \frac{D^\delta \alpha}{\alpha}$ from equation (2.93).

We will examine each term in equation (2.116) individually. Using the chain rule and the orthogonality relation $n^\nu K_{\alpha\nu} = 0$, the following relation can be established

$$n^\nu \nabla_\eta K_{\alpha\nu} = -K_{\alpha\nu} \nabla_\eta n^\nu \tag{2.117}$$

which allows the first term to be written as

$$\begin{aligned}
-\gamma^\eta_\beta n^\nu \nabla_\eta K_{\alpha\nu} &= \gamma^\eta_\beta K_{\alpha\nu} \nabla_\eta n^\nu \\
&= -K_{\alpha\nu} K_\beta{}^\nu.
\end{aligned} \tag{2.118}$$

Applying the chain rule again, the second term becomes in equation (2.116) becomes

$$\begin{aligned}
\gamma^\eta_\beta n^\nu \nabla_\eta \left(n_\nu \frac{D_\alpha \alpha}{\alpha} \right) &= \gamma^\eta_\beta n^\nu \left(n_\nu \nabla_\eta \frac{D_\alpha \alpha}{\alpha} + \frac{D_\alpha \alpha}{\alpha} \nabla_\eta n_\nu \right) \\
&= -\gamma^\eta_\beta \nabla_\eta \frac{D_\alpha \alpha}{\alpha} \\
&= -D_\beta \left(\frac{D_\alpha \alpha}{\alpha} \right) \\
&= -\frac{1}{\alpha} D_\alpha D_\beta \alpha + \frac{D_\alpha \alpha}{\alpha} \frac{D_\beta \alpha}{\alpha}
\end{aligned} \tag{2.119}$$

where we used $n^\nu \nabla_\eta n_\nu = 1/2 \nabla_\eta (n^\nu n_\nu) = 1/2 \nabla_\eta (-1) = 0$. The third term in equation (2.116) undergoes a simple transformation by applying the spatial projection operator to the extrinsic curvature tensor,

$$\gamma^\eta_\beta n^\nu \nabla_\nu K_{\alpha\eta} = n^\nu \nabla_\nu K_{\alpha\beta}. \tag{2.120}$$

We use equation (2.92) to rewrite the last term as

$$\begin{aligned}
\gamma^\eta_\beta n^\nu \nabla_\nu \left(n_\eta \frac{D_\alpha \alpha}{\alpha} \right) &= \gamma^\eta_\beta n^\nu \left(n_\eta \nabla_\nu \frac{D_\alpha \alpha}{\alpha} + \frac{D_\alpha \alpha}{\alpha} \nabla_\nu n_\eta \right) \\
&= \gamma^\eta_\beta n_\eta n^\nu \nabla_\nu \frac{D_\alpha \alpha}{\alpha} + \gamma^\eta_\beta n^\nu \nabla_\nu (n_\eta) \frac{D_\alpha \alpha}{\alpha} \\
&= 0 + \gamma^\eta_\beta n^\nu \frac{D_\eta \alpha}{\alpha} \frac{D_\alpha \alpha}{\alpha} \\
&= \frac{D_\alpha \alpha}{\alpha} \frac{D_\beta \alpha}{\alpha}.
\end{aligned} \tag{2.121}$$

Combining equations (2.118)-(2.121), Equation (2.116) becomes

$$\begin{aligned}
\gamma_{\alpha\delta} \gamma^\eta_\beta n^\nu R^\delta_{\mu\eta\nu} n^\mu &= \gamma^\eta_\beta n^\nu \left(-\nabla_\eta K_{\alpha\nu} - \nabla_\eta \left(n_\nu \frac{D_\alpha \alpha}{\alpha} \right) + \nabla_\nu K_{\alpha\eta} + \nabla_\nu \left(n_\eta \frac{D_\alpha \alpha}{\alpha} \right) \right) \\
&= -K_{\alpha\nu} K_\beta{}^\nu + \frac{1}{\alpha} D_\alpha D_\beta \alpha - \frac{D_\alpha \alpha}{\alpha} \frac{D_\beta \alpha}{\alpha} + n^\nu \nabla_\nu K_{\alpha\beta} + \frac{D_\alpha \alpha}{\alpha} \frac{D_\beta \alpha}{\alpha} \\
&= -K_{\alpha\nu} K_\beta{}^\nu + \frac{1}{\alpha} D_\alpha D_\beta \alpha + n^\nu \nabla_\nu K_{\alpha\beta}.
\end{aligned} \tag{2.122}$$

We can relate the last term in equation (2.122) to the Lie derivative of the extrinsic curvature, K along the time evolution vector, $\mathbf{m}$. We begin by using the definitions for the Lie derivative from equation (2.37), and the gradient of $\mathbf{m}$ in equation (2.95) to form the expression,

$$\begin{aligned}
\mathcal{L}_{\mathbf{m}}K_{\alpha\beta} &= m^\nu \nabla_\nu K_{\alpha\beta} + K_{\nu\beta} \nabla_\alpha m^\nu + K_{\alpha\nu} \nabla_\beta m^\nu \\
&= \alpha n^\nu \nabla_\nu K_{\alpha\beta} + K_{\nu\beta} (-\alpha K^\nu{}_\alpha - n_\nu D^\alpha \alpha + n^\alpha \nabla_\nu \alpha) \\
&\quad + K_{\alpha\nu} (-\alpha K^\nu{}_\beta - n_\nu D^\beta \alpha + n^\beta \nabla_\nu \alpha) \\
&= \alpha n^\nu \nabla_\nu K_{\alpha\beta} - 2\alpha K_{\alpha\nu} K^\nu{}_\beta + 2K_{\alpha\nu} n_\beta D^\nu \alpha.
\end{aligned} \tag{2.123}$$

Projecting this equation onto the hypersurface, we can write

$$\begin{aligned}
\mathcal{L}_{\mathbf{m}}K_{\alpha\beta} &= \gamma^\sigma{}_\alpha \gamma^\rho{}_\beta \mathcal{L}_{\mathbf{m}}K_{\sigma\rho} \\
&= \alpha \gamma^\sigma{}_\alpha \gamma^\rho{}_\beta n^\nu \nabla_\nu K_{\sigma\rho} - 2\alpha \gamma^\sigma{}_\alpha \gamma^\rho{}_\beta K_{\sigma\nu} K^\nu{}_\rho + 2\gamma^\sigma{}_\alpha \gamma^\rho{}_\beta K_{\sigma\nu} n_\rho D^\nu \alpha \\
&= \alpha n^\nu \nabla_\nu K_{\alpha\beta} - 2\alpha K_{\alpha\nu} K^\nu{}_\beta + 0.
\end{aligned} \tag{2.124}$$

In equation (2.124), $\gamma^\sigma{}_\alpha K_{\sigma\rho} = K_{\alpha\rho}$ as $K_{\sigma\rho}$ is tangent to the hypersurface, and $\gamma^\sigma{}_\alpha n_\sigma = 0$ as $\mathbf{n}$ is orthogonal to the hypersurface. Next, we solve equation (2.124) for $n^\nu \nabla_\nu K_{\alpha\beta}$,

$$n^\nu \nabla_\nu K_{\alpha\beta} = \frac{1}{\alpha} (\mathcal{L}_{\mathbf{m}}K_{\alpha\beta} + 2\alpha K_{\alpha\nu} K^\nu{}_\beta), \tag{2.125}$$

and put it into equation (2.122) to finally arrive at

$$\begin{aligned}
\gamma_{\alpha\delta} \gamma^\eta{}_\beta n^\nu R^\delta{}_{\mu\eta\nu} n^\mu &= -K_{\alpha\nu} K^\nu{}_\beta + \frac{1}{\alpha} D_\beta D_\alpha \alpha + n^\nu \nabla_\nu K_{\alpha\beta} \\
&= -K_{\alpha\nu} K^\nu{}_\beta + \frac{1}{\alpha} D_\beta D_\alpha \alpha + \frac{1}{\alpha} (\mathcal{L}_{\mathbf{m}}K_{\alpha\beta} + 2\alpha K_{\alpha\nu} K^\nu{}_\beta) \\
&= K_{\alpha\nu} K^\nu{}_\beta + \frac{1}{\alpha} D_\alpha D_\beta \alpha + \frac{1}{\alpha} \mathcal{L}_{\mathbf{m}}K_{\alpha\beta}.
\end{aligned} \tag{2.126}$$

As the covariant derivative D_α is torsion-free, we can freely exchange the α and β indices in the second term on the right-hand side.

Using equation (2.126) in the contracted Gauss equation, (2.107), we can write

$$\begin{aligned}
\gamma^\rho_\beta \gamma^\kappa_\alpha {}^{(4)}R_{\kappa\rho} - {}^{(3)}R_{\alpha\beta} - K K_{\beta\alpha} + K_{\sigma\alpha} K^\sigma_\beta \\
= -n^\rho n^\kappa \gamma^\mu_\beta \gamma_{\eta\sigma} {}^{(4)}R^\eta_{\kappa\mu\rho} \\
= -K_{\alpha\nu} K^\nu_\beta - \frac{1}{\alpha} D_\alpha D_\beta \alpha - \frac{1}{\alpha} \mathcal{L}_\mathbf{m} K_{\alpha\beta}. \tag{2.127}
\end{aligned}$$

Solving equation (2.127) for $\gamma^\rho_\beta \gamma^\kappa_\alpha {}^{(4)}R_{\kappa\rho}$, the above becomes

$$\gamma^\rho_\beta \gamma^\kappa_\alpha {}^{(4)}R_{\kappa\rho} = {}^{(3)}R_{\alpha\beta} + K K_{\beta\alpha} - 2K_{\sigma\alpha} K^\sigma_\beta - \frac{1}{\alpha} D_\alpha D_\beta \alpha - \frac{1}{\alpha} \mathcal{L}_\mathbf{m} K_{\alpha\beta}. \tag{2.128}$$

2.7 Time Evolution of the Spatial Metric, γ

We have explored the properties of a hypersurface, Σ_t . Now, we wish to consider how that hypersurface evolves into a neighboring hypersurface $\Sigma_{t+\delta t}$. We begin by defining the extrinsic curvature, $K_{\alpha\beta}$, as the projection of the gradient of n_β onto the hypersurface.

$$\begin{aligned}
K_{\alpha\beta} &:= -\gamma^\delta_\alpha \nabla_\delta n_\beta \\
&= -D_\alpha n_\beta \\
&= -\partial_\alpha^{(3)} n_\beta - {}^{(3)}\Gamma^\mu_{\alpha\beta} n_\mu \\
&= -{}^{(3)}\Gamma^\mu_{\alpha\beta} n_\mu. \tag{2.129}
\end{aligned}$$

The superscripts $^{(3)}$ denote that while the associated object uses 4-dimensional indices, it's only defined on the spatial hypersurface and therefore components related to the time index are zero, i.e. $\partial_\alpha^{(3)} = (0, \partial_1, \partial_2, \partial_3)$. As n_μ 's only non-zero component is

$n_0 = -\alpha$, we can drop the partial derivatives and simplify equation (2.129) to

$$\begin{aligned}
K_{\alpha\beta} &= -^{(3)}\Gamma_{\alpha\beta}^0 n_0 \\
&= \frac{\alpha}{2} \gamma^{0\nu} (\partial_\alpha^{(3)} \gamma_{\nu\beta} + \partial_\beta^{(3)} \gamma_{\alpha\nu} - \partial_\nu^{(3)} \gamma_{\alpha\beta}) \\
&= \frac{\alpha}{2} (\gamma^{0\nu} \partial_\alpha^{(3)} \gamma_{\nu\beta} + \gamma^{0\nu} \partial_\beta^{(3)} \gamma_{\alpha\nu} - \gamma^{0\nu} \partial_\nu^{(3)} \gamma_{\alpha\beta}). \tag{2.130}
\end{aligned}$$

We can decompose the first term in parenthesis in equation (2.130) as

$$\begin{aligned}
\gamma^{0\nu} \partial_\alpha^{(3)} \gamma_{\nu\beta} &= \gamma^{00} \partial_\alpha^{(3)} \gamma_{0\beta} + \gamma^{0\kappa} \partial_\alpha^{(3)} \gamma_{\kappa\beta} \\
&= \frac{-1}{\alpha^2} \partial_\alpha^{(3)} \beta_\beta + \frac{\beta^\kappa}{\alpha^2} \partial_\alpha^{(3)} \gamma_{\kappa\beta} \tag{2.131}
\end{aligned}$$

and by the symmetry of γ , the second term becomes

$$\gamma^{0\nu} \partial_\beta^{(3)} \gamma_{\alpha\nu} = \frac{-1}{\alpha^2} \partial_\beta^{(3)} \beta_\alpha + \frac{\beta^\kappa}{\alpha^2} \partial_\beta^{(3)} \gamma_{\alpha\kappa}. \tag{2.132}$$

Lastly, the third term becomes

$$\begin{aligned}
\gamma^{0\nu} \partial_\nu^{(3)} \gamma_{\alpha\beta} &= \gamma^{00} \partial_0 \gamma_{\alpha\beta} + \gamma^{0\kappa} \partial_\kappa \gamma_{\alpha\beta} \\
&= \frac{-1}{\alpha^2} \partial_0 \gamma_{\alpha\beta} + \frac{\beta^\kappa}{\alpha^2} \partial_\kappa \gamma_{\alpha\beta}. \tag{2.133}
\end{aligned}$$

If we now use equations (2.131) - (2.133) in equation (2.130), we can write

$$\begin{aligned}
K_{\alpha\beta} &= \frac{\alpha}{2} (\gamma^{0\nu} \partial_\alpha^{(3)} \gamma_{\nu\beta} + \gamma^{0\nu} \partial_\beta^{(3)} \gamma_{\alpha\nu} - \gamma^{0\nu} \partial_\nu^{(3)} \gamma_{\alpha\beta}) \\
&= \frac{\alpha}{2} \left[\left(\frac{-1}{\alpha^2} \partial_\alpha \beta_\beta + \frac{\beta^\kappa}{\alpha^2} \partial_\alpha \gamma_{\kappa\beta} \right) \right. \\
&\quad \left. + \left(\frac{-1}{\alpha^2} \partial_\beta \beta_\alpha + \frac{\beta^\kappa}{\alpha^2} \partial_\beta \gamma_{\alpha\kappa} \right) \right. \\
&\quad \left. - \left(\frac{-1}{\alpha^2} \partial_0 \gamma_{\alpha\beta} + \frac{\beta^\kappa}{\alpha^2} \partial_\kappa \gamma_{\alpha\beta} \right) \right] \\
&= \frac{1}{2\alpha} (-\partial_\alpha \beta_\beta - \partial_\beta \beta_\alpha + \partial_0 \gamma_{\alpha\beta}) \\
&\quad + \frac{\beta^\kappa}{2\alpha} (\partial_\alpha \gamma_{\kappa\beta} + \partial_\beta \gamma_{\alpha\kappa} - \partial_\kappa \gamma_{\alpha\beta}). \tag{2.134}
\end{aligned}$$

The terms lead by $\frac{\beta^\kappa}{2\alpha}$ equation (2.134) can be related to the 3-dimensional Christoffel symbol by

$$\begin{aligned}
\frac{\beta^\kappa}{2\alpha} (\partial_\alpha \gamma_{\kappa\beta} + \partial_\beta \gamma_{\alpha\kappa} - \partial_\kappa \gamma_{\alpha\beta}) &= \frac{1}{2\alpha} (\beta_\epsilon \gamma^{\epsilon\kappa} (\partial_\alpha \gamma_{\kappa\beta} + \partial_\beta \gamma_{\alpha\kappa} - \partial_\kappa \gamma_{\alpha\beta})) \\
&= \frac{\beta_\epsilon}{2\alpha} (2^{(3)} \Gamma^\epsilon_{\alpha\beta}) \\
&= \frac{\beta_\epsilon}{2\alpha} ({}^{(3)} \Gamma^\epsilon_{\alpha\beta} + {}^{(3)} \Gamma^\epsilon_{\beta\alpha}). \tag{2.135}
\end{aligned}$$

Putting this into equation (2.134), we can now write

$$\begin{aligned}
K_{\alpha\beta} &= \frac{1}{2\alpha} (\partial_0 \gamma_{\alpha\beta} - \partial_\alpha \beta_\beta - \partial_\beta \beta_\alpha) \\
&\quad + \frac{\beta^\kappa}{2\alpha} (\partial_\alpha \gamma_{\kappa\beta} + \partial_\beta \gamma_{\alpha\kappa} - \partial_\kappa \gamma_{\alpha\beta}) \\
&= \frac{1}{2\alpha} (\partial_0 \gamma_{\alpha\beta} - \partial_\alpha \beta_\beta - \partial_\beta \beta_\alpha) \\
&\quad + \frac{\beta_\epsilon}{2\alpha} ({}^{(3)} \Gamma^\epsilon_{\alpha\beta} + {}^{(3)} \Gamma^\epsilon_{\beta\alpha}) \\
&= \frac{1}{2\alpha} (\partial_0 \gamma_{\alpha\beta} - (\partial_\alpha \beta_\beta - \beta_\epsilon {}^{(3)} \Gamma^\epsilon_{\alpha\beta}) \\
&\quad - (\partial_\beta \beta_\alpha - \beta_\epsilon {}^{(3)} \Gamma^\epsilon_{\beta\alpha})) \\
&= \frac{1}{2\alpha} (\partial_0 \gamma_{\alpha\beta} - D_\alpha \beta_\beta - D_\beta \beta_\alpha). \tag{2.136}
\end{aligned}$$

When solved for $\partial_0\gamma_{\alpha\beta}$, equation (2.136) becomes

$$\partial_0\gamma_{\alpha\beta} = -2\alpha K_{\alpha\beta} + D_\alpha\beta_\beta + D_\beta\beta_\alpha \quad (2.137)$$

The Lie derivative of the spatial metric $\gamma_{\alpha\beta}$ along β , is defined as

$$\mathcal{L}_\beta\gamma_{\alpha\beta} = \beta^\kappa\nabla_\kappa\gamma_{\alpha\beta} + \gamma_{\alpha\kappa}\nabla_\beta\beta^\kappa + \gamma_{\kappa\beta}\nabla_\alpha\beta^\kappa \quad (2.138)$$

by equation (2.37). As $\beta^0 = 0$, the Lie derivative in equation (2.138) can be expressed with the three-dimensional covariant derivative, D , and thus the first term becomes zero, leaving us with the equation

$$\begin{aligned} \mathcal{L}_\beta\gamma_{\alpha\beta} &= \gamma_{\alpha\kappa}D_\beta\beta^\kappa + \gamma_{\kappa\beta}D_\alpha\beta^\kappa \\ &= D_\beta\beta_\alpha + D_\alpha\beta_\beta. \end{aligned} \quad (2.139)$$

Equation (2.139) allows us to write equation (2.137) in the more compact form,

$$\partial_0\gamma_{\alpha\beta} - \mathcal{L}_\beta\gamma_{\alpha\beta} = -2\alpha K_{\alpha\beta}. \quad (2.140)$$

Recalling equation (2.47), this can be further simplified to

$$\mathcal{L}_m\gamma_{\alpha\beta} = -2\alpha K_{\alpha\beta}. \quad (2.141)$$

A similar expression for the Lie derivative of the inverse metric, $\gamma^{\alpha\beta}$, along m can be derived. We start by taking the Lie along m of equation (2.43), which gives the relationship between the metric and its inverse,

$$0 = \mathcal{L}_m\delta^i_j = \mathcal{L}_m\gamma^{ik}\gamma_{kj} = \gamma^{ik}\mathcal{L}_m\gamma_{kj} + \gamma_{kj}\mathcal{L}_m\gamma^{ik}. \quad (2.142)$$

Next, the equation is multiplied by γ^{jl} and the first term is transformed using equation (2.141),

$$\begin{aligned}
0 &= \gamma^{jl}\gamma^{ik}\mathcal{L}_{\mathbf{m}}\gamma_{kj} + \gamma^{jl}\gamma_{kj}\mathcal{L}_{\mathbf{m}}\gamma^{ik} \\
&= -2\alpha\gamma^{jl}\gamma^{ik}K_{kj} + \delta_k^l\mathcal{L}_{\mathbf{m}}\gamma^{ik} \\
&= -2\alpha K^{il} + \mathcal{L}_{\mathbf{m}}\gamma^{il}.
\end{aligned} \tag{2.143}$$

Solving for $\mathcal{L}_{\mathbf{m}}\gamma^{il}$ and changing the index l to j for aesthetics, we have the expression

$$\mathcal{L}_{\mathbf{m}}\gamma^{ij} = 2\alpha K^{ij}. \tag{2.144}$$

2.8 Decomposition of Einstein Equation

We are now prepared to decompose the Einstein equations into their projections perpendicular to the hypersurface, Σ_t , onto Σ_t , as well as a mixed projection. To aid in this, we first form an alternative version of the Einstein equations by first taking the trace of equation (2.1),

$$\begin{aligned}
8\pi g^{\alpha\beta}T_{\alpha\beta} &= g^{\alpha\beta(4)}R_{\alpha\beta} - \frac{1}{2}{}^{(4)}Rg^{\alpha\beta}g_{\alpha\beta} \\
&= {}^{(4)}R - \frac{1}{2}\delta^\alpha_\alpha {}^{(4)}R \\
&= -{}^{(4)}R.
\end{aligned} \tag{2.145}$$

Realizing $g^{\alpha\beta}T_{\alpha\beta} = T$, we can put equation (2.145) into equation (2.1) and write

$$\begin{aligned}
{}^{(4)}R_{\alpha\beta} &= 8\pi T_{\alpha\beta} + \frac{1}{2}{}^{(4)}Rg_{\alpha\beta} \\
&= 8\pi T_{\alpha\beta} - \frac{1}{2}(8\pi T)g_{\alpha\beta} \\
&= 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right).
\end{aligned} \tag{2.146}$$

2.8.1 Decomposition of the Stress-Energy Tensor, T

We intend to project the Einstein equations onto various surfaces of the 3+1 geometry but to properly evaluate these projections, we will first need to decompose the stress-energy tensor, T , which is present on the left-hand side of equation (2.1).

The *Energy Density* comes from the projection of T perpendicular to Σ_t ,

$$E := n^\alpha n^\beta T_{\alpha\beta}. \quad (2.147)$$

The *Energy-Momentum tensor* is found by projecting T onto the hypersurface,

$$S_{\alpha\beta} := \gamma^\mu_\alpha \gamma^\nu_\beta T_{\mu\nu}, \quad (2.148)$$

which can be contracted with $\gamma^{\alpha\beta}$ to form the *momentum density scalar*,

$$S := S^\alpha_\alpha = \gamma^{\alpha\beta} S_{\alpha\beta}. \quad (2.149)$$

The *momentum density* is given by the mixed projection of T ,

$$S_\alpha := -\gamma^\mu_\alpha n^\nu T_{\mu\nu}. \quad (2.150)$$

One last useful relationship can be derived starting with equation (2.148) and expanding the projection operators using equation (2.64),

$$\begin{aligned} S_{\alpha\beta} &:= \gamma^\mu_\alpha \gamma^\nu_\beta T_{\mu\nu} \\ &= (\delta^\mu_\alpha + n^\mu n_\alpha) (\delta^\nu_\beta + n^\nu n_\beta) T_{\mu\nu} \\ &= (\delta^\mu_\alpha + n^\mu n_\alpha) (\delta^\nu_\beta + n^\nu n_\beta) T_{\mu\nu} \\ &= (\delta^\mu_\alpha \delta^\nu_\beta + 2\delta^\mu_\alpha n^\nu n_\beta + n_\alpha n_\beta n^\mu n^\nu) T_{\mu\nu} \\ &= T_{\alpha\beta} + 2n^\nu n_\beta T_{\alpha\nu} + n_\alpha n_\beta E. \end{aligned} \quad (2.151)$$

If we now contract equation (2.151) along α and β , we arrive at a relationship between the scalar quantities S , E , and T :

$$\begin{aligned}
S &= g^{\alpha\beta} S_{\alpha\beta} = g^{\alpha\beta} T_{\alpha\beta} + 2g^{\alpha\beta} n^\nu n_\beta T_{\alpha\nu} + g^{\alpha\beta} n_\alpha n_\beta E \\
&= T + 2n^\alpha n^\nu T_{\alpha\nu} + n^\alpha n_\alpha E \\
&= T + E.
\end{aligned} \tag{2.152}$$

2.8.2 Projection Perpendicular To Σ_t

Now, we want to project Einstein equations, equation (2.1), perpendicular to Σ_t by contracting it with $\mathbf{n}$. The projected left-hand side of (2.1) becomes,

$$n^\alpha n^\beta ({}^{(4)}R_{\alpha\beta} - \frac{1}{2} {}^{(4)}R g_{\alpha\beta}) = n^\alpha n^\beta R_{\alpha\beta} - \frac{1}{2} {}^{(4)}R n^\alpha n^\beta g_{\alpha\beta}. \tag{2.153}$$

Using the scalar Gauss relation, equation (2.108), we can rewrite this as

$$\begin{aligned}
n^\alpha n^\beta (R_{\alpha\beta} - \frac{1}{2} {}^{(4)}R g_{\alpha\beta}) &= n^\alpha n^\beta R_{\alpha\beta} - \frac{1}{2} {}^{(4)}R n^\alpha n^\beta g_{\alpha\beta} \\
&= \frac{1}{2} ({}^3R - {}^{(4)}R + K^2 - K_{\alpha\beta} K^{\alpha\beta}) + \frac{1}{2} {}^{(4)}R \\
&= \frac{1}{2} ({}^3R + K^2 - K_{\alpha\beta} K^{\alpha\beta}).
\end{aligned} \tag{2.154}$$

From the definition of the energy density, equation (2.147), we can readily see the projected left-hand side of equation (2.1) is

$$n^\alpha n^\beta 8\pi T_{\alpha\beta} = 8\pi E. \tag{2.155}$$

Using equation (2.154) and (2.155), we can write the perpendicular projection of equation (2.1) as

$${}^3R + K^2 - K_{\alpha\beta} K^{\alpha\beta} = 16\pi E. \tag{2.156}$$

This is known as the *Hamiltonian Constraint*.

2.8.3 Projection onto Σ_t

By applying γ^α_β to the Einstein Equations in the form of equation (2.146), we can project it onto the hypersurface Σ_t . The projected right-hand side of equation (2.146) is simply equation (2.128),

$$\gamma^\rho_\beta \gamma^\kappa_\alpha {}^{(4)}R_{\kappa\rho} = {}^{(3)}R_{\alpha\beta} + K K_{\beta\alpha} - 2K_{\sigma\alpha} K^\sigma_\beta - \frac{1}{\alpha} D_\alpha D_\beta \alpha - \frac{1}{\alpha} \mathcal{L}_\mathbf{m} K_{\alpha\beta}. \quad (2.157)$$

When projected, the left-hand side of equation (2.146) becomes

$$8\pi \left(\gamma^\rho_\beta \gamma^\kappa_\alpha T_{\rho\kappa} - \frac{1}{2} \gamma^\rho_\beta \gamma^\kappa_\alpha g_{\rho\kappa} T \right) = 8\pi \left(S_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} T \right). \quad (2.158)$$

Putting equations (2.157) and (2.158) together to form the projected Einstein equations, we can write

$${}^{(3)}R_{\alpha\beta} + K K_{\beta\alpha} - 2K_{\sigma\alpha} K^\sigma_\beta - \frac{1}{\alpha} D_\alpha D_\beta \alpha - \frac{1}{\alpha} \mathcal{L}_\mathbf{m} K_{\alpha\beta} = 8\pi \left(S_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} T \right), \quad (2.159)$$

which can be solved for $\mathcal{L}_\mathbf{m} K_{\alpha\beta}$,

$$\begin{aligned} \mathcal{L}_\mathbf{m} K_{\alpha\beta} &= \alpha {}^{(3)}R_{\alpha\beta} + \alpha K K_{\beta\alpha} - 2\alpha K_{\sigma\alpha} K^\sigma_\beta - D_\alpha D_\beta \alpha - 8\pi\alpha \left(S_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} T \right) \\ &= \alpha {}^{(3)}R_{\alpha\beta} + \alpha K K_{\beta\alpha} - 2\alpha K_{\sigma\alpha} K^\sigma_\beta - D_\alpha D_\beta \alpha - 8\pi\alpha \left(S_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} (S - E) \right). \end{aligned} \quad (2.160)$$

This gives us an expression for the time evolution of the extrinsic curvature.

2.8.4 Mixed Projection

Lastly, we wish to project the Einstein equations both onto the hypersurface and perpendicular to it. This is accomplished by applying $\gamma^\alpha_\mu n^\beta$ to both sides of equation

(2.1). The first term on the right-hand side once projected becomes the contracted Codazzi relation of equation (2.115) while the second term goes to zero as $n^\beta \gamma_\mu^\alpha g_{\alpha\beta} = n^\beta \gamma_{\mu\beta} = 0$, allowing us to write,

$$\gamma_\mu^\alpha n^\beta \left({}^{(4)}R_{\alpha\beta} - \frac{1}{2} {}^{(4)}R g_{\alpha\beta} \right) = D_\nu K - D_\rho K^\rho{}_\nu. \quad (2.161)$$

The left-hand side simplifies to the momentum density,

$$8\pi \gamma_\mu^\alpha n^\beta T_{\alpha\beta} = -8\pi S_\mu, \quad (2.162)$$

by equation (2.150). Putting this all together we arrive at the expression,

$$D_\sigma K^\sigma{}_\mu - D_\mu K = 8\pi S_\mu. \quad (2.163)$$

This is called the *momentum constraint*.

2.9 Summary

We have now completely described the 3+1 formalism of General Relativity. We began by foliating the four-dimensional metric into an infinite set of 3-dimension spatial hypersurfaces described by the 3-dimensional spatial metric, $\gamma_{\alpha\beta}$, and connected by a spatial shift vector, β_i , and a scalar lapse function α , such that the four-dimensional metric can be written

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \quad (2.164)$$

and the inverse

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix}. \quad (2.165)$$

By this definition of g , the proper distance becomes

$$\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= (-\alpha^2 + \beta^i \beta_i) dt^2 + \gamma_{ij} (dx^i + \beta_i dt) (dx^j + \beta_j dt) \\
&= -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j.
\end{aligned} \tag{2.166}$$

Next, we derived an expression for *The Time Evolution of the 3-Dimension Metric*, γ

$$\mathcal{L}_\beta \gamma_{ij} - \partial_0 \gamma_{ij} = 2\alpha K_{ij}. \tag{2.167}$$

The next step taken was to decompose the Einstein equations into their projections onto, perpendicular to, as well as a mixed projection that is both onto and perpendicular to the hypersurface Σ_t . This decomposition lead to the *Hamiltonian constraint*,

$${}^3R + K^2 - K_{ij} K^{ij} = 16\pi E, \tag{2.168}$$

the *Momentum constraint*,

$$D_i K^i_j - D_j K = 8\pi S_j, \tag{2.169}$$

and the *Time Evolution of the Extrinsic Curvature*, K ,

$$\mathcal{L}_\mathbf{m} K_{ij} = \alpha^{(3)} R_{ij} + \alpha K K_{ij} - 2\alpha K_{ki} K^k_j - D_i D_j \alpha - 8\pi \left(S_{ij} - \frac{1}{2} \gamma_{ij} (S - E) \right). \tag{2.170}$$

Chapter 3

Approximating General Relativistic Gravity: The Conformally Flat Condition, CFC

3.1 Introduction

Equations (2.167)-(2.170) constrain and evolve the 3+1 metric. Containing both hyperbolic and elliptic equations, the system of equations is a significant challenge to solve. To reduce the complexity different approximations have been devised to simplify the 3+1 system while attempting to preserve physical fidelity. One such approximation is known as the *Conformally Flat Condition* (CFC).

The CFC formalism reduces the 3+1 system of hyperbolic and elliptic equations to five elliptic equations, one each for the conformal factor, lapse function, and each component of the shift vector. This is done by first restricting the spatial 3-metric to be conformally flat, i.e. $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$, where $\bar{\gamma}_{ij}$ is a flat metric. Hence the name conformally flat condition. The second restriction is to adopt the maximal slicing gauge choice which restricts the trace of the extrinsic curvature tensor to zero for all

time,

$$K = \partial_t K = 0. \quad (3.1)$$

This chapter will show the effects of the CFC conditions on the 3+1 system.

3.2 Defining the Conformal Metric

The CFC formalism begins with a conformal transformation of the 3-metric. Conformal transformation takes the form of

$$\gamma = \psi^\alpha \tilde{\gamma} \quad (3.2)$$

where γ and $\tilde{\gamma}$ are the 3-metric and the *conformal 3-metric* on the hypersurface Σ_t respectively, ψ is a strictly positive scalar field called the *conformal factor*, and α is a real number. The set of all $\tilde{\gamma}$ that satisfies equation (3.2) is called a conformal equivalence class. One such class is the unit-determinant class that can be expressed as

$$\tilde{\gamma}_{ij} := \gamma^{-\frac{1}{3}} \gamma_{ij} \quad (3.3)$$

such that

$$\gamma := \det \gamma_{ij} \quad (3.4)$$

is the determinant of γ . For Cartesian coordinates, $\det \gamma_{ij} = 1$ but this does not hold for spherical coordinates. To accommodate for this, we introduce a background metric on the Σ_t hypersurfaces. This background metric, $\mathbf{f}$ shall possess the following qualities:

1. $\mathbf{f}$ is a flat metric and therefore has a vanishing Riemann tensor associated with it.
2. $\mathbf{f}$ is constant in time, i.e.

$$\mathcal{L}_t \mathbf{f} = 0 \tag{3.5}$$

where $\mathcal{L}_t$ is the Lie Derviative along the time axis $\mathbf{t}$. This requirement can also be written as

$$\frac{\partial f_{ij}}{\partial t} = 0 \tag{3.6}$$

3. $f_{ij} = \gamma_{ij}$ at spatial infinity.
4. The inverse background metric f^{ik} is defined

$$f^{ik} f_{kj} = \delta^i_j \tag{3.7}$$

If we now employ this background metric as a normalization factor in equation (3.3)

$$\bar{\gamma}_{ij} := \left(\frac{\gamma}{f}\right)^{-\frac{1}{3}} \gamma_{ij} \tag{3.8}$$

where $f := \det f_{ij}$, then

$$\det \bar{\gamma} = f. \tag{3.9}$$

This relation describes an equivalence class that relates all metrics that have the same determinant as the flat background metric $\mathbf{f}$, which allows us to include spherical coordinates without losing the ability to handle cartesian coordinates. If $\mathbf{f}$ is Cartesian then $\det f_{ij} = 1$ which reduces equation (3.8) to equation (3.3). If we

now put equation (3.8) in the form of equation (3.2), we find

$$\gamma_{ij} = \left(\frac{\gamma}{f}\right)^{\frac{1}{3}} \bar{\gamma}_{ij} = \psi^\alpha \bar{\gamma}_{ij}. \quad (3.10)$$

Picking $\alpha = 4$, we have a definition of the conformally flat equivalence relation

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \quad (3.11)$$

where

$$\psi = \left(\frac{\gamma}{f}\right)^{\frac{1}{12}}. \quad (3.12)$$

If we require the inverse conformal metric, $\bar{\gamma}^{ij}$, is related to the conformal metric in the usual way, i.e.

$$\bar{\gamma}^{ik} \bar{\gamma}_{kj} = \delta^i_j \quad (3.13)$$

then it follows

$$\gamma^{ij} = \psi^{-4} \bar{\gamma}^{ij}. \quad (3.14)$$

3.3 Conformal Transformations

3.3.1 Christoffel Symbol Transformation

With the relation between γ_{ij} and $\bar{\gamma}_{ij}$ established, we can now begin to define relations between the standard covariant derivative D_i and the conformal covariant derivative $\bar{D}_i$. (From here forward, we will denote anything that is defined using the conformal metric with a bar.) As the covariant derivative is defined in terms of the Christoffel symbol through equation (2.77), we will first examine how it transforms.

The Christoffel symbol on the hypersurface is defined in the 3+1 Formalism by equation (2.78),

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\partial_k\gamma_{lj} + \partial_j\gamma_{lk} - \partial_l\gamma_{jk}). \quad (3.15)$$

Using equations (3.11) and (3.14) in equation (3.15), we find

$$\begin{aligned} \Gamma^i_{jk} &= \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}(\partial_k\psi^4\bar{\gamma}_{lj} + \partial_j\psi^4\bar{\gamma}_{lk} - \partial_l\psi^4\bar{\gamma}_{jk}) \\ &= \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}[\psi^4(\partial_k\bar{\gamma}_{lj} + \partial_j\bar{\gamma}_{lk} - \partial_l\bar{\gamma}_{jk}) + (\bar{\gamma}_{lj}\partial_k\psi^4 + \bar{\gamma}_{lk}\partial_j\psi^4 - \bar{\gamma}_{jk}\partial_l\psi^4)] \\ &= \bar{\Gamma}^i_{jk} + \frac{2}{\psi}(\delta^i_j\partial_k\psi + \delta^i_k\partial_j\psi - \bar{\gamma}^{il}\bar{\gamma}_{jk}\partial_l\psi) \\ &= \bar{\Gamma}^i_{jk} + \hat{\Gamma}^i_{jk} \end{aligned} \quad (3.16)$$

where we have made the definitions

$$\bar{\Gamma}^i_{jk} := \frac{1}{2}\bar{\gamma}^{il}(\partial_k\bar{\gamma}_{lj} + \partial_j\bar{\gamma}_{lk} - \partial_l\bar{\gamma}_{jk}) \quad (3.17)$$

and

$$\hat{\Gamma}^i_{jk} := \frac{2}{\psi}(\delta^i_j\partial_k\psi + \delta^i_k\partial_j\psi - \bar{\gamma}^{il}\bar{\gamma}_{jk}\partial_l\psi) \quad (3.18)$$

Equation (3.17) is the *conformal Christoffel symbol* associated with the conformal metric $\bar{\gamma}_{ij}$. A brief inspection of these results reveals that both $\bar{\Gamma}^i_{jk}$ and $\hat{\Gamma}^i_{jk}$, are symmetric with respect to j and k or in other words $\bar{\Gamma}^i_{kj} = \bar{\Gamma}^i_{jk}$ and $\hat{\Gamma}^i_{jk} = \hat{\Gamma}^i_{kj}$.

3.3.2 Covariant Derivative Transformation

Now that it is known how the Christoffel symbols transform, it is straightforward to transform the covariant derivative. The standard covariant derivative is defined

$$\begin{aligned}
D_i T^{j_1, \dots, j_m}_{k_1, \dots, k_n} &= \partial_i T^{j_1, \dots, j_m}_{k_1, \dots, k_n} \\
&+ \sum_{h=1}^m \Gamma^{j_h}_{il} T^{j_1, \dots, j_{h-1}, l, j_{h+1}, \dots, j_m}_{k_1, \dots, k_n} \\
&- \sum_{h=1}^n \Gamma^l_{ik_h} T^{j_1, \dots, j_m}_{k_1, \dots, k_{h-1}, l, k_{h+1}, \dots, k_n} \quad (3.19)
\end{aligned}$$

We define the *conformal covariant derivative* in the same fashion but associated with conformal metric, $\bar{\gamma}$,

$$\begin{aligned}
\bar{D}_i T^{j_1, \dots, j_m}_{k_1, \dots, k_n} &= \partial_i T^{j_1, \dots, j_m}_{k_1, \dots, k_n} \\
&+ \sum_{h=1}^m \bar{\Gamma}^{j_h}_{il} T^{j_1, \dots, j_{h-1}, l, j_{h+1}, \dots, j_m}_{k_1, \dots, k_n} \\
&- \sum_{h=1}^n \bar{\Gamma}^l_{ik_h} T^{j_1, \dots, j_m}_{k_1, \dots, k_{h-1}, l, k_{h+1}, \dots, k_n} \cdot \quad (3.20)
\end{aligned}$$

The relationship between the covariant derivative and conformal covariant derivative is therefore given by

$$\begin{aligned}
D_i T^{j_1, \dots, j_m}_{k_1, \dots, k_n} &= \bar{D}_i T^{j_1, \dots, j_m}_{k_1, \dots, k_n} \\
&+ \sum_{h=1}^m \hat{\Gamma}^{j_h}_{il} T^{j_1, \dots, j_{h-1}, l, j_{h+1}, \dots, j_m}_{k_1, \dots, k_n} \\
&- \sum_{h=1}^n \hat{\Gamma}^l_{ik_h} T^{j_1, \dots, j_m}_{k_1, \dots, k_{h-1}, l, k_{h+1}, \dots, k_n} \cdot \quad (3.21)
\end{aligned}$$

where $\hat{\Gamma}^i_{jk}$ is defined by equation (3.18).

3.3.3 Ricci Conformal Transformations

Ricci Tensor Transformation

The Ricci tensor is defined as

$$R_{ij} := R^k_{ikj} \quad (3.22)$$

where R^k_{ikj} is the Riemann tensor on the spatial hypersurface, Σ_t given by equation (2.85),

$$R^l_{ikj}v^i = D_k D_j v^l - D_j D_k v^l. \quad (3.23)$$

We now use Roman rather than Latin indices, as the 0th components are zero due to the fact that all quantities live on the spatial hypersurface. Thus the Ricci tensor, obtained by contracting l and k , is given by

$$R_{ij}v^i = D_i D_j v^i - D_j D_i v^i \quad (3.24)$$

We wish to express a relationship between this and the conformal Ricci Tensor $\bar{R}$, so we use equation (3.21) to transform the derivatives in equation (3.23)

$$\begin{aligned} R_{ij}v^i &= D_i D_j v^i - D_j D_i v^i \\ &= \bar{D}_i(D_j v^i) + \hat{\Gamma}^i_{ik}(D_j v^k) - \bar{D}_j(D_i v^i) - \hat{\Gamma}^i_{jk}(D_i v^k) \\ &= \bar{D}_i(\bar{D}_j v^i + \hat{\Gamma}^i_{jk}v^k) + \hat{\Gamma}^i_{ik}(\bar{D}_j v^k + \hat{\Gamma}^k_{jl}v^l) - \bar{D}_j(\bar{D}_i v^i + \hat{\Gamma}^i_{im}v^m) \\ &\quad - \hat{\Gamma}^i_{jk}(\bar{D}_i v^k + \hat{\Gamma}^k_{im}v^m) \\ &= \bar{D}_i \bar{D}_j v^i + \bar{D}_i \hat{\Gamma}^i_{jk}v^k + \hat{\Gamma}^i_{jk} \bar{D}_i v^k + \hat{\Gamma}^i_{ik} \bar{D}_j v^k + \hat{\Gamma}^i_{ik} \hat{\Gamma}^k_{jl}v^l \\ &\quad - \bar{D}_j \bar{D}_i v^i - \bar{D}_j \hat{\Gamma}^i_{im}v^m - \hat{\Gamma}^i_{im} \bar{D}_j v^m - \hat{\Gamma}^i_{ik} \bar{D}_i v^k - \hat{\Gamma}^i_{jk} \hat{\Gamma}^k_{il}v^l \\ &= \bar{D}_i \bar{D}_j v^i - \bar{D}_j \bar{D}_i v^i + \bar{D}_i \hat{\Gamma}^i_{jk}v^k - \bar{D}_j \hat{\Gamma}^i_{im}v^m + \hat{\Gamma}^i_{ik} \hat{\Gamma}^k_{kl}v^l - \hat{\Gamma}^i_{jk} \hat{\Gamma}^k_{il}v^l \end{aligned} \quad (3.25)$$

Relabeling dummy indices and defining the *conformal Ricci tensor* as

$$\bar{R}_{ij}v^i := \bar{D}_i\bar{D}_jv^i - \bar{D}_j\bar{D}_iv^i, \quad (3.26)$$

we can rewrite equation (3.24) as

$$R_{ij}v^i = \bar{R}_{ij}v^i + \bar{D}_k\hat{\Gamma}_{ji}^kv^i - \bar{D}_j\hat{\Gamma}_{ki}^kv^i + \hat{\Gamma}_{lk}^l\hat{\Gamma}_{ji}^kv^i - \hat{\Gamma}_{jk}^l\hat{\Gamma}_{li}^kv^i. \quad (3.27)$$

Using the symmetry of $\hat{\Gamma}$ and dropping the general vector, v^i , we arrive at

$$R_{ij} = \bar{R}_{ij} + \bar{D}_k\hat{\Gamma}_{ij}^k - \bar{D}_j\hat{\Gamma}_{ki}^k + \hat{\Gamma}_{lk}^l\hat{\Gamma}_{ij}^k - \hat{\Gamma}_{jk}^l\hat{\Gamma}_{li}^k. \quad (3.28)$$

Ricci Scalar Transformation

The transformation of the Ricci scalar can now be shown using equation (3.28),

$$\begin{aligned} R &= R^i_i = \gamma^{ij}R_{ij} \\ &= \psi^{-4}\bar{\gamma}^{ij}(\bar{R}_{ij} + \bar{D}_k\hat{\Gamma}_{ij}^k - \bar{D}_j\hat{\Gamma}_{ki}^k + \hat{\Gamma}_{lk}^l\hat{\Gamma}_{ij}^k - \hat{\Gamma}_{jk}^l\hat{\Gamma}_{li}^k). \end{aligned} \quad (3.29)$$

The second term in parenthesis in equation (3.29) can be expanded as,

$$\begin{aligned} \bar{D}_k\hat{\Gamma}_{ij}^k &= \bar{D}_k\left(\frac{2}{\psi}(\delta^k_i\partial_j\psi + \delta^k_j\partial_i\psi - \bar{\gamma}^{kl}\bar{\gamma}_{ij}\partial_l\psi)\right) \\ &= \bar{D}_k(2(\delta^k_i\partial_j\ln\psi + \delta^k_j\partial_i\ln\psi - \bar{\gamma}^{kl}\bar{\gamma}_{ij}\partial_l\ln\psi)) \\ &= 2(\bar{D}_i\partial_j\ln\psi + \bar{D}_j\partial_i\ln\psi - \bar{\gamma}_{ij}\bar{D}^l\partial_l\ln\psi) \\ &= 4\bar{D}_i\bar{D}_j\ln\psi - 2\bar{\gamma}_{ij}\bar{D}^l\bar{D}_l\ln\psi \end{aligned} \quad (3.30)$$

and the third term can be rewritten as

$$\begin{aligned}
\bar{D}_j \hat{\Gamma}_{ki}^k &= \bar{D}_j \left(\frac{6}{\psi} \partial_i \psi \right) \\
&= \bar{D}_j (6 \partial_i \ln \psi) \\
&= 6 \bar{D}_i \bar{D}_j \ln \psi
\end{aligned} \tag{3.31}$$

where we have used the relation $\Gamma_{ki}^k = \frac{6}{\psi} \partial_i \psi$ established in equation (3.50). The fourth term becomes

$$\begin{aligned}
\hat{\Gamma}_{lk}^l \hat{\Gamma}_{ij}^k &= \frac{6}{\psi} \partial_k \psi \left(\frac{2}{\psi} (\delta_i^k \partial_j \psi + \delta_j^k \partial_i \psi - \bar{\gamma}^{kl} \bar{\gamma}_{ij} \partial_l \psi) \right) \\
&= \frac{12}{\psi^2} (\partial_i \psi \partial_j \psi + \partial_j \psi \partial_i \psi - \bar{\gamma}_{ij} \partial^l \psi \partial_l \psi) \\
&= 24 \bar{D}_i \ln \psi \bar{D}_j \ln \psi - 12 \bar{\gamma}_{ij} \bar{D}^l \ln \psi \bar{D}_l \ln \psi.
\end{aligned} \tag{3.32}$$

Lastly, the fifth term becomes

$$\begin{aligned}
\hat{\Gamma}_{jk}^l \hat{\Gamma}_{li}^k &= \left(\frac{2}{\psi} (\delta_j^l \partial_k \psi + \delta_k^l \partial_j \psi - \bar{\gamma}^{lm} \bar{\gamma}_{jk} \partial_m \psi) \right) \left(\frac{2}{\psi} (\delta_i^k \partial_l \psi + \delta_l^k \partial_i \psi - \bar{\gamma}^{km} \bar{\gamma}_{li} \partial_m \psi) \right) \\
&= \frac{4}{\psi^2} \left(\delta_j^l \delta_i^k \partial_k \psi \partial_l \psi + \delta_j^l \delta_i^k \partial_k \psi \partial_l \psi - \delta_j^l \bar{\gamma}^{km} \bar{\gamma}_{li} \partial_m \psi \partial_k \psi \right. \\
&\quad + \delta_k^l \delta_i^k \partial_k \psi \partial_l \psi + \delta_k^l \delta_i^k \partial_k \psi \partial_l \psi - \delta_k^l \bar{\gamma}^{km} \bar{\gamma}_{li} \partial_m \psi \partial_j \psi \\
&\quad \left. - \delta_i^k \bar{\gamma}^{lm} \bar{\gamma}_{jk} \partial_l \psi \partial_m \psi - \delta_i^k \bar{\gamma}^{lm} \bar{\gamma}_{jk} \partial_l \psi \partial_m \psi + \bar{\gamma}^{lm} \bar{\gamma}_{jk} \bar{\gamma}^{km} \bar{\gamma}_{li} \partial_m \psi \partial_m \psi \right) \\
&= \frac{4}{\psi^2} \left(\bar{D}_i \psi \bar{D}_j \psi + \bar{D}_i \psi \bar{D}_j \psi - \bar{\gamma}_{ij} \bar{D}^m \psi \bar{D}_m \psi \right. \\
&\quad + 3 \bar{D}_i \psi \bar{D}_j \psi + \bar{D}_i \psi \bar{D}_j \psi - \bar{D}_i \psi \bar{D}_j \psi \\
&\quad \left. - \bar{D}_i \psi \bar{D}_j \psi - \bar{\gamma}_{ij} \bar{D}^m \psi \bar{D}_m \psi + \bar{D}_i \psi \bar{D}_j \psi \right) \\
&= \frac{4}{\psi^2} \left(5 \bar{D}_i \psi \bar{D}_j \psi - 2 \bar{\gamma}_{ij} \bar{D}^m \psi \bar{D}_m \psi \right) \\
&= \left(20 \bar{D}_i \ln \psi \bar{D}_j \ln \psi - 8 \bar{\gamma}_{ij} \bar{D}^m \ln \psi \bar{D}_m \ln \psi \right).
\end{aligned} \tag{3.33}$$

If we now use equations (3.30)-(3.33) in equation (3.29), we can write

$$\begin{aligned}
R &= \psi^{-4} \bar{\gamma}^{ij} \bar{R}_{ij} + \psi^{-4} \bar{\gamma}^{ij} \left[\left(4 \bar{D}_i \bar{D}_j \ln \psi - 2 \bar{\gamma}_{ij} \bar{D}^l \bar{D}_l \ln \psi \right) - 6 \bar{D}_i \bar{D}_j \ln \psi \right. \\
&\quad \left. + \left(12 \bar{D}_i \ln \psi \bar{D}_j \ln \psi - 8 \bar{\gamma}_{ij} \bar{D}^l \ln \psi \bar{D}_l \ln \psi \right) \right. \\
&\quad \left. - \left(20 \bar{D}_i \ln \psi \bar{D}_j \ln \psi - 8 \bar{\gamma}_{ij} \bar{D}^l \ln \psi \bar{D}_l \ln \psi \right) \right] \\
&= \psi^{-4} \bar{R} + \psi^{-4} \left[\left(4 \bar{D}_i \bar{D}^i \ln \psi - 6 \bar{D}_i \bar{D}^i \ln \psi \right) - 6 \bar{D}_i \bar{D}^i \ln \psi \right. \\
&\quad \left. + \left(12 \bar{D}_i \ln \psi \bar{D}^i \ln \psi - 24 \bar{D}_i \ln \psi \bar{D}^i \ln \psi \right) \right. \\
&\quad \left. - \left(20 \bar{D}_i \ln \psi \bar{D}^i \ln \psi - 24 \bar{D}_i \ln \psi \bar{D}^i \ln \psi \right) \right] \\
&= \psi^{-4} \bar{R} - 8 \psi^{-4} \bar{D}_i \bar{D}^i \ln \psi - 8 \bar{D}_i \ln \psi \bar{D}^i \ln \psi \\
&= \psi^{-4} \bar{R} - 8 \psi^{-5} \bar{D}_i \bar{D}^i \psi.
\end{aligned} \tag{3.34}$$

Solving for $\bar{R}$, the relationship between the Ricci scalar and the conformal Ricci scalar can be compactly written as

$$\bar{R} = \psi^4 R + 8 \psi^{-1} \bar{D}_i \bar{D}^i \psi. \tag{3.35}$$

3.4 The Conformal Factor Equation

Next, we examine the Hamiltonian Constraint, equation (2.156),

$$R + K^2 - K_{ij} K^{ij} = 16\pi E. \tag{3.36}$$

The maximal slicing gauge choice, $K = \partial_t K = 0$, simplifies the Hamiltonian constraint to

$$R - K_{ij} K^{ij} = 16\pi E. \tag{3.37}$$

Using the relation between the Ricci Scalar and the Conformal Ricci Scalar established in equation (3.35), we can rewrite equation (3.37) as

$$\begin{aligned} 16\pi E &= R - K_{ij}K^{ij} \\ &= \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}_i\bar{D}^i\psi - K_{ij}K^{ij}. \end{aligned} \quad (3.38)$$

As $\bar{\gamma}_{ij}$ is a flat metric, $\bar{R} = 0$. Therefore

$$16\pi E = -8\psi^{-5}\bar{D}_i\bar{D}^i\psi - K_{ij}K^{ij}. \quad (3.39)$$

Rearranging equation (3.39), the Hamiltonian constraint is recast as an expression for the conformal factor,

$$\bar{\nabla}^2\psi = \bar{D}_i\bar{D}^i\psi = -2\pi\psi^5E - \frac{\psi^5}{8}K_{ij}K^{ij}. \quad (3.40)$$

3.5 The Lapse Function Equation

To derive an expression for the lapse function, we begin with the expression for the Lie derivative of the extrinsic curvature tensor along the time evolution vector given by equation (2.160),

$$\mathcal{L}_{\mathbf{m}}K_{ij} = \alpha R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k_j - D_i D_j \alpha - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)) \quad (3.41)$$

where we have adopted the 3-dimensional notation, dropping the references to the trivial terms related to time as all terms live on the hypersurface. The first action is to multiply both sides of this equation by γ^{ij} . The left-hand side becomes

$$\gamma^{ij}\mathcal{L}_{\mathbf{m}}K_{ij} = \mathcal{L}_{\mathbf{m}}K - 2\alpha K_{ij}K^{ij} = -2\alpha K_{ij}K^{ij} \quad (3.42)$$

by use of the chain rule, equation (2.144), and application of the maximal slicing condition. The transformation of the right-hand side of equation (3.41) begins by distributing γ^{ij} and applying the maximal slicing condition to remove terms,

$$\begin{aligned}
& \gamma^{ij} \left(\alpha R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k_j - D_i D_j \alpha - 8\pi\alpha (S_{ij} - \frac{1}{2}\gamma_{ij}(S - E)) \right) \\
&= \alpha R + \alpha K^2 - 2\alpha K_{ik} K^{ki} - D_i D^i \alpha - 8\pi\alpha \left(S - \frac{3}{2}(S - E) \right) \\
&= \alpha R - 2\alpha K_{ik} K^{ki} - D_i D^i \alpha - 4\pi\alpha (3E - S).
\end{aligned} \tag{3.43}$$

Next, equation (3.35) is used to rewrite the Ricci scalar in terms of the conformal Ricci scalar, and equation (3.21) to express the covariant derivatives in terms of their conformal counterparts,

$$\begin{aligned}
& \alpha R - 2\alpha K_{ik} K^{ki} - D_i D^i \alpha - 4\pi\alpha (3E - S) \\
&= \alpha\psi^{-4} \bar{R} - 8\alpha\psi^{-5} \bar{D}_i \bar{D}^i \psi - \psi^{-4} \bar{D}_i \bar{D}^i \alpha - 2\psi^5 \bar{D}_i \psi \bar{D}^i \alpha - 2\alpha K_{ij} K^{ij} - 4\pi\alpha (3E - S) \\
&= -8\alpha\psi^{-5} \bar{D}_i \bar{D}^i \psi - \psi^{-4} \bar{D}_i \bar{D}^i \alpha - 2\psi^{-5} \bar{D}_i \psi \bar{D}^i \alpha - 2\alpha K_{ij} K^{ij} - 4\pi\alpha (3E - S),
\end{aligned} \tag{3.44}$$

where we eliminated the conformal Ricci scalar as it is zero for a flat spatial metric. Using equations (3.42) and (3.44) in equation (3.41) and solving for $\bar{D}_i \bar{D}^i \alpha$, we can write

$$\bar{D}_i \bar{D}^i \alpha = -8\alpha\psi^{-1} \bar{D}_i \bar{D}^i \psi - 2\psi^{-1} \bar{D}_i \psi \bar{D}^i \alpha - 4\psi^4 \pi\alpha (3E - S). \tag{3.45}$$

While equation (3.45) constitutes a complete constraint equation for the lapse function, we wish to rewrite it in a different form. Consider the Laplace operator acting on the product of the conformal factor and the lapse function,

$$\begin{aligned}
\bar{D}_i \bar{D}^i (\alpha\psi) &= \bar{D}_i (\alpha \bar{D}^i \psi + \psi \bar{D}^i \alpha) \\
&= \alpha \bar{D}_i \bar{D}^i \psi + 2\bar{D}_i \psi \bar{D}^i \alpha + \psi \bar{D}_i \bar{D}^i \alpha.
\end{aligned} \tag{3.46}$$

Using this, equation (3.45) can be written as

$$\begin{aligned}
\bar{D}_i \bar{D}^i \alpha \psi &= -7\alpha \bar{D}_i \bar{D}^i \psi - 4\pi\alpha\psi^5(3E - S) \\
&= -7\alpha \left(-2\pi\psi^5 E - \frac{\psi^5}{8} K_{ij} K^{ij} \right) - 4\pi\alpha\psi^5(3E - S) \\
&= 14\pi\alpha\psi^5 E + \frac{7}{8}\alpha\psi^5 K_{ij} K^{ij} - 4\pi\alpha\psi^5(3E - S) \\
&= 2\pi\alpha\psi^5(E + 2S) + \frac{7}{8}\alpha\psi^5 K_{ij} K^{ij}.
\end{aligned} \tag{3.47}$$

Equation (3.47) is used as the governing equation for the lapse function. This form is preferable as there is only one differential operator. Treating $\alpha\psi$ as a single variable, equation (3.47) has a similar form to the equation that governs the conformal factor, equation (3.40) and therefore can be solved similarly. Once the entire CFC system of equations has been solved, the lapse function is easily recovered by division.

3.5.1 Extrinsic Curvature Transformation

Our next goal is to derive an equation for the shift vector using the momentum constraint, equation (2.163). We, therefore, need to know how the covariant derivative of the extrinsic curvature tensor will transform under the CFC conditions. To begin, equation (3.21) is used to write the covariant derivative, D_j , in terms of the conformal covariant derivative $\bar{D}_j$,

$$D_j K^{ij} = \bar{D}_j K^{ij} + \hat{\Gamma}^l_{ji} K^{ij} + \hat{\Gamma}^l_{lj} K^{ij}. \tag{3.48}$$

Equation (3.18) is used to expand the second to last term in equation (3.48). It can be written as

$$\begin{aligned}
\hat{\Gamma}_{ji}^l K^{ij} &= \frac{2}{\psi} (\delta_j^l \partial_i \psi + \delta_i^l \partial_j \psi - \bar{\gamma}^{lm} \bar{\gamma}_{ji} \partial_m \psi) K^{ij} \\
&= \frac{2}{\psi} (\partial_i \psi K^{il} + \partial_j \psi K^{lj} - K \bar{\gamma}^{lm} \partial_m \psi) \\
&= \frac{2}{\psi} (2\partial_j \psi K^{lj} - K \bar{\gamma}^{lm} \partial_m \psi)
\end{aligned} \tag{3.49}$$

where in the last step we have used the symmetry on K^{ij} to combine the positive terms. As

$$\begin{aligned}
\hat{\Gamma}_{lj}^l &= \frac{2}{\psi} (\delta_l^l \partial_j \psi + \delta_j^l \partial_l \psi - \bar{\gamma}^{lm} \bar{\gamma}_{lj} \partial_m \psi) \\
&= \frac{2}{\psi} (3\partial_j \psi + \partial_j \psi - \partial_j \psi) \\
&= \frac{6}{\psi} \partial_j \psi,
\end{aligned} \tag{3.50}$$

the last term in equation (3.48) becomes

$$\hat{\Gamma}_{lj}^l K^{ij} = \frac{6}{\psi} \partial_j \psi K^{ij}. \tag{3.51}$$

Using equations (3.49) and (3.51) in equation (3.48) and relabeling indices, we can conclude

$$\begin{aligned}
D_j K^{ij} &= \bar{D}_j K^{ij} + \frac{2}{\psi} (2\partial_j \psi K^{ij} - \bar{\Gamma}^{ij} \partial_j K) + \frac{6}{\psi} \partial_j \psi K^{ij} \\
&= \bar{D}_j K^{ij} + \frac{10}{\psi} \partial_j \psi K^{ij} - \frac{2}{\psi} K \bar{\gamma}^{ij} \partial_j \psi.
\end{aligned} \tag{3.52}$$

Under the maximal slicing conditions, equation (3.52) becomes

$$\begin{aligned} D_j K^{ij} &= \bar{D}_j K^{ij} + \frac{10}{\psi} K^{ij} \bar{D}_j \psi \\ &= \psi^{-10} \bar{D}_j (\psi^{10} K^{ij}). \end{aligned} \quad (3.53)$$

As the maximal slicing condition, restricts the trace of the extrinsic curvature to zero, it is convenient to define the traceless portion as

$$A_{ij} := K_{ij} - \frac{1}{3} \gamma_{ij} K. \quad (3.54)$$

Given this decomposition, we next consider the time evolution of the spatial metric, equation (2.141),

$$\mathcal{L}_{\mathbf{m}} \gamma_{ij} = -2\alpha K_{ij}. \quad (3.55)$$

Expanding the metric in its conformal form, equation (3.11), and using the traceless decomposition of the extrinsic curvature, equation (3.54) becomes

$$\mathcal{L}_{\mathbf{m}} (\psi^4 \bar{\gamma}_{ij}) = -2\alpha \left(A_{ij} + \frac{1}{3} \psi^4 \bar{\gamma}_{ij} K \right). \quad (3.56)$$

Isolating the Lie derivative of the conformal metric, we arrive at

$$\begin{aligned} \mathcal{L}_{\mathbf{m}} \bar{\gamma}_{ij} &= -2\alpha \psi^{-4} A_{ij} - \frac{2}{3} \alpha K \bar{\gamma}_{ij} - \psi^{-4} \bar{\gamma}_{ij} \mathcal{L}_{\mathbf{m}} \psi^4 \\ &= -2\alpha \psi^{-4} A_{ij} - \frac{\bar{\gamma}_{ij}}{3} (2\alpha K + 3\psi^{-4} \mathcal{L}_{\mathbf{m}} \psi^4). \end{aligned} \quad (3.57)$$

Taking the trace of equation (3.57), we find

$$\begin{aligned} \bar{\gamma}^{ij} \mathcal{L}_{\mathbf{m}} \bar{\gamma}_{ij} &= -2\alpha \psi^{-4} A_i^i - (2\alpha K + 3\psi^{-4} \mathcal{L}_{\mathbf{m}} \psi^4) \\ &= -2\alpha K - 3\psi^{-4} \mathcal{L}_{\mathbf{m}} \psi^4. \end{aligned} \quad (3.58)$$

The term on the left-hand side of equation (3.58), can be simplified,

$$\bar{\gamma}^{ij} \mathcal{L}_{\mathbf{m}} \bar{\gamma}_{ij} = \bar{\gamma}^{ij} (\mathcal{L}_{\mathbf{t}} \bar{\gamma}_{ij} - \mathcal{L}_{\beta} \bar{\gamma}_{ij}) = -\bar{\gamma}^{ij} \mathcal{L}_{\beta} \bar{\gamma}_{ij}. \quad (3.59)$$

as the conformal metric has been chosen to be the time-independent flat metric $\mathbf{f}$.

Using the definition of the Lie derivative in equation (2.37), this becomes

$$\begin{aligned} \bar{\gamma}^{ij} \mathcal{L}_{\mathbf{m}} \bar{\gamma}_{ij} &= -\bar{\gamma}^{ij} (\beta^k \bar{D}_k \bar{\gamma}_{ij} + \bar{\gamma}_{ik} \bar{D}_j \beta^k + \bar{\gamma}_{kj} \bar{D}_i \beta^k) \\ &= -\bar{\gamma}^{ij} \bar{\gamma}_{ik} \bar{D}_j \beta^k - \bar{\gamma}^{ij} \bar{\gamma}_{kj} \bar{D}_i \beta^k \\ &= -2 \bar{D}_j \beta^j. \end{aligned} \quad (3.60)$$

Equation (3.58) can be rewritten using equation (3.60) as

$$2\alpha K + 3\psi^{-4} \mathcal{L}_{\mathbf{m}} \psi^4 = 2 \bar{D}_j \beta^j, \quad (3.61)$$

which allows us to write equation (3.57) as

$$\mathcal{L}_{\mathbf{m}} \bar{\gamma}_{ij} = -2\alpha \psi^{-4} A_{ij} - \frac{2}{3} \bar{\gamma}_{ij} \bar{D}_k \beta^k. \quad (3.62)$$

Based on equation (3.62), we define the conformal traceless extrinsic curvature tensor as

$$\tilde{A}_{ij} := \psi^{-4} A_{ij}. \quad (3.63)$$

Notice that $\tilde{A}_{ij}$ retains the traceless property, $\bar{\gamma}^{ij} \tilde{A}_{ij} = 0$. Further, raising the indices using the conformal metric, $\tilde{A}^{ij} = \bar{\gamma}^{im} \bar{\gamma}^{jn} \tilde{A}_{mn}$ reveals

$$\tilde{A}^{ij} = \psi^4 A^{ij}. \quad (3.64)$$

Using the conformal traceless extrinsic curvature tensor, equation (3.62) becomes

$$\mathcal{L}_{\mathbf{m}}\bar{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k. \quad (3.65)$$

Lastly, we expand the Lie derivative on the left-hand side like equation (2.139) and then solve the equation for $\tilde{A}_{ij}$ to arrive at

$$\tilde{A}_{ij} = \frac{1}{2\alpha} \left(\gamma_{ik}\bar{D}_j\beta^k + \gamma_{kj}\bar{D}_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k \right). \quad (3.66)$$

Using equations (3.54), (3.63), and (3.66), we can write an expression relating the extrinsic curvature tensor to the shift vector,

$$K_{ij} = \frac{\psi^4}{2\alpha} \left(\gamma_{ik}\bar{D}_j\beta^k + \gamma_{kj}\bar{D}_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k \right) \quad (3.67)$$

3.6 The Shift Vector Equation

We now have enough information to derive an expression for the shift vector, β^i . We begin by applying the maximal slicing condition to the momentum constraint, equation (2.163) to write $D_i K^{ij} = 8\pi S^j$. Using equations (3.53), we see

$$\begin{aligned} 8\pi S^j &= D_i K^{ij} \\ &= \psi^{-10}\bar{D}_i(\psi^{10}K^{ij}). \end{aligned} \quad (3.68)$$

We prefer to express this using the double covariant extrinsic curvature tensor which is related to the double contravariant form by

$$\begin{aligned} K^{ij} &= \gamma^{im}\gamma^{nj}K_{mn} \\ &= \psi^{-8}\bar{\gamma}^{im}\bar{\gamma}^{nj}K_{mn}. \end{aligned} \quad (3.69)$$

Using this transformation in equation (3.70), the expression becomes

$$\begin{aligned} 8\pi S^j &= \psi^{-10} \bar{D}_i (\psi^{10} K^{ij}) \\ &= \psi^{-10} \bar{\gamma}^{im} \bar{\gamma}^{nj} K_{mn} \bar{D}_i (\psi^2) + \psi^{-8} \bar{\gamma}^{im} \bar{\gamma}^{nj} \bar{D}_i (K_{mn}). \end{aligned} \quad (3.70)$$

Applying maximal slicing to the conformal traceless decomposition of the extrinsic curvature tensor, we can write the relation

$$K_{mn} = \psi^4 \tilde{A}_{mn} + \frac{1}{3} \gamma_{mn} K = \psi^4 \tilde{A}_{mn}. \quad (3.71)$$

Equation (3.70) can therefore be written as

$$\begin{aligned} 8\pi S^j &= \psi^{-6} \bar{\gamma}^{im} \bar{\gamma}^{nj} \tilde{A}_{mn} \bar{D}_i (\psi^2) + \psi^{-8} \bar{\gamma}^{im} \bar{\gamma}^{nj} \bar{D}_i (\psi^4 \tilde{A}_{mn}) \\ &= \psi^{-6} \bar{\gamma}^{im} \bar{\gamma}^{nj} \tilde{A}_{mn} \bar{D}_i (\psi^2) + \psi^{-8} \bar{\gamma}^{im} \bar{\gamma}^{nj} \tilde{A}_{mn} \bar{D}_i (\psi^4) + \psi^{-4} \bar{\gamma}^{im} \bar{\gamma}^{nj} \bar{D}_i (\tilde{A}_{mn}) \\ &= 6\psi^{-5} \bar{\gamma}^{im} \bar{\gamma}^{nj} \tilde{A}_{mn} \bar{D}_i (\psi) + \psi^{-4} \bar{\gamma}^{im} \bar{\gamma}^{nj} \bar{D}_i (\tilde{A}_{mn}). \end{aligned} \quad (3.72)$$

The second term in equation (3.72) is expanded using equation (3.66),

$$\begin{aligned}
\psi^{-4}\bar{\gamma}^{im}\bar{\gamma}^{nj}\bar{D}_i\left(\tilde{A}_{mn}\right) &= \psi^{-4}\bar{\gamma}^{im}\bar{\gamma}^{nj}\bar{D}_i\left(\frac{1}{2\alpha}\left(\gamma_{mk}\bar{D}_n\beta^k+\gamma_{nj}\bar{D}_m\beta^k-\frac{2}{3}\bar{\gamma}_{mn}\bar{D}_k\beta^k\right)\right) \\
&= \psi^{-4}\bar{D}_i\left(\frac{1}{2\alpha}\left(\bar{D}^j\beta^i+\bar{D}^i\beta^j-\frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k\beta^k\right)\right) \\
&= \psi^{-4}\left(\bar{D}^h\beta^i+\bar{D}^i\beta^j-\frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k\beta^k\right)\bar{D}_i\left(\frac{1}{2\alpha}\right) \\
&\quad + \frac{\psi^{-4}}{2\alpha}\bar{D}_i\left(\bar{D}^j\beta^i+\bar{D}^i\beta^j-\frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k\beta^k\right) \\
&= -\frac{\psi^{-4}}{2\alpha^2}\left(\bar{D}^j\beta^i+\bar{D}^j\beta^i-\frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k\beta^k\right)\bar{D}_i(\alpha) \\
&\quad + \frac{\psi^{-4}}{2\alpha}\left(\bar{D}_i\bar{D}^j\beta^i+\bar{D}_i\bar{D}^i\beta^j-\frac{2}{3}\bar{\gamma}^{ij}\bar{D}_i\bar{D}_k\beta^k\right) \\
&= -\psi^{-4}\tilde{A}^{ij}\frac{D_i\alpha}{\alpha}+\frac{\psi^{-4}}{2\alpha}\left(\bar{D}_i\bar{D}^j\beta^i+\bar{D}_i\bar{D}^i\beta^j-\frac{2}{3}\bar{\gamma}^{ij}\bar{D}_i\bar{D}_k\beta^k\right) \\
&= -\psi^{-4}\tilde{A}^{ij}\frac{D_i\alpha}{\alpha}+\frac{\psi^{-4}}{2\alpha}\left(\bar{D}_i\bar{D}^i\beta^j+\frac{1}{3}\bar{D}^j\bar{D}_k\beta^k\right). \tag{3.73}
\end{aligned}$$

This allows equation (3.72) to be written as

$$\begin{aligned}
8\pi S^j &= 6\psi^{-5}\tilde{A}^{ij}\bar{D}_i(\psi)-\psi^{-4}\tilde{A}^{ij}\frac{D_i\alpha}{\alpha}+\frac{\psi^{-4}}{2\alpha}\left(\bar{D}_i\bar{D}^i\beta^j+\frac{1}{3}\bar{D}^j\bar{D}_k\beta^k\right) \\
&= \psi^{-4}\tilde{A}^{ij}\left(6\frac{D_i\psi}{\psi}-\frac{D_i\alpha}{\alpha}\right)+\frac{\psi^{-4}}{2\alpha}\left(\bar{D}_i\bar{D}^i\beta^j+\frac{1}{3}\bar{D}^j\bar{D}_k\beta^k\right) \\
&= \psi^{-4}\tilde{A}^{ij}\left(\frac{D_i(\alpha\psi^{-6})}{\alpha\psi^{-6}}\right)+\frac{\psi^{-4}}{2\alpha}\left(\bar{D}_i\bar{D}^i\beta^j+\frac{1}{3}\bar{D}^j\bar{D}_k\beta^k\right). \tag{3.74}
\end{aligned}$$

Equation (3.74) is rearranged to isolate the terms involving the shift vector,

$$\begin{aligned}
\bar{D}_i\bar{D}^i\beta^j+\frac{1}{3}\bar{D}^j\bar{D}_i\beta^i &= 16\pi\alpha\psi^{-4}S^i-2\psi^6\tilde{A}^{ij}D_i(\alpha\psi^{-6}) \\
&= 16\pi\alpha\psi^{-4}S^i-2\psi^{10}K^{ij}D_i(\alpha\psi^{-6}). \tag{3.75}
\end{aligned}$$

3.7 Summary

The conformally flat condition places two restrictions on the 3+1 formalism of General Relativity. First, it requires the spatial metric to be conformally flat, i.e. $\gamma = \psi^4 \bar{\gamma}$. Secondly, it imposes the maximal slicing condition, $K = \partial_t K = 0$. After applying those conditions to the 3+1 system, we can now write the CFC metric,

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \beta_j \\ \beta_i & \psi^4 \bar{\gamma}_{ij} \end{pmatrix} \quad (3.76)$$

where the metric terms are now governed by a set of five coupled elliptic equations,

$$\bar{\nabla}^2 \psi = -2\pi\psi^5 E - \frac{\psi^5}{8} K_{ij} K^{ij} \quad (3.77a)$$

$$\bar{\nabla}^2 (\alpha\psi) = 2\pi\alpha\psi^5 (E + 2S) + \frac{7}{8} \alpha\psi^5 K_{ij} K^{ij} \quad (3.77b)$$

$$\bar{\nabla}^2 \beta^j + \frac{1}{3} \bar{D}^j \bar{D}_i \beta^i = 16\pi\alpha\psi^{-4} S^i - 2\psi^{10} K^{ij} \bar{D}_i (\alpha\psi^{-6}), \quad (3.77c)$$

where K_{ij} is the extrinsic curvature tensor and is related to the shift vector by

$$K_{ij} = \frac{\psi^2}{2\alpha} (D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} D_k \beta^k). \quad (3.78)$$

The source variables are defined as

$$E := n_\mu n_\nu T^{\mu\nu}, \quad (3.79a)$$

$$S^i := -\gamma^{\mu i} n^\nu T_{\mu\nu}, \quad (3.79b)$$

$$S_{ij} := \gamma^\mu{}_i \gamma^\nu{}_j T_{\mu\nu}, \quad (3.79c)$$

$$S = \gamma^{ij} S_{ij}. \quad (3.79d)$$

Equations (3.77) constitute a fully constrained formulation of the Einstein equations and are an exact representation of general relativity in spherical symmetry. In multiple dimensions, CFC has been shown to be accurate for many astrophysical

applications (Cook et al., 1996; Ott et al., 2007a,b; Sekiguchi and Shibata, 2005; Dimmelmeier et al., 2007), but is not a good choice for a rapidly rotating deformed object such as black hole or neutron star mergers. In these instances, gravitational wave radiation can play an important part but CFC disallows for gravitational waves by requiring a conformally flat spatial metric. The CFC system of equations suffers from a further drawback of nonuniqueness, which will be discussed further in the next chapter.

Chapter 4

The eXtended Conformally Flat Condition, xCFC

4.1 Non-Uniqueness in the CFC Equations

A property of scalar elliptic equations is that they may admit nonunique solutions. This occurs when the differential operator has a nontrivial kernel, such as when the equation takes the form

$$\bar{\nabla}^2 u = hu^p + g, \tag{4.1}$$

where u and p are scalars and h and g are functions independent of u . If the signs of h and p are the same, then equation (4.1) can be shown to possess a locally unique solution by the maximum principle (Protter and Weinberger, 1984).

The conformal factor and lapse function are governed by equations (3.77a) and (3.77b) respectively which take the form of equation (4.1). Both terms on the right-hand side of equation (3.77a) have incompatible signs. At first glance, equation (3.77b) appears to have no sign issues. But once one expands the extrinsic curvature tensors using equation (3.78), the division by α^2 creates a sign conflict in the second term. Therefore both equations present a source of nonuniqueness.

4.2 Source Variable Scaling

To eliminate the sign conflict in the first term of equation (3.77a), a conformal scaling of the source variables is performed,

$$E^* := \psi^6 E \quad (4.2)$$

$$S^* := \psi^6 S \quad (4.3)$$

$$(S^*)^i := \psi^6 S^i. \quad (4.4)$$

Using these new variables, the CFC system becomes

$$\bar{\nabla}^2 \psi = -2\pi\psi^{-1}E^* - \frac{\psi^5}{8}K_{ij}K^{ij} \quad (4.5a)$$

$$\bar{\nabla}^2(\alpha\psi) = 2\pi\alpha\psi^{-1}(E^* + 2S^*) + \frac{7}{8}\alpha\psi^5 K_{ij}K^{ij} \quad (4.5b)$$

$$\bar{\nabla}^2 \beta^j + \frac{1}{3}\bar{D}^j \bar{D}_i \beta^i = 16\pi\alpha\psi^{-4}(S^*)^i - 2\psi^{10}K^{ij}D_i(\alpha\psi^{-6}), \quad (4.5c)$$

The first term in equation (4.5a) now posses matching signs between the function $-2\pi E^*$ and the exponent of ψ^{-1} .

4.3 Approximation of K_{ij}

To remove the remaining source of nonuniqueness from the CFC system, an approximate form for the extrinsic curvature tensor is used. In Section 3.5.1, we defined the conformal traceless form of the extrinsic curvature tensor

$$\tilde{A}_{ij} := \psi^4 \left(K_{ij} - \frac{1}{3}\bar{\gamma}_{ij}K \right), \quad (4.6)$$

where the scaling ψ^4 was chosen based on the equation governing the evolution of the conformal metric, equation (3.62). We now wish to consider a different decomposition of the extrinsic curvature tensor that is suggested by the transformation of $D_i K^{ij}$

given in equation (3.53),

$$D_j K^{ij} = \psi^{-10} \bar{D}_j (\psi^{10} K^{ij}). \quad (4.7)$$

Based on this relation, we make a new definition,

$$\hat{A}^{ij} := \psi^{10} K^{ij} \quad (4.8)$$

and consequently,

$$\hat{A}_{ij} := \gamma_{im} \gamma_{nj} \hat{A}^{mn} = \psi^2 K_{ij}. \quad (4.9)$$

A symmetric, traceless tensor, A^{ij} , can be decomposed into a transverse-traceless component, A^{ij}_{TT} , and a longitudinal component, A^{ij}_L as

$$A^{ij} = A^{ij}_{TT} + A^{ij}_L. \quad (4.10)$$

The transverse-traceless component is divergenceless, i.e. $\bar{D}_i A^{ij}_{TT} = 0$, while the longitudinal component satisfies

$$A^{ij}_L = (LW)^{ij} = D^i W^j + D^j W^i - \frac{2}{3} \bar{\gamma}^{ij} \bar{D}_k W^k \quad (4.11)$$

where W^k is a vector potential, and L is called the conformal killing operator.

In Section 3.5.1, an expression for the Lie derivative of $\bar{\gamma}_{ij}$ along $\mathbf{m}$ was derived. This expression, equation (3.65), can be rearranged and expressed as a decomposition of the conformal extrinsic curvature tensor, $\tilde{A}_{ij}$,

$$\tilde{A}_{ij} = \frac{1}{2\alpha} \left(\mathcal{L}_{\mathbf{m}} \bar{\gamma}_{ij} - \frac{2}{3} \bar{\gamma}_{ij} \bar{D}_k \beta^k \right). \quad (4.12)$$

Splitting $\mathcal{L}_{\mathbf{m}}\bar{\gamma}_{ij}$ into its temporal and spatial components, the above relationship becomes

$$\begin{aligned}\tilde{A}_{ij} &= \frac{1}{2\alpha} \left(\mathcal{L}_{\mathbf{t}}\bar{\gamma}_{ij} - \mathcal{L}_{\beta}\bar{\gamma}_{ij} - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k \right) \\ &= \frac{1}{2\alpha} \left(\mathcal{L}_{\mathbf{t}}\bar{\gamma}_{ij} - \bar{\gamma}_{ik}\bar{D}_j\beta^k - \bar{\gamma}_{kj}\bar{D}_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k \right) \\ &= \frac{1}{2\alpha} (\mathcal{L}_{\mathbf{t}}\bar{\gamma}_{ij} - (L\beta)^{ij}).\end{aligned}\tag{4.13}$$

Therefore the longitudinal part of the conformal extrinsic curvature tensor, $\tilde{A}_{ij}$, is

$$\tilde{A}^{ij}_L = -\frac{1}{2\alpha}(L\beta)^{ij} = -\frac{1}{2\alpha} \left(\bar{\gamma}_{ik}\bar{D}_j\beta^k + \bar{\gamma}_{kj}\bar{D}_i\beta^k + \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k\beta^k \right),\tag{4.14}$$

while the transverse-traceless component is proportional to the Lie derivative of the conformal metric along $\mathbf{t}$, which is zero by choice of the metric.

We now perform a similar decomposition of the new conformal extrinsic curvature tensor, $\hat{A}^{ij}$,

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + (LX)^{ij},\tag{4.15}$$

where X^i is an auxiliary vector field. The xCFC formalism chooses to ignore the transverse-traceless portion of this decomposition and therefore $\hat{A}^{ij}$ is given by

$$\hat{A}^{ij} \approx (LX)^{ij} = D^i X^j + D^j X^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k X^k.\tag{4.16}$$

Using equations (4.8) and (4.16), the relationship between the extrinsic curvature tensor and this conformal extrinsic curvature tensor can be written as

$$K^{ij} = \psi^{-10}\hat{A}^{ij} \approx \psi^{-10}(LX)^{ij}.\tag{4.17}$$

This approximation has been shown to be consistent at the accuracy level of the CFC approximation for slowly rotating neutron stars (Cordero-Carrión et al., 2009).

The auxiliary vector potential X^i is related to the momentum density through the momentum constraint, equation (2.163). By choice of maximal slicing, the $D_i K$ term disappears while the other differential term is transformed using equations (4.7) and (4.17),

$$\begin{aligned} D_i K^i_j &= \gamma_{jk} D_i K^{ik} \\ &= \psi^{-10} \gamma_{jk} \bar{D}_i (\psi^{10} K^{ik}) \\ &= \psi^{-6} \bar{\gamma}_{jk} \bar{D}_i (\hat{A}^{ik}). \end{aligned} \tag{4.18}$$

The right-hand side of equation (2.163) contains the momentum density which is scaled using equation (4.4),

$$\begin{aligned} 8\pi S_j &= 8\pi \gamma_{jk} S^k \\ &= 8\pi \psi^4 \bar{\gamma}_{jk} S^k \\ &= 8\pi \psi^{-2} \bar{\gamma}_{jk} (S^*)^k. \end{aligned} \tag{4.19}$$

Equations (4.18) and (4.19) are now used to rewrite the momentum constraint as

$$8\pi \psi^4 (S^*)^j = \bar{D}_i (\hat{A}^{ik}) = \bar{D}_i (LX)^{ij}. \tag{4.20}$$

Using the definition of the Killing operator, equation (4.11), and distributing the conformal covariant derivative, $\bar{D}_i$, the governing equation for X^i becomes

$$\bar{\Delta}_L X^i = \bar{D}_j \bar{D}^j X^i + \frac{1}{3} \bar{D}^i \bar{D}_j X^j = 8\pi \psi^4 (S^*)^i, \tag{4.21}$$

where the differential operator, $\bar{\Delta}_L$ is known as the *conformal vector Laplacian*.

To create an expression for the shift vector, an expression relating the two different conformal extrinsic curvature tensors is established using equations (3.63) and (4.8),

$$\hat{A}^{ij} = \psi^{10} K^{ij} = \psi^6 \tilde{A}^{ij}. \quad (4.22)$$

Expanding $\tilde{A}^{ij}$ in terms of the shift vector and taking the divergence, an elliptic expression relating the shift vector and the new conformal extrinsic curvature tensor is formed,

$$\bar{\Delta}_L \beta^i = \bar{D}_j \bar{D}^j \beta^i + \frac{1}{3} \bar{D}^i \bar{D}_j \beta^j = \bar{D}_j \left(2\alpha \psi^{-6} \hat{A}^{ij} \right). \quad (4.23)$$

Returning to the conformal factor and lapse function equations, the extrinsic curvature tensor terms in equations (4.5a) and (4.5b) are rewritten in terms of $\hat{A}^{ij}$,

$$K_{ij} K^{ij} = \psi^{-12} \hat{A}_{ij} \hat{A}^{ij}. \quad (4.24)$$

The second term in equation (4.5a) becomes

$$-\frac{\psi^5}{8} K_{ij} K^{ij} = -\frac{\psi^{-7}}{8} \hat{A}_{ij} \hat{A}^{ij}, \quad (4.25)$$

which allows the equation governing the xCFC conformal factor to be written as

$$\bar{D}_j \bar{D}^j \psi = -2\pi \psi^{-1} E^* - \frac{\psi^{-7}}{8} \hat{A}_{ij} \hat{A}^{ij}. \quad (4.26)$$

Likewise, the second term in equation (4.5b) becomes

$$\frac{7}{8} \alpha \psi^5 K_{ij} K^{ij} = \frac{7}{8} \alpha \psi^{-7} \hat{A}_{ij} \hat{A}^{ij}, \quad (4.27)$$

and the equation for the xCFC lapse function is given by

$$\bar{D}_j \bar{D}^j (\alpha \psi) = 2\pi \alpha \psi^{-1} (E^* + 2S^*) + \frac{7}{8} \alpha \psi^{-7} \hat{A}_{ij} \hat{A}^{ij}. \quad (4.28)$$

Now both terms on the right-hand side of equations (4.26) and (4.26) have signs that agree, thus removing the last potential for nonunique solutions.

4.4 Summary

Collecting equations, the xCFC system can now be expressed as

$$\bar{D}_j \bar{D}^j X^i + \frac{1}{3} \bar{D}^i \bar{D}_j X^j = 8\pi (S^*)^i \quad (4.29a)$$

$$\bar{D}_j \bar{D}^j \psi = -2\pi \psi^{-1} E^* - \frac{\psi^{-7}}{8} \hat{A}_{ij} \hat{A}^{ij} \quad (4.29b)$$

$$\bar{D}_j \bar{D}^j (\alpha \psi) = 2\pi \alpha \psi^{-1} (E^* + 2S^*) + \frac{7}{8} \alpha \psi^{-7} \hat{A}_{ij} \hat{A}^{ij} \quad (4.29c)$$

$$\bar{D}_j \bar{D}^j \beta^i + \frac{1}{3} \bar{D}^i \bar{D}_j \beta^j = \bar{D}_j \left(2\alpha \psi^{-6} \hat{A}^{ij} \right) \quad (4.29d)$$

where the scaled sources are given by

$$E^* := \psi^6 E \quad (4.30a)$$

$$S^* := \psi^6 S \quad (4.30b)$$

$$(S^*)^i := \psi^6 S^i. \quad (4.30c)$$

In addition to removing the potential for nonunique solution in the conformal factor and lapse function, the xCFC system has another benefit over the CFC system. While the xCFC equations are still coupled, they can be solved sequentially rather than needing to be solved simultaneously. The X^i equation is independent of the other variables and can be solved first. Once X^i is known, $\hat{A}^{ij}$ can be calculated and equation (4.29b) can then be solved for the conformal factor. With ψ and $\hat{A}^{ij}$ in hand, equation (4.29c) is solvable for the lapse function. With all other variables

calculated, the shift equation is solved last. This is a useful property that will be taken advantage of in the following chapter when the discretization of the system is discussed.

Chapter 5

Numerical Methods

To discretize the xCFC metric equations, (4.29), `Poseidon` uses a mixed Galerkin projection method with an approximation space spanned by real spherical harmonics and finite element basis functions. Galerkin methods approximate solutions using a finite number of basis functions and then demand the residual to vanish in the weak sense. This differs from the finite difference which solves for the solution directly by approximating differential operators, and the finite volume method calculates the solution's volume average. Appropriate choice of basis functions allows Galerkin methods to achieve accurate solutions with relatively few degrees of freedom.

As shown in Chapter 4, the xCFC metric variables are governed by a set of eight non-linear, coupled partial differential equations. Further, it was shown that these equations can be solved sequentially. Examination of equations (4.29a)-(4.29d) reveals the system consists of two types of equations: Equations (4.29b) and (4.29c) are scalar and non-linear while equations (4.29a) and (4.29d) are linear and involve the conformal vector Laplacian operator.

In this chapter, the method for the discretization of the xCFC equations will be given. The domain decomposition and basis functions will be presented first. Then the non-linear conformal factor and lapse function equations will be linearized and discretized. Finally, the auxiliary X and shift vector equations will be discretized.

5.1 Domain Decomposition

The mixed Galerkin projection can be viewed as a two-step operation. The first step uses the spherical harmonics to project the xCFC equations onto a radial domain while the second is the finite element method in that one dimension. Therefore the domain is discretized in a spherically symmetric fashion on a spherical polar domain, $Q = \mathbb{S}^2 \times \mathbb{D}$, where $\mathbb{S}^2 := \{\Omega = (\theta, \phi) \in \mathbb{R}^2 \mid \theta \in [0, \pi], \phi \in [0, 2\pi)\}$, and $\mathbb{D} := \{r \in \mathbb{R} \mid r \in [a, b]\}$, with $a, b \in \mathbb{R}_0^+$ being the radial domain boundaries.

The radial domain is divided into N_r nonoverlapping elements K^i such that $\mathcal{T} = \bigcup_{i=1}^{N_R} K^i$ covers the entire domain. Each element is defined by

$$K^i = \{x^i \in (x_L^i, x_H^i)\}, \quad (5.1)$$

where x_L^i and x_H^i are the lower and higher boundaries of the element. The element widths are defined as

$$\Delta x^i = (x_H^i - x_L^i), \quad (5.2)$$

while the element centers are given by

$$x_C^i = \frac{1}{2}(x_L^i + x_H^i). \quad (5.3)$$

Associated with each element K^i is a one-dimensional reference element,

$$\mathcal{I}^i = \{\xi^i \in (-1, +1)\}, \quad (5.4)$$

for performing mathematical operations. The reference element coordinate ξ maps to the radial domain by

$$r(\xi^i) = x_C^i + \xi^i \frac{\Delta x^i}{2}. \quad (5.5)$$

On the reference element, a set of interpolation points, $S^i = \{\xi_1^i, \dots, \xi_m^i, \dots, \xi_{k+1}^i\} \in \mathcal{I}^i$, is chosen where k is the degree of the radial polynomial approximation. The set of points includes the reference element boundaries, $\xi_1^i = -1$ and $\xi_{k+1}^i = +1$, and for $k > 1$, the interpolation points are chosen as the Lobatto (LGL) quadrature locations, $S^i = S_{LGL}(k)$. This choice of interpolation points combined with an appropriate choice of interpolating polynomials will ensure the $\mathcal{C}^0$ continuity of the radial solution.

5.2 Quadrature

Poseidon uses two quadrature rules to integrate functions numerically. The majority of the integrals are evaluated using the Legendre-Gauss (LG) quadrature rule with N_{quad} points on the reference element I^i . This rule integrates polynomials up to degree $2N_{quad} - 1$ exactly using a set of quadrature points $S_{quad} := \{\boldsymbol{\eta} : \eta_i \in S_{LG}(N_{quad})\}$ and weights $w_i \in W_{LG}(N_{quad})$.

As we will show, there are angular integrals in the stiffness matrices composed of products of the real spherical harmonics and their derivatives. For integrals of products of real spherical harmonics, the Kronecker delta property is applied. Integrals that do not obey a simple analytic expression, such as when one or more of the harmonic functions has been differentiated or multiplied by a physical source term, are mapped to a 2-dimensional reference element, $I_\Omega := \{\theta, \phi : \theta \in \{0, \pi\}, \phi \in \{0, 2\pi\}\}$. In each dimension of this space, the N_{Trap} -point trapezoid rule is applied, which evaluates trigonometric polynomials of degree n on a periodic domain exactly when $n < N_{Trap}$ ([Trefethen and Weideman, 2014](#)). This allows the angular integrals of the stiffness matrices to be calculated accurately using a few points regardless of the angular decomposition of the system.

5.3 Conformal Factor and Lapse Function Method

We are now ready to proceed with the discretization of the xCFC metric equations. The xCFC metric equations take two distinct forms, therefore we will present the method for each form separately. Equations (4.29b) and (4.29c) govern the conformal factor and lapse function, respectively. Both are non-linear and governed by the scalar Laplacian operator,

$$\bar{\Delta}u = \bar{D}_i \bar{D}^i u = \frac{1}{r^2} \partial_r (r^2 \partial_r u) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta u) + \frac{1}{r^2 \sin^2 \theta} \partial_{\phi\phi} u. \quad (5.6)$$

Therefore, we seek solutions to problems of the form,

$$\bar{\Delta}u = g(u, U, T) \quad (5.7a)$$

$$u = \gamma_{\mathcal{D}} \text{ at } r = b. \quad (5.7b)$$

where $g(u, U, T)$ is a source function that is dependent on the unknown metric quantity, u ; any known metric values, U , such as X^i or ψ in the case of the lapse function equation; and the physical sources contained in the stress-energy tensor, T . On the outer edge of the domain, a Dirichlet boundary value, $\gamma_{\mathcal{D}}$ is imposed. Currently, $\gamma_{\mathcal{D}}$ is restricted to a scalar quantity representing a spherically symmetric value at the outer boundary.

5.3.1 Galerkin Projection

We begin by defining approximation spaces. In the radial dimension, we define $\mathbb{Q}^k(K^i)$ as the set of 1D polynomials of maximum degree k with compact support on the element K^i . `Poseidon` uses Lagrange polynomials defined on the reference element

I^i ,

$$\ell_i(\xi) = \prod_{\substack{j=1 \\ j \neq i}}^{k+1} \frac{\xi - \xi_j}{\xi_i - \xi_j}. \quad (5.8)$$

where $\xi_i \in S_i$ are the interpolation points on I^i . The Lagrange polynomials and their defining interpolation points obey the relationship, $\ell_i(\xi_j) = \delta_{ij}$, and for $\xi \in \mathcal{I}^j$: $\sum_i \ell_i(\xi) = 1$.

By including the element boundaries in the set of interpolation points, adjacent elements will share interpolation points on their boundaries. We therefore chose to define $\mathbb{P}^k$ as the set of piecewise polynomials of maximum degree k as

$$\mathbb{P}^k = \{\lambda_n : n \in \mathcal{N} := \{1, \dots, N_{nodes}\}\} \quad (5.9)$$

where $N_{nodes} = kN_r + 1$ is the number of unique interpolating nodes on a domain with N_r radial elements. Using the mapping from the j th interpolating point of element K^i to its associated global node,

$$n_{node} = (i - 1)k + j, \quad (5.10)$$

we can define the functions λ_n as the product of all interpolating polynomials associated with a node,

$$\lambda_n(\xi) := \sum_{i \in N_r} \sum_{j=1}^{k+1} \ell_j^{K^i}(\xi) \delta_{n, (i-1)k+j}. \quad (5.11)$$

This way of combining the $\ell_j^{K^i}$ means that the λ_n have support on the elements which contain the associated node. In the case of a shared boundary node, the support of the function is over two elements, while λ_n associated nodes located in the interior of an element will only have support on that element. For $k = 1$, the λ_n functions are known as hat functions, see Figure 5.1. The Kronecker delta property of the

Lagrange polynomials ensures that the λ_n functions are continuous across element boundaries. This ensures that the approximate solution is also continuous across the element boundaries.

The function space $\mathbb{E}^{\mathcal{L}}$ is defined as

$$\mathbb{E}^{\mathcal{L}} = \{S_\ell^m(\Omega) : \ell \in \mathcal{L} = \{0, \dots, L\}, m \in \mathcal{M}_\ell := \{-\ell, \dots, +\ell\}\}, \quad (5.12)$$

where S_ℓ^m are the real spherical harmonic functions of degree ℓ and order m up to maximal degree L . The real spherical harmonic functions S_ℓ^m are combinations of the complex spherical harmonic functions Y_ℓ^m (Blanco et al., 1997),

$$S_\ell^m(\theta, \phi) = \begin{cases} \frac{(-1)^m}{\sqrt{2}} (Y_\ell^m + Y_\ell^{m*}) = N_{\ell m} P_\ell^m(\cos \theta) \sqrt{2} \cos(m\phi), & m > 0 \\ Y_\ell^0 = N_{\ell 0} P_\ell^0(\cos \theta), & m = 0 \\ \frac{(-1)^m}{i\sqrt{2}} (Y_\ell^{|m|} - Y_\ell^{|m|*}) = N_{\ell |m|} P_\ell^{|m|}(\cos \theta) \sqrt{2} \sin(|m|\phi), & m < 0. \end{cases} \quad (5.13)$$

where the P_ℓ^m are the associated Legendre polynomials and $N_{\ell m}$ is a normalization factor,

$$N_{\ell m} = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}}. \quad (5.14)$$

This combination of complex conjugates negates the complex portion of the harmonic functions while preserving all the qualities of the Y_ℓ^m functions that make them ideal for projections on a spherical polar domain. As the complex spherical harmonics are eigenfunctions of the flat-space Laplace operator, so are their real counterparts; i.e.,

$$\forall(\ell, m) \in \mathcal{L} \times \mathcal{M}_\ell : \quad r^2 \bar{\Delta} S_\ell^m = -\ell(\ell + 1) S_\ell^m. \quad (5.15)$$

Likewise, the real spherical harmonic functions are regular on the sphere, which prevents any issue with the coordinate singularity along the polar axis. The S_ℓ^m

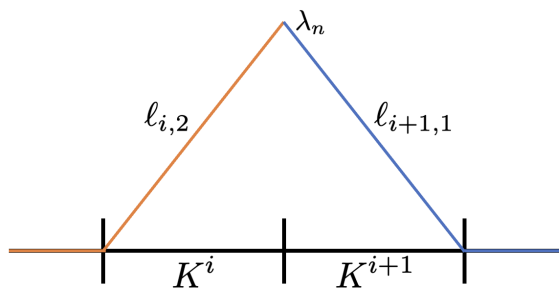


Figure 5.1: The hat function λ_n is the sum of the two linear Lagrange polynomials, $\ell_{i,2}$ and $\ell_{i+1,1}$.

also inherit the orthonormality of the Y_ℓ^m ,

$$\langle S_\ell^m, S_{\ell'}^{m'} \rangle_{\mathbb{S}^2} := \int_{\mathbb{S}^2} S_\ell^m(\Omega) S_{\ell'}^{m'*}(\Omega) d\Omega \quad (5.16)$$

$$= \int_{\mathbb{S}^2} S_\ell^m(\Omega) S_{\ell'}^{m'}(\Omega) d\Omega \quad (5.17)$$

$$= \delta_{\ell\ell'} \delta_{mm'}, \quad (5.18)$$

where δ_{ij} is the Kronecker delta symbol, and $d\Omega = \sin\theta d\theta d\phi$ is the solid angle element. The regularity and orthonormality of the S_ℓ^m allow any regular function $f \in \mathbb{S}^2$ to be approximated as

$$f(\Omega) = \sum_{\substack{\ell \in \mathcal{L} \\ m \in \mathcal{M}_\ell}} f_{\ell m} S_\ell^m(\Omega), \quad (5.19)$$

with

$$f_{\ell m} = \int_{\mathbb{S}^2} f(\Omega) S_\ell^m(\Omega) d\Omega. \quad (5.20)$$

We now define the global approximation space $\mathbb{V}_h^{k,\mathcal{L}}$ as

$$\mathbb{V}_h^{k,\mathcal{L}} := \mathbb{P}^k \bigotimes \mathbb{E}^\mathcal{L}. \quad (5.21)$$

The solution u is approximated using functions $\phi_{\ell mn} \in \mathbb{V}_h^{k,\mathcal{L}}$,

$$u \approx u_h := \sum_{\substack{\ell \in \mathcal{L} \\ m \in \mathcal{M}_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \phi_{\ell mn} = \sum_{\substack{\ell \in \mathcal{L} \\ m \in \mathcal{M}_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \lambda_n S_\ell^m. \quad (5.22)$$

The index i referring to the non-linear solve iteration is suppressed on the coefficients $u_{\ell mn}$ to simplify notation.

The Dirichlet boundary condition, Eq. (5.7b), is applied to Eq. (5.22),

$$\gamma_{\mathcal{D}} = u_h(b, \Omega) = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \lambda_n(b) S_\ell^m(\Omega). \quad (5.23)$$

The outer boundary of the domain is the outer element boundary of the outermost radial elements and is therefore an interpolation point allowing us to apply the Kronecker delta property of the Lagrange polynomials to eliminate the sum over n leaving only the terms associated with the last radial interpolation node. The spherical symmetry of $\gamma_{\mathcal{D}}$ is imposed by setting any coefficient of spherical harmonic order $\ell > 0$ to zero preserving only the monopole coefficients. Thus the Dirichlet boundary condition becomes

$$u_{\ell mn_{nodes}} = \begin{cases} \gamma_{\mathcal{D}} & \ell, m = 0, 0 \\ 0 & \text{else} \end{cases}. \quad (5.24)$$

The Galerkin method determines the remaining unknown coefficients by requiring the residual,

$$\mathcal{R} := \bar{\Delta} u_h - g(u_h, U, T) \quad (5.25)$$

be orthogonal to a set of test functions, $\phi'_{\ell' m' n'} \in \mathbb{V}_h^{k, \mathcal{L}}$. As the coefficients associated with the outer Dirichlet boundary have been fixed, the test functions associated with the outer boundary node can be omitted. Therefore, we define a new test function space, $\mathbb{W}_h^{k, \mathcal{L}}$, as

$$\mathbb{W}_h^{k, \mathcal{L}} := \mathbb{V}_h^{k, \mathcal{L}} \setminus \{ \lambda_{N_{nodes}} \times \mathbb{E}^{\mathcal{L}} \} \quad (5.26)$$

and

$$\mathcal{N}' := \mathcal{N} \setminus N_{nodes} = \{1, \dots, N_{nodes} - 1\} \quad (5.27)$$

is the set of indices for the radial functions in $\mathbb{W}_h^{k,\mathcal{L}}$.

Using the definition of the weighted $\mathcal{L}^2$ inner product,

$$(f, g) := \int_Q f g dV = \int_{\mathbb{S}^2 \times \mathbb{D}} f g r^2 \sin \theta dr d\theta d\phi, \quad (5.28)$$

the Galerkin requirement is written as

$$(\phi'_{\ell'm'n'}, \mathcal{R}) = \int_Q \phi'_{\ell'm'n'} (\bar{\Delta} u_h - g(u_h, U, T)) dV = 0. \quad (5.29)$$

for all $\phi'_{\ell'm'n'} \in \mathbb{W}_h^{k,\mathcal{L}}$. Equation (5.29) is known as the weak form of Eq. (5.25).

It is convenient to decompose the Laplace operator into its radial and angular components, $\bar{\Delta}_r$ and $\bar{\Delta}_\Omega$ respectively (i.e., $\bar{\Delta} = \bar{\Delta}_r + \bar{\Delta}_\Omega$), allowing Eq. (5.29) to be written as

$$\int_Q \phi'_{\ell'm'n'} \bar{\Delta}_r u_h dV + \int_Q \phi'_{\ell'm'n'} \bar{\Delta}_\Omega u_h dV = \int_Q \phi'_{\ell'm'n'} g(u_h, U, T) dV. \quad (5.30)$$

The first integral in Eq. (5.30) is evaluated using integration by parts,

$$\int_Q \phi'_{\ell'm'n'} \bar{\Delta}_r u_h dV = - \int_Q (\partial_r \lambda_n) (\partial_r u_h) S_\ell^m(\Omega) dV + \int_{\mathbb{S}^2} b^2 \lambda_n(b) \partial_r u_h S_\ell^m(\Omega) d\Omega \quad (5.31)$$

$$= - \int_Q (\partial_r \lambda_n) (\partial_r u_h) S_\ell^m(\Omega) dV, \quad (5.32)$$

where the second term in Eq. (5.31) vanishes because $\phi'_{\ell'm'n'} \in \mathbb{W}_h^{k,\mathcal{L}}$ for which $\lambda_n(b) = 0 \forall n \in \mathcal{N}'$. Using the expanded form of the approximate solution, Eq. (5.22), we can

now write

$$\int_Q \phi'_{\ell'm'n'} \bar{\Delta}_r u_h dV = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \left(- \int_Q (\partial_r \lambda_n) (\partial_r \lambda_{n'}) S_{\ell'}^{m'} S_\ell^m dV \right) \quad (5.33)$$

$$= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \delta_{\ell, \ell'} \delta_{m, m'} \left(- \int_D r^2 (\partial_r \lambda_n) (\partial_r \lambda_{n'}) dr \right). \quad (5.34)$$

Returning to Eq. (5.30), the approximate solution is now expanded in the angular Laplacian term,

$$\begin{aligned} \int_Q \phi'_{\ell'm'n'} \bar{\Delta}_\Omega u_h dV &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \int_Q \lambda_n S_\ell^m \left(\bar{\Delta}_\Omega \lambda_{n'} S_{\ell'}^{m'} \right) dV \\ &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \left(-\ell(\ell+1) \int_Q \lambda_n \lambda_{n'} S_\ell^m S_{\ell'}^{m'} dr d\Omega \right) \\ &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} \delta_{\ell, \ell'} \delta_{m, m'} \left(-\ell(\ell+1) \int_D \lambda_n \lambda_{n'} dr \right) \end{aligned} \quad (5.35)$$

where Eq. (5.15) was used to eliminate the differential operator. Combining Eqs. (5.34) and (5.35), Equation (5.30) can be written as

$$\sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} S_{\ell mn, \ell' m' n'} = \int_Q \phi'_{\ell'm'n'} g(u_h, U, T) dV \quad (5.36)$$

for $\phi'_{\ell'm'n'} \in \mathbb{W}_h^{k, \mathcal{L}}$ and

$$S_{\ell mn, \ell' m' n'} = -\delta_{\ell, \ell'} \delta_{m, m'} \int_D r^2 \left((\partial_r \lambda_n) (\partial_r \lambda_{n'}) + \ell(\ell+1) \lambda_n \lambda_{n'} \right) dr \quad (5.37)$$

is called the *stiffness matrix*.

The $u_{\ell mn}$ coefficients that have been fixed by the Dirichlet boundary can be isolated from the right-hand side summation of Eq. (5.36),

$$\begin{aligned} \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn} S_{\ell mn, l' m' n'} &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}'} u_{\ell mn} S_{\ell mn, l' m' n'} + \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} u_{\ell m N_{nodes}} S_{\ell m N_{nodes}, l' m' n'} \\ &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}'} u_{\ell mn} S_{\ell mn, l' m' n'} + u_{00 N_{nodes}} S_{00 N_{nodes}, l' m' n'} \quad (5.38) \end{aligned}$$

and moved to the right-hand side of the equation, allowing the discretized system to be expressed as

$$\sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}'} u_{\ell mn} S_{\ell mn, l' m' n'} = G_{l' m' n'} - u_{00 N_{nodes}} S_{00 N_{nodes}, l' m' n'} \quad , \quad (5.39)$$

$\forall (l', m', n') \in \mathcal{L} \times \mathcal{M}_\ell \times \mathcal{N}'$ and where

$$G_{l' m' n'} = \int_Q \phi_{\ell mn} g(u_h, U, T) dV. \quad (5.40)$$

5.3.2 Fixed Point Method and Anderson Acceleration

To transform Eq. (5.39) into a linear form that can be solved `Poseidon` uses the fixed point method with Anderson acceleration. The fixed-point iterative method requires an initial guess, $u_{[0]}$, which is used to calculate $g(u_{[0]}, U, T)$. By fixing the right-hand side, Eq. (5.7a) becomes linear and can be solved for $u_{[1]}$, which can then be used to re-calculate the right-hand side. The linear system is updated and solved until the approximate solution $u_{[i]}$ satisfies some criteria or $[i]$ exceeds some preset limit.

This sequence of successive approximations is known as the Picard iteration of $f(u) = \bar{\Delta}u - g(u)$ starting at $u_{[0]}$. The Picard, or fixed-point, iteration method's main advantage is that it only requires one evaluation of $\bar{\Delta}u = g(u)$ per iteration, whereas other methods, such as the secant method, require multiple evaluations, or the Newton method, which requires the evaluation of the Jacobian. However, the

Picard method is not guaranteed to converge, but requires $f(u)$ to be continuous and the initial guess to be chosen sufficiently close to the solution. Furthermore, when it does converge, it may do so at a linear rate, which may be too slow for practical use.

To help reduce the number of iterations required for convergence, extra steps can be added to the standard Picard iteration. One such alteration is known as Anderson acceleration, see algorithm 1, also known as Anderson Mixing (Walker and Ni, 2011). The superscript $\top$ in line 6 denotes the transpose while the $\|\cdot\|_2$ denotes the $\mathcal{L}^2$ -norm. Anderson acceleration uses information from the previous m steps when computing the next iteration, $u_{[m+1]}$, and therefore m is sometimes referred to as the memory parameter.

The choice of memory parameter is important to the efficiency of the method. If $m = 1$, Anderson acceleration is equivalent to the Picard iteration method and there is no acceleration. If it is chosen too small, not enough information is used and a significant reduction in iteration count is not achieved. On the other hand, if m is chosen too large, the least squares solve in line 6 can become costly and negate the gain from performing fewer iterations. The ideal memory parameter is problem-dependent. Poseidon uses $m = 3$ which was determined by testing with a source profile similar to those of CCSNe near core bounce and providing the flat space value of the conformal factor as an initial guess to maximize the unaccelerated number of iterations. See Section 6.5 for more details.

5.3.3 Properties of the Conformal Factor and Lapse Function Linear Systems

Utilizing the $\delta_{\ell,\ell'}\delta_{m,m'}$ present in the stiffness matrix, the linear system in Eq. (5.39) can be divided into a set of $(L+1)^2$ smaller systems of dimension $N_{nodes}-1 \times N_{nodes}-1$,

$$\sum_{n \in \mathcal{N}'} u_{\ell mn} S_{lnn'} = G_{\ell' m' n'}, \quad \forall n \in N_I : \forall \ell \in \mathcal{L} : \forall m \in \mathcal{M}_\ell. \quad (5.41)$$

```

1 Given  $u_0$  and  $m \geq 1$ .
2 Set  $u_1 = g(u_0)$ 
3 for  $i = 1, 2, \dots$  do
4    $m_i = \min\{m, i\}$ 
5    $F_i = (f_{i-m_i}, \dots, f_i)$ , where  $f_j = g(u_j) - u_j$ 
6   Find  $\alpha^{(i)} = (\alpha_0^{(i)}, \dots, \alpha_{m_i}^{(i)})^\top$  that solves  $\min_{\alpha=(\alpha_0, \dots, \alpha_{m_i})^T} \|F_i \alpha\|_2$  such
      that  $\sum_{j=0}^{m_i} \alpha_j = 1$ .
7    $u_{i+1} = \sum_{j=0}^{m_i} \alpha_j^{(i)} g(u_{i-m_i+j})$ .
end

```

Algorithm 1: Anderson Acceleration

where

$$S_{\ell nn'} = - \int_D r^2 \left((\partial_r \lambda_n) (\partial_r \lambda_{n'}) + \ell(\ell + 1) \lambda_n \lambda_{n'} \right) dr. \quad (5.42)$$

While the load vectors, $G_{\ell mn}$, are unique for every choice of ℓ and m , the stiffness matrices, $S_{\ell nn'}$, have no dependence on m . Therefore, only L unique matrices of dimension $N_{nodes} - 1 \times N_{nodes} - 1$ need to be created and stored.

During the derivation, the application of the Dirichlet boundary was used to reduce the dimension of the linear system. In practice, **Poseidon** uses matrices of dimension $N_{nodes} \times N_{nodes}$. This is achieved by modifying the last row and column to match those of the identity matrix and setting the corresponding entry in the load vector to enforce the boundary condition. Doing so gives the matrices a uniform overlapping block structure along the diagonal band; see Figure 5.2, where the overlapping element comes from a shared interpolation node on adjacent element boundaries. Each block is symmetric and has a bandwidth equal to the finite element method degree, k . Specifically, when $k = 1$, the matrix $S_{l nn'}$ is tridiagonal.

The non-zero pattern of one of the $S_{l nn'}$ matrices is well-defined and the sparsity of the matrices can be calculated as

$$\text{Sparsity} = \frac{N_r k^2 - N_r + 1}{N_r^2 k^2 + 2N_r k + 1}. \quad (5.43)$$

To reduce memory overhead, each $S_{l nn'}$ is stored in compressed column storage (CCS) format (Dongarra, 1995). Each linear system is then solved using Cholesky factorization with a Jacobi preconditioner.

5.4 X and Shift Vector Method

The X and Shift vectors are governed by equations (4.29a) and (4.29d), respectively. Both are linear equations controlled by the conformal vector Laplacian operator and

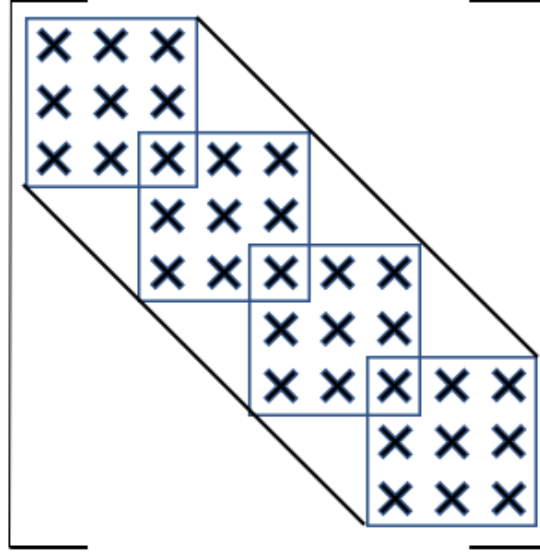


Figure 5.2: Nonzero structure of the matrix $S_{l_{nn'}}$ used in **Poseidon** for the case of $k = 2$ and $N_r = 4$. The diagonal lines denote the band structure of the matrix. The blue squares surrounding the crosses denote the overlapping block structure within the band, and the crosses denote individual, non-zero-valued, elements. The single overlapping element shared by the consecutive blocks comes from the shared interpolation node at element interfaces.

take the form,

$$\bar{\Delta}_L u^i = g(U, T), \quad (5.44)$$

where $g(U, T)$ is now a source function dependent only on known metric variables U and physical source terms from the stress-energy tensor T . The conformal vector Laplacian is defined as

$$\bar{\Delta}_L u^i := \bar{D}_j \bar{D}^j u^i + \frac{1}{3} \bar{D}^i \bar{D}_j u^j, \quad (5.45)$$

where $\bar{D}_i$ are the covariant derivatives associated with the flat metric, $\bar{\gamma}_{ij}$.

The vector Laplacian $\bar{\Delta}_L u^i$ can be expanded and the scalar Laplacian isolated. For each possible value of i , the first term $\bar{D}_j \bar{D}^j u^i$ expands to

$$\bar{D}^j \bar{D}_j u^1 = \bar{\Delta}(u^1) - \frac{2}{r} \partial_2 u^2 - \frac{2}{r} \partial_3 u^3 - \frac{2 \cot \theta}{r} u^2 - \frac{2}{r^2} u^1, \quad (5.46a)$$

$$\bar{D}^j \bar{D}_j u^2 = \bar{\Delta}(u^2) + \frac{2}{r} \partial_1 u^2 + \frac{2}{r^3} \partial_2 u^1 - \frac{2 \cot \theta}{r^2} \partial_3 u^3 + \frac{1 - \cot^2 \theta}{r^2} u^2, \quad (5.46b)$$

$$\bar{D}^j \bar{D}_j u^3 = \bar{\Delta}(u^3) + \frac{2}{r^3 \sin^2 \theta} \partial_3 u^1 + \frac{\cot \theta}{r^2 \sin^2 \theta} \partial_3 u^2 + \frac{2}{r} \partial_1 u^3 + \frac{2 \cot \theta}{r^2} \partial_2 u^3, \quad (5.46c)$$

where $\bar{\Delta}(u^i)$ denotes the Laplacian acting on u^i as if it were a scalar. The second term, $\frac{1}{3} \bar{D}^i \bar{D}_j u^j$, is likewise expanded for each index i ,

$$\frac{1}{3} \bar{D}^1 \bar{D}_j u^j = \frac{1}{3} \left(\partial_{11} u^1 + \frac{2}{r} \partial_1 u^1 - \frac{2}{r^2} u^1 + \partial_{12} u^2 + \cot \theta \partial_1 u^2 + \partial_{13} u^3 \right) \quad (5.47a)$$

$$\frac{1}{3} \bar{D}^2 \bar{D}_j u^j = \frac{1}{3r^2} \left(\partial_{21} u^1 + \frac{2}{r} \partial_2 u^1 + \partial_{22} u^2 + \cot \theta \partial_2 u^2 - \frac{1}{\sin^2 \theta} u^2 + \partial_{23} u^3 \right) \quad (5.47b)$$

$$\frac{1}{3} \bar{D}^3 \bar{D}_j u^j = \frac{1}{3r^2 \sin^2 \theta} \left(\partial_{31} u^1 + \frac{2}{r} \partial_3 u^1 + \partial_{32} u^2 + \cot \theta \partial_3 u^2 + \partial_{33} u^3 \right). \quad (5.47c)$$

Eq. (5.45) can now be compactly expressed by

$$\bar{\Delta}_L u^i = \bar{\Delta}(u^i) + F^i(u) \quad (5.48)$$

where the functions $F^i(u)$ are given by

$$\begin{aligned} F^1 = & \frac{1}{3} \partial_{11} u^1 + \frac{2}{3r} \partial_1 u^1 - \frac{4}{3r^2} u^1 + \frac{1}{3} \partial_{12} u^2 + \frac{1}{3} \cot \theta \partial_1 u^2 - \frac{2}{r} \partial_2 u^2 \\ & - \frac{2 \cot \theta}{r} u^2 - \frac{2}{r} \partial_3 u^3 + \frac{1}{3} \partial_{13} u^3 \end{aligned} \quad (5.49a)$$

$$\begin{aligned} F^2 = & + \frac{1}{3r^2} \partial_{21} u^1 + \frac{8}{3r^3} \partial_2 u^1 + \frac{1}{3r^2} \partial_{22} u^2 + \frac{2}{r} \partial_1 u^2 + \frac{1}{3r^2} \cot \theta \partial_2 u^2 \\ & - \frac{1}{3r^2 \sin^2 \theta} u^2 + \frac{1 - \cot^2 \theta}{r^2} u^2 + \frac{1}{3r^2} \partial_{23} u^3 - \frac{2 \cot \theta}{r^2} \partial_3 u^3 \end{aligned} \quad (5.49b)$$

$$\begin{aligned} F^3 = & \frac{1}{r^2 \sin^2 \theta} \left(\frac{1}{3} \partial_{31} u^1 + \frac{8}{3r} \partial_3 u^1 + \frac{1}{3} \partial_{32} u^2 + 2 \cot \theta \partial_3 u^2 \right) \\ & + \frac{1}{r^2 \sin^2 \theta} \partial_{33} u^3 + \frac{2}{r} \partial_1 u^3 + \frac{2 \cot \theta}{r^2} \partial_2 u^3. \end{aligned} \quad (5.49c)$$

The F^i terms reveal coupling between the equations for each component of the $\mathbf{u}$, therefore we seek an approximate solution to the problem

$$\bar{\Delta}(u^i) + F^i(u) = g^i(U, T) \quad (5.50)$$

$$u^i = \gamma_D^i \text{ ar } r = b \quad (5.51)$$

for $i = 1, 2, 3$. Following the steps to discretize the conformal factor and lapse function iterative equations, we choose the approximation space for each component of $\mathbf{u}$ to be $\mathbb{V}_h^{k, \mathcal{L}}$. Thus the total approximation space is given by $\mathbb{V}_h^{k, \mathcal{L}} \times \mathbb{V}_h^{k, \mathcal{L}} \times \mathbb{V}_h^{k, \mathcal{L}}$. The solution is then approximated by

$$u^i \approx u_h^i = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn}^i \phi_{\ell mn}^i = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn}^i \lambda_n^i S_\ell^{i, m}, \quad (5.52)$$

and the Dirichlet boundary condition, Eq. (5.51), is imposed by setting

$$u_{\ell mn_{nodes}}^i = \begin{cases} \gamma_{\mathcal{D}}^i & l, m = 0, 0 \\ 0 & \text{else} \end{cases}. \quad (5.53)$$

Once again, the functions associated with the Dirichlet node are extracted from the approximation space, $\mathbb{U}_h^{k,\mathcal{L}} := \mathbb{W}_h^{k,\mathcal{L}} \times \mathbb{W}_h^{k,\mathcal{L}} \times \mathbb{W}_h^{k,\mathcal{L}}$.

Next, the residual is defined as

$$\mathcal{R}^i := \bar{\Delta}(u^i) + F^i(u) - g^i(U, T). \quad (5.54)$$

The $\mathcal{L}^2$ -norm of the residual is taken with respect to test functions $\phi_{\ell' m' n'}^{i' i'} \in \mathbb{U}_h^{k,\mathcal{L}}$ and set equal to zero,

$$0 = \int_Q \phi_{\ell' m' n'}^{i' i'} (\bar{\Delta}(u^i) + F^i(u) - g^i(U, T)) dV. \quad (5.55)$$

This is the weak form of Eq. (5.50).

Rearranging Eq. (5.55),

$$\int_Q \phi_{\ell' m' n'}^{i' i'} \bar{\Delta}(u^i) dV + \int_Q \phi_{\ell' m' n'}^{i' i'} F^i(u) dV = \int_Q \phi_{\ell' m' n'}^{i' i'} g^i(U, T) dV, \quad (5.56)$$

we see that the first integral has the same structure as the scalar Laplacian equations that were discretized in Section 5.3. Therefore, it is discretized in the same fashion resulting in

$$\int_Q \phi_{\ell' m' n'}^{i' i'} \bar{\Delta}(u^i) dV = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} u_{\ell mn}^i T_{i \ell mn, i' \ell' m' n'} \quad (5.57)$$

where the matrix $t_{i\ell mn, i'\ell' m'n'}$ is defined like those in Eq. (5.37),

$$t_{i\ell mn, i'\ell' m'n'} := -\delta_{\ell, \ell'} \delta_{m, m'} \delta_{i, i'} \int_D r^2 \left((\partial_r \lambda_n^i) (\partial_r \lambda_{n'}^{i'}) + \ell(\ell+1) \lambda_n^i \lambda_{n'}^{i'} \right) dr. \quad (5.58)$$

The Kronecker deltas, $\delta_{i, i'}$, were introduced to prevent undesired coupling between components.

The approximate solution is expanded in the second term of Eq. (5.56) and the terms are collected with respect to the coefficients of each vector component. $F^1(u)$ becomes

$$F^1(u_{\ell mn}) = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} \left(u_{\ell mn}^1 F_{\ell mn}^{11} + u_{\ell mn}^2 F_{\ell mn}^{12} + u_{\ell mn}^3 F_{\ell mn}^{13} \right). \quad (5.59)$$

where

$$F_{\ell mn}^{11} = \frac{1}{3} \partial_{11} \phi_{\ell mn}^1 + \frac{2}{3r} \partial_1 \phi_{\ell mn}^1 - \frac{4}{3r^2} \phi_{\ell mn}^1, \quad (5.60a)$$

$$F_{\ell mn}^{12} = \frac{1}{3} \partial_{12} \phi_{\ell mn}^2 + \frac{1}{3} \cot \theta \partial_1 \phi_{\ell mn}^2 - \frac{2}{r} \partial_2 \phi_{\ell mn}^2 - \frac{2 \cot \theta}{r} \phi_{\ell mn}^2, \quad (5.60b)$$

$$F_{\ell mn}^{13} = \frac{1}{3} \partial_{13} \phi_{\ell mn}^3 - \frac{2}{r} \partial_3 \phi_{\ell mn}^3. \quad (5.60c)$$

Likewise, $F^2(u)$ becomes

$$F^2(u_{\ell mn}) = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} \left(u_{\ell mn}^1 F_{\ell mn}^{21} + u_{\ell mn}^2 F_{\ell mn}^{22} + u_{\ell mn}^3 F_{\ell mn}^{23} \right), \quad (5.61)$$

such that

$$F_{\ell mn}^{21} = \frac{1}{3r^2} \partial_{21} \phi_{\ell mn}^1 + \frac{8}{3r^3} \partial_2 \phi_{\ell mn}^1, \quad (5.62a)$$

$$F_{\ell mn}^{22} = \frac{1}{3r^2} \partial_{22} \phi_{\ell mn}^2 + \frac{2}{r} \partial_1 \phi_{\ell mn}^2 + \frac{1}{3r^2} \cot \theta \partial_2 \phi_{\ell mn}^2 - \frac{1}{3r^2 \sin^2 \theta} \phi_{\ell mn}^2 + \frac{1 - \cot^2 \theta}{r^2} \phi_{\ell mn}^2, \quad (5.62b)$$

$$F_{\ell mn}^{23} = \frac{1}{3r^2} \partial_{23} \phi_{\ell mn}^3 - \frac{2 \cot \theta}{r^2} \partial_3 \phi_{\ell mn}^3. \quad (5.62c)$$

$$(5.62d)$$

Lastly, $F^3(u)$ becomes

$$F^3 = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} \left(u_{\ell mn}^1 F_{\ell mn}^{31} + u_{\ell mn}^2 F_{\ell mn}^{32} + u_{\ell mn}^3 F_{\ell mn}^{33} \right), \quad (5.63)$$

where

$$F_{\ell mn}^{31} = \frac{1}{r^2 \sin^2 \theta} \left(\frac{1}{3} \partial_{31} \phi_{\ell mn}^1 + \frac{8}{3r} \partial_3 \phi_{\ell mn}^1 \right), \quad (5.64a)$$

$$F_{\ell mn}^{32} = \frac{1}{r^2 \sin^2 \theta} \left(\frac{1}{3} \partial_{32} \phi_{\ell mn}^2 + 2 \cot \theta \partial_3 \phi_{\ell mn}^2 \right), \quad (5.64b)$$

$$F_{\ell mn}^{33} = \frac{1}{r^2 \sin^2 \theta} \left(\partial_{33} \phi_{\ell mn}^3 + \frac{2}{r} \partial_1 \phi_{\ell mn}^3 + \frac{2 \cot \theta}{r^2} \partial_2 \phi_{\ell mn}^3 \right). \quad (5.64c)$$

$$(5.64d)$$

Given these expansions of the $F^i(u)$ functions, the second intergral in Eq. (5.56) can be compactly written as

$$\begin{aligned} \int_Q \phi_{\ell' m' n'}^{i'} F^i(u) dV &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} \int_Q \phi_{\ell' m' n'}^{i'} \left(u_{\ell mn}^1 F_{\ell mn}^{i1} + u_{\ell mn}^2 F_{\ell mn}^{i2} + u_{\ell mn}^3 F_{\ell mn}^{i3} \right) dV \\ &= \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} \sum_{j=1}^3 u_{\ell mn}^j T_{i \ell mn, i' \ell' m' n'}^j, \end{aligned} \quad (5.65)$$

where

$$T_{i\ell mn, i'\ell' m' n'}^j := \int_Q \phi_{\ell' m' n'}^{i'} F_{\ell mn}^{ij} dV. \quad (5.66)$$

We now combine Eq. (5.65) with Eq. (5.57) to write left-hand side of Eq. (5.56) as,

$$\int_Q \phi_{\ell' m' n'}^{i'} \bar{\Delta}(u^i) dV + \int_Q \phi_{\ell' m' n'}^{i'} F^i(u) dV = \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}} \left(u_{\ell mn}^i T_{i\ell mn, i'\ell' m' n'} + \sum_{j=1}^3 u_{\ell mn}^j T_{i\ell mn, i'\ell' m' n'}^j \right). \quad (5.67)$$

Extracting the terms associated with the fixed outer boundary and moving them to the right-hand side, the final system can be written as

$$\begin{aligned} & \sum_{\substack{\ell \in \mathcal{L} \\ m \in M_\ell}} \sum_{n \in \mathcal{N}'} \left(u_{\ell mn}^i T_{i\ell mn, i'\ell' m' n'} + \sum_{j=1}^3 u_{\ell mn}^j T_{i\ell mn, i'\ell' m' n'}^j \right) \\ &= \int_Q \phi_{\ell' m' n'}^{i'} g^i(U, T) dV - \left(u_{00N_{nodes}}^i T_{i00N_{nodes}, i'\ell' m' n'} + \sum_{j=1}^3 u_{00N_{nodes}}^j T_{i00N_{nodes}, i'\ell' m' n'}^j \right). \end{aligned} \quad (5.68)$$

$$(5.69)$$

The matrix resulting from Eq. (5.67) has dimensions $3N_{\mathcal{L}}N_{nodes} \times 3N_{\mathcal{L}}N_{nodes}$, where $N_{\mathcal{L}} := |\mathbb{E}^{\mathcal{L}}|$ is the cardinality of $\mathbb{E}^{\mathcal{L}}$. The non-zero pattern for this system is complex and an analytic expression for the sparsity is not attempted. But, if we define the map from between expansion indices, (i, ℓ, m, n) and the matrix index, a , as

$$a := 3N_{\mathcal{L}}n + N_{\mathcal{L}}i + \ell(\ell + 1) + m + 1, \quad (5.70)$$

the matrix adopts an overlapping block diagonal similar to the lapse function and conformal factor matrices. Each block corresponds to all terms associated with a radial element, and will therefore have a size of $3N_{\mathcal{L}}(k + 1) \times 3N_{\mathcal{L}}(k + 1)$. If we

assume each block is entirely non-zero, which in general it isn't, we can formulate an expression for the upper bound of the matrix's sparsity,

$$\text{Sparsity} \leq \frac{9N_{\mathcal{L}}^2 (k^2 + 2k) N_R + 1}{(3N_{\mathcal{L}}(kN_r + 1))^2}. \quad (5.71)$$

Given the size, sparseness, and layout of the stiffness matrix, it is stored in band matrix format (Blackford, 1999). In this storage scheme, an m -by- n band matrix with kl subdiagonals and ku superdiagonals is stored in a 2-dimensional array with n columns and $kl + ku + 1$ rows. The values are mapped from the original matrix indices (i, j) to the storage indices $(ku + kl + 1 + i - j, j)$. The above mapping scheme requires kl more rows than is necessary to store the original matrix. These spaces are used by LAPACK's band matrix LU factorization routines DGBTRF and DGBTRS which are used to solve the linear system in Eq. (5.69).

Before the system is solved, a Jacobi preconditioner is applied to reduce the condition number of the system. Jacobi preconditioning uses the diagonal of the matrix as a preconditioner to reduce the condition number of the linear system. This method was chosen for its low computational cost.

5.5 Hydrodynamics Coupling

Poseidon has been incorporated into the toolkit for high-order neutrino radiation hydrodynamics (`thornado`) framework, which aims to simulate relativistic astrophysical systems such as core-collapse supernovae (Endeve et al., 2019; Chu et al., 2019; Laiu et al., 2020; Pochik et al., 2021; Dunham et al., 2021). `thornado` evolves the hydrodynamic variables

$$\mathbf{U} := (D, S_j, \tau, N_e)^T \quad (5.72)$$

where T denotes transposition and

$$D := \rho W, \quad (5.73)$$

$$S_j := \rho h W^2 v_j, \quad (5.74)$$

$$\tau := \rho h W^2 - p - \rho W, \quad (5.75)$$

$$N_e := n_e W, \quad (5.76)$$

where $W := (1 - v_i v^i)^{-1/2}$ is the Lorentz factor, v_j is the j th component of the covariant three-velocity of the fluid as measured by an Eulerian observer. The specific enthalpy as measured by an observer moving with the fluid is given by $h := 1 + (e + p)/\rho$ where e is the internal energy density, p is the thermal pressure, and ρ is the baryon mass density, all as measured by the comoving observer. Finally, n_e is the electron number density.

The evolution is governed by the equation

$$\partial_t (\psi^6 \mathbf{U}) + \frac{\psi^6}{\sqrt{\gamma}} \partial_i (\alpha \sqrt{\gamma} \mathbf{F}^i(\mathbf{U})) = \alpha \psi^6 \mathbf{S}(\mathbf{U}). \quad (5.77)$$

The $\mathbf{F}^i(\mathbf{U})$ vector contains the fluxes of the i th spatial dimension,

$$\mathbf{F}^i(\mathbf{U}) = \begin{pmatrix} D v^i - \alpha^{-1} \beta^i D \\ P_j^i - \alpha^{-1} \beta^i S_j \\ S^i - D v^i - \alpha^{-1} \beta^i \tau \\ N_e v^i - \alpha^{-1} \beta^i N_e \end{pmatrix}. \quad (5.78)$$

And, the $\mathbf{S}(\mathbf{U})$ is the vector containing the sources,

$$\mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ \frac{1}{2} P^{ik} \partial_j \gamma_{ij} - \alpha^{-1} S_j \partial_j \beta^i - \alpha^{-1} (\tau + D) \partial_j \alpha \\ P^{ij} K_{ij} - \alpha^{-1} S^j \partial_j \alpha \\ 0 \end{pmatrix}. \quad (5.79)$$

thornado uses the discontinuous Galerkin (DG) method to discretize space and a multi-stage explicit strong-stability-preserving Runge-Kutta (SSPRK) method (Gottlieb et al., 2001) to evolve the densitized quantities, $\mathbf{U}^* = \psi^6 \mathbf{U}$. This method achieves high-order temporal accuracy by using a convex combination of forward-Euler timesteps to create an update. **thornado** begins a stage by calculating the conformal factor, ψ using **Poseidon** based on the current state of the system. With ψ , the unscaled variables can be recovered as well as the primitive variables, such as the rest-mass density and internal energy per unit volume. Using those, the pressure, P , can be calculated and used to form S^* which is used to solve for the remaining metric variables, α and β^i . Once the full metric is known, the fluid update can be calculated and used to update the system.

On each element, **thornado** stores values of quantities at Legendre-Gauss nodes, S_{LG} , while **Poseidon** performs its calculations on the Lobatto points, S_{LGL} . Therefore, the values received from **thornado** are interpolated to the **Poseidon** grid upon input. Once the xCFC coefficients are known, the solution can be calculated at any point within the domain that the calling code requests.

5.6 AMReX Incorporation

The **thornado** framework uses block-structured adaptive mesh refinement (AMR) through coupling with **AMReX** (Zhang et al., 2019). To enable refinement, **AMReX** uses a multi-level mesh with each level being a refinement of the level below it with the coarsest level covering the entire domain. To refine the domain, a set of tagging criteria is chosen and any element found satisfying the criteria is refined and the data mapped to the corresponding finer elements on the next level.

To handle source data contained on a multi-level mesh, **Poseidon** begins by choosing a single-level mesh constructed of the leaf elements. Leaf elements are the finest representation of source data for a given location in the domain. By

construction, the union of leaf elements will always cover the entire domain and can be used to create a single-level mesh.

AMReX distributes among multiple processes by assigning each a logically cartesian portion of the domain. During each linear solve, **Poseidon** instructs each process to move through the locally owned leaf elements and calculates the contribution to the load vector. These contributions are collected via a **MPI_REDUCE** command to a single process where they are solved. The solution coefficients are then distributed to all processes using **MPI_BCAST**.

A key feature of AMR is its ability to alter the mesh as the system evolves focusing computational resources. After the initial solve, **Poseidon** compares the refinement pattern of the input source variables against the pattern from the previous call. This is done by comparing the collection of grids, known as a box array, on each level. If the refinement pattern has changed, **Poseidon** will destroy and recreate the stiffness matrices to match the new domain. The solutions for the conformal factor and lapse function from the previous call are interpolated to the new domain for use as the initial guess in their non-linear solves.

Chapter 6

Code Tests

6.1 Introduction

We have thus far given an account of the xCFC metric and the system of equations that govern it. Further, we have presented the methods that we have implemented to solve said system of equations. Now, we will present the results of several tests to validate the implementation presented.

As discussed in the previous chapter, there are two types of equations that govern the xCFC metric equations and we test each separately. Beginning with the non-linear scalar Laplacian-type equations, we demonstrate their ability to converge to a known solution in spherical symmetry. We also show the code’s ability to solve problems of this form with an axis-symmetric source using a homogenous ellipsoid. Next, we demonstrate the ability to solve the scalar vector Laplacian equations. Finally, we present the evolution of the Yahil profile to show the full xCFC system solve as well as to demonstrate the coupling with `thornado` hydrodynamics.

6.2 Hamiltonian Constraint Test

To test Poseidon’s ability to solve the conformal factor equation, and similarly the lapse function equation, we explore a problem based on the Hamiltonian constraint

found in [Baumgarte et al. \(2007\)](#). On a time-symmetric hyper-surface, i.e a hyper-surface where $K_{ij} = 0 \rightarrow \hat{A}^{ij} = 0$ everywhere, the Hamiltonian constraint, Eq. (3.40), simplifies to

$$\nabla^2 \psi = -2\pi\psi^5 E. \quad (6.1)$$

If E is limited to a homogeneous mass density ρ of star with radius R , and the conditions,

$$\psi > 0 \quad (6.2)$$

$$\psi \rightarrow 1, \quad r \rightarrow \infty, \quad (6.3)$$

are imposed, equation (6.1) has the analytic solution,

$$\psi(r) = \begin{cases} Cu_\alpha(r), & r \leq R \\ \frac{\beta}{r} + 1, & r > R \end{cases} \quad (6.4)$$

where

$$C = \left(\frac{2}{3}\pi\rho \right)^{-\frac{1}{4}}, \quad (6.5)$$

and

$$u_\alpha(r) = \sqrt{\frac{\alpha R}{r^2 + (\alpha R)^2}}. \quad (6.6)$$

The variables α and β are determined by the continuity of values and derivatives at the boundary, R . β is related to α by

$$\frac{\beta}{R} + 1 = Cu_\alpha(R), \quad (6.7)$$

and α is given by

$$\rho R^2 = \frac{3}{2\pi} f^2(\alpha), \quad (6.8)$$

where

$$f(\alpha) = \frac{\alpha^5}{(1 + \alpha^2)^3}. \quad (6.9)$$

$f(\alpha)$ is not globally unique but does take a unique maximum value at $\alpha_c = \sqrt{5}$. Therefore ρ is set by choosing using α_c and choosing R ,

$$\rho_c = \frac{3}{2\pi R^2} \frac{5^5}{6^6}. \quad (6.10)$$

Rather than solving Eq. (6.1), **Poseidon** solves the problem using the scaled source $E^* = \psi^6 E$ using some initial guess for ψ and iterates until the system converges. The solution is considered converged if the maximum amount the conformal factor or lapse function varies between iterations is less than 10^{-13} . Comparing results once convergence has been achieved, Figure 6.1 shows that the solver displays the expected rate of convergence for expansion degree.

6.3 Homogenous Ellipsoid Test

Next, we used a homogenous oblate ellipsoid as a source profile to test **Poseidon**'s ability to capture details of a two-dimensional problem. For a flattened, or oblate, ellipsoid with semi-major axes, a_1 and a_3 , and semi-minor axis a_2 , the gravitational potential inside is given by [Chandrasekhar \(1969\)](#) :

$$\Phi(r, \theta, \phi) = \pi G \rho \left[A_1 (2a_1^2 - r^2 \sin^2 \theta) + A_3 (a_3^2 - r^2 \cos \theta) \right], \quad (6.11)$$

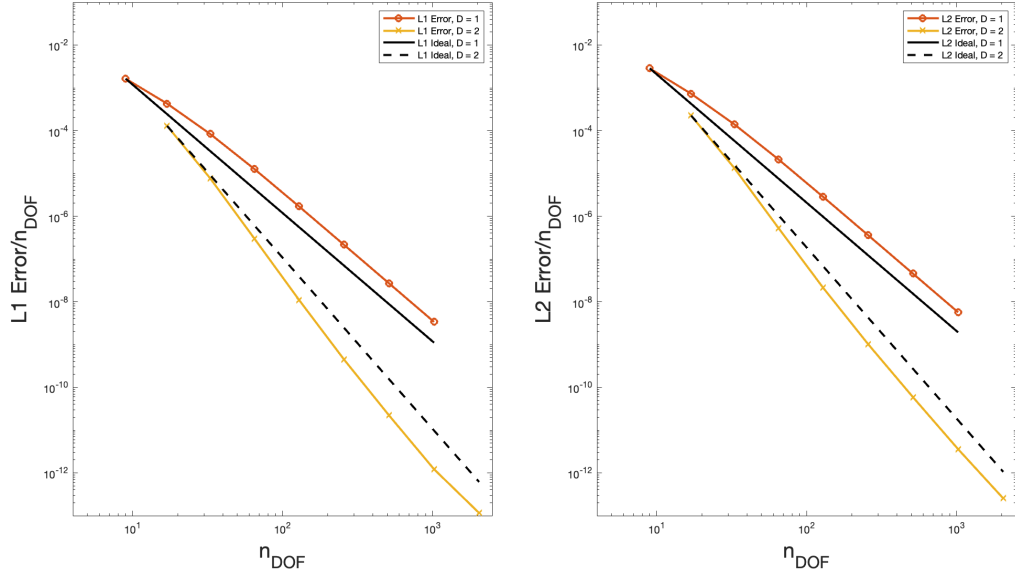


Figure 6.1: L^1 (left panel) and L^2 (right panel) errors between the analytic and numerical solution of the conformal factor for a uniform density sphere. The error norms are scaled by the number of degrees of freedom to obtain an average error per node. The dashed lines serve as a reference for the convergence rates of the numerical solutions.

where

$$A_1 = \frac{\sqrt{1-e^2}}{e^3} \sin^{-1} e - \frac{1-e^2}{e^2}, \quad (6.12)$$

$$A_3 = \frac{2}{e^2} - \frac{2\sqrt{1-e^2}}{e^3} \sin^{-1} e, \quad (6.13)$$

and

$$e = \sqrt{1 - \left(\frac{a_3}{a_1}\right)^2}. \quad (6.14)$$

Outside the surface of the spheroid, the potential is given by

$$\Phi(r, \theta, \phi) = \pi G \rho a_1 a_2 a_3 \int_{\lambda}^{\infty} \Psi^{-1} \left(1 - \frac{r^2 \sin^2 \theta \cos^2 \phi}{a_1^2 + u} - \frac{r^2 \sin^2 \theta \sin^2 \phi}{a_2^2 + u} - \frac{r^2 \cos^2 \theta}{a_3^2 + u} \right) du, \quad (6.15)$$

where

$$\Psi = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u), \quad (6.16)$$

and λ is the positive root of

$$\frac{r^2 \sin^2 \theta \cos^2 \phi}{a_1^2 + \lambda} + \frac{r^2 \sin^2 \theta \sin^2 \phi}{a_2^2 + \lambda} + \frac{r^2 \cos^2 \theta}{a_3^2 + \lambda} = 1. \quad (6.17)$$

The solutions given in equations (6.11) and (6.15) are for the Newtonian potential. To compare the results from **Poseidon**'s xCFC metric solve with this solution, we need to establish a relation between the conformal factor, lapse function, and Newtonian potential. When $\beta^i = 0$, the xCFC metric can be written isotropically. In this form and in the weak field limit, $|\Phi|/c^2 \ll 1$, the xCFC metric can be compared with the

isotropic Schwarzschild metric to define the desired relations, (Baumgarte, 2010)

$$\alpha(r) := \frac{1 + \frac{\Phi(r)}{2c^2}}{1 - \frac{\Phi(r)}{2c^2}} \quad (6.18)$$

and

$$\psi(r) := 1 - \frac{\Phi(r)}{2c^2}. \quad (6.19)$$

Figure 6.2 and Figure 6.3 show the results and relative errors of tests us using $a_1 = a_3 = 2500$ cm and $a_2 = 250$ cm and uniform density $\rho = 10^{16}$ g cm³. These values were chosen to keep the gravitational potential small so that approximation errors caused by mapping the gravitational potential to the conformal are minimal. As the limit on the spherical harmonic expansion increases, the code better captures the shape of the solution which can be seen in Table 6.1

Lastly, we show the benefits of mesh refinement by comparing the results using a uniform mesh with those that have been refined. The middle plot of Figure 6.4 shows the relative error maps for a uniform grid with 1024 uniform radial elements and 256 theta elements per radial shell. The right-most plot shows the results using a refined grid with 156 total radial leaf elements across three levels of refinement and 256 theta elements on the finest level. Given these configurations, the refined grid uses 6.5 times fewer degrees of freedom to capture the solution with comparable accuracy to the uniform grid solution.

6.4 Conformal Vector Laplacian Test

To test Poseidon's ability to solve the equations governed by the conformal vector Laplacian, a resolution study was performed using the X vector equation, 4.29a using the Yahil self-similar collapse profile (see Section 6.5 for details.) For this test, we began by choosing a domain decomposition and solver method that we felt confident

Table 6.1: Error norms for the Homogenous Ellipsoid problem run with 706 radial leaf elements on 4 levels of refinement and 256 theta elements at the finest resolution.

Effects of Increasing Spherical Harmonic Expansion			
$\mathcal{L}$	L^1 Error	L^2 Error	L^∞ Error
0	7.64E-08	1.13E-07	3.50E-07
2	1.97E-08	3.57E-08	1.64E-07
4	8.24E-09	1.63E-08	9.77E-08
6	4.41E-09	9.11E-09	6.53E-08
8	2.72E-09	5.52E-09	4.67E-08

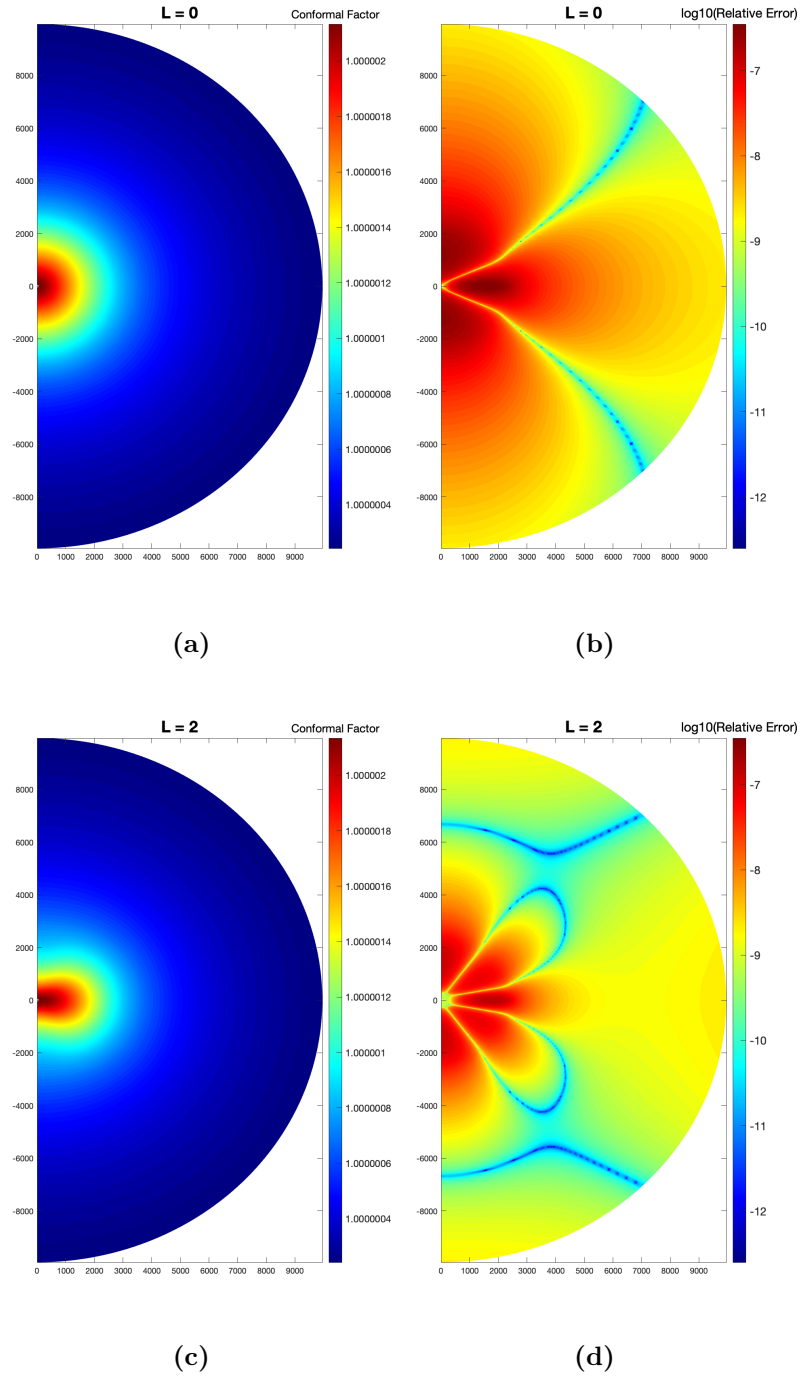
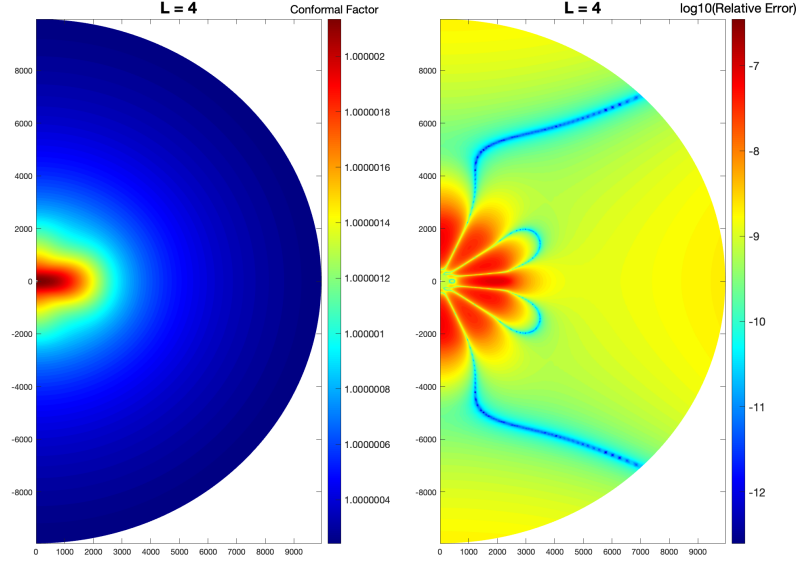
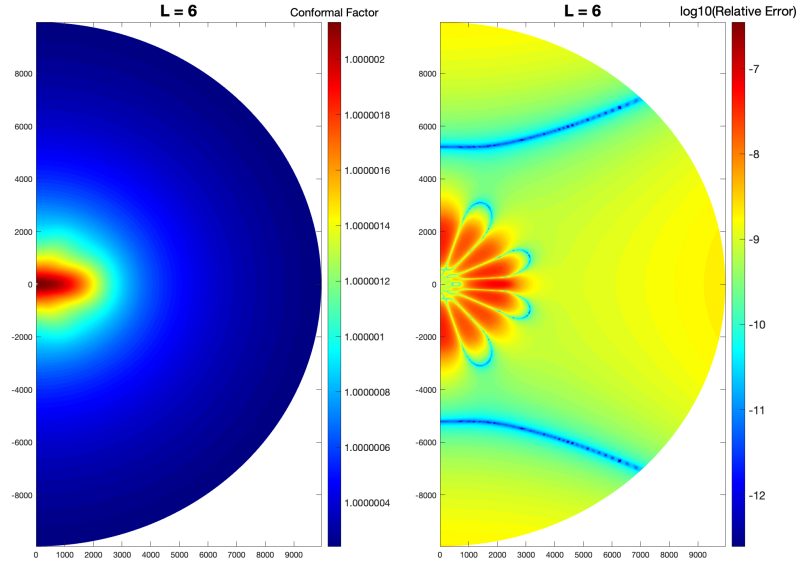


Figure 6.2: Results from Poseidon using a homogenous ellipsoid as the source profile. This test demonstrates the code's ability to capture multidimensional details using different limits on the spherical harmonic expansion. The left column shows the Conformal Factor as calculated by Poseidon while the right column shows the relative error.



(a)

(b)



(c)

(d)

Figure 6.3: Results from *Poseidon* using a homogenous ellipsoid as the source profile. This test demonstrates the code's ability to capture multidimensional details using different limits on the spherical harmonic expansion. The left column shows the Conformal Factor as calculated by *Poseidon* while the right column shows the relative error.

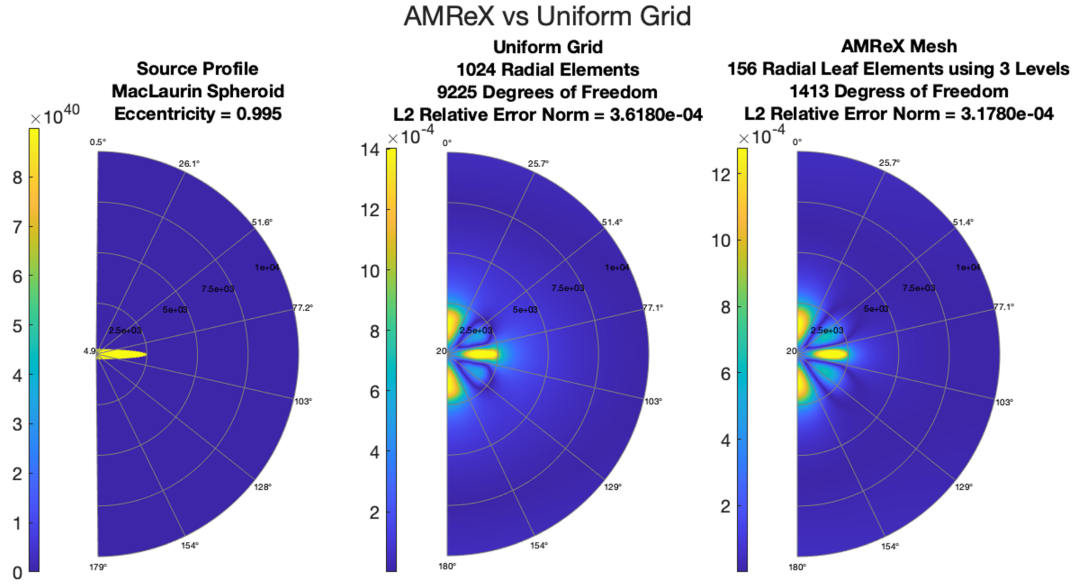


Figure 6.4: Comparable results can be achieved with a refined mesh using far fewer elements than with a uniform mesh.

captured the solution. This reference resolution used 524,288 radial elements and second-order finite element polynomials for a total of 1,048,577 degrees of freedom. Using this resolution, a reference solution for X^i was calculated using **Poseidon**.

Next, the solution was calculated at lower resolutions, starting with 512 elements and increasing to 16,384, using both first and second-order finite element polynomials. Figure 6.5 shows the relative L^1 and L^2 norms of the difference, $\Delta u = |u_h - u_R|$, between the low resolution solutions and the reference solution. Based on these results, we are confident in the conformal vector Laplacian solve.

6.5 Yahil Self-Similar Collapse Test

The first application is the collapse of a polytropic star; i.e. a star where the pressure p and density ρ obey the relation $p = \kappa \rho^\gamma$, where κ is the polytropic constant. We use the Yahil self-similar collapse profile (Yahil, 1983; Yahil and Lattimer, 1982). This profile is representative of CCSN profiles as it smoothly connects a supersonically collapsing outer core to a subsonically collapsing inner core. The profile is determined by choosing the polytropic constant κ , the adiabatic index γ , and a time parameter, t , from which a dimensionless similarity variable is calculated,

$$X = \kappa^{-1/2} G^{(\gamma-1)/2} r(-t)^{\gamma-2}, \quad (6.20)$$

where G is the gravitational constant, and $t = 0$ is the moment of infinite central density. From X , the density ρ and velocity v can be calculated and used to generate the xCFC source variables. For our test, the polytropic constant κ is determined by choosing a reference pressure $p = 6 \times 10^{27}$ erg cm⁻³ and density $\rho = 7 \times 10^9$ g cm⁻³ (values typical for a pre-collapse massive star) and the adiabatic index is chosen to be $\gamma = 1.3$.

Figure 6.6 shows the results of the Yahil collapse simulation. Each set of lines corresponds to a time when the central density first arrives at a new decade of density,

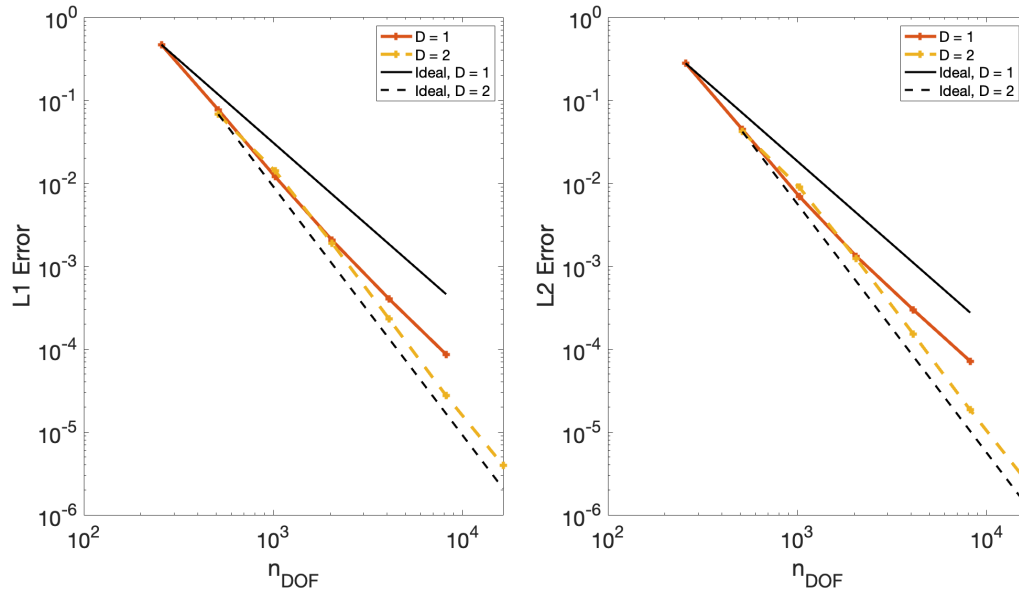


Figure 6.5: The L^1 (Left) and L^2 (Right) norms of the relative difference between low-resolution solutions and a very fine-resolution reference solution are represented by the solid lines. The dashed lines give the norms that one would expect based on the convergence properties of the finite element method. Different colors represent the degrees of finite element polynomials used to approximate the solution. In both cases, we see the expected convergence rate for this method.

i.e. when $\rho = [10^9, 10^{10}, 10^{11}, 10^{12}, 10^{13}, 10^{14}] \text{ g cm}^{-3}$. The domain is spherical polar and extends from the origin to $1 \times 10^5 \text{ km}$. The domain is decomposed using adaptive mesh refinement through the coupling with **AMReX**. The coarsest level has 512 elements with a width of 195 km. Mesh refinement is done by density decade, starting at 10^6 g cm^{-3} and continuing up to $10^{14} \text{ g cm}^{-3}$, allowing for a total of 9 levels of refinement. Refinement in this fashion provides higher resolution closer to the origin with the finest elements having a width of 0.381 km.

As discussed in Section 6.3, when $\beta^i = 0$ and $|\Phi|/c^2 \ll 1$, a relationship between the Newtonian potential and conformal factor exists,

$$\psi(r) := 1 - \frac{\Phi(r)}{2c^2}. \quad (6.21)$$

Figure 6.7 shows results from early in the Yahil collapse. At these times, the Newtonian potential is small and although the shift is non-zero, it is sufficiently small that its impact is negligible. These results show that when the system satisfies the limiting assumptions, there is good agreement between the simulated results and the predicted solution. As the system intensifies, the approximation begins to break and we start to see the simulated results become more extreme than the weak-field limit predicts.

As discussed in Section 5.3.2, the choice of memory parameter, M , is important to the efficiency of the Anderson acceleration fixed point method. Generally, the choice of M is dependent on the problem being solved. Therefore, to determine a suitable value for M , **Poseidon** was run using the Yahil source profile with varying values for M . To maximize the number of iterations, a late time, $t = -1.5 \text{ ms}$, in the Yahil collapse was chosen and the flat space values, $\alpha\psi = \psi = 1.0$ were used as an initial guess. Figure 6.8 shows the number of fixed point iterations required for a given residual under these conditions. A significant reduction in iterations occurs by using information from just one previous iteration, but little improvement is observed

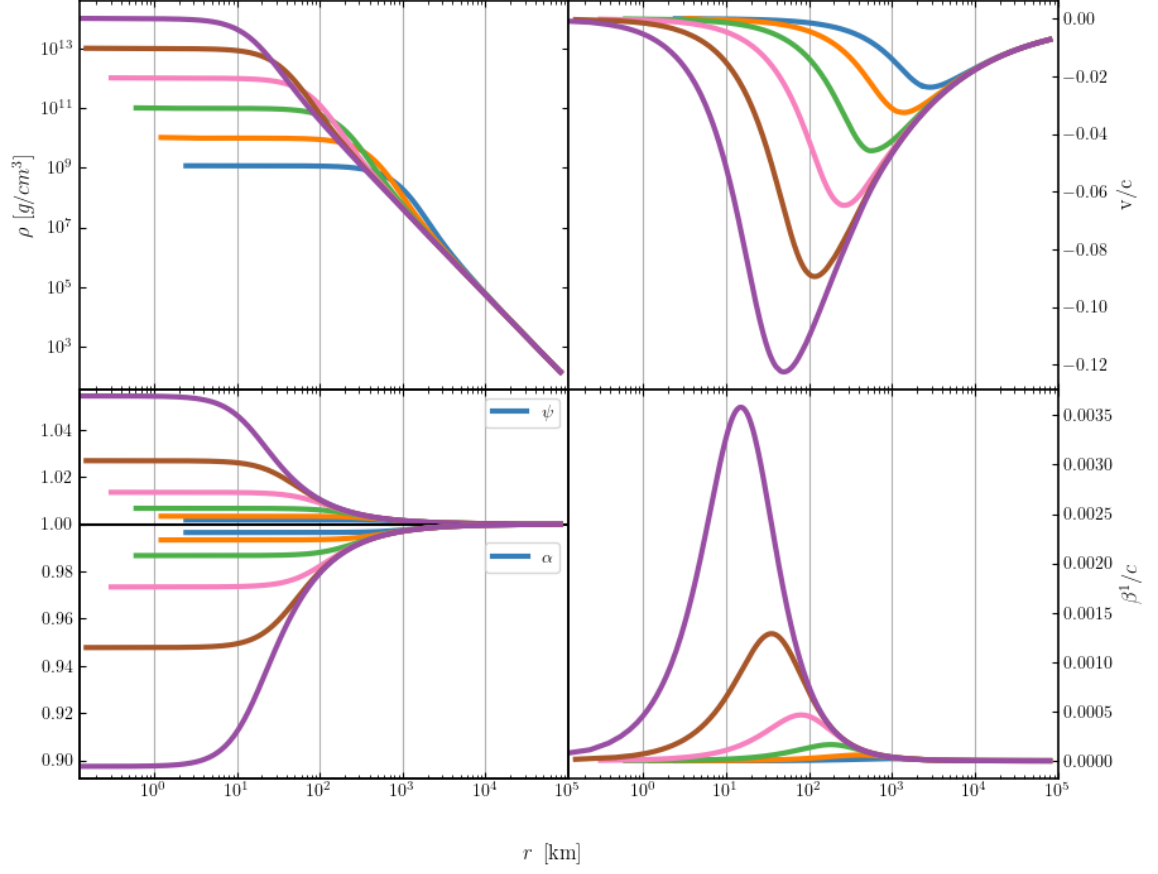


Figure 6.6: Results from `thornado-Poseidon` showing the collapse of the Yahil progenitor using an ideal equation of state. The upper left and upper right panels show the fluid density and velocity respectively. The lower left panel shows the conformal factor, ψ , and lapse function, α , while the lower right presents the radial component of the shift vector.

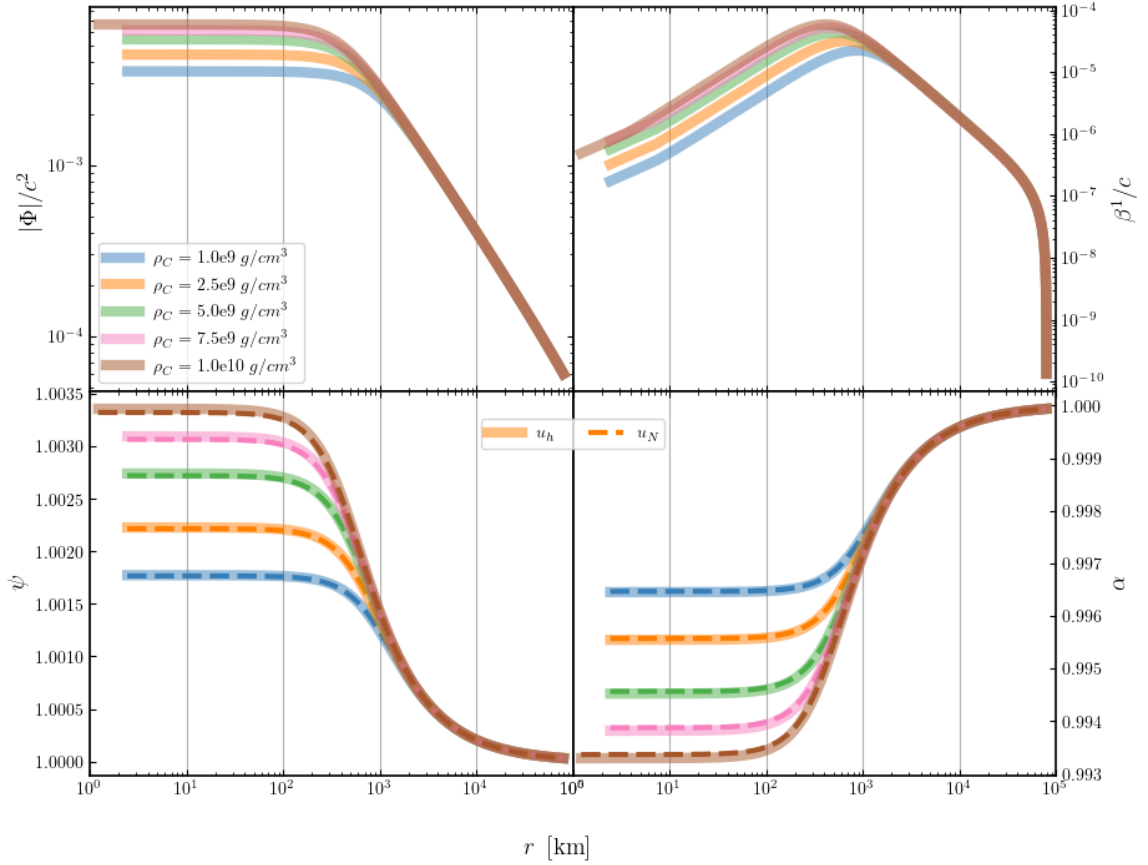


Figure 6.7: Here, we compare the simulated conformal factor and lapse function (solid lines) with the values predicted by the Schwarzschild metric in Isotropic coordinates (dashed lines). This comparison is only appropriate in the weak-field limit and while $\beta/c \ll 1$ as can be seen by the increasing difference between predicted and simulated results.

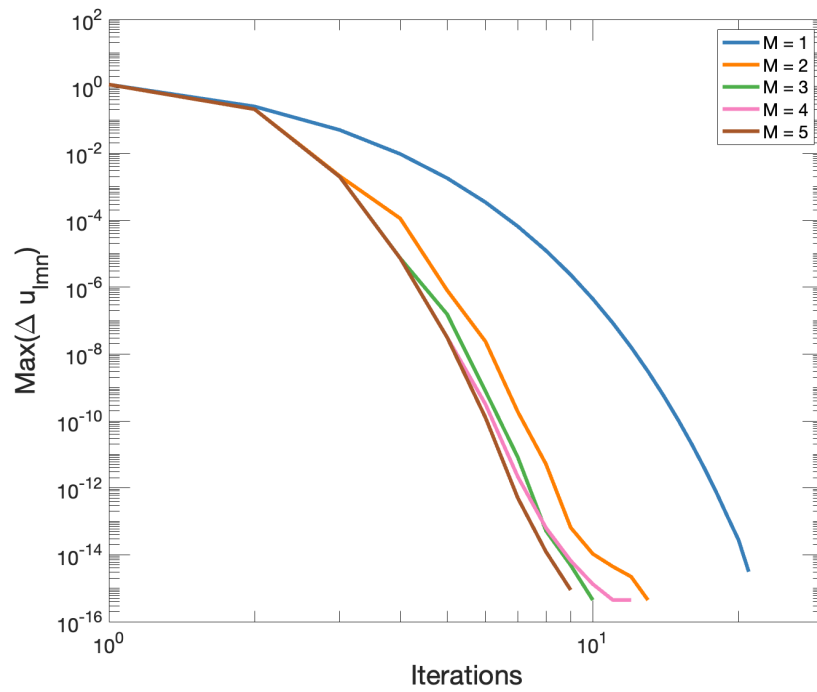


Figure 6.8: Anderson acceleration is used to reduce the number of fixed point iterations needed to converge to a solution. This figure shows the effects of using information from the previous M iterates to update the solution.

if more than three previous steps are used. Therefore `Poseidon` uses $M = 3$ as the default memory parameter for Anderson acceleration.

6.6 Conclusion

In this chapter, we have presented the results from tests that demonstrate `Poseidon`'s abilities. We have shown the ability to solve both types of equations governing the xCFC metric equations in spherical symmetry. Likewise, we have demonstrated the code's ability to work with axis-symmetric data using a static homogenous ellipsoid. These tests validate `Poseidon`'s ability to perform the desired solves.

Next, we presented results from with `Poseidon` incorporated into `thornado` using the Yahil profile as the source. This test demonstrated the coupling of the two codes and their ability to evolve physical systems using relativistic gravity and hydrodynamics. In the early stages of the collapse, we were able to use the weak field relations between the Newtonian potential and xCFC metric variables to show that the returned solutions matched expectations in that limit. Further, we used a single moment of the Yahil collapse to explore the ideal memory parameter for Anderson acceleration of our fixed point iterator. In all, we feel that these results show that the `Poseidon` code is capable of solving the xCFCmetric system accurately and can function within the `thornado` framework.

Chapter 7

General Relativistic Adiabatic Collapse in Spherical Symmetry

In this section, we will present the results from the simulation of the general relativistic adiabatic collapse of $15 M_{\odot}$ and $37 M_{\odot}$ progenitors. The simulation of a core-collapse supernova is a complex endeavor requiring relativistic gravity, hydrodynamics, neutrino transport, and nuclear physics. The relativistic neutrino transport module of **thornado** is still in development, therefore we can not present a true CCSN simulation at this time. In the absence of neutrino radiation, the entropy per fluid element remains constant and the gravitational collapse can be classified as adiabatic. These simulations test the full current capabilities of the **thornado** and **Poseidon** codes. In the future, the data from these simulations will be used as a base of comparison for runs with neutrinos, which will help identify what aspects of the CCSN mechanism are affected by their inclusion.

Similar to the Yahil collapse test, these simulations are performed using **thornado**'s xCFC gravity and hydrodynamics. Unlike the Yahil collapse, which uses an ideal equation of state (EOS), these adiabatic collapse simulations use a tabulated equation of state from the **WeakLib** library. Further, these simulations use realistic progenitor

profiles from [Woosley and Heger \(2007\)](#) and [Heger and Woosley \(2010\)](#). These results therefore represent the culmination of several significant astrophysical endeavors.

7.1 Collapse of a 15 $M_{\odot}$ Star

First, we examine the evolution of a non-rotating 15 solar mass, $M_{\odot}$, solar metallicity progenitor from [Woosley and Heger \(2007\)](#) as it progresses through collapse, core bounce, and post-bounce shock propagation. The progenitor data is linearly interpolated onto a spherical polar grid that includes the origin and out to a maximum radius of 8,192 km covering the inner region of the star around the iron core. Linear polynomials are used to approximate variables in both `thornado` and `Poseidon`, and a second-order SSPRK method is used for time integration.

7.1.1 Validation of AMR Implementation

Here we will compare the results from a single-level mesh simulation to a six-level mesh simulation. The meshes are chosen such that the uniform resolution of the single-level mesh, 0.5 km, is the same as the finest resolution of the six-level mesh, 0.5 km. Given these choices in resolution, the uniform grid divides the radial domain into 16,384 elements while the AMR grid uses a variable number of elements. Initially, the multi-level system begins with 548 leaf elements across 2 levels of refinement. As the system evolves and becomes more centrally compact, refinement boundaries move outward and new levels are added at the origin until a maximum of 624 leaf elements are present on 6 levels. The maximum occurs near bounce and relaxes afterward but never returns to its initial configuration, see [Figure 7.1](#).

Given the reduced number of elements used to discretize the system, a reduced computational cost and run time is expected. For both meshes, the simulation was evolved for 150 ms. The uniform mesh took 26 hours, 2 minutes, and 16 seconds using 6 ranks on Oak Ridge National Laboratory’s Summit supercomputer, while the AMR

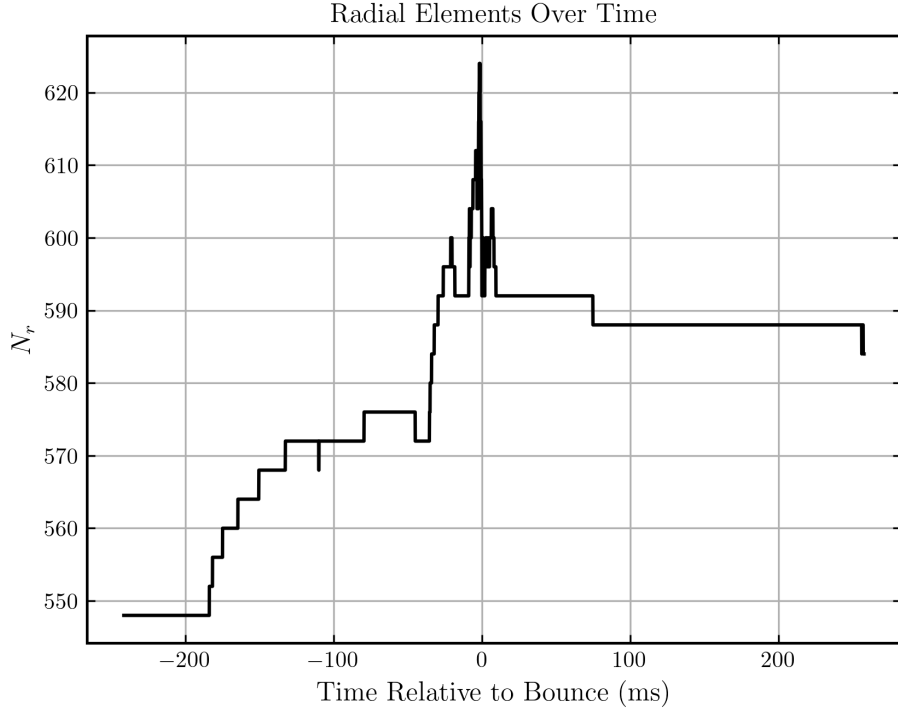


Figure 7.1: The number of radial elements during collapse and post-bounce. The domain is refined based on density values therefore as the star collapses and becomes more compact, more cells are refined to higher degrees. This leads to an increasing number of leaf elements during collapse which peak at bounce. In the post-bounce regime, there is an initial drop followed by long periods when the mesh is unchanging.

mesh took 14 hours, 40 minutes, and 44 seconds to complete. The ratio between the runtimes times is 0.56 AMR/Uniform.

Figure 7.2 shows the results during the initial collapse. The results are similar, with the primary difference being slightly more extreme values occurring in the six-level system as the simulation approaches bounce. The results continue to agree well in the post-bounce regime shown in Figure 7.3. The disagreements between the earliest two times, most notably seen in the radial shift plot, are likely due to minor differences in sample time during the most dynamic period of the simulation.

7.1.2 Resolution Study

Confident in `thornado`'s AMR implementation, a resolution study was performed next by varying the maximum level of refinement used to simulate the evolution of the $15 M_{\odot}$ progenitor. Beginning with 512 radial elements on the coarsest level, the refinement of the domain was then limited to 5, 6, and 7 levels. The width of the finest elements were then 1 km, 0.5 km, and 0.25 km respectively. Figure 7.4 contains snapshots of the collapse phase for the three simulations overlaid. We see excellent agreement throughout the collapse with minor differences in the innermost temperature values just before bounce.

Figure 7.5 shows the post-bounce evolution of the three cases. Here we also see good agreement between the finer two resolutions. The coarsest simulation shows differences in the late post-bounce regime that we believe originate during bounce and become more obvious as they evolve. We conclude from these results that a minimum resolution, less than 1km per element, is needed in the core to capture the dynamics of core bounce accurately.

Adiabatic Collapse of a $15 M_{\odot}$ Star

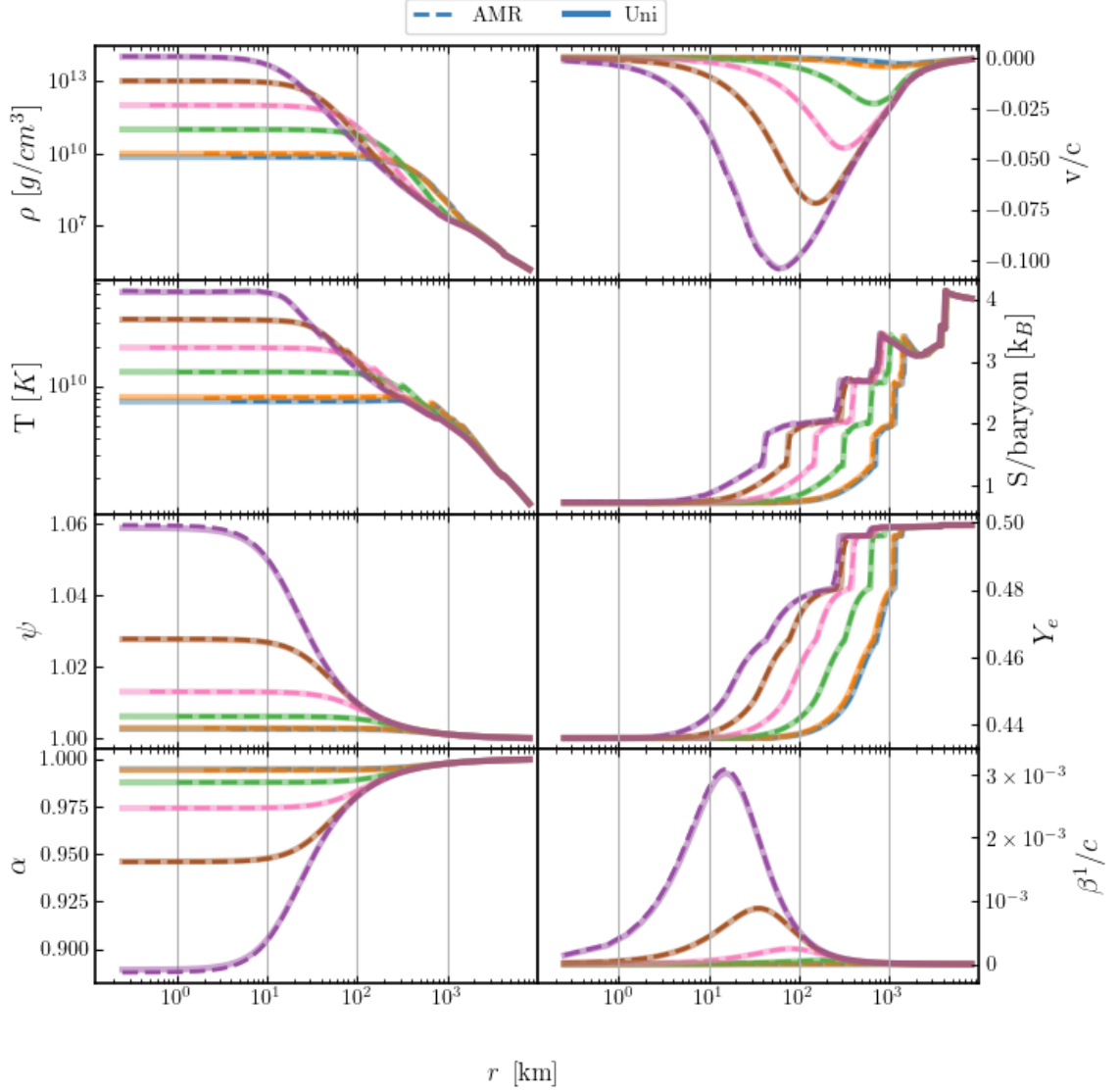


Figure 7.2: Adiabatic collapse of a $15 M_{\odot}$ star using a uniform grid and an AMR grid. (Top Left) Mass density in g cm^{-3} , (Top Right) fluid velocity normalized to the speed of light, (Upper Middle Left) temperature (K), (Upper Middle Right) entropy per baryon in k_B , (Lower Middle Left) conformal factor, (Lower Middle Right) electron fraction, (Lowest Left) lapse function, (Lowest Right) radial component of the shift vector are plotted as functions of radius in km. The solutions are compared during collapse when the central density is approximately $[10^{10}, 10^{11}, 10^{12}, 10^{13}, 10^{14}] \text{ g cm}^{-3}$.

Post-Bounce Phase of a $15 M_{\odot}$ Star

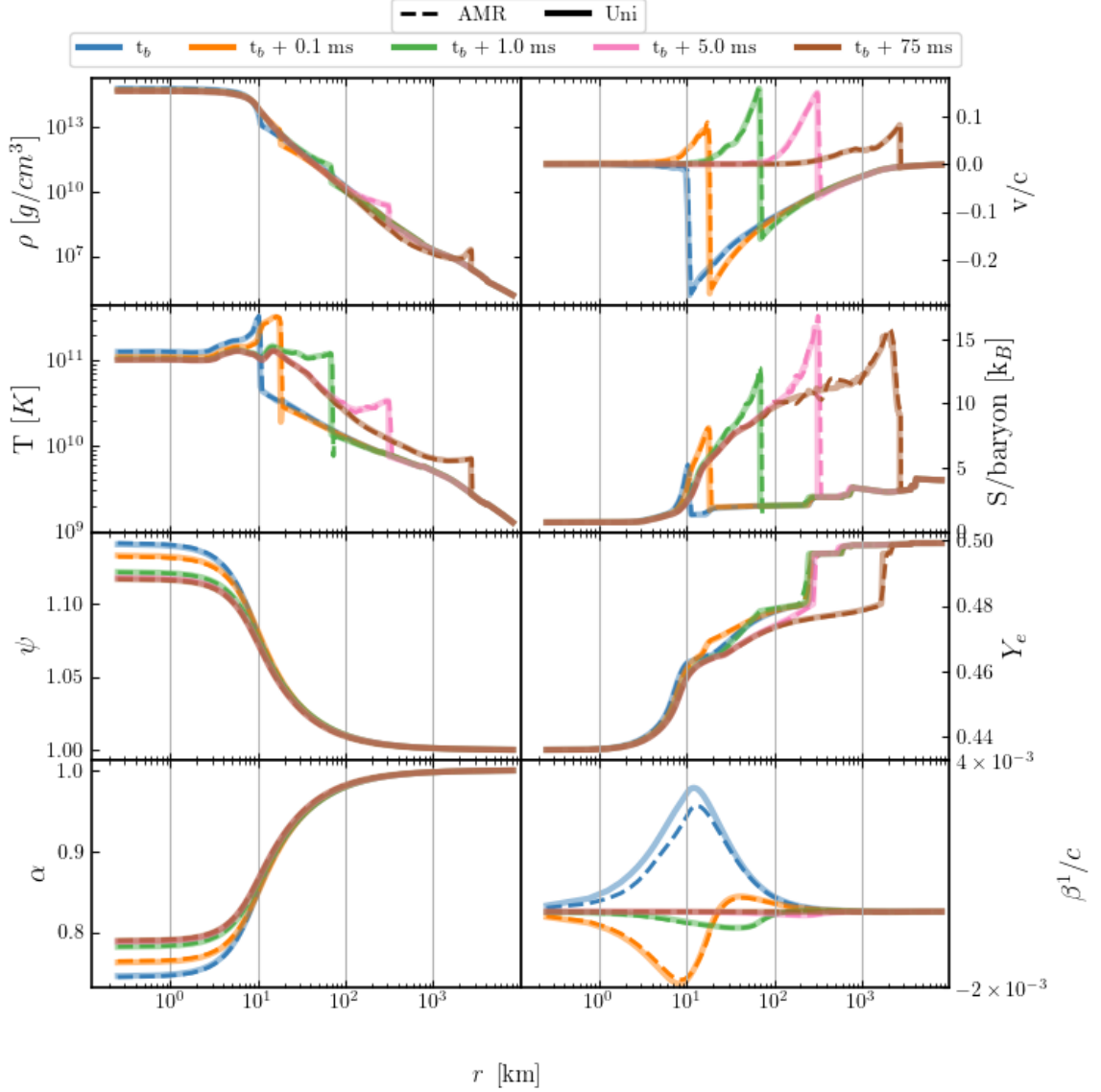


Figure 7.3: Post-bounce evolution of a $15 M_{\odot}$ star using a uniform grid and an AMR grid. (Top Left) Mass density in g cm^{-3} , (Top Right) fluid velocity normalized to the speed of light, (Upper Middle Left) temperature (K), (Upper Middle Right) entropy per baryon in k_B , (Lower Middle Left) conformal factor, (Lower Middle Right) electron fraction, (Lowest Left) lapse function, (Lowest Right) radial component of the shift vector are plotted as functions of radius in km. The solutions are compared at intervals measured relative to core bounce.

Adiabatic Collapse of a $15 M_{\odot}$ Star

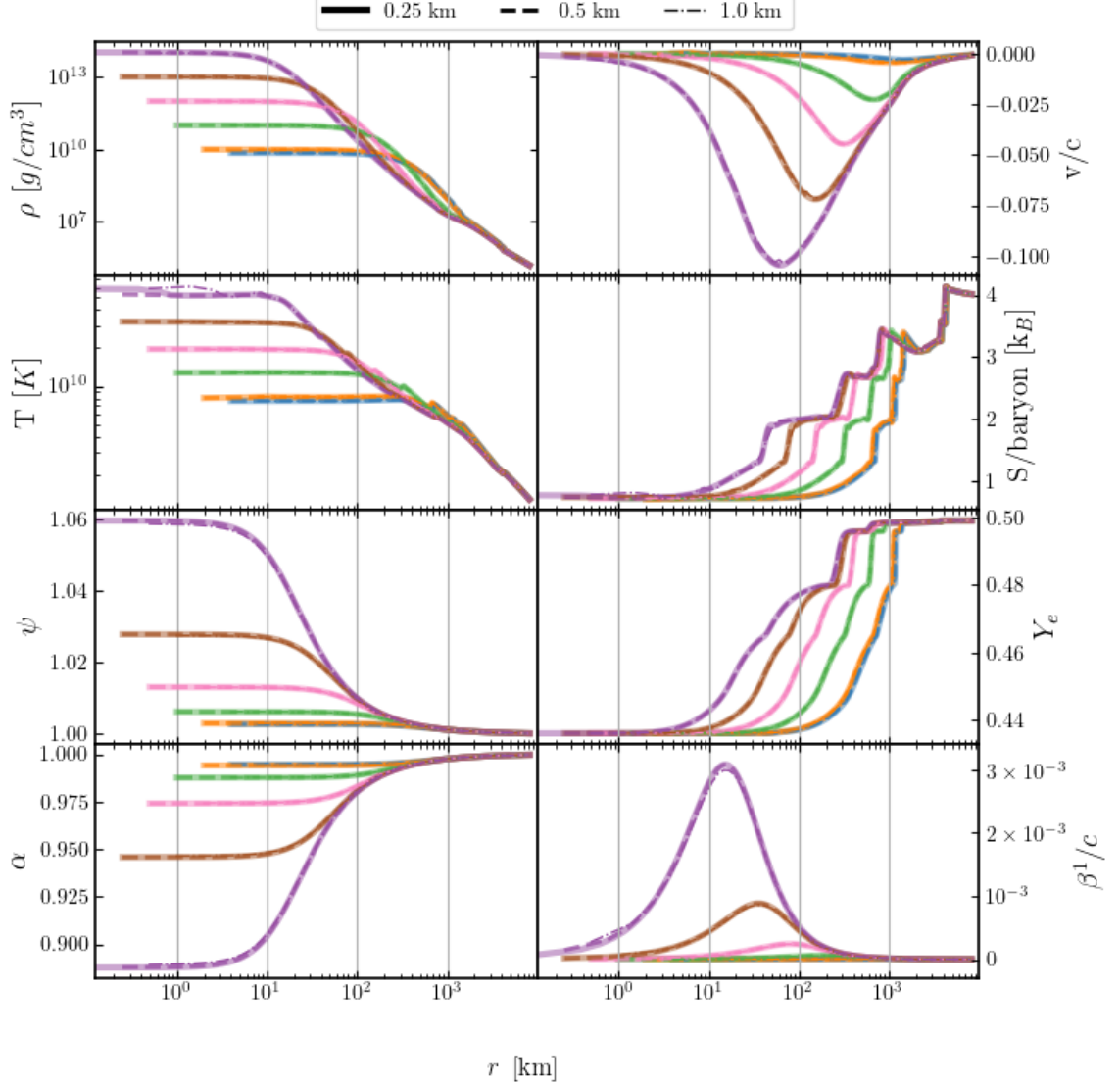


Figure 7.4: Results from the adiabatic collapse of a $15 M_{\odot}$ star run with different maximum levels of refinement. The resolution of the simulations on the finest levels are 1km, 0.5km, and 0.25km given by the different line styles. The different colors represent different moments during collapse corresponding to the first instant the central density reaches a new decade. The panels show the same variables as in Figure 7.2

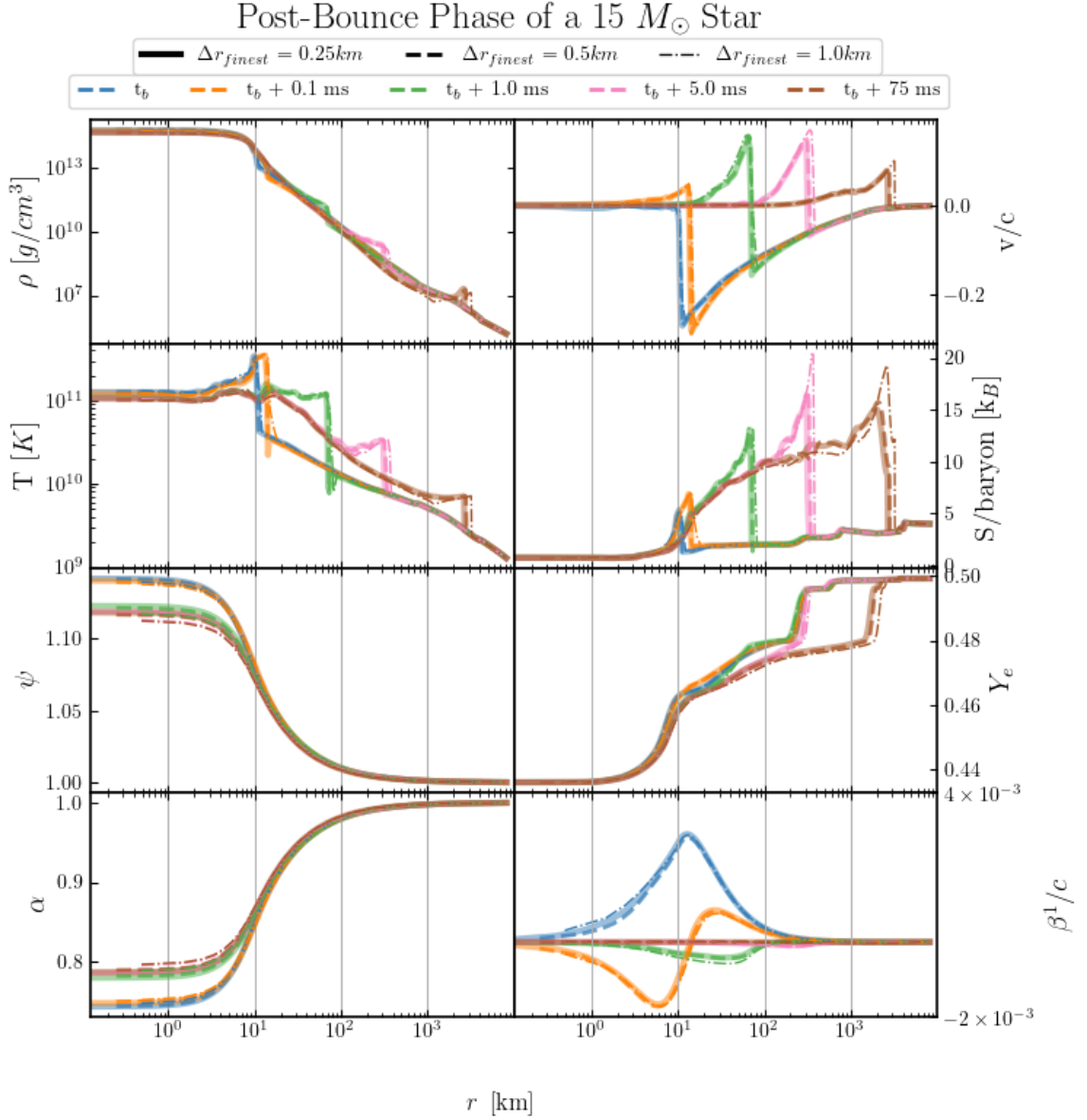


Figure 7.5: Post-bounce phase of the collapse of a $15 M_{\odot}$ star run with different maximum levels of refinement. The resolution of the simulations on the finest levels are 1 km, 0.5 km, and 0.25 km given by the different line styles. The different colors represent different moments during collapse corresponding to the first instant the central density reaches a new decade.

7.2 Collapse of a $37 M_{\odot}$ Star

In addition to the 15 solar mass star, we evolved a $37 M_{\odot}$ metal-free progenitor from [Heger and Woosley \(2010\)](#). In [Figure 7.6](#), we see the state of the fluid and metric variables at two times in the post-bounce regime. The first time is at core bounce when the system is at its most extreme, while the second sample was taken 100 ms after bounce. At both times, we see similar fluid profiles but the large star shows greater values for all metric variables.

Both the hydrodynamics of **thornado** and the gravity of **Poseidon** have been developed to run using different degrees of finite element polynomials. To test the combined frameworks' ability to simulate gravitational collapse with different degree expansions, the $37 M_{\odot}$ simulation was performed with second- and third-order methods. [Figure 7.7](#) shows the results from these two runs. In general, we see good agreement between the two sets of results with the main difference being the higher-order showing finer details in the entropy plots.

These simulations fall short of true core-collapse supernova simulations due to the lack of the still-in-development neutrino transport module. Once that capability has been coupled to the relativistic gravity and hydrodynamics, **thornado** will be capable of fully simulating CCSNe. These adiabatic collapse runs will serve as fiducial points of comparison at that time, allowing the impact of neutrinos to be observed. Additionally, multi-dimensional capabilities have been developed and are currently being tested, which will allow future simulations to explore the effects of the degrees of freedom higher dimensions allow.

7.3 Performace

Solving the xCFC metric equations requires

For the simulations presented above, the gravity solve takes between 38% and 47% of the computational time. One of the earliest efforts to reduce the cost of

Post-Bounce Comparison of $15 M_{\odot}$ and $37 M_{\odot}$ Stars

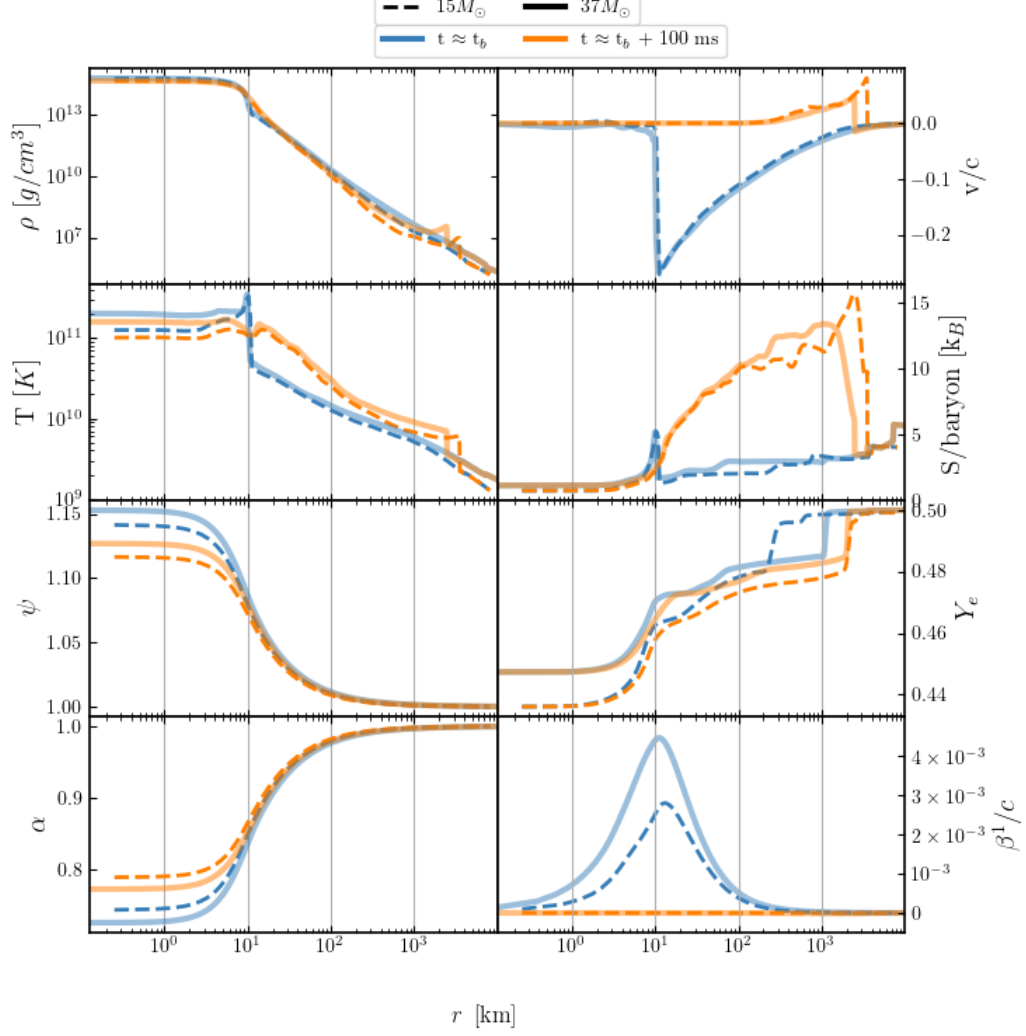


Figure 7.6: Post-bounce evolution of a 15 solar mass star and 37 solar mass star using an AMR grid. (Top Left) Mass density in grams per cubic centimeter, (Top Right) fluid velocity normalized to the speed of light, (Upper Middle Left) temperature, (Upper Middle Right) entropy, (Lower Middle Left) conformal factor, (Lower Middle Right) electron fraction, (Lowest Left) lapse function, (Lowest Right) radial component of the shift vector are plotted as functions of radius in kilometers. The solutions are compared at bounce and after 100 ms of post-bounce evolution.

Adiabatic Collapse of a $37 M_{\odot}$ Star

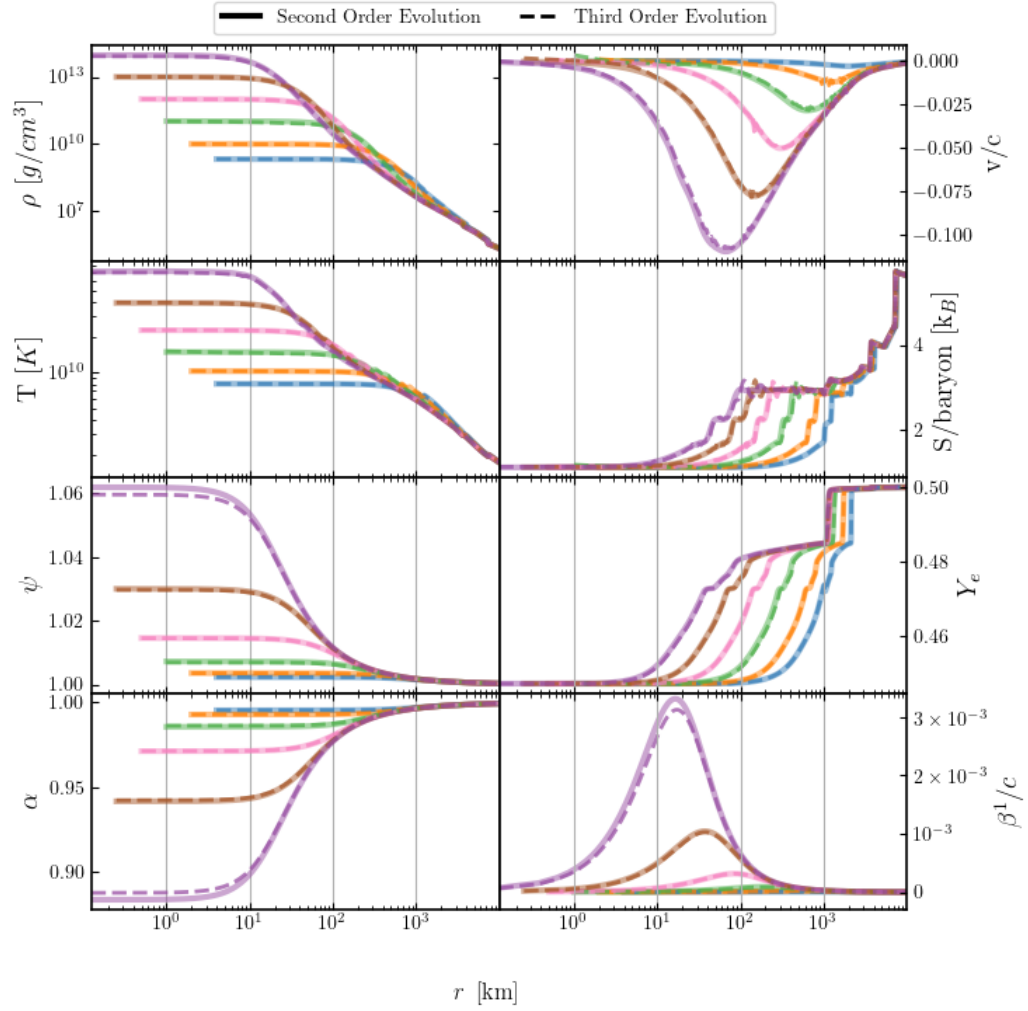


Figure 7.7: Collapse of a 37 solar mass star using different order methods. The panels relate the same variables as in Figure 7.2. The solutions are plotted as each run reaches a new central density decade.

the gravity solve was the conversion from complex spherical harmonic functions to real-valued spherical harmonic functions discussed in Section 5.3.1. Using the real spherical harmonics allows the stiffness matrices, load vectors, and coefficient vectors to be stored as double-precision reals rather than double-precision complex numbers which provides a memory and computational savings. Furthermore, efforts to adapt routines within *Poseidon* to GPUs have begun through the use of OpenMP and OpenACC pragmas.

Another method for reducing the cost of gravity calculations during a collapse simulation is to perform intermittent solves. Using the adiabatic collapse, we explored the effects of performing the gravity solve after a fixed number of steps. During the collapse and post-bounce phases, skipping the gravity solve for up to 50 hydrodynamics time steps was shown to have minimal impact on the evolution as shown in Figure 7.8.

As the simulation approaches bounce, the rate of change in the gravity fields becomes so fast that performing even one hydrodynamics timestep without a corresponding gravity solve will lead to discrepancies. Clearly, a static spacing of the gravity solve other than every cycle would be inappropriate. Therefore, a scheme was developed that calculates an approximate rate of change in the conformal factor based on the change in central density. The number of steps between solves is varied according to the rate of change of the conformal factor. This method has been implemented but is currently being tested. Once verified, this will reduce the cost of gravity solves throughout the simulation.

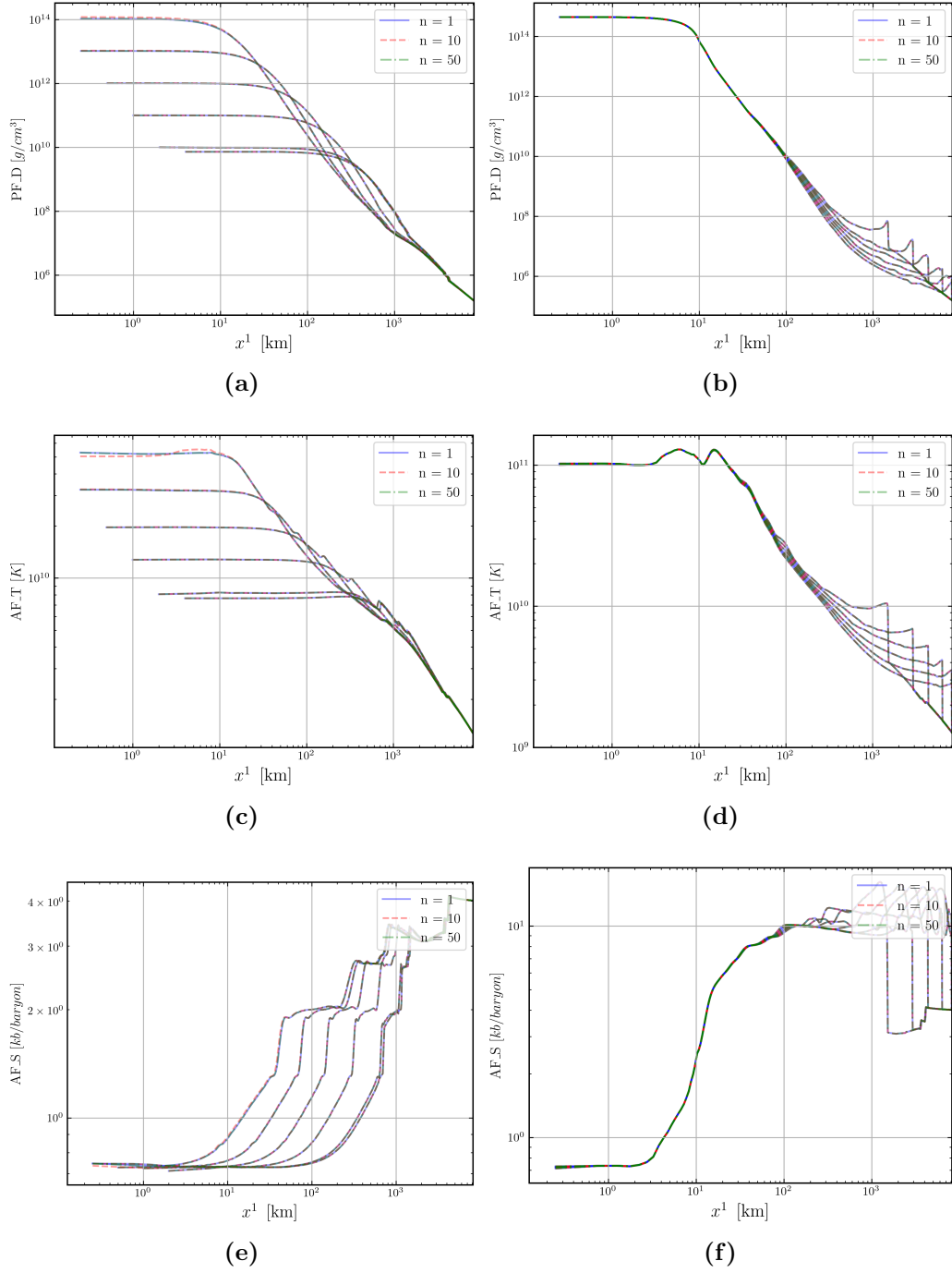


Figure 7.8: Comparisons of the density (panels A and B), Temperature, (panels C and D), and Entropy (panels E and F) during the pre-bounce collapse (left panels) and post-bounce shock propagation (right panels) for the adiabatic collapse of a $15 M_{\odot}$ star. We see good agreement between the runs. Spacing out gravity solves without impacting the evolution of the star will help reduce computational costs.

Chapter 8

Conclusion

8.1 Summary

Herein, we have presented the means and methods of the `Poseidon` gravity solver code, developed as part of the next-generation, state-of-the-art `thornado` framework for simulating core-collapse supernovae. CCSNe are cosmic refineries, turning hydrogen into heavier elements and scattering the ash across the universe. Understanding the mechanisms of CCSNe requires simulating complex multi-physics interplay across several orders of magnitude of space and energies. Together `thornado` with `Poseidon` will be capable of simulating CCSNe, giving us insight into these events.

The first challenge in developing `Poseidon` was working through the formulation of the physics. In Chapter 2, we followed [Gourgoulhon \(2012\)](#) recasting of the Einstein equations in the 3+1 formalism. We then made approximations of [Isenberg \(2008\)](#) and [Wilson and Mathews \(1989\)](#) in Chapter 3, limiting the spatial metric to be conformally flat and restricting the evolution of the extrinsic curvature tensor, and arrived at the set of equations for the CFC metric. Next, we made a further approximation on the extrinsic curvature tensor in Chapter 4 and derived the xCFC metric equations following [Cordero-Carrión et al. \(2009\)](#).

With the metric equations in hand, we next faced the challenges of discretization and development in Chapter 5. The xCFC equations come in two distinct forms, non-linear scalar Laplacians and linear vector Laplacians, therefore we had to develop methods for solving each. To begin, the conformal factor and lapse function equations were linearized using fixed point iterations with Anderson acceleration. Galerkin projections using real spherical harmonics and radial finite element polynomials were used to discretize both types of equations, resulting in diagonal linear systems. Next, we incorporated **Poseidon** into the **thornado** framework allowing the relativistic simulation of gravitational collapse. Further, both codes were adapted to use adaptive mesh refinement through the **AMReX** framework.

To test **Poseidon**’s abilities, we presented results from tests in Chapter 6. First, we showed the code’s ability to solve both equation types in 1 dimension with the expected convergence rate. Two-dimensional results of the conformal factor were also presented, demonstrating the ability to capture multi-dimensional details. Next, we simulated the gravitational collapse of the Yahil profile with a tabulated equation of state using the combined capabilities of **thornado** with the **Poseidon** gravity solver.

In Chapter 7, we presented results from the simulation of the relativistic adiabatic collapse of a $15 M_{\odot}$ star and a $37 M_{\odot}$ star. These simulations took a realistic stellar progenitor and evolved it through collapse, core-bounce, and post-bounce shock propagation using xCFC gravity and hydrodynamics coupled with the tabulated equation of state, **WeakLib**. These results represent a significant achievement and will provide fiducial data for comparison when the neutrino transport module of **thornado** is complete.

8.2 My Contributions

The author’s primary contribution is being the sole developer of the **Poseidon** gravity solver. Initially, this included developing and implementing the numerical method for the CFC metric equations. After the CFC method was developed, the author’s testing

revealed that the non-uniqueness issues would affect the simulations and he therefore suggested pivoting to xCFC. The differences between the CFC and xCFC metric system of equations required the author to develop new methods for solving the new systems and coupling those methods with **AMReX**.

The author also aided in the development of the **thornado** framework. Their primary contribution was aiding in the coupling of **thornado** hydrodynamics with the **Poseidon** gravity solver. This coupling provided the **thornado** framework with the ability to use the xCFC metric variables as well as the Newtonian gravitational potential for use in CCSNe simulation in the Newtonian limit. Further contributions include aiding in the coupling of **thornado**'s hydrodynamics module with **AMReX**, and the development of data processing and visualizing scripts. The author is also responsible for the 1D adiabatic collapse simulations using the $15 M_{\odot}$ and $37 M_{\odot}$ progenitors run with the **thornado-Poseidon** framework in parallel with adaptive mesh refinement on the Oak Ridge Leadership Computing Facility (OLCF) - Summit supercomputer.

8.3 Outlook

A goal of the **thornado** framework is to simulate core-collapse supernovae in three dimensions. Currently, a relativistic neutrino transport module is being developed which will allow full CCSNe simulations to be performed. While that module is being developed, the coupled gravity and hydrodynamics modules are being tested in axis-symmetry. Multi-dimensional simulations will allow for more complex hydrodynamic phenomena such as convection and the SASI which have been shown to impact CCSNe evolution. Multi-dimensional simulations bring new challenges as the additional dimension greatly increases the degrees of freedom. To prepare for the extra computational cost, future work will include porting more of **Poseidon**'s routines to GPUs and the coupling of a distributed memory linear solver. Additionally, efforts have already been made to explore the effects of updating the metric terms

intermittently during simulations. These steps will hopefully enable **thornado-Poseidon** to simulate stellar collapse in two dimensions efficiently and set the stage for eventual three-dimensional simulations. In addition to preparing **Poseidon** for the complexity of multi-dimensional simulations, potential future work would include adding the ability to handle non-uniform boundary conditions and increasing the compatibility with multi-dimensional **AMReX** meshes.

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106

Vita

Born in Birmingham, Alabama, Nick Roberts lived in several cities in Alabama before his family moved to Dover, Tennessee. There, he attended high school taking courses from Nashville State Community College in his junior year, and dual enrolling with the University of Tennessee, Knoxville for his senior year. He continued his undergraduate career at UTK and became an active member of the campus community by becoming a teaching assistant for Professor Stuart Elston's introductory physics course; becoming the editor-in-chief of the campus art and literary magazine, *The Phoenix*; and engaging behind the scenes with multiple theatre productions. After graduating, he chose to continue his education at the University of Tennessee as a graduate student in the physics department. He continued as a teaching assistant until joining the UT-ORNL core-collapse supernova group. After graduation, he will begin his new position as a mathematics lecturer at the University of Tennessee, Knoxville.