Some Classification Results for Hadamard Matrices of Order 6

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April 15, 2010

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Abstract

Hadamard matrices are easy to define, yet quite meaningful objects which arise in many branches of mathematics and physics. Those of dimension 6 are of major interest in quantum information theory. This is also the smallest dimension for which a classification of Hadamard matrices is not known. I will obtain some interesting classes of 6x6 Hadamard matrices, satisfying some extra restrictions.

1 Introduction

Complex Hadamard Matrices are an interesting type of matrices that appear in several branches of mathematics, physics and computer science. Despite their various applications in these fields, there are actually quite easy to define.

Definition. *A matrix $H \in \mathbb{M}_n(\mathbb{C})$ is a Hadamard matrix if and only if*

1. $HH^* = nI_n$, *where* $H^* = \overline{H}^T$
2. $|H_{i,j}| = 1$, *for* $1 \leq i, j \leq n$

In other words, all the entries of an $n \times n$ Hadamard matrix are from the complex unit circle, while the rows are made up of orthogonal vectors from $\mathbb{C}^n$.

From these conditions, it is easy to see that if one takes a Hadamard matrix and permutes the rows or columns, the new matrix will still satisfy the Hadamard conditions. In addition, one can multiply a row or column by a number from the complex unit circle and still obtain a Hadamard matrix. Any matrix that can be obtain from a Hadamard matrix using these operations is said to be equivalent to the original one. Therefore, by multiplying the rows and columns by the appropriate values, it is possible to normalize any Hadamard matrix into the following form:

$$H = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & . & . & \cdots & . \\ 1 & . & . & \cdots & . \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & . & . & \cdots & . \end{pmatrix}$$

From this normalized form and the orthogonality of the rows and columns, solving for a general $n \times n$ Hadamard matrix is equivalent to solving a $n(n-1)$ nonlinear system of equations in $(n-1)^2$ variables. Therefore, finding and classifying Hadamard matrices quickly becomes difficult for large n. Despite this, it is known that there does exist at least one Hadamard matrix of order n for any $n \geq 1$, the so-called Fourier Matrix.

$$(F_n)_{k,l} = \epsilon^{kl}, \text{ for } 0 \leq k, l \leq n-1, \text{ with } \epsilon = e^{2\pi i/n}, \text{ the } n^{th} \text{ root of unity}$$

$$F_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \epsilon & \epsilon^2 & \cdots & \epsilon^{n-1} \\ 1 & \epsilon^2 & \epsilon^4 & \cdots & \epsilon^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \epsilon^{n-1} & \epsilon^{2(n-1)} & \cdots & \epsilon^{(n-1)(n-1)} \end{pmatrix}$$

2 Classifying Hadamard Matrices

While the Fourier Matrix is a Hadamard matrix for every n, it is generally unknown if it is the only Hadamard matrix of size n. In fact, the only orders that have been completely classified are those less than 6. For $n = 2$, 3, or 5, it turns out that the Fourier matrix is the only Hadamard Matrix of that size up to equivalence [1]. However, for $n = 4$, the Fourier matrix is actually part of a one parameter family, meaning that there are, in fact, an infinite number of non-equivalent Hadamard matrices of order 4.

Another important result is that for n prime, F_n is actually an isolated matrix, meaning there is no family of matrices that contains F_n like in the $n = 4$ case. Because of this, it was conjectured that for n prime, F_n was the only Hadamard matrix of order n. However, this was shown to be false for $n = 7$, as there actually exists a Hadamard matrix whose entries are based upon $e^{2\pi i/6}$, the sixth root of unity [2].

In particular, Hadamard Matrices of order 6 are of interest, since they are the smallest order for which a complete classification is unknown. Furthermore, they are of particular interest in Quantum Information Theory, specifically in how they relate to Mutually Unbiased Bases [3]. Because of this connection, any new Hadamard matrix of order 6 is important, as it helps reveal insights into the related problems in this field.

3 Order 6 Hadamard Matrices

In order to easily classify a subset of the order 6 matrices, we will assume that the Hadamard matrix we are solving for is in a normalized form and will have four -1's along the diagonal. From this simple requirement, we will get the following classification theorem.

Theorem 1. *Let $H \in M_6(\mathbb{C})$ be a normalized Hadamard matrix with four -1's along the diagonal. Then H is equivalent to one of the following matrices:*

$$H_1(b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i & b & -b \\ 1 & i & -1 & -i & -b & b \\ 1 & -i & -i & -1 & i & i \\ 1 & -\bar{b} & \bar{b} & i & -1 & -i \\ 1 & \bar{b} & -\bar{b} & i & -i & -1 \end{pmatrix}$$

$$H_2(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i & x & -x \\ 1 & i & -1 & -i & ix & -ix \\ 1 & x & -ix & -1 & -x & ix \\ 1 & -i & i & 1 & -1 & -1 \\ 1 & -x & ix & -1 & -ix & x \end{pmatrix}$$

where $x, b \in \mathbb{C}$ and $|x| = |b| = 1$.

Remark: Note that $H_1(1)$ is actually equivalent to $H_2(i)$ after permuting the appropriate rows and columns.

This reason that the requirement about the diagonal is so useful in actually classifying these matrices arises from the following lemma.

Lemma. *If $x, y, z, t \in \mathbb{C}$ such that $|x| = |y| = |z| = |t| = 1$ and $x + y + z + t = 0$, then $x \in \{-y, -z, -t\}$.*

Proof. We have $(x + y)(x + z)(x + t) = x^2(x + y + z + t) + xyz + xyt + xzt + yzt = xyz + xyt + xzt + yzt = xyzt(\bar{x} + \bar{y} + \bar{z} + \bar{t}) = 0$. □

Since the matrix is in a normalized form, we know from the orthogonality condition that the sum of the elements in any row must be 0. Therefore, if a row has a -1 in it, applying the lemma tells us that the remaining four unknown elements in that row will contain two pairs of opposite complex numbers. A similar result holds for the columns of the matrix as well.

In particular, the second row of the matrix will be based upon two variables, a and b. Furthermore, because of the ability to permute columns and rows, we can place these a's and b's as we wish without removing the -1's from the diagonal. Therefore, we will assume that b is in the fifth column, and by looking at the position of $-b$ in column five, we get two main cases:

Case 1.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & b & -b \\ 1 & . & -1 & . & -b & . \\ 1 & . & . & -1 & d & . \\ 1 & . & . & . & -1 & . \\ 1 & . & . & . & -d & x \end{pmatrix}$$

Case 2.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & b & -b \\ 1 & . & -1 & . & d & . \\ 1 & . & . & -1 & -d & . \\ 1 & . & . & . & -1 & . \\ 1 & . & . & . & -b & x \end{pmatrix}$$

Building on these initial cases, it is possible to continue creating cases based upon the location of opposite pairs of entries in the rows and columns containing a -1. By using the Hadamard conditions, one can eventually solve for particular variables until one of the two matrices from Theorem 1 is reached, or a contraction is reached.

4 Example Calculations

For the full calculations used to prove Theorem 1, see Appendix 1 for Case 1, and Appendix 2 for Case 2. Below is a sample derivation of one matrix from Case 1.

To begin, note that because of the placement of $-b$ in the fifth column, there are three choices for Row 3:

$$R_3 = \begin{cases} 1) & \begin{pmatrix} 1 & c & -1 & -c & -b & b \end{pmatrix} \\ 2) & \begin{pmatrix} 1 & c & -1 & b & -b & -c \end{pmatrix} \\ 3) & \begin{pmatrix} 1 & b & -1 & c & -b & -c \end{pmatrix} \end{cases}$$

In the first case, the orthogonality of rows two and three gives us

$$\begin{aligned} (C_2) \cdot (C_3) &= 1 - \bar{c} - a + a\bar{c} - 1 - 1 \\ 0 &= 1 + \bar{c} + a - a\bar{c} \end{aligned}$$

Therefore, by applying the lemma, we see that $1 = a\bar{c}$, as otherwise, $a = 0$ or $c = 0$. Hence, because $|a| = |c| = 1$, then $a = c$, and since $\bar{c} = -a$, then $a = c = \pm i$. This then forces the last two entries of column four to be $\pm i$, so we can now use the orthogonality of columns four and five to further reduce the variables:

$$(C_5) \cdot (C_4) = 1 \pm ib \mp ib - d \pm i \pm id$$
$$0 = 1 \pm i - d \pm id$$

Again, by using the lemma, we can conclude that $d = \pm i$, so our matrix is of the following form:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & b & -b \\ 1 & \pm i & -1 & \mp i & -b & b \\ 1 & . & . & -1 & \pm i & . \\ 1 & . & . & \pm i & -1 & . \\ 1 & . & . & \pm i & \mp i & x \end{pmatrix}$$

Again, there are now three choices for row four:

$$R_4 = \begin{cases} 1) & \begin{pmatrix} 1 & e & -e & -1 & \pm i & \mp i \end{pmatrix} \\ 2) & \begin{pmatrix} 1 & e & \mp i & -1 & \pm i & -e \end{pmatrix} \\ 3) & \begin{pmatrix} 1 & \mp i & e & -1 & \pm i & -e \end{pmatrix} \end{cases}$$

If we look at the first choice, and take the dot products of row two with row four and of row three with row four, we get the following:

$$(R_2) \cdot (R_4) = 1 - \bar{e} \mp i\bar{e} \pm i \mp ib \mp ib$$
$$(R_3) \cdot (R_4) = 1 \pm i\bar{e} + e \pm i \pm ib \pm ib$$

Adding these gives us $0 = 2 \pm 2i$, so this can not produce a Hadamard matrix.

On the other hand, looking at case two, when we we take the dot product of rows two and four, and of rows three and four, we get

$$(R_2) \cdot (R_4) = 1 - \bar{e} - 1\bar{e} \pm i \mp ib + b\bar{e}$$
$$(R_3) \cdot (R_4) = 1 - \pm i\bar{e} \mp i \pm i \pm ib - b\bar{e}$$

Adding these gives us the following:

$$0 = 1 \pm i - \bar{e} \pm i\bar{e}$$

Using the lemma we know that $e = \mp i$. This leaves only six unknown entries in the matrix. Performing only a few more calculations, we discover that the matrix must be of the form

Appendix I: Case 1 Calculations

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & b & -b \\ 1 & & -1 & & -b & \\ 1 & & & -1 & d & \\ 1 & & & & -1 & \\ 1 & & & & -d & x \end{bmatrix} \Rightarrow R_3 = \begin{cases} 1)\ [1\ \ c\ \ -1\ \ -c\ \ -b\ \ b] \\ 2)\ [1\ \ c\ \ -1\ \ b\ \ -b\ \ -c] \\ 3)\ [1\ \ b\ \ -1\ \ c\ \ -b\ \ -c] \end{cases}$$

1) $(R_2)(R_3)^* = 1 - \bar{c} - a + a\bar{c} - 1 - 1 = 0$

$\Rightarrow a\bar{c} - a - \bar{c} - 1 = 0$

$\Rightarrow a\bar{c} = -1$ (if a or $c = \pm 1$, then the other is 0)

$\Rightarrow a = c = \pm i$

So, $C_4 = \begin{bmatrix} 1 \\ \mp i \\ \mp i \\ -1 \\ \pm i \\ \pm i \end{bmatrix} \Rightarrow (C_4^*)(C_5) = 1 \pm bi \mp bi - d \pm i \pm di = 0$

$\Rightarrow d(1 \mp i) = 1 \pm i$

$\Rightarrow d = \pm i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & b & -b \\ 1 & \pm i & -1 & \mp i & -b & b \\ 1 & & & -1 & \pm i & \\ 1 & & & \pm i & -1 & \\ 1 & & & \pm i & \mp i & x \end{bmatrix} \Rightarrow R_4 = \begin{cases} 1)\ [1\ \ e\ \ -e\ \ -1\ \ \pm i\ \ \mp i] \\ 2)\ [1\ \ e\ \ \mp i\ \ -1\ \ \pm i\ \ -e] \\ 3)\ [1\ \ \mp i\ \ e\ \ -1\ \ \pm i\ \ -e] \end{cases}$$

1.1) $(R_2)(R_4)^* = 1 - \bar{e} \mp i\bar{e} \pm i \mp ib \mp ib = 0$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{e} \pm i \pm ib \pm ib = 0$

adding gives: $2 \pm 2i = 0$ $\quad \divideontimes$

1.2) $(R_2)(R_4)^* = 1 - \bar{e} - 1 \pm i \mp ib + b\bar{e} = 0$

$(R_3)(R_4)^* = 1 \pm i\bar{e} \mp i \pm i \pm ib - b\bar{e} = 0$

adding gives: $1 \pm i - \bar{e} \pm i\bar{e} = 0$

$\Rightarrow \bar{e} = \pm i, \quad e = \mp i$

$$So,\ H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & b & -b \\ 1 & \pm i & -1 & \mp i & -b & b \\ 1 & \mp i & \mp i & -1 & \pm i & \pm i \\ 1 & f & g & \pm i & -1 & h \\ 1 & -f & -g & \pm i & \mp i & x \end{bmatrix}$$

$(R_5)(R_6)^* = 1 - 1 - f + 1 \mp i + h\bar{x} = 0 \Rightarrow h = \pm i x$

$(R_5)(R_1)^* + (R_6)(R_1)^* = -1 \mp f + g \pm i - 1 \pm ix + 1 - f - g \pm i \mp i + x = 0$

$= 1 \pm i \pm ix + x = 0$

$\Rightarrow x = -1,\ h = \mp i$

$(R_5)(R_1)^* = 1 + f + g \pm i - 1 \mp i = 0 \Rightarrow f = -g$

$(C_5)^*(C_2) = 1 - \bar{b} \mp i\bar{b} - 1 - f \mp if = 0$

$\Rightarrow (\bar{b} + f)(1 \pm i) = 0$

$\Rightarrow f = -\bar{b},\ g = \bar{b}$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & b & -b \\ 1 & \pm i & -1 & \mp i & -b & b \\ 1 & \mp i & \mp i & -1 & \pm i & \pm i \\ 1 & -\bar{b} & \bar{b} & \pm i & -1 & \mp i \\ 1 & \bar{b} & -\bar{b} & \pm i & \mp i & -1 \end{bmatrix} = H_f(b)$$

$H^*H = HH^* = 6I_6$

So H is Hadamard

1.3) Switch R_2 with R_3 and C_2 with C_3
Then H is the same form as 1.2

2) $(R_2)(R_3)^* = 1 - \bar{c} - a - a\bar{b} - 1 + b\bar{c} = 0$

$\Rightarrow \bar{c}(b - 1) = a(\bar{b} + 1) = a(1 + b)/b$

$\Rightarrow |b + 1| = |b - 1| \Rightarrow b = \pm i$

and $\bar{c} = a(b+1)/b(b-1) =$

$= a(\pm i + 1)/(\pm i(\pm i - 1)) = -a,\ \Rightarrow c = -\bar{a}$

Thus

$$(C_4)^T = \begin{cases} 1) \; [\,1 \quad -a \quad \pm i \quad -1 \quad a \quad \mp i\,] \\ 2) \; [\,1 \quad -a \quad \pm i \quad -1 \quad \mp i \quad a\,] \\ 3) \; [\,1 \quad -(\pm i) \quad \pm i \quad -1 \quad K \quad -K\,] \end{cases}$$

2.1) $(C_5)^*(C_4) = 1 \pm ai - 1 - \bar{d} - a \pm i\bar{d} = 0$

$$\Rightarrow \bar{d}(\pm i - 1) = a(1 \mp i) \Rightarrow d = -\bar{a}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & \pm i & \mp i \\ 1 & -\bar{a} & -1 & \pm i & \mp i & \bar{a} \\ 1 & & & -1 & -\bar{a} & \\ 1 & & & a & -1 & \\ 1 & & & \mp i & \bar{a} & x \end{bmatrix} \Rightarrow R_4 = \begin{cases} 1) \; [\,1 \quad e \quad -e \quad -1 \quad -\bar{a} \quad \bar{a}\,] \\ 2) \; [\,1 \quad e \quad \bar{a} \quad -1 \quad -\bar{a} \quad -e\,] \\ 3) \; [\,1 \quad \bar{a} \quad e \quad -1 \quad -\bar{a} \quad -e\,] \end{cases}$$

2.1.1) $(R_2)(R_4)^* = 1 - \bar{e} - a\bar{e} + a \mp \bar{a}i \mp ai = 0$

$$\Rightarrow \pm 2ai = (1 - \bar{e})(1 + a) \Rightarrow [\,a \neq -1,\; e \neq 1\,]$$

$$\Rightarrow \bar{e} = \mp 2ai / 1+a + 1 = 1 + a \mp 2ai / 1+a$$

$$\Rightarrow e = 1 + \bar{a} \mp 2i\bar{a} / \cancel{1+\bar{a}}$$

$$= a + 1 \pm 2i / a + 1$$

But, $\bar{e} = 1/e$ for $|e| = 1$, so

$$1 + a \mp 2ai / 1+a = a+1 / \cancel{1+a\pm 2i}$$

$$\Rightarrow (a+1)^2 = (1 + a \pm 2i)(1 + a \mp 2ia)$$

$$= (1+a)^2 + (\pm 2i \mp 2ia)(1+a) + (\pm 2i)(\mp 2ia)$$

$$\Rightarrow \quad 0 = \pm 2i \mp 2ia^2 + 4a = \mp 2i(a \pm i)^2$$

$$\Rightarrow a = \mp i \;;\; e = (\mp i + 1 \pm 2i)/(\mp i + 1) = \pm i$$

$(R_4)(R_3)^* = 1 - ae + e \pm i \mp i\bar{a} + 1 = 0$

$$\Rightarrow 2 - (\mp i)(\pm i) \pm i \pm i \mp i(\pm i) = 0$$

$$\Rightarrow \pm 2i + 2 = 0 \quad \cancel{\ast}$$

2.1.2) $(R_3)(R_4)^* = 1 - \bar{a}\bar{e} - a \mp i \pm ia - \bar{a}\bar{e} = 0$

$\Rightarrow 2\bar{a}\bar{e} = (1-a)(1 \mp i)$

$\Rightarrow 2 = |1-a| \cdot \sqrt{2} \Rightarrow a = \pm i$

So: i) $\bar{e} = \frac{1}{2} a(1-a)(1 \mp i) = \frac{1}{2}(\pm i)(1 \mp i)(1 \mp i) = \mp 1$

ii) $\bar{e} = \frac{1}{2} a(1-a)(1 \mp i) = \frac{1}{2}(\mp i)(1 \pm i)(1 \mp i) = \mp i$

$(R_2)(R_4)^* = 1 - \bar{e} + a^2 + a \mp ia \pm i\bar{e} = 0$

i) $1 - (\mp 1) + (-1) \pm i \mp i(\pm i) \pm i(\mp 1) = \pm 2i \neq 0$ ✗

ii) $1 - (\mp i) + (-1) \mp i \mp i(\mp i) \pm i(\mp i) = 0$

ii) $H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \mp i & -1 & \pm i & \mp i & \pm i \\ 1 & \pm i & \pm i & -1 & \mp i & \mp i \\ 1 & f & g & \mp i & -1 & h \\ 1 & -f & -g & \mp i & \pm i & x \end{bmatrix}$ $(C_6)^*(C_5) = 1 - 1 - 1 + 1 - \bar{h} \pm i\bar{x} = 0$

$\sim \Rightarrow h = \mp i x$

$(R_5 + R_6)(R_1)^* = 1 \mp i \mp ix + x = 0$

$\Rightarrow x = -1, h = \pm i$

$(R_5)(R_1)^* = f + g = 0 \Rightarrow f = -g$

$(C_5)^*(C_2) = 1 \pm i + 1 - 1 - f \pm if = 0 \Rightarrow f = \pm i$

So $H \sim H_1(\pm i)$ $\begin{bmatrix} C_2 \leftrightarrow C_5, \; C_3 \leftrightarrow C_6, \\ R_2 \leftrightarrow R_5, \; R_3 \leftrightarrow R_6 \end{bmatrix}$

2.1.3) $(R_2)(R_4)^* = 1 \mp i - \bar{e} \pm i \pm ia - \bar{a}\bar{e} = 0$

$\Rightarrow \pm i(a-1) = \bar{e}(1+\bar{a}) = \bar{e}\bar{a}(1+a)$

$\Rightarrow |a-1| = |1+a| \Rightarrow a = \pm i$

i) $\bar{e} = \pm i(a-1)\bar{a}/(1+a) = (\pm i)(\pm i - 1)(\mp i)/(1 \pm i) = \mp i$

ii) $\bar{e} = \pm i(\mp i - 1)(\pm i)/(1 \mp i) = \mp i$

$(R_2)(R_4)^* = 1 + a + a\bar{e} + a \mp ia \pm i\bar{e} = 0$

i) $1 + (\pm i)(\mp i) \mp i(\pm i) \pm i(\mp i) = 4 \neq 0$

ii) $1 + (\mp i)(\mp i) \mp i(\mp i) \pm i(\mp i) = 0$

$$\Rightarrow H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \mp i & -1 & \pm i & \mp i & \pm i \\ 1 & \pm i & \pm i & -1 & \mp i & \mp i \\ 1 & f & g & \mp i & -1 & h \\ 1 & -f & -g & \mp i & \pm i & \chi \end{bmatrix}$$

which is the same as 2.1.2) above, so $H \sim H_1(\pm i)$

2.2) $(C_5)^*(C_4) = 1 \pm ia - 1 - \bar{d} \pm i - a\bar{d} = 0$

$\Rightarrow (1+a)(\pm i - \bar{d}) = 0$

$\Rightarrow$ i) $a = -1$ or ii) $d = \mp i$

$$R_4 = \begin{cases} 1)\ [1 \quad e \quad -e \quad -1 \quad d \quad -d] \\ 2)\ [1 \quad e \quad -d \quad -1 \quad d \quad -e] \\ 3)\ [1 \quad -d \quad e \quad -1 \quad d \quad -e] \end{cases}$$

2.2.1) $(R_2)(R_4)^* = 1 - \bar{e} - a\bar{e} + a \pm i\bar{d} \pm i\bar{d} = 0$

i) $\Rightarrow 1 - \bar{e} + \bar{e} - 1 \pm 2i\bar{d} = 0 \Rightarrow \bar{d} = 0$ ~~✕~~

ii) $\Rightarrow 1 - \bar{e} - a\bar{e} + a \pm 2i(\pm i) = -1 - \bar{e} - a\bar{e} + a = 0$

$\Rightarrow \bar{e}(1+a) = a - 1 \Rightarrow a = \pm i$ or $\mp i$; and $\bar{e} = a$

$(R_3)(R_4)^* = 1 + (-a)(\bar{e}) + \bar{e} \mp i \mp i\bar{d} \mp e\bar{d} = 0$

$\Rightarrow 1 + (-\bar{a})(a) + (a) \mp i \mp i(\pm i) - a(\pm i) = 0$

$\Rightarrow \mp a \mp i + 1 \mp i\bar{a} = 0 \Rightarrow a = \pm i$

$$\Rightarrow H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \pm i & \mp i & \mp i \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & f & \mp i & \mp i & -1 & h \\ 1 & -f & \mp i & \pm i & \pm i & \chi \end{bmatrix}$$

note: switching Columns 3 and 4, and Rows 3 and 4, H is the same as 2.1.2) above, so $H \sim H_1(\pm i)$

$2.2.2)\ (R_2)(R_4)^* = 1 - \bar{e} - a\bar{d} + a \pm i\bar{d} \pm i\bar{e} = 0$

$\Rightarrow$ i) $1 - \bar{e} + \bar{d} - 1 \pm i\bar{d} \pm i\bar{e} = 0$

$\Rightarrow \bar{e}(1 \mp i) = \bar{d}(1 \pm i) \Rightarrow e = \mp i d =$

ii) $1 - \bar{e} \mp ai + a - 1 \pm i\bar{e} = 0$

$\Rightarrow (a - \bar{e})(1 \mp i) = 0 \Rightarrow a = \bar{e}$

$(R_3)(R_4)^* = 1 + c\bar{e} + \bar{d} - b \pm db + \bar{e}\bar{c} = 0$

$= 1 - \bar{a}\bar{e} + \bar{d} \mp i \mp i\bar{d} - \bar{a}\bar{e} = 0$

$\Rightarrow$ i) $1 + (\pm i\bar{d}) + \bar{d} \mp i \mp i\bar{d} \pm i\bar{d} \Rightarrow d = \mp i,\ e = \mp 1$

or ii) $1 - e\bar{e} \pm i \mp i \mp i(\pm i) - e\bar{e} = 0$

$2.2.2.$ ii) $H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & \pm i & \mp i \\ 1 & -\bar{a} & -1 & \pm i & \mp i & \bar{a} \\ 1 & \bar{a} & \pm i & -1 & \mp i & -\bar{a} \\ 1 & f & g_1 & \mp i & -1 & h \\ 1 & -f & g_2 & a & \pm i & x \end{bmatrix}$ where 1) $g_1 = -a,\ g_2 = \mp i$
2) $g_1 = \mp i,\ g_2 = -a$
3) $a = \mp i$ (see 2.1.2)

$(C_5)^*(C_2) = 1 \pm i \mp i\bar{a} \pm i\bar{a} - f \pm i f = 0 \Rightarrow f = \pm i$

$(C_6)^*(C_5) = 1 - 1 \mp ia \pm ia - \bar{h} \pm i\bar{x} = 0 \Rightarrow h = \mp i x$

$(R_5)(R_1)^* + (R_6)(R_1)^* = \mp i + 1 \mp ix + x = 0 \Rightarrow x = -1,\ h = \pm i$

$(R_5)(R_6)^* = 1 - 1 \pm g_1\bar{g}_2 \mp i\bar{a} \pm i \mp i = 0$

$\Rightarrow g_1 = \mp i,\ g_2 = -a \Rightarrow H \sim H_2(a)$

$g_2 = \mp i,\ g_1 = -a = 1 \Rightarrow (R_6)(R_1)^* \neq 0$ *

$2.2.2.$ i) $H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cancel{-1} & \cancel{\ } & \cancel{\ } & \cancel{\ } & \mp i \\ 1 & \cancel{\ } & \cancel{\ } & \cancel{\ } & \mp i & -1 \\ 1 & \mp id & -d & -1 & d & \pm id \\ 1 & f_1 & g_1 & \mp i & -1 & h \\ 1 & f_2 & g_2 & -1 & -d & x \end{bmatrix}$

2.2.2.i.ii) $f_1=-1,\ f_2=\pm i\partial$

$$(C_4)^*(C_2) = 1-1\mp i \pm i\partial \mp i \mp i\partial = \mp 2i \neq 0 \ \text{✱}$$

2.2.2.i.iii) $f_1=\pm i\partial,\ f_2=-1$

$$(C_4)^*(C_2) = 1-1\mp i \pm i\partial - \partial + 1 = 0$$

$$\Rightarrow \partial = 1$$

$$(C_5)^*(C_2) = 1 \pm i \pm i \mp i \mp i\partial + \partial = 2 \neq 0 \ \text{✱}$$

2.2.3) $(R_2)(R_4)^* = 1+\bar\partial + a\bar e + a \pm i\bar\partial \pm i\bar e = 0$

i) $\Rightarrow 1+\bar\partial - \bar e - 1 \pm i\bar\partial \pm i\bar e = 0$

$$\Rightarrow \bar\partial(1\pm i) = \bar e(1\mp i) \Rightarrow \bar e = \pm i\bar\partial$$

ii) $1 \pm i + a\bar e + a - 1 \pm i\bar e = 0$

$$\Rightarrow (a\pm i)(1+\bar e) = 0$$

$$\Rightarrow \text{i) } a = \mp i \quad \text{or} \quad \text{ii) } e=-1$$

$$(R_3)(R_4)^* = 1+\bar a\bar\partial - \bar e \mp i \mp i\bar\partial - \bar a\bar e = 0$$

i) $1-\bar\partial \mp i\bar\partial \mp i \mp i\bar\partial \pm i\bar\partial = 0$

$$\Rightarrow \partial = \pm i,\quad e=1$$

ii)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & \pm i & \mp i \\ 1 & 1 & -1 & \pm i & \mp i & -1 \\ 1 & \mp i & 1 & -1 & \pm i & -1 \\ 1 & f_1 & g & \mp i & -1 & h \\ 1 & f_2 & -g & -1 & \mp i & x \end{bmatrix}$$

$(C_2)^*(C_1) = 0 \Rightarrow$
$f_1=-1,\ f_2=\pm i$
or $f_2=\pm i,\ f_1=-1$

$$(C_5)^*(C_6) = 1-1\mp i \pm i - h \pm ix = 0 \Rightarrow h=\pm ix$$

$$(R_5)^*(R_1) + (R_6)^*(R_1) = -1 \mp i \pm ix + x = 0$$

$$\Rightarrow x=1,\ h=\pm i$$

$$(C_4)^*(C_3) = 1-1\pm i - 1 \pm ig + g = 0 \Rightarrow g = \mp i$$

$\Rightarrow f_1 = \pm i, \quad f_2 = -1$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & \pm i & \mp i \\ 1 & 1 & -1 & \pm i & \mp i & -1 \\ 1 & \mp i & 1 & -1 & \pm i & -1 \\ 1 & \pm i & \mp i & \mp i & -1 & \pm i \\ 1 & -1 & \pm i & -1 & \mp i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & -1 & 1 \\ 1 & \pm i & -1 & \mp i & 1 & -1 \\ 1 & \mp i & \mp i & -1 & \pm i & \pm i \\ 1 & -1 & 1 & \pm i & -1 & \mp i \\ 1 & 1 & -1 & \pm i & \mp i & -1 \end{bmatrix} = H_1(-1) \text{ or } H_1(1)$$

ii) $1 \pm i\bar{a} - \bar{e} \mp i + 1 - \bar{a}\bar{e} = 0$

i) $1 - 1 - \bar{e} \mp i + 1 \mp i\bar{e} = 0 \Rightarrow \bar{e} = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \mp i & -1 & \pm i & \mp i & \pm i \\ 1 & \pm i & \pm i & -1 & \mp i & \mp i \\ 1 & f & g & \mp i & -1 & h \\ 1 & -f & -g & \mp i & \pm i & x \end{bmatrix}$$

see 2.1.2)

$H \sim H_2(\pm i)$

ii) $1 - \pm i\bar{a} + 1 \mp i + 1 + \bar{a} = 0 \Rightarrow \bar{a} = {}^{-3 \pm i}/{(1 \pm i)}$

$\Rightarrow |a| \neq 1 \quad \times$

2.3) $(C_5)^*(C_4) = 1 \pm 1 - 1 - \bar{d} - k + k\bar{d} = 0$

$\Rightarrow k = d = \pm i$ or $\mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \pm i & \mp i & \mp i \\ 1 & & & -1 & d & \\ 1 & & & k & -1 & \\ 1 & & & -k & -d & x \end{bmatrix}$$

$$R_4 = \begin{cases} 1)\ [1 \quad e \quad -e \quad -1 \quad d \quad -d] \\ 2)\ [1 \quad e \quad -d \quad -1 \quad d \quad -e] \\ 3)\ [1 \quad -d \quad e \quad -1 \quad d \quad -e] \end{cases}$$

2.3.1) $(R_2)(R_4)^* = 1 - \bar{e} \mp i\bar{e} \pm i \pm i\bar{d} \pm i\bar{d} = 0$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{e} \mp i \mp i\bar{d} \pm i\bar{d} = 0$

add: $2 \pm 2i\bar{d} = 0 \Rightarrow k = d = \mp i$

$\Rightarrow e = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \pm i & \mp i & \mp i \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & f & \mp i & \mp i & -1 & h \\ 1 & -f & \mp i & \pm i & \pm i & x \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \mp i & -1 & \pm i & \mp i & \pm i \\ 1 & \pm i & \pm i & -1 & \mp i & \mp i \\ 1 & f & \mp i & \mp i & -1 & h \\ 1 & -f & \pm i & \mp i & \pm i & x \end{bmatrix} \boxed{\begin{matrix} C_3 \leftrightarrow C_4 \\ R_3 \leftrightarrow R_4 \end{matrix}}$$

So $H \sim H_1(\pm i)$ by 2.1.2)

2.3.2) $(R_2)(R_4)^* = 1 - \bar{e} \mp i\bar{d} \pm i \pm i\bar{d} \pm i\bar{d} = 0 \Rightarrow$

$\Rightarrow e = \mp i,\ d = \mp i = k$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \pm i & \mp i & \mp i \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & f & \mp i & \mp i & -1 & h \\ 1 & -f & \mp i & \pm i & \pm i & x \end{bmatrix} \sim H_1(\pm i) \text{ by 2.1.2)}$$

$[C_3 \leftrightarrow C_4,\ R_3 \leftrightarrow R_4]$

2.3.3) $(R_2)(R_4)^* = 1 + \bar{d} \pm i\bar{e} \pm i \pm i\bar{d} \pm i\bar{e} = 0$

i) $d = \pm i$: $1 \mp i \pm i\bar{e} \pm i + 1 \pm i\bar{e} = 0 \Rightarrow e = \mp i$

ii) $d = \mp i$: $1 \pm i \pm i\bar{e} \pm i - 1 \pm i\bar{e} = 0 \Rightarrow e = -1$

$(R_3)(R_4)^* = 1 \mp i\bar{d} - \bar{e} \mp i \mp i\bar{d} \pm i\bar{e} = 0$

i) $1 - 1 \mp i \mp i + 1 - 1 = \mp 2i - 2 \neq 0$ ✗

ii) $1 + 1 + 1 \mp i + 1 \mp i = 4 \mp 2i \neq 0$ ✗

3) $(R_2)(R_3)^* = 1 - \bar{b} - a - a\bar{c} - 1 + b\bar{c} = 0$

1) $-\bar{b} = -a \;;\; a\bar{c} = b\bar{c} \Rightarrow a = b, \quad a = b = \pm i$

2) $\bar{b} = -a\bar{c} \;;\; a = b\bar{c}$

$\Rightarrow \bar{b} = -b\bar{c}^2 \Rightarrow -b^2 = c^2 \Rightarrow c = \pm i b, \; a = \mp i$

3) $\bar{b} = b\bar{c} \;;\; a = -a\bar{c} \Rightarrow c = -1$

$\Rightarrow \bar{b} = -b \Rightarrow b = \pm i$

3.1)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & c & \mp i & -c \\ 1 & & & -1 & d & \\ 1 & & & & \mp 1 & \\ 1 & & & & -d & X \end{bmatrix}$$

$$(C_4)^T = \begin{cases} 1)\,[1 \;\; \mp i \;\; c \;\; -1 \;\; -c \;\; \pm i] \\ 2)\,[1 \;\; \mp i \;\; c \;\; -1 \;\; \pm i \;\; -c] \\ 3)\,[1 \;\; \mp i \;\; \pm i \;\; -1 \;\; K \;\; -K] \end{cases}$$

3.1.1) $(C_5)^*(C_4) = 1 - 1 \pm ic - \bar{d} + c \mp i\bar{d} = 0 \Rightarrow c = \bar{d}$

$$R_4 = \begin{cases} 1)\,[1 \;\; e \;\; -e \;\; -1 \;\; \bar{c} \;\; -\bar{c}] \\ 2)\,[1 \;\; e \;\; -\bar{c} \;\; -1 \;\; \bar{c} \;\; -e] \\ 3)\,[1 \;\; -\bar{c} \;\; e \;\; -1 \;\; \bar{c} \;\; -e] \end{cases}$$

3.1.1.1) $(R_2)(R_4)^* = 1 - \bar{e} \mp i\bar{e} \pm i \pm i\bar{c} \pm i\bar{c} = 0$

$\Rightarrow \mp 2i\bar{c} = (1 \pm i)(1 - \bar{e})$

$\Rightarrow |2ic| / |1 \pm i| = \sqrt{2} = |1 - \bar{e}| \Rightarrow e = \pm i$ or $\mp i$

$\Rightarrow$ [illegible] $\sqrt{\ }$ $= |1 - \bar{e}|$ $\Rightarrow$ $e =$ [illegible]

3.1.1. $e = \pm i$: $c = \pm i(1 \pm i)(1 - \bar{e})/2 = \mp i(1 \pm i)^2/2 = -1$

$e = \mp i$: $c = \pm i(1 \pm i)(1 - \bar{e})/2 = \pm i(1 \pm i)(1 \mp i)/2 = \pm i$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{e} - c \pm ic + c^2 = 0$

$e = \pm i : 1 + 1 \mp i + 1 \mp i + 1 \neq 0$

$e = \mp i : 1 - 1 \pm i \mp i + 1 - 1 \neq 0$

3.1.1.2) $(R_2)(R_4)^* = 1 - \bar{e} \mp ic \pm i \pm ic \pm i\bar{e} = 0$

$\Rightarrow 1 - \bar{e} \pm i \pm i\bar{e} = 0 \Rightarrow e = \mp i$

$\Rightarrow (C_2)^T = [1 \quad -1 \quad \pm i \quad \mp i \quad f \quad -f]$

$(C_2)^*(C_4) = 1 \mp i \pm i\bar{c} \pm i - f\bar{c} = 0$

$\Rightarrow (\bar{c} \pm i)(\pm i - f) = 0$

• $c = \pm i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \pm i & \mp i & \mp i \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & f & \mp i & \mp i & -1 & h \\ 1 & -f & \mp i & \pm i & \pm i & x \end{bmatrix}$$

$(R_5)(R_1)^* = 0 \Rightarrow f = h = \pm i$

$(R_6)(R_1)^* = 0 \Rightarrow x = -1$

So $H \sim H_1(\pm i)$ $[R_2 \leftrightarrow R_3,\ C_2 \leftrightarrow C_3 + *]$

• $f = \pm i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & c & \mp i & -c \\ 1 & \mp i & -\bar{c} & -1 & \bar{c} & \pm i \\ 1 & \pm i & g_1 & -c & -1 & h \\ 1 & \mp i & g_2 & \pm i & -\bar{c} & x \end{bmatrix}$$

$g_1, g_2 \in \{\mp i, \bar{c}\}$ and $g_1 \neq g_2$

$(C_3)^*(C_5) = 1 \pm i \mp i \pm ic \pm i\bar{g}_1 \mp i\bar{g}_2 = 0$

1) $-1 = \pm ic \Rightarrow c = \pm i$, see above

2) $-1 = \pm i\bar{g}_1 \Rightarrow g_1 = \mp i,\ g_2 = \bar{c}$

$\Rightarrow h = c,\ x = -1$, and $H \sim H_1(c) =$

3) $-1 = \mp i\bar{g}_2 \Rightarrow g_2 = \pm i,\ g_1 = -\bar{c}$

$\Rightarrow c = \mp i,\ h = \mp i,\ x = -1$

and $H \sim H_1(\pm i)$ (see 2.1.2)

3.1.1.3) $(R_2)(R_4)^* = 1 + c \pm i\bar{e} \pm i \pm ic \pm i\bar{e} = 0$

$(R_3)(R_4)^* = 1 \mp ic - \bar{e} - c \mp ic + c\bar{e} = 0$

add: $2 \pm 2i\bar{e} \pm i \mp ic - \bar{e} + c\bar{e} = 0$

$\Rightarrow 2 \pm i \mp ic \pm i\bar{e}(2 \pm i \mp ic) = 0$

$\Rightarrow (2 \pm i \mp ic) = 0 \quad$ or $\quad (1 \pm i\bar{e}) = 0$

$\Rightarrow |c| = |2 \pm i| \cdot |\pm i| \neq 1 \qquad \Rightarrow e = \mp i$

$\times$

$(R_2)(R_4)^* = 1 + c - 1 \pm i \pm ic - 1 = 0 \Rightarrow c = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \mp i & \mp i & \pm i \\ 1 & \mp i & \mp i & -1 & \pm i & \pm i \\ 1 & f & g & \pm i & -1 & h \\ 1 & -f & -g & \pm i & \mp i & x \end{bmatrix} \sim H_1(\pm i), \text{ see } 1.2)$$

with $b = \pm i$

3.1.2) $(C_5)^*(C_4) = 1 - 1 \pm ic - \bar{d} \mp i + \bar{d}c = 0$

$\Rightarrow (\bar{d} \pm i)(c - 1) = 0$

$\Rightarrow d = \pm i \quad$ or $\quad c = 1$

$$R_4 = \begin{cases} 1) \; [1 \;\; e \;\; -e \;\; -1 \;\; d \;\; -d] \\ 2) \; [1 \;\; e \;\; -d \;\; -1 \;\; d \;\; -e] \\ 3) \; [1 \;\; -d \;\; e \;\; -1 \;\; d \;\; -e] \end{cases}$$

3.1.2.1) $(R_2)(R_4)^* = 1 - \bar{e} \mp i\bar{e} \pm i \pm i\bar{d} \pm i\bar{d} = 0$

$d = \pm i$: $\quad 1 - \bar{e} \mp i\bar{e} \pm i + 2 = 0$

$\Rightarrow |\bar{e}| = |3 \pm i| / |1 \pm i| \neq 1 \;\; \times$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{e} - c \mp i\bar{d} + e\bar{d} = 0$

$c = 1$: $\quad 1 \pm i\bar{e} + \bar{e} - 1 \mp i\bar{d} + \bar{d} = 0$

add $(R_2)(R_4)^*$: $1 \pm i \pm i\bar{d} + \bar{d} = 0 \Rightarrow d = -1$

$(R_3)(R_4)^*$: $\pm i\bar{e} \mp \bar{e} \mp i(-1) + (-1) = 0 \Rightarrow e = \pm i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & 1 & \mp i & -1 \\ 1 & \pm i & \mp i & -1 & -1 & 1 \\ 1 & \mp i & g & \pm i & -1 & g \\ 1 & \mp i & -g & -1 & 1 & x \end{bmatrix}$$

$(C_4)^*(C_3) = 1 - 1 - 1 \pm i \mp ig + g = 0$
$\Rightarrow g = 1$

$(R_6)(R_1)^* = 0 \Rightarrow x = \pm i$

$$H \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \mp i & 1 & -1 \\ 1 & \mp i & \mp i & -1 & \pm i & \pm i \\ 1 & -1 & -1 & \pm i & -1 & \mp 1 \\ 1 & -1 & -1 & \pm i & \mp i & -1 \end{bmatrix} = H_1(-1)$$

$\boxed{\begin{array}{c} R_2 \leftrightarrow R_5,\ R_3 \leftrightarrow R_5, \\ R_2 \leftrightarrow R_6 \\ C_2 \leftrightarrow C_4,\ C_3 \leftrightarrow C_5 \\ C_5 \leftrightarrow C_6 \end{array}}$

3.1.2.2) $(R_2)(R_4)^* = 1 - \bar{e} \mp i\bar{d} \pm i \pm i\bar{d} \pm i\bar{e} = 0$

$\Rightarrow 1 - \bar{e} \pm i \pm i\bar{e} = 0 \Rightarrow e = \mp i$

$(R_3)(R_4)^* = 1 \pm i(\pm i) + \bar{d} - c \mp i\bar{d} \pm i\bar{c} = 0$

i) $d = \pm i$: $\mp i - c \mp i(\mp i) \pm ic = 0 \Rightarrow c = \mp i$

ii) $c = 1$: $\mp \bar{d} - 1 \mp i\bar{d} \pm i = 0 \Rightarrow \bar{d} = 1$

3.1.2.2.i) $$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \mp i & \mp i & \pm i \\ 1 & \mp i & \mp i & -1 & \pm i & \pm i \\ 1 & f & g & \pm i & -1 & h \\ 1 & -f & -g & \pm i & \mp i & x \end{bmatrix} \sim H_1(\pm i)$$, see 1.2 with $b = \pm i$

3.1.2.2.ii)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & 1 & \mp i & -1 \\ 1 & \mp i & -1 & -1 & 1 & \pm i \\ 1 & f & g_1 & \pm i & -1 & h \\ 1 & -f & g_2 & -1 & 1 & x \end{bmatrix}$$

$(C_4)^*(C_2) = 1 \mp i \pm i \pm i \pm i f + f$
$= 0$
$\Rightarrow f = -1$

$g_1, g_2 \in \{1, \mp i\}$, and $g_1 \neq g_2$

$(C_5)^*(C_3) = 1 + 1 \mp i - 1 - g_1 + g_2 = 0$

$g_1 = 1 \Rightarrow g_2 = \pm i$ ✗

$g_1 = \mp i \Rightarrow g_2 = -1$ ✗

3.1.2.3) $(R_2)(R_4)^* = 1 + \bar{d} \pm i\bar{e} \pm i \pm i\bar{d} \pm i\bar{e} = 0$

$d = \pm i$: $1 \mp i \pm i\bar{e} \pm i + 1 \pm i\bar{e} = 0 \Rightarrow \bar{e} = \mp i$

$(R_3)(R_4)^* = 1 \mp i\bar{d} - \bar{e} - c \mp i\bar{d} + c\bar{e} = 0$

- $d = \pm i$: $1 - 1 \mp i - c - 1 \pm i c = 0 \Rightarrow c = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \mp i & \mp i & \pm i \\ 1 & \mp i & \mp i & -1 & \pm i & \mp i \\ 1 & \mp i & g & \pm i & -1 & -g \\ 1 & -f & -g & \pm i & \mp i & x \end{bmatrix} \sim H_1(\pm i)$$

, see 1.2) with $b = \pm i$

- $c = 1$: $1 \mp i\bar{d} - \bar{e} - 1 \mp i\bar{d} + \bar{e} = 0$; $\bar{d} \neq 0$

$\mp 2i\bar{d} \neq 0$ ✗

3.1.3)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & \pm i & \mp i & \mp i \\ 1 & & & -1 & d & \\ 1 & & & k & -1 & \\ 1 & & & -k & -d & x \end{bmatrix}$$

See 2.3)

3.2)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & b & -b \\ 1 & -b & -1 & \pm ib & -b & \mp ib \\ 1 & & & -1 & \partial & \\ 1 & & & & \mp 1 & \\ 1 & & & & -\partial & \chi \end{bmatrix}$$

$$(C_4)^T = \begin{cases} 1)\ [\,1 \quad \pm i \quad \pm ib \quad -1 \quad \mp i \quad \mp ib\,] \\ 2)\ [\,1 \quad \pm i \quad \pm ib \quad -1 \quad \mp ib \quad \mp i\,] \\ 3)\ [\,1 \quad \pm i \quad \mp i \quad -1 \quad K \quad -K\,] \end{cases}$$

3.2.1) $(C_5)^T(C_4) = 1 \pm i\bar{b} \mp i - \bar{\partial} \pm i \pm i\bar{\partial} b = 0$

$\Rightarrow 1 \mp i\bar{b} - \bar{\partial}(\pm \mp i\bar{b}) = 0$

$\Rightarrow \bar{b}(b \pm i) \pm i\bar{\partial}(\pm i + b) = 0$

$\Rightarrow b = \mp i \quad \text{or} \quad b = \pm i\partial$

$$R_4 = \begin{cases} 1)\ [\,1 \quad e \quad -e \quad -1 \quad \partial \quad -\partial\,] \\ 2)\ [\,1 \quad e \quad -\partial \quad -1 \quad \partial \quad -e\,] \\ 3)\ [\,1 \quad -\partial \quad e \quad -1 \quad \partial \quad -e\,] \end{cases}$$

3.2.1.1) $(R_2)(R_4)^* = 1 - \bar{e} \pm i\bar{e} \mp i + b\bar{\partial} + b\bar{\partial} = 0$

- $b = \mp i$: $1 - \bar{e} \pm i\bar{e} \mp i \mp 2i\bar{\partial} = 0$

  $\Rightarrow |2i\bar{\partial}| = 2 = |1 \pm i||1 - \bar{e}|$

  $\Rightarrow \sqrt{2} = |1 - \bar{e}| \Rightarrow e = \pm i$, or $\mp i$

- $b = \pm i\partial$: $1 - \bar{e} \pm i\bar{e} \mp i \pm i \pm i = 0$

  $\Rightarrow e = \mp i$

$(R_3)(R_4)^* = 1 + b\bar{e} - \bar{e} \mp ib - b\bar{d} \pm ib\bar{d} = 0$

3.2.1.1.1) $b = \mp i$: $1 \mp i\bar{e} - \bar{e} - 1 \pm i\bar{d} + \bar{d} = 0$

- $e = \pm i \Rightarrow d = \pm i$
- $e = \mp i \Rightarrow d = \mp i$

• $e = \pm i$:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & 1 & \pm i & \mp 1 \\ 1 & \pm i & \mp i & -1 & \pm i & \mp i \\ 1 & f & \pm i & \mp i & -1 & -f \\ 1 & -f & \pm i & -1 & \mp i & x \end{bmatrix}$$

$(C_4)^T(C_3) = 1 - 1 - 1 \pm i - 1 \mp i$
$\neq 0$
$\times$

• $e = \mp i$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & \pm i & g & \mp i & -1 & -g \\ 1 & \pm i & -g & -1 & \pm i & x \end{bmatrix}$$

$(C_5)^T(C_2) = 1 \mp i - 1 + 1 \mp i + 1$
$\neq 0$
$\times$

3.2.1.1.2) $b = \pm i d$: $1 \pm i d(\pm i) \pm i \mp i(\pm i d) \mp i - 1 = 0$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & b & -b \\ 1 & b & -1 & \pm ib & -b & \mp ib \\ 1 & \mp i & \pm i & -1 & \mp ib & \pm ib \\ 1 & f_1 & g & \mp i & -1 & h \\ 1 & f_2 & -g & \mp ib & \pm ib & x \end{bmatrix}$$

$$\begin{cases} 1,2,3) - f_1, f_2 \in \{\pm i, -b\} \\ \quad f_1 \neq f_2,\ b \neq \pm i \\ 4)\ b = \pm i,\ f_1 = -f_2 \end{cases}$$

$(C_4)^*(C_2) = 1 - 1 \pm i\bar{b} \mp i \pm ig \mp i\bar{b}g = 0$

3.2.1.1.2.1) $\pm i\bar{b} = \pm i \Rightarrow \bar{b} = 1$, $\pm ig = \pm i\bar{b}g \Rightarrow g = g$

$(C_5)^*(C_2) = 1 \mp i\bar{b} + \bar{b} - \bar{b} - g \pm i\bar{b}g$
$= 1 \mp i + 1 - 1 - g \pm ig = 0$
$\Rightarrow g = 1$

$(C_3)^*(C_2) = 1 \mp i - 1 - 1 + f_1 - f_2 = 0$

$f_1, f_2 \in \{-1, \pm i\}$, and $f_1 \neq f_2$

$f_1 = -1 \Rightarrow -1 \mp i - 1 - f_2 = 0 \Rightarrow f_2 = -2 \mp i$ $\times$

$f_1 = \pm i \Rightarrow -1 \mp i \pm i - f_2 = 0 \Rightarrow f_2 = -1$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & 1 & -1 \\ 1 & 1 & -1 & \pm i & -1 & \mp i \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & \pm i & 1 & \mp i & -1 & -1 \\ 1 & -1 & -1 & \mp i & \pm i & 1 \end{bmatrix} \sim H_1(-1) \text{ or } H(1)$$

[$C_2 \leftrightarrow C_6$ and *]

3.2.1.1.2.2) $\pm i \bar{b} = \mp i g \Rightarrow \bar{b} = -g$,

$\pm i = \mp i \bar{b} g \Rightarrow 1 = g^2 \Rightarrow g = \pm 1,\ b = \mp 1$

$(C_5)^*(C_2) = 1 \mp \bar{b} \mp 1 + \bar{b} - f_1 \pm i \bar{b} f_2 = 0$

- $f_1 = -b \Rightarrow b = \pm i \bar{b} (\pm i) \Rightarrow -b = \bar{b}$ $\times$
- $f_1 = \pm i \Rightarrow \pm i = \pm i \bar{b}(-b) = \mp i$ $\times$

3.2.1.1.2.3) $\pm i \bar{b} = \pm i \bar{b} g \Rightarrow g = 1$; $\mp i = \pm i g = \pm i$

$(C_3)^*(C_2) = 1 \mp i - b - 1 + f_1 - f_2 = 0$

- $f_1 = \pm i \Rightarrow -b - (-b) = 0$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & b & -b \\ 1 & b & -1 & \pm ib & -b & \mp ib \\ 1 & \mp i & \pm i & -1 & \mp ib & \pm ib \\ 1 & \pm i & 1 & \mp i & -1 & -1 \\ 1 & -b & -1 & \mp ib & \pm ib & b \end{bmatrix} \sim H_2(b)$$

[$C_3 \leftrightarrow C_4$, $R_3 \leftrightarrow R_4$,

• $f_1 = -b \Rightarrow \mp i - b - b \mp i = 0 =$

$\Rightarrow b = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \mp i & \pm i & -1 & -1 & 1 \\ 1 & \pm i & 1 & \mp i & 1 & -1 \\ 1 & \pm i & -1 & -1 & 1 & \mp i \end{bmatrix} \sim H_1(-1) \text{ or } H_1(1)$$

3.2.1.1.2.4)
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & \mp i & \pm i & -1 & -1 & -1 \\ 1 & f & g & \mp i & -1 & h \\ 1 & -f & -g & 1 & -1 & x \end{bmatrix}$$

$(C_4)^*(C_2) = 1 \pm i \mp i \pm i \pm i f - f = 0 \Rightarrow f = \pm i$

$(C_4)^*(C_3) = 1 - 1 + 1 \mp i \pm i g - g = 0 \Rightarrow g = 1$

$\Rightarrow h = -1,\ x = \pm i$

$H \sim H_1(1)$ or $H_1(-1)$

3.2.1.2) $(R_2)(R_4)^* = 1 - \bar{e} \pm i \bar{d} \mp i + b\bar{d} + b\bar{e} = 0$

1)• $b = \mp i$: $1 - \bar{e} \pm i\bar{d} \mp i \mp i\bar{d} \mp i\bar{e} = 0$

$\Rightarrow e = \pm i$

2)• $b = \pm i d$: $1 - \bar{e} \pm i\bar{d} \mp i \pm i \pm i d\bar{e} = 0$

$\Rightarrow 1 \pm i\bar{d} = \pm\bar{e} \mp i d\bar{e} = \bar{e}(1 \mp i d)$

$\Rightarrow \bar{d}(d \pm i) = \mp i\bar{e}(\pm i \pm d)$

$\Rightarrow d = \mp i$, or $d = \pm i\bar{e}$

$$(R_3)(R_4)^* = 1 + b\bar{e} + \bar{d} \mp ib - b\bar{d} \pm ib\bar{e} = 0$$

3.2.1.2.1) $b = \mp i$: $1 - 1 + \bar{d} - 1 \pm i\bar{d} \mp i = 0$

$$\Rightarrow d = 1$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \pm i & \mp i & -1 & 1 & -1 \\ 1 & f & \pm i & \mp i & -1 & -f \\ 1 & -f & \pm i & -1 & -f & x \end{bmatrix}$$

$$(C_4)^*(C_3) = 1 - 1 - 1 \pm i - 1 \mp i \neq 0$$

$$\times$$

3.2.1.2.2) $b = \pm id \Rightarrow 1 \pm id\bar{e} + \bar{d} + d \mp i - d\bar{e} = 0$

- $d = \mp i$: $1 + \bar{e} \pm i \mp i \mp i \pm i\bar{e} = 0 \Rightarrow e = \mp i$, $b = 1$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & 1 & -1 \\ 1 & 1 & -1 & \pm i & -1 & \mp i \\ 1 & \mp i & \pm i & -1 & \mp i & \pm i \\ 1 & f_1 & g & \mp i & -1 & h \\ 1 & f_2 & -g & \mp i & \pm i & x \end{bmatrix}$$

$$f_1, f_2 \in \{-1, \pm i\}$$
$$f_1 \neq f_2$$

$$(C_5)^*(C_2) = 1 - 1 - 1 + 1 - f_1 \mp if_2 = 0 \Rightarrow$$

$$f_1 = -1 \Rightarrow -(-1) \mp i(\pm i) = 2 \neq 0$$

$$f_1 = \pm i \Rightarrow -(\pm i) \mp i(-1) = 0$$

$$(C_5)(C_3) = 1 \mp i + 1 - 1 - g \pm ig = 0 \Rightarrow g = 1$$

$$h = -1, \ x = 1$$

$$\Rightarrow H \sim H_1(-1) \text{ or } H_1(1)$$

- $d = \pm ie \Rightarrow 1 - 1 \mp i\bar{e} \pm ie \mp i \mp i = 0$

$$\Rightarrow \cancel{\bar{e} - e =} \ 2I_m e = -2 \ \times$$

3.2.1.3) $(R_2)(R_4)^* = 1 + \bar{d} \mp i\bar{e} \mp i + b\bar{d} + b\bar{e} = 0$

$\bullet\ b = \mp i :\ 1 + \bar{d} \mp i\bar{e} \mp i \mp i\bar{d} \mp i\bar{e} = 0$

$\Rightarrow \pm 2i\bar{e} = (1 \mp i)(1 + \bar{d})$

$\bullet\ b = \pm id :\ 1 + \bar{d} \mp i\bar{e} \mp i \pm i \pm id\bar{e} = 0$

$\bullet\ 1 = -\bar{d}\ ,\ \pm i\bar{e} = \pm id\bar{e}$ $\times$

$\bullet\ 1 = \mp id\bar{e}\ ,\ \bar{d} = \pm i\bar{e} \Rightarrow 1 = \mp i(\mp ie)\bar{e}$ $\times$

$\bullet\ 1 = \pm i\bar{e}\ ,\ \bar{d} = \mp id\bar{e} = -d$

$\Rightarrow d = \pm i$ or $\mp i$; $\bar{e} = \pm i$

$(R_3)(R_4)^* = 1 - b\bar{d} - \bar{e} \mp ib - b\bar{d} \pm ib\bar{e} = 0$

$\bullet\ b = \mp i :\ 1 \pm i\bar{d} - \bar{e} - 1 \pm i\bar{d} + \bar{e} = 0$

$\Rightarrow \pm 2i\bar{d} = 0$ $\times$

$\bullet\ b = \pm id :\ 1 \mp i - \bar{e} + d \mp i - d\bar{e} = 0$

$d = \pm i \Rightarrow 1 \mp i - \bar{e} \mp i \mp i \mp i\bar{e} = 0$

$\Rightarrow |1 \mp 3i| = |\bar{e}||1 \pm i|$ $\times$

$d = \mp i \Rightarrow 1 \mp i - \bar{e} \pm i \mp i \pm i\bar{e} = 0$

$\Rightarrow |\mp i \pm i \pm i(\mp i)| = 2 \neq 0$ $\times$

3.2.2) $(C_5)^T(C_4) = 1 \pm i\bar{b} \mp i - \bar{d} \pm ib \pm i\bar{d} = 0$

$\Rightarrow (1 \mp i)(1 - \bar{d}) = \mp i(\bar{b} + b)$

$\Rightarrow 1 - \bar{d} = (1/2 \mp 1/2 i)(2\,\mathrm{Re}\,b)$

$\Rightarrow |\bar{d}| = 1 = |1 - (1 \mp i)\mathrm{Re}\,b|$

$\Rightarrow \mathrm{Re}\,b = 0$ or 1

$\Rightarrow b = \pm i, \mp i,$ or 1 ; $d = 1 - (1 \pm i)\mathrm{Re}\,b$

$$R_4 = \begin{cases} 1)\ [\,1 \quad e \quad -e \quad -1 \quad d \quad -d \\ 2)\ [\,1 \quad e \quad -d \quad -1 \quad d \quad -e \\ 3)\ [\,1 \quad -d \quad e \quad -1 \quad d \quad -e \end{cases}$$

3.2.2.1) $(R_2)(R_4)^* = 1 - \bar{e} \pm i\bar{e} \mp i + b\bar{d} + b\bar{d} = 0$

- $b = \pm i,\ d = 1$: $1 - \bar{e} \pm i\bar{e} \mp i \pm i \pm i = 0$

$$\Rightarrow \bar{e} = \mp i$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & -1 \\ 1 & \mp i & \pm i & -1 & 1 & -1 \\ 1 & f & g & 1 & -1 & h \\ 1 & -f & -g & \mp i & -1 & x \end{bmatrix} \sim H_1(1) \text{ or } H_1(-1)$$

[$R_5 \longleftrightarrow R_6$ and see 3.2.1.1.2.4]

- $b = \mp i,\ d = 1$: $1 - \bar{e} \pm i\bar{e} \mp i \mp i \mp i = 0$

$$\Rightarrow |\bar{e}| \cdot |1 \mp i| = |1 \mp 3i| \quad \times$$

- $b = 1,\ d = \pm i$: $1 - \bar{e} \pm i\bar{e} \mp i \mp i \mp i = 0$

$$\Rightarrow |\bar{e}| \cdot |1 \mp i| = |1 \mp 3i| \quad \times$$

3.2.2.2) $(R_2)(R_4)^* = 1 - \bar{e} \pm i\bar{d} \mp i + b\bar{d} + b\bar{e} = 0$

$(R_3)(R_4)^* = 1 + b\bar{e} + \bar{d} \mp ib - b\bar{d} \pm ib\bar{e} = 0$

- $b = \pm i,\ d = 1$: $(R_2)(R_4)^* = 1 - \bar{e} \pm i \mp i \pm i \pm i\bar{e} = 0$

$$\Rightarrow e = \mp i$$

$$(R_3)(R_4)^* = 1 \pm i(\pm i) + 1 \mp 1 \mp i \pm i(\pm i)(\pm i)$$

$$= 2 \mp 2i \neq 0 \quad \times$$

• $b = \mp i,\ d = 1:\ (R_2)(R_4)^* = 1 - \bar{e} \pm i \mp i \mp i \mp i\bar{e} = 0$

$\Rightarrow e = \pm i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \pm i & 1 & -1 & 1 & \mp i \\ 1 & f & g_1 & -1 & -1 & x \\ 1 & -f & g_2 & \mp i & -1 & x \end{bmatrix} \qquad \begin{array}{c} g_1, g_2 \in \{1, \pm i\} \\ g_1 \neq g_2 \end{array}$$

$(C_4)^*(C_2) = 1 \pm i \mp i \mp i - f \mp if = 0 \Rightarrow f = \mp i$

$(R_5)(R_6)^* = 1 - 1 + g_1\bar{g}_2 \mp i + 1 - 1 = 0$

$\Rightarrow g_1 = \pm i\, g_2 \Rightarrow g_1 = \pm i,\ g_2 = 1$

$\Rightarrow x = -1$

$\Rightarrow H \sim H_1(1)$ or $H_1(-1)$

• $b = 1,\ d = \pm i:\ 1 - \bar{e} + 1 \mp i \mp i + \bar{e} = 0$ $\times$

3.2.2.3) $(R_2)(R_4)^* = 1 + \bar{d} \mp i\bar{e} \mp i + b\bar{d} + b\bar{e} = 0$

• $b = \pm i,\ d = 1:\ 1 + 1 \mp i\bar{e} \mp i \pm i \pm i\bar{e} = 2 \neq 0$ $\times$

• $b = \mp i,\ d = 1:\ 1 + 1 \mp i\bar{e} \mp i \mp i \mp i\bar{e} = 0$

$\Rightarrow 2(1 \mp i \mp i\bar{e}) = 0$

$\Rightarrow \bar{e} = \pm i(-1 \pm i) \Rightarrow |e| \neq 1$ $\times$

• $b = 1,\ d = \pm i:\ 1 \mp i \mp i\bar{e} \mp i \mp i + \bar{e} = 0$

$\Rightarrow |\bar{e}||1 \mp i| = |-1 \pm 3i|$ $\times$

3.2.3) $b=-1$

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & -1 & 1 \\ 1 & -1 & -1 & \mp i & 1 & \pm i \\ 1 & & & -1 & d & \\ 1 & & & k & -1 & \\ 1 & & & -k & -d & x \end{bmatrix}$$

$$R_4=\begin{cases} 1)\ [1 \quad e \quad -e \quad -1 \quad d \quad -d\ ] \\ 2)\ [1 \quad e \quad -d \quad -1 \quad d \quad -e\ ] \\ 3)\ [1 \quad -d \quad e \quad -1 \quad d \quad -e\ ] \end{cases}$$

3.2.3.1) $(R_3)(R_4)^*=1-\bar{e}+\bar{e}\pm i+\bar{d}\mp i\bar{d}=0$

$\Rightarrow d=\pm i$

$(R_2)(R_4)^*=1-\bar{e}\pm i\bar{e}\mp i-\bar{d}-\bar{d}$

$=1-\bar{e}\pm i\bar{e}\mp i\pm i\pm i=0 \Rightarrow e=\mp i$

$((C_5)^*(C_4))^*=1\mp i\mp i-\bar{d}-k+k\bar{d}$

$=1\mp i\mp i\pm i-k\mp ik=0 \Rightarrow k=\mp i$

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & -1 & 1 \\ 1 & -1 & -1 & \mp i & 1 & \pm i \\ 1 & \mp i & \pm i & -1 & \pm i & \mp i \\ 1 & f_1 & g & \mp i & -1 & -1 \\ 1 & f_2 & -g & \pm i & \mp i & -1 \end{bmatrix}$$

$f_1, f_2 \in \{1, \pm i\}$

$f_1 \neq f_2$

$(C_5)^*(C_3)=1\pm i-1+1-g\mp ig=0 \Rightarrow g=1$

$\Rightarrow f_1=\pm i,\ f_2=1$

$H \sim H_1(1)$ or $H_1(-1)$

3.2.3.2) $(R_2)(R_4)^* = 1 - \bar{e} \pm i\bar{d} \mp i - \bar{d} - \bar{e} = 0$

$\Rightarrow 2\bar{e} = (1-\bar{d})(1 \mp i)$

$\Rightarrow |1-\bar{d}| = 2 \cdot |e| \cdot |1/2(1 \pm i)| = \sqrt{2}$

$\Rightarrow d = \pm i$ or $\mp i$, $e = 1/2(1-d)(1 \pm i)$

$\Rightarrow d = \pm i, e = 1$ or $d = \mp i, e = \pm i$

$(R_3)(R_4)^* = 1 - \bar{e} + \bar{d} \pm i + \bar{d} \mp i\bar{e} = 0$

- $d = \pm i, e = 1$: $1 - 1 \mp i \pm i \mp i \mp i = \mp 2i \neq 0$ ✗
- $d = \mp i, e = \pm i$: $1 \pm i \pm i \pm i \pm i - 1 = \pm 4i \neq 0$ ✗

3.2.3.3) $(R_2)(R_4)^* = 1 + \bar{d} \mp i\bar{e} \mp i - \bar{d} - \bar{e} = 0$

$\Rightarrow e = \pm i$

$(R_3)(R_4)^* = 1 + \bar{d} - \bar{e} \pm i + \bar{d} \mp i\bar{e}$

$= 1 + \bar{d} \pm i \pm i + \bar{d} - 1 = 0 \Rightarrow d = \pm i$

$(C_5)^*(C_4) = 1 \mp i \mp i - \bar{d} - k + k\bar{d}$

$= 1 \mp i \mp i \pm i - k \mp ik = 0 \Rightarrow k = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & -1 & 1 \\ 1 & -1 & -1 & \mp i & 1 & \pm i \\ 1 & \mp i & \pm 1 & -1 & \pm i & \mp i \\ 1 & f_1 & g & \mp i & -1 & -1 \\ 1 & f_2 & -g & \pm i & \mp i & -1 \end{bmatrix} \sim H_1(1) \text{ or } H_1(-1)$$

[see 3.2.3.1]

3.3)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & & & -1 & d & \\ 1 & & & & -1 & \\ 1 & & & & -d & x \end{bmatrix}$$

$$(C_4)^T = \begin{cases} 1) \; [\,1 \;\; -a \;\; -1 \;\; -1 \;\; a \;\; 1\,] \\ 2) \; [\,1 \;\; -a \;\; -1 \;\; -1 \;\; 1 \;\; a\,] \\ 3) \; [\,1 \;\; 1 \;\; -1 \;\; -1 \;\; K \;\; -K\,] \end{cases}$$

3.3.1) $(C_5)^*(C_4) = 1 \pm ia \mp i - \bar{d} - a - \bar{d} = 0$

$\Rightarrow 2\bar{d} = (1-a)(1 \mp i)$

$\Rightarrow |1-a| = |2\bar{d}| \cdot |1/2(1 \pm i)| = \sqrt{2}$

$\Rightarrow a = \pm i$ or $\mp i$, $d = 1/2(1-\bar{a})(1 \pm i)$

$\Rightarrow a = \pm i, d = \pm i$ or $a = \mp i, d = 1$

$$(R_4) = \begin{cases} 1) \; [\,1 \;\; e \;\; -e \;\; -1 \;\; d \;\; -d\,] \\ 2) \; [\,1 \;\; e \;\; -d \;\; -1 \;\; d \;\; -e\,] \\ 3) \; [\,1 \;\; -d \;\; e \;\; -1 \;\; d \;\; -e\,] \end{cases}$$

3.3.1.1) $(R_2)(R_4)^* = 1 - \bar{e} - a\bar{e} + a \pm i\bar{d} \pm i\bar{d} = 0$

- $a = \pm i, d = \pm i$: $1 - \bar{e} \mp i\bar{e} \pm i + 1 + 1 =$

$\Rightarrow |\bar{e}| \, |1 \pm i| = |-3 \mp i|$ $\cancel{\;\;}$

- $a = \mp i, d = 1$: $1 - \bar{e} \pm i\bar{e} \mp i \pm i \pm i = 0 \Rightarrow e = \mp i$

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & \mp i & \pm i & -1 & 1 & -1 \\ 1 & f & g & \mp i & -1 & h \\ 1 & f & -g & 1 & -1 & x \end{bmatrix} \sim H_1(1) \text{ or } H_1(-1)$$

[see 3.2.1.1.2.4]

3.3.1.2) $(R_2)(R_4)^* = 1 - \bar{e} - a\bar{d} + a \pm i\bar{d} \pm i\bar{e} = 0$

- $a=\pm i,\ d=\pm i$: $1-\bar{e} - 1 \pm i + 1 \pm i\bar{e} = 0 \Rightarrow e = \mp i$
- $a=\mp i,\ d=1$: $1-\bar{e} \pm i \mp i \pm i \pm i\bar{e} = 0 \Rightarrow e = \mp i$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{d} + 1 \mp i\bar{d} - \bar{e} = 0$

$= 1 - 1 + \bar{d} + 1 \mp i\bar{d} \mp i = 0$

$\Rightarrow d = -1$ $\times$

3.3.1.3) $(R_3)(R_4)^* = 1 \mp i\bar{d} - \bar{e} + 1 \mp i\bar{d} - \bar{e} = 0$

- $a=\pm i,\ d=\pm i$: $1-1-\bar{e}+1-1-\bar{e}=0$

  $\Rightarrow \bar{e} = 0$ $\times$

- $a=\mp i,\ d=1$: $1 \mp i - \bar{e} + 1 \mp i - \bar{e} = 0$

  $\Rightarrow -\bar{e} = 1 \mp i$ $\times$

3.3.2) $(C_5)^*(C_4) = 1 \pm ia \mp i - \bar{d} - 1 - a\bar{d} = 0$

$\Rightarrow \bar{d} \pm i \mp ia + a\bar{d} = 0$

- $\bar{d} = \mp i,\ \pm ia = a\bar{d} = \mp ia$ $\times$
- $\bar{d} = \pm ia,\ \pm i = -a\bar{d} = \mp ia^2 \Rightarrow a = \pm i$ or $\mp i$
- $\bar{d} = -a\bar{d} \Rightarrow a=-1,\ \pm i = \pm ia \Rightarrow a = 1$ $\times$

$\Rightarrow a = \pm i,\ d = -1$ or $a = \mp i,\ d = 1$

$$R_4 = \begin{cases} 1)\ [\,1 \quad e \quad -e \quad -1 \quad \partial \quad -\partial\,] \\ 2)\ [\,1 \quad e \quad -\partial \quad -1 \quad \partial \quad -e\,] \\ 3)\ [\,1 \quad -\partial \quad e \quad -1 \quad \partial \quad -e\,] \end{cases}$$

3.3.2.1) $(R_2)(R_4)^* = 1 - \bar{e} - a\bar{e} + a \pm i\bar{\partial} \pm i\bar{\partial} = 0$

- $a = \pm i,\ \partial = -1$: $1 - \bar{e} \mp i\bar{e} \pm i \mp i \mp i = 0 \Rightarrow e = \pm i$
- $a = \mp i,\ \partial = 1$: $1 - \bar{e} \pm i\bar{e} \mp i \pm i \pm i = 0 \Rightarrow e = \mp i$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{e} + 1 \mp i\bar{\partial} - \bar{\partial} = 0$

- $a = \pm i$: $1 + 1 \mp i + 1 \pm i + 1 = 4 \neq 0$ $\cancel{\ }$

- $a = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & \mp i & \pm i & -1 & 1 & -1 \\ 1 & f & g & 1 & -1 & h \\ 1 & -f & -g & \mp i & -1 & x \end{bmatrix} \sim H_1(1) \text{ or } H_1(-1)$$

[$R_5 \leftrightarrow R_6$ + see 3.2.1.1.2.4]

3.3.2.2) $(R_2)(R_4)^* = 1 - \bar{e} - a\bar{\partial} + a \pm i\bar{\partial} \pm i\bar{e} = 0$

- $a = \pm i,\ \partial = -1$: $1 - \bar{e} \pm i \pm i \mp i \pm i\bar{e} = 0 \Rightarrow e = \mp i$
- $a = \mp i,\ \partial = 1$: $1 - \bar{e} \pm i \mp i \pm i \pm i\bar{e} = 0 \Rightarrow e = \mp i$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{\partial} + 1 \mp i\bar{\partial} - \bar{e}$

$= 1 - 1 + \bar{\partial} + 1 \mp i\bar{\partial} \mp i = 0$

$\Rightarrow \partial = -1$

$\Rightarrow a = \pm i,\ \partial = -1,\ e = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \mp i & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & \mp i & 1 & -1 & -1 & \pm i \\ 1 & f & g_1 & 1 & -1 & -1 \\ 1 & -f & g_2 & \pm i & 1 & -1 \end{bmatrix} \qquad \begin{array}{c} g_1, g_2 \in \{-1, \mp i\} \\ g_1 \neq g_2 \end{array}$$

$$(C_5)^*(C_2) = 1 \pm i - 1 \pm i - f - f = 0 \Rightarrow f = \pm i$$

$$\Rightarrow g_1 = \mp i, \; g_2 = -1$$

$$\Rightarrow H \sim H_1(1) \text{ or } H_1(-1)$$

3.3.2.3) $(R_2)(R_4)^* = 1 + \bar{d} \mp a\bar{e} \pm a \pm i\bar{d} \pm i\bar{e} = 0$

- $a = \pm i, d = -1 : 1 - 1 \pm i\bar{e} \pm i \mp i \pm i\bar{e} = 0$

$$\Rightarrow \pm 2i\bar{e} = 0 \quad \times$$

- $a = \mp i, d = 1 : 1 + 1 \mp i\bar{e} \mp i \pm i \pm i\bar{e} = 2 \neq 0$

$$\times$$

3.3.3) $$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & & & -1 & d & \\ 1 & & & k & -1 & \\ 1 & & & -k & -d & \end{bmatrix}$$

$$R_4 = \begin{cases} 1) \; [1 \;\; e \;\; -e \;\; -1 \;\; d \;\; -d] \\ 2) \; [1 \;\; e \;\; -d \;\; -1 \;\; d \;\; -e] \\ 3) \; [1 \;\; -d \;\; e \;\; -1 \;\; d \;\; -e] \end{cases}$$

3.3.3.1) $(R_2)(R_4)^* = 1 - \bar{e} + \bar{e} - 1 \pm i\bar{d} \pm i\bar{d} = 0$

$\Rightarrow \bar{d} = 0$

3.3.3.2) $(R_2)(R_4)^* = 1 - \bar{e} + \bar{d} - 1 \pm i\bar{d} \pm i\bar{e} = 0$

$\Rightarrow \bar{e}(1 \mp i) = \bar{d}(1 \pm i)$

$\Rightarrow e = \mp i d$

$(R_3)(R_4)^* = 1 \pm i\bar{e} + \bar{d} + 1 \mp i\bar{d} - \bar{e}$

$= 1 - \bar{d} + \bar{d} + 1 \mp i\bar{d} \mp i\bar{d} = 0$

$\Rightarrow d = \pm i \;,\; e = 1$

$(C_5)^*(C_4) = 1 \mp i \mp i - \bar{d} - k + k\bar{d}$

$= 1 \mp i \mp i \pm i - k \mp ik = 0$

$\Rightarrow k = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & \pm i & \mp i \\ 1 & \pm i & -1 & -1 & \mp i & 1 \\ 1 & 1 & \mp i & -1 & \pm i & -1 \\ 1 & f_1 & g_1 & \mp i & -1 & h \\ 1 & f_2 & g_2 & \pm i & \mp i & x \end{bmatrix}$$

$f_1, f_2 \in \{-1, \mp i\}$

$f_1 \neq f_2$

$g_1, g_2 \in \{1, \pm i\}$

$g_1 \neq g_2$

$(C_4)^*(C_2) = 1 - 1 + 1 \pm i \pm i g_1 \mp i g_2 = 0$

$\Rightarrow g_1 = \pm i \;,\; g_2 = 1$

$(C_5)^*(C_3) = 1 \pm i \mp i + 1 - g_1 \pm i g_2 = 0$

$\Rightarrow g_1 = 1, \; g_2 = \pm i$

3.3.3.3) $(R_2)(R_4)^* = 1 + \bar{d} - \bar{e} - 1 \pm i\bar{d} \pm i\bar{e} = 0$

$\Rightarrow \bar{d}(1 \pm i) = \bar{e}(1 \mp i) \Rightarrow d = \pm i e$

$(R_3)(R_4)^* = 1 \mp i\bar{d} - \bar{e} + 1 \mp i\bar{d} - \bar{e}$

$= 1 - \bar{e} - \bar{e} + 1 - \bar{e} - \bar{e} = 0$

Appendix II: Case 2 calculations

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & b & -b \\ 1 & & -1 & & d & \\ 1 & & & -1 & -d & \\ 1 & & & & -1 & \\ 1 & & & & -b & x \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & & & -d & \\ 1 & & -1 & & d & \\ 1 & -a & a & -1 & b & -b \\ 1 & & & & -1 & \\ 1 & & & & -b & x \end{bmatrix}$$

$$(R_2) = \begin{cases} 1)\ [1\ \ -1\ \ e\ \ -e\ \ -d\ \ d\ ] \\ 2)\ [1\ \ -1\ \ e\ \ d\ \ -d\ \ -e\ ] \\ 3)\ [1\ \ -1\ \ d\ \ e\ \ -d\ \ -e\ ] \end{cases}$$

1) This is the same as the calculations done in Appendix I, when $R_4 \sim [1\ -e\ e\ -1\ -d\ d\ ]$

2)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & e & d & -d & -e \\ 1 & & -1 & & d & \\ 1 & -a & a & -1 & b & -b \\ 1 & & & & -1 & \\ 1 & & & & -b & x \end{bmatrix} \qquad R_3 = \begin{cases} 1)\ [1\ \ c\ \ -1\ \ -c\ \ d\ \ -d] \\ 2)\ [1\ \ c\ \ -1\ \ -d\ \ d\ \ -c] \\ 3)\ [1\ \ -d\ \ -1\ \ c\ \ d\ \ -c] \end{cases}$$

2.1) $(R_2)(R_3)^* = 1 - \bar{c} - e - \bar{c}d - 1 + e\bar{d} = 0$

$$\Rightarrow \bar{c}(1+d) = e(\bar{d}-1) = e\bar{d}(1-d)$$

$$\Rightarrow |1+d| = |1-d| \quad \Rightarrow d = \pm i$$

$$e = \bar{c}\,d(1+d)/(1-d) = -\bar{c}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -\bar{c} & \pm i & \mp i & \bar{c} \\ 1 & c & -1 & -c & \pm i & \mp i \\ 1 & -a & a & -1 & b & -b \\ 1 & & & & -1 & \\ 1 & & & & -b & X \end{bmatrix}$$

$$(C_4)^T = \begin{cases} 1)\ [1 \ \ \pm i \ \ -c \ \ -1 \ \ \mp i \ \ c\,] \\ 2)\ [1 \ \ \pm i \ \ -c \ \ -1 \ \ c \ \ \mp i\,] \\ 3)\ [1 \ \ \pm i \ \ \mp i \ \ -1 \ \ K \ \ -K\,] \end{cases}$$

2.1.1) $(C_5)^*(C_4) = 1 - 1 \pm ic - \bar{b} \pm i - \bar{b}c = 0$

$\Rightarrow (\pm i - \bar{b})(1 + c) = 0$

$\Rightarrow b = \mp i$ or $c = -1$

$(R_3)(R_4)^* = 1 - c\bar{a} - \bar{a} + c \pm i\bar{b} \pm i\bar{b} = 0$

- $c = -1: \ 1 + \bar{a} - a - 1 \pm 2i\bar{b} = \pm 2i\bar{b}$

$\Rightarrow b = 0$ $\times$

- $b = \mp i: \ 1 - c\bar{a} - \bar{a} + c - 1 - 1 = 0$

$\Rightarrow -c\bar{a} = 1 \Rightarrow \bar{a} = \pm c$, and $\bar{a} = c$

$\Rightarrow -a = \bar{a} \Rightarrow a = \pm i$ or $\mp i$

$(R_2)(R_4)^* = 1 + \bar{a} - \bar{a}\bar{c} \mp i \mp i\bar{b} - \bar{c}\bar{b}$

- $a = \pm i: \ 1 \mp i - 1 \mp i + 1 + 1 \neq 0$ $\times$
- $a = \mp i: \ 1 \pm i - 1 \mp i + 1 - 1 = 0$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm i & \mp i & \mp i \\ 1 & \pm i & -1 & \mp i & \pm i & \mp i \\ 1 & \pm i & \mp i & -1 & \mp i & \pm i \\ 1 & \mp i & g & \mp i & -1 & h \\ 1 & \mp i & -g & \pm i & \pm i & x \end{bmatrix}$$

$(R_5)(R_1)^* = 1 \mp i + g \mp i - 1 + h = 0 \Rightarrow g = h = \pm i$

$\Rightarrow x = -1$

So $H \sim H_1(\pm i)$

$$2.1.2)\quad (C_5)^*(C_4) = 1 - 1 \pm ic - \bar{b} - c \pm i\bar{b} = 0$$
$$\Rightarrow (\bar{b} + c)(-1 \pm i) = 0$$
$$\Rightarrow b = -\bar{c}$$

$$(R_2)(R_4)^* = 1 + \bar{a} - \bar{c}\bar{a} \mp i \mp i\bar{b} - \bar{c}\bar{b}$$
$$= 1 + \bar{a} - \bar{c}\bar{a} \mp i \pm ic + 1 = 0$$
$$\Rightarrow \bar{a}(\bar{c} - 1) = 2 \mp i \pm ic \qquad [\text{note}: c = 1 \Rightarrow 0 = 2]$$
$$\Rightarrow a = (2 \pm i \mp i\bar{c})/(c-1)$$
$$\text{also, } a = 1/\bar{a} = (\bar{c}-1)/(2 \mp i \pm ic)$$
$$\Rightarrow (\bar{c}-1)(c-1) = (2 \pm i \mp i\bar{c})(2 \mp i \pm ic)$$
$$\Rightarrow 1 - \bar{c} - c + 1 = 4 \mp 2i \pm 2ic \pm 2i + 1 - c$$
$$\mp 2i\bar{c} - \bar{c} + 1$$
$$\Rightarrow 0 = 4 \pm 2ic \mp 2i\bar{c}$$
$$\Rightarrow (-4/2)(\mp i) = c - \bar{c} = 2\,\mathrm{Im}\, c$$
$$\Rightarrow c = \pm i, \quad a = \mp i$$

$$(R_3)(R_4) = 1 - c\bar{a} - \bar{a} + c \pm i\bar{b} \pm i\bar{b}$$
$$= 1 - c\bar{a} - \bar{a} + c \mp 2ic$$
$$= 1 - (\pm i)(\pm i) \mp i \pm i \mp 2i(\pm i)$$
$$\neq 0$$

$$2.1.3)\quad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm i & \mp i & \mp i \\ 1 & \pm i & -1 & \mp i & \pm i & \mp i \\ 1 & -a & a & -1 & b & -b \\ 1 & & & K & -1 & h \\ 1 & & & -K & -b & x \end{bmatrix} \sim H_1(\pm i)$$

[see I.2.3]

2.2) $(R_2)(R_3)^* = 1 - \bar{c} - e - 1 - 1 + e\bar{c} = 0$

$$\Rightarrow e\bar{c} = 1,\ \bar{c} = -e \Rightarrow c = e = \pm i$$

$$(R_2)(R_4)^* = 1 + \bar{a} + \bar{a}e - d - d\bar{b} + e\bar{b}$$
$$= 1 + \bar{a} \pm i\bar{a} - d - d\bar{b} \pm i\bar{b} = 0$$
$$(R_3)(R_4)^* = 1 - c\bar{a} - \bar{a} + d + d\bar{b} + c\bar{b}$$
$$= 1 \mp i\bar{a} - \bar{a} + d + d\bar{b} \pm i\bar{b} = 0$$

add: $2 \pm 2i\bar{b} = 0 \Rightarrow b = \mp i$

$$\Rightarrow (R_2)(R_4)^* = 1 + \bar{a} \pm i\bar{a} - d \mp id - 1 = 0$$
$$\Rightarrow (\bar{a} - d)(1 \pm i) = 0 \Rightarrow \bar{a} = d$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & d & -d & \mp i \\ 1 & \pm i & -1 & -d & d & \mp i \\ 1 & -\bar{d} & \bar{d} & -1 & \mp i & \pm i \\ 1 & f_1 & g_1 & K & -1 & h \\ 1 & f_2 & g_2 & -K & \pm i & X \end{bmatrix}$$

$f_1, f_2 \in \{\mp i, \bar{d}\}$

$f_1 \neq f_2$

$g_1, g_2 \in \{\mp i, -\bar{d}\}$

$g_1 \neq g_2$

[if $d = \pm i$ or $\mp i$, $H \sim H_1(\pm i)$, see I.23]

$$(C_5)^*(C_4) = 1 - 1 - 1 \mp i - K \pm iK = 0 \Rightarrow K = \mp i$$

$$(C_5)^*(C_2) = 1 + \bar{d} \pm i\bar{d} \mp i\bar{d} - f_1 \mp if_2 = 0$$
$$\Rightarrow f_1 = \bar{d},\ f_2 = \mp i$$
$$(C_5)^*(C_3) = 1 \mp i\bar{d} - \bar{d} \pm i\bar{d} - g_1 \mp ig_2 = 0$$
$$\Rightarrow g_1 = -\bar{d},\ g_2 = \mp i$$
$$(R_5)(R_1)^* = 0 \Rightarrow h = \pm i$$
$$(R_6)(R_1)^* = 0 \Rightarrow X = -1$$

So $H \sim H_1(d)$

2.3) $(R_2)(R_3)^* = 1 + \bar{d} - e + d\bar{c} - 1 + e\bar{c} = 0$

$$= \begin{cases} 1)\ \bar{d} = e,\ \ d\bar{c} = -e\bar{c} = -\bar{d}\bar{c} \Rightarrow -d = \bar{d} \Rightarrow d = \pm i,\ e = \mp i \\ 2)\ \bar{d} = -d\bar{c},\ \ e = e\bar{c} \Rightarrow c = 1,\ d = \pm i \\ 3)\ \bar{d} = -e\bar{c},\ \ e = d\bar{c} = -\bar{e} \Rightarrow e = \pm i,\ d = \pm ic \end{cases}$$

2.3.1) $(R_2)(R_4)^* = 1 + \bar{a} + e\bar{a} - d - d\bar{b} + e\bar{b}$

$$= 1 + \bar{a} \mp i\bar{a} \mp i \mp i\bar{b} \mp i\bar{b} = 0$$

$$\Rightarrow (\bar{a}+1)(1 \mp i) = \pm 2i\bar{b}$$

$$\Rightarrow |\bar{a}+1| = |2i\bar{b}| / |1 \mp i| = \sqrt{2}$$

$$\Rightarrow a = \pm i \text{ or } a = \mp i,\ b = \pm \tfrac{i}{2}(1 \pm i)(1+a)$$

$$\Rightarrow a = \pm i,\ b = -1 \quad \text{or} \quad a = \mp i,\ b = \pm i$$

$$(R_3)(R_4) = 1 + d\bar{a} - \bar{a} - c + d\bar{b} + c\bar{b}$$

$$= 1 \pm i\bar{a} - \bar{a} - c \pm i\bar{b} + c\bar{b} = 0$$

- $a = \pm i:\ 1 + 1 \pm i - c \mp i - c = 0 \Rightarrow c = 1$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \mp i & \pm i & -1 & -1 & 1 \\ 1 & \pm i & g & K_1 & -1 & -K_1 \\ 1 & \pm i & -g & K_2 & 1 & -\chi \end{bmatrix} \qquad \begin{array}{c} K_1, K_2 \in \{-1, \mp i\} \\ K_1 \neq K_2 \end{array}$$

$$(C_5)^*(C_3) = 1 + 1 \pm i \mp i - g - g = 0 \Rightarrow g = 1$$

$$(C_5)^*(C_4) = 1 - 1 \mp i + 1 - K_1 + K_2 = 0$$

$$\Rightarrow K_1 = \mp i,\ K_2 = -1$$

$$\Rightarrow h = -1,\ \chi = \mp i$$

So $H \sim H_1(1)$ or $H_1(-1)$

• $a=\mp i$: $1-1\mp i-c+1\mp ic=0$

$\Rightarrow c=\mp i$

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mp i & \pm i & \mp i & \pm i \\ 1 & \mp i & -1 & \mp i & \pm i & \pm i \\ 1 & \pm i & \mp i & -1 & \pm i & \mp i \\ 1 & f & \pm i & K & -1 & h \\ 1 & -f & \pm i & -K & \mp i & x \end{bmatrix}$$

$(C_5)^*(C_4) = 1-1=1\pm i-K\mp iK=0 \Rightarrow K=\pm i$

$(C_5)^*(C_4) = 1\mp i-1+1-f\mp if=0 \Rightarrow f=\mp i$

$\Rightarrow h=\mp i,\ x=\mp 1$

So $H\sim H_1(\pm i)$

2.3.2) $(R_3)(R_4)^* = 1+d\bar a\mp\bar a-c\mp d\bar b+c\bar b$

$= 1\pm i\bar a\mp\bar a-1\pm i\bar b+\bar b=0$

$\Rightarrow \bar a(1\mp i)=\bar b(1\pm i) \Rightarrow a=\mp ib$

$(R_2)(R_4)^* = 1+\bar a+e\bar a-d-d\bar b+e\bar b$

$= 1\pm i\bar b\pm i\bar b e\mp i\mp i\bar b+e\bar b=0$

$\Rightarrow (1\mp i)=-e\bar b(1\pm i)$

$\Rightarrow \pm ib=e$

[b = -1 - see 2.3.1]

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm ib & \pm i & \mp i & \mp ib \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \pm ib & \mp ib & -1 & b & -b \\ 1 & f_1 & g & k_1 & -1 & h \\ 1 & f_2 & -g & k_2 & -b & x \end{bmatrix}$$

$f_1,f_2\in\{\pm i,\mp ib\},\ f_1\neq f_2$

$h,x\in\{b,\pm ib\},\ h\neq x$

$k_1,k_2\in\{-1,\mp i\}$

$k_1\neq k_2$

$$(C_5)^*(C_4) = 1 - 1 \mp i - \bar{b} - K_1 - K_2\bar{b} = 0$$

$$\Rightarrow K_1 = \mp i, \; K_2 = -1$$

$$(C_5)^*(C_2) = 1 \mp i - 1 \pm i - f_1 - \bar{b} f_2 = 0$$

$$\Rightarrow f_1 = \pm i, \; f_2 = \mp i b$$

$$(C_5)^*(C_6) = 1 + b \pm i - 1 - h - \bar{b} x = 0$$

$$\Rightarrow h = b, \; x = \pm i b$$

$$(R_5)(R_1)^* = 0 \Rightarrow g = -b$$

So $H \sim H_2(b)$

2.3.3)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm ic & \mp ic & \mp i \\ 1 & \mp ic & -1 & c & \pm ic & -c \\ 1 & -a & a & -1 & b & -b \\ 1 & & & & 1 & \\ 1 & & & & -b & x \end{bmatrix}$$

$$(C_4)^T = \begin{cases} 1) \; [\,1 \;\; \pm ic \;\; c \;\; -1 \;\; -c \;\; \mp ic\,] \\ 2) \; [\,1 \;\; \pm ic \;\; c \;\; -1 \;\; \mp ic \;\; -c\,] \end{cases}$$

2.3.3.1) $(C_5)^*(C_4) = 1 - 1 \mp i - \bar{b} + c \pm ic\bar{b} = 0$

$$\Rightarrow \mp i = \mp ic\bar{b} \Rightarrow c = b$$

$$c = \bar{b} \Rightarrow b = 1$$

$$(R_3)(R_4)^* = 1 \pm ic\bar{a} - \bar{a} - c \pm i + 1$$

$$= 1 \pm i\bar{a} - \bar{a} - 1 \pm i + 1 = 0 \Rightarrow a = \mp i$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm i & \mp i & \mp i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \pm i & \mp i & -1 & 1 & -1 \\ 1 & f & g & -1 & -1 & h \\ 1 & -f & -g & \mp i & -1 & x \end{bmatrix}$$

$$(C_4)^*(C_3) = 1 + 1 - 1 \pm i - g \mp i g = 0 \Rightarrow g = 1$$

$$(C_4)^*(C_2) = 1 \pm i \mp i \mp i - f \mp i f = 0 \Rightarrow f = \mp i$$

$$\Rightarrow h = \pm i, \quad x = 1$$

So $H \sim H_1(1)$ or $H_1(-1)$

2.3.3.2) $(C_5)^*(C_4) = 1 - 1 \mp i - \bar b \pm i c + c\bar b = 0$

$$\begin{cases} 1)\ \mp i = \bar b, \quad \mp ic = c\bar b = \mp ic \\ 2)\ \mp i = \mp ic \Rightarrow c = 1, \quad \bar b = c\bar b = \bar b \\ 3)\ \mp i = -c\bar b \Rightarrow \pm ib = c, \quad \bar b = \pm ic = -b \Rightarrow b = \pm i \text{ or } \mp i, \quad c = -1,\ 1 \end{cases}$$

2.3.3.2.1) $(R_3)^*(R_4) = 1 \pm ic\bar a \mp \bar a - c \pm ic\bar b + c\bar b$

$$= 1 \pm ic\bar a \mp \bar a - c + c \mp ic = 0$$

$$\begin{cases} 1)\ 1 = \mp ic\bar a \Rightarrow a = \mp ic, \quad \bar a = \mp ic = a \Rightarrow a = 1, -1; \quad c = \pm i, \mp i \\ 2)\ 1 = \bar a, \quad \pm ic = \pm ica = \pm ic \\ 3)\ 1 = \pm ic \Rightarrow c = \mp i, \quad \bar a = \pm ic\bar a = \bar a \end{cases}$$

$$(R_2)^*(R_4) = 1 + \bar a \pm i\bar a \mp ic \mp ic\bar b \pm i\bar b$$

$$= 1 + \bar a \pm i\bar a \mp ic - c + 1 = 0$$

1) $1 + 1 \pm i + 1 \mp i + 1 \neq 0 \quad (a = 1)$

$1 - 1 \mp i - 1 \pm i + 1 = 0 \quad (a = -1)$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & 1 & -1 & \mp i \\ 1 & -1 & -1 & \mp i & 1 & \pm i \\ 1 & 1 & -1 & -1 & \pm i & \mp i \\ 1 & f & g_1 & -1 & -1 & h \\ 1 & -f & g_2 & \pm i & \mp i & x \end{bmatrix}$$

$g_1, g_2 \in \{1, \mp i\}$ $g_1 \neq g_2$

$h, x \in \{-1, \pm i\}$

$$(C_5)^*(C_2) = 1 + 1 - 1 \mp i - f \mp i f = 0 \Rightarrow f = \mp i$$

$$(C_5)^*(C_3) = 1 \mp i - 1 \pm i - g_1 \pm i g_2 = 0 \Rightarrow g_1 = \mp i, \ g_2 = 1$$

$(C_3)^*(C_2) = 1 \pm i + 1 - 1 + 1 \pm i \neq 0$ $\cancel{\ }$

2) $(R_2)^*(R_4) = 1 + \bar{a} \pm i\bar{a} \mp ic - c + 1$
$= 1 + 1 \pm i \mp ic - c + 1 = 0$
$\Rightarrow |c| \cdot |1 \pm i| = |-3 \mp i|$ $\cancel{\ }$

3) $(R_2)^*(R_4) = 1 + \bar{a} \pm i\bar{a} \mp ic - c + 1$
$= 1 + \bar{a} \pm i\bar{a} - 1 \pm i + 1 = 0$
$\Rightarrow a = -1$

Same as 1 above

2.3.3.2.2) $(R_3)(R_4)^* = 1 \pm ic\bar{a} - \bar{a} - c \pm ic\bar{b} + c\bar{b}$
$= 1 \pm i\bar{a} - \bar{a} - 1 \pm i\bar{b} + \bar{b} = 0$
$\Rightarrow \bar{a}(1 \mp i) = \bar{b}(1 \pm i) \Rightarrow a = \mp ib$

$(R_2)(R_4)^* = 1 + \bar{a} \pm i\bar{a} \mp ic \mp ic\bar{b} \pm i\bar{b}$
$= 1 \pm i\bar{b} - \bar{b} \mp i \mp i\bar{b} \pm i\bar{b} = 0$
$\Rightarrow b = 1,\ a = \mp i$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm i & \mp i & \mp i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & \pm i & \mp i & -1 & 1 & -1 \\ 1 & f & g & \mp i & -1 & h \\ 1 & -f & -g & 1 & -1 & x \end{bmatrix} \sim H_1(1) \text{ or } H_1(-1)$$

[$R_4 \leftrightarrow R_5$ and see 2.3.3.1]

2.3.3.2.3) $(R_3)(R_4) = 1 \pm ic\bar{a} - \bar{a} - c \pm ic\bar{b} + c\bar{b} = 0$

- $b = \pm i$: $1 \mp i\bar{a} - \bar{a} + 1 - 1 \pm i = 0$

$\Rightarrow a = 1$

$\bullet\ b = \mp i$: $1 \pm i\bar{a} - \bar{a} - 1 - 1 \pm i = 0$

$\Rightarrow a = \mp i$

$(R_2)(R_4)^* = 1 + \bar{a} \pm i\bar{a} \mp ic \mp ic\bar{b} \pm i\bar{b} = 0$

$\bullet\ b = \pm i$: $-1 + 1 \pm i \pm i + 1 + 1 \neq 0$ $\times$

$\bullet\ b = \mp i$: $1 \pm i - 1 \mp i + 1 - 1 = 0$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm i & \mp i & \mp i \\ 1 & \mp i & -1 & 1 & \pm i & -1 \\ 1 & -1 & 1 & -1 & \mp i & \pm i \\ 1 & f_1 & g_1 & \mp i & -1 & -x \\ 1 & f_2 & g_2 & -1 & \pm i & x \end{bmatrix}$$

$f_1, f_2 \in \{1, \pm i\}$; $f_1 \neq f_2$

$g_1, g_2 \in \{-1, \mp i\}$; $g_1 \neq g_2$

$(C_5)^*(C_2) = 1 \mp i - 1 \mp i - f_1 \mp i f_2 = 0$

$\Rightarrow f_2 = -1,\ f_1 = \mp i$ $\times$

3)
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & d & e & -d & -e \\ 1 & & -1 & & d & \\ 1 & -a & a & -1 & b & -b \\ 1 & & & & -1 & \\ 1 & & & & -b & x \end{bmatrix}$$

$$(C_4)^T = \begin{cases} 1)\ [\,1 \quad e \quad -e \quad -1 \quad K \quad -K\,] \\ 2)\ [\,1 \quad e \quad K \quad -1 \quad -e \quad -K\,] \\ 3)\ [\,1 \quad e \quad K \quad -1 \quad -K \quad -e\,] \end{cases}$$

3.1) $$H \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & d & -d & e & -e \\ 1 & & -1 & d & -e & \\ 1 & & & -1 & k & \\ 1 & -a & a & b & -1 & -b \\ 1 & & & -b & -k & x \end{bmatrix}$$

[after $C_4 \longleftrightarrow C_5$, $R_4 \longleftrightarrow R_5$]

This is covered in I.

3.2) $$H \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -d & d & e & -e \\ 1 & & -1 & & -e & \\ 1 & & d & -1 & k & \\ 1 & -a & b & a & -1 & -b \\ 1 & & -b & & -k & x \end{bmatrix}$$

[$C_4 \longleftrightarrow C_5$, $R_4 \longleftrightarrow R_5$, $C_3 \longleftrightarrow C_4$, $R_3 \longleftrightarrow R_4$]

Covered in I.

3.3) $$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & d & e & -d & -e \\ 1 & & -1 & k & d & \\ 1 & -a & a & -1 & b & -b \\ 1 & & & -k & -1 & \\ 1 & & & -e & -b & x \end{bmatrix}$$

$$R_3 = \begin{cases} 1)\ [1 \quad -k \quad -1 \quad k \quad d \quad -d] \\ 2)\ [1 \quad -d \quad -1 \quad k \quad d \quad -k] \\ 3)\ [1 \quad c \quad -1 \quad -d \quad d \quad -c] \end{cases}$$

3.3.1) $$H \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -k & k & d & -d \\ 1 & d & -1 & e & -d & -e \\ 1 & -a & a & -1 & b & -b \\ 1 & & & -k & -1 & \\ 1 & & & -e & -b & x \end{bmatrix}$$

[$R_2 \longleftrightarrow R_3$, $C_2 \longleftrightarrow C_3$]

Covered in I.

3.3.2)

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & d & e & -d & -e \\ 1 & -d & -1 & K & d & -K \\ 1 & -a & a & -1 & b & -b \\ 1 & & & -K & -1 & \\ 1 & & & -e & -b & X \end{bmatrix}$$

$$(R_2)(R_3)^* = 1 + \bar{d} - d + e\bar{K} - 1 + e\bar{K} = 0$$
$$\Rightarrow 2e\bar{K} = d - \bar{d} = -2\,\mathrm{Im}\, d$$
$$\Rightarrow |2e\bar{K}| = 2 = 2\cdot|\mathrm{Im}\, d| \Rightarrow d = \pm i$$
$$\Rightarrow 2e\bar{K} = \pm 2i \Rightarrow e = \pm iK$$
$$(C_5)^*(C_4) = 1 - \bar{d}e + K\bar{d} - \bar{b} + K + e\bar{b}$$
$$= 1 - K \mp iK - \bar{b} + K \pm iK\bar{b} = 0$$
$$\Rightarrow 0 = (1-\bar{b})(1 \mp iK)$$
$$\Rightarrow 1)\ b = 1 \quad \text{or} \quad 2)\ K = \mp i,\ e = 1$$

3.3.2.1) $(R_2)(R_4)^* = 1 + \bar{a} + d\bar{a} - e - d\bar{b} + e\bar{b}$
$$= 1 + \bar{a} \pm i\bar{a} - e \mp i + e = 0$$
$$\Rightarrow a = \mp i$$

$$H=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & \pm iK & \mp i & \mp iK \\ 1 & \mp i & -1 & K & \pm i & -K \\ 1 & \pm i & \mp i & -1 & 1 & -1 \\ 1 & f & g & -K & -1 & h \\ 1 & -f & -g & \mp iK & -1 & X \end{bmatrix}$$

$$(C_4)^*(C_2) = 1 \pm i\bar{K} \mp i\bar{K} \mp i - \bar{K}f \mp i\bar{K}f = 0$$
$$\Rightarrow 1 = \pm i\bar{K}f \Rightarrow f = \mp iK$$
$$(C_4)^*(C_3) = 1 + \bar{K} - \bar{K} \pm i - \bar{K}g \mp ig\bar{K} = 0$$
$$\Rightarrow g = K$$

$\Rightarrow h = \pm i K, \ \chi = K$

So $H \sim H_2(K)$

3.3.2.2) $(R_3)(R_4)^* = 1 + \partial\bar{a} - \bar{a} - K + \partial\bar{b} + K\bar{b}$

$= 1 \pm i\bar{a} - \bar{a} \pm i \pm i\bar{b} \mp i\bar{b} = 0$

$\Rightarrow a = \mp i$

$(R_2)(R_4)^* = 1 + \bar{a} + \partial\bar{a} - e - \partial\bar{b} + e\bar{b}$

$= 1 \pm i - 1 \mp 1 \mp i\bar{b} \pm \bar{b} = 0$

$\Rightarrow b = 1$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & 1 & \mp i & -1 \\ 1 & \mp i & -1 & \mp i & \pm i & \pm i \\ 1 & \pm i & \mp i & -1 & 1 & -1 \\ 1 & f & g & \pm i & -1 & h \\ 1 & -f & -g & -1 & -1 & x \end{bmatrix}$$

$(C_4)^*(C_2) = 1 - 1 + 1 \mp i \mp if + f = 0 \Rightarrow f = -1$

$(C_4)^*(C_3) = 1 \pm i \mp i \pm i \mp ig + g = 0 \Rightarrow g = \mp i$

$\Rightarrow h = 1, \ \chi = \mp i$

So $H \sim H_1(1)$ or $H_1(-1)$

3.3.3) $$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & d & e & -d & -e \\ 1 & c & -1 & -d & d & -c \\ 1 & -a & a & -1 & b & -b \\ 1 & & & d & -1 & \\ 1 & & & -e & -b & x \end{bmatrix} \sim$$

$(R_2)(R_3)^* = 1 - \bar{c} - \partial - e\bar{\partial} - 1 + e\bar{c} = 0$

$$\begin{cases} 1)\ \bar{c} = e\bar{c} \Rightarrow e=1,\ d=-e\bar{d} = -\bar{d} \Rightarrow d=\pm i \\ 2)\ \bar{c} = -e\bar{d},\ d = e\bar{c} = -e^2\bar{d} \Rightarrow d^2 = -e^2 \Rightarrow d = \pm ie,\ c = \mp i \\ 3)\ \bar{c} = -d,\ e\bar{d} = e\bar{c} = -ed \Rightarrow d = \pm i,\ c = \pm i \end{cases}$$

3.3.3.1) $(R_2)(R_4)^* = 1 + \bar{a} + d\bar{a} - e - d\bar{b} + e\bar{b}$

$= 1 + \bar{a} \pm i\bar{a} - 1 \mp i\bar{b} + \bar{b} = 0$

$\Rightarrow \bar{a}(1 \pm i) = -\bar{b}(1 \mp i) \Rightarrow b = \pm ia$

$(R_3)(R_4)^* = 1 - c\bar{a} - \bar{a} + d + d\bar{b} + c\bar{b}$

$= 1 - c\bar{a} - \bar{a} \pm i + \bar{a} \mp i\bar{a}c = 0$

$\Rightarrow c = a$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & 1 & \mp i & -1 \\ 1 & a & -1 & \mp i & \pm i & -a \\ 1 & -a & a & -1 & \pm ia & \mp ia \\ 1 & f & g_1 & \pm i & -1 & h \\ 1 & -f & g_2 & -1 & \mp ia & x \end{bmatrix} \qquad \begin{array}{c} g_1, g_2 \in \{\mp i, -a\} \\ g_1 \neq g_2 \end{array}$$

$(C_4)^*(C_2) = 1 - 1 \pm ia + a \mp if + f = 0 \Rightarrow f = \mp ia$

$(C_4)^*(C_3) = 1 \pm i \mp i - a \mp ig_1 - g_2 = 0$

$\Rightarrow g_1 = \mp i,\ g_2 = -a$

$\Rightarrow h = \pm ia,\ x = a$

So $H \sim H_2(a)$

3.3.3.2) $(C_5)^*(C_4) = 1 - e\bar{d} - 1 - \bar{b} - d + e\bar{b}$

$= 1 \mp i - \bar{b} \mp ie + e\bar{b} = 0$

$\Rightarrow \pm i = e\bar{b} \Rightarrow b = \mp ie$

$(R_2)(R_4)^* = 1 + \bar{a} + \bar{a}d - e - d\bar{b} + e\bar{b}$

$= 1 + \bar{a} \pm i\bar{a}e - e + 1 \pm i = 0$

$\Rightarrow \bar{a}(1 \pm ie) = e - 2 \mp i \quad [e = \pm i \Rightarrow 0 = -2]$

$\Rightarrow \bar{a} = e - 2 \mp i / (1 \pm ie)$

$\Rightarrow a = \bar{e} - 2 \pm i / 1 \mp i\bar{e}$

and $|a| = 1 \Rightarrow a = 1/\bar{a} = 1 \pm ie / e - 2 \mp i$

$\Rightarrow (1 \mp i\bar{e})(1 \pm ie) = (\bar{e} - 2 \pm i)(e - 2 \mp i)$

$\Rightarrow 1 \pm ie \mp i\bar{e} + 1 = 1 - 2\bar{e} \mp i\bar{e} - 2e + 4 \pm 2i$

$\pm ie \mp 2i + 1$

$\Rightarrow 0 = -2\bar{e} - 2e + 4$

$\Rightarrow 2 = \bar{e} + e = 2\,\mathrm{Re}\, e \Rightarrow e = \pm 1$ or $= 1$

- $e = 1$, $a = -1 \pm i / 1 \mp i = -1$
- $e = -1$, $a = -3 \pm i / 1 \pm i \Rightarrow |a| \neq 1$ $\times$

$(R_3)(R_4) = 1 - c\bar{a} - \bar{a} + d + d\bar{b} + c\bar{b}$

$= 1 \mp i + 1 \pm i - 1 + 1 \neq 0$ $\times$

3.3.3.3) $(C_5)^*(C_4) = 1 - e\bar{d} - 1 - \bar{b} - d + e\bar{b}$

$= \pm ie - \bar{b} \mp i + e\bar{b} = 0$

$\Rightarrow (\bar{b} \pm i)(e - 1) = 0$

$\Rightarrow$ 1) $b = \pm i$ or 2) $e = 1$

3.3.3.3.1) $(R_3)(R_4) = 1 - c\bar{a} - \bar{a} + d + d\bar{b} + c\bar{b}$

$= 1 \mp i\bar{a} - \bar{a} \pm i + 1 + 1 = 0$

$\Rightarrow |\bar{a}||1 \pm i| = |3 \pm i|$ $\times$

3.3.3.3.2) $(R_2)(R_4) = 1 + \bar{a} + d\bar{a} - e - d\bar{b} + e\bar{b}$

$$= 1 + \bar{a} \pm i\bar{a} - 1 \mp i\bar{b} + \bar{b} = 0$$

$$\Rightarrow \bar{a}(1 \pm i) = -\bar{b}(1 \mp i) \Rightarrow a = \mp i b$$

$$(R_3)(R_4) = 1 - c\bar{a} - \bar{a} + d + d\bar{b} + c\bar{b}$$

$$= 1 - (\pm i)(\pm i\bar{b}) \mp i\bar{b} \pm i \pm i\bar{b} \pm i\bar{b} = 0$$

$$\Rightarrow b = -1, \ a = \pm i$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \pm i & 1 & \mp i & -1 \\ 1 & \pm i & -1 & \mp i & \pm i & \mp i \\ 1 & \mp i & \pm i & -1 & -1 & 1 \\ 1 & f & \mp i & \pm i & -1 & -f \\ 1 & -f & \mp i & -1 & 1 & x \end{bmatrix}$$

$$(C_4)^*(C_2) = 1 - 1 - 1 \pm i \mp if + f = 0 \Rightarrow f = 1$$

$$\Rightarrow x = \pm i$$

So $H \sim H_1(1)$ or $H_1(-1)$