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A THEORETICAL MODEL FOR THE REYNOLDS STRESSES NEAR SEPARATION

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Forrest Wade Callicutt

December 1978

137460

DEDICATION

This thesis is dedicated to my parents, Betty and Wade Callicutt. They have unselfishly provided me with many educational opportunities as well as given me the timely encouragements necessary to complete this task. Their constant love has made many things possible and I wish to present this as a token of my love and appreciation.

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ABSTRACT

The region of separation in a turbulent boundary layer flow is considered. The complexity of the flow and its corresponding equations is noted. A model, valid for a slowly separating region, with constant mean velocities and relatively small fluctuation intensities is developed. This model is expressed as a set of four linear partial differential equations.

An approximation for the Reynolds stress tensor is derived. It is found by first solving for the fluctuating velocities and then forming the appropriate correlations. This is accomplished by uncoupling the four governing equations via the vorticity components. The fluctuating velocities are then determined from the vorticity components.

Approximations to the fluctuating velocities are made and then correlations are constructed to form the Reynolds stresses. It is found that the Reynolds stresses can be represented by a polynomial in y. Recent experimental work may be used to define the coefficients of the polynomial approximations and thus allow Reynolds stress evaluations throughout the separated region.

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LIST OF SYMBOLS

A,B,C,D	Arbitrary constants
A_n,B_n	Arbitrary constants
$E[\ ]$	Correlation operator
$e^x,\exp[\ ]$	Exponential function
f,g,h	Boundary values of the vorticity components
$\bar{f},\bar{g},\bar{h}$	Weighted mean average of the boundary values
F_u,F_v,F_w	Velocity components at edge of separation
$g_{m,n}(y,\eta)$	y component of Green's function
$G(x,y,z,\xi,\eta,\zeta)$	Green's function
i	Index notation and the complex number $\sqrt{-1}$
I	Identity matrix
I_3	In Chapter I, set of integers $\{1,2,3\}$
I_i,J_i	Integrals used in defining the solutions
m,n	Indices
$\mathcal{O}$	Landau's notation
p	Pressure
$\bar{P}$	Mean pressure
p'	Fluctuating pressure
r	Dummy coordinate
$\vec{r}$	Position vector
S	Boundary surface of solution domain
t	Time coordinate

T	Time constant
u_i, u, v, w	Velocity components
$\overline{U}_i, \overline{U}, \overline{V}, \overline{W}$	Mean velocity components
u_i', u', v', w'	Fluctuating velocity components
x_i, x, y, z	Rectangular coordinates
Y	Edge of separation
α_n	Arbitrary constant
$\delta(x - \xi)$	Dirac delta function
δ	Boundary layer thickness
$\sum$	Summation
ρ	Density
ν	Kinematic viscosity
ω	Vorticity
ξ, η, ζ	Dummy coordinates
ξ^*	Fixed dummy coordinate
∇	Differential operator, $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$
∇^2	Laplace operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$< >$	Temporal average operator

CHAPTER I

INTRODUCTION

The turbulent boundary layer resulting from low speed fluid flow near a wall is mathematically characterized by the Navier-Stokes equations for a constant density fluid:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = \nu \frac{\partial u_i}{\partial x_j \partial x_j} \qquad (1.1)$$

for $i \in I_3$ and the continuity equation:

$$\frac{\partial u_j}{\partial x_j} = 0 \quad .$$

It is assumed that temperature fluctuations are not of interest in the present study so that it is not necessary to introduce the energy equation. The intricate nature of turbulent motion prohibits a complete description of the motion in terms of time and space, either experimentally or theoretically. Visual data by both Hinze [1][1] and Schlichting [2] demonstrate the general randomness and complexity of turbulent flow. Derivations of solutions to the exact Navier-Stokes equations are

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[1]Numbers in brackets refer to similarly numbered references in the List of References.

in general impossible. The nonlinearity of the equations has proved to make analytical solutions unattainable, even for laminar flow (except in a few particular cases, Chapter V, Schlichting [2]). Numerical solutions appear to be as limited because of the enormous computer run times involved (Chapter I, Cebeci, Tuncer and Smith [3]).

Thus, although it is impossible to describe occurrences at a specific time and point in space, averaging of velocities and pressures yield quantities which can be used to statistically characterize the flow. Now consider the Reynolds decompositions of velocities and pressure:

$$u_i(x,y,z,t) = \overline{U}_i(x,y,z) + u_i'(x,y,z,t), \ \forall\ i \in I_3$$

$$p(x,y,z,t) = \overline{P}(x,y,z) + p'(x,y,z,t)$$

where the dashed values are fluctuations and the over-scored quantities represent mean values. Here the mean values (the flow is assumed to be statistically stationary) are defined by the temporal average operator:

$$\overline{U}_i(x_1,x_2,x_3) = \langle u_i(x_1,x_2,x_3,t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T u_i(x_1,x_2,x_3,t)\, dt \quad .$$

From the continuity equation it follows that $u_i \frac{\partial u_j}{\partial x_j} = 0$, for $i \in I_3$, which when added to the Navier-Stokes equation produces the equation:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i\ u_j)}{x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = \nu \frac{\partial^2 u_i}{\partial x_j\ \partial x_j} \quad , \quad i \in I_3 \quad .$$

Replacing the instantaneous velocities by their Reynolds decomposition counterparts and applying the mean value operation to the above equations produces the Reynolds averaged equations:

$$\overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \langle u_i'\ u_j' \rangle}{\partial x_j} + \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} = \nu \frac{\partial^2 \overline{U}_i}{\partial x_j\ \partial x_j} \quad , \quad i \in I_3 \quad ,$$

where certain elementary operations with the mean value operation have been applied. Similarly from the continuity equation it follows that

$$\frac{\partial \overline{U}_j}{\partial x_j} = 0 \qquad \text{and} \qquad \frac{\partial u_j'}{\partial x_j} = 0$$

and represents a constraint that must be imposed upon the velocity fluctuations, u_i' , and consequently has implications for the spatial variation of the Reynolds stresses $\langle u_i'\ u_j' \rangle$.

Thus, by the introduction of statistical methods, i.e., Reynolds decompositions and the mean value operator, it is seen that the original equations, which could

theoretically be solved, have now been replaced by an indeterminate system. The unknown quantities, the Reynolds stress tensor $\langle u_i' u_j' \rangle \equiv \langle u_j' u_i' \rangle$, have increased the number of unknown functions from 4 to 10 while the number of equations remains unchanged. In addition, no physical principles are available to provide guidance for the determination of the Reynolds stresses, so heuristic ideas must be followed.

Since only a limited number of cases allow solution of the exact Navier-Stokes equations, empirical methods have been developed for calculating turbulent flows. Using these methods, calculation techniques have been developed for turbulent flow but as the flow approaches separation their validity and usefulness rapidly diminish. Reasons for the breakdown of these methods include inadequacies in the turbulent modeling and the unavailability of reliable assumptions to use in the development of such models. Experimental works are few in number because of the difficulties, time consumption, and expense involved in deriving useful data.

In the light of these difficulties an attempt is made to develop a viable calculation method using recent experimental work by Simpson, Strickland, and Barr [4] for empirical input, which will model the Reynolds averaged Navier-Stokes and continuity equations in the region of separation.

CHAPTER II

MODELING THE FLOW

A model of the Navier-Stokes equations which will be compatible with the physical attributes of the flow and which will be more amenable to analysis is needed. As pointed out in Chapter I the complex nature of the Navier-Stokes equations suggests that some simplification is required before much progress can be made towards an analytic solution. The most satisfactory approach would be simplifications based upon the precepts of matched asymptotic expansion analysis. The use of perturbation methods in turbulence problems is currently not well developed, nor understood, and therefore a less formal method based on empirical data [4] will be employed in what follows.

The area of interest in the present analysis is the thin region of the boundary layer next to the wall. This region is known to be highly viscous due to the need to satisfy the no-slip boundary condition at the wall (i.e., $\bar{U}_i = u_i' = 0$ at $x_2 = 0$). The dominating viscous stress term is $\frac{\partial^2 u_i}{\partial x_2^2}$, $i = 1,2,3$. This simplification of the momentum equations in Equation (1.1) results in

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = \nu \frac{\partial^2 u_i}{\partial x_2^2} \quad , \; i \in I_3 \quad .$$

Experimental work by Reynolds [5] has shown that it is this no-slip boundary condition and the resulting large mean velocity gradient close to the wall that is responsible for the turbulent energy production. In separated flows where $\frac{\partial \overline{U}_1}{\partial x_2}$ is usually quite small close to the wall (Figure 1), there is a corresponding reduction in the turbulent energy term, $\langle u_i' \; u_j' \rangle \frac{\partial \overline{U}_i}{\partial x_j}$. There is no need to suppose, however, that gradients of the fluctuations will be small close to the wall near separation.

Additional simplifications are possible once the flow is considered to be slowly separating, which is done in the present case. This basic assumption made for the fluid flow in this problem corresponds to the situation that the mean pressure is essentially a function of x_1 , alone, i.e., $\overline{P} = \overline{P}(x_1)$, which implies that the mean streamline curvature is small and that $\overline{U}_1$ = constant, $\overline{U}_2 = \overline{U}_3 = 0$; the mean motion is essentially one-dimensional. These assumptions further imply that the statistical properties of the motion are independent of x_3 , that is the fluid motion is two-dimensional in a statistical sense, but it is important that the fluctuating quantities still remain

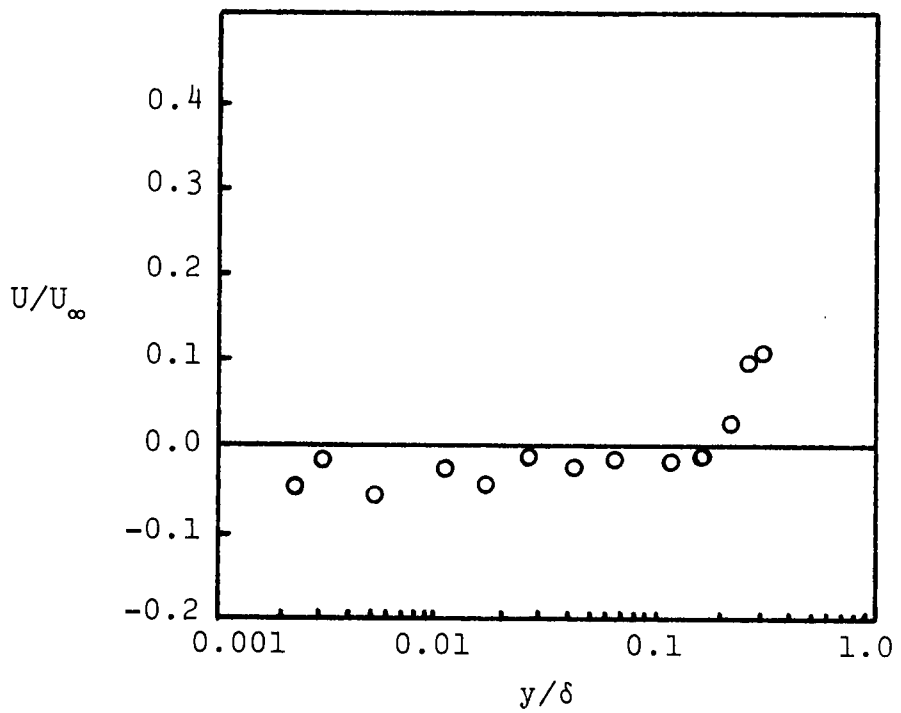


Figure 1. Mean Velocity Profiles Near the Wall in the Separation Region. Data from Simpson, et al. [4].

functions of x_1, x_2, x_3, and t . This mean flow field that has been assumed is the simplest model available for a boundary layer that is slowly separating and allows further simplification of the momentum equations:

$$\frac{\partial u_i'}{\partial t} + (\bar{U}_1 + u_1') \frac{\partial u_i'}{\partial x_1} + u_2' \frac{\partial u_i'}{\partial x_2} + u_3' \frac{\partial u_i'}{\partial x_3} + \frac{1}{\rho} \frac{(\bar{P} + p')}{\partial x_i}$$

$$= \nu \frac{\partial^2 u_i'}{\partial x_2^2} \quad .$$

The final assumptions about the separated region of the boundary layer flow close to the wall can now be stated. In this region the inertia forces are assumed to be small (i.e., $u_j' \frac{\partial u_i'}{\partial x_j}$ can be neglected in comparison to the product $\bar{U}_j \frac{\partial u_i'}{\partial x_j}$), and all of the turbulent velocity fluctuation intensities are considered to be of comparable magnitudes, thus all three equations of momentum are treated equally. It is known physically that in the region of separation the fluctuation gradients in the x_1 direction as well as the time gradients are a dominant part of the overall behavior of the flow. Also it is supposed that the pressure fluctuations make a contribution to the dynamics of the flow, even though the pressure-velocity correlations are usually considered to be small.

With regard to these last simplifications and introducing a new notation where $x = x_1$, $y = x_2$, $z = x_3$, $u = u_1$, $v = u_2$ and $w = u_3$, the final form of the simplified momentum equations is:

$$\frac{\partial u'}{\partial t} + \overline{U} \frac{\partial u'}{\partial x} + \frac{1}{\rho} \frac{d\overline{P}}{dx} + \frac{1}{\rho} \frac{\partial p'}{\partial x} = \nu \frac{\partial^2 u'}{\partial y^2}$$

$$\frac{\partial v'}{\partial t} + \overline{U} \frac{\partial v'}{\partial x} + \frac{1}{\rho} \frac{\partial p'}{\partial y} = \nu \frac{\partial^2 v'}{\partial y^2}$$

$$\frac{\partial w'}{\partial t} + \overline{U} \frac{\partial w'}{\partial x} + \frac{1}{\rho} \frac{\partial p'}{\partial z} = \frac{\partial^2 w'}{\partial y^2} \quad .$$

These equations along with the continuity equation comprise the complete set of equations defining a model of the slowly separating region of the boundary layer. The object is to calculate the fluctuations u' , v' and w' from these equations as functions of space and time and then form the required correlations for the Reynolds stress tensor.

A crucial feature of these equations is that they are now linear in the velocity components. While it is known that linear equations give a good representation of the attached flow sublayer, it is the assumption that the turbulent intensities are small that gives some validity to the use of linear equations in the present case.

Several features of the present study represent departures from previous work. One previous study

by van Driest [3], pp. 213-221, concerned an approach that was very simple but which ignored the continuity equation and resulted in questionable solutions. In his discussion of the attached flow sublayer, van Driest solved the linear diffusion equation

$$\frac{\partial u'}{\partial t} = \nu \frac{\partial^2 u'}{\partial y^2}$$

for a sinusoidally oscillating boundary. The solution indicated that $u' = u'_o(1 - e^{-ny})$. Now, instead of imposing the continuity equation as a restriction on the velocity fluctuations, van Driest also supposes that $v' = v'_o(1 - e^{-ny})$ which gives for the velocity correlation in the sublayer:

$$\langle u'v' \rangle = \langle u'_o v'_o \rangle (1 - e^{-ny})^2 \quad ,$$

a result that must be called in question.

CHAPTER III

SOLUTIONS OF THE GOVERNING EQUATIONS

A. Outline of the Solution Procedure

A method for developing a solution of the following linear second order partial differential equations derived in Chapter II is discussed. The equations are restated here for completeness:

$$\frac{\partial u'}{\partial t} + \overline{U}\frac{\partial u'}{\partial x} + \frac{1}{\rho}\frac{d\overline{P}}{dx} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = \nu\frac{\partial^2 u'}{\partial y^2}$$

$$\frac{\partial v'}{\partial t} + \overline{U}\frac{\partial v'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial y} = \nu\frac{\partial^2 v'}{\partial y^2}$$

$$\frac{\partial w'}{\partial t} + \overline{U}\frac{\partial w'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial z} = \nu\frac{\partial^2 w'}{\partial y^2}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \ . \qquad (3.1)$$

The domain of the solution is chosen to represent the separated region of a turbulent boundary layer flow as shown in Figure 2. The fluctuating velocities and pressure are assumed to be described in terms of stationary random processes throughout the domain and on its boundaries. At the wall, both the mean and fluctuating velocities are zero which implies that the corresponding gradients in

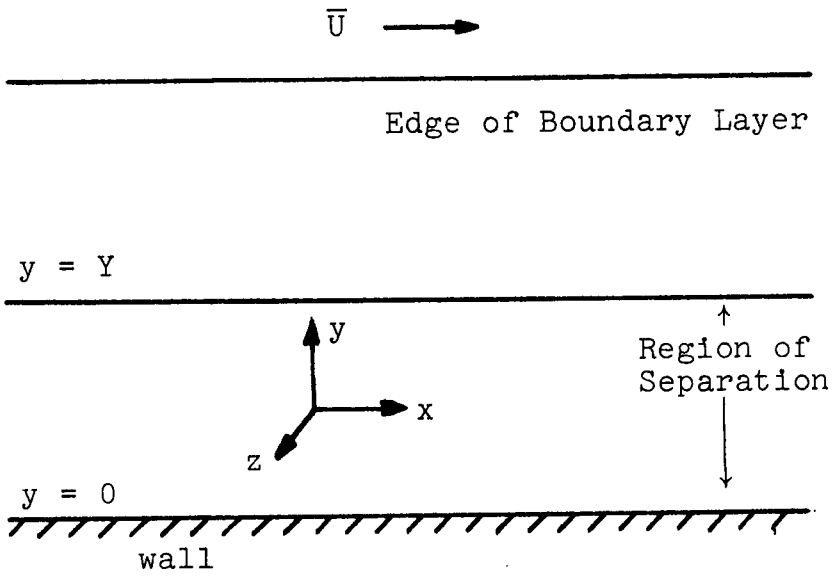


Figure 2. Domain of the Solution is the Region Where the Coordinates are Defined as Follows: $-\infty < x < \infty$
$0 < y < Y$
$-\infty < z < \infty$.

the x and z directions are also zero. At $y = Y$, the edge of the separated region within the boundary layer, u', v' and w' are expressed in terms of the random functions $F_u(x,z,t)$, $F_v(x,z,t)$ and $F_w(x,z,t)$ respectively. These functions are used to denote the turbulent activity associated with the flow external to the separation region.

Equations (3.1) can be uncoupled by use of the vorticity components:

$$\omega_x = \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}$$

$$\omega_y = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}$$

$$\omega_z = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \quad .$$

The momentum equations in (3.1) may be expressed in vector notation:

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial y^2}\right) I \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho}\left(\frac{d\bar{P}}{dx} + \frac{\partial p'}{\partial x}\right) \\ -\frac{1}{\rho}\frac{\partial p'}{\partial y} \\ -\frac{1}{\rho}\frac{\partial p'}{\partial z} \end{pmatrix}$$

where I is the identity matrix. The curl of this expression results in the equation:

$$\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}-\nu\frac{\partial^2}{\partial y^2}\right) I \begin{bmatrix} \frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z} \\ \frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x} \\ \frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\rho}\left(\frac{\partial}{\partial y}\left(\frac{p'}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial p'}{\partial y}\right)\right) \\ -\frac{1}{\rho}\left(\frac{\partial}{\partial z}\left(\frac{d\overline{P}}{dx}+\frac{\partial p'}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial p'}{\partial z}\right)\right) \\ -\frac{1}{\rho}\left(-\frac{\partial}{\partial y}\left(\frac{d\overline{P}}{dX}+\frac{\partial p'}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial p'}{\partial y}\right)\right) \end{bmatrix}$$

Since the mean pressure $\overline{P}$ is a function of x alone, i.e., $\overline{P} = \overline{P}(X)$, both $\frac{\partial}{\partial z}\left(\frac{\partial \overline{P}}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial \overline{P}}{\partial x}\right)$ are zero. Thus the vorticity components are seen to satisfy identical equations of the form:

$$\frac{\partial \omega}{\partial t} + \overline{U}\,\frac{\partial \omega}{\partial x} = \nu\,\frac{\partial^2 \omega}{\partial y^2}$$

where ω denotes one of the quantities ω_x, ω_y ω_z . At the wall the vorticity components are assumed to be known and are given the values $f_o(x,z,t)$, $g_o(x,z,t)$ and $h_o(x,z,t)$ (corresponding respectively to u', v' and w'). Likewise at the boundary, y = Y , the vorticity components are assumed to be known and are given the values $f_1(x,z,t)$, $g_1(x,z,t)$ and $h_1(x,z,t)$ (respective again to u', v' and w'). These quantities will relate to the spatial gradients of the functions F_u , F_v and F_w which were defined above.

Once ω_x, ω_y and ω_z are known as functions of x, y, z and t , equations for u', v' and w' can be found by manipulation of the vorticity equations and the continuity

equation. A solution of the resulting partial differential equation then gives u', v' and w' . For example, to find u' , consider the following derivatives of ω_y, ω_z and the continuity equation:

$$\frac{\partial}{\partial z}(\omega_y) = \frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)$$

$$-\frac{\partial}{\partial y}(\omega_z) = -\frac{\partial}{\partial y}\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)$$

$$0 = \frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) .$$

Addition of these three equations yields Poisson's equation for u' :

$$\nabla^2 u' = \frac{\partial \omega_y}{\partial z} - \frac{\partial \omega_z}{\partial y}$$

where ∇^2 is the Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. ω_y and ω_z are known from the previous problem and u' is assumed to be known at the boundaries. The corresponding equations for v' and w' are

$$\nabla^2 v' = \frac{\partial \omega_z}{\partial x} - \frac{\partial \omega_x}{\partial z} ,$$

$$\nabla^2 w' = \frac{\partial \omega_x}{\partial y} - \frac{\partial \omega_y}{\partial z} .$$

Thus the original system of linear equations has been uncoupled and the fluctuating velocities may be found by solving two independent second-order partial differential equations; the diffusion equation, and Poisson's equation.

B. Solution for the Fluctuating Vorticity Components

Separation of variables is used to develop a solution of the equation for the vorticity components:

$$\nu \frac{\partial^2 \omega}{\partial y^2} = \overline{U} \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial t} \quad ,$$

where ω can be ω_x, ω_y or ω_z . Let $\xi = \frac{1}{2}\left(\frac{X}{\overline{U}} + t\right)$ and notice that and $\frac{\partial \xi}{\partial t} = \frac{1}{2}$. The above equation can then be rewritten:

$$\nu \frac{\partial^2 \omega}{\partial y^2} = \overline{U} \frac{\partial \omega}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \omega}{\partial \xi} \frac{\partial \xi}{\partial t}$$

$$= \frac{\partial \omega}{\partial \xi} \quad .$$

To find the solution for the vorticity component ω_x , for example, subject to nonhomogeneous boundary conditions at $y = 0$ and $y = Y$, the problem is divided into three simpler subproblems. Introduce the decomposition $\omega_x = \omega_H + \omega_B + \omega_D$ so that

$$\nu \frac{\partial^2}{\partial y^2} (\omega_H + \omega_B + \omega_D) = \frac{\partial}{\partial \xi} (\omega_H + \omega_B + \omega_D)$$

with

$$f_o(x,z,t) = \omega_H + \omega_B + \omega_D \qquad \text{at } y = 0$$
$$f_1(x,z,t) = \omega_H + \omega_B + \omega_D \qquad \text{at } y = Y$$
$$f_2(y,z) = \omega_H + \omega_B + \omega_D \qquad \text{at } \xi = \xi^* \ .$$

The vorticity distribution is assumed to be known across the boundary layer for some value of $\xi = \xi^*$. The quantities f_o, f_1 and f_2 are stationary random functions representing the vorticity on the boundaries. Further let the quantities ω_H, ω_B and ω_D satisfy the following boundary value problems:

i) $$\nu \frac{\partial^2 \omega_H}{\partial y^2} = \frac{\partial \omega_H}{\partial \xi}$$ with $\omega_H = 0$ at $y = 0$; $\omega_H = 0$ at $y = Y$; $\omega_H = f_2(y,z)$ at $\xi = \xi^*$.

ii) $$\nu \frac{\partial^2 \omega_B}{\partial y^2} = 0$$ with $\omega_B = f_o(\xi,z)$ at $y = 0$; $\omega_B = f_1(\xi,z)$ at $y = Y$.

iii) $$\nu \frac{\partial^2 \omega_D}{\partial y^2} = \frac{\partial \omega_B}{\partial \xi} + \frac{\partial \omega_D}{\partial \xi}$$ with $\omega_D = 0$ at $y = 0$; $\omega_D = 0$ at $y = Y$.

Thus the quantity ω_H satisfies the homogeneous part of the problem, ω_B satisfies the boundary conditions and ω_D is

a particular solution that forces the sum $\omega_B + \omega_D$ to satisfy the diffusion equation.

Using the separation of variables technique the solution to i) is:

$$\omega_H = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi}{Y} y \, e^{-\nu(\frac{n\pi}{Y})^2 \xi} ,$$

where once the complete solution, ω_x , is known, α_n is found by evaluating ω_x at $\xi = \xi^*$ and will be described in terms of $f_2(y,z)$ and the rest of the solution which is to be determined. The solution to ii) which satisfies the boundary conditions at $y = 0$ and $y = Y$, is simply the first degree polynomial in y :

$$\omega_B = \frac{f_1(\xi,z) - f_2(\xi,z)}{Y} y + f_o(\xi,z) .$$

Lastly in finding a particular solution of boundary value problem iii) an expansion for $\frac{\partial \omega_B}{\partial \xi}$ in terms of the eigenfunctions, $\sin \frac{n\pi}{Y} y$, of the homogeneous part, $\frac{\partial \omega_H}{\partial \xi} = \nu \frac{\partial^2 \omega_H}{\partial y^2}$, is assumed. ω_D is then of the form:

$$\omega_D = \sum_{n=1}^{\infty} e^{-\nu\left(\frac{n\pi}{Y}\right)^2 \xi}\left[\int_{-\infty}^{\xi} - \frac{\int_0^Y \left(\frac{\partial}{\partial\eta}\left[\frac{f_1(\xi,z)-f_o(\xi,z)}{Y}y+f_o(\xi,z)\right]\right)\sin\frac{n\pi}{Y}y\,dy}{\int_0^Y \sin^2\frac{n\pi}{Y}y\,dy}\right.$$

$$\left. \cdot\, e^{\nu\left(\frac{n\pi}{Y}\right)^2 \eta}\, d\eta\right] \sin\frac{n\pi}{Y}y$$

which simplifies to give the expression:

$$\omega_D = \sum_{n=1}^{\infty} e^{-\nu\left(\frac{n\pi}{Y}\right)^2 \xi}\left[\frac{2}{n\pi}\int_{-\infty}^{\xi} \frac{\partial}{\partial\eta}\left((-1)^n f_1(\xi,z)+f_o(\xi,z)\right)e^{\nu\left(\frac{n\pi}{Y}\right)^2 \eta}\, d\eta\right]\sin\frac{n\pi}{Y}y$$

The integral within the summation is integrated by parts and then a change of variables, $r = \xi - \eta$, is introduced to yield

$$e^{-\nu\left(\frac{n\pi}{Y}\right)^2\xi}\int_{-\infty}^{\xi}\frac{\partial}{\partial\eta}\left[(-1)^n f_1(\xi,z)+f_o(\xi,z)\right]e^{\nu\left(\frac{n\pi}{Y}\right)^2\eta}\,d\eta$$

$$=\left[(-1)^n f_1(\xi,z)-f_o(\xi,z)\right]$$

$$-\int_o^{\infty}\left[(-1)^n f_1(\xi-r,z)-f_o(\xi-r,z)\right]\nu\left(\frac{n\pi}{Y}\right)^2 e^{-\nu\left(\frac{n\pi}{Y}\right)^2 r}\,dr\ .$$

The form of this last integral is that of a low pass filter which acts upon the random function, $\left[(-1)^n f_1(\xi,z) - f_o(\xi,z)\right]$ and returns a weighted mean value of the function represented here by $\left[(-1)^n \bar{f}_1(z) - \bar{f}_o(z)\right]$, Bendat [6] p. 15. Hence ω_D is now of the form:

$$\omega_D = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{Y}\,y\left[\left[(-1)^n f_1(\xi,z) - f_o(\xi,z)\right] - \left[(-1)^n \bar{f}_1(z) - \bar{f}_o(z)\right]\right]$$

where $\bar{f}_1(z)$ and $\bar{f}_o(z)$ are the weighted mean values, with respect to ξ , of $\bar{f}_1(\xi,z)$ and $\bar{f}_o(\xi,z)$ respectively. This form obtained for ω_D may be summed exactly to give:

$$\omega_D = \frac{1}{2}\left[\left(-f_1(\xi,z) - f_o(\xi,z)\right) - \left((-\bar{f}_1(z) - \bar{f}_o(z)\right)\right]$$

$$+ \left(\frac{1}{2} - \frac{y}{Y}\right)\left[\left(f_1(\xi,z) - f_o(\xi,z)\right) - \left(f_1(z) - \bar{f}_o(z)\right)\right] .$$

Thus the complete solution to the fluctuating vorticity component with respect to the x direction is of the form:

$$\omega_X = \sum_{n=1}^{\infty}\left[\alpha_n \sin\frac{n\pi}{Y}\, y\, e^{-\nu\left(\frac{n\pi}{Y}\right)^2 \xi}\right]$$

$$+ \left[\frac{f_1(\xi,z) - f_o(\xi,z)}{Y}\, y + f_o(\xi,z)\right]$$

$$- \frac{1}{2}\left[\left(f_1(\xi,z) + f_o(\xi,z)\right) + \left(-f_1(z) - f_o(z)\right)\right]$$

$$+ \left(\frac{1}{2} - \frac{y}{Y}\right)\left[\left(f_1(\xi,z) - f_o(\xi,z)\right) - \left(\bar{f}_1(z) - \bar{f}_o(z)\right)\right]$$

$$= \sum_{n=1}^{\infty}\left[\alpha_n \sin\frac{n\pi}{Y} y\, e^{-\nu\left(\frac{n\pi}{Y}\right)^2 \xi}\right] + \left[\frac{\bar{f}_1(z) - \bar{f}_o(z)}{Y} y + \bar{f}_o(z)\right] .$$

The coefficients α_n are evaluated by inserting $\xi = \xi^*$, multiplying by six $\frac{n\pi}{Y}\, y$ and integrating with respect to y on the interval [0, Y]:

$$\alpha_n = e^{\nu\left(\frac{n\pi}{Y}\right)^2 \xi^*}\left[\frac{2}{Y}\int_0^Y f_2(y,z)\sin\frac{n\pi}{Y}\,y\,dy\right.$$

$$\left. + (-1)^n \frac{2}{n\pi}\bar{f}_1(z) - \frac{2}{n\pi}\bar{f}_o(z)\right] \quad .$$

ξ^* represents a fixed value of the quantity $\xi \equiv t + \frac{x}{\bar{U}}$ so that when $t \to \infty$ the contribution from ω_H vanishes because α_n is only expressed in terms of the finite random functions $f_2(y,z)$, $\bar{f}_1(z)$ and $\bar{f}_o(z)$. In the present analysis, the transient behavior of the velocities, resulting from the term involving f_2 , is not of interest.

As a result of the analysis it is found that the fluctuating vorticity expression involves the weighted mean average of its values on the boundaries, i.e.,

$$\omega_x = \frac{\bar{f}_1(z) - \bar{f}_o(z)}{Y}\,y - \bar{f}_o(z) \quad .$$

A similar analysis will show that:

$$\omega_y = \frac{\bar{g}_1(z) - \bar{g}_o(z)}{Y}\,y - \bar{g}_o(z) \quad ,$$

$$\omega_z = \frac{\bar{h}_1(z) - \bar{h}_o(z)}{Y}\,y - \bar{h}_o(z) \quad .$$

This analysis suggests that it is only the mean values of the vorticity components on the boundaries that influence the velocity fluctuations inside the separated flow region. This has been justified experimentally by Eckelmann [7] in the attached flow sublayer. He showed that the fluctuating values of u', v' and w' within the viscous sublayer of a turbulent flow are highly correlated with their fluctuating derivatives at the wall. Introduction of another fluctuating influence from the vorticity would be unnecessary. Therefore the appearance of a weighted mean average in the vorticity expressions is acceptable for the present problem, but it can be anticipated that the structure of the separated region is less regular than that of the viscous sublayer and a more detailed analysis should take such effects into account and also admit non-stationary statistical properties in the flow.

C. Solution of Poisson's Equation for the Velocity Fluctuations

The fluctuating velocities can now be determined from the solution of Poisson's equation. Specifically u' is governed by the equation:

$$\nabla^2 u' = \frac{\partial \omega_y}{\partial z} - \frac{\partial \omega}{\partial y} \quad .$$

This equation will be solved by the use of a Green's function for the Laplacian operator. The details for u' are worked out below with similar procedures necessary for evaluating v' and w' . A Green's function for the domain

$$-\infty < x < \infty \ , \ -\infty < z < \infty \ , \ 0 < y < Y$$

and subject to appropriate boundary conditions is developed. From Arfken [8] p. 758, the solution of u' can be expressed in terms of $G(x,y,z,\xi,\eta,\zeta)$, the Laplacian Green's function:

$$u'(x,y,z,t) = \iiint \left(\frac{\partial \omega_y}{\partial \zeta} - \frac{\partial \omega_z}{\partial \eta} \right) \cdot G(x,y,z,\xi,\eta,\zeta) d\xi d\eta d\zeta$$

$$+ \iint_S \Big[G(x,y,z,\xi,\eta,\zeta) \cdot \nabla_{(\xi,\eta,\zeta)} u'(\xi,\eta,\zeta,t)$$

$$- u'(\xi,\eta,\zeta,t) \nabla_{(\xi,\eta,\zeta)} G(x,y,z,\xi,\eta,\zeta) \Big] dS$$

where $u'(\xi,\eta,\zeta,t)$ is the value of u' on the boundaries.

The first step in evaluating u' is to develop the Green's function, $G(x,y,z,\xi,\eta,\zeta)$. Consider the continuous expansion of $G(x,y,z,\xi,\eta,\zeta)$ in the form:

$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn \; g_{m,n}(y,\eta) \; e^{im(x-\xi)} \; e^{in(z-\zeta)} \ ,$$

where $g_{m,n}$ is the y component of the complete Green's function. It is required that G satisfy the equation:

$$\nabla^2_{(x,y,z)}\, G(x,y,z,\xi,\eta,\zeta) = -\,\delta(\vec{r}_1 - \vec{r}_2)$$

where $\delta(\vec{r}_1 - \vec{r}_2)$ is the three-dimensional Dirac delta function, $\delta(\vec{r}_1 - \vec{r}_2) = \delta(x-\xi)\,\delta(y-\eta)\,\delta(z-\zeta)$. From Arfken [8] p. 673, the Dirac functions $\delta(x-\xi)$ and $\delta(x-\zeta)$ can be expressed in the form:

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dm\; e^{im(x-\xi)}$$

$$\delta(z-\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dn\; e^{in(z-\zeta)}$$

Inserting these expansions into the equation for G yields:

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn \left[\frac{d^2}{dy^2}\, g_{m,n}(y,\eta) - (m^2+n^2) g_{m,n}(y,\eta)\right] e^{im(x-\xi)}\, e^{in(z-\zeta)}$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn\; -\,\delta(y-\eta)\; e^{im(x-\xi)}\, e^{in(z-\zeta)}$$

This equation implies that;

$$\frac{d^2\, g_{m,n}}{dy^2} - (m^2+n^2) g_{m,n} = -\,\delta(y-\eta)\ .$$

Following standard procedures [8], the solution of $g_{m,n}$ for the region $0 < y < \eta$ is:

$$g^{(1)}_{m,n} = \frac{\exp[-(2Y-\eta+y)M]-\exp[-(2Y-\eta-y)M]-\exp[-(\eta-y)M]-\exp[-(\eta+y)M]}{2M(1-\exp[-2YM])}$$

where $M \equiv \sqrt{m^2 + n^2}$, and for the region $\eta < y < Y$ is:

$$g^{(2)}_{m,n} = \frac{\exp[-(2Y-y+\eta)M]-\exp[-(2Y-y-\eta)M]+\exp[-(y-\eta)M]-\exp[-(y+\eta)M]}{2M(1-\exp[-2YM])}$$

Define the functions $G^{(1)}$ and $G^{(2)}$ corresponding to the values $g^{(1)}_{m,n}$ and $g^{(2)}_{m,n}$. It is then seen that

$$u' = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \left[\int_{0}^{y} G^{(2)} \left(\frac{\partial \omega_y}{\partial \zeta} - \frac{\partial \omega_z}{\partial \eta} \right) d\eta + \int_{y}^{Y} G^{(1)} \left(\frac{\partial \omega_y}{\partial \zeta} - \frac{\partial \omega_z}{\partial \eta} \right) d\eta \right]$$

+ Surface Terms.

Introducing the expansions for $G^{(1)}$ and $G^{(2)}$ results in the identity:

$$u' = \frac{1}{16\pi^4} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn \left[\int_{0}^{y} g^{(2)}_{m,n} \left(\frac{\partial \omega_y}{\partial \zeta} - \frac{\partial \omega_z}{\partial \eta} \right) d\eta \right.$$

$$\left. + \int_{y}^{Y} g^{(1)}_{m,n} \left(\frac{\partial \omega_y}{\partial \zeta} - \frac{\partial \omega_z}{\partial \eta} \right) d\eta \right] e^{im(x-\xi)} e^{in(z-\zeta)}$$

+ Surface Terms.

The surface terms have been defined as:

$$\text{Surface Terms} = \iiint_{S(\xi,\eta,\zeta)} \nabla_{(\xi,\eta,\zeta)}[G(x,y,z,\xi,\eta,\zeta) - u'(\xi,\eta,\zeta)]dS_{(\xi,\eta,\zeta)} \cdot$$

It should be noted that $G(x,y,z,\xi,\eta,\zeta)$ was developed in such a way that it attains the value of zero on the boundary surfaces. The fluctuating velocity, u', is also zero at the wall, $y = 0$, and is assumed to have a known value, $u' = F_u(x,z,t)$ at $y = Y$. The only contribution to the surface term is therefore:

$$\text{Surface Term} = -\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \, F_u(\xi,\zeta,t) \frac{\partial}{\partial \eta} G^{(1)}(x,y,z,\xi,\eta,\zeta)$$

$$= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn \left[\frac{\exp[-(Y+y)M]-\exp[-(Y-y)M}{1-2\exp[-2YM]}\right] F_u(\xi,\zeta,t)\exp[im(x-\xi)+in(z-\zeta)] .$$

The nonhomogeneous part of Poisson's equation for u' is:

$$\frac{\partial \omega_y}{\partial \zeta} - \frac{\partial \omega_z}{\partial \eta} = \left[\frac{\partial \bar{g}_o(\zeta)}{\partial \zeta} - \frac{\bar{h}_1(\zeta)-\bar{h}_o(\zeta)}{Y}\right] + \left[\frac{\partial\left(\bar{g}_1(\zeta)-\bar{g}_o(\zeta)\right)}{Y\ \partial \zeta}\right]\eta$$

which can be integrated with respect to η where it occurs in the solution. Now u' can be written as

$$u'(x,y,z,t) =$$

$$\frac{1}{16\pi^4}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}d\Bigg\{\frac{\left[(1-\frac{y}{Y})(1-\exp[-2YM]+(\exp[-(2Y-y)M]-\exp[-yM]\right]}{M^2(1-\exp[-2YM])}$$

$$\cdot\frac{\partial\bar{g}_o(\zeta)}{\partial\zeta}$$

$$+\frac{\left[\frac{y}{Y}(1-\exp[-2YM]+(\exp[-(Y+y)M]-\exp[-(Y-y)M])\right]}{M^2(1-\exp[-2YM])}\cdot\frac{\partial\bar{g}_1(\zeta)}{\partial\zeta}$$

$$+\frac{(1-\exp[-2YM]+(\exp[-(Y-y)M]+\exp[-yM])(\exp[-YM]-1)}{M^2(1-\exp[-2YM])}$$

$$\cdot(\bar{h}_o(\zeta)-\bar{h}_1(\zeta))$$

$$-4\pi^2\frac{\left[\exp[-(Y+y)M]-\exp[-(Y-y)M]\right]}{(1-2\exp[-2YM])}$$

$$\cdot F_u(\xi,\zeta,t)\Bigg\}e^{im(x-\xi)}\,e^{in(z-\zeta)}$$

where $M = \sqrt{m^2+n^2}$. A similar expression is obtained for w'(x,y,z,t). Since the nonhomogeneous part of the v' equation is slightly different from that of either u' and

and w', the solution to v' varies accordingly. It is found that:

$$v'(x,y,z,t) =$$

$$\frac{1}{16\pi^4}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}dn\left\{\frac{\left[(1-\frac{y}{Y})(1-\exp[-2YM])+(\exp[-2Y-y)M]-\exp[-yM])\right]}{M^2(1-\exp[-2YM])}\right.$$

$$\cdot\left(\frac{\partial\bar{h}_o(\xi)}{\partial\xi}-\frac{\partial\bar{f}_o(\xi)}{\partial\zeta}\right)$$

$$+\frac{\left[\frac{y}{Y}(1-\exp[-2YM])+(\exp[-(Y+y)M]-\exp[-(Y-y)M])\right]}{M^2(1-\exp[-2YM])}$$

$$\cdot\left(\frac{\partial\bar{h}_1(\zeta)}{\partial\xi}-\frac{\partial\bar{f}_1(\zeta)}{\partial\zeta}\right)$$

$$\left.-4\pi^2\frac{\left[\exp[-(Y+y)M]-\exp[-(Y-y)M]\right]}{(1-\exp[-2YM])}\right\}e^{im(x-\xi)}\,e^{in(z-\zeta)}$$

Excluding the exponentials with imaginary arguments, the expressions for the fluctuating velocities are even functions with regard to m and n . This allows some simplification, i.e., the integrals over m and n from $-\infty$ to ∞ can be changed to integrals over $[0,\infty]$. In addition $e^{im(x-\xi)}$ can be replaced by $\cos m(x-\xi)$ and $e^{in(z-\zeta)}$ by $\cos n(z-\zeta)$. When both m and n are zero an indeterminate

value results. However, successive application of l'Hopital's rule indicates that each term in the integrals attains a finite value.

It has been shown that viable solutions for the fluctuation velocities in the region of separation of a turbulent boundary layer can be obtained. The equations for the fluctuation velocities will now be examined and used to develop the correlations necessary for defining the Reynolds stresses.

CHAPTER IV

RESULTS

Using the results obtained in Chapter III for the fluctuating velocities, u' and v', a theoretical expression can be formed for the Reynolds stresses, $\langle u'^2 \rangle$, $\langle u'v' \rangle$ and $\langle v'^2 \rangle$. The empirical data of Simpson, et al. 1977 [4], is used to verify the consistency of the theory developed and to make further observations about the data.

The flow model as developed in Chapter II for the region of separated flow in a boundary layer includes time averaged quantities which are independent of z (i.e., $\overline{W} = 0$. As a consequence the quantities in the expressions for u' and v', which are mean values over $\xi = \frac{x}{\overline{U}} + t$, are also independent of the z coordinate and hence are constant. Subsequently any derivatives of these terms are identically zero and the corresponding values of u' and v' are:

$$u' = \frac{1}{4\pi^4} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dn$$

$$\left\{ \frac{\left[(1-\exp[-2YM]) + (\exp[-(Y-y)M] + \exp(-yM))(\exp[-YM]-1) \right]}{M^2(1-\exp[-2YM])} (\overline{h}_o - \overline{h}_1) \right.$$

$$\left. - 4\pi^2 \frac{\left[\exp[-(Y+y)M] - \exp[-(Y-y)M] \right]}{(1-\exp[-2YM])} F_u \right\} \cos m(x-\xi) \cos n(z-\zeta)$$

$$v' = -\frac{1}{\pi^2}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}dn\left\{\frac{\left[\exp[-(Y+y)M]-\exp[-(Y-y)M]\right]}{(1-\exp[-2YM])}F_v\right\}$$

$$\cos m(x-\xi)\cos n(z-\zeta)$$

where $M = \sqrt{m^2+n^2}$.

By expanding the exponential terms, collecting terms of identical powers in y, and then truncating the expansions to $O(y^4)$, the expressions for u' and v' are:

$$u'(x,y,z,t) = \frac{1}{4\pi^4}\left\{I_1(x,z) + y[I_2(x,z) + I_3(x,z,t)] + y^2 I_4(x,z) + y^3 I_5(x,z,t)\right\} + O(y^4)$$

$$v'(x,y,z,t) = \frac{1}{4\pi^4}\left\{y\, J_1(x,z,t) + y^3 J_2(x,z,t)\right\} + O(y^4) .$$

These expressions are in the form

$$u'(x,y,z,t) = \sum_{n=0}^{3} A_n(x,z,t)y^n \quad \text{and}$$

$$v'(x,y,z,t) = \sum_{n=0}^{3} B_n(x,z,t)y^n ,$$

and display the y dependence of the velocity fluctuations directly. The coefficients A_n , B_n are essentially unknown and reflect the basic nature of the boundary value problem being considered: boundary conditions at y = 0 and y = Y are imposed without being explicitly described. In other

words, it is supposed that if the statistical properties of the turbulence on the boundaries is known, then the present model will describe its decay within the solution domain. The basically indeterminate nature of the turbulence problem is just removed to the boundary conditions.

The values of I_i , i = 1,5 , and J_1 , J_2 which define u' and v' are given in the Appendix. They contain function values representing the vorticity boundary conditions which were averaged with respect to x and t , and also functions $F_u(x,z,t)$ and $F_v(x,z,t)$, corresponding to u' and v' , which define the fluctuations at the outer edge of separated region. As mentioned in Chapter III.A, the present model justifiably implies that the turbulence near the wall is more influenced by velocity fluctuations than it is by vorticity fluctuations. The vorticity entering as a mean value is essentially in conformity with the physical picture of turbulent energy production being given by $\langle u'v' \rangle \frac{\partial \overline{U}}{\partial y}$ for a two-dimensional boundary layer. $\frac{\partial \overline{U}}{\partial y}$ is then the major component of the mean vorticity. Defining the correlation operator

$$E[(\ )] \equiv \langle (\ ) \rangle \equiv \lim_{T} \frac{1}{T} \int_{o}^{T} (\ )\, di$$

and forming $E[u'^2]$, $E[v'^2]$ and $E[u'v']$ truncated to $O(y^4)$ yields:

$$E[u'^2] = E[I_1^2] + 2E[I_1(I_2 + I_3)]y \tag{4.1}$$

$$+ \left\{2E[I_1 I_4] + E[(I_2 + I_3)^2]\right\}y^2$$

$$+ \left\{2E[I_1 I_5] + 2E[(I_2 + I_3)I_4]\right\}y^3 + O(y^4)$$

$$E[v'^2] = E[J_1^2]y^2 + O(y^4) \tag{4.2}$$

$$E[u'v'] = E[I_1 J_1]y + E[(I_2 + I_3)J_1]y^2 \tag{4.3}$$

$$+ E[(I_1 J_2 + I_4 J_1)]y^3 + O(y^4) \quad .$$

These can now be matched to the experimental data in Figures 3, 4 and 5 from Simpson, et al., 1977 [4].

Looking at the data for the intensity $E[u'^2]$ in Figure 3, a third order polynomial in y can be used to closely match the data from $y/\delta = 0$ to $y/\delta = 0.63$. The assumption $\overline{U}$ = constant is valid only up to $y/\delta \simeq 0.2$. Since experimental data for $\langle v'^2 \rangle$ and $\langle u'v' \rangle$ is only available to $y/\delta = 0.67$ and $y/\delta = 0.71$ respectively, the theoretical expansion of $\langle u'^2 \rangle$ is extended to $y/\delta = 0.63$ to demonstrate the plausibility of using extended theoretical expansions for $\langle v'^2 \rangle$ and $\langle u'v' \rangle$. Using either the laser data or the hot-film data it is seen that:

$$E\left[\frac{u'^2}{\overline{U}_\infty^2}\right] \cdot 10^3 \simeq A + By + Cy^2 + Dy^3$$

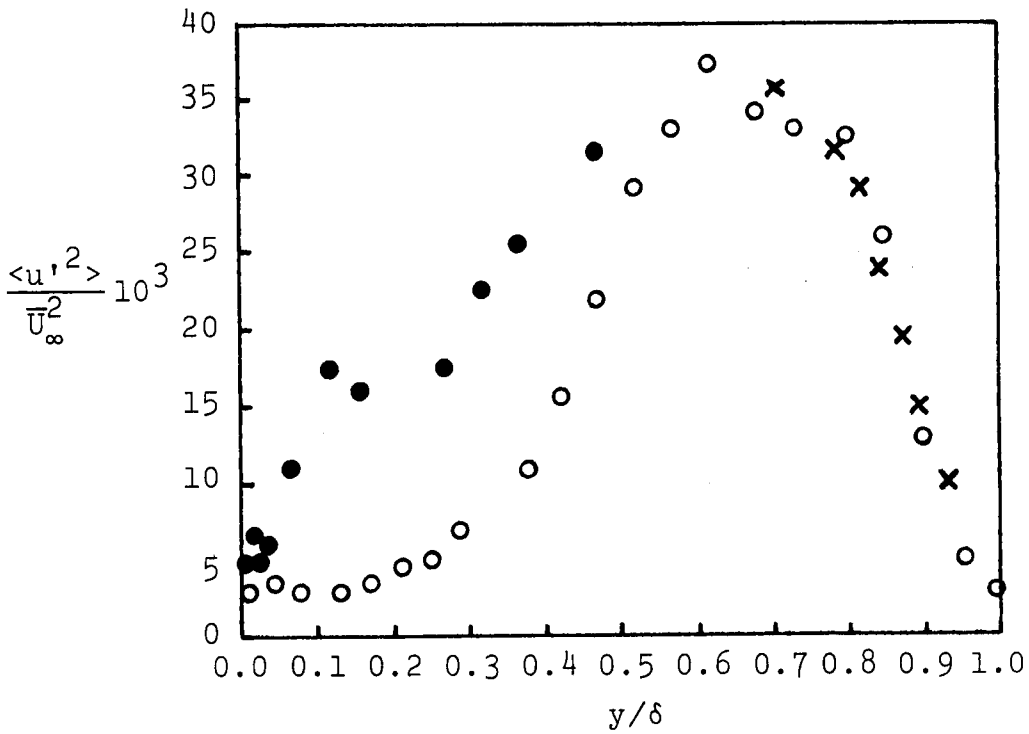


Figure 3. $\frac{\langle u'^2 \rangle}{\overline{U}_\infty^2} 10^3$ Distribution. (From Simpson, et al. [4].)

- o Normal hot film
- ● Laser anemometer
- × Cross-film

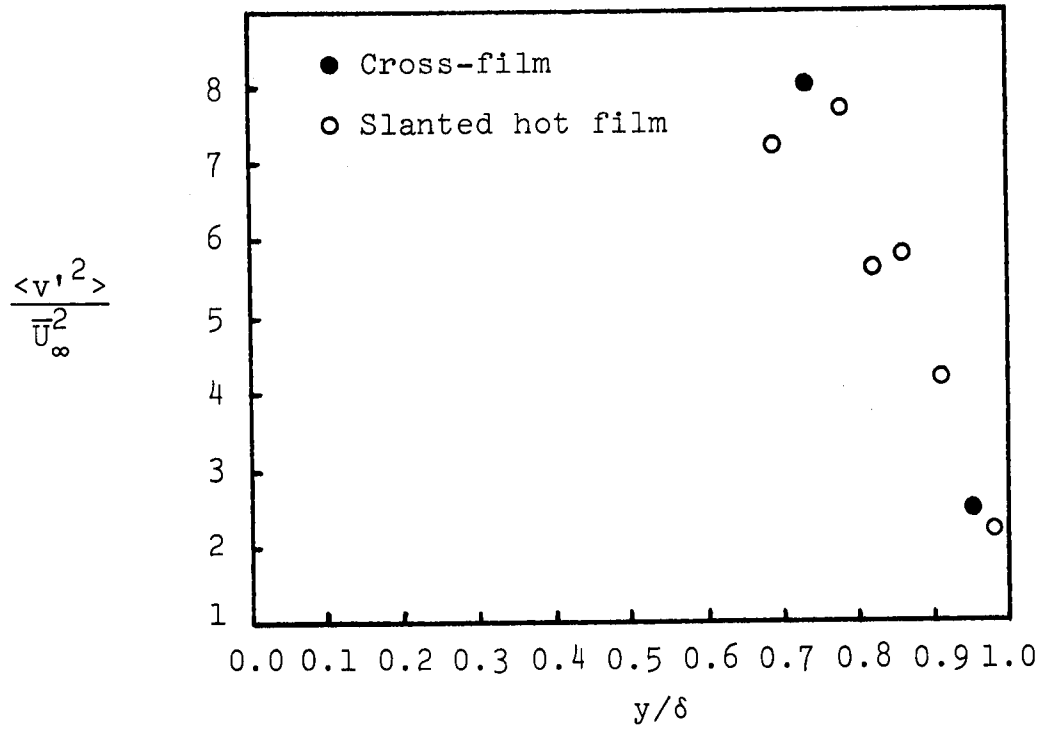


Figure 4. $\frac{\langle v'^2 \rangle}{\bar{U}_\infty^2} 10^3$ Distribution. (From Simpson, et al. [4].)

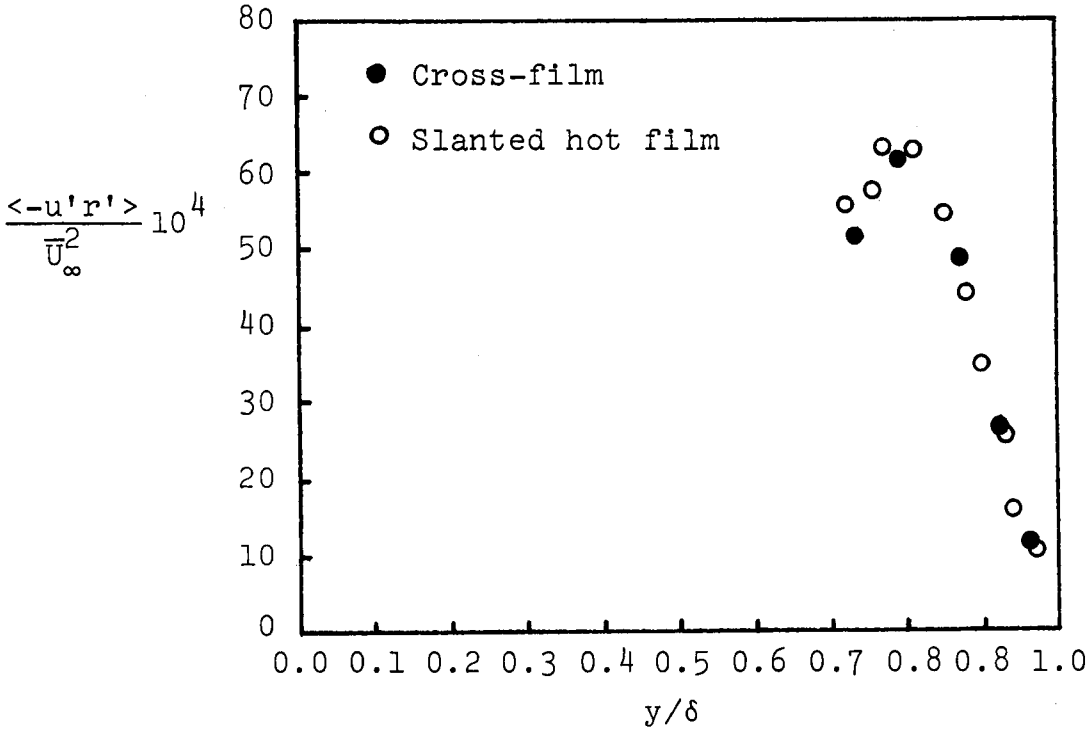


Figure 5. $\frac{<-u'v'>}{U_\infty^2} 10^4$ Distribution. (From Simpson, et al. [4].)

where A is the value of $E\left[\frac{u'^2}{\bar{U}_\infty^2}\right] \cdot 10^3$ at $y = 0$ and B is its slope at $y = 0$. Irregardless of the data used to match with, or the point in the boundary layer where the matching is done, it is seen that all of the coefficients are non-zero. Thus when matching the polynomial of the theory with the polynomial of the experimental data, four matching conditions result, each corresponding to the value of the coefficients in the third degree polynomials.

The mean value operator defined earlier when applied to a function which is independent of t simply returns the original function. Utilizing this fact when examining $E[u'^2]$, which was previously developed from the theory, reveals four unknown integrand quantities:

$$\bar{h}_o \ , \ \bar{h}_1 \ , \ E[F_u(\xi,\zeta,t)] \quad \text{and} \quad E[F_u{}^2(\xi,\zeta,t)] \ .$$

These unknown integrand quantities can theoretically be obtained given four conditions from the experimental data with which to match. Hence the resulting polynomial approximations to the experimental data for $E\left[\frac{u'^2}{\bar{U}_\infty^2}\right] \cdot 10^3$ is shown in Figure 6 after resolving the unknown quantities A, B, C and D of the polynomial from the function and slope values at $y/\delta = 0$ and $y/\delta = 0.63$ for both the laser data and the hot-film data. The results indicate a better matching of the theory with the laser data (from the new

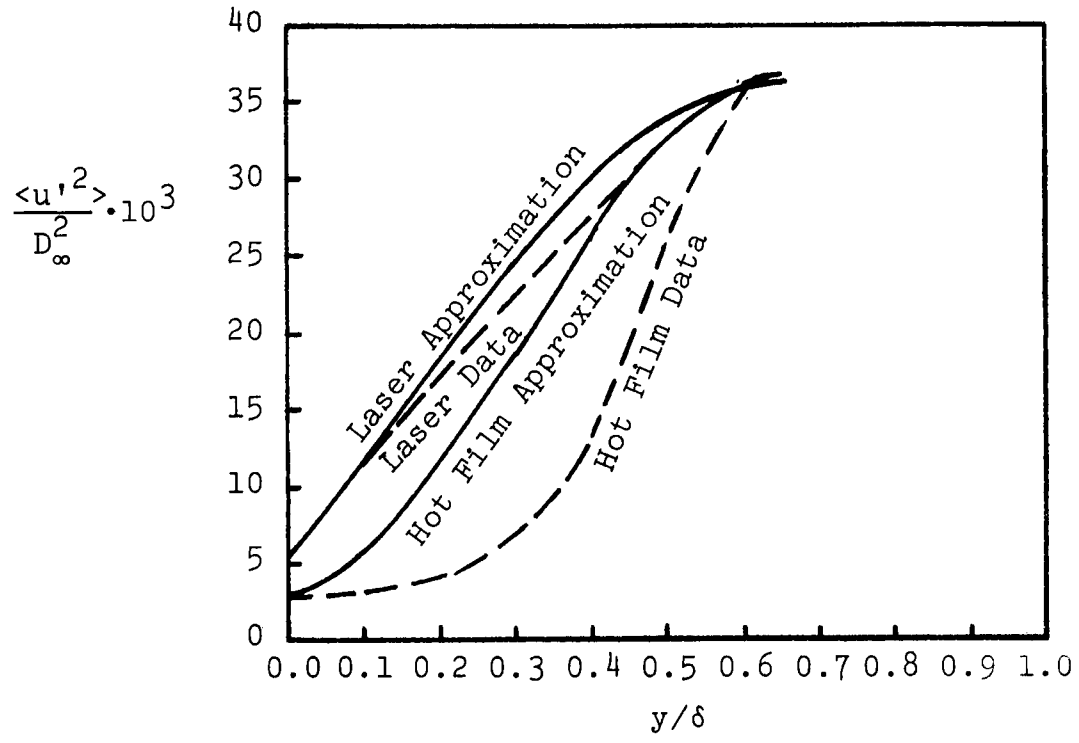


Figure 6. Approximations to $\frac{\langle u'^2 \rangle}{\bar{U}_\infty^2} \cdot 10^3$.
Data from Simpson, et al. [4].

laser anemometer of Simpson, et al., 1977 [4]) than with the normal hot-film data. Hence the usefulness of normal hot-film data in estimating the correlation intensities near the wall is questionable.

The approximation to $E[v'^2]$ in Equation (4.2) is truncated to $O(y^4)$ and results in only a single term which reflects a dependence on y^2 . The absence of a constant and y term is physically correct since at $y = 0$, $v' \equiv 0$ and $\frac{dv'}{dy} \equiv 0$ (from the continuity equation) for all x and z. Matching the equation for $E[v'^2]$, which has the single new unknown quantity $E[F_v^2]$, with the nearest available data at $y/\delta = 0.67$ gives the curve in Figure 7. The single unknown quantity can again theoretically be obtained and now an estimate to $O(y^4)$ of $E[v'^2]$ next to the wall is available.

The final Reynolds stress term considered is E[u'v'] whose approximation is given in Equation (4.3). At this point it has been shown that the five unknowns in the two previous Reynolds stress terms examined can theoretically be obtained. Excluding these there are now only two additional unknown quantities occurring in E[u'v']: $E[F_v]$ and $E(F_uF_v]$. Equation (4.3) is of the form $By + Cy^2 + Dy^3$ and only two matching conditions are needed to uniquely define $E[F_v]$ and $E[F_uF_v]$. One matching condition at the wall ($y/\delta = 0$) will uniquely define B which will yield a theoretical expression for $E[F_v]$. To evaluate $E[F_uF_v]$

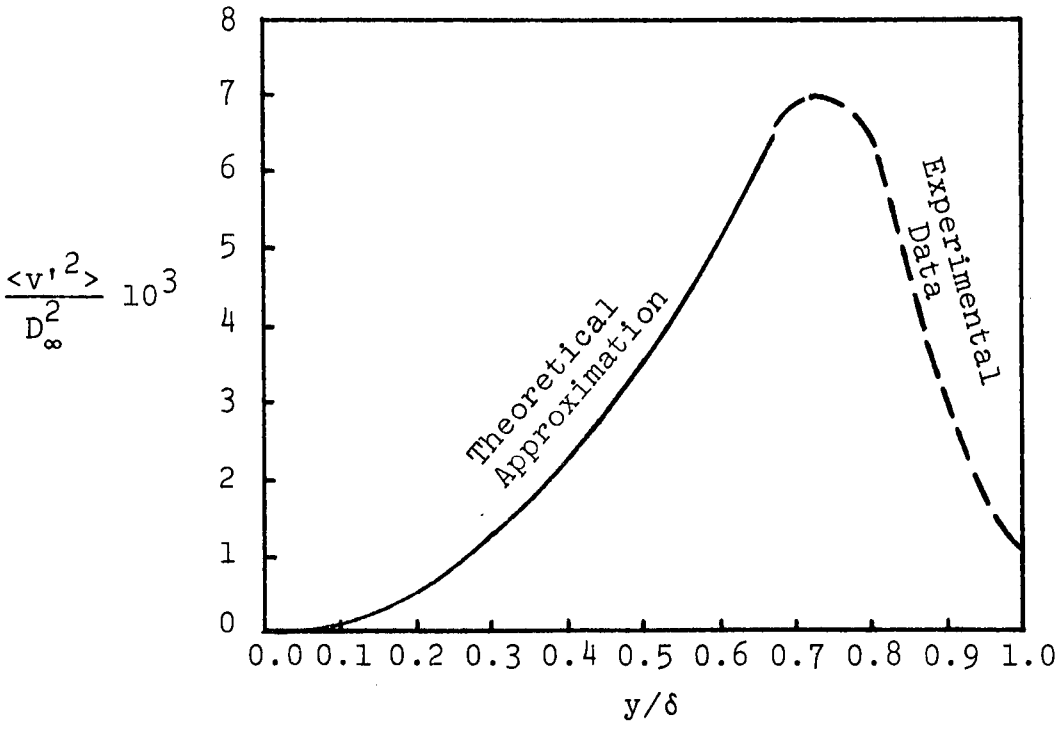


Figure 7. Approximation to $\frac{\langle v'^2 \rangle}{\overline{U}_\infty^2}\ 10^3$.
Data from Simpson, et al. [4].

a matching condition at $y/\delta = 0.71$ must be used. The value of D is only implicitly known so that the estimate of $E[u'v']$ must be truncated to $O(y^3)$ with a degree of uncertainty associated with C . The two matching conditions used are the slope at $y = 0$ which is proportional to the mean pressure at the wall and the function value at $y/\delta = 0.71$ which is the closest available data to the wall. The resulting approximation to $E[u'v']$ is given in Figure 8. Hence an estimate to order y^3 is available for $E[u'v']$ next to the wall in the separated region of the boundary layer.

Thus it is seen that through empirically determined constants appropriate equations which describe the region of separation in turbulent flow near a wall can be formed and corresponding solutions for the fluctuating velocities obtained. An expression for the Reynolds stresses may be formed and when matched with experimental data the unknown quantities on the boundaries, which are assumed to govern the behavior of the flow within the solution domain, may be theoretically solved although their solution is not readily seen. As shown, the implications which evolve from the matching of the theory with the experiment allow approximations to be made for the Reynolds stresses close to the wall in the separated region. It must be noted that the assumption $\bar{U}$ = constant is only valid for $y/\delta \leq 0.2$ and the extension of the theory during matching beyond this value

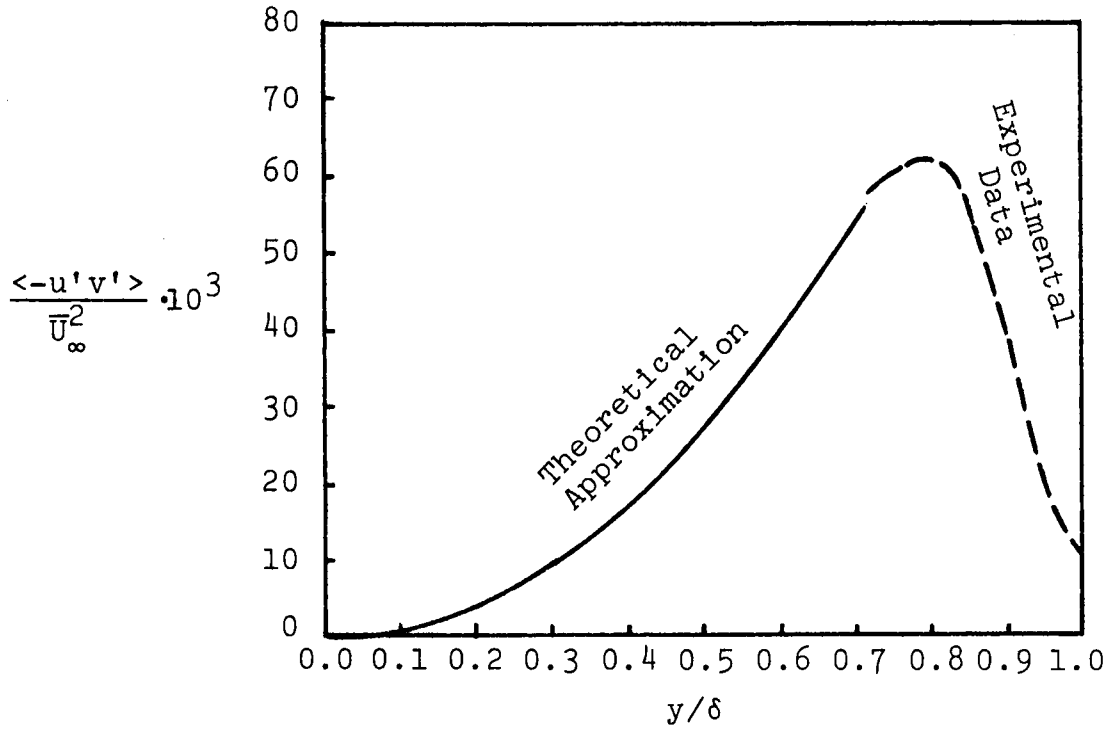


Figure 8. Approximation to $\frac{<-u'v'>}{\overline{U}_\infty^2} \cdot 10^3$.
Data from Simpson, et al. [4].

introduces a degree of uncertainty for these extended values.

CHAPTER V

CONCLUSIONS

A theory has been developed for estimating the Reynolds stress tensor components in the near-wall region of a slowly separating boundary layer. The basic indeterminancy of the turbulence problem is left unresolved but only enters into the present formulation as boundary conditions which are determined empirically. The need for experimental data to close the problem is not removed by the present theory.

Because of the lack of sufficient experimental data--particularly very close to the wall--it is not possible to make a full comparison of the above results with data. In addition, measurements made close to a solid surface are not too reliable due to interference effects. The large difference between hot-wire and laser data in Figure 6 attests to this fact.

While the above theory is encouraging due to the simplicity of the final equations, it is clear that additional study and experimental data is needed.

LIST OF REFERENCES

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1. Hinze, J. O. Turbulence, McGraw-Hill, Inc., 1975.

2. Schlichting, Herman. Boundary-Layer Theory, McGraw-Hill, Inc., 1968.

3. Cebeci, Tuncer and Smith, A.M.O. Analysis of Turbulent Boundary Layers, Academic Press, 1974.

4. Simpson, R. L., Strickland, J. H., and Barr, P. W. "Features of a Separating Turbulent Boundary Layer in the Vicinity of Separation," Journal of Fluid Mechanics, Vol. 79, Part 3, pp. 553-594, 1977.

5. Uzkan, T. and Reynolds, W. C. "A Shear-Free Turbulent Boundary Layer," Journal of Fluid Mechanics, Vol. 28, Part 4, pp. 803-821, 1967.

6. Bendat, J. S. Principles and Applications of Random Noise Theory, John Wiley and Sons, Inc., 1958.

7. Eckelmann, Helmut. "The Structure of the Viscous Sublayer and the Adjacent Wall Region in a Turbulent Channel Flow," Journal of Fluid Mechanics, Vol. 65, Part 3, pp. 439-459, 1974.

8. Arfken, George. Mathematical Methods for Physicists, Academic Press, 1970.

APPENDIX

APPENDIX

The following integrals are used in the approximation to $u'(x,y,z,t)$ where $M = \sqrt{m^2 + n^2}$.

$$I_1(x,z) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{(1-\exp[-2YM]) - 2\exp[-YM](YM + \frac{1}{6}Y^3M^3)}{M^2(1-\exp[-2YM])} \right] \cdot (\bar{h}_o - \bar{h}_1) \cos m(x-\xi) \cos n(x-\zeta) \right\}$$

$$I_2(x,z) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{-\frac{1}{2}Y^2M^3}{M^2(1-\exp[-2YM])} \right] (\bar{h}_o - \bar{h}_1) \cos m(x-\xi) \cos n(z-\zeta) \right\}$$

$$I_3(x,z,t) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{-2M\exp[-YM]}{(1-\exp[-2YM])} \right] F_u(\xi,\zeta,t) \cdot \cos m(x-\xi) \cos n(z-\zeta) \right\} (-4\pi^2)$$

$$I_4(x,z) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{\frac{1}{2}YM^3}{M^2(1-\exp[-2YM])} \right] (\bar{h}_o - \bar{h}_1) \cos m(x-\xi) \cos n(z-\zeta) \right\}$$

$$I_5(x,z,t) = -4\pi^2 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{-\frac{1}{3}M^3 \exp[-YM]}{(1-\exp[-2YM])} \right] F_u(\xi,\zeta,t) \right.$$

$$\left. \cdot \cos m(x-\xi) \cos m(z-\zeta) \right\}$$

These two integrals are used in the approximations to $v'(x,y,z,t)$ where again $M = \sqrt{m^2 + n^2}$.

$$J_1(x,z,t) = -4\pi^2 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{-2M\exp[-YM]}{(1-\exp[-2YM])} \right] F_v(\xi,\zeta,t) \right.$$

$$\left. \cdot \cos m(x-\xi) \cos n(z-\zeta) \right\}$$

$$J_2(x,z,t) = -4\pi^2 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} dm \int_{0}^{\infty} dn \left\{ \left[\frac{-\frac{1}{3}M^3 \exp[-YM]}{(1-\exp[-2YM])} \right] F_v(\xi,\zeta,t) \right.$$

$$\left. \cdot \cos m(x-\xi) \cos n(z-\zeta) \right\}$$

VITA

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