

A STUDY OF THE ELEMENTARY THEORY  
OF TRANSMISSION LINES

by

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## FOREWORD

The study here presented is an attempt to achieve simplicity and clarity rather than exhaustiveness. It deals with the most important points in the theory of transmission lines calculations, stressing particularly the mathematical derivations involved. A problem has been included to illustrate the principles discussed. It is such a study as would furnish a student unfamiliar with the subject basic information in each of its most important phases.

Work of this character is necessarily a compilation. It has been based largely on notes taken in class under Professor J. G. Tarboux in Electrical Engineering 443, an extra-curricular course. The writer acknowledges his indebtedness to Professor Tarboux for this, and for his assistance in general.

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# A STUDY OF THE ELEMENTARY THEORY OF TRANSMISSION LINES

## Chapter I

### INDUCTANCE

1. Ohm's Law Applied to the Magnetic Circuit.- In a non-inductive electric circuit, Ohm's law is expressed by the equation

$$E = IR$$

where  $E$  = Electromotive force in volts,

$I$  = Current in amperes, due to  $E$ ,

$R$  = Resistance of the circuit in ohms.

If the length of the conductor carrying the current is  $l$  and its cross-sectional area is  $A$ , then

$$R = \frac{l}{k A}$$

where  $k$  is a constant, the value of which depends on the material of the conductor, and is called the conductivity of that material. It follows, then, that

$$E = I \frac{l}{k A}$$

or

$$\frac{E}{l} = \frac{I}{k A}$$

That is, the voltage drop per unit length of conductor is equal to a constant times the current density.

Ohm's law may be applied to the magnetic circuit also, and is expressed by the equation

$$F = \Phi R$$

where  $F$  = Magnetomotive force,

$\phi$  = Magnetic flux,

$\mathcal{R}$  = Reluctance of the circuit.

The c.g.s. unit of magnetomotive force is the gilbert. The practical unit is the ampere-turn. The unit of magnetic flux is the maxwell, also called line of force. The unit of reluctance is the oersted.

Reluctance may be expressed thus

$$\mathcal{R} = \frac{1}{\mu} \frac{1}{A}$$

therefore  $F = \phi \frac{1}{\mu} \frac{1}{A}$

or  $\frac{F}{l} = \frac{1}{\mu} \frac{1}{A}$

or  $H = \frac{1}{\mu} B$

Where  $H$  = magnetic field intensity in ampere turns per cm.

$B$  = flux density in lines per square cm., or gaussses,

$\mu$  = a constant called the permeability of the circuit.

For non-magnetic materials  $\mu = 1.257$ .

## 2. Magnetic Field Around a Circular Conductor Carrying a Current.-

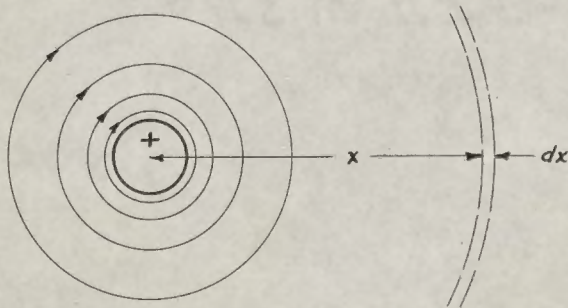


Figure 1

Uniform distribution of current in the conductor is assumed. The total ampere-turns produced by the current in the conductor is equal to the current in the conductor. Call this current  $I$ . If  $H$  represents the ampere-turns of m.m.f. drop per cm., in other words, the magnetic potential gradient in the direction of the field, then the total drop around a circular ring of radius  $x$ , figure 1, is

$$I = Hl = 2\pi xH$$

or 
$$H = \frac{I}{2\pi x} \quad \text{ampere turns per cm.}$$

The flux density

$$B_0 = \mu H = \frac{I}{2\pi x} = 0.2 \frac{I}{x} \quad \text{gausses.} \quad (1)$$

3. Magnetic Field Within the Conductor.— Figure 2 represents the cross-section of the conductor.

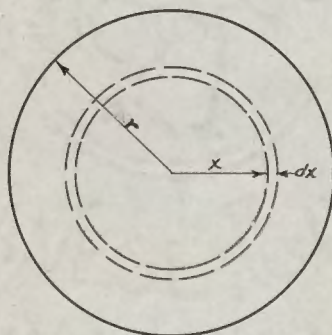


Figure 2

Since uniform distribution of current has been assumed, the current flowing in the partial area of radius  $x$  is proportional to this area or to the square of the radius, that is

$$I_x = I \frac{x^2}{r^2}$$

and the ampere-turns this current produces are also

$$= I \frac{x^2}{r^2}$$

Therefore  $2\pi xH = I \frac{x^2}{r^2}$

or  $H = \frac{Ix}{2\pi r^2}$

The density  $B_c = \frac{Ix}{2\pi r^2}$  gaussess. (2)

Equations (1) and (2) show that the flux density varies as the first power of  $x$  within the conductor, and as the  $-1$  power of  $x$  outside of it.

At  $x = r$ , and for  $\mu = 1.257$ , equation (1) reduces to

$$B_o = 0.2 \frac{I}{r}$$

and equation (2) reduces to

$$B_c = \frac{1.257 Ir}{2\pi r^2} = 0.2 \frac{I}{r}$$

or  $B_o = B_c$

If flux density is plotted against distance, the distribution would be represented by the curve of figure 3.

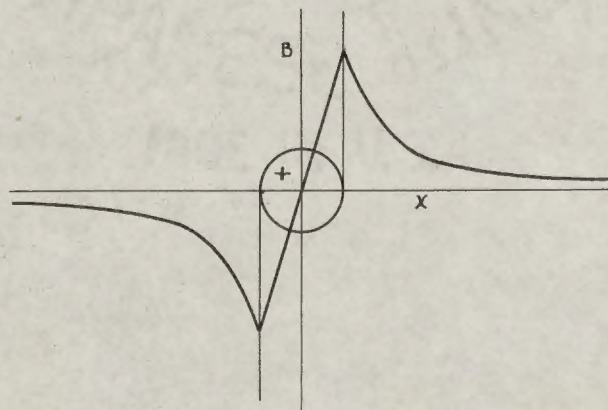


Figure 3.

4. Flux Linkages.— The voltage induced in a conductor by a changing field surrounding it is proportional to the time rate of change of flux, or in the form of an equation

$$e = - \frac{d\phi}{dt} 10^{-8}$$

Since flux is proportional to the current causing it, the voltage may be expressed in terms of current

$$e = -L \frac{di}{dt}$$

where the constant of proportionality  $L$  is the inductance of the circuit. It follows that

$$L = \frac{d\phi}{di} 10^{-8} \quad \text{henries.}$$

For a constant current  $\frac{d\phi}{di}$  becomes  $\frac{\phi}{i}$ , therefore

$$L = \frac{\phi}{i} 10^{-8}$$

Evidently this equation is correct when the total flux links the total current, as in the case represented by equation (1). In equation 2, however, where the flux does not link the entire current, it becomes necessary to introduce a modification, which consists in the use of an equivalent flux of such value that when considered linking the entire current would produce the same voltage of self induction as the partial flux linkages produce. That is

$$L = \frac{\phi_e}{i} 10^{-8} \quad \text{henries} \quad (3)$$

where  $\phi_e$  = equivalent flux linkages in maxwells.

5. Inductance of Single-phase Line.— From figure 1, the flux per unit length of conductor in the circular band of width

dx, outside of the wire

$$d\phi_o = B dx$$

and the total flux outside of the conductor per centimeter length of line

$$\phi_o = \int_r^D B dx$$

Substituting the value of B from equation (1)

$$\phi_o = \int_r^D \frac{0.2I}{x} dx = 0.2 I \int_r^D \frac{1}{x} dx$$

$$\phi_o = 0.2 I \ln \frac{D}{r} \quad \text{lines per cm.} \quad (4)$$

Now consider figure 2. The flux within the wire in element dx, equation (2), is

$$d\phi_c = \frac{\mu I x}{2\pi r^2} dx \quad \text{lines per cm. length.}$$

The equivalent flux linking the current I is less by the ratio of the areas or the ratio of the square of the radii

$$d\phi_{ce} = \frac{\mu I x}{2\pi r^2} \cdot \frac{x^2}{r^2} = \frac{\mu I x^3}{2\pi r^4}$$

And the total equivalent flux linkages

$$\phi_{ce} = \int_0^r \frac{\mu I x^3}{2\pi r^4} dx = \frac{\mu I}{2\pi r^4} \cdot \frac{r^4}{4}$$

$$\phi_{ce} = \frac{\mu I}{8\pi} \quad \text{lines per cm.} \quad (5)$$

Evidently, the total flux linkages of one conductor due to its own current is the sum of equations (4) and (5), that is

$$\phi_t = \phi_o + \phi_{ce} = 0.2 I \ln \frac{D}{r} + \frac{\mu I}{8\pi} \quad \text{lines/cm.} \quad (6)$$

Therefore  $L_c = \frac{\phi_t}{I} 10^{-8} = (0.2 \ln \frac{D}{r} + \frac{\mu}{8\pi}) 10^{-8}$  henries/cm.

But since both conductors of the single-phase line act together in producing flux, the inductance of the line as a whole is

$$L = 2L_c = \left(0.4 \ln \frac{D}{r} + \frac{\mu}{4\pi}\right) 10^{-8} \text{ henries per cm.}$$

And reducing to the more practical units of millihenries per mile

$$L = \frac{2.54 \times 12 \times 5280 \times 10^3}{10^8} \left(0.4 \ln \frac{D}{r} + \frac{\mu}{4\pi}\right)$$

$$L = 1.61 \left(0.4 \ln \frac{D}{r} + \frac{\mu}{4\pi}\right) \text{ millihenries per mile (7)}$$

#### 6. Inductance of Three-phase Lines. Unsymmetrical Spacing.

When considering polyphase lines all values are obtained per phase. In the general case of a three-phase line that we shall take up presently, the spacing of the wires may be any, and the currents may be unbalanced to any degree, the only limitation being that the vector sum of the three currents must equal zero. That is

$$I_a + I_b + I_c = 0 \text{ vectorially.}$$

Since the currents are out of phase with each other, so will the fluxes be also. It is, therefore, necessary to consider the currents, as well as the fluxes, as vector quantities.

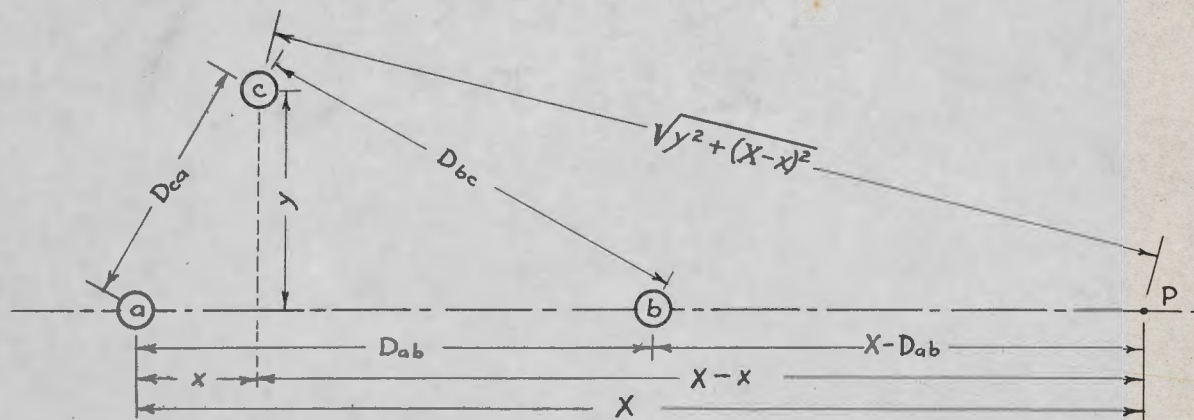


Figure 4.

Assume the line of figure 4, made up of three equal conductors of radius  $r$ , and any spacing whatever. Assume the point P at a large, finite distance  $X$  from conductor a.

The flux linkages of a due to its own current, from equation (6)

$$\Phi_{aa} = 0.2 I_a \ln. \frac{X}{r} - \frac{\mu I}{8\pi}$$

The flux linkages of a due to current in b

$$\Phi_{ba} = 0.2 I_c \ln. \frac{X - D_{ab}}{D_{ab}}$$

The flux linkages of a due to current in c

$$\Phi_{ca} = 0.2 I_c \ln. \frac{\sqrt{y^2 + (X-x)^2}}{D_{ca}}$$

And the total flux linkages acting on a

$$\begin{aligned} \Phi_{ta} &= \Phi_{aa} + \Phi_{ca} + \Phi_{ba} \\ \text{or } \Phi_{ta} &= \frac{\mu I_a}{8\pi} + 0.2 I_c \ln. \frac{\sqrt{y^2 + (X-x)^2}}{D_{ca}} + 0.2 I_a \ln. \frac{X}{r} \\ &\quad + 0.2 I_b \ln. \frac{X - D_{ab}}{D_{ab}} \end{aligned}$$

But  $I_a + I_b + I_c = 0$

or  $I_c = -I_a - I_b$

Therefore

$$\Phi_{ca} = -0.2 I_a \ln. \frac{\sqrt{y^2 + (X-x)^2}}{D_{ca}} - 0.2 I_b \ln. \frac{\sqrt{y^2 + (X-x)^2}}{D_{ca}}$$

and

$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln. \frac{X D_{ca}}{r \sqrt{y^2 + (X-x)^2}} + 0.2 I_b \ln. \frac{(X - D_{ab}) D_{ca}}{D_{ab} \sqrt{y^2 + (X-x)^2}}$$

Now move the point P away and let  $X$  approach infinity,

then  $\frac{X}{\sqrt{y^2 + (X-x)^2}}$  approaches 1

and  $\frac{X-D_{ab}}{\sqrt{y^2 + (X-x)^2}}$  approaches 1

Therefore

$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{D_{ca}}{r} + 0.2 I_b \ln \frac{D_{ca}}{D_{ab}}$$

$$= \frac{\mu I_a}{8\pi} + 0.2(I_a + I_b) \ln D_{ca} + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D_{ab}}$$

or

$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D_{ab}} + 0.2 I_c \ln \frac{1}{D_{ca}} \quad (8)$$

Noting the sequence of terms, we can generalize and write a general equation for the flux linkages of a due to any number of conductors by adding terms of the general form  $0.2 I_n \ln \frac{1}{D_{na}}$  for any conductor n.

For any other conductor, such as b, we may write the value of  $\Phi_{tb}$  from inspection

$$\begin{aligned} \Phi_{tb} = & \frac{\mu I_b}{8\pi} + 0.2 I_b \ln \frac{1}{r} + 0.2 I_a \ln \frac{1}{D_{ba}} + 0.2 I_c \ln \frac{1}{D_{cb}} \\ & \dots + 0.2 I_n \ln \frac{1}{D_{nb}} \end{aligned} \quad (9)$$

This is the final form of the equation for most convenient use. For any specific conditions the flux linkages should be calculated from equation (8) or (9) and the inductance found from equation (3).

#### 7. Equilateral Spacing. - In this case

$$D_{ab} = D_{bc} = D_{ca} = D$$

And from eq. (8)

$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{1}{r} + 0.2 I_n \ln \frac{1}{D} (I_b + I_c)$$

or 
$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{1}{r} - 0.2 I_a \ln \frac{1}{D}$$

or 
$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{D}{r}$$

And since, from eq.(3)

$$L = \frac{\Phi}{I} 10^{-8}$$

It follows that for equilateral spacing

$$L = \left( \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{D}{r} \right) 10^{-8} \text{ henries per cm.} \quad (10)$$

8. Horizontal or "Flat" Spacing.- In this case

$$D_{ab} = D_{bc} = \frac{D_{ac}}{2} = D$$

From equation (8)

$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D} + 0.2 I_c \ln \frac{1}{2D}$$

Substituting for  $I_c$  its value  $-I_a - I_b$

$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D} - 0.2 I_a \ln \frac{1}{2D} - 0.2 I_b \ln \frac{1}{2D}$$

hence 
$$\Phi_{ta} = \frac{\mu I_a}{8\pi} + 0.2 I_a \ln \frac{2D}{r} + 0.2 I_b \ln 2$$

and 
$$L_a = \left( \frac{\mu}{8\pi} + 0.2 \ln \frac{2D}{r} + 0.2 \frac{I_b}{I_a} \ln 2 \right) 10^{-8} \text{ henries per cm.}$$

For the middle conductor b the result is the same as for equilateral spacing, since  $D_{ab} = D_{bc} = D$ . In other words

$$L_b = \left( \frac{\mu}{8\pi} + 0.2 \ln \frac{D}{r} \right) 10^{-8} \text{ henries per cm.}$$

9. Transposition.- The conductors of a power line are transposed, first, in order to minimize inductive interference

on communication circuits running parallel to it, and second, in order to equalize the reactance of all phases. The latter is not necessary in the case of equilateral spacing. Evidently the transposition should be made so that each conductor occupies each position an equal portion of the length of the line. For a three-phase line, therefore, each conductor should occupy each position one third of the length of the line, and the minimum number of complete transpositions is two.

10. Inductance of Three-phase, Transposed Line.- Call the spaces between conductors  $D_1$ ,  $D_2$ , and  $D_3$ . The flux linkages on conductor a, if considered occupying each position the entire length of the line, are

$$\Phi_{a1} = \frac{\mu}{8\pi} I_a + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D_1} + 0.2 I_c \ln \frac{1}{D_3}$$

$$\Phi_{a2} = \frac{\mu}{8\pi} I_a + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D_2} + 0.2 I_c \ln \frac{1}{D_1}$$

$$\Phi_{a3} = \frac{\mu}{8\pi} I_a + 0.2 I_a \ln \frac{1}{r} + 0.2 I_b \ln \frac{1}{D_3} + 0.2 I_c \ln \frac{1}{D_2}$$

The average flux linkages on conductor a should be one third of the sum of these three equations, or

$$\Phi_{av.} = \frac{\mu}{8\pi} I_a + 0.2 I_a \ln \frac{1}{r} + \frac{0.2}{3} I_b \ln \frac{1}{D_1 D_2 D_3} + \frac{0.2}{3} I_c \ln \frac{1}{D_1 D_2 D_3}$$

Combining the last two terms and remembering that  $I_b + I_c = -I_a$

$$\Phi_{av.} = \frac{\mu}{8\pi} I_a + 0.2 I_a \ln \frac{1}{r} + \frac{0.2}{3} I_a \ln \frac{1}{D_1 D_2 D_3}$$

$$\text{or } \Phi_{av.} = \frac{\mu}{8\pi} I_a + 0.2 I_a \ln \frac{\sqrt[3]{D_1 D_2 D_3}}{r}$$

$$\text{and } L = \left\{ \frac{\mu}{8\pi} + 0.2 \ln \frac{\sqrt[3]{D_1 D_2 D_3}}{r} \right\} 10^{-8} \text{ henries per cm. (11)}$$

This is the general equation for the inductance of a three-phase transposed line regardless of the spacing. For equilateral spacing, where  $D_1 = D_2 = D_3$ , the expression  $\sqrt[3]{D_1 D_2 D_3}$  becomes  $= D$ , and equation (11) becomes equation (10):  $\sqrt[3]{D_1 D_2 D_3}$ , or  $D'$ , is called the equivalent separation. For horizontal or flat spacing where  $D_1 = D_2 = \frac{D_3}{2}$

$$D' = \sqrt[3]{2D^3} = D \sqrt[3]{2} = 1.26 D$$

11.- Double Circuit Line.- Figure 5 shows a common arrangement of conductors in two parallel, three-phase circuits. The same method is followed here to calculate the inductance of one wire.

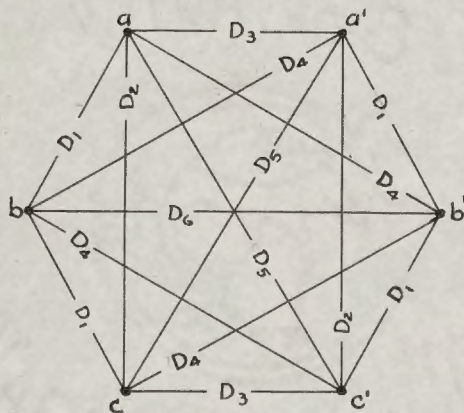


Figure 5

The conditions are that

$$I_a + I_b + I_c + I_{a'} + I_{b'} + I_{c'} = 0$$

Also

$$I_a = I_{a'}$$

$$I_b = I_{b'}$$

$$I_c = I_{c'}$$

The inductance per wire with transposition is

$$L = \left\{ \frac{\mu}{8\pi} + 0.2 \ln \frac{1}{r} \sqrt[3]{\frac{D_4^2 D_1^2 D_2 D_3}{D_5^2 D_6}} \right\} 10^{-8} \text{ henries/cm.}$$

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## Chapter II

## CAPACITANCE

1. The Dielectric Circuit.- Consider two parallel plates of conductive material separated by air or any other insulating material, as represented in figure 6.



Figure 6

If a direct-current voltage is impressed across the plates a current flows during one instant in the circuit. If the source of potential is removed and the electrical circuit around the condenser is short-circuited a current will flow again, but in the opposite direction. The quantity of electricity that flows when the potential is applied is called the charge of the condenser, and is measured in ampere-seconds or coulombs. Evidently, in order that this current may flow a disturbance must be operated on the insulating material between the plates (called the dielectric). The charge may be thought of as lines of dielectric flux flowing from one plate to the other across the dielectric. This is a similar conception to the magnetic lines of flux, but the lines of dielectric flux

are always normal to the lines of magnetic flux and represent lines of constant magnetic potential.

Ohm's law, as discussed in Chapter I, may be applied also to the dielectric circuit. Algebraically

$$E = SQ$$

where  $E$  = potential across condenser in volts  
 $Q$  = the dielectric flux, or charge in coulombs.

The proportionality constant  $S$  is called the elastance of the circuit.

$$S = \frac{1}{C}$$

where  $C$  is the capacitance in farads. The unit of  $S$  is the daraf. Also

$$C = k \frac{A}{l}$$

and

$$S = \frac{1}{k} \frac{l}{A}$$

where  $k$  is a constant the value of which depends on the properties of the dielectric. Then

$$E = \frac{1}{k} \frac{l}{A} Q$$

or

$$\frac{E}{l} = \frac{1}{k} \frac{Q}{A} \quad (13)$$

that is, the potential gradient or voltage per unit length of path is equal to a constant times the dielectric flux-density, which corresponds to the equations already found for the electric and magnetic circuits. If  $l$  is centimeters,  $A$  in square centimeters and the other quantities in the units mentioned

$$k = 0.08842 \times 10^{-8} \text{ farads per cubic cm. for air.}$$

Since the capacitance of materials is given relative to that of air, another constant must be introduced in the equa-

tion to represent this ratio. Call it  $k_a$ . Then for any material other than air

$$C = k k_a \frac{A}{l}$$

2. Capacitance of Single-phase Line.- Consider two paral-

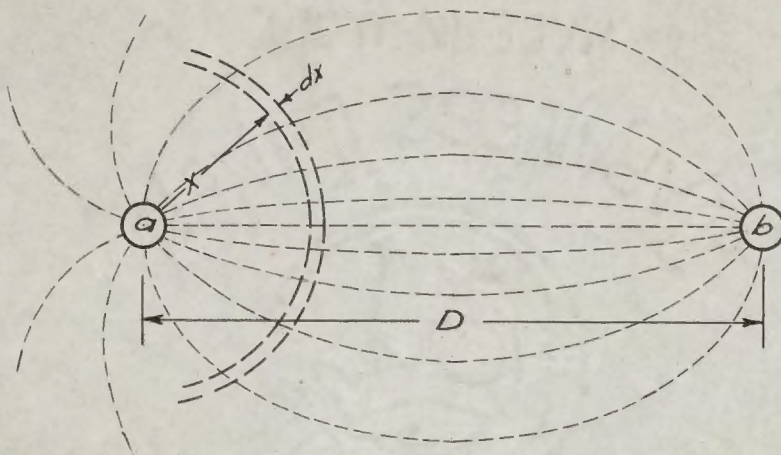


Figure 7

lel cylinders a and b, figure 7, with a difference of potential between them. They will act as a condenser, establishing a dielectric flux between them. The flux density on the surface of the cylinders will be greatest on the inside faces, and decrease to a minimum on the outside. In order to simplify the results this density is assumed constant over the conductor surface. This condition is approached when  $D$  becomes very large relative to  $r$ , which is generally the case in practice.

Let  $G$  be the potential gradient  $\frac{E}{l}$  from a to b due to charge of a. The dielectric flux density

$$\frac{Q}{A} = \Delta$$

Then, since

$$\frac{E}{l} = \frac{1}{k} \frac{Q}{A}$$

$$G = \frac{1}{k} \Delta$$

Consider a circle of radius  $x$  around the conductor, figure 7.

The dielectric flux density around the circumference

$$\Delta = \frac{Q}{A} = \frac{Q}{2\pi x}$$

or

$$G_x = \frac{1}{k} \frac{Q}{2\pi x}$$

Now assume an element of width  $dx$ , figure 7. The potential across this element

$$de = G_x dx = \frac{1}{k} \frac{Q}{2\pi x} dx$$

and the total potential from  $a$  to  $b$  due to charge of  $a$

$$E_{aba} = \int_r^{D-r} \frac{1}{k} \frac{Q_a}{2\pi x} dx = \frac{Q_a}{2\pi k} \ln. \frac{D-r}{r} \text{ vector volts. (14)}$$

The same reasoning may be applied to conductor  $b$ , but here  $x$  becomes  $D-x$  and the limits of integration are reversed. The total potential due to charge of  $b$

$$E_{abb} = \int_{D-r}^r -\frac{1}{k} \frac{Q_b}{2\pi(D-x)} dx = -\frac{Q_b}{2\pi k} \ln. \frac{D-r}{r}$$

But

$$Q_b = -Q_a = Q$$

and the total potential from  $a$  to  $b$  due to both charges

$$E_{abt} = \frac{Q}{\pi k} \ln. \frac{D-r}{r} \text{ vector volts (15)}$$

The capacitance between  $a$  and  $b$

$$C_{ab} = \frac{Q}{E_{ab}} = \frac{\pi k}{\ln. \frac{D-r}{r}} \text{ (16)}$$

For air

$$C = \frac{0.728 \times 10^{-6}}{\ln \frac{D-r}{r}} \text{ microfarads per cm.} \quad (17)$$

Usually  $r$  is small relative to  $D$  and  $D-r$  is then practically equal to  $D$ . The fraction  $\frac{D-r}{r}$  is then written  $\frac{D}{r}$ .

3. Capacitance of Three-phase Line.— The capacitance of a three-phase line is defined as the per-phase capacitance between one wire and the neutral.

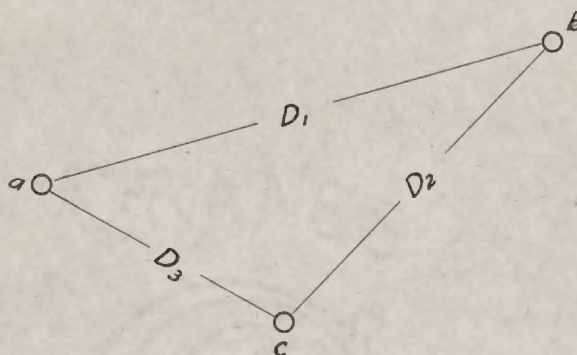


Figure 8

Assume any unsymmetrical spacing such as that of figure 8. The drop of potential from  $a$  to  $b$  due to the charge of  $a$ , from equation (14)

$$E_{aba} = \frac{Q_a}{2\pi k} \ln \frac{D_1}{r} \quad (18)$$

That due to charge of  $b$

$$E_{abb} = - \frac{Q_b}{2\pi k} \ln \frac{D_1}{r} \quad (19)$$

and that due to charge of  $c$

$$E_{abc} = \frac{Q_c}{2\pi k} \ln \frac{D_2}{D_3} \quad (20)$$

The total potential drop from  $a$  to  $b$  will be the sum of equa-

tions (18), (19) and (20)

$$E_{ab} = \frac{1}{2\pi k} \left\{ (Q_a - Q_b) \ln. \frac{D_1}{r} + Q_c \ln. \frac{D_2}{D_3} \right\} \quad (21)$$

By the same procedure it is found that

$$E_{bc} = \frac{1}{2\pi k} \left\{ (Q_b - Q_c) \ln. \frac{D_2}{r} + Q_a \ln. \frac{D_3}{D_1} \right\} \quad (22)$$

It is also assumed that

$$Q_a + Q_b + Q_c = 0 \quad (23)$$

Equations (21), (22) and (23) are three simultaneous equations with three unknowns, namely  $Q_a$ ,  $Q_b$ , and  $Q_c$ . It is most convenient to substitute the values of  $D_1$ ,  $D_2$ , and  $D_3$  for any particular spacing and then solve for each  $Q$ .

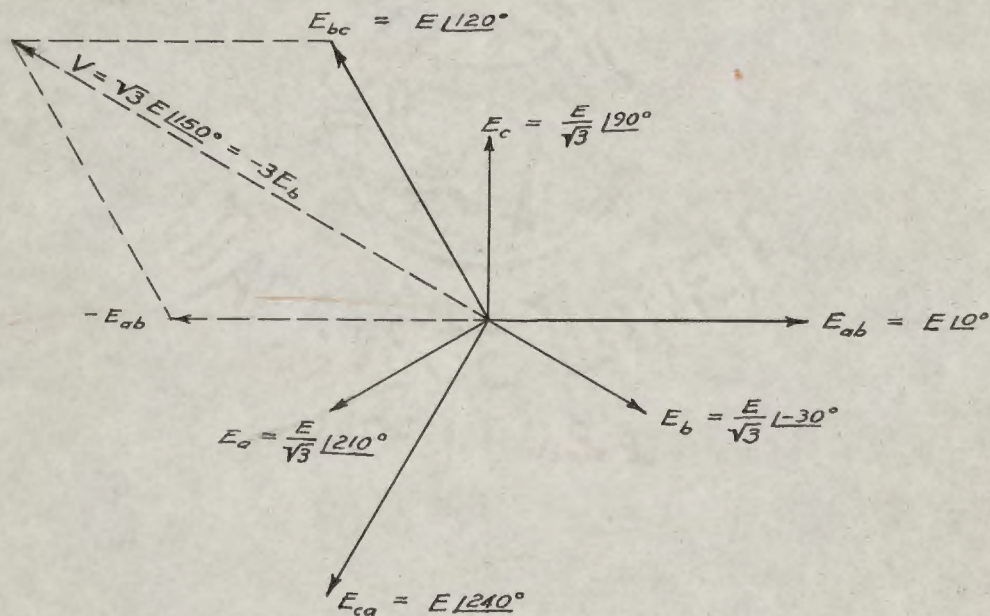


Figure 9

Figure 9 is the vector diagram of voltages of a balanced three-phase line. According to the definition of capacity of

a three-phase line (page 21)

$$C_a = \frac{Q_a}{\frac{E}{\sqrt{3}} / 210^\circ} \quad (24a)$$

$$C_b = \frac{Q_b}{\frac{E}{\sqrt{3}} / -30^\circ} \quad (24b)$$

$$C_c = \frac{Q_c}{\frac{E}{\sqrt{3}} / 90^\circ} \quad (24c)$$

4. Symmetrical Spacing.- For symmetrical spacing

$$D_1 = D_2 = D_3 = D$$

Equations (21) and (22) become

$$E_{ab} = \frac{1}{2\pi k} (Q_a - Q_b) \ln. \frac{D}{r} \quad (25)$$

$$E_{bc} = \frac{1}{2\pi k} (Q_b - Q_c) \ln. \frac{D}{r} \quad (26)$$

Substituting the value of  $-Q_c$  from (23) in (26)

$$E_{bc} = \frac{1}{2\pi k} (Q_a + 2Q_b) \ln. \frac{D}{r} \quad (27)$$

Subtracting (27) from (25)

$$E_{ab} - E_{bc} = \frac{1}{2\pi k} (-3Q_b) \ln. \frac{D}{r}$$

or

$$-Q_b = \frac{2\pi k (E_{ab} - E_{bc})}{3 \ln. \frac{D}{r}}$$

But  $E_{ab} - E_{bc} = -3E_b$  (See dotted portion of fig.9)

Therefore

$$Q_b = \frac{2\pi k E_b}{\ln. \frac{D}{r}}$$

and from (24)

$$C_b = \frac{2\pi k}{\ln. \frac{D}{r}} \quad \text{farads per cm.} \quad (28)$$

Converting to more practical units, and using the value of  $k$

for air

$$C_b = \frac{2\pi \times 0.08842 \times 10^{-6}}{\ln \frac{D}{r}} \text{ microfarads/cm.} \quad (29)$$

5. Transposition.— As discussed in page 14, a three-phase line is transposed in such a way that each conductor occupies each position one third of the total length of the line. Evidently the average capacity to neutral is the same for the three conductors. The average vector voltage  $E_{ab}$  will be one third of the sum of equations (21), (22), and the similar equation for  $E_{ca}$  with the proper substitutions.

$$E_{ab} = \frac{1}{3} \frac{1}{2\pi k} \left\{ (Q_a - Q_b) \ln \frac{D_1}{r} - Q_c \ln \frac{D_3}{D_2} + (Q_a - Q_b) \ln \frac{D_2}{r} - Q_c \ln \frac{D_1}{D_3} - (Q_a - Q_b) \ln \frac{D_3}{r} - Q_c \ln \frac{D_2}{D_1} \right\}$$

$$E_{ab} = \frac{1}{6\pi k} (Q_a - Q_b) \ln \frac{D_1 D_2 D_3}{r^3}$$

or

$$E_{ab} = \frac{1}{2\pi k} (Q_a - Q_b) \ln \frac{\sqrt[3]{D_1 D_2 D_3}}{r} \text{ vector volts} \quad (30)$$

Which is equation (25) with the geometric mean of the distances in place of  $D$ .

As mentioned above, the mean capacitance to neutral is the same for all conductors, or

$$\left. \begin{aligned} Q_a &= C_o E_a \\ Q_b &= C_o E_b \\ Q_c &= C_o E_c \end{aligned} \right\} \text{ vector lines}$$

Where  $E_a$ ,  $E_b$  and  $E_c$  are voltages to neutral and  $C_o$  is the aver-

age capacitance to neutral.

Substituting these values in (29) and remembering that

$$E_{ab} = E_a - E_b \quad \text{vectorially,}$$

$$E_a - E_b = \frac{1}{2\pi k} C_0 (E_a - E_b) \ln. \frac{\sqrt[3]{D_1 D_2 D_3}}{r}$$

and

$$C_0 = \frac{2\pi k}{\ln \frac{\sqrt[3]{D_1 D_2 D_3}}{r}} \quad \text{farads per cm.} \quad (31)$$

which is equivalent to equation (28) with  $\sqrt[3]{D_1 D_2 D_3}$  replacing  $D$ . If  $D_1 = D_2 = D_3 = D$ , equation (31) reduces to equation (28).

6. The Effect of Ground.- In the preceding discussion the effect of ground has not been considered. Actually the ground affects the capacitance of the line considerably. To take account of this effect a surface of zero potential is assumed which coincides with the surface of the earth, and below it, at a distance equal to the height of the line, another system of conductors is assumed, the exact image of the line itself. The calculations involved will not be taken up here.

-----

## Chapter III

### SHORT LINE CALCULATIONS

1. Introductory.- Transmission lines are homogeneous, that is, the material of the conductors, their size and their spacing are uniform. Therefore the resistance, inductance and capacitance per unit length of line are constant, and are called the constants of the line. When these are known, and the current and voltage are sinusoidal, it is possible to derive equations that permit the calculation of voltage, current, phase-angle, etc. at one end of the line if the voltage, current, phase-angle, etc. at the other end are known.

The exact solution of lines must take account of the uniform distribution of capacitance, as well as resistance and inductance. Calculations may be greatly simplified, however, in the case of short lines where the effect of capacitance is small, by assuming the capacitance concentrated at one or more points or by disregarding capacitance altogether. The different methods used for these approximate, short-line calculations will be treated very briefly.

2. Simple Impedance Line.- (In this, as in the following examples, only one phase and the neutral are considered.)

This is the simplest approximation that may be assumed. It neglects capacitance entirely, and therefore should not be

used in any but very short lines, say twenty-five miles or less.

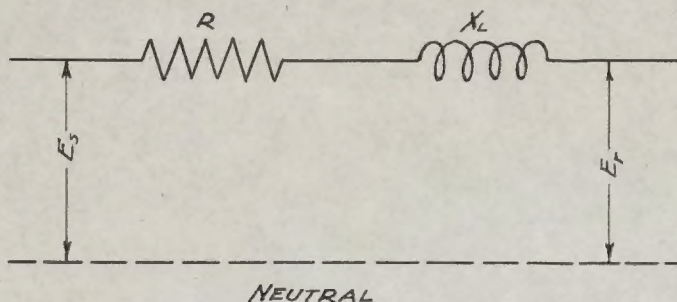


Figure 10

If the receiving end conditions are known, the sending conditions are easily found. Let subscript r stand for receiver end, and s for sending end,

$$I_r = I_s = I$$

$$E_s = E_r + IZ \quad \text{vectorially}$$

The angle between  $E_r$  and  $E_s$

$$\alpha = \tan^{-1} \frac{e_{s2}}{e_{s1}}$$

where  $e_{s1}$  and  $e_{s2}$  are the real and imaginary part of  $E_s$  respectively, then

$$\theta_s = \alpha \pm \theta_r$$

where  $\theta_s$  and  $\theta_r$  are the power-factor angles .

The vector diagram of the simple impedance line with  $E_r$  at the origin is shown in Figure 11 for a lagging current.

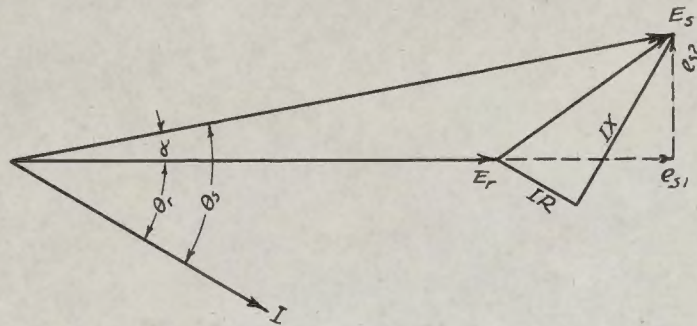


Figure 11

3. Load-end Condenser Method.- In the simple impedance line no current would flow at no-load. In the actual line, however, due to the capacitance, a current flows on open circuit which is a maximum at the generator end, and decreases to zero at the load end. This is called the charging current of the line. Due to this current the voltage rises toward the receiver end on open circuit. The simplest approximation to account for this current is that<sup>of</sup> assuming a condenser in parallel with the load, as shown in figure 12.

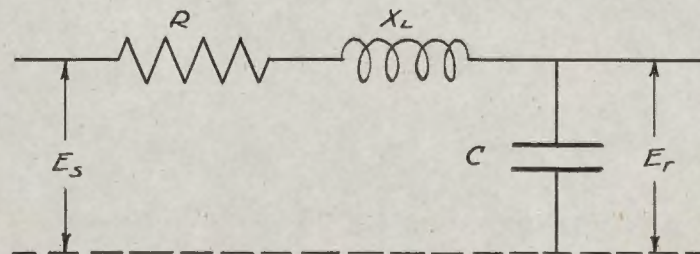


Figure 12

If the admittance of the condenser is  $Y$ , the charging current

$$I_c = E_r Y \text{ vectorially.}$$

$I_c$  is constant throughout the line, and is assumed to lead  $E_r$  by 90 degrees. Therefore the total current

$$I = I_r + I_c \text{ vectorially,}$$

where  $I_r$  is the load current. The rest of the calculations are the same as for the simple impedance line.

4. Nominal  $\pi$  Line.- The nominal  $\pi$  line, figure 13, is a

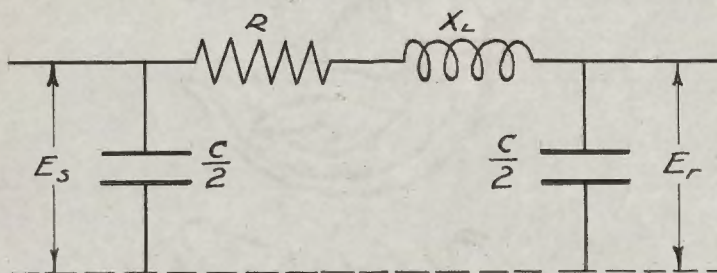


Figure 13

closer approximation. It assumes one half of the capacity of the line concentrated at each end. In this case only one half of the charging current circulates through the impedance, therefore the impedance drop

$$= (I_r + \frac{1}{2}E_r Y) Z \text{ vectorially}$$

and

$$E_s = E_r + (I_r + \frac{1}{2}E_r Y) Z$$

or 
$$E_s = E_r \frac{1}{2}(2 + ZY) + I_r Z \text{ vectorially}$$

Also 
$$I_s = I_r + \frac{1}{2}E_r Y + \frac{1}{2}E_s Y$$

or 
$$I_s = E_r \frac{1}{4}(4 + ZY)Y + I_r \frac{1}{2}(2 + ZY)$$

The vector diagram of this line for a lagging current load is given in figure 14.

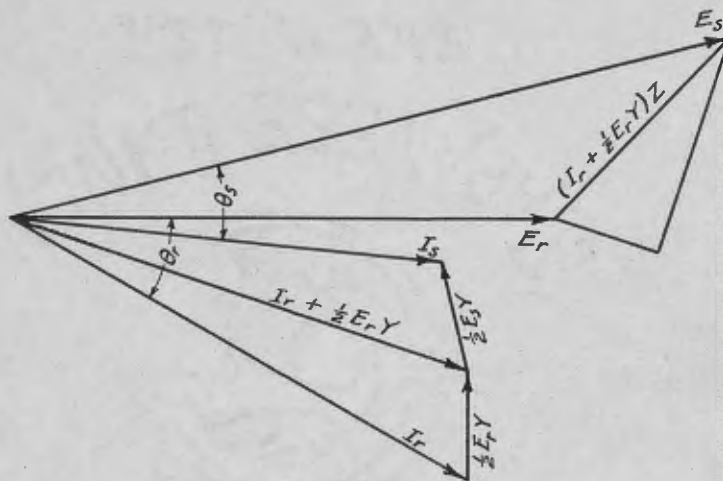


Figure 14

5. Nominal T Line.— Another good approximation is the so-called nominal T line, figure 15, in which the impedance is divided in two equal portions, and the capacity shunted between them.

Here the charging current circulates only through one half of the total impedance.

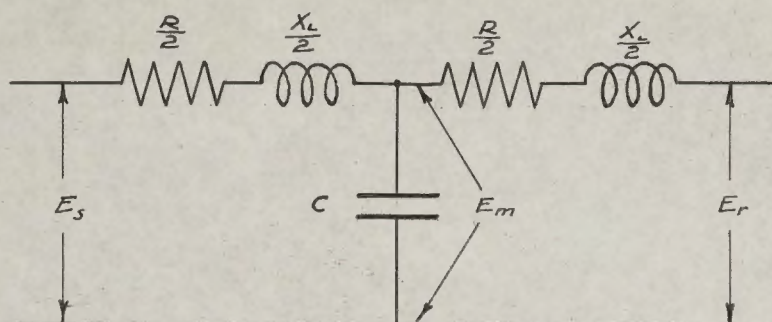


Figure 15

All the operations that follow are done vectorially.

$$E_m = E_r + \frac{1}{2}I_r Z$$

$$I_c = E_m Y = (E_r + \frac{1}{2}I_r Z)Y$$

Then

$$I_s = I_r + I_c = I_r + (E_r + \frac{1}{2}I_r Z)Y$$

or

$$I_s = I_r(1 + \frac{1}{2}ZY) + E_r Y$$

Also

$$E_s = E_m + \frac{1}{2}I_s Z$$

$$= E_r + \frac{1}{2}I_r Z + \frac{1}{2}I_r Z + \frac{1}{2}E_r YZ + \frac{1}{4}I_r ZY$$

or

$$E_s = E_r(1 + \frac{1}{2}ZY) + I_r Z(1 + \frac{1}{4}Y)$$

The vector diagram for a lagging power factor is shown in figure 16.

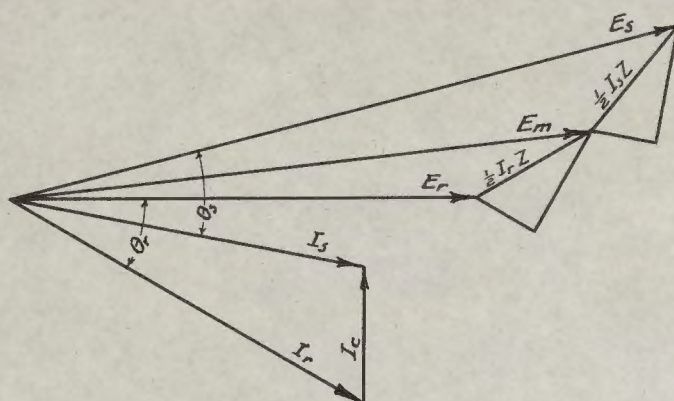


Figure 16

6. Steinmetz Split-condenser Method.— This method adds another refinement to the approximations preceding, and gives accurate results for longer lines, but the calculations are more involved. The assumptions are as shown in figure 17.

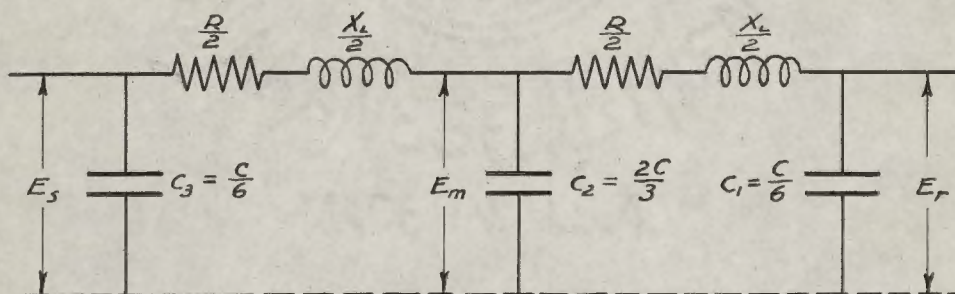


Figure 17

Call the currents in the right and left halves  $I_a$  and  $I_b$  respectively. All operations are done vectorially.

$$I_a = I_r + E_r Y_1$$

$$E_m = E_r + \frac{1}{2} I_a Z = E_r + \frac{1}{2} I_r Z + \frac{1}{2} E_r Z Y_1$$

$$I_b = I_a + E_m Y_2 = I_r + E_r Y_1 + E_r Y_2 + \frac{1}{2} I_r Z Y_2 + E_r Z Y_1 Y_2$$

$$E_s = E_m + \frac{1}{2} I_b Z = E_r + \frac{1}{2} I_r Z + \frac{1}{2} E_r Z Y_1 + \frac{1}{2} I_r Z + \frac{1}{2} E_r Y_1 Z \\ + \frac{1}{2} E_r Y_2 Z + \frac{1}{4} I_r Z^2 Y_2 + \frac{1}{4} E_r Z^2 Y_1 Y_2$$

$$E_s = E_r \left( 1 + \frac{ZY}{6} + \frac{ZY}{3} + \frac{Z^2 Y^2}{36} \right) + I_r \left( Z + \frac{YZ^2}{6} \right)$$

$$E_s = E_r \left\{ 1 + \frac{1}{2} ZY + \left( \frac{ZY}{6} \right)^2 \right\} + I_r \left( Z + \frac{YZ^2}{6} \right)$$

-----

$$I_s = I_b + E_s Y_3 = I_r + E_r Y_1 + E_r Y_2 + \frac{1}{2} I_r Z Y_2 + \frac{1}{2} E_r Z Y_1 Y_2 \\ + E_r Y_3 + \frac{1}{2} I_r Z Y_3 + \frac{1}{2} E_r Z Y_1 Y_3 + \frac{1}{2} I_r Z Y_3 \\ + \frac{1}{2} E_r Z Y_1 Y_3 + \frac{1}{2} E_r Z Y_2 Y_3 + \frac{1}{4} I_r Z^2 Y_2 Y_3 \\ + \frac{1}{4} E_r Z^2 Y_1 Y_2 Y_3$$

$$I_s = E_r (Y_1 + Y_2 + Y_3 + \frac{1}{2} Z Y_1 Y_2 + \frac{1}{2} Z Y_2 Y_3 + Z Y_1 Y_3 + \frac{1}{4} Z^2 Y_1 Y_2 Y_3) \\ + I_r (1 + \frac{1}{2} Z Y_2 + Y_3 + \frac{1}{4} Z^2 Y_2 Y_3)$$

$$I_s = E_r \left( Y + \frac{ZY^2}{18} + \frac{ZY^2}{36} + \frac{ZY^2}{18} + \frac{Z^2 Y^3}{216} \right) + I_r \left( 1 + \frac{ZY}{3} + \frac{ZY}{3} + \frac{Z^2 Y^2}{36} \right)$$

$$I_s = E_r \left\{ Y + 5Z \left( \frac{Y}{6} \right)^2 + Z^2 \left( \frac{Y}{6} \right)^3 \right\} + I_r \left\{ 1 + \frac{2ZY}{3} + \left( \frac{ZY}{6} \right)^2 \right\}$$

The vector diagram is shown in figure 17.

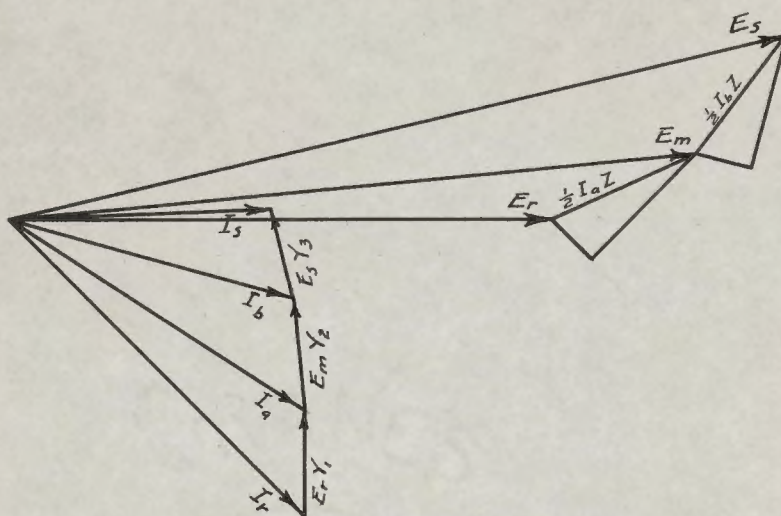


Figure 17

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## Chapter IV

### LONG LINE CALCULATIONS

1. Exact Method. Derivation of General Equation.- For high voltages and long distances more exact solutions than those treated in Chapter III become necessary. As stated before, the capacitance of the line is not concentrated at any particular points, but distributed throughout the total length. The same may be said of the resistance and the inductance. The actual line

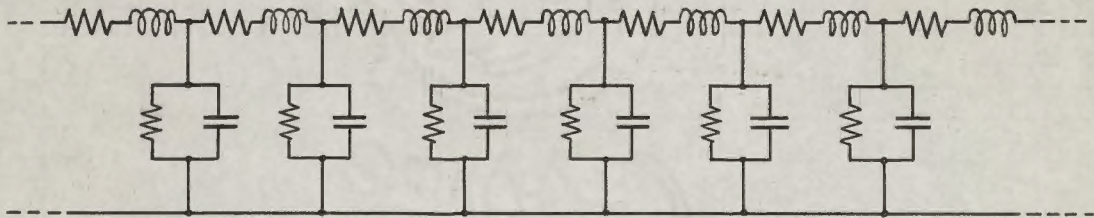


Figure 18

might be thought of as made up of an infinite number of series impedances and shunt admittances as represented in figure 18. Obviously, any exact method of solution must build up from an infinitesimal element of the line. Such an element ( $dl$ ) is shown in figure 19 at a distance  $l$  from the load.

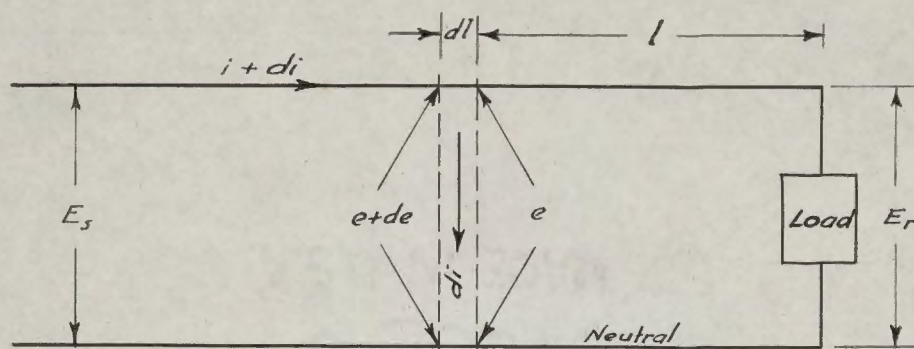


Figure 19

Let  $z = r + jx$  = series impedance per unit length, and  
 $y = g + jb$  = shunt admittance per unit length.

The series impedance in length  $dl$  is  $zdl$ , and the shunt admittance in length  $dl$  is  $ydl$ .

The drop along  $dl$  must be the product of average current and impedance, that is

$$de = \frac{(i + di) + i}{2} zdl$$

$$de = \left(1 + \frac{dl}{2}\right) z$$

$$\text{or} \quad \frac{de}{dl} = \left(1 + \frac{dl}{2}\right) z \quad (32)$$

The shunt current in element  $l$  must equal the product of average voltage and admittance, that is

$$di = \left(e + \frac{de}{2}\right) ydl$$

or  $\frac{dl}{dl} = (e + \frac{de}{2})y$  (33)

As the element approaches zero equations (32) and (33) approach the limits

$$\frac{de}{dl} = iz \quad (34)$$

$$\frac{dl}{dl} = ey \quad (35)$$

Equations (34) and (35) are fundamental.

From (35)

$$e = \frac{1}{y} \frac{dl}{dl}$$

Differentiating with respect to  $l$

$$\frac{de}{dl} = \frac{1}{y} \frac{d^2 l}{dl^2}$$

Substituting the value of  $\frac{de}{dl}$  from (34)

$$iz = \frac{d^2 l}{dl^2} \frac{1}{y}$$

or

$$\frac{d^2 l}{dl^2} = izy = \alpha^2 l \quad (36)$$

Similarly, from (34)

$$l = \frac{1}{z} \frac{de}{dl}$$

Differentiating with respect to  $l$

$$\frac{dl}{dl} = \frac{1}{z} \frac{d^2 e}{dl^2}$$

From (35)

$$ey = \frac{1}{z} \frac{d^2 e}{dl^2}$$

or

$$\frac{d^2 e}{dl^2} = ezy = \alpha^2 e \quad (37)$$

where  $\alpha = \sqrt{zy}$ .

It may be noticed from (36) and (37) that any algebraic expression for 1 or e must be such that when differentiated twice with respect to 1 will yield the same expression multiplied by the constant  $\alpha^2$ . A little thought will reveal that only an infinite series of increasing powers of 1 will satisfy this requisite. The Maclaurin series for both the hyperbolic sine and the hyperbolic cosine of 1 give the required result:

$$\begin{aligned}\sinh \alpha 1 &= \alpha 1 + \frac{\alpha^3 1^3}{3!} + \frac{\alpha^5 1^5}{5!} + \frac{\alpha^7 1^7}{7!} \dots \dots \text{etc.} \\ \frac{d \sinh \alpha 1}{d1} &= \alpha + \frac{\alpha^3 1^2}{2!} + \frac{\alpha^5 1^4}{4!} + \frac{\alpha^7 1^6}{7!} \dots \dots \text{etc.} \\ \frac{d^2 \sinh \alpha 1}{d1^2} &= \alpha^3 1 + \frac{\alpha^5 1^3}{3!} + \frac{\alpha^7 1^5}{5!} + \frac{\alpha^9 1^7}{7!} \dots \dots \text{etc.} \\ &= \alpha^2 \left( \alpha 1 + \frac{\alpha^3 1^3}{3!} + \frac{\alpha^5 1^5}{5!} + \frac{\alpha^7 1^7}{7!} \dots \dots \text{etc.} \right)\end{aligned}\tag{37a}$$

That is

$$\frac{d^2}{d1^2} \sinh \alpha 1 = \alpha^2 \sinh \alpha 1$$

which satisfies the conditions.

Similarly for the cosine

$$\cosh \alpha 1 = 1 + \frac{\alpha^2 1^2}{2!} - \frac{\alpha^4 1^4}{4!} + \frac{\alpha^6 1^6}{6!} - \dots \dots \dots \text{etc.}$$

Following the same procedure it is found that

$$\frac{d^2}{d1^2} \cosh \alpha 1 = \alpha^2 \left( 1 + \frac{\alpha^2 1^2}{2!} - \frac{\alpha^4 1^4}{4!} + \dots \dots \dots \text{etc.} \right)$$

or

$$\frac{d^2}{d1^2} \cosh \alpha 1 = \alpha^2 \cosh \alpha 1$$

Therefore, we may express e and 1 in terms of  $\sinh \alpha 1$  and

$\cosh \alpha l$  as follows:

$$e = A_1 \cosh \alpha l + B_1 \sinh \alpha l \quad (38)$$

$$i = C_1 \cosh \alpha l + D_1 \sinh \alpha l \quad (39)$$

where  $A_1$ ,  $B_1$ ,  $C_1$ , and  $D_1$  are constants. The evaluation of these constants follows.

At  $l = 0$ ,  $e = E_r$  and  $i = I_r$   
also  $\cosh \alpha l = 1$  and  $\sinh \alpha l = 0$   
therefore

$$A_1 = e = E_r$$

and

$$C_1 = i = I_r$$

Substituting in (38) and (39)

$$e = E_r \cosh \alpha l + B_1 \sinh \alpha l \quad (40)$$

$$i = I_r \cosh \alpha l - D_1 \sinh \alpha l \quad (41)$$

From equation (37a)

$$\begin{aligned} \frac{d}{dl} \sinh \alpha l &= \alpha \left( 1 + \frac{\alpha^2 l^2}{2!} + \frac{\alpha^4 l^4}{4!} + \dots \dots \dots \text{etc} \right) \\ &= \alpha \cosh \alpha l \end{aligned}$$

$$\text{Similarly } \frac{d}{dl} \cosh \alpha l = \alpha \sinh \alpha l$$

Differentiating (41)

$$\frac{di}{dl} = I_r \alpha \sinh \alpha l + D_1 \alpha \cosh \alpha l$$

From equation (35)  $\frac{di}{dl} = ey$ , and substituting the value of  $e$  from equation (38)

$$I_r \alpha \sinh \alpha l + D_1 \alpha \cosh \alpha l = E_r \gamma \cosh \alpha l + B_1 \gamma \sinh \alpha l$$

Therefore  $B_1 = I_r \alpha$

and  $D_1 \alpha = E_r \gamma$

Remembering that  $\alpha = \sqrt{zy}$

$$B_1 = \frac{I_r \sqrt{zy}}{\gamma} = \sqrt{\frac{z}{y}} I_r$$

and  $D_1 = \frac{E_r \gamma}{\sqrt{zy}} = \sqrt{\frac{y}{z}} E_r$

Substituting back in (40) and (41)

$$e = E_r \cosh \alpha l + \sqrt{\frac{z}{y}} I_r \sinh \alpha l \quad (42)$$

$$i = I_r \cosh \alpha l + \sqrt{\frac{y}{z}} E_r \sinh \alpha l \quad (43)$$

These are the general transmission line equations. They give the values of voltage and current at any distance  $l$  from the receiver end in terms of the constants of the line and the distance. The angle  $\alpha = \sqrt{zy}$  is called the complex angle of the line, and the expression  $\sqrt{\frac{z}{y}}$  is called the surge impedance.

A simpler way to express equations (42) and (43) is as follows:

$$E_s = AE_r + BI_r \quad (44)$$

$$I_s = CE_r + DI_r \quad (44a)$$

Where

$$A = \cosh \alpha l = D$$

$$B = \sqrt{\frac{z}{y}} \sinh \alpha l$$

and

$$C = \sqrt{\frac{Y}{Z}} \sinh \alpha l$$

And where  $\alpha = \sqrt{ZY} = \sqrt{(r+jx)(g+jb)}$

The conductance  $g$  is assumed equal zero. The capacity susceptance

$$b = \frac{1}{X_c} = 2\pi fC$$

As they stand, the evaluation of the constants  $A$ ,  $B$  and  $C$  is not obvious because the quantity  $\alpha l$  is a complex, say  $k + jp$ , and therefore hyperbolic sines and cosines of complex quantities are involved. It becomes necessary, then, to evaluate the expressions

$$\cosh(k + jp)$$

and

$$\sinh(k + jp)$$

The expansions of  $\sinh x$  and  $\cosh x$  give

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \text{etc.}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \text{etc.}$$

Also the expansions of  $e$  (base of the Napierian logarithms) to the  $x$  and  $-x$  powers give

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \text{etc.}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \text{etc.}$$

It follows that

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad (45)$$

and 
$$\cosh x = \frac{e^x + e^{-x}}{2} \quad (46)$$

The expansions of  $\sin x$  and  $\cos x$  give

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \text{etc.}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{etc.}$$

and the expansions of  $e^{jx}$  and  $e^{-jx}$  give

$$e^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + \dots \text{etc.}$$

$$e^{-jx} = 1 - jx - \frac{x^2}{2!} + j\frac{x^3}{3!} + \frac{x^4}{4!} - \dots \text{etc.}$$

It follows that

$$j\sin x = \frac{e^{jx} - e^{-jx}}{2} \quad (47)$$

and 
$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \quad (48)$$

From equation (45) for  $x = k + jp$

$$\begin{aligned} \sinh(k + jp) &= \frac{e^{k+jp} - e^{-k-jp}}{2} = \frac{e^k e^{jp} - e^{-k} e^{-jp}}{2} \\ &= \frac{e^k e^{jp} - e^{-k} e^{-jp}}{4} + \frac{e^k e^{jp} - e^{-k} e^{-jp}}{4} \end{aligned} \quad (49)$$

From equations (45) and (48)

$$\sinh k \cosh p = \frac{e^k e^{jp} - e^{-k} e^{-jp} - e^{-k} e^{jp} + e^k e^{-jp}}{4} \quad (50)$$

From equations (46) and (47)

$$j \cosh k \sin p = \frac{e^k e^{jp} - e^{-k} e^{-jp} + e^{-k} e^{jp} - e^k e^{-jp}}{4} \quad (51)$$

It may be readily seen that (50) + (51) = (49), therefore

$$\sinh(k + p) = \sinh k \cosh p - j \cosh k \sin p \quad (52)$$

Now, from equation (46)

$$\begin{aligned} \cosh(k + jp) &= \frac{e^{k+jp} + e^{-k-jp}}{2} \\ &= \frac{e^k e^{jp} + e^{-k} e^{-jp}}{4} + \frac{e^k e^{jp} + e^{-k} e^{-jp}}{4} \end{aligned} \quad (53)$$

Taking the product of (46) and (48)

$$\cosh k \cos p = \frac{e^k e^{jp} + e^{-k} e^{-jp} + e^{-k} e^{jp} + e^k e^{-jp}}{4} \quad (54)$$

and from (45) and (47)

$$j \sinh k \sin p = \frac{e^k e^{jp} + e^{-k} e^{-jp} - e^{-k} e^{jp} - e^k e^{-jp}}{4} \quad (55)$$

Then (55) + (54) = (53) or

$$\cosh(k + jp) = \cosh k \cosh p + j \sinh k \sin p \quad (56)$$

For convenience all the equations are grouped below.

$$E_s = A E_r + B I_r \quad (44)$$

$$I_s = C E_r + D I_r \quad (44a)$$

---


$$A = D = \cosh \alpha l = \cosh(k + jp)$$

$$\text{or } A = \cosh k \cosh p + j \sinh k \sinh p \quad (57)$$


---

$$B = \sqrt{\frac{z}{y}} \sinh \alpha l = \sqrt{\frac{z}{y}} \sinh(k + jp)$$

$$\text{or } B = \sqrt{\frac{z}{y}} (\sinh k \cosh p + j \cosh k \sinh p) \quad (58)$$


---

$$C = \sqrt{\frac{y}{z}} \sinh \alpha l = \sqrt{\frac{y}{z}} \sinh(k + jp)$$

$$\text{or } C = \sqrt{\frac{y}{z}} (\sinh k \cosh p + j \cosh k \sinh p) \quad (59)$$


---

$$\alpha l = \sqrt{zy} l = l \sqrt{(r+jx)(g+jb)} = k + jp \quad (60)$$

-----

2. Solution of Practical Problem.- Assume a line with the following requisites:

Line voltage - 220,000 volts, three-phase.

Frequency - 60 cycles.

Length - 400 miles.

Spacing - transposed, flat horizontal, 24 feet between conductors.

Conductor - 660,600 circular mil aluminum, 420,000 circular mil equivalent copper, diameter 1 inch, resistance = 0.026 ohms per 1000 ft. at 20 deg. C.

1.- Determine  $E_s$  and  $I_s$  for  $E_r = \frac{220,000}{\sqrt{3}}$  and a load of

100,000 kilowatts at 80 percent power-factor, lagging current.

2.- Determine  $E_r$  and  $I_r$  for zero load and  $E_g$  constant as per part 1. Compute charging kilovolt-amperes.

---

Solution.

---

a. Calculation of inductance and inductive reactance.

Consider equation (11), that is

$$L = \left\{ \frac{\mu}{8\pi} + 0.2 \ln. \frac{\sqrt[3]{D_1 D_2 D_3}}{r} \right\} 10^{-8} \text{ henries per cm.}$$

Substituting the corresponding values

$$L = \left( \frac{1.257}{8\pi} + 0.2 \ln. 24 \sqrt[3]{2 \times 24} \right) 10^{-8} \times 2.54 \times 12 \times 5,280 \text{ henries}$$

(per mile.

$$= \left\{ 0.05 + 0.2(2 \ln. 24 + \frac{1}{3} \ln. 2) \right\} 1.61 \times 10^{-3}$$

$$L = 2.2 \times 10^{-3} \text{ henries per mile}$$


---

$$X_L = 2\pi fL = 2\pi \times 60 \times 2.2 \times 10^{-3}$$

$$X_L = 0.83 \text{ ohms per mile}$$


---

b. -Calculation of capacitance and capacity-susceptance.

Take equation (31)

$$C = \frac{2\pi k}{\ln. \frac{\sqrt[3]{D_1 D_2 D_3}}{r}} = \frac{6\pi k}{\ln. \frac{D_1 D_2 D_3}{r^3}} \text{ farads per cm.}$$

$$C = \frac{6 \times 0.278 \times 10^{-6}}{6 \ln 2.4 + \ln 2} \times 1.61 \times 10^5 \text{ microfarads per mile.}$$

$$C = 0.0136 \text{ microfarads per mile.}$$


---

$$B_c = 2\pi fC = 2\pi \times 60 \times 0.0136 \times 10^{-6}$$

$$B_c = 5.13 \times 10^{-6} \text{ mhos per mile.}$$


---

#### c. Evaluation of A.

Take equation (60)

$$\alpha = \sqrt{zy} = \sqrt{(r+jx)(g+jb)}$$

$$r = 0.026 \times 5.28 = 0.1373 \text{ ohms per mile.}$$

$$g = 0$$

x and b as found for  $X_L$  and  $B_c$

$$\begin{aligned} \alpha &= \sqrt{(0.1373 + j0.83) j5.13 \times 10^{-6}} \\ &= 10^{-3} / \sqrt{-4.26 + j0.705} = 10^{-3} / 4.32 / 170^\circ 36' \\ &= 10^{-3} \times 2.075 / 85^\circ 18' \end{aligned}$$

$$\begin{aligned} 1 &= 400 = 0.83 / 85^\circ 18' = 0.068 + j0.827 \\ &= k + jp \end{aligned}$$

$$A = \cosh(k + jp) = \cosh 0.068 \cos 0.827 + j \sinh 0.068 \sin 0.827$$

$$A = 0.678 + j0.0504 = 0.68 / 4^\circ 15' = D$$


---

#### d. Evaluation of B.

From equation (58)

$$B = \sqrt{\frac{z}{y}} \sinh(k+jp)$$

$$\begin{aligned} \sqrt{\frac{z}{y}} &= \sqrt{\frac{0.1373 + j0.83}{j5.13 \times 10^{-6}}} = 10^3 \sqrt{\frac{0.841/80^\circ 37'}{5.13/90^\circ}} \\ &= 405/-4^\circ 41' \end{aligned}$$

$$\begin{aligned} \sinh(k+jp) &= \sinh 0.068 \cos 0.827 + j \cosh 0.068 \sin 0.827 \\ &= 0.0462 + j0.738 = 0.74/86^\circ 27' \end{aligned}$$

$$B = 405/-4^\circ 41' \times 0.74/86^\circ 27'$$

$$B = 299.6/81^\circ 46' = 42.9 - j296.7$$


---

#### e. Evaluation of C.

From equation (59)

$$C = \sqrt{\frac{y}{z}} \sinh(k+jp)$$

$$\sqrt{\frac{y}{z}} = \frac{1}{\sqrt{\frac{z}{y}}} = \frac{1}{405/-4^\circ 41'} = 2.47 \times 10^{-3}/4^\circ 41'$$

$$C = 2.47 \times 10^{-3}/4^\circ 41' \times 0.74/86^\circ 27'$$

$$C = 1.83 \times 10^{-3}/91^\circ 8' = -0.0348 \times 10^{-3} + j1.825 \times 10^{-3}$$


---

#### f. Calculation of $E_s$ .

From equation (44)

$$E_s = AE_r + BI_r$$

$$E_r = \frac{220,000}{\sqrt{3}} = 127,000/0^\circ \text{ vector volts}$$

$$I_r = \frac{P}{\sqrt{3}E\cos\theta} = \frac{100,000,000}{\sqrt{3} \times 220,000 \times 0.8} = 328.03$$

$$= 328.03/-36^\circ 52' \text{ vector amperes}$$

$$AE_r = (0.678 + j0.0504) 127,000 = 86,100 + j6,400$$

$$BI_r = 299.6/81^\circ 46' \times 328.03/-36^\circ 52'$$

$$= 98,300/44^\circ 54' = 69,600 + j69,400$$

Substituting

$$E_s = 155,700 + j75,800 = 172,800/26^\circ 3' \text{ vector volts.}$$


---

g. Calculation of  $I_s$ .

From equation (44a)

$$I_s = CE_r + DI_r$$

$$CE_r = 1.83 \times 10^{-3} /91^\circ 8' \times 127,000 = 232.5/91^\circ 8'$$

$$= -4.65 + j232.4$$

$$DI_r = 0.68/4^\circ 15' \times 328/-36^\circ 52' = 222/-32^\circ 37'$$

$$= 186.5 - j119.5$$

Substituting

$$I_s = 181.95 + j112.9 = 214/32^\circ \text{ vector amperes.}$$


---

Therefore, to deliver 100,000 kilowatts at 80% power-factor and 220,000 line voltage, the voltage at the generator end must be 172,800. The voltage at the sending end is rotated 26 degrees and 3 minutes from the voltage at the load end, leading the latter by that angle. The current at the sending end is less in magnitude than the receiver current, and leads it by 36 degrees 52 minutes plus 32 degrees, or 68 degrees 52 minutes. The power-factor angle at the sending end is 32 degrees minus 26 degrees 3 minutes, or 5 degrees 57 minutes. The vector diagram for these conditions is shown in figure 20.

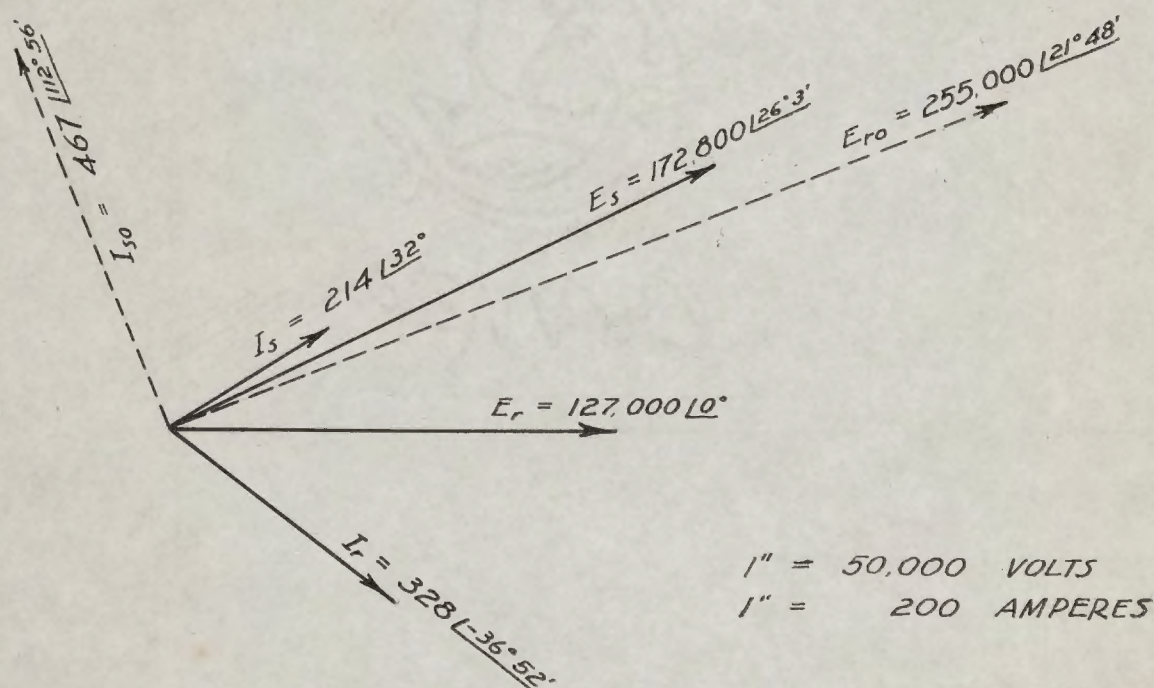


Figure 20

h. Calculation of no-load receiver voltage and sending current.

$$\begin{array}{ll} \text{At no load} & I_r = 0 \\ \text{and} & E_s = 172,800/26^\circ 3' \end{array}$$

$$\begin{array}{l} \text{But also} \\ E_s = AE_r + BI_r = AE_r \\ E_r = \frac{E_s}{A} = \frac{172,800/26^\circ 3'}{0.676/4^\circ 15'} \end{array}$$

$$E_r = 255,000/21^\circ 48'$$


---

$$I_s = CE_r + DI_r = CE_r$$

$$I_s = 1.83 \times 10^{-3} / 91^\circ 8' \times 255,000 / 21^\circ 48'$$

$$I_s = 467 / 112^\circ 56'$$


---

These vectors are plotted in figure 20 as broken lines.

Obviously, it would not be practicable to operate this line under the given load conditions without a synchronous condenser to supply the necessary reactive voltamperes and stabilize the voltage.

The charging kilovolt-amperes

$$Kva_0 = E_s I_{s0} = 172.8 \times 467 = 80,698$$


---

Transformers have not been taken into consideration in this discussion. When considered, their constants, referred to the line side, are incorporated with the line constants.

## Chapter V

### CIRCLE DIAGRAM

1. Introductory.— Circle diagrams to represent the performance of a line may be developed which, if plotted to sufficiently large scale, are useful in the calculation of important relations, besides giving an idea of the line characteristics with the clarity and conciseness peculiar to graphical methods.

The type of diagram here discussed was applied and popularized by R.D. Evans and H.K. Sels, of the Westinghouse Company in 1921.<sup>1</sup>

Hereafter all vector quantities will be underscored, and all conjugates<sup>2</sup> of vectors will be overscored. Quantities not distinguished in this manner must be interpreted as numerics.

2. Power Circle Diagram for Receiving End.— The development of the power circle equation for receiving end conditions follows.

Take the general equation

$$\underline{E}_s = \underline{A}\underline{E}_r + \underline{B}\underline{I}_r \quad (44)$$

-----

1. The Electric Journal, Dec. 1921, p. 530.

2. The conjugate of a complex quantity is the same quantity with the sign of the  $j$  term reversed.

The conjugate of (44)

$$\bar{E}_s = \bar{A}E_r + \bar{B}\bar{I}_r \quad (61)$$

The product of (44) and (61)

$$E_s \bar{E}_s = \bar{A}A E_r \bar{E}_r + \bar{A}B \bar{I}_r \bar{E}_r + \bar{A}\bar{B} E_r \bar{I}_r + \bar{B}B \bar{I}_r \bar{I}_r$$

Multiplying through by  $E_r \bar{E}_r$  and remembering that  $E \bar{E} = E^2$

$$E_s^2 E_r^2 = A^2 E_r^4 + \bar{A}B E_r \bar{I}_r E_r^2 + \bar{A}\bar{B} E_r \bar{I}_r E_r^2 + B^2 E_r \bar{I}_r \bar{E}_r \bar{I}_r \quad (62)$$

$$\begin{aligned} \text{But } \bar{E}_r \bar{I}_r &= \text{Real power} + j \text{ reactive volt-amperes} \\ &= P_r + jQ_r \end{aligned}$$

$$\text{And } E_r \bar{I}_r = P_r - jQ_r$$

Substituting these values in (62)

$$\begin{aligned} E_s^2 E_r^2 &= A^2 E_r^4 + \bar{A}B(P_r + jQ_r)E_r^2 + \bar{A}\bar{B}(P_r - jQ_r)E_r^2 \\ &\quad + B^2(P_r^2 + Q_r^2) \\ E_s^2 E_r^2 - A^2 E_r^4 &= (\bar{A}B + \bar{A}\bar{B})P_r E_r^2 - j(\bar{A}B - \bar{A}\bar{B})Q_r E_r^2 \\ &\quad + B^2(P_r^2 + Q_r^2) \end{aligned}$$

Dividing through by  $B^2$  and rearranging

$$P_r^2 + \frac{\bar{A}B + \bar{A}\bar{B}}{B^2} P_r E_r^2 + Q_r^2 - j \frac{\bar{A}B - \bar{A}\bar{B}}{B^2} Q_r E_r^2 = \frac{E_s^2 E_r^2 - A^2 E_r^4}{B^2}$$

Completing the squares

$$\begin{aligned} \left\{ P_r^2 + \frac{\bar{A}B + \bar{A}\bar{B}}{B^2} E_r^2 P_r + \left( \frac{\bar{A}B + \bar{A}\bar{B}}{2B^2} E_r^2 \right)^2 \right\} &+ \left\{ Q_r^2 - j \frac{\bar{A}B - \bar{A}\bar{B}}{B^2} Q_r E_r^2 \right. \\ &\left. + \left( j \frac{\bar{A}B - \bar{A}\bar{B}}{2B^2} E_r^2 \right)^2 \right\} = \frac{E_s^2 E_r^2 - A^2 E_r^4}{B^2} + \left\{ \frac{\bar{A}B + \bar{A}\bar{B}}{2B^2} \right\}^2 + \left\{ j \frac{\bar{A}B - \bar{A}\bar{B}}{2B^2} E_r^2 \right\}^2 \end{aligned} \quad (63)$$

Let  $\underline{A} = a_1 + ja_2$

and  $\underline{B} = b_1 + jb_2$

Then  $\underline{A}\bar{\underline{B}} = (a_1 + ja_2)(b_1 - jb_2) = (a_1b_1 + a_2b_2) + j(a_2b_1 - a_1b_2)$

and  $\bar{\underline{A}}\underline{B} = (a_1 - ja_2)(b_1 + jb_2) = (a_1b_1 + a_2b_2) - j(a_2b_1 - a_1b_2)$

Therefore, the term

$$\frac{\underline{A}\bar{\underline{B}} + \bar{\underline{A}}\underline{B}}{2B^2} = \frac{a_1b_1 + a_2b_2}{B^2} \quad (64)$$

and the term

$$\frac{\underline{A}\bar{\underline{B}} - \bar{\underline{A}}\underline{B}}{2B^2} = \frac{a_1b_2 - a_2b_1}{B^2} \quad (65)$$

The expansion of the second member of equation (63) gives

$$\begin{aligned} & \frac{E_s^2 E_r^2 - A^2 E_r^4}{B^2} + \frac{A^2 B^2 + 2\bar{\underline{A}}\underline{B}\underline{A}\bar{\underline{B}} + \bar{\underline{A}}^2 \underline{B}^2}{4B^4} E_r^4 - \frac{A^2 B^2 - 2\bar{\underline{A}}\underline{B}\underline{A}\bar{\underline{B}} + \bar{\underline{A}}^2 \underline{B}^2}{4B^4} E_r^4 \\ &= \frac{E_s^2 E_r^2 - A^2 E_r^4}{B^2} + \frac{4\bar{\underline{A}}\underline{B}\underline{A}\bar{\underline{B}}}{4B^4} E_r^4 = \frac{E_s^2 E_r^2}{B^2} - \frac{A^2 E_r^4}{B^2} + \frac{A^2 B^2}{B^4} E_r^4 \\ &= \frac{E_s^2 E_r^2}{B^2} \quad (66) \end{aligned}$$

Substituting (64), (65), and (66) back in (63)

$$\left\{ P_r + \frac{a_1b_1 + a_2b_2}{B^2} E_r^2 \right\}^2 + \left\{ Q_r - \frac{a_1b_2 - a_2b_1}{B^2} E_r^2 \right\}^2 = \frac{E_s^2 E_r^2}{B^2}$$

Or say

$$(P_r + lE_r^2)^2 + (Q_r - mE_r^2)^2 = \frac{E_s^2 E_r^2}{B^2} \quad (67)$$


---

Where  $l = \frac{a_1 b_1 + a_2 b_2}{B^2} \quad (68)$

and  $m = \frac{a_1 b_2 - a_2 b_1}{B^2} \quad (69)$

Equation (67) may be recognized as the equation of a circle of radius  $\frac{E_s E_r}{B}$ , the center of which has the coordinates  $-lE_r^2$  and  $+mE_r^2$ . Such a circle is shown plotted in figure 21. It should be noticed that the coordinates of this diagram are in the units of real power in watts horizontally and reactive volt-amperes vertically. For any given constant values of  $E_s$  and  $E_r$  the circle will give the conditions of operation of the line. Assume a load of  $P$  watts at a power factor  $= \cos\theta$ , represented by the point  $P'$ . The conditions of operation of the line are represented by  $P''$ , the power factor being  $\cos\theta'$ . Therefore, if the voltage is to be maintained the reactive leading volt -amperes represented by  $P'P''$  must be supplied to the line from a synchronous condenser. The point of tangency of the circle with the vertical marks the "static stability limit", or the theoretical point beyond which the line would cease to operate. In practice a large factor of safety must be introduced to allow for surges, so that the normal operation of the line is well below this maximum limit. All quantities in the diagram are per phase.

3. Power Circle Diagram for Sending End.- Obviously, if the same procedure is to be followed here  $E_r$  must be expressed in terms of  $E_s$  and  $I_s$ .

From equation (44-a)

$$DI_r = I_s - CE_r$$

$$\text{or} \quad I_r = \frac{I_s}{D} - \frac{C}{D}E_r \quad (70)$$

From equation (44)

$$AE_r = E_s - BI_r \quad (71)$$

Substituting (70) in (71)

$$AE_r = E_s - \frac{B}{D}I_s + \frac{BC}{D}E_r$$

$$\text{or} \quad E_r = \frac{DE_s - BI_s}{AD - BC} \quad (72)$$

But referring back to equations (57), (58) and (59)

$$A = D = \cosh \alpha l$$

$$B = \sqrt{\frac{Z}{Y}} \sinh \alpha l$$

$$C = \sqrt{\frac{Y}{Z}} \sinh \alpha l$$

Therefore

$$AD - BC = \cosh^2 \alpha l + \sinh^2 \alpha l = 1$$

-----

1. To prove this identity express  $\cosh \alpha l$  and  $\sinh \alpha l$  in terms of  $e^{\alpha l}$ , equations (45) and (46).

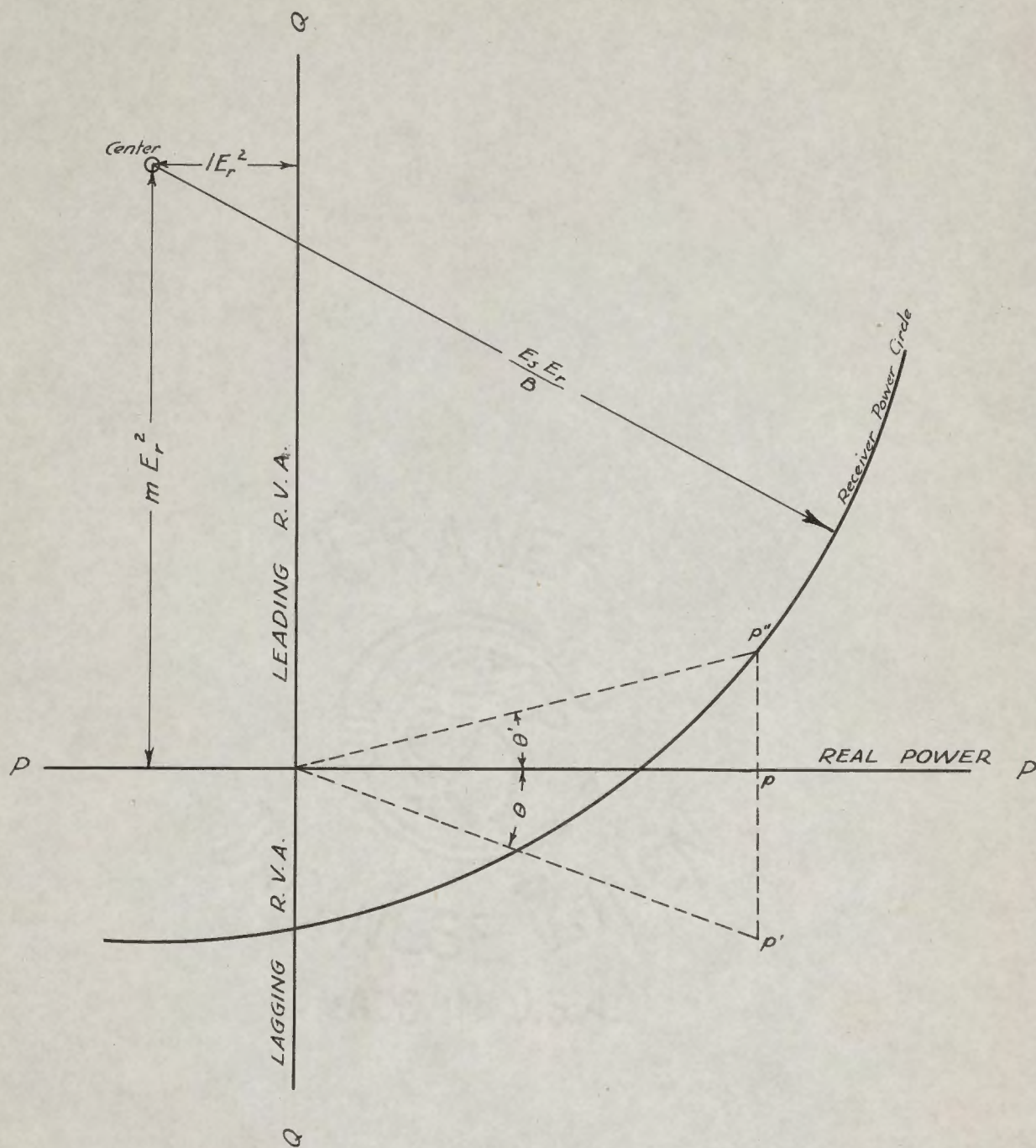


Figure 21

Equation (72) then becomes

$$E_r = DE_s - BI_s \quad (73)$$

Equation (73) is of the same form as equation (44), with the exception of the sign of the last term. The same procedure used before may be used in developing the circle equation, that is, (73) is multiplied by its conjugate, then by  $E_s \bar{E}_s$ , the squares completed and the result simplified. The details will not be repeated here. The final result is

$$(P_s - l'E_s^2)^2 + (Q_s + m'E_s^2)^2 = \frac{E_s^2 E_r^2}{B^2} \quad (74)$$


---

$$\text{where } l' = \frac{d_1 b_1 + d_2 b_2}{B^2} \quad (75)$$

$$\text{and } m' = \frac{d_1 b_2 - d_2 b_1}{B^2} \quad (76)$$

Equation (74) is that of a circle of radius  $\frac{E_s E_r}{B}$  and center at  $+l'E_s^2, -m'E_s^2$ . The units of the coordinates of this diagram are the same as for the receiving circle, namely real power and reactive volt-amperes, therefore, they may be drawn together as in figure 22.

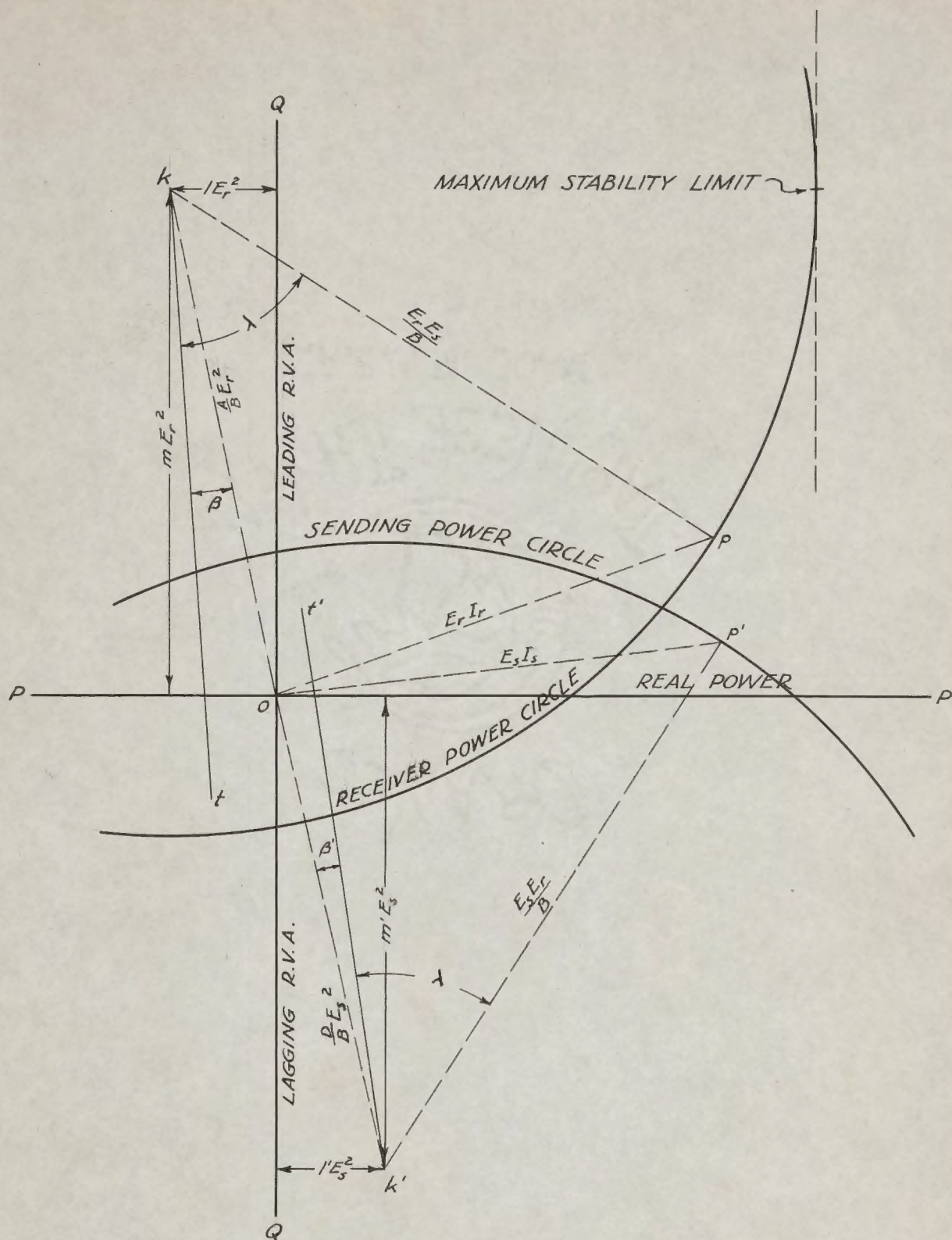


Figure 22

On observing figure 22 the question suggests itself, what are the sending conditions corresponding to any particular receiving conditions? But thus far there is no reference in the diagram to serve as a link of connection between the two circles. To obtain this reference consider first the distance  $kO$ , figure 22.

$$\begin{aligned} kO &= \sqrt{l^2 E_r^4 + m^2 E_r^4} \\ &= E_r^2 \sqrt{l^2 + m^2} \end{aligned}$$

From (68) and (69)

$$\begin{aligned} l^2 &= \frac{a_1^2 b_1^2 + 2a_1 a_2 b_1 b_2 + a_2^2 b_2^2}{B^4} \\ m^2 &= \frac{a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2}{B^4} \\ l^2 - m^2 &= \frac{a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_2^2 + a_2^2 b_1^2}{B^4} \\ &= \frac{a_1^2 (b_1^2 + b_2^2) + a_2^2 (b_2^2 + b_1^2)}{B^4} \\ &= \frac{a_1^2 B^2 + a_2^2 B^2}{B^4} = \frac{A^2}{B^2} \end{aligned}$$

Therefore

$$kO = \frac{A}{B} E_r^2$$

Now consider the vector diagram of figure 23, drawn with  $E_r$  at the origin. The triangle  $kOp$  of figure 22 is the triangle  $KOP$  of figure 23 with all sides multiplied by  $\frac{E_r}{B}$ ,

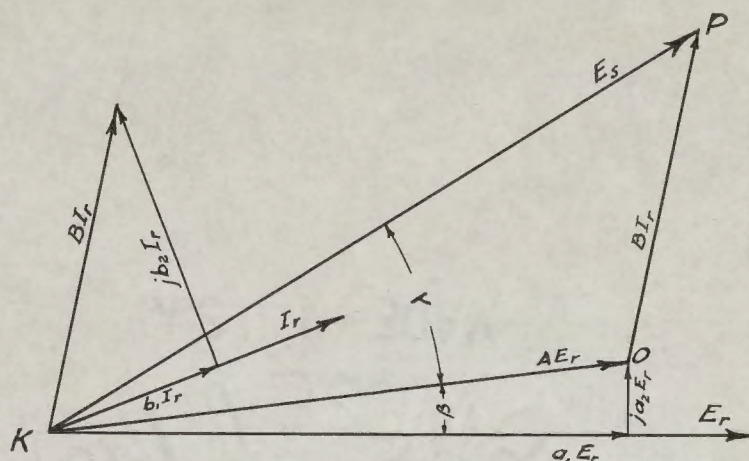


Figure 23

therefore these triangles are similar. The angle  $\lambda$  in figure 23 is the angle between sending and receiving voltages for any particular conditions. To locate this angle on the receiver circle the origin of vectors is transferred to it by means of the angle  $\beta$ , the tangent of which is the imaginary part of A divided by the real part, or  $\frac{a_2}{a_1}$ . This follows from inspection. The line  $kt$ , figure 22, is this transferred origin. Having the angle  $\lambda$  from the receiver circle for any specific conditions, the position of  $E_s$  must be found on the sending circle, that is, the origin of vectors of the sending circle must be found, so that the angle  $\lambda$  may be laid off and the corresponding sending conditions found. If the same procedure is followed for the sending circle, it is found that the distance

$$k'O = \frac{D}{B} E_s^2$$

A vector diagram for the sending circle may also be drawn, as in figure 24, which corresponds to figure 23.

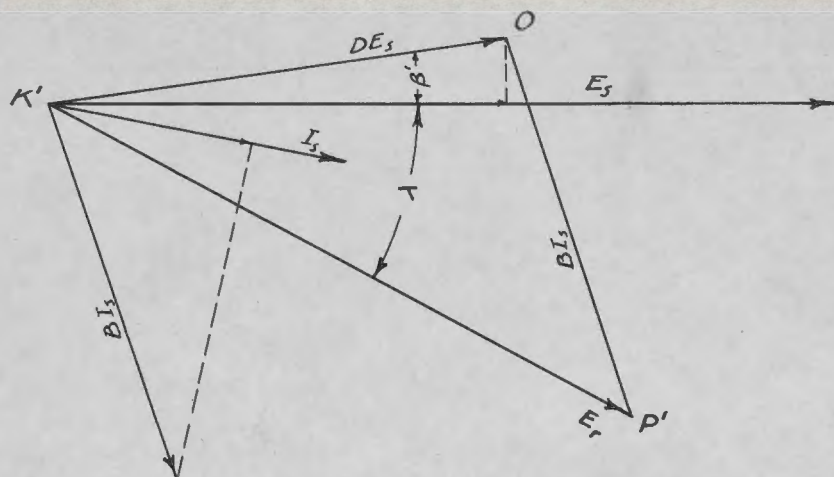


Figure 24

The triangle  $K'OP'$  of figure 24 is similar to triangle  $k'Op'$  of figure 22, the latter having all sides multiplied by  $\frac{E_s}{B}$ . Again the angle  $\lambda$  appears. The new origin of vectors is found from the angle  $\beta'$ , the tangent of which is  $\frac{d_2}{d_1}$ , locating the line  $k't'$  in figure 22. Since  $A = D$  it happens here that  $\beta = \beta'$ . Now if the sending conditions corresponding to the receiving conditions given by point  $p$  are desired, the angle  $\lambda$  measured to  $kt$  is laid off from  $k't'$ , locating the point  $p'$ .

In the following pages the circle diagram is applied to the problem of Chapter IV, and various relations obtained and curves plotted to illustrate the utility of the diagram. One more observation here seems pertinent. If the power circles from the equations just derived are drawn for different values of voltage, the various circles are not concentric. This follows

from the fact that voltage enters in the expression for the co-ordinates of the centers. Another circle diagram may be obtained by dividing the circle equation (67) by  $E_r^4$ . The resultant equation

$$\left\{ \frac{P_r}{E_r^2} + 1 \right\}^2 + \left\{ \frac{Q_r}{E_r^2} - m \right\}^2 = \frac{E_s^2}{B^2 E_r^2}$$

is still the equation of a circle, but the voltage has been eliminated from the coordinates of the center, these being simply 1 and m. Circles for different voltages are now concentric, but the units of the diagram are no longer in power and reactive volt-amperes, but in ratios of power and reactive voltamperes to the square of the voltage, or in the nature of conductance and susceptance. This type of diagram is useful for some purposes, and still others may be developed which are applicable to different classes of problems.

4. Application to Problem of Chapter IV.- The same data will be assumed. The receiving voltage  $E_r$  will be assumed constant and circles drawn for three values of sending voltage, namely

115,000 volts,  
127,000 volts, and  
127,000 volts.

From the diagram, and for an 80% power-factor load the following information will be determined,

1. The synchronous condenser capacity required for different values of power received.
2. The generated power and power-factor at the sending end for

different values of power delivered.

3. The angle between  $E_s$  and  $E_r$  for different values of receiver power.

4. The efficiency of the line.

5. The maximum stability limit for the three different voltages.

### CALCULATIONS

From equations (68) and (69)

$$l = \frac{a_1 b_1 + a_2 b_2}{B^2} = l'$$

$$m = \frac{a_1 b_2 + a_2 b_1}{B^2} = m'$$

Substituting the values found

$$l = \frac{0.678 \times 42.9 + 0.0504 \times 296.7}{42.9^2 + 296.7^2}$$

$$l = 0.0004887$$


---

$$m = \frac{0.678 \times 296.7 - 0.0504 \times 42.9}{42.9^2 + 296.7^2}$$

$$m = 0.002205$$


---

$$l E_r^2 = 7.88 \times 10^6$$


---

$$m E_r^2 = 35.6 \times 10^6$$


---

Other results are tabulated below.

| $E_s$   | $l'E_s^2$          | $m'E_s^2$          | $\frac{E_s E_r}{B}$ |
|---------|--------------------|--------------------|---------------------|
| 140,000 | $9.57 \times 10^6$ | $43.2 \times 10^6$ | $59.27 \times 10^6$ |
| 127,000 | $7.88 \times 10^6$ | $35.6 \times 10^6$ | $53.75 \times 10^6$ |
| 115,000 | $6.45 \times 10^6$ | $29.2 \times 10^6$ | $48.68 \times 10^6$ |

To locate reference lines

$$\tan \beta = \frac{a_2}{a_1} = \frac{0.0505}{0.676} = 0.0748$$

$$\beta = 4.3^\circ = \beta'$$

The diagram is shown in figure 25. The line OF, drawn at 36 degrees and 53 minutes with the P axis represents the conditions of loading. The ordinates from this line to the receiving circle represent the synchronous condenser capacity necessary for operation at each particular load. These are plotted for all three circles in figure 26. Figure 27 shows the variation of the angle  $\lambda$  between  $E_s$  and  $E_r$  as measured between the radius of the receiver circle for any particular load and the reference line kt.

The sending power corresponding to each load, as plotted in figure 28, is found by laying off the angles from figure 27 from the reference line in the sending circle. The power factor at the sending end is given by the cosine of the angle between the P axis and the line joining each point of the circle with the origin O. It is plotted in figure 29. In this

figure the efficiency is also plotted. This, of course, is the ratio of receiving to sending powers. The maximum stability limits are found by drawing vertical tangents to each circle. They are plotted in figure 30.

These curves illustrate some of the utilities of the circle diagram.

-----



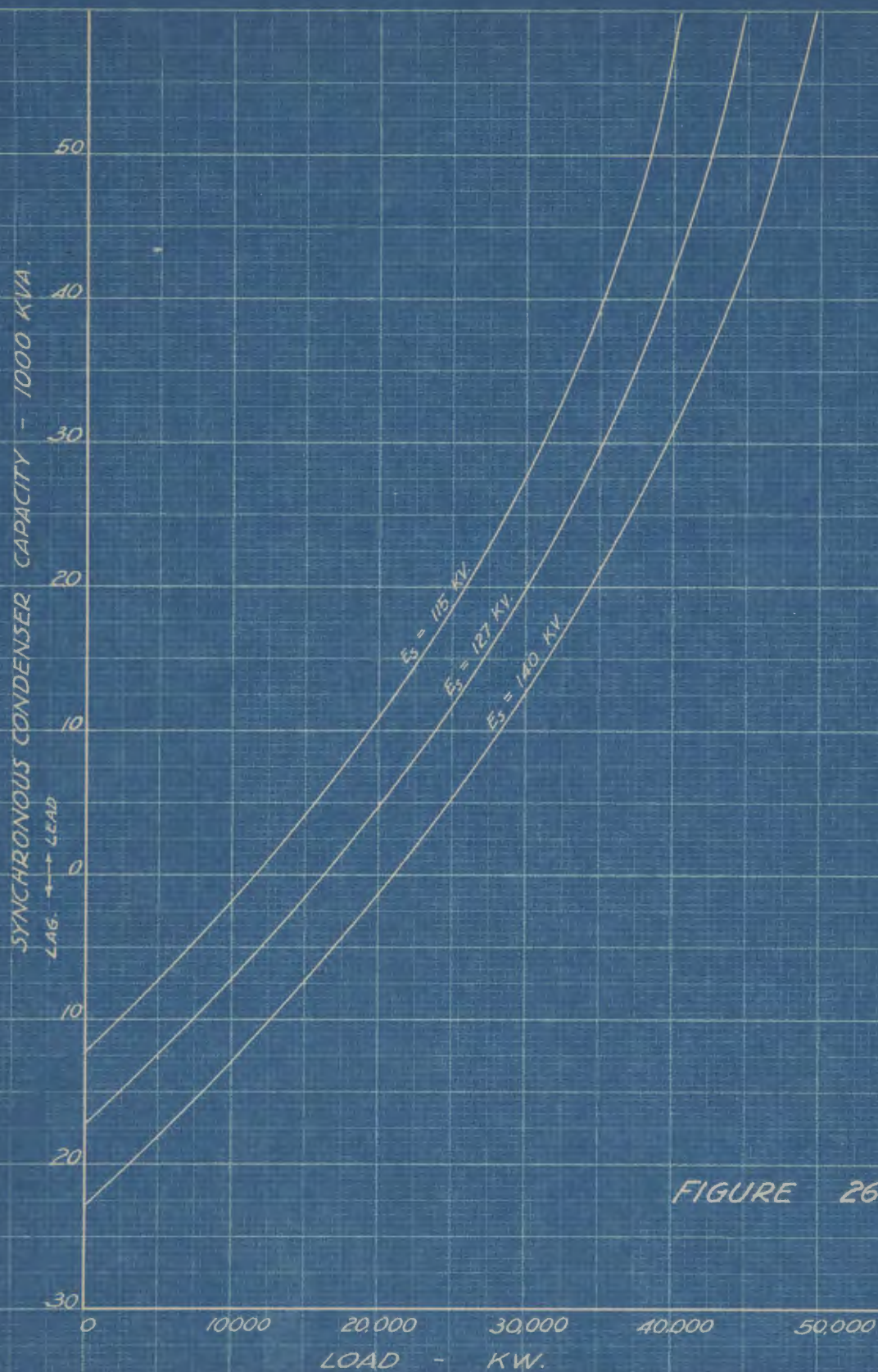


FIGURE 26

ANGLE BETWEEN  $E_s$  AND  $E_r$  - DEGREES

80

70

60

50

40

30

20

10

0

0

10,000

20,000

30,000

40,000

50,000

LOAD - KW.

$E_s = 115 \text{ KV.}$

$E_s = 127 \text{ KV.}$

$E_s = 140 \text{ KV.}$

FIGURE 27

GENERATED POWER - 1000 KW

70

60

50

40

30

20

10

0

10

20

30

40

50

LOAD - 1000 KW.

$E_s = 115 \text{ KV.}$

$E_s = 127 \text{ KV.}$

$E_s = 140 \text{ KV.}$

FIGURE 28

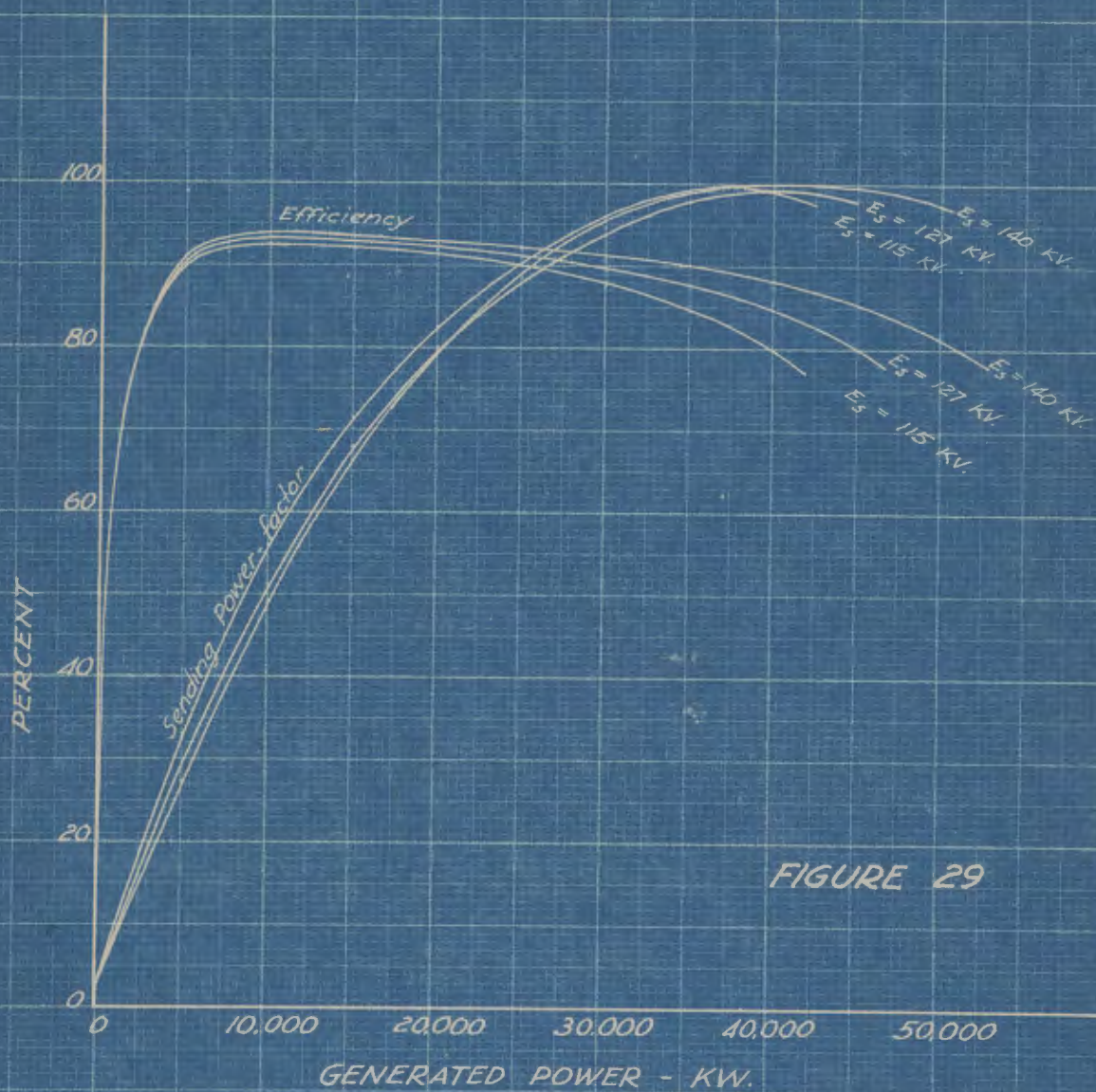


FIGURE 29

MAXIMUM STABILITY LIMIT - 1000 KW

50  
40  
30

100 110 120 130 140 150

SENDING VOLTAGE - KV.

FIGURE 30

## BIBLIOGRAPHY

Electrical Power Transmission by E.A. Loew.

Transmission Line Theory by Wm.S. Franklin and F.E. Terman.

Transmission Line Formulas by H.B. Dwight.

Circle Diagrams for Transmission Lines by R.D. Evans and

H.K. Sels, The Electric Journal, Dec.1921, p.530.

# TABULATIONS

|         | Pr<br>1000 kw. | Syn.Cond.<br>Cap.<br>1000 kva. | $\lambda$<br>deg. | P <sub>s</sub><br>1000 kw. | Kva.<br>x<br>1000 | Send.<br>p.f.<br>% | Eff.<br>% |
|---------|----------------|--------------------------------|-------------------|----------------------------|-------------------|--------------------|-----------|
| 115 KV. | 0              | -12.2                          | 1.2               | 0.5                        | 19.1              | 2.62               | 0         |
|         | 5              | - 7.3                          | 7.1               | 5.5                        | 20.2              | 27.2               | 90        |
|         | 10             | - 1.8                          | 13.4              | 10.8                       | 22.0              | 49.1               | 92.5      |
|         | 15             | 4.2                            | 19.7              | 16.3                       | 24.6              | 56.3               | 92.0      |
|         | 20             | 11.0                           | 26.8              | 22.0                       | 27.7              | 79.4               | 91.9      |
|         | 25             | 18.7                           | 34.0              | 27.8                       | 31.4              | 88.5               | 90.0      |
|         | 30             | 27.7                           | 42.7              | 34.2                       | 35.8              | 95.5               | 87.7      |
|         | 35             | 39.0                           | 53.4              | 41.1                       | 41.4              | 99.2               | 85.1      |
|         | 40             | 57.4                           | 71.8              | 50.0                       | -50.5             | 99.0               | 80.0      |
| 127 KV. | 0              | -17.3                          | 0.2               | 0.4                        | 17.5              | 2.5                | 0         |
|         | 5              | -12.6                          | 5.7               | 5.5                        | 18.8              | 29.2               | 90.5      |
|         | 10             | - 7.3                          | 11.2              | 10.7                       | 20.9              | 51.2               | 93.5      |
|         | 15             | - 1.5                          | 17.1              | 16.1                       | 23.7              | 58.0               | 93.2      |
|         | 20             | 5.0                            | 22.8              | 21.7                       | 27.1              | 80.0               | 92.2      |
|         | 25             | 12.1                           | 29.3              | 27.5                       | 31.0              | 88.8               | 91.0      |
|         | 30             | 20.3                           | 36.4              | 33.5                       | 35.4              | 94.7               | 89.5      |
|         | 35             | 29.6                           | 44.5              | 39.8                       | 40.7              | 97.8               | 87.9      |
|         | 40             | 41.4                           | 54.4              | 46.9                       | 46.9              | 100                | 85.3      |
|         | 45             | 60.5                           | 71.3              | 56.0                       | -57.2             | 97.9               | 80.3      |
| 140 KV. | 0              | -22.8                          | -0.7              | 0.4                        | 15.2              | 2.6                | 0         |
|         | 5              | -18.2                          | 4.4               | 5.5                        | 16.6              | 33.1               | 91.5      |
|         | 10             | -13.0                          | 9.3               | 10.7                       | 19.1              | 56.0               | 93.5      |
|         | 15             | - 7.5                          | 14.5              | 16.0                       | 22.3              | 71.7               | 93.7      |
|         | 20             | - 1.4                          | 19.7              | 21.5                       | 26.0              | 82.7               | 93.0      |
|         | 25             | 5.4                            | 25.2              | 27.2                       | 30.2              | 90.0               | 92.0      |
|         | 30             | 12.8                           | 31.4              | 33.0                       | 34.8              | 94.8               | 90.9      |
|         | 35             | 21.3                           | 37.9              | 39.1                       | 39.9              | 98.0               | 89.5      |
|         | 40             | 31.0                           | 45.5              | 45.6                       | 45.7              | 99.8               | 87.7      |
|         | 45             | 42.8                           | 54.5              | 52.7                       | -52.8             | 99.8               | 85.4      |
|         | 50             | 61.3                           | 68.5              | 51.5                       | -63.3             | 97.2               | 81.3      |

# APPROVAL

The foregoing thesis is hereby approved as a creditable study of an engineering subject, carried out and presented in a manner sufficiently satisfactory to warrant its acceptance as a prerequisite to the degree for which it has been submitted. It is to be understood that by this approval the undersigned does not necessarily endorse or approve any statement made, opinions expressed, or conclusions drawn therein, but approves the thesis only for the purpose for which it is submitted.

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