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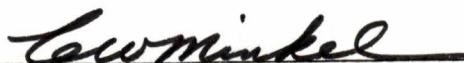
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CHARACTERIZATION OF GROUPS OF INVOLUTIONS AND  
PSEUDO-INVOLUTIONS AND CONSTRUCTION  
OF GROUPS OF INVOLUTIONS

A Thesis  
Presented for the  
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## ABSTRACT

An  $n \times n$  matrix  $A$  over a field  $F$  is an involution if  $A^2 = I_n$ . If the characteristic of  $F = p$  then  $A$  is a pseudo-involution if  $A + A^{-1} = 2I_n$ . If the characteristic of  $F = 2$  then  $A$  is a pseudo-involution if and only if  $A$  is an involution.

This paper deals with the groups formed by the products of involutions and pseudo-involutions, the diagonalization of a group of involutions, and the construction of groups of involutions.

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## CHAPTER I

## INTRODUCTION

Let  $A$  denote an  $n \times n$  matrix over a field  $F$  with characteristic  $p$ .  $A$  is involutory if  $A^2 = I_n$ , where  $I_n$  is the  $n \times n$  identity matrix. In this paper we will denote the characteristic of the field  $F$  by  $\text{char } p$ , and the  $n \times n$  identity matrix by  $I$ . If  $F$  is a field of  $\text{char } p \neq 2$ , it is known that  $A$  is involutory if and only if  $A$  is similar to a diagonal matrix of the form  $\text{Diag}(-I_s, I_{n-s})$ ,  $0 \leq s \leq n$ . If  $F$  is a field of  $\text{char } p = 2$ , then  $A$  is involutory if and only if  $A$  is similar to a diagonal matrix of the form  $\text{Diag}(I_{n-2s}, E_1, \dots, E_s)$ ,  $0 \leq s \leq [\frac{n}{2}]$ , where  $[\frac{n}{2}]$  is the greatest integer in  $\frac{n}{2}$ , and each

$$E_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, \dots, s. \quad (1)$$

In either case the integer  $s$  is called the signature of the involution  $A$  and will be written as  $\sigma(A) = s$ .

An  $n \times n$  matrix  $A$  over any field is a pseudo-involution if  $A + A^{-1} = 2I$ . If  $\text{char } p = 2$ , then  $A = -A^{-1} = A^{-1}$  and  $A$  is a pseudo-involution if and only if  $A$  is an involution [4].

We will state definitions and results by Brawley [1] for involutions and extend some of these results to pseudo-involutions. We consider products of involutions and pseudo-involutions and the groups thus formed. If  $G$  denotes a finite group of involutions under multiplication over a field  $F$  with  $\text{char } p \neq 2$ , and  $P$  a nonsingular

$n \times n$  matrix, we show that there is a  $P$  which diagonalizes every involution  $A$  in  $G$ . The diagonalization gives nonzero blocks of size  $s \times s$  and  $(n-s) \times (n-s)$ , where  $s$  is determined by  $G$ .

Furthermore, we obtain restrictions on the signature of the  $(n-s) \times (n-s)$  blocks if  $A \neq -I$ . Finally using these results we form a group  $H$  of involutions under multiplication over a field  $F$  with  $\text{char } p \neq 2$  using two groups of involutions.

## CHAPTER II

CHARACTERIZATION OF GROUPS GENERATED BY SETS  
OF INVOLUTIONS AND PSEUDO-INVOLUTIONS

Results have been given by Brawley [1] concerning certain sets of involutory matrices and their groups. We will state some of these results and prove some analogous results for pseudo-involutions. Although the proof of some of these results for pseudo-involutions mimics the proof for involutions, the proofs are included for completeness. Result 2.1 and Definitions 2.2 and 2.3 are given in [1].

Result 2.1. If  $A$  is an  $n \times n$  matrix over an arbitrary field  $F$ , then  $A$  is involutory of signature  $s > 0$  if and only if  $A$  can be written as  $A = I + cQP$  where  $Q$  and  $P^T$  are  $n \times s$  and of rank  $s$ ,  $c = -2$  and  $PQ = I_s$  if  $\text{char } p \neq 2$  and  $c = 1$  and  $PQ = 0_{s \times s}$  if  $\text{char } p = 2$ .

Definition 2.2. If  $P$  is  $s \times n$  and rank  $s$  over a field  $F$ , then the  $Q$ -set,  $(P)$ , of  $P$  is

$$(P) = \{Q: Q \text{ is } n \times s, \text{ rank } s \text{ and } PQ = -(1+c)I_s\},$$

where  $c = 1$  when  $\text{char } p = 2$  and  $c = -2$  when  $\text{char } p \neq 2$ .

Thus for any  $Q \in (P)$ ,  $PQ = I_s$  when  $\text{char } p \neq 2$  and  $PQ = 0_{s \times s}$  when  $\text{char } p = 2$ . Hence for  $Q \in (P)$ ,  $I_n + cQP$  is an involution. Moreover,  $I + cQP$  has signature  $= s$ .

Definition 2.3. The P-class,  $[P]$ , of  $P$  is the set of all involutory matrices determined by  $P$ , that is,

$$[P] = \{I + cQP : Q \in (P)\} .$$

Result 2.4. If the  $n \times n$  matrix  $A$ , where  $p$  is arbitrary, is a pseudo-involution then  $A$  can be written as  $A = I - QP$  where  $PQ = 0_{s \times s}$ , with  $\sigma(A) = s$  if  $\text{rank } Q = \text{rank } P = s$  [4] .

We next show that the product of an odd number of involutions from  $[P]$  is again in  $[P]$  and the product of an even number of involutions from  $[P]$  is a pseudo-involution.

Lemma 2.5. The product of  $k$  involutory matrices from  $[P]$  is of the form  $I + c f_k(c, Q_1, \dots, Q_k)P$  where  $f_k(c, Q_1, \dots, Q_k)$  is defined recursively by  $f_1(c, Q_1) = Q_1$  and

$$f_{j+1}(c, Q_1, \dots, Q_{j+1}) = Q_{j+1} + [1 - c(1 + c)] f_j(c, Q_1, \dots, Q_j)$$

for all  $j \geq 1$  .

Proof. We give an inductive argument on  $k$  . Note  $f_1(c, Q_1) = Q_1$  since  $A = I + cQ_1P$  when  $k = 1$  . Let  $A_1 \cdot A_2 \cdot \dots \cdot A_k = I + c f_k(c, Q_1, \dots, Q_k)P$  and let  $A_{k+1} = I + cQ_{k+1}P$  . Now the product

$$\begin{aligned} (A_1 \cdot A_2 \cdot \dots \cdot A_k) \cdot A_{k+1} &= [I + c f_k(c, Q_1, \dots, Q_k)P] \\ &\cdot [I + cQ_{k+1}P] \end{aligned}$$

$$\begin{aligned}
&= I + c[Q_{k+1} + f_k(c, Q_1, \dots, Q_k) + c f_k(c, Q_1, \dots, Q_k) P Q_{k+1}]^P \\
&= I + c[Q_{k+1} + f_k(c, Q_1, \dots, Q_k) - c(1 + c)f_k(c, Q_1, \dots, Q_k)]^P \\
&= I + c[Q_{k+1} + (1 - c(1 + c)) f_k(c, Q_1, \dots, Q_k)]^P \\
&= I + c f_{k+1}(c, Q_1, \dots, Q_{k+1}) P .
\end{aligned}$$

For the remainder of this discussion  $f_k(c, Q_1, \dots, Q_k)$  will be denoted  $f_k$ .

Theorem 2.6. The product of an odd number of involutions from  $[P]$  is an involution in  $[P]$ , while the product of an even number of elements from  $[P]$  is a pseudo-involution.

Proof. For a product of  $k$  involutions from  $[P]$ , let  $f_k$  be defined as in Lemma 2.5. By induction we show

$$P f_k = c^{-1}([1 - c(1 + c)]^k - 1) I_S .$$

By Lemma 2.5,  $f_1 = Q_1$  and  $f_k = Q_k + [1 - c(1 + c)] f_{k-1}$ , for  $k \geq 2$ . For  $k = 1$  the result is clear since  $P f_1 = P Q_1 = -(1 + c) I_S$ . So assume valid for  $P f_k$ ,  $k \geq 1$ . Then

$$\begin{aligned}
P f_{k+1} &= P[Q_{k+1} + f_k - c(1 + c)f_k] \\
&= -(1 + c) I_S + P f_k - c(1 + c) P f_k
\end{aligned}$$

$$\begin{aligned}
&= [1 - c(1 + c)][c^{-1}([1 - c(1 + c)]^k - 1)I_s - (1 + c)I_s] \\
&= c^{-1}([1 - c(1 + c)]^{k+1} - 1)I_s .
\end{aligned}$$

Since  $\text{Pf}_k = c^{-1}([1 - c(1 + c)]^k - 1)I_s$ , Theorem 2.6 follows by noting that for

$$\text{char } p = 2 \text{ and } c = 1, \quad \text{Pf}_k = \frac{(1 - 2)^k - 1}{1} I_s = 0 = -(1 + c)I_s ,$$

and for

$$\text{char } p \neq 2 \text{ and } c = -2 ,$$

$$\text{Pf}_k = \frac{(1 - 2)^k - 1}{-2} I_s = \frac{(-1)^k - 1}{-2} I_s = \begin{cases} 0, & k \text{ even} \\ I_s, & k \text{ odd} . \end{cases}$$

Hence if  $k$  is odd, then  $\text{Pf}_k = -(1 + c)I_s$  and  $I + c f_k^P$  is an involution, whereas if  $k$  is even,  $\text{Pf}_k = 0$  and  $I + c f_k^P$  is a pseudo-involution by Theorem 2.7.

Theorem 2.7. For any  $s \times n$  matrix  $P$  of rank  $s$ , set  $\bar{P} = \{I - XP: X \text{ is } n \times s \text{ and } PX = 0\}$ . Then every element of  $\bar{P}$  is a pseudo-involution of signature  $\leq s$ . Moreover  $\bar{P}$  is a group under multiplication.

Proof. Note that for any  $A$  in  $\bar{P}$ , if  $A = I - XP$ , where  $PX = 0$ , then  $(XP)^2 = 0$  and  $A$  is a pseudo-involution whose signature is the rank of  $XP$  [4]. Thus  $\sigma(A) \leq s$ . Let  $A_1, A_2$

be pseudo-involutions in  $\bar{P}$ . Then  $A_1 = I - X_1P$  and  $A_2 = I - X_2P$ , with  $PX_1 = 0 = PX_2$ . Hence

$$\begin{aligned} A_1 A_2 &= (I - X_1P)(I - X_2P) = I - (X_1 + X_2)P + X_1P X_2P \\ &= I - (X_1 + X_2)P, \quad P(X_1 + X_2) = 0. \end{aligned}$$

Thus  $\bar{P}$  is closed. Since  $I = I - OP$  and  $PO = 0$ ,  $I$  is in  $\bar{P}$ . If  $A = I - XP$ , with  $PX = 0$ , then  $(I - XP)(I - (-X)P) = (I - XP)(I + XP) = I - XPXP = I$  and thus  $A^{-1} = I + XP$  is in  $\bar{P}$  as  $P(-X) = 0$ . Therefore  $\bar{P}$  is a group.

Brawley [1] defines the group  $G[P]$  of a  $P$ -class  $[P]$  as the subgroup generated by  $[P]$  in the general linear group  $GL_n(F)$ . He shows that if  $\text{char } p \neq 2$ ,  $0 < s \leq n$  and  $P$  is an  $s \times n$  rank  $s$  matrix, then

$$\begin{aligned} G[P] &= \{A_1 A_2: A_i \in [P]\} \dot{\cup} [P] \\ &= \{I - 2XP: PX = 0\} \dot{\cup} \{I - 2XP: PX = I_s\}, \end{aligned}$$

where  $\dot{\cup}$  denotes disjoint union. Under these assumptions, Theorem 2.7 implies that

$$G[P] = \bar{P} \dot{\cup} [P].$$

Moreover, since  $G[P]$  is the group generated by  $[P]$ ,  $G[P]$  consists of all finite products of matrices from  $[P]$ , where the vacuous

product is  $I$ . If  $G_1[P]$  is the set of all even products of matrices from  $[P]$ , then  $G_1[P]$  is a subgroup of  $G[P]$ . Clearly the product of two elements of  $G_1[P]$  is again a product of an even number of elements from  $[P]$ ,  $I$  is in  $G_1[P]$  as  $[P] \neq \emptyset$ , and, for  $A \in G_1[P]$  then  $A = A_1 \cdot \dots \cdot A_k$  with  $k$  even and  $A_i \in [P]$ ,  $1 \leq i \leq k$ , the inverse of  $A$  in  $G_1[P]$  is  $A_k \cdot \dots \cdot A_1$ . Again Theorem 2.7 implies that

$$G[P] = G_1[P] \cup [P] .$$

Furthermore, we have

$$\{A_1 A_2 : A_i \in [P]\} = \{I - 2XP : PX = 0\} = \bar{P} = G_1[P]$$

and

$$[P] = \{I - 2XP : PX = I_s\} .$$

Definition 2.8. Let  $P$  be an  $s \times n$  rank  $s$  matrix over  $F$  where  $0 < s \leq [n/2]$ . The group generated by  $\{I - QP : Q \text{ is } n \times s, \text{ rank } Q \text{ is } s, \text{ and } PQ = 0\}$  in  $GL_n(F)$  is denoted by  $\bar{G}[P]$ .

Brawley showed that  $G[P] = \{I - XP : X \text{ is } n \times s \text{ and } PX = 0\}$  when  $\text{char } p = 2$ . When  $\text{char } p = 2$ , since  $A$  is a pseudo-involution of signature  $s$  if and only if  $A$  is an involution of signature  $s$ , we have  $\{I - QP : Q \text{ is } n \times s, \text{ rank } Q \text{ is } s \text{ and } PQ = 0\} = [P]$ . We extend this result to matrices over a field of arbitrary characteristic

by showing  $\overline{G}[P] = \{I - XP : X \text{ is } n \times s \text{ and } PX = 0\}$ . The following lemma from Brawley [2] is needed. Let  $V(n, n)$  denote the vector space of all  $n \times n$  matrices over a field  $F$ . Then  $B = \{A_1, A_2, \dots, A_r\}$ , where  $r = n^2$ , is a rank  $n$  basis for  $V(n, n)$ , if  $B$  is a basis for  $V(n, n)$  and each matrix in  $B$  has rank  $n$ .

Lemma 2.9. [2] For an arbitrary field  $F$ , the space  $V(n, n)$  contains a rank  $n$  basis.

Proof. The proof in [2] is by induction on  $n$ . We show here that the particular selection of  $n^2$  matrices in  $V(n, n)$  formed from a rank  $n - 1$  basis for  $V(n-1, n-1)$  leads to a linearly independent set of nonsingular matrices. Consider the  $n \times n$  matrices,  $n^2$  in number, defined by

$$\begin{aligned}
 B_i &= \left( \begin{array}{c|c} A_i & 0 \\ \hline 0 & 1 \end{array} \right) ; \quad i = 1, 2, \dots, (n-1)^2 \\
 C_j &= \left( \begin{array}{c|c} A_1 & 0 \\ \hline c_j & 1 \end{array} \right) ; \quad j = 1, 2, \dots, n-1 \\
 D_j &= \left( \begin{array}{c|c} A_1 & c_j^T \\ \hline 0 & 1 \end{array} \right) ; \quad j = 1, 2, \dots, n-1 \\
 E &= \left( \begin{array}{c|c} A_1 & \overline{1}^T \\ \hline \overline{1} & 0 \end{array} \right) ,
 \end{aligned}
 \tag{2.1}$$

where  $\{A_i : 1 \leq i \leq (n-1)^2\}$  is a rank  $n-1$  basis for  $V(n-1, n-1)$ ,  $c_j$  denotes a  $(n-1)$  row vector with 1 in position  $j$  and zeros

elsewhere and  $\bar{1}$  is the  $n-1$  row vector whose components are all 1's .  
 $A_1$  can be chosen as an arbitrary nonsingular matrix and will be assumed to be  $I$  if characteristic  $F$  is not a divisor of  $n-1$  and  $I + E_{12}$  , where  $E_{ij}$  is the matrix with 1 in the  $(i,j)^{th}$  position and zeros elsewhere, when characteristic  $F$  is a divisor of  $n-1$  .

Assume that

$$\sum_{i=1}^{(n-1)^2} \alpha_i B_i + \sum_{j=1}^{n-1} \beta_j C_j + \sum_{j=1}^{n-1} \gamma_j D_j + \phi E = 0 ,$$

where  $\alpha_i, \beta_j, \gamma_j, \phi$  are scalars. Thus, by equation (2.1),

$$(i) \quad \left( \sum_{j=1}^{n-1} \beta_j + \sum_{j=1}^{n-1} \gamma_j + \phi + \alpha_1 \right) A_1 + \sum_{i=2}^{(n-1)^2} \alpha_i A_i = 0$$

$$(ii) \quad \sum_{j=1}^{n-1} \gamma_j C_j^T + \phi \cdot \bar{1}^T = 0$$

$$(iii) \quad \sum_{j=1}^{n-1} \beta_j C_j + \phi \cdot \bar{1} = 0$$

$$(iv) \quad \sum_{i=1}^{(n-1)^2} \alpha_i + \sum_{j=1}^{n-1} \beta_j + \sum_{j=1}^{n-1} \gamma_j = 0 .$$

By using (ii),  $\sum_{j=1}^{n-1} \gamma_j C_j^T + \phi \cdot \bar{1}^T = 0$  implies  $\sum_{j=1}^{n-1} \gamma_j C_j^T = -\phi \cdot \bar{1}^T$  .

Then each component of  $\sum_{j=1}^{n-1} \gamma_j C_j^T$  is  $-\phi$  and thus  $\sum_{j=1}^{n-1} \gamma_j = -\phi(n-1)$  .

Similarly in (iii),  $\sum_{j=1}^{n-1} \beta_j = -\phi(n-1)$  and so in (iv),  $\sum_{i=1}^{(n-1)^2} \alpha_i = 2\phi(n-1)$  .

Now as  $\{A_i: 1 \leq i \leq (n-1)^2\}$  is a basis for  $V(n-1, n-1)$  , then

$\{A_i: 1 \leq i \leq (n-1)^2\}$  is linearly independent. Then from (i)

$\alpha_i = 0$  ,  $i = 2, 3, \dots, (n-1)^2$  and also  $-2\phi(n-1) + \phi + \alpha_1 = 0$  . But

then  $2\phi(n-1) = \sum_{i=1}^{(n-1)^2} \alpha_i = \alpha_1 + \sum_{i=2}^{(n-1)^2} \alpha_i = \alpha_1$  . So  $\alpha_1 = 2\phi(n-1)$

and  $0 = -2\phi(n-1) + \phi + 2\phi(n-1) = \phi$  and hence  $\alpha_1 = 0$  . As each compo-

nent of  $\sum_{j=1}^{n-1} \gamma_j c_j^T$  and  $\sum_{j=1}^{n-1} \beta_j c_j$  is  $-\phi$  , then  $\gamma_j = \beta_j = 0$  for

$j = 1, 2, \dots, n-1$  . Thus the  $n^2$  matrices in (2.1) are linearly independent. It is clear that  $B_i, C_j, D_j$  are nonsingular since the  $A_i$ 's are nonsingular. If the characteristic of  $F$  is not a divisor of  $n-1$  , then

$$E = \left( \begin{array}{c|c} I & \bar{1}^T \\ \hline \bar{1} & 0 \end{array} \right)$$

and has determinant  $-(n-1) \neq 0$  . If the characteristic of  $F$  is a divisor of  $n-1$  , then

$$E = \left( \begin{array}{c|c} I + E_{12} & \bar{1}^T \\ \hline \bar{1} & 0 \end{array} \right)$$

and has determinant  $-(n-2) \neq 0$  .

Using Lemma 2.9 and further results in [2], Brawley obtains the following result.

Result 2.10. Let  $P$  be an  $s \times n$  rank  $s$  matrix over a field  $F$  where  $0 < s \leq [n/2]$  , and let  $V$  denote the vector space consisting

of all  $n \times s$  solutions  $X$  to  $PX = 0$ . Then  $V$  has a rank  $s$  basis.

Theorem 2.11. For  $0 < s \leq [n/2]$  let  $P$  be an  $s \times n$  rank  $s$  matrix over an arbitrary field. Then any matrix in  $\{P\} = \{I - QP : Q \text{ is } n \times s, \text{rank } Q \text{ is } s, \text{ and } PQ = 0\}$  is a pseudo-involution of signature  $s$ . Moreover, the group  $\overline{G}[P]$  generated by  $\{P\}$  is  $\overline{P} = \{I - XP : X \text{ is } n \times s \text{ and } PX = 0\}$ .

Proof. It is clear that  $\{P\}$  contains only pseudo-involutions of signature  $s$ . From Theorem 2.7  $\overline{P}$  is a group. Clearly  $\{P\}$  is contained in  $\overline{P}$ . It remains to be shown that any element in  $\overline{P}$  is a finite product of elements from  $\{P\}$ . Consider  $I - XP$ , where  $X$  is  $n \times s$  and  $PX = 0$ . By Lemma 2.10,  $X$  is 0 or a sum of a finite number of  $Q_i$ 's where each  $Q_i$  is  $n \times s$ , rank  $s$ . If  $X = 0$ , then  $I = (I - QP)(I + QP)$ , where  $I - QP$  in  $\{P\}$  implies  $I + QP$  is in  $\{P\}$ . Otherwise  $X = \sum_i Q_i$ , and hence  $I - XP = I - (\sum_i Q_i)P = I - \sum_i Q_i P = \prod_i (I - Q_i P)$  as  $PQ_i = 0$ . As each  $I - Q_i P$  is in  $\{P\}$ ,  $I - XP$  is in  $\overline{G}[P]$ .

We now state two results from Brawley [1], which characterize  $G[P]$ , first when  $\text{char } p \neq 2$  and then when  $\text{char } p = 2$ .

Result 2.12. If  $P$  is  $s \times n$ , rank  $s$  over  $F$  where  $\text{char } p \neq 2$ , there exists an  $n \times n$  nonsingular  $K$  over  $F$  such that  $KG[P]K^{-1} = G_0$ , where  $G_0$  consists of all matrices of the form

$$\begin{pmatrix} I_{n-s} & Z \\ 0 & \pm I_s \end{pmatrix}, \text{ where } Z \text{ is an arbitrary } (n-s) \times s \text{ matrix.}$$

Result 2.13. Let  $P$  be an  $s \times n$  rank  $s$  matrix over a field of characteristic 2 with  $0 < s \leq [n/2]$ . Then

- (a) every matrix in  $G[P]$  is involutory,
- (b)  $G[P] = \{I + XP : X \text{ is } n \times s \text{ and } PX = 0\}$ ,
- (c) there exists an  $n \times n$  nonsingular matrix  $K$  such that  $K G[P] K^{-1} = G_0$  where

$$G_0 = \left\{ \begin{pmatrix} I_{n-s} & Z \\ 0 & I_s \end{pmatrix} : Z \text{ is arbitrary } (n-s) \times s \text{ matrix} \right\}.$$

In order to give a characterization for  $\overline{G}[P]$  when the characteristic of  $F$  is arbitrary which is similar to the characterization of  $G[P]$  in Result 2.13, we characterize  $\overline{P}$  and hence also  $\overline{G}[P]$ .

Theorem 2.14. If  $P$  is an  $s \times n$  rank  $s$  matrix over a field  $F$ , then there exists an  $n \times n$  nonsingular matrix  $K$  over  $F$  such that  $K \overline{P} K^{-1} = G_0$  where

$$G_0 = \left\{ \begin{pmatrix} I_s & 0 \\ Z & I_{n-s} \end{pmatrix} : Z \text{ is arbitrary } (n-s) \times s \text{ over } F \right\}.$$

Proof. Let  $A \in \overline{P}$ , where  $A = I - XP$  with  $PX = 0$ . As  $P$  has rank  $s$ , there is an  $(n-s) \times n$  matrix  $P^*$  such that

$$K = \begin{bmatrix} P \\ P^* \end{bmatrix} \text{ is nonsingular.}$$

Set  $K^{-1} = [K_1, K_2]$ , where  $K_1$  is  $n \times s$  and  $K_2$  is  $n \times (n-s)$ , and note that  $A = I - XP = I - [X, 0]K$ , where  $0$  is the  $n \times (n-s)$  zero matrix. Thus

$$\begin{bmatrix} P \\ P^* \end{bmatrix} [K_1, K_2] = \begin{bmatrix} PK_1 & PK_2 \\ P^*K_1 & P^*K_2 \end{bmatrix} = \begin{bmatrix} I_s & 0 \\ 0 & I_{n-s} \end{bmatrix},$$

and so  $PK_1 = I$ ,  $PK_2 = 0$ ,  $P^*K_1 = 0$  and  $P^*K_2 = I$ . With  $PX = 0$ ,

$$\begin{aligned} KAK^{-1} &= \begin{bmatrix} P \\ P^* \end{bmatrix} (I - [X, 0]K) K^{-1} \\ &= \begin{bmatrix} P \\ P^* \end{bmatrix} ([K_1, K_2] - [X, 0]) \\ &= \begin{bmatrix} P \\ P^* \end{bmatrix} [K_1 - X, K_2] \\ &= \begin{bmatrix} PK_1 - PX & PK_2 \\ P^*K_1 - P^*X & P^*K_2 \end{bmatrix} \\ &= \begin{bmatrix} I - PX & 0 \\ -P^*X & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -P^*X & I \end{bmatrix}. \end{aligned}$$

Hence  $KAK^{-1}$  is in  $G_0$ , with  $Z = -P^*X$ . If  $Z$  is an arbitrary  $(n-s) \times s$  matrix over  $F$ , let  $X$  be the unique  $n \times s$  solution of

$$KX = \begin{bmatrix} 0 \\ -Z \end{bmatrix},$$

where  $0$  is the  $s \times s$  zero matrix. Then  $P^*X = -Z$  and  $PX = 0$  and so  $A = I - XP$  is in  $\bar{P}$  and

$$KAK^{-1} = \begin{bmatrix} I_s & 0 \\ Z & I_{(n-s)} \end{bmatrix}.$$

Corollary 2.15. Let  $P$  be an  $s \times n$  rank  $s$  matrix over an arbitrary field with  $0 < s \leq [n/2]$ . Then

- (a) every matrix in  $\bar{G}[P]$  is a pseudo-involution,
- (b)  $\bar{G}[P] = \{I - XP : X \text{ is } n \times s \text{ and } PX = 0\}$ ,
- (c) there is an  $n \times n$  nonsingular matrix  $K$  such that  $K \bar{G}[P] K^{-1} = G_0$ , where

$$G_0 = \left\{ \begin{pmatrix} I_s & 0 \\ Z & I_{n-s} \end{pmatrix} : Z \text{ is an } (n-s) \times s \text{ arbitrary matrix} \right\}.$$

Proof. This result follows immediately from Theorems 2.14, 2.11, 2.7, and the observation that  $\bar{G}[P] = \bar{P}$ .

## CHAPTER III

## DIAGONALIZATION OF A GROUP OF INVOLUTIONS

We now consider a group  $G$  of involutions,  $G = \{I_n, A_1, A_2, \dots, A_t\}$ , over a field  $F$  of  $\text{char } p \neq 2$ . It is known that each involution of  $G$  is similar to a matrix of the form  $\text{Diag}(-I_s, I_{n-s})$  for a unique  $s$ ,  $0 \leq s \leq n$ .

Let  $P$  be a nonsingular  $n \times n$  matrix such that for a particular involution of  $G$ , denoted  $A_0$ ,  $P^{-1} A_0 P = \text{Diag}(-I_s, I_{n-s})$ , where  $\sigma(A_0) = s$ . If  $s = 0$  then  $P^{-1} A_0 P = I$  and if  $s = n$  then  $P^{-1} A_0 P = -I_n$ , so we will be concerned with the case where  $0 < \sigma(A_0) < n$ .

It should be noted that if  $G$  is a group of involutions over a field  $F$  with  $\text{char } p \neq 2$ , then  $P^{-1} G P$  is also a group of involutions over the field  $F$  and furthermore  $P^{-1} G P$  is isomorphic to  $G$ .

Theorem 3.1. Given a group  $G$  of  $n \times n$  involutions which contains an  $A \neq \pm I$ , set  $s = \max\{\sigma(A) : A \in G, 0 < \sigma(A) < n\}$ . Let  $A_0 \in G$  such that  $\sigma(A_0) = s$  and let  $P$  be a fixed nonsingular  $n \times n$  matrix such that  $P^{-1} A_0 P = \text{Diag}(-I_s, I_{n-s})$ . Then  $P^{-1} A P = \text{Diag}(A_{11}, A_{22})$  with  $A_{11}^2 = I_s$  and  $A_{22}^2 = I_{n-s}$  for any  $A \in G$ . Moreover, if  $P^{-1} A P = \text{Diag}(A_{11}, A_{22})$ , then  $P^{-1} A_0 A P = \text{Diag}(-A_{11}, A_{22})$ .

Proof. For  $A \in G$ , set

$$P^{-1} AP = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$  are  $s \times s$ ,  $s \times (n-s)$ ,  $(n-s) \times s$ , and  $(n-s) \times (n-s)$  respectively. Then

$$\begin{aligned} P^{-1} A_0 AP &= P^{-1} A_0 PP^{-1} AP = \\ &= \text{Diag}(-I_s, I_{n-s}) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} -A_{11} & -A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \end{aligned}$$

$$(P^{-1} AP)^2 = \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix},$$

and

$$(P^{-1} A_0 AP)^2 = \begin{pmatrix} A_{11}^2 - A_{12}A_{21} & A_{11}A_{12} - A_{12}A_{22} \\ -A_{21}A_{11} + A_{22}A_{21} & -A_{21}A_{12} + A_{22}^2 \end{pmatrix}.$$

But  $(P^{-1} AP)^2 = \text{Diag}(I_s, I_{n-s}) = (P^{-1} A_0 AP)^2$ , so equating corresponding blocks in the partitioned matrices and adding gives,

- (i)  $A_{11}^2 = I_s$
- (ii)  $A_{11}A_{12} = 0_{s \times (n-s)}$

$$(iii) \quad A_{22}^2 = I_{n-s},$$

$$(iv) \quad A_{22}A_{21} = 0_{(n-s) \times s}.$$

From (ii) and (iv)  $A_{12} = 0_{s \times (n-s)}$  and  $A_{21} = 0_{(n-s) \times s}$ . Thus for any  $A \in G$ ,  $P^{-1}AP = \text{Diag}(A_{11}, A_{22})$  with  $A_{11}^2 = I_s$  and  $A_{22}^2 = I_{n-s}$ . Moreover,  $P^{-1}A_0AP = \text{Diag}(-A_{11}, A_{22})$ .

For the rest of this paper  $G$  is a group of  $n \times n$  involutions over a field  $F$  with  $\text{char } p \neq 2$  and  $A_0$  denotes a fixed involution with singature  $s = \max\{\sigma(A) : A \in G, 0 < \sigma(A) < n\}$ . Moreover,  $P^{-1}A_0P = \text{Diag}(-I_s, I_{n-s})$  for a fixed nonsingular  $n \times n$  matrix  $P$ . Also  $A$  will denote any involution in  $G$ .

Proposition 3.2. If  $P^{-1}A_0P = \text{Diag}(-I_s, I_{n-s})$ ,  $P^{-1}AP = \text{Diag}(A_{11}, A_{22})$ ,  $A_{11}^2 = I_s$  and  $A_{22}^2 = I_{n-s}$ , then

$$(i) \quad \sigma(A_{11}) + \sigma(-A_{11}) = s,$$

$$(ii) \quad \sigma(A_{22}) \leq [s/2], \text{ for all } A \neq -I, -A_0.$$

Proof. Let  $\sigma(A_{11}) = k$ ,  $0 \leq k \leq s$ . Then  $\sigma(-A_{11}) = s - k$  and  $\sigma(A_{11}) + \sigma(-A_{11}) = k + s - k = s$ . As  $A \neq -I$

$$\sigma(P^{-1}AP) = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} = \sigma(A_{11}) + \sigma(A_{22}) \leq s,$$

and

$$\sigma(P^{-1}A_0AP) = \begin{pmatrix} -A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} = \sigma(-A_{11}) + \sigma(A_{22}) \leq s,$$

as  $A \neq A_0$ . Then  $\sigma(P^{-1}AP) = \sigma(A_{11}) + \sigma(A_{22}) \leq s$  and  
 $\sigma(P^{-1}A_0AP) = \sigma(-A_{11}) + \sigma(A_{22}) \leq s$ . Adding gives  
 $\sigma(A_{11}) + \sigma(-A_{11}) + 2\sigma(A_{22}) = s + 2\sigma(A_{22}) \leq 2s$  from (i). Thus  
 $\sigma(A_{22}) \leq \frac{s}{2}$  and, since  $\sigma(A_{22})$  is an integer,  $\sigma(A_{22}) \leq [s/2]$ .

Proposition 3.3. For  $A \neq -I$ ,  $-A_0$ , set  $P^{-1}AP = \text{Diag}(A_{11}, A_{12})$ .  
 Then  $\sigma(A_{11}) \geq \sigma(A_{22})$ .

Proof. By contradiction. Suppose  $\sigma(A_{11}) < \sigma(A_{22})$ . By  
 Proposition 3.2,  $\sigma(A_{22}) \leq [s/2]$ , so then  $\sigma(A_{11}) < [s/2]$ . Let  
 $\sigma(A_{22}) = u$  and  $\sigma(A_{11}) = v$ . Then  $v < u$  and for  $u = v + \alpha$ ,  
 $\alpha > 0$ . Consider  $P^{-1}A_0AP = \text{Diag}(-A_{11}, A_{22})$ . Then  
 $\sigma(P^{-1}A_0AP) = \sigma(-A_{11}) + \sigma(A_{22}) = (s - v) + u = s + \alpha > s$ , a contra-  
 diction as  $A_0A \neq -I$ . Thus  $\sigma(A_{11}) \geq \sigma(A_{22})$ .

Corollary 3.4. Suppose  $A$  satisfies the hypotheses of Proposition  
 3.3 and  $\sigma(A_{22}) = [s/2]$ . If  $s$  is even, then  $\sigma(A_{11}) = [s/2]$  and  
 if  $s$  is odd, then  $\sigma(A_{11}) = [s/2]$  or  $[s/2] + 1$ .

Proof. Let  $s$  be even so that  $[s/2] = s/2$ . Let  $\sigma(A_{22}) = s/2$ .  
 By Proposition 3.3,  $\sigma(A_{11}) \geq \sigma(A_{22})$  so that  $\sigma(A_{11}) \geq s/2$ . If  
 $\sigma(A_{11}) > s/2$ , then  $\sigma(A) = \sigma(A_{11}) + \sigma(A_{22}) > s$ , a contradiction as  
 $\sigma(A) \leq s$ . So if  $\sigma(A_{22}) = s/2$  then  $\sigma(A_{11}) = s/2 = [s/2]$ . Now  
 let  $s$  be odd and  $\sigma(A_{22}) = [s/2]$ . Again by Proposition 3.3,  
 $\sigma(A_{11}) \geq \sigma(A_{22})$ . So let  $\sigma(A_{22}) = [s/2]$  and observe  $\sigma(A_{11}) \geq [s/2]$ .  
 Thus, if  $\sigma(A_{11}) > [s/2] + 1$ , then  $\sigma(A) = \sigma(A_{11}) + \sigma(A_{22}) > ([s/2] + 1)$   
 $+ [s/2] = s$ , a contradiction.

Proposition 3.5. Let  $G$  be a group of involutions containing an  $A$  such that  $0 < \sigma(A) < n$ . The sets  $G_{11} = \{A_{11} : \text{Diag}(A_{11}, A_{22}) \in P^{-1}GP\}$  and  $G_{22} = \{A_{22} : \text{Diag}(A_{11}, A_{22}) \in P^{-1}GP\}$  are groups of involutions under multiplication.

Proof. As  $s < n$ , then  $I_s \in G_{11}$ ,  $I_{n-s} \in G_{22}$  since  $I \in G$  and  $P^{-1} \text{Diag}(I_s, I_{n-s})P = \text{Diag}(I_s, I_{n-s})$ . For  $A \in G$ ,  $A = A^{-1}$  since  $A^2 = I$  and  $P^{-1}AP = \text{Diag}(A_{11}, A_{22})$  and, by Theorem 3.1,  $A_{11}^2 = I_s$  and  $A_{22}^2 = I_{n-s}$ . Thus for each  $A_{11} \in G_{11}$ ,  $A_{11} = A_{11}^{-1}$  and for each  $A_{22} \in G_{22}$  we have  $A_{22} = A_{22}^{-1}$ . Now let  $A, B \in G$  where  $P^{-1}AP = \text{Diag}(A_{11}, A_{22})$  and  $P^{-1}BP = \text{Diag}(B_{11}, B_{22})$  with  $A_{11}^2 = I_s = B_{11}^2$  and  $A_{22}^2 = I_{n-s} = B_{22}^2$ . Consider  $P^{-1}ABP = \text{Diag}(A_{11}B_{11}, A_{22}B_{22})$ . As  $A, B, AB \in G$  since  $G$  is a group, then  $P^{-1}AP$ ,  $P^{-1}BP$ , and  $P^{-1}ABP \in P^{-1}GP$ . Thus  $A_{11}B_{11} \in G_{11}$  and  $A_{22}B_{22} \in G_{22}$  and  $G_{11}, G_{22}$  are closed. Therefore  $G_{11}$  and  $G_{22}$  are groups of involutions under multiplication.

Let  $\theta_1$  denote the function which maps  $\text{Diag}(A_{11}, A_{22})$  to  $A_{11}$ . Thus  $\theta_1: P^{-1}GP \rightarrow G_{11}$  and is clearly a homomorphism. Similarly for  $\theta_2$ . These functions will be used in Chapter IV.

## CHAPTER IV

## CONSTRUCTION OF GROUPS OF INVOLUTIONS

Let  $G_1$  be a finite group of involutions consisting of  $n \times n$  matrices with  $-I_n \in G_1$ . Every element of  $G_1$  has order two and thus any two elements of  $G_1$  commute. Hence  $G_1$  is an Abelian group of order  $2^k$  for some  $k$  and  $G_1$  can be thought of as a direct product,  $G_1 \cong \overline{G}_1 \times \overline{G}_2 \times \dots \times \overline{G}_k$ , where each  $\overline{G}_i$  has order two and  $\overline{G}_i = \{I_n, A_i\}$  with  $\sigma(A_i) = s_i \leq n$ . Then for  $A \in G$ , we have  $A = A_1^{i_1} \cdot A_2^{i_2} \cdot \dots \cdot A_k^{i_k}$ , where  $i_j = 0$  or  $1$ .

Let  $G_2 = \{I_m, L\}$  where  $L^2 = I_m$ . Thus  $G_2$  is a group of involutions of order two. Note, since  $|G_1| = 2^k$  and  $|G_2| = 2$ , then  $|G_1|/|G_2| = 2^{k-1}$ .

Using groups  $G_1$  and  $G_2$  of involutions, we will construct another group of involutions where each element is a diagonal block matrix with two nonzero blocks and these blocks are chosen from groups  $G_1$  and  $G_2$ .

Lemma 4.1. Let  $H$  be a group of order  $2^k$  of  $(n+m) \times (n+m)$  involutions, where

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \text{ is } n \times n \text{ and } B \text{ is } m \times m \right\}.$$

Let  $G_1 = \theta_1 H$  and  $G_2 = \theta_2 H$ . Suppose  $\theta_2 H = \{I_m, L\}$ , then

$$|\theta_2^{-1}(I_m)| = |\theta_2^{-1}(L)| = 2^{k-1}.$$

Proof. As  $\theta_2$  is a homomorphism of  $H$  onto  $\theta_2 H$ , then  $H/K \cong \theta_2 H$ , where  $K$  is the kernel of  $\theta_2$ . Thus  $\theta_2^{-1}(I_m) = K$ . Since  $|\theta_2 H| = 2$ ,  $H/K$  has two cosets and each coset contains  $2^{k/2} = 2^{k-1}$  elements. Therefore  $|\theta_2^{-1}(I_m)| = |\theta_2^{-1}(L)| = 2^{k-1}$ .

Theorem 4.2. Let  $G_1$  be a group of order  $2^k$  ( $k \geq 1$ ) of  $n \times n$  involutions with  $-I_n$  in  $G_1$  and let  $G_2 = \{I_m, L\}$  be a group of two  $m \times m$  involutions. There is a group  $H$  of order  $2^k$  such that  $\theta_1 H = G_1$  and  $\theta_2 H = G_2$ .

Proof. We show by induction on  $i$  that for  $1 \leq i \leq k$  there is a subgroup  $G_1^{(i)}$  of  $G_1$  of order  $2^i$  and a group  $H^{(i)}$  of order  $2^i$  of  $(n+m) \times (n+m)$  involutions with  $\theta_1 H^{(i)} = G_1^{(i)}$  and  $\theta_2 H^{(i)} = G_2$ . For  $i = 1$ , set  $G_1^{(1)} = \{I, -I\}$  and with

$$H^{(1)} = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} -I_n & 0 \\ 0 & L \end{pmatrix} \right\},$$

$H^{(1)}$  has order 2 and  $\theta_1 H^{(1)} = G_1^{(1)}$ ,  $\theta_2 H^{(1)} = G_2$ , where  $|H^{(1)}| = |G_1^{(1)}| = 2$ . For  $i = 2$ , set  $G_1^{(2)} = \{I, -I, A_1, -A_1\}$ , where  $A_1 \in G_1$ ,  $A_1 \neq \pm I$ .  $G_1^{(2)}$  is a subgroup of order 2 of  $G_1$ . Then set

$$\begin{aligned} H^{(2)} &= \left\{ \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} -I_n & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} -A_1 & 0 \\ 0 & L \end{pmatrix} \right\} \\ &= H^{(1)} \cup \begin{pmatrix} A_1 & 0 \\ 0 & I_m \end{pmatrix} H^{(1)} \end{aligned}$$

or

$$H^{(2)} = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} -I_n & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} -A_1 & 0 \\ 0 & I_m \end{pmatrix} \right\}$$

$$= H^{(1)} \cup \begin{pmatrix} A_1 & 0 \\ 0 & L \end{pmatrix} H^{(1)}.$$

Then for either choice of  $H^{(2)}$ ,  $H^{(2)}$  is a group of order  $2^2$  and  $\theta_1 H^{(2)} = G_1^{(2)}$  and  $\theta_2 H^{(2)} = G_1$ , where  $|H^{(2)}| = |G_1^{(2)}| = 2^2$ . Now suppose that for  $i < k$ ,  $G_1^{(i)}$  is a subgroup of  $G_1$  of order  $2^i$  and  $H^{(i)}$  is a group of order  $2^i$  such that  $\theta_1 H^{(i)} = G_1^{(i)}$  and  $\theta_2 H^{(i)} = G_2$ . Then  $|G_1^{(i)}| = |H^{(i)}| = 2^i$ . As  $i < k$ , and  $H$  has order  $2^k$ , then  $|G_1^{(i)}| = 2^i < 2^k = |G_1|$  and there is an  $A_{i-1} \in G_1$  such that  $A_{i-1} \notin G_1^{(i)}$ . Set

$$G_1^{(i+1)} = G_1^{(i)} \cup A_{i-1} G_1^{(i)}.$$

Then as  $G_1$  is a commutative group,  $G_1^{(i+1)}$  is a subgroup of  $G_1$  of order  $2^{i+1}$ . Set

$$H^{(i+1)} = H^{(i)} \cup \begin{pmatrix} A_{i-1} & 0 \\ 0 & I_m \end{pmatrix} H^{(i)}$$

or

$$H^{(i+1)} = H^{(i)} \cup \begin{pmatrix} A_{i-1} & 0 \\ 0 & L \end{pmatrix} H^{(i)}.$$

We assert that  $H^{(i+1)}$  is a group. Since  $H^{(i)}$  is a group, the product of two elements from  $H^{(i)}$  is in  $H^{(i)}$ . Since all elements in  $H^{(i+1)}$  are in diagonal block form,  $\theta_1 H^{(i+1)} = G_1^{(i+1)}$ , a commutative group, and  $\theta_2 H^{(i+1)} = G_2$ , a commutative group, the product of elements in  $H^{(i+1)}$  is commutative. Hence, the product of an element from  $H^{(i)}$  and an element from either coset is again in this coset and the product of two elements from either coset is in  $H^{(i)}$ . Moreover, each element in  $H^{(i+1)}$  is its own inverse.

Clearly,  $\theta_1 H^{(i+1)} = G_1^{(i+1)}$  and  $\theta_2 H^{(i+1)} = G_2$ , where  $|H^{(i+1)}| = |G_1^{(i+1)}| = 2^{i+1}$ . Now set  $H = H^{(k)}$ . Then

$\theta_1 H = \theta_1 H^{(k)} = G_1^{(k)} = G_1$  as  $|G_1^{(k)}| = 2^k = |G_1|$  and  $\theta_2 H = \theta_2 H^{(k)} = G_2$ . Moreover,  $|H| = |H^{(k)}| = 2^k$ .

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