To the Graduate Council:

I am submitting herewith a dissertation written by Michael Scott Gilbert entitled "Extensions of Commutative Rings With Linearly Ordered Intermediate Rings". I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

David E. Dobbs

David E. Dobbs, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Associate Vice Chancellor and
Dean of The Graduate School

EXTENSIONS OF COMMUTATIVE RINGS WITH LINEARLY ORDERED INTERMEDIATE RINGS

A Dissertation
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Degree
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Michael Scott Gilbert
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This dissertation is dedicated to my parents

Robert and Carol Gilbert

whose support and encouragement
made this achievement possible.

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ABSTRACT

This work concerns λ-extensions of (commutative) rings, which are defined as ring extensions $R \subsetneq T$ whose set of intermediate rings is linearly ordered by inclusion. λ-extensions form a subclass of the Δ-extensions of Gilmer-Huckaba and generalize the adjacent extensions of Ferrand-Olivier, Modica, and Dechéne.

In Chapter I, we characterize λ-domains (i.e., integral domains R with quotient field K such that $R \subseteq K$ is a λ-extension), showing that they form a subclass of quasilocal i-domains. These results parallel results of Gilmer-Huckaba for Δ-domains. We relate λ-domains to divided domains and pseudovaluation domains.

Chapter II concerns λ-extensions $R \subsetneq T$ such that T is decomposable as a ring direct product. We show that a nontrivial direct product decomposition of such a T has only two factors and characterize λ-extensions of this form, extending the corresponding result of Ferrand-Olivier for adjacent extensions.

Chapter III treats λ-extensions $K \subsetneq T$ for K a field. The case where T has a nontrivial direct product decomposition is covered using a result from Chapter II. Substantial results are obtained if T is indecomposable but not a field, generalizing results of Ferrand-Olivier and Modica for an adjacent extension T of a field K. If T is a field, we obtain good characterizations for T either purely inseparable or Galois over K. To facilitate proofs, the notion of a μ-extension of fields is introduced and related to λ-extensions of fields. We also relate infinite-dimensional λ-extensions of fields and the J-extensions of fields studied by Gilmer-Heinzer.

Chapter IV begins by studying the conductor of a λ-extension, obtaining analogues of the results of Ferrand-Olivier and Modica for adjacent extensions. We then consider λ-extensions $R \subsetneq T$ of integral domains and show that, under certain conditions (but not in general), the ring T is necessarily an overring of R. We classify the λ-extension overrings for two classes of integral domains and end with a useful class of examples of λ-extensions.

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INTRODUCTION

Throughout this work, we observe the following conventions. All rings are commutative with unity. By an overring of an integral domain R, we mean a ring contained between R and its quotient field. The symbols $\subseteq$ and $\supseteq$ are used for inclusion and containment, while $<$ and $>$ are used for proper inclusion and proper containment.

The reader is assumed to be familiar with basic commutative algebra, as in [Bo1], [G2] and [K].

(I.1) DEFINITION. An extension $R \subseteq T$ of rings is said to be a *λ-extension* (alternately, T is a *λ-extension* of R) if the set of rings contained between R and T (the "intermediate rings") is linearly ordered by inclusion.

It is clear from the definition that if $R \subseteq T$ is a λ-extension and if A and B are rings such that $R \subseteq A \subseteq B \subseteq T$, then $A \subseteq B$ is also a λ-extension.

The notion of a λ-extension of rings is motivated by the consideration of two other classes of rings. Gilmer and Huckaba [GHu] call an extension $R \subseteq T$ of rings a *Δ-extension* if, for any two intermediate rings A and B, the sum $A + B$ is again a ring. Any λ-extension is a Δ-extension since, given any two intermediate rings A and B, either $A \subseteq B$ or $B \subseteq A$, and so either $A + B = B$ or $A + B = A$. A number of results in this work sharpen the results on Δ-extensions of Gilmer and Huckaba for the special case of λ-extensions.

In addition, λ-extensions are a generalization of the adjacent extensions considered (under various names) by Ferrand and Olivier [FO], Modica [Mo] and Dechéne [De]. We use the term "adjacent extension" of Dechéne. An extension $R \subseteq T$ of rings is an *adjacent extension* if there are no rings properly contained between R and T. It is clear that any adjacent extension is a λ-extension. As with the Δ-extensions, a number of results on adjacent extensions are generalized to λ-extensions.

A number of properties of rings are considered in Chapter I. We include here a summary of the definitions and basic properties which are used below. An extension $R \subseteq T$ of rings is called a *P-extension* if each element of T satisfies some polynomial in $R[X]$ having unit content (a polynomial in $R[X]$ has *unit content* if the ideal generated by its coefficients is R). An extension $R \subseteq T$ of rings is called an *INC-pair* if, for any two intermediate rings A and B such that $A \subseteq B$, the extension $A \subseteq B$ satisfies INC, in the sense of [K, page 28]. We have (cf. [D5, Corollary 4]):

(I.2) A ring extension $R \subseteq T$ is a P-extension if and only if it is an INC-pair.

An integral domain R is called a *treed domain* (see [D1]) if, for any prime ideal P of R, the set of prime ideals of R contained in P is linearly ordered by inclusion. An integral domain R is called an *i-domain* (see [P]) if, for each overring T of R, the canonical contraction map $\mathrm{Spec}(T) \rightarrow \mathrm{Spec}(R)$ is injective. The following properties of i-domains will be useful to us:

(I.3) (i) Any i-domain is a treed ring (combine [P, Corollary 2.13] and [D1, Theorem 2.2]);
(ii) An integral domain R is a quasilocal i-domain if and only if the integral closure R' of R is a valuation domain [P, Corollary 2.15];
(iii) Each overring of a quasilocal i-domain is again a quasilocal i-domain (this is a consequence of (ii)).

If $R \subseteq T$ is a ring extension, then R is said to be *root-closed in* T if, for any $x \in T$ and $n \geq 0$, the containment $x^n \in R$ implies that $x \in R$.

An integral domain R with quotient field K is said to be *seminormal* (see [ADHu]) if, for any $x \in K$, the conditions $x^2 \in R$ and $x^3 \in R$ imply that $x \in R$. Then (cf. [ADHu, Lemma 1.2(c)]):

(I.4) An integral domain R is seminormal if and only if R_P is seminormal for each prime ideal P of R (i.e., seminormality is a local property).

An integral domain R is a *going-down domain* (see [D1] and [DP]) if $R \subseteq T$ satisfies the going-down property, in the sense of [K, page 28], for each integral domain T containing R. Then the property of being a going-down domain is a local property:

(I.5) (i) An integral domain R is going-down if and only if R_P is going-down for each prime ideal P of R [D1, Lemma 2.1];
(ii) Any i-domain is a going-down domain [P, Corollary 2.13].

A prime ideal P of an integral domain R is said to be a *divided prime* if $P = PR_P$; and an integral domain R is said to be a *divided domain* (see [D2]) if each prime ideal of R is divided. Then:

(I.6) The maximal ideal M of any quasilocal domain R is a divided prime.

A quasilocal integral domain (R, M) is a *pseudovaluation domain* (see [HH1]) if there is a valuation overring V of R with maximal ideal M. An integral domain R is *coherent* if the intersection of any two finitely-generated ideals of R is again a finitely-generated ideal (cf. [Bo1]). Finally, an integral domain R with quotient field K is a *finite conductor domain* if, for each $x \in K$, the ideal $I_R(x) = \{r \in R : rx \in R\}$ is finitely-generated (cf. [Mc]).

In Chapter I, it is shown that if $R \subseteq K$ is a λ-extension with K a field, then either R is a field or else R is a quasilocal i-domain with quotient field K (Proposition (1.3)). The first case leads to the study of λ-extensions of fields, which are considered in Chapter III. In the second case, we call R a *λ-domain.* These domains are the object of study in Chapter I. Theorem (1.9) characterizes λ-domains as a certain subclass of the quasilocal i-domains. Corollary (1.12) contains another condition on a quasilocal i-domain R which implies that R is a λ-domain, namely, the condition that the integral closure R' of R is an adjacent extension of R. Theorem (1.9) introduces the condition (**) for an integral domain: an integral domain R with integral closure R' satisfies (**) if each of the overrings of R compares with R' under inclusion. Some results concerning (**) are obtained in Propositions (1.15) and (1.16). Theorems (1.22) and (1.25) are proved for a class of domains larger than the λ-domains, namely, the quasilocal i-domains R which satisfy (**). Theorem (1.22) shows that if such a domain R is either seminormal or has finite integral closure, then it is a divided domain. If R is a divided quasilocal i-domain satisfying (**), then Theorem (1.25) shows that the nonmaximal prime ideals of R coincide exactly with the nonmaximal prime ideals of the integral closure R'. This conclusion closely parallels the definition of a pseudo-valuation domain (PVD) and, in fact, Theorem (1.25) gives a condition under which such an R is necessarily a PVD. Proposition (1.27) characterizes which PVDs are (quasilocal) i-domains, and Proposition (1.28) characterizes which PVD i-domains are λ-domains. Finally, Theorem (1.31) determines which PVDs satisfy the property (**).

The goal in Chapter II is to study the λ-extensions $R \subsetneq T$ such that T has a nontrivial ring direct product decomposition. This is motivated by the need, in Chapter III, to classify the decomposable λ-extensions T of a field K. The notion of a λ-extension T of a ring R with respect to an ideal J of R is introduced (Definition (2.3)). Such an extension is a (usual) λ-extension and this definition reduces to the original definition of a λ-extension when $J = R$. In Proposition (2.8), it is shown that if $R \subseteq T$ is a λ-extension such that T has a nontrivial direct product decomposition, then there are only two factors in such a decomposition. (The special case of this for the adjacent extensions has apparently not been noticed earlier.) Using the above generalized notion of a λ-extension, Theorem (2.12) characterizes the λ-extensions $R \subseteq T$ such that T is decomposable. Corollary (2.14) considers the special case in which R is a field (this result is needed in Chapter III).

Chapter III is concerned with λ-extensions $K \subseteq T$ such that K is a field. General results about such extensions are obtained in Theorems (3.1) and (3.2) and Proposition (3.3); in particular, it is shown that the intermediate rings of such an extension are countable in number and well-ordered by inclusion. Moreover, the ring T is necessarily Artinian. Three cases are then distinguished. The case in which T is decomposable as a nontrivial direct product of rings is dealt with in Theorem (3.4), using the work of Chapter II. The second case is that in which T is indecomposable but not a field. The main results for this case are contained in Theorems (3.8) and (3.11). The bulk of the chapter is devoted to the remaining case, in which T is a field (in this section, we use the notation $K \subseteq L$ for an extension of fields). Some of the techniques employed are extensions of the technique used by Quigley [Q]. In particular, the notion of a μ-extension of fields is introduced (Definition (3.14)) and it shown that any λ-extension of fields is a μ-extension (Proposition (3.15)). In addition, it is noted that any infinite-dimensional λ-extension of fields is a J-extension of fields, in the sense of [GHe3]. Two special subcases are considered for the case of a field extension $K \subseteq L$, namely, the cases in which L is a purely inseparable extension or a Galois extension of K. In the purely inseparable case, characterizations of the adjacent, λ-, and μ-extensions are obtained in Theorems (3.19), (3.22) and (3.25), respectively. Examples (3.31) and (3.34) gives explicit examples of such extensions. The chapter concludes with a study of the case of a Galois field extension. For this case, the classification of the adjacent, λ-, and μ-extensions is contained in Theorems (3.35) through (3.38), in which it is shown that all three properties can be characterized by the Galois group. It turns out that in the Galois case, the notions of λ-extension and μ-extension are equivalent. Moreover, for an infinite-dimensional Galois field extension, all three notions of λ-extension, μ-extension and J-extension are equivalent.

The final Chapter begins with results on the conductor ideal of an arbitrary λ-extension $R \subseteq T$ (Theorem (4.2)), which parallel corresponding results for adjacent extensions obtained by Ferrand and Olivier [FO] and Modica [Mo]. Proposiion (4.3) and Theorems (4.5) and (4.7) contain results on λ-extensions $R \subseteq T$ of integral domains which indicate the importance of the case in which T is an overring of R. By way of contrast, Example (4.6) shows that an integral domain λ-extension T of an integral domain R need not, in general, be an overring of R. The general problem of classifying the λ-extension overrings T of an arbitrary integral domain R remains open, but such a classification is obtained for two classes of integral domains (Corollary (4.10) and Proposition (4.11)): one-dimensional Prüfer domains satisfying property (#), in the sense of [G1], and principal ideal domains. The chapter ends with the consideration of a useful class of examples of λ-extensions (Propositions (4.12) and (4.14) and Corollary (4.13)), which is used to show the difficulty of generalizing the μ-property for field extensions to arbitrary ring extensions.

CHAPTER I

THE TOP RING IS A FIELD

In this chapter, we consider λ-extensions $R \subseteq K$ such that K is a field. The main motivating example for this chapter arises in case R is a valuation domain, since it is well known that the set of overrings of any valuation domain is linearly ordered by inclusion. The analogous question, of which extensions $R \subseteq K$ (where K is any field containing R) are adjacent extensions, has a well-known answer: R is a rank one valuation domain with quotient field K or $R \subseteq K$ is an adjacent extension of fields. In [GHu], Gilmer and Huckaba discuss related questions concerning Δ-domains and show, in particular, that the integral closure of any Δ-domain is a Prüfer domain ([GHu, Theorem 4]). One would hope to say more for λ-extensions $R \subseteq K$ such that K is a field, since any λ-extension is a Δ-extension. In fact, we do so in Proposition (1.3), which is the λ-analogue of [GHu, Corollary 1 and Theorem 4]. Although part of (1.3) is a special case of [GHu, Corollary 1], we provide a complete proof based on the λ-property. For this proof, we need the following two lemmas.

(1.1) LEMMA. *Let $R \subseteq T$ be a λ-extension of rings. Then T is a P-extension of R and so, by (I.2), $R \subseteq T$ is an INC-pair. More generally, the conclusion holds for any Δ-extension $R \subseteq T$. In particular, if $R \subseteq T$ is a Δ-extension, then T is an algebraic extension of R.*

Proof. An examination of the proof of [GHu, Lemma 3] reveals that each element of T satisfies a polynomial in $R[X]$ with unit content (in fact the coefficient of X^5 is 1). ■

(1.2) LEMMA. *Let $R \subseteq T$ be a λ-extension of rings such that R is integrally closed in T. If u is a unit of T, then either $u \in R$ or $u^{-1} \in R$.*

Proof. The rings between R and T are comparable under inclusion, and so either $R[u] \subseteq R[u^{-1}]$ or $R[u^{-1}] \subseteq R[u]$. By [Bo1, Exercise 5, p.355], $R[u] \cap R[u^{-1}]$ is an integral extension of R contained in T. As R is integrally closed in T, we have $R[u] \cap R[u^{-1}] = R$. Thus, either $R[u] = R$ or $R[u^{-1}] = R$. Hence either $u \in R$ or $u^{-1} \in R$. ■

(1.3) PROPOSITION. *Let $R \subseteq K$ be a λ-extension such that K is a field. Then either*

(1) R is a field; or

(2) R is not a field, K is the quotient field of R and the integral closure R' of R is a valuation domain (and so, by (I.3)(ii), R is a quasilocal i-domain).

Proof. Assume that R is not a field. Let A be the integral closure of R in K and let F be the quotient field of R, viewed inside K. By hypothesis, either $A \subseteq F$ or $F \subseteq A$. In the latter case F is an integral extension of R, which is impossible for an integral domain R which is not a field. Hence $A \subseteq F$.

By (1.1), K is an algebraic extension of R and so $K = A_{R\setminus\{0\}}$, whence the quotient field of A is K. Since $A \subseteq F$, F is a field between A and its quotient field. Hence $K = F$. Thus K is the quotient field of R, and $A = R'$, where R' is the integral closure of R. Since $R \subseteq K$ is a λ-extension, the subextension $R' \subseteq K$ is also a λ-extension. Since R' is integrally closed in K, (1.2) implies that, for every nonzero element u of K, either $u \in R$ or $u^{-1} \in R$. Hence R' is a valuation domain. ■

The first case of Proposition (1.3), in which both rings of the λ-extension are fields, will be studied further in Chapter III. In the present chapter, we restrict our attention to the second case of Proposition (1.3). This case motivates the following definition.

(1.4) DEFINITION. If R is an integral domain, we say that R is a *λ-domain* if the set of all the overrings of R is linearly ordered by inclusion.

Note that any field is a λ-domain.The next result is the analogue of [GHu, Corollary 2] for λ-domains.

(1.5) COROLLARY. *An integrally closed domain R is a λ-domain if and only if it is a valuation domain.*

Proof. Any valuation domain is a λ-domain. Conversely, (1.3) implies that any integrally closed λ-domain R is a valuation domain, since $R = R'$. ■

A natural question is whether the class of λ-domains properly contains the class of valuation domains and is properly contained in the class of quasilocal i-domains. In order to answer this question (in (1.7)), we first consider the classical D+M construction.

(1.6) PROPOSITION. *Let (V,M) be a valuation domain containing a field F such that $V = F+M$. Let D be a proper subring of F and set $R = D+M$. Then:*
(a) R is a valuation domain if and only if D is a valuation domain with quotient field F.
(b) R is a quasilocal i-domain if and only if D is quasilocal and $D \subseteq E$ is an i-extension for each ring E contained between D and F.
(c) R is a λ-domain if and only if $D \subseteq F$ is a λ-extension.
If, in addition, D is a proper subfield of F then: (i) R is not a valuation domain;
(ii) R is a quasilocal i-domain if and only if F/D is an algebraic field extension; and
(iii) R is a λ-domain if and only if $D \subseteq F$ is a (necessarily algebraic) λ-extension of fields.

Proof. (a) is a standard result; see [G2, Appendix 2]. [P, Proposition 2.22] states the assertions in (b) without the quasilocality conditions. But, by [G2, Appendix 2], R is quasilocal if and only if D is quasilocal.

By [BG,Theorem 3.1], the overrings of R are of two types: the overrings of V; and the rings $B+M$ where B ranges over the rings contained between D and F. Since each of the rings $B+M$ is contained in V and the overrings of V are linearly ordered by inclusion, R is a λ-domain if and only if the set $\{B+M : D \subseteq B \subseteq F\}$ is linearly ordered by inclusion. If $D \subseteq B_1, B_2 \subseteq F$ then, since the sums B_i+M are direct, $B_1+M \subseteq B_2+M$ if and only if $B_1 \subseteq B_2$. Hence the set $\{B+M : D \subseteq B \subseteq F\}$ is linearly ordered by inclusion if and only if the set $\{B : D \subseteq B \subseteq F\}$ is linearly ordered by inclusion. This proves (c).

Suppose next that D is a proper subfield of F. Statement (i) follows immediately from (a). Suppose that F is an algebraic field extension of D. Then every ring E between D and F is a field. Since any extension $D \subseteq E$ of fields is an i-extension, (b) implies that R is a quasilocal i-domain. Conversely, suppose R is a quasilocal i-domain. Then, for each ring E contained between the fields D and F, $D \subseteq E$ is an i-extension; since Spec(D) = $\{0\}$, it follows that Spec(E) = $\{0\}$, and so E is a field. This shows that every ring between D and F is a field. Thus F is an algebraic extension of D. This proves (ii). In view of (1.1), (iii) follows immediately from (c). ■

(1.7) EXAMPLES. (a) Let $F = \mathbf{Q}(\sqrt{2})$ and $V = F[[X]] = F+M$, where $M = XF[[X]]$. Let $D = \mathbf{Q}$ and set $R = D+M$. Since $[F:D] = 2$, F is a λ-extension of D (in fact, an adjacent extension of D). By (i) and (iii) of (1.6), R is a λ-domain which is not a valuation domain.

(b) Let $F = \mathbf{Q}(\sqrt{2}, \sqrt{3})$ and let $V = F[[X]] = F+M$, with $M = XV$; $D = \mathbf{Q}$; and $R = D + M$. Then F is an algebraic extension of D, but is not a λ-extension of D since $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ are intermediate fields which are not comparable. By (ii) and (iii) of (1.6), R is a quasilocal i-domain which is not a λ-domain.

(1.8) REMARK. The generalized D+M construction of Brewer and Rutter [BR] does not yield a larger class of examples of λ-domains when D is a field. Indeed, let T be an integral domain, M a maximal ideal of T, and F a subfield of T such that $T = F+M$. Let D be a proper subring of F and set $R = D+M$. If T is integrally closed then, by [BR, Remark, p.35], the integral closure of R is $J+M$, where J is the integral closure of D in F. Hence, R is a quasilocal i-domain if and only if $J+M$ is a valuation domain. Since a valuation domain is precisely a quasilocal Bézout domain, [BR, Theorem 7 and Proposition 8] together show that $J+M$ is a valuation domain if and only if T and D are valuation domains with F the quotient field of D. Thus, if R is a λ-domain (more generally, if R is a quasilocal i-domain), then T is a classical D+M construction.

The next result provides a characterization of λ-domains and indicates what conditions need to be imposed on a quasilocal i-domain R to ensure that R is a λ-domain.

(1.9) THEOREM. *Let R be an integral domain with integral closure R'. Then R is a λ-domain if and only if :*
(*) R is a quasilocal i-domain.
(**) All the overrings of R are comparable to R' under inclusion.
(***) The set of rings between R and R' is linearly ordered by inclusion.

Proof. ($\Rightarrow$) Any field is a λ-domain and satisfies the properties (*), (**) and (***). Hence, the implication follows immediately from (1.3) and the definition of a λ-domain.

($\Leftarrow$) Suppose that (*), (**) and (***) hold. Let A and B be overrings of R. If A and B are each contained in R' then, by (***), they are comparable. If A and B each contain R' then A and B are comparable, since R' is a valuation domain by (*) and any valuation domain is a λ-domain, . By virtue of (**), the only other option is that one of A,B is contained in R' and the other contains R'; in this case A and B are comparable. Thus, the set of overrings of R is linearly ordered by inclusion; that is, R is a λ-domain. ■

(1.10) REMARK. A quasilocal integrally closed domain R may fail to be a λ-domain and yet have a nontrivial overring T such that $R \subseteq T$ is a λ-extension. Indeed, let D be a valuation domain properly contained in its quotient field L and let $F = L(X)$, where X is an indeterminate. Let $V = F[[X]]$, $M = XV$ and $R = D + M$. By [G2, Appendix 2], R is a quasilocal integrally closed domain, since D is quasilocal and integrally closed in F. Since F is not the quotient field of D, (1.3) implies that $D \subseteq F$ is not a λ-extension; hence, by (1.6)(c), R is not a λ-domain.

Now, set $T = L + M$. By [BG, Theorem 3.1], the rings contained between R and T are in one-to-one inclusion-preserving correspondence with the rings contained between D and L. The latter set of rings is linearly ordered by inclusion, as D is a valuation domain with quotient field L, and so $R \subseteq T$ is a λ–extension.

Corollary (1.12) provides another condition which may be imposed on a quasilocal i-domain to ensure that it is a λ-domain. Its proof depends on a result of Gilmer and Heinzer, which is paraphrased, using the notation and terminology of this paper, in the next proposition.

(1.11) PROPOSITION. (Gilmer-Heinzer, [GHe2, Theorem 2.4]) *Let (R,M) be a quasilocal integral domain with integral closure R'. If $R \subseteq R'$ is an adjacent extension and R' is a Prüfer domain, then R' is contained in every proper overring of R.*

(1.12) COROLLARY. *If R is a quasilocal i-domain and $R \subseteq R'$ is an adjacent extension, then R is a λ-domain.*

Proof. R clearly satisfies the conditions (*) and (***) of (1.9). Since R' is a valuation domain and hence a Prüfer domain, (1.11) implies that R also satisfies condition (**) of (1.9). Thus R is a λ-domain. ■

The converse of (1.12) is false. Indeed, as (1.13)(a) shows, if R is a λ-domain, then $R \subseteq R'$ need not be an adjacent extension.

(1.13) EXAMPLES. (a) Let F be a cyclic field extension of $\mathbf{Q}$ of degree 4 and let $V = F[[X]]$. Let $D = \mathbf{Q}$ and set $R = D+M$, where $M = XF[[X]]$. By the Galois correspondence, $D \subseteq F$ is a λ-extension with one proper intermediate field. Then, by (1.6)(iii), R is a λ-domain with $R' = V$ and $R \subseteq R'$ is not an adjacent extension.

(b) The ring R constructed in Example (1.7)(a) is an example of a ring satisfying the hypotheses of Corollary (1.12).

(1.14) REMARKS. It is natural to ask whether any of the three conditions (*), (**) and (***) of (1.9) is redundant, that is, implied by the other two conditions.

(i) (*) is not redundant, since any integrally closed domain which is not a valuation domain trivially satisfies (**) and (***), but fails to satisfy (*).

(ii) In [GHu, pp. 429-430], Gilmer and Huckaba discuss the following class of examples. Let K be a field. For each $n \geq 0$, let $D_n = K + X^nK[[X]]$. They show that each D_n satisfies (*) and (**). By [GHu, Proposition 11], D_n is a λ-domain if and only if $n \leq 3$. Thus, for $n \geq 4$, (1.9) ensures that D_n fails to satisfy (***), and so (***) is not redundant.

Moreover, by [GHu, Proposition 10], D_n is a Δ-domain if and only if $n \leq 5$. Thus, D_4 and D_5 are examples of integral domains which are quasilocal i-domains and Δ-domains, but which are not λ-domains. Observe that each D_n is Noetherian by Eakin's Theorem [Ea, Theorem 2], since $K[[X]] = D_n + XD_n + X^2D_n + \ldots + X^nD_n$ is a Noetherian extension ring of D_n which is finitely generated as a D_n-module.

(iii) We do not know an example showing that (**) is not redundant. However, the next two propositions contain more information about the property (**).

(1.15) PROPOSITION. *Let R be an integral domain not equal to its integral closure R'. If R satisfies (**) of (1.9), then any overring of R which is properly contained in R' is quasilocal.*

Proof. Any such ring S inherits the property (**) from R, since $S' = R'$, and so S satisfies the same hypotheses as R. Hence it suffices to prove that R is quasilocal. Assume not. Then for any maximal ideal M of R, R_M is not an integral overring of R (cf. Kaplansky [K, Exercise 10, p. 24]), and so $R_M \not\subset R'$. Since R satisfies (**), it follows that $R' \subseteq R_M$. Then $R' \subseteq \cap\{R_M : M \in \operatorname{Max} R\} = R$, the desired contradiction. ■

(1.16) PROPOSITION. *(1) Any Noetherian treed domain has dimension ≤ 1.*
(2) Any Noetherian i-domain has dimension ≤ 1.

(3) If a quasilocal i-domain has dimension ≤ 1, then it satifies (**) of (1.9). Thus, if R is a quasilocal i-domain, then: R Noetherian $\Rightarrow$ dim R $\leq 1 \Rightarrow$ R satisfies (**).

Proof. (1) As observed in the proof of [D1, Corollary 2.3], this assertion follows from the Principal Ideal Theorem (cf. Kaplansky [K, Theorem 144]).

(2) By (I.3)(i), any i-domain is treed. Hence, (2) follows from (1).

(3) Let R be a quasilocal i-domain of dimension ≤ 1. If dim $R = 0$, then R is a field and (**) holds trivially. Suppose that dim $R = 1$. Let $V = R'$, the integral closure of R. Since dim V = dim R, V is a rank one valuation domain. Let T be an arbitrary overring of R other than the quotient field. Then the integral closure of T is a valuation domain containing V. Since V has rank one, its only overrings are itself and the quotient field. Hence the integral closure of T must be V. It follows that $T \subseteq V$, thus proving the first assertion in (3). The final assertion follows immediately from (2) and (3). ■

(1.17) REMARK. By the preceding proposition, any Noetherian local i-domain R satisfies (*) and (**) of (1.9). However, such a ring R need not satisfy (***), and so need not be a λ-domain. Indeed, the rings D_n (n≥4) discussed in (1.13) provide a counterexample.

Additional counterexamples may be obtained from the classical D+M construction, as follows. In terms of the notation of (1.6), if D is a proper subfield of F, it is well-known that R is Noetherian if and only if V is Noetherian and $[F{:}D] < \infty$ (see [G2, Appendix 2]). Combining this with (1.6)(ii), we see that if D is a field, then R is a Noetherian local i-domain if and only if V is Noetherian and $[F{:}D] < \infty$. Thus, Example (1.7)(b) provides an example of a Noetherian local i-domain which is not a λ-domain.

By (I.3)(ii) and (1.5), the properties "quasilocal i-domain" and "λ-domain" reduce to the property "valuation domain" for integrally closed domains R. It is natural to consider the effect of weaker integrality-type conditions when combined with these two properties. In particular, we shall investigate the property of seminormality (see (I.4)) in this context. The next example shows that seminormality is not enough to ensure that a quasilocal i-domain is a λ-domain.

(1.18) EXAMPLE. Let V, M, F, D and $R = D + M$ be as in (1.6). According to [ADHu, Lemma 2.1(g)], R is seminormal if and only if D is seminormal. In particular, R is seminormal if D is a field. Suppose that F/D is an algebraic field extension. By (1.6)(ii), R is a quasilocal i-domain. Moreover, all the overrings of R are seminormal. This can be seen by applying [ADHu, Theorem 2.3]. For a more direct proof, notice by [BG, Theorem 3.1], that the overrings of R are of two types: the overrings of V; and the rings $B+M$ where $D \subseteq B \subseteq F$. Now, the overrings of V are valuation domains and so are seminormal. Since F/D is an algebraic extension, each ring B such that $D \subseteq B \subseteq F$ is a field. Hence $B+M$ is seminormal. It follows that the ring R in example (1.7)(b) is a quasilocal i-domain all of whose overrings are seminormal, but which is not a λ-domain.

Since any quasilocal i-domain is a quasilocal going-down domain (see (I.5)(ii)), we are motivated to consider some results of Dobbs connecting seminormality, going-down domains and divided domains (see (I.6)).

The proof of [D2, Corollary 2.8] actually establishes the following more general results. Any locally divided integral domain is a going-down domain and any seminormal going-down domain is locally divided. Since, by (I.4) and (I.5)(i), the properties of seminormality and going-down domains are local properties of integral domains, the above results may be restated in local form. We record this reformulation as part (a) of the next proposition. We will also need [D2, Theorem 2.10(b)], which we record as part (b) of the next proposition.

(1.19) PROPOSITION. *(a) Any divided domain is a quasilocal going-down domain and any seminormal quasilocal going-down domain is a divided domain.*
(b) Let R be a going-down domain and suppose that R has an overring T which is a divided domain and which is a finitely generated R-module. If P is a prime ideal of R such that R_P is integrally closed, then P is a divided prime of R.

(1.20) REMARK. The converse of the first statement in (1.19)(a) is false; that is, a quasilocal going-down domain need not be a divided domain. In fact, [D2, Example 2.9] gives an example of a quasilocal i-domain which is not a divided domain.

(1.21) LEMMA. *Let R be a quasilocal i-domain which satisfies (**) of (1.9). If P is a nonmaximal prime ideal of R, then $R' \subseteq R_P$ and so R_P is a valuation domain.*

Proof. R_P is a proper overring of R which is not integral over R, and so $R_P \not\subset R'$. Since R satisfies (**), $R' \subseteq R_P$. As R' is a valuation domain, its overring R_P is also a valuation domain. ∎

(1.22) THEOREM. *Let R be a quasilocal i-domain which satisfies (**) of (1.9). If R satisfies either of the following two conditions then R is a divided domain:*
(i) R is seminormal;
(ii) The integral closure R' of R is a finitely generated R-module.

Proof. By (I.5)(ii), R is a quasilocal going-down domain. If R satisfies (i), then (1.19)(a) gives the desired conclusion.

Assume that R satisfies (ii). We must show that each prime ideal of R is a divided prime of R. Since R is quasilocal, its maximal ideal is divided (by (I.6)). Let P be a nonmaximal prime ideal of R. By (1.21), R_P is a valuation domain and so is integrally closed. Also, R' is a divided domain, since it is a valuation domain. Hence, by (1.19)(b), P is a divided prime of R. Since every prime ideal of R is divided in R, R is a divided domain. ∎

(1.23) REMARK. (i) The most natural application of (1.22) asserts that if R is a λ-domain which either is seminormal or has finitely generated integral closure, then R is a divided domain. In the same way, (1.25) leads to a result on divided λ-domains, whose formulation is left to the reader.

(ii) A divided domain, though necessarily a quasilocal going-down domain, need not be an i-domain. Using the notation of (1.6), we may construct an example using the D+M construction. By [D2, Lemma 2.2], $R = D + M$ is a divided domain if and only if D is divided. In particular, R is divided if D is a field. Hence, by (1.6)(iii), if D is a field and F/D is a transcendental field extension, then R is a divided domain which is not an i-domain.

(iii) We can use the classical D+M construction to show that condition (ii) of (1.22) is not necessary in order that a quasilocal i-domain R be divided. By the preceding paragraph, $R = D + M$ is a divided domain if D is a field. If F/D is an infinite-dimensional algebraic field extension, then R is a divided quasilocal i-domain with integral closure $V = F + M$. By [BG, Theorem 3.1], R satisfies property (**) of (1.9). However, $V = R'$ is not a finitely generated R-module since F is not a finite extension of D, and so (ii) of (1.22) is not satisfied.

Pseudovaluation domains (PVDs) are a special class of divided domains. The connections between PVDs and quasilocal i-domains have been explored implicitly in several papers, including [HH2], [D3] and [ADHu]. The next result is a special case of [ADHu, Lemma 3.4]. For completeness, we provide a direct proof.

(1.24) LEMMA. *Let (R,M) be a quasilocal i-domain with (quasilocal) integral closure (R',M'). Then R is a pseudovaluation domain if and only if $M = M'$.*

Proof. If $M = M'$, then M is the maximal ideal of the valuation overring R' of R. By [HH1, Theorem 2.7], R is a PVD. Conversely, suppose that R is a PVD. Since R' is a valuation domain, [D3, Remark 4.8(a)] implies that R' is the associated valuation domain of R. Hence, R and R' have the same maximal ideal; that is, $M = M'$. ■

(1.25) THEOREM. *Let (R,M) be a quasilocal i-domain which satisfies (**) of (1.9). Suppose that R is a divided domain. Let (R',M') denote the integral closure of R. Then the set of all nonmaximal prime ideals of R coincides with the set of all nonmaximal prime ideals of R'. Moreover, if R is not a field, then exactly one of the following two conditions holds:*

(i) Both R and R' have a (unique) largest nonmaximal prime ideal; or

(ii) In both R and R′, the maximal ideal is the union of the nonmaximal prime ideals.

If condition (ii) holds, then R is a pseudovaluation domain.

Proof. Let P be a nonmaximal prime ideal of R. By (1.21), R_P is a valuation domain containing R'. Since P is a divided prime of R, the maximal ideal of R_P is $PR_P = P$. It follows that $P = P \cap R' = PR_P \cap R'$ is a prime ideal of R'. Since R is an i-domain and R' is an integral overring of R, R' is a unibranched extension of R; that is, there is a unique prime ideal of R' lying over any given prime ideal of R. This argument shows that, for each nonmaximal prime ideal P of R, P is the unique prime ideal of R' lying over P. Since integrality ensures that each nonmaximal prime ideal of R' must contract to a nonmaximal prime ideal of R, we have shown that the nonmaximal primes of R and R' coincide.

Note that this common set of nonmaximal prime ideals forms a chain, since R is a valuation domain. Thus the union Q of all the nonmaximal prime ideals is a prime ideal of both R and R'. Clearly, at most one of (i) and (ii) holds. To show that one of (i) and (ii) holds, it is enough to show that if $Q = M$, then $Q = M'$. If $Q = M$, then M is a prime ideal of R' which contracts to M, whence $M = M'$ and $Q = M'$.

If (ii) holds, then $Q = M = M'$. It then follows, by (1.24), that R is a pseudovaluation domain. ■

(1.26) REMARKS. (i) Any pseudovaluation domain has exactly the same prime ideals as its associated valuation domain. Theorem (1.25) shows that an integral domain satisfying the hypotheses of the theorem satisfies a slightly more general condition. A PVD need not satisfy the hypotheses of the theorem (although any PVD is a divided domain). For example, the ring R discussed in Remark (1.23)(ii) is a PVD which is not a quasilocal i-domain The question of which PVD's are quasilocal i-domains and which satisfy (**) of (1.9) is considered in Proposition (1.27) and Theorem (1.31).

(ii) The hypotheses of (1.25) do not imply that R is a PVD. Perhaps the simplest example to illustrate this is given by $R = D_n$, for $n \geq 2$, in the notation of (1.14)(ii) and [GHu, pp. 429-430].

(1.27) PROPOSITION. *Let R be a pseudovaluation domain (PVD) which is not a valuation domain. Let K be the quotient field of R, M the maximal ideal of R, and V the associated valuation domain of R. Then:*

(A) The following four conditions are equivalent:

(a) R is coherent;

(b) R is a finite conductor domain;

(c) M is a finitely generated ideal of R;

(d) Every overring of R is a coherent PVD.

(B) The following four conditions are equivalent:

(i) R is a (quasilocal) i-domain (and so $V = R'$);

(ii) Every overring of R is a PVD i-domain;

(iii) Every overring of R is a PVD;

(iv) R satisfies the condition: for all $x \in K \backslash R$, $x^{-1} \in R'$.

Moreover, the equivalent conditions of (A) imply the equivalent conditions of (B).

Proof. (A) The equivalence of (a), (b) and (c) is given by [HH2, Theorem 1.6] and [D4, Proposition 3.5]. The implication (a) $\Rightarrow$ (d) is given by [HH2, Theorem 1.9] and [D4, Corollary 3.6]. The implication (d) $\Rightarrow$ (a) is obvious.

(B) The equivalence (i) $\Leftrightarrow$ (iii) is given by [HH2, Proposition 2.7]. Clearly, (ii) $\Rightarrow$ (i). If (i) holds and S is an overring of R, then S is a quasilocal i-domain by (I.3)(iii); and since $R \subseteq S$ is an i- (hence, INC-) extension, it follows from [HH1, Theorem 1.7] that S is a PVD. Hence (i) $\Rightarrow$ (ii).

If (iv) holds and $x \in K \backslash R'$, then $x \in K \backslash R$ and so $x^{-1} \in R'$. It follows that (iv) implies that R' is a valuation domain; that is, (iv) $\Rightarrow$ (i). Conversely, suppose that (i) holds. By [D3, Remark 4.8(a)], $V = R'$, and so M is the maximal ideal of R'. If $x \in K \backslash R$, then $x \notin M$, whence $x^{-1} \in R'$. Thus (i) $\Rightarrow$ (iv).

For the "moreover" assertion, it is enough to observe that condition (A)(d) trivially implies condition (B)(iii). ■

The following proposition characterizes which PVD i-domains are λ-domains. It is stated using the terminology and notation of this paper.

(1.28) PROPOSITION. (Gilmer-Huckaba [GHu, Proposition 9]) *Let R be a PVD i-domain. Then the following conditions are equivalent:*
(a) R is a λ-domain;
(b) R' is a Δ-extension of R;
(c) R'/M is a λ-extension of R/M, where M is the maximal ideal of R and R'.

(1.29) REMARK. In the proof of [D4, Corollary 3.6], Dobbs observes that if a PVD, R, which is not a valuation domain satisfies the equivalent conditions in (A) above, then R satisfies (**) of (1.9). This fact can also be derived from [HH2, Theorem 1.9] by observing that $V = R'$.

This leads one to ask which PVD's satisfy (**). An answer to this question is provided by Theorem (1.31), whose proof uses the following lemma.

(1.30) LEMMA. *Let $K \subseteq L$ be an extension of fields and let F be the algebraic closure of K in L. Then every ring contained between K and L compares with F under inclusion if and only if either $F = K$ or $F = L$.*

Proof. ($\Leftarrow$) is clear. Conversely, suppose that every intermediate ring compares with F. Assume that $F \neq L$ and choose $x \in L \backslash F$. Then x is transcendental over K. By hypothesis, since $x \notin F$, it must be the case that $F \subseteq K[x]$. Since F is algebraic over K and $K[x]$ is a purely transcendental extension of K, it follows that $F = K$. ∎

(1.31) THEOREM. *Let R be a pseudovaluation domain. Then R has property (**) of (1.9) if and only if either R is integrally closed or R is an i-domain.*

Proof. Let V be the associated valuation domain of R, M the maximal ideal of R, and R' the integral closure of R.

($\Rightarrow$) Suppose that R has property (**). Then $R \subseteq R' \subseteq V$ and M is the maximal ideal of all three rings. Since R has the property (**), all the rings between R and V compare to R'. Thus, all the rings between R/M and V/M compare to R'/M. But R'/M is the algebraic closure of R/M in V/M. Hence, by (1.30), either $R'/M = R/M$ or $R'/M = V/M$. In other words, either $R' = R$ or $R' = V$; thus, either R is integrally closed or R is an i-domain.

($\Leftarrow$) If R is integrally closed, then it satisfies (**) trivially. Suppose, instead, that R is a (quasilocal) i-domain and, without loss of generality, not a valuation domain. Then $V = R'$, and so M is the maximal ideal of both R and R'. Then $M = (R{:}R')$, the conductor of R in R'. In particular, $M = rad_{R'}(R{:}R')$, and so, by [GHu, Proposition 8], R has the property (**). ■

CHAPTER II

DIRECT PRODUCTS AND λ-EXTENSIONS

The main concern of this chapter is the following question. If $R \subseteq T$ is a λ-extension of rings and T can be written nontrivially as a direct product of nonzero rings, then what can be said about the number and the behavior of the factor rings of T? This question is answered in Proposition (2.8) and Theorem (2.12).

The original motivation for the above question was our need for a result, contained in Corollary (2.14), which is needed in the proof of Theorem (3.4) in Chapter III. As a side benefit, we recover, in Theorem (2.13), a result of Ferrand and Olivier ([FO, Lemme 1.5]). The Definitions (2.1) and (2.3) and the Propositions (2.2) and (2.5) introduce some basic terminology which will be useful in the statements of the subsequent results in this chapter.

(2.1) DEFINITION. Let $R \subseteq T$ be a ring extension. A subset $I \subseteq T$ will be called an *ideal of (the extension)* $R \subseteq T$ if there is a ring A contained between R and T such that I is an ideal of A. If I is a <u>proper</u> ideal of some intermediate ring, then I will be called a *proper ideal of (the extension)* $R \subseteq T$.

Notice that the rings contained between R and T are included among the ideals of $R \subseteq T$. In fact, they are exactly the ideals of $R \subseteq T$ which contain 1. A useful characterization of the ideals of $R \subseteq T$ is given by the following proposition.

(2.2) PROPOSITION. *Let $R \subseteq T$ be a ring extension. A subset $I \subseteq T$ is an ideal of $R \subseteq T$ if and only if I is an R-submodule of T which is closed under multiplication.*

Proof. If I is an ideal of a ring A such that $R \subseteq A \subseteq T$, then I is an A-module and hence an R-module. The "only if" assertion now follows, since any ideal of a ring is closed under multiplication. Conversely, if I is an R-submodule of T which is closed under multiplication, then $A = R + I$ is a ring contained between R and T and I is an ideal of A. ■

(2.3) DEFINITION. Given an extension $R \subseteq T$ of rings and an ideal J of R, we say that T is a *λ-extension of R with respect to J* if the collection of all the ideals of the extension $R \subseteq T$ which contain J is linearly ordered by inclusion.

(2.4) REMARK. The above terminology is justified by the following observations.

(i) Let J_1 and J_2 be ideals of R such that $J_1 \subseteq J_2$. It follows from (2.3) that if T is a λ-extension of R with respect to J_1, then T is a λ-extension of R with respect to J_2.

(ii) An ideal I of $R \subseteq T$ contains R precisely if I is a ring contained between R and T. Hence, T is a λ-extension of R with respect to R if and only if T is a λ-extension of R.

(iii) Let J be an ideal of R. It follows from (i) and (ii) that if T is a λ-extension of R with respect to J, then T is a λ-extension of R.

(2.5) PROPOSITION. *Let $R \subseteq T$ be a ring extension and let J be an ideal of R. Then T is a λ-extension of R with respect to J if and only if the following three conditions hold:*

(1) T is a λ-extension of R;

(2) The set of all the ideals of R containing J is linearly ordered by inclusion (that is, R/J is a chained ring);

(3) If I is a proper ideal of $R \subseteq T$ and if $J \subseteq I$, then $I \subseteq R$.

Proof. ($\Rightarrow$) Let T be a λ-extension of R with respect to J. By (2.4)(iii), T is a λ-extension of R. Condition (2) follows from Definition (2.3). For (3), let I be a proper ideal of $R \subseteq T$ containing J. The ring R is an ideal of $R \subseteq T$ containing J and $R \not\subseteq I$; hence, by hypothesis, $I \subseteq R$.

($\Leftarrow$) Suppose that conditions (1), (2) and (3) hold. Let I_1 and I_2 be any two ideals of $R \subseteq T$ containing J. If I_1 and I_2 are each proper ideals of $R \subseteq T$, then (3) implies that I_1 and I_2 are ideals of R containing J; hence, by (2), I_1 and I_2 are comparable. If neither I_1 nor I_2 is proper, then each of them is a ring contained between R and T; hence, by (1), I_1 and I_2 are comparable. If, say, I_1 is proper and I_2 is not proper, then I_2 is a ring contained between R and T and, by (3), $I_1 \subseteq R$; then

$I_1 \subseteq R \subseteq I_2$. Thus T is a λ-extension of R with respect to J. ∎

(2.6) REMARK. (i) Note that conditions (2) and (3) of (2.5) hold trivially in case $J = R$ (cf. Remark (2.4)(ii)).

(ii) Each of the conditions (1), (2) and (3) of (2.5) is independent of the other two conditions, as is illustrated by the following examples.

(a) Example (1.7)(b) provides an example showing that (1) is independent. Indeed, let F, V, M, D and R be as in (1.7)(b), and set $T = V$ and $J = M$. Then condition (2) of (2.5) holds automatically, since J is a maximal ideal of R. As for condition (3), note that, by [BG, Theorem 3.1], the rings contained between R and T are the rings $B + M$ where B is a ring contained between D and F. Any such ring B is a field, and so M is the unique maximal ideal of each ring contained between R and T. Consequently, any proper ideal of the extension $R \subseteq T$ is contained in M and so is contained in R. Hence, condition (3) of (2.5) is satisfied. Finally, note that condition (1) of (2.5) fails to hold since, for example, $\mathbf{Q}(\sqrt{2}) + M$ and $\mathbf{Q}(\sqrt{3}) + M$ are rings contained between R and T which are not comparable under inclusion.

(b) Example (1.7)(a) contains an extension $R \subseteq T$ of rings which, for some ideal J of R, satisfies conditions (1) and (3) of (2.5), but fails to satisfy condition (2). In fact, let R be as in (1.7)(a), let T be the ring V in (1.7)(a) and let J be the zero ideal of R. It follows from (1.7)(a) and (1.24) that R is a PVD λ-domain, T is the integral closure and associated valuation domain of R, and T is an adjacent extension of R. The extension $R \subseteq T$ satisfies conditions (1) and (3) of (2.5), since any adjacent extension is a λ-extension and any proper ideal of T is contained in R. However, condition (3) of (2.5) fails since $R/J = R$ is not a valuation domain (it is not integrally closed).

(c) Let K be a field and let $R = K$, $T = K \times K$ and $J = 0$. If we regard T as an extension ring of R by the diagonal map ($r \mapsto (r,r)$ for all $r \in R$), then by [FO, Lemme 1.2(b)], T is an adjacent extension of R. Hence, T is a λ-extension of R, and so $R \subseteq T$ satisfies condition (1) of (2.5). Since R is a field, condition (2) holds trivially. However, condition (3) fails to hold since, for example, $K \times 0$ is a proper ideal of T containing J which is not contained in R.

Because of Remark (2.4)(ii), we already have examples of extensions $R \subseteq T$ satisfying Definition (2.3) for $J = R$, namely, any λ-extension $R \subseteq T$. As Example (2.7) shows, there are nontrivial examples of ring extensions $R \subseteq T$ satisfying Definition (2.3) for some proper ideal J of R.

(2.7) EXAMPLE. Let V be a valuation domain and W a proper overring of V. It is well known that any proper ideal of an overring of V is contained in V. Since any valuation domain is a λ-domain and a chained ring, it follows from Proposition (2.5) that W is a λ-extension of V with respect to $\{0\}$. Hence, by Remark (2.4)(i), W is a λ-extension of V with respect to J, for every ideal J of V.

The next result contains a partial answer to the question posed at the beginning of this chapter. Specifically, it shows that under certain conditions on a λ-extension $R \subseteq T$, the number of nonzero factors in a direct product decomposition of T is greatly restricted.

(2.8) PROPOSITION. *Let $R \subseteq T$ be a λ-extension and suppose that $T = \prod T_i$, where there are at least two factors. Let π_i: $T \to T_i$ be the canonical surjection and let $I_i = \ker(\pi_i) \cap R$, for each i. Suppose that $I_i + I_j$ is a proper ideal of R for each pair i,j of indices. Then the product $\prod T_i$ contains exactly two factors and $R \neq T$.*

Proof. If $T_l = 0$ for some index l, then $I_l = R$ and, in particular, $I_l + I_m = R$ for any index m, contrary to hypothesis. Thus, each ring T_i is nonzero.

Suppose that the hypotheses hold, but the product contains at least three factors. Let i, j and k be three pairwise distinct indices. Define $A = \{t \in T : \exists\, r \in R$ such that $\pi_i(t) = \pi_i(r)$ and $\pi_j(t) = \pi_j(r)\}$ and $B = \{t \in T : \exists\, r \in R$ such that $\pi_i(t) = \pi_i(r)$ and $\pi_k(t) = \pi_k(r)\}$. Then A and B are rings contained between R and T, and so, by hypothesis, either $A \subseteq B$ or $B \subseteq A$. Without loss of generality, suppose that $A \subseteq B$. For an arbitrary element s of R, consider the element t of T which is $\pi_k(s)$ in its k^{th} coordinate and 0 in all its other coordinates. Since $\pi_i(t) = 0$ and $\pi_j(t) = 0$, we have $t \in A$, and so $t \in B$. Thus, there is an element r of R such that $\pi_i(t) = \pi_i(r)$ and $\pi_k(t) = \pi_k(r)$; that is, $\pi_i(r) = 0$ and $\pi_k(r) = \pi_k(s)$. Then, $r \in I_i$ and

$s\text{-}r \in I_k$, and so $s = (s\text{-}r)+r \in I_k + I_i$. As s is arbitrary, $I_k + I_i = R$, the desired contradiction.

Next, suppose that the hypotheses hold, but $R = T$. By the preceding paragraph, the product $\prod T_i$ contains exactly two factors, say, T_1 and T_2. Then $I_1 = 0 \times T_2$ and $I_2 = T_1 \times 0$, and so $I_1 + I_2 = R$, the desired contradiction. ■

(2.9) REMARK. (i) Proposition (2.8) is false without the hypothesis on the sums $I_i + I_j$, even if we assume that each ring T_i is nonzero. Indeed, if we let $R = T = T_1 \times T_2 \times T_3$ for arbitrary nonzero rings T_i, then all the hypotheses of (2.8) except the hypothesis on the sums $I_i + I_j$ are satisfied, but the conclusion fails to hold. (The verification is analogous to the reasoning in the last paragraph of the proof of (2.8).)

(ii) If $R = T$ is a nonzero indecomposable ring, then both conclusions of (2.8) fail. If each ring T_i is assumed nonzero, then the index set is a singleton set, say $\{1\}$, and $I_1 = \{0\} \neq R$; in particular, $I_1 + I_1 = \{0\}$ is a proper ideal of R. Thus, if we insist that each ring T_i is nonzero, the hypothesis that there are at least two factors cannot be removed from (2.8).

Lemma (2.11) and Theorems (2.12) and (2.13) consider extensions $R \subseteq T$ such that T is a product of two nonzero rings. For convenience, we first set up some standard notation.

(2.10) NOTATION. Let $R \subseteq T$ be an extension of rings; let $T = T_1 \times T_2$, where T_1 and T_2 are nonzero rings; and let $\pi_i : T \to T_i$ be the canonical surjections, $i = 1,2$. For $i = 1,2$, set $R_i = \pi_i(R)$ and $I_i = ker\ (\pi_i) \cap R$. Let $J_2 = \pi_2(I_1)$ and $J_1 = \pi_1(I_2)$. It is straightforward to verify that J_2 is an ideal of R_2 and that $\pi_2^{-1}(J_2) \cap R = I_1 + I_2$, with the analogous assertions for J_1 also holding. Observe that R_1 and R_2 are nonzero, since $1 = (1,1) \in R$.

(2.11) LEMMA. *Let the notation be as in (2.10) and suppose that $R_1 = T_1$.*

(1) If A is a ring between R and T, then $A^{\dagger} := \pi_2(ker\ (\pi_1) \cap A)$ is an ideal of the extension $R_2 \subseteq T_2$ containing J_2.

(2) If I is an ideal of $R_2 \subseteq T_2$ containing J_2, then $I^{\dagger} := \{t \in T : \exists\, r \in R$ such that $\pi_1(t-r) = 0$ and $\pi_2(t-r) \in I\}$ is a ring contained between R and T.

(3) The maps $A \mapsto A^{\dagger}$ and $I \mapsto I^{\dagger}$ are inverse maps and are inclusion-preserving. Thus, they define a one-to-one inclusion-preserving correspondence between the set of all rings contained between R and T and the set of all ideals of the extension $R_2 \subseteq T_2$ which contain J_2.

Proof. (1) It is easy to check that $A^{\dagger}$ is an ideal of $\pi_2(A)$ which contains J_2; hence, $A^{\dagger}$ is an ideal of the extension $R_2 \subseteq T_2$ containing J_2.

(2) First note that $I^{\dagger}$ contains R, since for any $r \in R$, $\pi_1(r-r) = 0$ and $\pi_2(r-r) \in I$. Now, suppose that $t, u \in I^{\dagger}$; then, there are elements r, s of R such that $\pi_1(t-r) = 0$ and $\pi_2(t-r) \in I$ and $\pi_1(u-s) = 0$ and $\pi_2(u-s) \in I$. Then, $r+s \in R$ and $\pi_1((t+u)-(r+s)) = \pi_1(t-r) + \pi_1(u-s) = 0$ and $\pi_2((t+u)-(r+s)) = \pi_2(t-r) + \pi_2(u-s) \in I$; hence $t+u \in I^{\dagger}$. Thus $I^{\dagger}$ is closed under addition. For any $a \in R$, $ar \in R$ and $\pi_1(at-ar) = \pi_1(a)\pi_1(t-r) = 0$ and $\pi_2(at-ar) = \pi_2(a)\pi_2(t-r) \in I$; hence $at \in I^{\dagger}$. Thus $I^{\dagger}$ is an R-module.

Let t, u, r and s be as in the preceding paragraph. Then $\pi_1((t-r)(u-s)) = 0$ and $\pi_2((t-r)(u-s)) \in I$, and so $\pi_1((tu-ru-ts)-(-rs)) = 0$ and $\pi_2((tu-ru-ts)-(-rs)) \in I$. Since $-rs \in R$, $tu-ru-ts \in I^{\dagger}$. But, since $I^{\dagger}$ is an R-module, $ru, ts \in I^{\dagger}$, and so $tu \in I^{\dagger}$. This shows that $I^{\dagger}$ is closed under multiplication; hence, $I^{\dagger}$ is a ring contained between R and T.

(3) It is clear from the definitions of $I^{\dagger}$ and $A^{\dagger}$ that the maps $A \mapsto A^{\dagger}$ and $I \mapsto I^{\dagger}$ are each inclusion-preserving. To show that the maps are inverses, we show that $A = A^{\dagger\dagger}$ for each ring A contained between R and T and $I = I^{\dagger\dagger}$ for each ideal of $R_2 \subseteq T_2$ containing J_2.

If $t \in A$, then, since $R_1 = T_1$, there exists $r \in R$ such that $\pi_1(t) = \pi_1(r)$. Then $t-r \in ker\,(\pi_1) \cap A$, and so $\pi_2(t-r) \in A^{\dagger}$. Thus, by definition, $t \in A^{\dagger\dagger}$, and so $A \subseteq A^{\dagger\dagger}$.

Next, suppose that $t \in A^{\dagger\dagger}$. Then there is an element $r \in R$ such that $\pi_1(t-r) = 0$ and $\pi_2(t-r) \in A^{\dagger}$. Thus, by the definition of $A^{\dagger}$, $\pi_2(t-r) = \pi_2(s)$ for some

$s \in ker(\pi_1) \cap A$. Since $\pi_2(t-r) = \pi_2(s)$ and $\pi_1(t-r) = 0 = \pi_1(s)$, it follows that $t-r = s$. Therefore $t = r+s \in A$, whence $A^{\dagger\dagger} \subseteq A$. This implies, by the preceding paragraph, that $A = A^{\dagger\dagger}$.

If $x \in I$, let $t = (0, x) \in T$. Then, $\pi_1(t-0) = 0$ and $\pi_2(t-0) = x \in I$, and so $t \in I^{\dagger}$. Since $t \in ker(\pi_1) \cap I^{\dagger}$, $x = \pi_2(t)$ is an element of $I^{\dagger\dagger}$; hence, $I \subseteq I^{\dagger\dagger}$.

Now, suppose that $x \in I^{\dagger\dagger}$. Then, $x = \pi_2(s)$ for some $s \in ker(\pi_1) \cap I^{\dagger}$. As $s \in I^{\dagger}$, there is an element $r \in R$ such that $\pi_1(s-r) = 0$ and $\pi_2(s-r) \in I$. Note that $\pi_1(r) = \pi_1(s-(s-r)) = \pi_1(s) - \pi_1(s-r) = 0 - 0 = 0$, whence $r \in ker(\pi_1) \cap R = I_1$ and $\pi_2(r) \in \pi_2(I_1) = J_2$. Since, by assumption, $J_2 \subseteq I$, we have $\pi_2(r) \in I$, and so $x = \pi_2(s) = \pi_2(s-r) + \pi_2(r) \in I + I = I$, whence $I^{\dagger\dagger} \subseteq I$. Therefore, by the preceding paragraph, we conclude that $I = I^{\dagger\dagger}$. ■

(2.12) THEOREM. *Let the notation be as in (2.10). Then:*

(a) Suppose that $R \subseteq T$ is a λ-extension. Then (by reindexing, if needed), $R_1 = T_1$ and T_2 is a λ-extension of R_2 with respect to J_2.

(b) If $R_1 = T_1$ and T_2 is a λ-extension of R_2 with respect to J_2, then $R \subseteq T$ is a λ-extension.

Proof. First, suppose that $R \subseteq T$ is a λ-extension. The rings $R_1 \times T_2$ and $T_1 \times R_2$ are contained between R and T, and so either $R_1 \times T_2 \subseteq T_1 \times R_2$ or $T_1 \times R_2 \subseteq R_1 \times T_2$. Hence, either $R_2 = T_2$ or $R_1 = T_1$. By reindexing, if needed, we may assume that $R_1 = T_1$. It then follows immediately from (2.11)(3) and (2.3) that T_2 is a λ-extension of R_2 with respect to J_2. The proof of the converse also follows immediately from (2.3) and (2.11)(3). ■

Theorem (2.13) is the analogue of Theorem (2.12) for the special case of an adjacent extension. This result coincides, modulo notation, with the result [FO, Lemme 1.5] of Ferrand and Olivier.

(2.13) THEOREM. *Let the notation be as in (2.10). Then:*

(a) Suppose that $R \subseteq T$ is an adjacent extension. Then exactly one of the following two sets of conditions holds:

(i) $R_1 = T_1$, $R_2 = T_2$ *and* I_1+I_2 *is a maximal ideal of* R*;*

(ii) Up to reindexing, $R_1 = T_1$ *and* $R_2 \subseteq T_2$ *is an adjacent extension and* $I_1+I_2 = R$.

(b) If either of the above sets of conditions (i) or (ii) holds for $R \subseteq T$*, then* $R \subseteq T$ *is an adjacent extension.*

Proof. (a) Suppose that $R \subseteq T$ is an adjacent extension. Then $R \subseteq T$ is a λ-extension and so, by (2.12)(a), we may assume (possibly after reindexing) that $R_1 = T_1$. By (2.11)(3), the only ideals of $R_2 \subseteq T_2$ containing J_2 are J_2 and T_2. But R_2 is an ideal of $R_2 \subseteq T_2$, and so either $R_2 = J_2$ or $R_2 = T_2$. If $R_2 = T_2$, then J_2 must be a maximal ideal of $R_2 = T_2$. Then, $\pi_2^{-1}(J_2) \cap R$ is a maximal ideal of R; that is, by a remark in (2.10), I_1+I_2 is a maximal ideal of R. If $R_2 = J_2$, then $R_2 \subseteq T_2$ must be an adjacent extension and $I_1+I_2 = \pi_2^{-1}(J_2) \cap R = R$.

(b) Suppose that one of the sets of conditions (i) or (ii) holds for $R \subseteq T$. In either case, $R_1 = T_1$ and one can verify that $J_2 \neq T_2$. Hence, by (2.11)(3), we need only show that there are no ideals of $R_2 \subseteq T_2$ containing J_2 other than J_2 and T_2. If (i) holds, then $J_2 = \pi_2(I_1+I_2)$ is a maximal ideal of $R_2 = T_2$, and so the desired conclusion follows. If (ii) holds, then $J_2 = \pi_2(I_1+I_2) = R_2$ and so, since $R_2 \subseteq T_2$ is an adjacent extension, the conclusion follows. ■

A special case of Theorems (2.12) and (2.13) which will be useful in Chapter III is the case in which the base ring R is a field. This case is the subject of the next result.

(2.14) COROLLARY. *Let* $K \subseteq T$ *be a ring extension with* K *a field. Suppose that* $T = \prod T_i$*, where the product contains at least two factors and each factor ring* T_i *is nonzero. Then:*

(a) If $K \subseteq T$ *is a* λ*-extension (for instance, an adjacent extension), then the product* $\prod T_i$ *contains exactly two factors.*

(b) $K \subseteq T$ *is a* λ*-extension if and only if* T *is isomorphic as a* K*-algebra to* $K \times L$*, where* $K \subseteq L$ *is a* λ*-extension of fields.*

(c) $K \subseteq T$ *is an adjacent extension if and only if* T *is isomorphic as a* K*-algebra to* $K \times K$.

Proof. (a) We may apply Proposition (2.8) with $R = K$. Indeed, in the notation of (2.8), since $K \not\subset ker\,(\pi_i)$ for any index i, each ideal I_i must be a proper ideal of K; that is, $I_i = 0$ for each i. Thus the hypotheses, and hence the conclusion, of (2.8) hold.

(b) Suppose first that $K \subseteq T$ is a λ-extension. By (a), we may write $T = T_1 \times T_2$. Then, letting $R = K$ in the notation of (2.10), we have that $J_2 = 0$, since we saw in the proof of (a) that $I_1 = 0$. Moreover, since each ring R_i is a nonzero homomorphic image of K, each R_i is isomorphic to K as a K-algebra, via the map $\pi_i|_K : K \to R_i$. Applying Theorem (2.12)(a), we have, up to reindexing, that $R_1 = T_1$ and T_2 is a λ-extension of R_2 with respect to $J_2 = 0$. Then, by (2.5)(3), any proper ideal of T_2 is a proper ideal of R_2; hence, since R_2 is a field, any proper ideal of T_2 is 0, and so T_2 is also a field. If we let $L = T_2$, then $K \subseteq L$ is a λ-extension of fields and $T = T_1 \times T_2$ is isomorphic to $K \times L$ as K-algebras.

Now suppose that T is K-algebra isomorphic to $K \times L$, where $K \subseteq L$ is a λ-extension of fields. Let $R = T_1 = K$ and $T_2 = L$ and consider the notation of (2.10). It is clear that $I_1 = 0$, and so $J_2 = 0$. Also, $K \cong \pi_2(R) \subseteq T_2 = L$ is a λ-extension of fields, and so is an algebraic extension (by (1.1)). Thus any intermediate ring of $\pi_2(R) \subseteq T_2$ is a field, and so any proper ideal of $\pi_2(R) \subseteq T_2$ is 0. By verifying each of the three conditions in (2.5), we see that T_2 is a λ-extension of $\pi_2(R)$ with respect to J_2; hence, since $R_1 = K = T_1$, it follows from (2.12)(b) that $K = R \subseteq T$ is a λ-extension.

(c) Suppose first that $K \subseteq T$ is an adjacent extension. As in (b), write $T = T_1 \times T_2$ and observe that $J_2 = 0$ and each ring R_i is K-algebra isomorphic to K. Also observe that $I_1+I_2 = 0 \neq R$. Applying (2.13)(a), we have that $R_1 = T_1$ and $R_2 = T_2$. Thus, $T = T_1 \times T_2$ is K-algebra isomorphic to $K \times K$.

Next, suppose that T is isomorphic to $K \times K$. Then, by [FO, Lemme (1.2)(b)], $K \subseteq T$ is an adjacent extension. ■

The next result is a corollary of all the main results of this chapter. Note that (2.15)(b) provides a new class of λ-extensions of rings and, thus, a new class of Δ-extensions.

(2.15) COROLLARY. *(a) Let R be a ring and regard $R \times R$ as an extension ring of R by the diagonal map ($r \mapsto (r,r)$ for all $r \in R$). Then, the extension $R \subseteq R \times R$ is a λ-extension if and only if the ideals of R are linearly ordered by inclusion (that is, if and only if R is a chained ring). Moreover, $R \subseteq R \times R$ is an adjacent extension if and only if R is a field.*

(b) Let R be a valuation domain and W an overring of R. Embed R in $R \times W$ by the diagonal map, as in (a). Then the extension $R \subseteq R \times W$ is a λ-extension.

Proof. (a) Without loss of generality, $R \neq \{0\}$. In the notation of (2.10), let $T_1 = T_2 = R$. Then $R_1 = T_1$ and $R_2 = T_2$, while I_1, I_2, J_1 and J_2 are each the zero ideal of (the appropriate copy of) R. By (2.12), $R \subseteq R \times R$ is a λ-extension if and only if $R_i = T_i$ is a λ-extension of R_i with respect to 0, for some i = 1,2. The latter condition holds, by (2.5), if and only if the ideals of $R_2 = R$ are linearly ordered by inclusion. As for the second assertion, observe that $I_1 + I_2 = 0 \neq R$. If $R \subseteq R \times R$ is an adjacent extension then, by (2.13)(a)(i), $I_1 + I_2 = 0$ is a maximal ideal of R; that is, R is a field. Conversely, if R is a field then, by (2.14)(c), $R \subseteq R \times R$ is an adjacent extension.

(b) In the notation of (2.10), let $T_1 = R$ and $T_2 = W$. Then $R_1 = T_1$ and I_1, I_2 and J_2 are each the zero ideal. Also, the extension $R_2 \subseteq T_2$ is essentially the extension $R \subseteq W$. Since R is a valuation domain and W is an overring of R then, by Example (2.7), W is a λ-extension of R with respect to 0. Hence, by (2.12)(b), $R \subseteq R \times W$ is a λ-extension. ■

We complete this chapter by taking a closer look at extensions $R \subseteq T$ such that T is a λ-extension of R with respect to $J = 0$. In particular, Theorem (2.17) characterizes such extensions in the case that R is an integral domain. The proof of (2.17) depends on the next result and the dichotomy established in (1.3).

(2.16) LEMMA. *Suppose that $R \subseteq T$ is a ring extension such that any proper ideal of T is contained in R. Then R is an integral domain if and only if T is an integral domain.*

Proof. The "if" implication is obvious. Suppose that R is an integral domain. To show that T is an integral domain, it is enough to show that if $x, y \in T$ and $xy = 0$ and $x \neq 0$, then $y = 0$. If x is a unit of T, then we are done, and so we may assume that x is a nonunit of T. Then xT is a proper ideal of T and so, by hypothesis, $xT \subseteq R$; in particular, $x \in R$. Since $x \neq 0$, the ideal $Ann_T(x)$ is a proper ideal of T, and so $Ann_T(x) \subseteq R$. Since $y \in Ann_T(x)$, we have $y \in R$. Thus, $x, y \in R$ and $xy = 0$ and $x \neq 0$; hence, $y = 0$. ■

(2.17) THEOREM. *Suppose that $R \subseteq T$ is a ring extension. Then:*

(a) If T is a λ-extension of R with respect to 0, then R is an integral domain if and only if T is an integral domain.

(b) Suppose that T is a λ-extension of R with respect to 0 and that R is an integral domain. Then either (i) $R \subseteq T$ is a λ-extension of fields or (ii) R is a valuation domain which is not a field and T is an overring of R.

(c) If either of the conditions (i) or (ii) of (b) holds, then T is a λ-extension of R with respect to 0.

Proof. (a) Apply (2.5)(3) and (2.16).

(b) If R is a field then, by (2.5)(3), 0 is the only proper ideal of T; then T is a field which, by (2.5)(1), is a λ-extension of R. Thus, (i) holds if R is a field.

Assume that R is not a field. By (2.5)(2), the ideals of R are linearly ordered by inclusion; that is, R is a valuation domain. In case T is a field, (2.5)(1) and (1.3)(2) imply that T is the quotient field of R; in particular, T is an overring of R.

In case T is not a field, T has a nonzero proper ideal I. Then $I \subseteq R$, by (2.5)(3), and so R and T have a nonzero ideal in common. Note, by (a), that T is an integral domain. Hence R and T have the same quotient field, and so T is an overring of R.

(c) If (i) holds, then the assertion follows from (2.5). If (ii) holds, then the assertion is proved in Example (2.7). ■

CHAPTER III

THE BASE RING IS A FIELD

As stated following the proof of Proposition (1.3), this chapter chapter concerns λ-extensions $K \subseteq L$ of fields. This is the second main theme based on the dichotomy introduced in (1.3), the first being the λ-domains considered in Chapter I. Actually, we consider here the more general λ-extensions $K \subseteq T$ in which only the base ring K is assumed to be a field.

The more focused question of which proper ring extensions T of a field K are adjacent extensions of K has been considered both by Ferrand-Olivier [FO, Lemme 1.2] and by Modica [Mo, Theorem 3 and subsequent remarks]. The adjacent extensions of K are of three types, up to K-algebra isomorphism: (i) adjacent field extensions $K \subseteq L$; (ii) the extension $K \subseteq K \times K$ where K is embedded in $K \times K$ via the diagonal map $x \mapsto (x,x)$; and (iii) the extension $K \subseteq K[X]/(X^2)$. While (ii) and (iii) each give only one adjacent extension of K, (i) gives a class of such extensions. The existence and classification of adjacent field extensions of a field K appears to be an open problem, although Ferrand and Olivier comment (after the proof of [FO, Lemme 1.2]) on an unpublished partial existence result.

The three types of adjacent extensions of a field K mentioned in the previous paragraph may be thought of in the following way. Type (ii) represents the decomposable adjacent extensions, whereas types (i) and (iii) represent the indecomposable adjacent extensions (field and non-field, respectively). By analogy, in Theorems (3.4), (3.8) and (3.11), we obtain information about the decomposable λ-extensions of a field K and about the indecomposable λ-extensions of a field K which are not necessarily fields. The part of the chapter following Remark (3.12) is devoted to λ-extensions of fields.

We begin the chapter with three general results about λ-extensions $K \subseteq T$ of a field K.

(3.1) THEOREM. *Suppose that K is a field and that $K \subseteq T$ is a λ-extension. Then the following assertions hold:*

(1) T is an algebraic extension of K and the Krull dimension of T is 0.

(2) Each ring contained between K and T, except possibly T itself, is a finite dimensional K-vector space.

(3) (i) If $\dim_K T < \infty$, then the rings between K and T form a finite chain

$$K = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n = T$$

where $T_i \subseteq T_{i+1}$ is an adjacent extension for $i = 0, 1, \ldots, n-1$.

(ii) If $\dim_K T = \infty$, then the rings between K and T form an increasing denumerable chain

$$K = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n \subseteq \ldots \subseteq T$$

where $\dim_K T_n < \infty$ for each $n \geq 0$ and $T_i \subseteq T_{i+1}$ is an adjacent extension for each $i \geq 0$. Moreover, $T = \cup_{n \geq 0} T_n$.

Proof. (1) Lemma (1.1) implies that T is an algebraic extension of K. Since K is a field, T is, in fact, an integral extension of K; hence the Krull dimension of T is equal to the Krull dimension of K, which is 0.

(2) Let A be a ring contained between K and T and suppose that $A \neq T$. Let $x \in T \setminus A$. By hypothesis, the intermediate rings A and $K[x]$ are comparable under inclusion. As $x \notin A$, it must be the case that $A \subseteq K[x]$. Since, by (1), x is algebraic over K, it follows that $K[x]$, and hence A, are finite dimensional K-vector spaces.

(3) Let A and x be as in the previous paragraph. The rings between A and $K[x]$ are linearly ordered by inclusion and are finite dimensional K-vector spaces; so there can be only finitely many of them. Thus, there is a smallest ring between A and $K[x]$ properly containing A; call it B. Then B is the smallest of the rings between A and T which properly contain A. Indeed, any such ring either contains $K[x]$, in which case it contains B, or else is contained in $K[x]$, in which case it contains B (by the way B was chosen). Note that B is necessarily an adjacent extension of A and, since $B \subseteq K[x]$, B is a finite dimensional K-vector space.

Successive application of the construction in the previous paragraph beginning at $K = T_0$ yields a chain $K = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ of adjacent extensions with each T_n a finite dimensional K-vector space. In case T is a finite dimensional K-vector space, a dimension argument shows that $T_n = T$ for some n. In case T is an infinite dimensional K-vector space, the chain does not terminate. In the latter case, we must show that $T = \cup_{n\geq 0} T_n$; and in both cases, we must show that all rings between K and T lie in the chain.

Suppose that $\dim_K T = \infty$ and that $T \neq \cup_{n\geq 0} T_n$. If $x \in T \setminus \cup_{n\geq 0} T_n$, then $K[x] \not\subset T_n$ for any n, and so $T_n \subseteq K[x]$ for all $n \geq 0$. As $n+1 \leq \dim_K T_n \leq \dim_K K[x] < \infty$ for all n, we obtain the desired contradiction.

Finally, suppose that S is a ring between K and T. First, assume that $S \not\subset T_n$ for any n. Then $T_n \subseteq S$ for all $n \geq 0$, and so $\cup_{n\geq 0} T_n \subseteq S$, whence $S = T$. Next, assume that $S \subseteq T_k$ for some k, and suppose that k is minimal. If $k = 0$, then $S = K$. Otherwise, $k \geq 1$ and $S \not\subset T_{k-1}$, and so $T_{k-1} \subseteq S \subseteq T_k$ with $T_{k-1} \neq S$. Since T_k is an adjacent extension of T_{k-1}, it must be the case that $S = T_k$. ∎

After this research was completed and this dissertation nearly completely written, it was kindly suggested by William Heinzer that the theory of J-extensions (= Jónsson extensions) might overlap some part of this work (see [GHe3] and [GHe4]). If $K \subseteq T$ is an extension with K a field, then T is a *J-extension* of K if $\dim_K T = \infty$ and $\dim_K S < \infty$ for each proper K-subalgebra S of T. It is clear from (3.1)(2) that any infinite-dimensional λ-extension T of a field K is a J-extension of K. There are examples of such extensions. See, for instance, the λ-extensions discussed in Remark (3.13), Example (3.31)(b) and Remark (3.39). Further examples may be found in [GHe3, Examples (E1) and (E2) (pp. 81-82), Example 2.7 and Example 2.15].

(3.2) THEOREM. *Suppose that K is a field and that $K \subseteq T$ is a λ-extension. Let the rings T_i be as in (3) of the previous theorem. Then the following four conditions are equivalent:*

(a) $\dim_K T < \infty$.

(b) $T = T_n$ for some $n \geq 0$.

(c) $T = K[u]$ for some $u \in T$.

(d) $T = K[u_1, u_2, \ldots, u_n]$ for some elements $u_i \in T$.

Proof. (a) $\Rightarrow$ (b) This implication is contained in (3.1)(3)(i).

(b) $\Rightarrow$ (c) There is nothing to prove if $T = T_0 = K$, and so we may suppose that n $\geq$ 1. If $x \in T\backslash T_{n-1}$, then $T_{n-1} \subseteq K[x] \subseteq T_n = T$ with $T_{n-1} \neq K[x]$. Since $T_n = T$ is an adjacent extension of T_{n-1}, we have $K[x] = T$.

(c) $\Rightarrow$ (d) Trivial.

(d) $\Rightarrow$ (a) T is a finitely-generated K-algebra and is integral over K (see (3.1)(1)), and so T is a finitely-generated K-module. ■

(3.3) PROPOSITION. *Let $K \subseteq T$ be an extension with K a field. If the set of rings between K and T satisfies the descending chain condition, then T is an Artinian ring. In particular, if T is a λ-extension of a field K, then T is an Artinian ring.*

Proof. Suppose that the rings between K and T satisfy the descending chain condition, but T is not Artinian. Then there exists an infinite strictly descending chain of proper ideals of T : $I_1 > I_2 > I_3 > \ldots$. We claim that $K + I_1 \supseteq K + I_2 \supseteq K + I_3 \supseteq \ldots$ is a strictly descending chain of rings between K and T, contrary to hypothesis. Indeed, suppose $K + I_i = K + I_{i+1}$ for some i. For arbitrary $x \in I_i$, $x \in K + I_i = K + I_{i+1}$, and so $x = a + y$ for some $a \in K$ and $y \in I_{i+1}$. Then $a = x - y \in K \cap I_i = 0$ (since I_i is a proper ideal). Thus $x = y \in I_{i+1}$, and so $I_i = I_{i+1}$, a contradiction.

The last assertion follows from the first assertion and (3.1)(3). ■

As discussed in the introduction to this chapter, let us turn our attention to the decomposable (resp., indecomposable) λ-extensions of a field K. The next result concerns the decomposable case and is nothing more that a paraphrase of parts (a) and (b) of Corollary (2.14). The analogous result for J-extensions is [GHe4, Theorem 3.1]. Notice that Theorem (3.4) reduces the study of the decomposable case to the study of λ-extensions of fields.

(3.4) THEOREM. *Let $K \subseteq T$ be a ring extension with K a field. Then T is a decomposable λ-extension of K if and only if T is K-algebra isomorphic to $K \times L$, where $K \subseteq L$ is a λ-extension of fields.* ■

Next we consider the indecomposable case. Theorems (3.8) and (3.11) contain the main results concerning the indecomposable λ-extensions T of a field K which are not necessarily fields. We first need three preliminary results.

(3.5) PROPOSITION. *Let K be a field and n a positive integer. Then the ring extension $K \subseteq K[X]/(X^n)$ is a λ-extension if and only if $n \leq 3$.*

Proof. First, suppose that $n \geq 4$. Note that $2(n-1) \geq 2(n-2) \geq n$. We may write $K[X]/(X^n)$ as $K[u]$ where $u = X + (X^n)$. Observe that u is nilpotent of index n and that $\{1, u, u^2, ..., u^{n-1}\}$ is a K-vector space basis of $K[u]$. Set $v = u^{n-1}$ and $w = u^{n-2}$, and note that v and w are nonzero elements satisfying $v^2 = w^2 = 0$. We next show that $K \subseteq K[u]$ is not a λ-extension by showing that the intermediate rings $K[v]$ and $K[w]$ are incomparable under inclusion. But $K[v] = K + Kv = K + Ku^{n-1}$ and $K[w] = K + Kw = K + Ku^{n-2}$; these sets are clearly incomparable because of the linear independence of $\{1, u, u^2, ..., u^{n-1}\}$.

If n=1, then $K \subseteq K[X]/(X^1) \cong K$ is the identity extension, which is trivially a λ-extension. In case n=2, we know that $K \subseteq K[X]/(X^2)$ is an adjacent extension and thus is a λ-extension.

Finally, suppose that n = 3 and let u be as in the first paragraph of the proof. Observe that $u^2 \neq 0$ and $(u^2)^2 = 0$. Thus, $A := K[u^2] = K + Ku^2$ is a ring contained between K and $K[u]$ with $\dim_K A = 2$. It is enough to show that A is the only ring properly contained between K and $K[u]$. To this end, we let $x \in K[u]\setminus K$ and, by a dimension argument, it is enough to show that either $K[x] = K[u]$ or $K[x] = A$. Now, $x = a + bu + cu^2$ for some $a, b, c \in K$. Clearly, $K[x] = K[bu + cu^2]$. Since $x \notin K$, at least one of b or c is nonzero. If $b = 0$, then $c \neq 0$ and $K[x] = K[cu^2] = K[u^2] = A$. Hence, we may assume that $b \neq 0$. In that case, $K[x] = K[bu + cu^2] = K[u + du^2]$, where $d = c/b$. Since $u^3 = 0$, we have $u^2 = (u + du^2)^2 \in K[x]$, and so $A \subseteq K[x] \subseteq K[u]$. A dimension argument shows that either $K[x] = K[u]$ or $K[x] = A$, as desired. Thus $K \subseteq K[X]/(X^3)$ is a λ-extension. ■

(3.6) COROLLARY. *Let $K \subseteq T$ be a λ-extension with K a field, and suppose that u is a nilpotent element of T. Then the index n of nilpotency of u satisfies $n \leq 3$.*

Proof. The subextension $K \subseteq K[u]$ is a λ-extension since $K \subseteq T$ is a λ-extension. But $K[u]$ is K-algebra isomorphic to $K[X]/(X^n)$. Thus, by Proposition (3.5), n $\leq$ 3. ■

(3.7) LEMMA. *Let $R \subseteq T$ be an extension of rings. If T is an indecomposable ring, then R is also an indecomposable ring.*

Proof. This follows from the fact that a ring is decomposable if and only if it contains a pair of nontrivial orthogonal idempotents whose sum is 1. Thus, if R is decomposable, then R contains such a pair of idempotents. Then T also contains this pair of idempotents, and so T is decomposable. ■

(3.8) THEOREM. *Let $K \subseteq T$ be a λ-extension with K a field and suppose that T is indecomposable. Let N be the nilradical of T. Then:*
(1) Each ring between K and T has a unique prime ideal.
(2) N is the unique prime ideal of each ring between $K+N$ and T.
(3) N is a principal nilpotent ideal of T with index of nilpotency r, $1 \leq r \leq 3$.
(4) The ring $K+N$ is K-algebra isomorphic to $K[X]/(X^r)$, where r is as in (3). Also, $K+N = T_{r-1}$, using the notation of (3.1)(3).
(5) If T is not a field, then the extension $K \subseteq T/N$ is a purely inseparable λ-extension of fields.

Proof. (1) Let $K \subseteq A \subseteq T$. By (3.7), since T is indecomposable, A is also indecomposable. Suppose that A is properly contained in T. By (3.1)(2), A is a finite-dimensional K-algebra and so is Artinian. Since A is indecomposable, it is a local Artinian ring, and so it has a unique prime ideal.

To show that T has a unique prime ideal, it suffices to show that each nonunit x of T is nilpotent. By (3.1)(1), x is algebraic over K, and so $K[x]$ is a finite-dimensional K-algebra between K and T. Thus, by the previous paragraph, $K[x]$ has a unique prime ideal, which is necessarily its nilradical. Since x is not a unit of T, it is also not a unit of $K[x]$. Then x lies in the prime ideal of $K[x]$ and so is nilpotent.

An alternative proof that T has a unique prime ideal, which also relies on the first paragraph of this proof, is available via [GD, Propositions (6.1.2)(ii) and (6.1.6)(i)], since T is the direct limit (in fact, the directed union) of the proper K-subalgebras of T which contain K.

(2) Suppose that $K + N \subseteq A \subseteq T$. The maximal ideal M of A, being the nilradical of A, is contained in the nilradical N of T. But, since $K + N \subseteq A$, we have $N \subseteq M$ and so $N = M$.

(3) By (3.6), the index of nilpotency of any nilpotent element of T is ≤ 3. Let v be a nilpotent element of T with maximal index r of nilpotency. We claim that $K + N = K[v]$. The containment ($\supseteq$) is clear, since $v \in N$. To prove the containment ($\subseteq$), we show that $N \subseteq K[v]$. Let $u \in N$. Then, by the choice of v, u is nilpotent of index $i \leq r$. It follows that $\dim_K K[u] = i \leq r = \dim_K K[v]$. Since $K[u]$ and $K[v]$ must be comparable under inclusion, we have $K[u] \subseteq K[v]$ and so $u \in K[v]$. This proves the claim.

By (2), since $K + N = K[v]$, N is the nilradical of $K[v]$. But $K[v] \cong K[X]/(X^r)$ as K-algebras, and so the maximal ideal of $K[v]$ is $vK[v]$. Hence N is nilpotent of index $r \leq 3$. Moreover, since $N = vK[v]$, we have $N = vT$ and so N is a principal ideal of T.

(4) The first assertion was proved above in the proof of (3). As for the second assertion, if r=1, then $N = 0$ and $K + N = K = T_0$. If r=2, then $K + N \cong K[X]/(X^r)$ is an adjacent extension of K ([FO, Lemme 1.2]) and so must be equal to T_1. If r=3, then $K + N \cong K[X]/(X^r)$ and so the proof of (3.5) shows that there is only one ring properly contained between K and $K + N = K[v]$, namely, $K + Kv^2$. It follows that $K + N = T_2$.

(5) It is easy to see (cf. (3.9)) that $K \subseteq T/N$ is a λ-extension. Let L be the separable closure of K in T/N and let S be the ring between $K + N$ and T corresponding to L. By (1) and Proposition (3.3), S is a local Artinian K-algebra; hence, S is an equicharacteristic complete local ring. It follows from the structure theorem for complete local rings (see [Ma, Theorem 28.3]) that S contains a field F such that $F \supseteq K$ and F maps onto the field L under the canonical surjection $S \rightarrow S/N$. If T is not a field, it follows easily from (2), (4) and the last paragraph of the proof of (3.5) that the only field contained between K and T is K. Thus, $F = K$ and so $L =$

K. Hence the separable closure of K in T/N is K, and so T/N is a purely inseparable extension of K. ■

Corollary (3.10) is a corollary of Theorems (3.4) and (3.8). Its proof uses the following proposition.

(3.9) PROPOSITION. *Let $R \subseteq T$ be an extension of rings and J be an ideal of T. Set $I = J \cap R$. Then the extension $R/I \subseteq T/J$ is a λ-extension of rings if and only if the extension $R + J \subseteq T$ is a λ-extension of rings.*

Proof. The assertion is an immediate consequence of the easily verified fact that the mapping $A \mapsto A/J$ gives a one-to-one inclusion-preserving and inclusion-reflecting correspondence from the set of all rings contained between $R + J$ and T onto the set of all rings contained between R/I and T/J. ■

(3.10) COROLLARY. *If $R \subseteq T$ is a λ-extension of rings such that the conductor $I = (R:T)$ is a maximal ideal of R, then there are at most two prime ideals of T containing I, and any such prime ideal of T is necessarily maximal.*

Proof. An application of Lemma (3.6) with $J = I = (R:T)$ shows that $R/I \subseteq T/I$ is a λ-extension. By hypothesis, R/I is a field. It follows from Theorems (3.4) and (3.8) that, depending on whether T/I is decomposable or indecomposable, T/I has either two or one prime ideals, which are necessarily maximal. The assertion follows immediately from this fact. ■

The indecomposable adjacent extensions T of a field K discussed in the introduction to this chapter are of two types (up to K-algebra isomorphism): (i) $T = K[X]/(X^2)$ when T is not a field; and (ii) $T = K[X]/(p(X))$, for some irreducible polynomial $p(X)$, when T is a field. Our next goal is to show that λ-extensions T of a field K, such that $\dim_K T < \infty$, are similarly restricted in form.

(3.11) THEOREM. *Let $K \subseteq T$ be a λ-extension with K a field and suppose that T is indecomposable and $\dim_K T < \infty$. Then:*

(1) T is K-algebra isomorphic to $K[X]/(p(X)^n)$ for some irreducible polynomial $p(X)$ and some positive integer $n \leq 3$.

(2) The integer n and the degree of the polynomial $p(X)$ in (1) are uniquely determined by T.

(3) Let S be a ring contained between K and T. By (1), S is K-algebra isomorphic to $K[X]/(q(X)^m)$ for some irreducible polynomial $q(X)$ and some positive integer m. Then $m \leq n$ and $\deg q(X) \leq \deg p(X)$, where n and $p(X)$ are as in (1).

Proof. (1) If $K = T$, then the assertion holds by taking $p(X) = X$ and n=1. Thus, we may assume that $K \neq T$. By (3.2), $T = K[u]$ for some $u \in T$. Let $f(X)$ be a generator of the (proper principal) ideal of polynomials in $K[X]$ having u as a root and let $f(X) = p_1(X)^{e_1} p_2(X)^{e_2} \dots p_r(X)^{e_r}$ be a factorization of $f(X)$ as a product of irreducibles. By the Chinese Remainder Theorem, we have $T \cong K[X]/(f(X)) \cong \prod_{i=1}^{r} K[x]/(p_i(X)^{e_i})$. Since T is indecomposable, r = 1 and $f(X) = p(X)^n$ for some irreducible polynomial $p(X)$ and some positive integer n. By the choice of $f(X)$, we have $p(u)^n = 0$ and $p(u)^{n-1} \neq 0$. Applying (3.6) to the nilpotent element $p(u)$ shows that $n \leq 3$, whence (1) is proved.

(2) Let $p(X)$ and n be as in (1). Then the maximal ideal N of T corresponds to the ideal $(p(X))/(p(X)^n)$. It is then clear that n is equal to the index of nilpotency of N, a number determined by T. Moreover, $\dim_K T = \deg(p^n) = n \deg(p)$, and so $\deg(p) = \dim_K T/n$, which is also determined by T.

(3) By (3.7), S is indecomposable. Thus assertion (1) of this theorem applies to S and so we obtain the polynomial $q(X)$ and the integer m. By (3.8)(1), the maximal ideal of S is contained in the maximal ideal of T and so, by the proof of (2), the index of nilpotency m of the maximal ideal of S is less than or equal to the index of nilpotency n of the maximal ideal N of T.

If $K + N \subseteq S$ then, by (3.8)(2), N is the maximal ideal of S and so m = n by the proof of (2). In this case, $n \deg(q) = m \deg(q) = \dim_K S \leq \dim_K T = n \deg(p)$, and so $\deg(q) \leq \deg(p)$.

By the proof of (3.8)(4), the only other possibilities for S are $S = K$, when n = 2; and $S = K$ or $S = T_1 \cong K[X]/(X^2)$, when n = 3. If $S = K$, then $\deg(q) = 1 \leq \deg(p)$. Assume that n = 3 and $S = T_1 \cong K[X]/(X^2)$. Then $2 = \dim_K S = \mathrm{m} \deg(q)$ and m = 2, whence $\deg(q) = 1 \leq \deg(p)$. ■

(3.12) REMARKS. (i) Let K be a field, $p(X)$ an irreducible element of $K[x]$, and n is a positive integer ≤ 3. The ring $K[X]/(p(X)^n)$ is not necessarily a λ-extension of K, and so the converse of Theorem (3.11)(1) is false. Indeed, in the case n = 1, $K[X]/(p(X))$ is a typical simple algebraic field extension of K. Such an extension need not be a λ-extension of K, as is shown by the example $\mathbf{Q} \subseteq \mathbf{Q}(\sqrt{2}, \sqrt{3}) \cong \mathbf{Q}[X]/(X^4 - 10X^2 + 1)$. Part (ii) of this remark contains a counterexample in the case n = 2.

(ii) Suppose that K,T satisfy the hypotheses of (3.11), and let $p(X)$ and n be as in (3.11)(1). Note that the maximal ideal N of T corresponds to the ideal $p(X)/(p(X)^n)$ of $K[X]/(p(X)^n)$. Thus $T/N \cong K[X]/(p(X))$ as K-algebras. Since $K + N \subsetneq T$ is a λ-extension, (3.9) implies that $K \subseteq T/N$ (and so $K \subseteq K[X]/(p(X))$) is a λ-extension of fields. Thus, given an irreducible polynomial $p(X)$ in $K[x]$ and a positive integer $n \leq 3$, a necessary condition for $K[X]/(p(X)^n)$ to be a λ-extension of K is that $K \subseteq K[X]/(p(X))$ is a λ-extension of fields. We now give an example to show that this condition is not sufficient when n = 2.

Let n = 2 and let $p(X) = X^2 + 1 \in \mathbf{Q}[X]$. The field $\mathbf{Q}[X]/(X^2 + 1)$ is certainly a λ-extension (in fact, an adjacent extension) of $\mathbf{Q}$, but we show that $T := \mathbf{Q}[X]/((X^2 + 1)^2)$ is not a λ-extension of $\mathbf{Q}$. Let $u = X + ((X^2 + 1)^2) \in T$ and let $v = u^2 + 1$. Observe that $T = \mathbf{Q}[u]$ and $\{1, u, u^2, u^3\}$ is a basis of T as a $\mathbf{Q}$-vector space. As v and uv are each nilpotent elements of T of index 2, $\mathbf{Q}[v] = \mathbf{Q} + \mathbf{Q}v$ and $\mathbf{Q}[uv] = \mathbf{Q} + \mathbf{Q}uv$ are rings between $\mathbf{Q}$ and T. We show that these two rings are incomparable under inclusion. Note that $\mathbf{Q}[v] = \mathbf{Q} + \mathbf{Q}(u^2 + 1) = \mathbf{Q} + \mathbf{Q}u^2$ and $\mathbf{Q}[uv] = \mathbf{Q} + \mathbf{Q}(u^3 + u) \subseteq \mathbf{Q} + \mathbf{Q}u + \mathbf{Q}u^3$. Then it is clear from the linear independence of $\{1, u, u^2, u^3\}$ that $v \notin \mathbf{Q}[uv]$ and $uv \notin \mathbf{Q}[v]$, the desired conclusion.

Now we turn our attention to λ-extensions $K \subseteq L$ such that both K and L are fields. Recall from Chapter 0 that any λ-extension is a Δ-extension, but, as can be seen in (1.14)(ii), not every Δ-extension is a λ-extension. For field extensions, however, the notion of Δ-extension is not more general than the notion of a λ-extension. Indeed, Gilmer and Huckaba have shown ([GHu, Theorem 1]) that an extension $K \subseteq L$ of fields is a Δ-extension if and only if it is a λ-extension.

(3.13) REMARK. A special case of λ-extensions $K \subseteq L$ of fields arises when K is the prime subfield of L and all the subfields of L are linearly ordered by inclusion. Examples of such fields have been considered by Gilmer and Huckaba ([GHu, Proposition 6]) and, later and independently, by Curtin ([C1, Theorem 2] and [C2, Classification Theorem]). In fact, Curtin shows that a field L is a λ-extension of its prime subring if and only if L is one of the following two types of fields: (i) finite fields GF(p^{q^n}), for prime numbers p, q and positive integer n; and (ii) GF(p^{q^∞}) := $\bigcup_{n \geq 0}$GF(p^{q^n}), for some prime numbers p and q, in some fixed algebraic closure of GF(p). On the other hand, Gilmer and Huckaba show that a field L is a Δ-extension of its prime subring if and only if it is either $\mathbf{Q}$ or of one of the types (i) or (ii) above. (Curtin asks the more general question of which commutative rings have the property that their subrings are linearly ordered by inclusion. In [C1] and [C2], he answers this question for infinite rings and indecomposable finite rings, but leaves the question open for decomposable finite rings.)

To study λ-extensions of fields, it will be useful to consider a property of field extensions which is suggested by a paper of Quigley [Q]. In this paper, Quigley classifies the subfields of an algebraically closed field L which are maximal among the subfields of L not containing a given element of L. This notion is generalized in the next definition.

(3.14) DEFINITION. We say that an extension $K \subseteq L$ of fields is a *μ-extension*, or that L is a *μ-extension of* K, if there is an element x of L such that K is maximal among the subfields of L not containing x.

(3.15) PROPOSITION. *(1) If $K \subseteq L$ is a μ-extension, then L is an algebraic extension of K.*
(2) If L is a proper λ-extension of the field K, then L is a μ-extension of K.

Proof. (1) This follows from the proof of [Q, Lemma 1], which does not use Quigley's standing assumption that the top field L is algebraically closed.

(2) Let the rings T_i be as in (3.1)(3). By hypothesis, $T_1 \neq K$, and so we may choose an element $x \in T_1 \backslash K$. If F is a subfield of L properly containing K, then, by the nature of the rings T_i, $T_1 \subseteq F$ and so $x \in F$. It follows that K is maximal among the subfields of L not containing x. ■

(3.16) REMARKS. (i) A μ-extension need not be a λ-extension. An example may be found in [GHe3, page 96, lines 6-11].

(ii) The significance of (3.15)(2) is that Quigley obtains its converse under the assumption that L is algebraically closed ([Q, Theorem 3]). Theorem (3.29) establishes the converse under different conditions on the field L.

Given an algebraic field extension $K \subseteq L$, let L_s (resp., L_i) denote the set of elements of L which are separable (resp., purely inseparable) over K. The notations $L_s(K)$ and $L_i(K)$ will be used when the base field is not clear. It is well known that L_s and L_i are fields contained between K and L whose intersection is K. In general, L is purely inseparable over L_s. So, if $L_s = K$, then $L_i = L$ and L is a purely inseparable extension of K. If $L_i = K$, then we say that K is *purely inseparably closed in L.*

(3.17) PROPOSITION. *Let L be a proper field extension of the field K. Then:*

(1) If $K \subseteq L$ is an adjacent extension, then L is algebraic over K and L is either a separable or a purely inseparable extension of K.

(2) $K \subseteq L$ is a λ-extension (i.e., the rings between K and L are linearly ordered by inclusion) if and only if the fields between K and L are linearly ordered by inclusion. If $K \subseteq L$ is a λ-extension, then L is algebraic over K and either L is purely inseparable over K or K is purely inseparably closed in L.

(3) $K \subseteq L$ is a μ-extension if and only if there is a (necessarily unique) intermediate field K_0 properly containing K which is contained in every intermediate field properly containing K (in which case, K_0 is the unique minimal proper field extension of K inside L). If $K \subseteq L$ is a μ-extension and $x \in L$, then K is maximal as a sub-field of L not containing x if and only if $x \in K_0 \backslash K$; for such elements x, we have $K_0 = K(x) = K[x]$. Moreover, if $K \subseteq L$ is a μ-extension, then L is algebraic over K and either L is purely inseparable over K or K is purely inseparably closed in L.

Proof. (1) If $K \subseteq L$ is an adjacent extension, then $K \subseteq L$ is a λ-extension and so, by (1.1), L is an algebraic extension of K. Since the extension is adjacent, we have that either $L_s = L$ or $L_s = K$. In the former case, L is a separable extension of K. In the latter case, according to the paragraph preceding this proposition, L is a purely inseparable extension of K.

We prove (3) next. Observe that the field K_0 is clearly unique, if it exists. First, suppose that $K \subseteq L$ is a μ-extension. Let x be an element of L such that K is maximal as a subfield of L not containing x. Note that $K[x]$ properly contains K. If F is any subfield of L that properly contains K then, by the maximality of K, we have that $x \in F$, whence $K[x] \subseteq F$. Thus, $K(x)$ is a suitable K_0. The equality $K(x) = K[x]$ follows from (3.15)(1). This argument proves the "only if" implications of the first two assertions of (3).

Next, suppose that the field K_0 exists. Let $x \in K_0 \backslash K$. Then, $x \notin K$ and, for any subfield F of L properly containing K, we have $x \in K_0 \subseteq F$. Thus, K is maximal as a subfield of L not containing x, and so $K \subseteq L$ is a μ-extension. This argument proves the "if" implications of the first two assertions of (3).

As for the final assertion of (3), we have shown in (3.15)(1) that L is algebraic over K when $K \subseteq L$ is a μ-extension. Assume that $K \subseteq L$ is a μ-extension. Since $L_s \cap L_i = K$ and K_0 is contained in each proper field extension of K inside L, either $L_s = K$ or $L_i = K$. In the former case, by the paragraph preceding this proposition, L is a purely inseparable extension of K. In the latter case, K is, by definition, purely inseparably closed in L.

(2) The "only if" implication in the first assertion is clear. Suppose that the fields between K and L are linearly ordered by inclusion. We first show that L is an algebraic extension of K. Assume not and let $x \in L$ be transcendental over K. Then, it is easy to verify by a degree argument that $x^2 \notin K(x^3)$ and $x^3 \notin K(x^2)$. Hence, the intermediate fields $K(x^2)$ and $K(x^3)$ are incomparable under inclusion, contrary to hypothesis. Since $K \subsetneq L$ is therefore an algebraic extension, all the intermediate rings between K and L are, in fact, fields. The "if" implication then follows.

As for the second assertion of (2), we have shown in (1.1) that L is algebraic over K when $K \subsetneq L$ is a λ-extension. The remainder of the second assertion is proved by combining (3) with Proposition (3.15)(2). ∎

It follows from (3.17) that if L is an adjacent (resp. proper λ-, resp. μ-) field extension of a field K, then either L is purely inseparable over K or K is purely inseparably closed in L. The next few results deal with the case in which L is purely inseparable over K. Our methodology used to handle this case is based on the methods of Quigley [Q]. After these results, we consider next a special subcase of the case in which K is purely inseparably closed in L, namely, the case in which L is a Galois extension of K.

The next result appears in Bourbaki [Bo2, Proposition 1 of A. V.§5] and is included here for reference purposes.

(3.18) LEMMA. *Let L be a purely inseparable algebraic field extension of the field K, with $char(K) = p > 0$. If $x \in L$, then $[K(x):K] = p^n$, where p^n is the least power of p such that $x^{p^n} \in K$.*

Theorem (3.19) contains both a characterization of, and existence results for, purely inseparable adjacent field extensions. The purely inseparable λ- (resp., μ-) field extensions are characterized in Theorems (3.22) and (3.25), after a few preliminary results.

Suppose that L is a purely inseparable algebraic field extension of a field K of characteristic p > 0. Let us recall some notation of Bourbaki. For each integer r ≥ 0, define

$$K^{p^{-r}} = \{x \in L : x^{p^r} \in K\}.$$

Each of these sets is a field contained between K and L. Moreover, $K^{p^{-r}} \subseteq K^{p^{-(r+1)}}$ for each r ≥ 0 and $\bigcup_{r \geq 0} K^{p^{-r}} = L$. In addition, given an element $v \in K$ and an integer r ≥ 0, there is at most one element $u \in L$ such that $u^{p^r} = v$. This element u, if it exists, is denoted $v^{p^{-r}}$.

(3.19) THEOREM. *(a) Let L be a purely inseparable algebraic field extension of a proper subfield K and let p = char(K). Then L is an adjacent extension of K if and only if [L:K] = p.*
(b) Let K be a field of characteristic p > 0. Then K has a purely inseparable adjacent field extension L if and only if K is not perfect. In this case, the purely inseparable adjacent field extensions of K are, up to K-algebra isomorphism, precisely the fields $K[u^{p^{-1}}]$ *where* $u \in K \setminus K^p$.
(c) Let L be a field of characteristic p > 0. Then there is a subfield K of L such that L is a purely inseparable adjacent field extension of K if and only if L is not perfect.

Proof. (a)(⇐) Since p is prime, a degree argument shows that there can be no fields properly contained between K and L. Thus, since L is an algebraic extension of K, there are no rings properly contained between K and L.

(⇒) Suppose that L is an adjacent extension of K. The conclusion will follow from (3.18) once we show that there is an element $x \in L \setminus K$ such that $x^p \in K$. But, if y is any element of $L \setminus K$ and if y^{p^n} is the least power of y lying in K, then $x = y^{p^{n-1}}$ has the desired properties.

(b)(⇒) If such a field L exists, then K is not a perfect field.

(⇐) Suppose that K is not perfect and let $u \in K \setminus K^p$ and $L = K[u^{p^{-1}}]$, viewed as a proper field extension of K inside some algebraic closure of K. Observe that L is a purely inseparable field extension of K. Since $(u^{p^{-1}})^p = u \in K$, it follows from (3.18) that $[L:K]$ = p, and so, by (a), L is a purely inseparable adjacent field extension of K.

Now suppose that L is any purely inseparable adjacent field extension of K. By (a), we have that $[L:K] = p$. If $v \in L \setminus K$, then $L = K[v]$ (since the extension is adjacent) and $v^p \in K$ (by (3.18)). Then clearly $u := v^p \in K \setminus K^p$ and $L = K[v] = K[u^{p^{-1}}]$.

(c)($\Rightarrow$) If such a subfield K exists then K is not perfect, by (b). Then, since L is a finite-dimensional field extension of K, L is also not perfect ([J, Exercise 7, §8.7, page 491]).

($\Leftarrow$) Suppose that L is not perfect and let $v \in L \setminus L^p$. Using Zorn's Lemma, expand L^p to a subfield K of L which is maximal among the subfields of L not containing v. Since $L^p \subseteq K$, we have that L is a purely inseparable extension of K. To show that L is an adjacent extension of K, it is sufficient, by (a) and (3.18), to show that $L = K(v)$. Consider any $u \in L \setminus K$. As $u^p \in L^p \subseteq K$, we see by (3.18) that $[K(u):K] = p$; in particular, $[K(v):K] = p$. By the maximality of K, we must have $v \in K(u)$. Thus, $K(u) = K(v)$ for each $u \in L \setminus K$, whence $L = K(v)$. ■

(3.20) LEMMA. *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic $p > 0$. If M is a field contained between K and L and $[M:K] < \infty$, then for some integer r, $[M:K] = p^r$ and $M \subseteq K^{p^{-r}}$.*

Proof. The first assertion is given by [Bo2, Proposition 4 of A.V.§5]. Let M be as in the statement and let $u \in M$. By (3.18), there is an integer s such that $[K(u):K] = p^s$ and $u^{p^s} \in K$. Since $p^s = [K(u):K] \leq [M:K] = p^r$, we have $u^{p^r} \in K$ and so $u \in K^{p^{-r}}$. Hence $M \subseteq K^{p^{-r}}$. ■

(3.21) LEMMA. *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic $p > 0$ and suppose that $K^{p^{-(r+1)}}$ is a simple extension of $K^{p^{-r}}$ for all $r \geq 0$.*

(1) For any integer $r \geq 1$, $[K^{p^{-r}}:K] = p^r$ if and only if $K^{p^{-r}} \neq K^{p^{-(r-1)}}$. If this is the case, then $[K^{p^{-i}}:K] = p^i$ for $0 \leq i \leq r$.

(2) Consider any integer $r \geq 0$. Then, there is at most one intermediate field M such that $[M:K] = p^r$. If M is such a field, then $M = K^{p^{-r}}$.

Proof. We first prove that if $K^{p^{-r}} \neq K^{p^{-(r-1)}}$, then $[K^{p^{-r}} : K^{p^{-(r-1)}}] = p$. Indeed, there is, by hypothesis, an element u such that $K^{p^{-r}} = K^{p^{-(r-1)}}(u)$. Since $u^p \in K^{p^{-(r-1)}}$, it follows from (3.18) that $[K^{p^{-r}} : K^{p^{-(r-1)}}] = [K^{p^{-(r-1)}}(u) : K^{p^{-(r-1)}}] = p$.

From the preceding paragraph, it follows that for each $r \geq 1$, $[K^{p^{-r}} : K^{p^{-(r-1)}}]$ is either 1 or p. For a given $r \geq 1$, $[K^{p^{-r}} : K] = [K^{p^{-r}} : K^{p^{-(r-1)}}][K^{p^{-(r-1)}} : K^{p^{-(r-2)}}] \ldots [K^{p^{-1}} : K] \leq p^r$. If equality holds, then each factor in the product is p, and so $K^{p^{-r}} \neq K^{p^{-(r-1)}}$ and $[K^{p^{-i}} : K] = p^i$ for $0 \leq i \leq r$. This establishes the ($\Rightarrow$) implication of the first assertion in (1) and also the second assertion of (1).

We prove (2) next. Without loss of generality, $r \geq 1$. If M is an intermediate field such that $[M:K] = p^r$, then by (3.20), $M \subseteq K^{p^{-r}}$. It follows from the preceding paragraph that $p^r = [M:K] \leq [K^{p^{-r}} : K] \leq p^r$ and so $M = K^{p^{-r}}$. This establishes (2).

Finally, we establish the ($\Leftarrow$) implication of the first assertion of (1). Suppose that $K^{p^{-r}} \neq K^{p^{-(r-1)}}$. By (3.20) and the inequality in the second paragraph of this proof, $[K^{p^{-r}} : K] = p^s$ and $K^{p^{-r}} \subseteq K^{p^{-s}}$, for some $s \leq r$. If $s \leq r - 1$, then $K^{p^{-r}} \subseteq K^{p^{-(r-1)}}$ and so $K^{p^{-r}} = K^{p^{-(r-1)}}$, contrary to assumption. Hence $s = r$ and $[K^{p^{-r}} : K] = p^r$. ■

(3.22) THEOREM. *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic $p > 0$. Then the following six statements are equivalent:*

(a) $K = K^{p^{-0}}, K^{p^{-1}}, K^{p^{-2}}, \ldots,$ *and L are all the fields between K and L;*

(b) $K \subseteq L$ *is a λ-extension;*

(c) $K^{p^{-r}}$ *is a simple field extension of K for all* $r \geq 0$*;*

(d) $K^{p^{-r}}$ *is a simple field extension of* $K^{p^{-(r-1)}}$ *for all* $r \geq 1$*;*

(e) $[K^{p^{-r}} : K] \leq p^r$ *for all* $r \geq 0$*;*

(f) $[K^{p^{-r}} : K^{p^{-(r-1)}}] \leq p$ *for all* $r \geq 1$.

Proof. The plan of the proof is (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (d) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b) This is clear since the listed fields form a chain under inclusion.

(b) $\Rightarrow$ (c) We proceed by induction on r. The case r = 0 is clear since $K^{p^{-0}} = K$. Suppose that r $\geq$ 1 and that $K^{p^{-(r-1)}}$ is a simple extension of K. If $K^{p^{-r}} = K^{p^{-(r-1)}}$, then there is nothing to prove; hence we assume that $K^{p^{-r}} \neq K^{p^{-(r-1)}}$. Let $u \in K^{p^{-r}} \setminus K^{p^{-(r-1)}}$. We show that $K^{p^{-r}} = K(u)$. By (3.18) and the choice of u, we have $[K(u):K] = p^r$; in fact, $[K(w):K] = p^r$ for any $w \in K^{p^{-r}} \setminus K^{p^{-(r-1)}}$. For any such w, either $K(u) \subseteq K(w)$ or $K(w) \subseteq K(u)$ (since $K \subseteq L$ is a λ-extension). Since these two fields are comparable and have the same finite dimension over K, they must be equal. It follows that $w \in K(u)$ for all $w \in K^{p^{-r}} \setminus K^{p^{-(r-1)}}$. But we also have $K^{p^{-(r-1)}} \subseteq K(u)$, since $u \notin K^{p^{-(r-1)}}$ and $K \subseteq L$ is a λ-extension. Hence $K^{p^{-r}} = K(u)$.

(c) $\Rightarrow$ (d) This is clear.

To simplify the proof of (d) $\Rightarrow$ (a), we next note that the implications (d) $\Rightarrow$ (f) $\Rightarrow$ (e) are proved in the first two paragraphs of the proof of (3.21).

(d) $\Rightarrow$ (a) Let M be a field contained between K and L. First, suppose that $M \not\subseteq K^{p^{-r}}$ for any r $\geq$ 0. Fix r $\geq$ 0 and choose an element $a \in M \setminus K^{p^{-r}}$. By (3.18) and the choice of a, there is an integer i > r such that $[K(a):K] = p^i$. By (d) and (3.21)(2), $K(a) = K^{p^{-i}}$ and so $K^{p^{-r}} \subseteq K^{p^{-i}} \subseteq M$. Since r is arbitrary, we have $L = \bigcup_{r \geq 0} K^{p^{-r}} \subseteq M$; hence $M = L$.

Next, suppose that $M \subseteq K^{p^{-r}}$ for some r $\geq$ 0. Since (d) $\Rightarrow$ (e), $[M:K] \leq [K^{p^{-r}}:K] \leq p^r < \infty$. It follows from (3.20), (d), and (3.21)(2) that $M = K^{p^{-i}}$ for some i $\leq$ r. This completes the proof of (d) $\Rightarrow$ (a).

It remains to prove (e) $\Rightarrow$ (c). We proceed by induction on r. The case r = 0 is clear since $K^{p^{-0}} = K$. Suppose that r $\geq$ 1 and that $K^{p^{-(r-1)}}$ is a simple extension of K. If $K^{p^{-r}} = K^{p^{-(r-1)}}$, then there is nothing to prove; hence we assume that $K^{p^{-r}} \neq K^{p^{-(r-1)}}$. Let $u \in K^{p^{-r}} \setminus K^{p^{-(r-1)}}$. By (3.18) and the choice of u, we have $[K(u):K] = p^r$. Then $p^r = [K(u):K] \leq [K^{p^{-r}}:K] \leq p^r$ by (e), and so $K^{p^{-r}} = K(u)$. ∎

Examples of nontrivial purely inseparable λ-extensions of fields may be found in (3.31).

Proposition (3.24) is a refinement of the statement of Theorem (3.1)(3) in the case of a purely inseparable field extension. Its proof depends on the next lemma.

(3.23) LEMMA. *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic p > 0. If* $K^{p^{-r}} = K^{p^{-(r+1)}}$ *for some* $r \geq 0$*, then* $K^{p^{-r}} = K^{p^{-(r+k)}}$ *for all* $k \geq 0$ *and* $K^{p^{-r}} = L$.

Proof. We proceed by induction on $k \geq 1$. Note that we need only show that $K^{p^{-(r+k)}} \subseteq K^{p^{-r}}$ or, equivalently, that for $u \in L$, $u^{p^{r+k}} \in K$ implies $u^{p^r} \in K$. The case k = 1 is given. Suppose that $k \geq 2$ and that the result holds for k-1. Let $u \in L$ satisfy $u^{p^{r+k}} \in K$. If $v = u^p$, then $v^{p^{r+k-1}} \in K$. By the induction hupothesis, $v^{p^r} \in K$ and so $u^{p^{r+1}} \in K$. It follows from the k = 1 case that $u^{p^r} \in K$. ■

(3.24) PROPOSITION. *Let L be a purely inseparable λ-field extension of a proper subfield K of characteristic p > 0. Then one of the following conditions holds:*

(i) $[L{:}K] < \infty$ *and for some* $s \geq 0$,

$$K = K^{p^{-0}} < K^{p^{-1}} < \ldots < K^{p^{-s}} = L$$

are all the fields between K and L; or

(ii) $[L{:}K] = \infty$ *and*

$$K = K^{p^{-0}} < K^{p^{-1}} < \ldots < K^{p^{-r}} < \ldots . < L$$

are all the fields between K and L. In this case, $K^{p^{-r}} \neq L$ *for all* $r \geq 0$ *and* $\bigcup_{r \geq 0} K^{p^{-r}} = L$.

Moreover, in either case (i) or (ii), $[K^{p^{-r}} : K] = p^r$ *for each r.*

Proof. Suppose that $[L{:}K] = \infty$. By (3.22)(e), $K^{p^{-r}} \neq L$ for all $r \geq 0$. This fact and (3.23) imply that $K^{p^{-r}} \neq K^{p^{-(r+1)}}$ for all $r \geq 0$. This, together with (3.22)(f) and (3.18), implies that $[K^{p^{-r}} : K^{p^{-(r-1)}}] = p$ for all $r \geq 1$. It then follows by induction that $[K^{p^{-r}} : K] = p^r$ for each $r \geq 0$. The last assertion of (ii) follows by the pure inseparability of L over K. Finally, the asserted description of the fields between K and L follows from (3.22)(a).

If $[L:K] < \infty$ then, by (3.20) and (3.22)(a), $L = K^{p^{-s}}$ for some $s \geq 0$. We may suppose that s is minimal with respect to this property. It follows from (3.23) that the intermediate fields $K^{p^{-0}}, K^{p^{-1}}, \ldots, K^{p^{-s}}$ are distinct. By (3.22)(d) and (3.21)(1), we see that $[K^{p^{-r}} : K] = p^r$ for each r, $0 \leq r \leq s$. Another appeal to (3.22)(a) completes the proof. ■

The next theorem characterizes the purely inseparable μ-extensions of fields. Since any λ-extension is a μ-extension, it is not surprising that the conditions (a) and (b) of Theorem (3.25) are weakened versions of the conditions (c) and (e) of Theorem (3.22).

(3.25) THEOREM. *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic p > 0. Then the following four conditions are equivalent:*

(a) $K^{p^{-1}}$ *is a simple extension of K;*

(b) $[K^{p^{-1}} : K] = p$*;*

(c) $K \subseteq L$ *is a μ-extension and* $K^{p^{-1}} = K_0$ *(where* K_0 *is the field in (3.17)(3));*

(d) $K \subseteq L$ *is a μ-extension.*

Proof. First note that since $K = K^{p^{-0}} \neq L$, (3.23) implies that $K \neq K^{p^{-1}}$.

(a) ⇒ (b) If $K^{p^{-1}} = K(u)$ then, by (3.18), $[K(u):K] = p$, i.e., $[K^{p^{-1}}:K] = p$.

(b) ⇒ (c) To show that $K^{p^{-1}}$ satisfies the conditions for K_0 in (3.17)(3), we show that, given any subfield M of L properly containing K, we have $K^{p^{-1}} \subseteq M$. Let $u \in M \setminus K$. If $[K(u):K] = p^r$, then $u^{p^{r-1}} \in K^{p^{-r}} \setminus K$, and so $[K(u^{p^{r-1}}):K] = p$ by (3.18). Since $[K^{p^{-1}}:K] = p$ and $K(u^{p^{r-1}}) \subseteq K^{p^{-1}}$, we have $K(u^{p^{r-1}}) = K^{p^{-1}}$. Since $u \in M$, it follows that $K^{p^{-1}} \subseteq M$, whence the conclusion.

(c) ⇒ (d) Trivial.

(d) ⇒ (c) Since $K \neq K^{p^{-1}}$, the definition of K_0 in (3.17)(3) implies that $K_0 \subseteq K^{p^{-1}}$. If $u \in K^{p^{-1}} \setminus K$, then $K_0 \subseteq K(u)$ and, by (3.18), $[K(u):K] = p$. Hence $K(u)$ is an adjacent extension of K and, since $K_0 \neq K$, we have $K_0 = K(u)$. Thus, $u \in K_0$ for an arbitrary $u \in K^{p^{-1}} \setminus K$ and so $K^{p^{-1}} \subseteq K_0$.

(c) $\Rightarrow$ (a) Since $K^{p^{-1}} = K_0$, we have that $K^{p^{-1}}$ is an adjacent, and so a simple, extension of K. ■

Examples of nontrivial purely inseparable μ-extensions of fields may be found in (3.31).

The next goal is to obtain a partial converse of Proposition (3.15)(2) (see Remark (3.16)(ii)). This partial converse is contained in Theorem (3.29), after some preliminary results. The next result follows from the Propositions of [Bo2, A.V.§5¶2]. Bourbaki calls the field Π in (3.26) the *perfect closure* of K.

(3.26) PROPOSITION. *If K is a field of characteristic $p > 0$, then K has a perfect purely inseparable algebraic extension, which is unique up to K-algebra isomorphism. In particular, if $\overline{K}$ is an algebraic closure of K, then $\Pi := \{u \in \overline{K} : u$ is purely inseparable over $K\}$ is such a field.*

(3.27) PROPOSITION. *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic $p > 0$. Then L is a perfect field if and only if $(K^{p^{-(r+1)}})^p = K^{p^{-r}}$ for all $r \geq 0$. In this case, $[L:K] = \infty$, $K^{p^{-r}} \neq K^{p^{-(r+1)}}$ for all $r \geq 0$, and, moreover, $v^{p^{-r}}$ exists in L for all $v \in K$ and all $r \geq 0$.*

Proof. ($\Rightarrow$) The containment $(K^{p^{-(r+1)}})^p \subseteq K^{p^{-r}}$ is clear. Let $u \in K^{p^{-r}}$. Since L is perfect, $L = L^p$; hence, the element u^{-p} exists in L. Note that $u^{-p} \in K^{p^{-(r+1)}}$. Thus, $u = (u^{-p})^p \in (K^{p^{-(r+1)}})^p$, and so $(K^{p^{-(r+1)}})^p = K^{p^{-r}}$.

($\Leftarrow$) We show that $L \subseteq L^p$. This follows since $\bigcup_{r \geq 0} K^{p^{-r}} = L$ and $K^{p^{-r}} = (K^{p^{-(r+1)}})^p$ for all $r \geq 0$.

Now suppose that L is perfect. If either $[L:K] < \infty$ or $K^{p^{-r}} = K^{p^{-(r+1)}}$ for some $r \geq 0$, then by (3.20) or (3.23) (resp.), $L \subseteq K^{p^{-r}}$ for some $r \geq 0$. Then $L = L^{p^r} \subseteq K$, a contradiction. Hence $[L:K] = \infty$ and $K^{p^{-r}} \neq K^{p^{-(r+1)}}$ for all $r \geq 0$. The last assertion follows from the fact that $L = L^{p^r}$. ■

(3.28) LEMMA. *Let K be a non-perfect field of characteristic $p > 0$. Then $[K:K^p] = p$ if and only if K is a simple extension of K^p. Moreover, if these two equivalent conditions hold, then any purely inseparable algebraic extension L of K such that $L \neq K$ is a μ-extension of K.*

Proof. ($\Rightarrow$) Since $[K:K^p] = p$, K is an adjacent, and so a simple, extension of K^p.

($\Leftarrow$) If $K = K^p(u)$ then, by (3.18), $[K:K^p] = [K^p(u):K^p] = p$.

Now, suppose that these two conditions hold for K. Let L be a purely inseparable algebraic extension of K such that $L \neq K$. To show that L is a μ-extension of K it suffices, by (3.25), to show that $K^{p^{-1}}$ is a simple extension of K. By (3.23), we can choose $u \in K^{p^{-1}} \setminus K$. Then $u^p \in K \setminus K^p$ and so, since K is an adjacent extension of K^p, we have $K^p(u^p) = K$. We show that $K^{p^{-1}} = K(u)$. If $v \in K^{p^{-1}}$ then $v^p \in K = K^p(u^p)$, and so $v^p = f(u^p)$ where, say, $f(X) = a_n{}^pX^n + \ldots + a_0{}^p$, with each $a_i \in K$. Then, $v^p = f(u^p) = (a_nu^n + \ldots + a_0)^p$, whence $v = a_nu^n + \ldots + a_0 \in K(u)$. ■

(3.29) THEOREM. *Let K be a non-perfect field of characteristic $p > 0$ and let Π be the perfect closure of K. Then the following four statements are equivalent:*

(a) $K \subseteq \Pi$ is a λ-extension;

(b) $K \subseteq \Pi$ is a μ-extension;

(c) $[K:K^p] = p$;

(d) K is a simple extension of K^p.

Proof. It follows from (3.28) that (c) $\Leftrightarrow$ (d) $\Rightarrow$ (b), while the implication (a) $\Rightarrow$ (b) follows from (3.15)(2). It suffices to prove that (b) implies both (d) and (a).

(b) $\Rightarrow$ (d) By (3.25), $K^{p^{-1}}$ is a simple extension of K, say $K^{p^{-1}} = K(u)$. Note that $u^p \in K \setminus K^p$. We show that $K = K^p(u^p)$. If $v \in K$ then, since Π is perfect, $v^{p^{-1}}$ exists in Π and lies in $K^{p^{-1}} = K(u)$. Thus $v^{p^{-1}} = a_nu^n + \ldots + a_0$ for some $a_i \in K$, and so $v = (a_nu^n + \ldots + a_0)^p = a_n{}^p(u^p)^n + \ldots + a_0{}^p \in K^p(u^p)$.

(b) $\Rightarrow$ (a) Let $u \in K \setminus K^p$. By (3.26) and (3.27), $u^{p^{-r}}$ exists in Π for all r $\geq$ 0. We prove by induction that $K^{p^{-r}} = K(u^{p^{-r}})$ for all r $\geq$ 0. It will then follow from (3.22)((c) $\Rightarrow$ (b)) that $K \subseteq \Pi$ is a λ-extension.

The base case r = 0 holds since $K^{p^{-0}} = K = K(u) = K(u^{p^{-0}})$. We establish the case r = 1 using the μ-property. By (3.25), $K^{p^{-1}} = K_0$ and so, by (3.17)(3), $K^{p^{-1}}$ is generated as a field extension of K by any one of its elements which is not in K. In particular, $u^{p^{-1}}$ is such an element, since $u \in K \setminus K^p$. Hence $K^{p^{-1}} = K(u^{p^{-1}})$.

Now, suppose r $\geq$ 2 and $K^{p^{-(r-1)}} = K(u^{p^{-(r-1)}})$. Clearly $K(u^{p^{-r}}) \subseteq K^{p^{-r}}$. Suppose $v \in K^{p^{-r}}$. Then $v^p \in K^{p^{-(r-1)}} = K(u^{p^{-(r-1)}})$ and so $v^p = a_n(u^{p^{-(r-1)}})^n + \ldots + a_0$ for some elements $a_i \in K$. The elements $a_i^{p^{-1}}$ exist in Π and lie in $K^{p^{-1}}$. Hence $v^p = (a_n^{p^{-1}})^p((u^{p^{-r}})^p)^n + \ldots + (a_0^{p^{-1}})^p = [\, a_n^{p^{-1}}(u^{p^{-r}})^n + \ldots + a_0^{p^{-1}} \,]^p$, and so $v = a_n^{p^{-1}}(u^{p^{-r}})^n + \ldots + a_0^{p^{-1}} \in K^{p^{-1}}(u^{p^{-r}}) = K(u^{p^{-1}}, u^{p^{-r}}) = K(u^{p^{-r}})$. Thus $K^{p^{-r}} \subseteq K(u^{p^{-r}})$. ■

(3.30) COROLLARY. *Suppose that the field K is nonperfect and that $K \subseteq \Pi$ is a λ-extension, where Π is the perfect closure of K. Then for each integer r ≥ 0 there is a unique (up to K-algebra isomorphism) purely inseparable λ-extension of K of dimension p^r, namely, the field $K^{p^{-r}}$.*

Proof. Up to K-algebra isomorphism, any purely inseparable algebraic extension of K may be regarded as a subfield of Π. By hypothesis and (3.27), Π is an infinite-dimensional λ-extension of K. It then follows by combining (3.22), (3.21) and (3.24) that, for each r $\geq$ 0, there is a unique subfield $K^{p^{-r}}$ of Π of dimension p^r and this field is a purely inseparable λ-extension of K. ■

It is natural to ask whether any fields satisfying the hypotheses of (3.30) exist. Examples of such fields are contained in (3.31)(a).

(3.31) EXAMPLES. (a) Let F be a perfect field (for example, a finite field) of characteristic p > 0 and X an indeterminate over F. Then $K := F(X)$ is a field satisfying $[K:K^p] = p$. Indeed, a degree argument shows that $X \notin K^p$ (see [Bo2, page A.V.7]). Moreover, $K = K^p(X)$, since $K^p \supseteq F^p = F$. As $X^p \in K^p$, (3.18) implies that $[K:K^p] = p$. Hence, by (3.29), K satisfies the hypotheses of (3.30).

(b) Let K be any field of characteristic $p > 0$ satisfying $[K:K^p] = p$, and let Π be the perfect closure of K. Then by (3.27) and (3.29), the field extension $K \subseteq \Pi$ is an infinite-dimensional purely inseparable λ- (and therefore μ-) extension. Moreover, by (3.30), there is a (unique, up to K-algebra isomorphism) purely inseparable λ- (and therefore μ-) field extension of K of each possible finite dimension p^r, $r \geq 0$.

Further examples of purely inseparable λ-extensions are provided by Theorem (3.33), in view of (3.31)(a). The proof of (3.33) depends on the next lemma.

(3.32) LEMMA. *Let L be a nonperfect field of characteristic $p > 0$. Then:*
(1) $L^{p^r} \neq L^{p^{r+1}}$ for all $r \geq 0$.
(2) Relative to the extension $L^{p^r} \subseteq L$, $(L^{p^r})^{p^{-k}} = L^{p^{r-k}}$ for $0 \leq k \leq r$.

Proof. (1) Assume that $L^{p^r} = L^{p^{r+1}}$ for some $r \geq 0$. For any $u \in L$, $u^{p^r} \in L^{p^r} = L^{p^{r+1}}$ and so $u^{p^r} = v^{p^{r+1}}$ for some $v \in L$. Hence $u = v^p \in L^p$. Since u is arbitrary, $L = L^p$, contradicting the assumption that L is not perfect.

(2) The containment $(L^{p^r})^{p^{-k}} \supseteq L^{p^{r-k}}$ is clear. If $u \in (L^{p^r})^{p^{-k}}$, then $u^{p^k} \in L^{p^r}$ and so there is some $v \in L$ such that $u^{p^k} = v^{p^r} = (v^{p^{r-k}})^{p^k}$, whence $u = v^{p^{r-k}} \in L^{p^{r-k}}$. ■

(3.33) THEOREM. *Let L be a nonperfect field of characteristic $p > 0$. Then the following three conditions are equivalent:*
(a) $L^{p^r} \subseteq L$ is a λ-extension for all $r \geq 1$;
(b) $L^{p^r} \subseteq L$ is a λ-extension for some $r \geq 1$;
(c) $[L:L^p] = p$.

Proof. (a) $\Rightarrow$ (b) Trivial.

(b) $\Rightarrow$ (c) Suppose $L^{p^r} \subseteq L$ is a λ-extension with r $\geq$ 1. By (3.22)((b) $\Rightarrow$ (f)), $[(L^{p^r})^{p^{-r}} : (L^{p^r})^{p^{-(r-1)}}] \leq$ p. In other words, by (3.32), $[L{:}L^p] \leq$ p. Since $L \neq L^p$, we must have $[L{:}L^p]$ = p.

(c) $\Rightarrow$ (a) Suppose that $[L{:}L^p]$ = p. Fix r $\geq$ 1. According to (3.22), in order to show that $L^{p^r} \subseteq L$ is a λ-extension, it suffices to show that $[(L^{p^r})^{p^{-k}} : (L^{p^r})^{p^{-(k-1)}}]$ $\leq$ p for 1 $\leq$ k $\leq$ r. In other words, by (3.32)(2), we must show that $[L^{p^{r-k}} : L^{p^{r-k+1}}]$ $\leq$ p for 1 $\leq$ k $\leq$ r. Clearly, it is enough to show that $[L^{p^n} : L^{p^{n+1}}]$ = p for all n $\geq$ 0.

We proceed by induction on n. The case n = 0 is the hypothesis of (c). Suppose that n $\geq$ 1 and $[L^{p^{n-1}} : L^{p^n}]$ = p. According to (3.33)(1), we may choose $u \in L^{p^n} \setminus L^{p^{n+1}}$. We show that $L^{p^n} = L^{p^{n+1}}(u)$. By the choice of u, there is a $v \in L^{p^{n-1}} \setminus L^{p^n}$ such that $v^p = u$. Since, by (3.19)(a), $L^{p^{n-1}}$ is an adjacent extension of L^{p^n}, we have $L^{p^{n-1}} = L^{p^n}(v)$. Then, $L^{p^n} = (L^{p^{n-1}})^p = (L^{p^n}(v))^p = L^{p^{n+1}}(v^p) = L^{p^{n+1}}(u)$. As $u^p \in L^{p^{n+1}}$, it follows from (3.18) that $[L^{p^n} : L^{p^{n+1}}]$ = p. This completes the induction argument and the proof. ■

(3.34) EXAMPLE. By (3.31)(a), we can let the field L in (3.33) be equal to $F(X)$, the field of rational functions in one indeterminate over a perfect field F. In this case, $L^{p^r} = F(X)^{p^r} = F^{p^r}(X^{p^r}) = F(X^{p^r})$. Thus by (3.33), for each r $\geq$ 0, the field extension $F(X^{p^r}) \subseteq F(X)$ is a λ-extension. According to (3.22)(a) and (3.32)(2), the list of intermediate fields looks like

$$F(X^{p^r}) \subseteq F(X^{p^{r-1}}) \subseteq \ldots \subseteq F(X^{p^2}) \subseteq F(X^p) \subseteq F(X).$$

The above completes the discussion of the purely inseparable case. As discussed after the proof of (3.17), the other case we must consider is that of field extensions $K \subseteq L$ such that K is purely inseparably closed in L. We shall, in fact, consider only the special case in which $K \subseteq L$ is an (algebraic) Galois extension. The next result classifies the adjacent Galois extensions.

(3.35) THEOREM. *Let $K \subseteq L$ be an algebraic Galois extension of fields. Then L is an adjacent extension of K if and only if $[L{:}K] = p$ for some prime number p.*

Proof. ($\Leftarrow$) This follows by a degree argument - see the proof of (a)($\Leftarrow$) of Theorem (3.19).

($\Rightarrow$) Since $K \subseteq L$ is an adjacent extension, it follows that L is a nontrivial finite-dimensional Galois extension of K. Therefore, by the Fundamental Theorem of Galois Theory, Gal(L/K) is a finite nontrivial group with no subgroups other than itself and the trivial subgroup. It follows that Gal(L/K) is cyclic of order p, for some prime number p. Hence $[L{:}K] = |\text{Gal}(L/K)| = \text{p}$. ■

It turns out that for Galois field extensions, λ-extensions and μ-extensions coincide. Theorem (3.36) concerns finite-dimensional Galois extensions and Theorems (3.37) and (3.38) concern infinite-dimensional Galois extensions. Note that (3.36) and (3.38) provide a complete characterization of Galois λ-extensions in terms of Galois groups.

(3.36) THEOREM. *Let L be a proper finite-dimensional Galois field extension of a field K. Then the following three conditions are equivalent:*
(a) $K \subseteq L$ is a μ-extension;
(b) $K \subseteq L$ is a λ-extension;
(c) Gal(L/K) is cyclic of prime power order.

Proof. (b)$\Rightarrow$(a) Apply (3.15)(2).

(c)$\Rightarrow$(b) This implication follows from the Fundamental Theorem of Galois Theory and the fact that the subgroups of a cyclic group of prime power order are linearly ordered by inclusion.

(a)$\Rightarrow$(c) Suppose that $K \subseteq L$ is a μ-extension. By (3.17)(3), there is a unique minimal proper field extension K_0 of K inside L. Thus, by the Fundamental Theorem of Galois Theory, $G :=$ Gal(L/K) has a (unique) proper subgroup, say H, which contains every proper subgroup of G. Choose $g \in G \setminus H$; then the cyclic subgroup of G generated by g is not contained in H, and so is equal to G. Thus, G is a cyclic group. For each prime p dividing $|G|$, G has a subgroup, say G(p), of index p. As G(p) is necessarily a maximal subgroup of G, it follows that G(p) =

H. Then, by the Fundamental Theorem of Galois Theory, H determines $G(\mathrm{p})$ and, thus, p. Therefore $|G|$ has only one prime divisor and the conclusion follows. ■

(3.37) THEOREM. *Let L be an infinite-dimensional algebraic Galois field extension of the field K. Then L is a μ-extension of K if and only if L is a λ-extension of K.*

Proof. ($\Leftarrow$) This implication follows from (3.15)(2).

($\Rightarrow$) Assume that L is a μ-extension, but not a λ-extension, of K. Choose intermediate fields E and F which are not comparable under inclusion. Choose elements $x \in E \setminus F$ and $y \in F \setminus E$. Then $K(x)$ and $K(y)$ are incomparable intermediate fields which are finite-dimensional over K. It follows that, in order to prove that L is a λ-extension of K (and thus obtain the desired contradiction), it is sufficient to prove that the set of intermediate fields which are finite-dimensional over K is linearly ordered by inclusion. Clearly, we need only consider the intermediate fields which properly contain K.

Let K_0 be the unique minimal proper field extension of K inside L, which is guaranteed by (3.17)(3). Let E and F be intermediate fields which properly contain K and are finite-dimensional over K. The composite field EF is a finite-dimensional intermediate field properly containing K (hence, containing K_0). Hence, so is N, the normal closure of EF over K. It follows from (3.17)(3) that N is a μ-extension of K. Moreover, N is a finite-dimensional Galois extension of K. Thus, by (3.36), N is a λ-extension of K. It follows that the intermediate fields E and F are comparable under inclusion. ■

The next result is essentially contained in [GHe3, Theorem 2.5], since any infinite-dimensional λ-extension of fields is a J-extension of fields. A detailed proof is given next for completeness.

(3.38) THEOREM. *Let L be an infinite-dimensional algebraic Galois field extension of the field K. Then $K \subseteq L$ is a λ-extension if and only if $Gal(L / K)$ is isomorphic* *(as a topological group) to the additive group $\hat{Z}_p$ of p-adic integers for some prime number p.*

Proof. ($\Leftarrow$) By the Fundamental Theorem of Galois Theory, it is sufficient to show that the closed subgroups of $\hat{Z}_p$ are linearly ordered by inclusion. But this is an immediate consequence of [FJ, (1b) and (1c), page 8].

($\Rightarrow$) Suppose that $K \subseteq L$ is a λ-extension. We first show that each of the intermediate fields F of the extension is a Galois extension of K. Without loss of generality, we may assume that F is a proper subfield of L. By (3.1)(2), F is a finite-dimensional extension of K. To show that F is a Galois extension of K it is sufficient, by [H, Lemma V.2.12], to show that F is stable in $L \setminus K$; i.e., that $\sigma(F) \subseteq F$ for all $\sigma \in \mathrm{Gal}(L / K)$. But in the λ-extension $K \subseteq L$, the proper subfields of L containing K are uniquely determined by their vector space dimension over K. Since $\sigma(F)$ has the same finite vector space dimension as F for all $\sigma \in \mathrm{Gal}(L / K)$, it follows that $\sigma(F) = F$ for all $\sigma \in \mathrm{Gal}(L / K)$.

By (3.1)(3), we may list of intermediate fields of the extension $K \subsetneq L$ as follows:

$$K = L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq L.$$

It follows from (3.36) and the fact that each L_i is a finite Galois λ-extension of K, that each group $\mathrm{Gal}(L_i / K)$ is cyclic of prime power order, for $i \geq 1$. It is clear that there is a prime number p such that, for each i, $\mathrm{Gal}(L_i / K) \cong \mathbf{Z}/p^{a_i}\mathbf{Z}$ for some a_i. In addition, $a_1 < a_2 < a_3 < \ldots$ and $a_i \rightarrow \infty$.

$\mathrm{Gal}(L / K)$ is the inverse limit, in the category of topological groups, of the inductive system

$$\{1\} = \mathrm{Gal}(L_0 / K) \leftarrow \mathrm{Gal}(L_1 / K) \leftarrow \mathrm{Gal}(L_2 / K) \leftarrow \ldots.$$

where each of the mappings is the canonical surjection induced by the restriction map. Similarly, $\hat{Z}_p$ is the inverse limit, in the category of topological groups, of any inductive system of the form

$$\{0\} \leftarrow \mathbf{Z}/p^{a_1}\mathbf{Z} \leftarrow \mathbf{Z}/p^{a_2}\mathbf{Z} \leftarrow \mathbf{Z}/p^{a_3}\mathbf{Z} \leftarrow \ldots.$$

where $a_1 < a_2 < a_3 < \dots$, $a_n \to \infty$, and each of the maps is the canonical surjection. (Each of the groups in both the above inductive systems is a finite group endowed with the discrete topology.)

To prove that $\mathrm{Gal}(L/K)$ is isomorphic to $\hat{Z}_p$ as a topological group, it suffices to show that we may choose isomorphisms $\mathrm{Gal}(L_i/K) \to \mathbf{Z}/p^{a_i}\mathbf{Z}$ for each i in such a way that the following diagram commutes:

$$(*)\qquad \begin{tikzcd} \{1\} & \mathrm{Gal}(L_1/K) \arrow[l] \arrow[d] & \mathrm{Gal}(L_2/K) \arrow[l] \arrow[d] & \mathrm{Gal}(L_3/K) \arrow[l] \arrow[d] & \dots \arrow[l] \\ \{0\} & \mathbf{Z}/p^{a_1}\mathbf{Z} \arrow[l] & \mathbf{Z}/p^{a_2}\mathbf{Z} \arrow[l] & \mathbf{Z}/p^{a_3}\mathbf{Z} \arrow[l] & \dots \arrow[l] \end{tikzcd}$$

We build this diagram recursively. It suffices to consider the general square

$$\begin{tikzcd} \mathrm{Gal}(L_i/K) \arrow[d] & \mathrm{Gal}(L_{i+1}/K) \arrow[l] \arrow[d] \\ \mathbf{Z}/p^{a_i}\mathbf{Z} & \mathbf{Z}/p^{a_{i+1}}\mathbf{Z} \arrow[l] \end{tikzcd} .$$

We may assume that $i \geq 1$. Suppose that the lefthand isomorphism has been determined by $\sigma \mapsto 1 + p^{a_i}\mathbf{Z}$, where σ is a specific generator of $\mathrm{Gal}(L_i/K)$. To define the righthand isomorphism in such a way that the diagram commutes, it is sufficient to show that σ may be extended to some $\tau \in \mathrm{Gal}(L_{i+1}/K)$ such that $\langle\tau\rangle = \mathrm{Gal}(L_{i+1}/K)$ (since the righthand map can then be determined by $\tau \mapsto 1 + p^{a_{i+1}}\mathbf{Z}$). We claim that <u>any</u> extension τ of σ has this property.

Indeed, let F be the fixed field of τ. As $K \subseteq L$ is a λ-extension, either $F \subseteq L_i$ or $L_i \subseteq F$. If $L_i \subseteq F$, then $\sigma = \tau|L_i = 1$, a contradiction since $L_i \neq K$. Hence, $F \subseteq L_i$. Then $\sigma = \tau|L_i$ also fixes F, and so F is contained in the fixed field of σ. But, since $\langle\sigma\rangle = \mathrm{Gal}(L_i/K)$, the fixed field of σ is K. Thus F, the fixed field of τ, is equal to K. It then follows from the Fundamental Theorem of Galois Theory that $\langle\tau\rangle = \mathrm{Gal}(L_{i+1}/K)$.

Therefore, a commuting diagram of the form (*) exists and so the result follows. ■

(3.39) REMARK. It is known that, for each prime p, there are fields K of arbitrary characteristic such that $\hat{Z}_p$ is realizable as the Galois group of some algebraic Galois extension L of K (see [W, Theorem 2] and note that $\hat{Z}_p$ is a profinite group). Indeed, much of [GHe3, §3] is devoted to producing examples of algebraic Galois field extensions with Galois group $\hat{Z}_p$. Thus by (3.38), there are examples of infinite-dimensional Galois λ-extensions in every characteristic. By an argument similar to that in (3.31)(b), it follows via Galois theory that there are examples of Galois λ-extensions of arbitrarily large prime-power dimension in every characteristic (use the fact that $p^n\hat{Z}_p$ is an open (and hence closed) subgroup of $\hat{Z}_p$ of index p^n [FJ, (1a), page 8]).

CHAPTER IV

ADDITIONAL RESULTS

This chapter contains additional results concerning λ-extensions of rings. We first consider, in Theorem (4.2), the conductor ideal $(R:T)$ for a λ-extension $R \subseteq T$. This result, as well as Theorem (4.5), suggests the consideration of λ-extensions $R \subseteq T$ of integral domains, especially those such that T is an overring of R. Example (4.6) shows that for a λ-extension $R \subseteq T$ of integral domains, T need not, in general, be an overring of R. Theorem (4.7), on the other hand, gives a class of integral domains R such that any integral domain λ-extension of R is necessarily an overring of R. In Corollary (4.10) and Proposition (4.11), we classify the λ-overrings of two classes of integral domains. Finally, in (4.12) to (4.15), we consider a general class of λ-extensions of rings.

We begin by considering the conductor $(R:T)$ of a λ-extension $R \subseteq T$ of rings. In general, this ideal need not be a prime ideal of either R or T; for an example of this, see Remark (4.15)(a) below. However, if $R \subseteq T$ is an adjacent extension, Modica [Mo, Theorems 1 and 7] and Ferrand and Olivier [FO, Théorème 2.2 and Lemme 3.2] show that $(R:T)$ is a prime ideal of R. If, moreover, R is integrally closed in an adjacent extension T, they show that $(R:T)$ is also a prime ideal of T. We obtain analogues of these results for λ-extensions in Theorem (4.2). The proof of this Proposition relies on the next Lemma.

(4.1) LEMMA. *(1) If $R \subseteq T$ is a ring extension such that R is root-closed in T, then the conductor $(R:T)$ is a radical ideal of R and also a radical ideal of T.*
(2) Suppose that $R \subseteq T$ is a λ-extension. If $x,y \in R$ and $xy \in (R:T)$, then either $x^2 \in (R:T)$ or $y^2 \in (R:T)$.
(3) Suppose that $R \subseteq T$ is a λ-extension such that R is integrally closed in T. If $x,y \in T$ and $xy \in (R:T)$, then either $x \in R$ with $x^2 \in (R:T)$ or $y \in R$ with $y^2 \in (R:T)$.

Proof. (1) Let $I = \mathrm{Rad}_R((R{:}T))$, the radical of $(R{:}T)$ in the ring R, and let $J = \mathrm{Rad}_T((R{:}T))$. Let $x \in J$. Then $x^n \in (R{:}T)$ for some n ≥ 1. As $(R{:}T) \subseteq R$ and R is root-closed in T, it follows that $x \in R$. Hence, $J \subseteq R$ and so J is an ideal of both R and T. Since $(R{:}T)$ is the largest of the common ideals of R and T, we have $J \subseteq (R{:}T)$. The reverse containment is obvious and so $J = (R{:}T)$. Moreover, since $I = J \cap R$, it is clear that $I = J = (R{:}T)$. Thus $(R{:}T)$ is a radical ideal of both R and T. (For an alternative proof of (1), use [T, Lemma 1.3], which applies because R root-closed in T implies that R is seminormal in T.)

(2) The intermediate rings $R + xT$ and $R + yT$ are comparable. Suppose that $R + xT \subseteq R + yT$. Then $xR + x^2T \subseteq xR + xyT \subseteq R$, since $xy \in (R{:}T)$. Since $xR + x^2T \subseteq R$ and $0 \in xR$, it follows that $x^2T \subseteq R$; hence, $x^2 \in (R{:}T)$. Similarly, if $R + yT \subseteq R + xT$, then $y^2 \in (R{:}T)$.

(3) Suppose, as in the previous paragraph, that $R + xT \subseteq R + yT$. Then $x = r + yt$ for some $r \in R$ and $t \in T$. Multiplying this equality by x and rearranging terms yields the equality $x^2 - rx - xyt = 0$. But $xyt \in R$, since $xy \in (R{:}T)$. Hence this equality is an equation of integrality for x over R. As R is integrally closed in T, we have $x \in R$. The argument in the previous paragraph then shows that $x^2 \in (R{:}T)$. Similarly, if $R + yT \subseteq R + xT$, then $y \in R$ and $y^2 \in (R{:}T)$. ■

(4.2) THEOREM. *Suppose that $R \subseteq T$ is a λ-extension with $R \neq T$. Then:*
(1) $\mathrm{Rad}_R((R{:}T))$ is a prime ideal of R.

Suppose, moreover, that R is integrally closed in T. Then the following conditions also hold:
(2) $(R{:}T)$ is a prime ideal of both R and T. If $R \neq T$, then $(R{:}T)$ is not a maximal ideal of R.
(3) $R/(R{:}T) \subseteq T/(R{:}T)$ is a λ-extension of integral domains and $R/(R{:}T)$ is integrally closed in $T/(R{:}T)$. In addition, $T/(R{:}T)$ is an overring of $R/(R{:}T)$.

Proof. (1) Suppose that $x,y \in R$ and $xy \in \mathrm{Rad}_R((R{:}T))$. Then $x^ny^n = (xy)^n \in (R{:}T)$, for some n ≥ 1. By (4.1)(2), either $x^{2n} = (x^n)^2 \in (R{:}T)$ or $y^{2n} = (y^n)^2 \in (R{:}T)$. Thus, either $x \in \mathrm{Rad}_R((R{:}T))$ or $y \in \mathrm{Rad}_R((R{:}T))$, and so $\mathrm{Rad}_R((R{:}T))$ is a prime ideal of R.

(2) To prove the first assertion of (2), it suffices to show that $(R{:}T)$ is a prime ideal of T, since $(R{:}T) = (R{:}T) \cap R$ is then a prime ideal of R. By (4.1)(1), $(R{:}T)$ is a radical ideal of T. Suppose that $x,y \in T$ and $xy \in (R{:}T)$. By (4.1)(3), either $x^2 \in (R{:}T)$ or $y^2 \in (R{:}T)$. Hence, since $(R{:}T)$ is a radical ideal of T, either $x \in (R{:}T)$ or $y \in (R{:}T)$, as desired. The proof of the second assertion of (2) is postponed until the end of the proof of (3).

(3) It follows from Proposition (3.9) that $R/(R{:}T) \subseteq T/(R{:}T)$ is a λ-extension, and it is clear from (2) that both these rings are integral domains. Moreover, $R/(R{:}T)$ can be shown to be integrally closed in $T/(R{:}T)$ either by invoking [F, Corollary 1.5(5)] and the pullback description $R = R/(R{:}T) \times_{T/(R:T)} T$ or by a direct calculation.

The proof of the last assertion of (3) follows immediately from the next Proposition.

Finally, we prove the last assertion of (2). Suppose that $R \neq T$ and that $(R{:}T)$ is a maximal ideal of R. Then $R/(R{:}T)$ is a field and, as was just asserted in (3), $T/(R{:}T)$ is an overring of $R/(R{:}T)$. Thus, $T/(R{:}T) = R/(R{:}T)$ and so $T = R$, a contradiction. ■

(4.3) PROPOSITION. *Suppose that $R \subseteq T$ is a λ-extension of integral domains such that R is integrally closed in T. Then T is an overring of R.*

Proof. Let L denote the quotient field of T and let K be the copy of the quotient field of R contained in L. Let $u \in T$. By (1.1), $R \subseteq T$ is an INC-pair, and so $R \subseteq R[u]$ satisfies INC. Thus, if Q is any prime ideal of $R[u]$ and $P = Q \cap R$, then Q is isolated in its fiber over P. By Evans' generalization of Zariski's Main Theorem [Ev, Theorem], there exists an $s \in R \setminus P$ such that $R[u]_s = R_s$ in L. It follows that $u \in R_s \subseteq K$. Hence $T \subseteq K$ and so T is an overring of R. ■

Theorem (4.2)(3) indicates how a λ-extension of integral domains may arise from another λ-extension. Theorem (4.5) demonstrates another way in which λ-extensions of integral domains can arise from a general class of λ-extensions. This Theorem uses the construction described in the next Lemma.

(4.4) LEMMA. *Let $R \subseteq T$ be a ring extension such that R is an integral domain and the conductor $(R:T)$ is nonzero. Let K be the quotient field of R.*

(1) Let r be a nonzero element of $(R:T)$. The map $\phi_r : T \to K$ defined by $\phi_r(t) = rt/r$ is an R-algebra homomorphism from T into K. The kernel of ϕ_r is $Ann_T(r)$ and is a prime ideal of T. The image $\phi_r(T)$ is an overring of R and there is a one-to-one inclusion-preserving and inclusion-reflecting correspondence between the set of rings contained between R and $\phi_r(T)$ and the set of rings contained between $R + Ann_T(r)$ and T.

(2) If r_1 and r_2 are any two nonzero elements of $(R:T)$, then the corresponding maps ϕ_{r_1} and ϕ_{r_2} are equal, and so the map ϕ_r is independent of the choice of element r. Let ϕ: $T \to K$ denote this map.

(3) If ϕ is as in (2) and T is an integral domain, then ϕ is an injection. Hence, under these conditions, T is R-algebra isomorphic to an overring of R.

Proof. (1) ϕ_r is clearly R-linear. For $s,t \in T$, $\phi_r(st) = r(st)/r = r^2(st)/r^2 = (rs/r)(rt/r) = \phi_r(s)\phi_r(t)$; hence, ϕ_r is an R-algebra homomorphism. The assertions concerning the kernel are clear from the definition of ϕ_r and the fact that $T/\ker(\phi_r) \cong \phi_r(T)$ is a subring of K. As $\phi_r(x) = x$ for each $x \in R$, it follows that $\phi_r(T)$ contains R and thus is an overring of R. The assertion about the correspondence follows by applying a standard homomorphism theorem to ϕ_r, observing that $\phi_r(R + Ann_T(r)) = R$.

(2) For any $t \in T$, $r_1t/r_1 = r_1r_2t/r_1r_2 = r_2t/r_2$.

(3) If T is an integral domain and r is any nonzero element of $(R:T)$, then $\ker(\phi) = Ann_T(r) = 0$. ■

(4.5) THEOREM. *Let the notation and hypotheses be as in Lemma (4.4) and let ϕ be the map defined in (4.4)(2).*

(1) If $R \subseteq T$ is an adjacent extension then either (i) ϕ is injective and $R \subseteq \phi(T)$ is an adjacent extension; or (ii) $R + Ann_T(r) = T$ and $\phi(T) = R$.

(2) If $R \subseteq T$ is a λ-extension, then $R \subseteq \phi(T)$ is an λ-extension (possibly $\phi(T) = R$).

Proof. (1) $R + Ann_T(r)$ is a ring between R and T, and so either $R + Ann_T(r) = R$ or $R + Ann_T(r) = T$. In the former case, $\ker(\phi) = Ann_T(r) = 0$, since $Ann_T(r) \subseteq R$ and R is an integral domain. Thus ϕ is injective and $R \subseteq \phi(T)$ is an adjacent extension, by the bijection in (4.4)(1). In the latter case, $\phi(T) = \phi(R + Ann_T(r)) = \phi(R + \ker(\phi)) = R$.

(2) This assertion is a consequence of the bijection in (4.4)(1). ■

We now turn our attention to λ-extensions of integral domains. Modica has shown [Mo, Corollary 2 and Theorem 10] that if $R \subsetneq T$ is an adjacent extension of integral domains, then T is an overring of R. Proposition (4.3) extends this result to a λ-extension $R \subsetneq T$ of integral domains, as long as R is integrally closed in T. The next example shows that, for a λ-extension $R \subseteq T$ of integral domains, T need not be an overring of R.

(4.6) EXAMPLE. Let p be a prime number, d a square-free integer and set $L := \mathbf{Q}(\sqrt{d})$. Let $\omega := (1 + \sqrt{d})/2$ if $d \equiv 1 \pmod 4$ and $\omega := \sqrt{d}$ if $d \equiv 2,3 \pmod 4$. Then $\mathbf{Z} + \mathbf{Z}\omega$ is the ring of (algebraic) integers of L. Let $S := \mathbf{Z}\setminus(p)$ and set $R := \mathbf{Z}_S$ and $T := (\mathbf{Z} + \mathbf{Z}\omega)_S$, the integral closure of R in L.

For each $n \geq 0$, let $B_n := (\mathbf{Z} + \mathbf{Z}p^n\omega)_S$. The rings B_n form a decreasing chain of intermediate rings between R and T, each properly containing R, since $p^n\omega \notin \mathbf{Q}$. Moreover, the rings B_n are pairwise distinct. Indeed, if not, then $B_n = B_{n+1}$ for some $n \geq 0$. Then $p^n\omega \in B_{n+1}$; that is, $p^n\omega = (a + p^{n+1}b\omega)/m$, for some integers a,b,m with $m \neq 0$ and $\gcd(m,p) = 1$. Since $\{1, \omega\}$ is linearly independent over $\mathbf{Q}$, we have $p^n = p^{n+1}b/m$, and so $m = pb$, a contradiction.

We next show that the rings B_n are the only rings between R and T which properly contain R. Note that it will then follow that $R \subseteq T$ is an integral λ-extension of integral domains, with an infinite number of intermediate rings, such that T is not an overring of R.

Let A be a ring between R and T which properly contains R. Let $x \in A \setminus R$. Then $x = (a + b\omega)/m$, for some integers a,b,m such that $m \neq 0$ and $\gcd(m,p) = 1$. Write b in the form $b = p^e b'$ with $e \geq 0$ and $\gcd(b',p) = 1$. Then $b' \in S$, and so $p^e\omega$

$= (mx - a)/b' \in A$. Among the elements of A of the form $p^f\omega$, let $p^n\omega$ be the one with the smallest possible exponent n. Then the ring $A \cap (\mathbf{Z} + \mathbf{Z}\omega)$ lies between the rings $\mathbf{Z} + \mathbf{Z}p^n\omega$ and $\mathbf{Z} + \mathbf{Z}\omega$. By a well-known result of number theory, the subrings of $\mathbf{Z} + \mathbf{Z}\omega$ containing $\mathbf{Z} + \mathbf{Z}p^n\omega$ are of the form $\mathbf{Z} + \mathbf{Z}p^k\omega$, where $0 \leq k \leq n$. By the choice of n, $A \cap (\mathbf{Z} + \mathbf{Z}\omega)$ cannot contain any $p^k\omega$ with $k < n$. It follows that $A \cap (\mathbf{Z} + \mathbf{Z}\omega) = \mathbf{Z} + \mathbf{Z}p^n\omega$.

We claim that $A = (\mathbf{Z} + \mathbf{Z}p^n\omega)_S = B_n$. The containment $A \supseteq (\mathbf{Z} + \mathbf{Z}p^n\omega)_S$ is clear. Suppose $y = (a + b\omega)/m \in A$, where $a,b,m \in \mathbf{Z}$, $m \neq 0$ and $\gcd(m,p) = 1$. If p^d is the highest power of p dividing b, then the argument in the previous paragraph shows that $p^d\omega \in A$, and so $d \geq n$, by the choice of n. Thus $b = p^n b'$ for some integer b' and $y = (a + p^n b'\omega)/m \in (\mathbf{Z} + \mathbf{Z}p^n\omega)_S = B_n$.

In the context of the preceding example, it is worth noting the following result of Gilmer and Huckaba [GHu, Theorem 2(3)]. If $R \subseteq T$ is a Δ-extension of integral domains (in particular, if $R \subseteq T$ is a λ-extension of integral domains) and if $[L:K] > 2$, where K and L are the quotient fields of R and T, respectively, then the integral closure of R is a Prüfer domain.

The next result yields a class of integral domains R such that any integral domain λ-extension of R is necessarily an overring of R. This result shows that, in a sense, the success of the preceding example depends on the fact that the base ring is a valuation domain.

(4.7) THEOREM. *Let R be an integrally closed domain, with quotient field K, which is not a valuation domain. If $R \subseteq T$ is a λ-extension such that T is an integral domain, then T is an overring of R.*

Proof. In view of (4.3), we need only show that R is integrally closed in T. Assume, to the contrary, that there is an $x \in T \setminus R$ which is integral over R. Since R is not a valuation domain, there are elements $r,s \in R$ such that the ideals rR and sR are incomparable under inclusion. We show that the intermediate rings $R[1+rx]$ and $R[1+sx]$ are incomparable, and thus achieve a contradiction.

As x is integral over R, the element $1 + sx$ is also integral over R and therefore algebraic over K. Let $f(X) \in K[X]$ be the minimum polynomial of $1 + sx$ over K. Since R is an integrally closed domain, $f(X) \in R[X]$ (cf. [AM, Proposition 5.15]). Let n := deg f. Then $R[1 + sx] = R + R(1 + sx) + \ldots + R(1 + sx)^{n-1}$. Moreover, we claim that the set $\{1, x, x^2, ..., x^{n-1}\}$ is linearly independent over K. Indeed, $K(1+sx) \subseteq K(x)$, and so n = $[K(1+sx):K] \leq [K(x):K]$. Thus, the minimum polynomial of x over K has degree $\geq$ n, and so the linear independence of the above set follows.

We can now show that $R[1 + rx]$ is not a subset of $R[1 + sx]$. Suppose, to the contrary, that $1 + rx \in R[1 + sx] = R + R(1 + sx) + \ldots + R(1 + sx)^{n-1}$; say, $1 + rx = b_0 + b_1(1 + sx) + b_2(1 + sx)^2 + \ldots + b_{n-1}(1 + sx)^{n-1}$, where $b_0, ..., b_{n-1} \in R$. By using the linear independence of the powers of x over K and comparing the coefficients of x on either side of this equation, we see that $r = b_1s + 2b_2s + ... + (n-1)b_{n-1}s \in sR$, contradicting the assumption that rR and sR are incomparable.

An argument analogous to that of the preceding two paragraphs shows that $R[1 + sx]$ is not a subset of $R[1 + rx]$. Thus, the intermediate rings $R[1 + rx]$ and $R[1 + sx]$ are incomparable and we have the desired contradiction. ■

Now we consider λ-extensions $R \subseteq T$ of integral domains such that T is an overring of R. For a general integral domain R, it is difficult to classify the overrings T of R such that $R \subseteq T$ is a λ-extension. In Corollary (4.10), however, we consider a class of integral domains for which we can obtain such a classification. In Proposition (4.11), we obtain a more explicit classification of the λ-overrings of any principal ideal domain.

Following Gilmer [G1], we say that an integral domain R has *property (#)* if, for any two distinct subsets Ω_1 and Ω_2 of the set of maximal ideals of R, the intersections $\bigcap_{M\in\Omega_1} R_M$ and $\bigcap_{M\in\Omega_2} R_M$ are distinct. We need two results, one due to Gilmer and the other to Gilmer and Heinzer, which are stated for convenience as the next Proposition.

(4.8) PROPOSITION. *(1)* [G1, Corollary 2] *Suppose that R is a one-dimensional Prüfer domain. If R has property (#), then each overring of R has property (#).*

(2) [GHe1, Theorem 1] *Let R be a Prüfer domain. Then the following three conditions are equivalent:*

(a) R has property (#);

(b) For any maximal ideal M of R, there is a finitely generated ideal I of R such that M is the only maximal ideal of R containing I;

(c) R is representable in a unique manner as the intersection of a family of mutually incomparable valuation overrings of R.

For the purpose of (4.9) and (4.10), we do <u>not</u> consider the quotient field of an integral domain D to be a valuation overring of D.

(4.9) PROPOSITION. *Let R be a one-dimensional Prüfer domain with property (#). Define the map Φ: {overrings of R} $\to$ {subsets of the set of valuation overrings of R} by $\Phi(T)$ = {valuation overrings of T}. Define the map Ψ: {subsets of the set of valuation overrings of R} $\to$ {overrings of R} by $\Psi(\{V_\alpha\}) = \cap V_\alpha$. Then Φ and Ψ are inverse maps and are both inclusion-reversing.*

Proof. Since R is a Prüfer domain of dimension one, the proper valuation overrings of R are mutually incomparable (they are the localizations of R at its prime ideals of height one) and all have rank one.

$\Psi\Phi = 1$: Any overring T of R is again a Prüfer domain. Hence T is equal to the intersection of its valuation overrings, since it is integrally closed.

$\Phi\Psi = 1$: Without loss of generality, R is not a valuation domain. Let $\{V_\alpha\}$ be a set of valuation overrings of R and set $T := \cap V_\alpha$. We must show that $\{V_\alpha\}$ is equal to the set Γ of valuation overrings of T. Clearly, $\{V_\alpha\} \subseteq \Gamma$. On the other hand, T is a Prüfer domain and so is integrally closed, whence $T = \bigcap\{V : V \in \Gamma\}$. By (4.8)(1), T has property (#). Thus, by (4.8)(2)[(a)$\Rightarrow$(c)] and the fact that the valuation overrings of R are mutually incomparable, we have $\{V_\alpha\} = \Gamma$ (since $\bigcap\{V : V \in \Gamma\} = T = \cap V_\alpha$).

Finally, it is clear from the definitions of Φ and Ψ that they are both inclusion-reversing. ■

(4.10) COROLLARY. *Let R be a one-dimensional Prüfer domain with property (#). Then:*

(1) The overrings of R which are adjacent extensions of R are precisely those overrings which are the intersection of all but one of the valuation overrings of R.

(2) Let T be a proper overring of R. Then T is a λ-extension of R if and only if T is an adjacent extension of R.

Proof. Without loss of generality, R is not a valuation domain.

(1) It is clear from (4.9) that an intersection of all but one of the valuation overrings of R is an adjacent extension of R. Conversely, suppose that an overring T of R is an adjacent extension of R, and let Γ denote the set of valuation overrings of T. By (4.9), since $T \neq R$, Γ does not contain all the valuation overrings of R. If there are two distinct valuation overrings V and W of R which are not contained in Γ, then by (4.9), $\cap(\Gamma \cup \{V\})$ and $\cap(\Gamma \cup \{W\})$ are incomparable rings properly contained between R and T, a contradiction. Thus Γ contains all but one of the valuation overrings of R.

(2) The "if" assertion is immediate. For the converse, we once again let Γ denote the set of valuation overrings of T. As in the proof of (1), Γ does not contain all the valuation overrings of R. As was also shown in the proof of (1), if Γ fails to contain two distinct valuation overrings of R, then there exist two incomparable rings contained between R and T, a contradiction. Thus Γ contains all but one of the valuation overrings of R, and so, by (1), T is an adjacent extension of R. ■

There is a plentiful supply of integral domains satisfying the hypotheses of (4.10). In fact, any Dedekind domain is a one-dimensional Noetherian Prüfer domain and so, by (4.8)(2), has the property (#). In the next result, we consider a special class of integral domains satisfying the hypotheses of (4.10), namely, the principal ideal domains. For these rings, we can obtain a more explicit description of the adjacent overrings.

(4.11) PROPOSITION. *Let R be a principal ideal domain not equal to its quotient field K. Then the adjacent overrings of R are precisely the rings $R[1/p]$, where p is an irreducible element of R.*

Proof. Let T be an adjacent extension of R, and choose $a/b \in T \setminus R$, with $a,b \in R$ and $\gcd(a,b) = 1$. Since T is an adjacent extension of R, we have $T = R[a/b]$. Since a and b are relatively prime, there are elements x,y of R such that $ax + by = 1$. Then $1/b = y + (a/b)x \in T$. As $a/b \in R[1/b]$, it follows that $T = R[1/b]$. Since $T \neq R$, the element b is a nonunit of R. Let p be any irreducible factor of b. If we write $b = pb'$, then $1/p = b'/b \in R[1/b] = T$. Thus $R < R[1/p] \subseteq T$ and so, since T is an adjacent extension of R, we have $T = R[1/p]$.

Conversely, suppose that p is an irreducible element of R and set $T := R[1/p]$. To show that T is an adjacent extension of R, it suffices to show that $T = R[z]$ for an arbitrary element $z \in T \setminus R$. We may write $z = r/p^e$ for some $e \geq 1$ and some $r \in R$ such that $\gcd(r,p) = 1$. Arguing as in the previous paragraph, with an equation of the form $rx + p^e y = 1$, one can show that $1/p^e \in R[z]$. Then $1/p = p^{e-1}(1/p^e) \in R[z]$. Thus $T = R[1/p] = R[z]$. ■

We finish this chapter by discussing a class of examples. In [GHu, Lemma 4], Gilmer and Huckaba consider ring extensions $R \subseteq S$ such that $S = R + Rs$ for some $s \in S$ and show that any such extension is a Δ-extension. We next consider such extensions in more detail, in order to determine the conditions under which such an extension is an adjacent extension or a λ-extension.

(4.12) PROPOSITION. *Let $R \subseteq T$ be a ring extension such that $T = R + Rt$ for some $t \in T$. Let $I := (R:T)$, the conductor of R in T. Then there is a one-to-one inclusion-preserving and inclusion-reflecting correspondence between the set of R-modules contained between R and T and the set of ideals of R containing I. Moreover, each of the R-modules between R and T is a ring (and so the set of rings contained between R and T is in one-to-one correspondence with the ideals of R containing I).*

Proof. The R-linear mapping $R \to T/R$ defined by $x \mapsto xt + R$ is surjective, since $T = R + Rt$, and has kernel $(R:T) = I$. Thus the R-modules R/I and T/R are isomorphic, from which the first assertion follows.

Let M be an R-module between R and T. We show that M is a ring. Let $x,y \in M$. It suffices to show that $xy \in M$. Since $T = R + Rt$, we may write $x = r_1 + r_2t$, $y = s_1 + s_2t$ and $t^2 = a + bt$, for some elements $r_1, r_2, s_1, s_2, a, b \in R$. Note that $r_2t = x - r_1 \in M$ and $s_2t = y - s_1 \in M$, since $R \subseteq M$. Then $xy = r_1s_1 + r_2s_2t^2 + (r_2s_1 + r_1s_2)t = r_1s_1 + r_2s_2a + s_2b(r_2t) + s_1(r_2t) + r_1(s_2t)$, an R-linear combination of elements of M. Hence, $xy \in M$ and the proof is complete. ■

The next result follows immediately from Proposition (4.12) and so its proof is omitted. Notice that, for the special class of ring extensions considered in the Corollary (4.13), assertion (1) below contains the converse of a result due to Ferrand and Olivier [FO, Théorème 2.2(ii)] and Modica [Mo, Theorem 1]; namely, if $R \subseteq T$ is an integral adjacent extension of rings, then the conductor $(R:T)$ is a maximal ideal of R.

(4.13) COROLLARY. *Let R, T and I be as in Proposition (4.12). Then the following assertions hold:*
(1) $R \subseteq T$ is an adjacent extension if and only if I is a maximal ideal of R.
(2) $R \subseteq T$ is a λ-extension if and only if the set of all ideals of R containing I is linearly ordered by inclusion (equivalently, R/I is a chained ring).

Given a ring R and an ideal I of R, the question arises as to whether there exists an extension ring T of R of the type described in (4.12) with $(R:T) = I$. That question is answered affirmatively by the next proposition.

(4.14) PROPOSITION. *(1) Let R be a ring. Then a proper extension ring T of R has the form $T = R + Rt$ for some $t \in T$ if and only if T is R-algebra isomorphic to $R[X]/J$ for some ideal J of $R[X]$ which satisfies the following three conditions:*
(i) $J \cap R = \{0\}$;
(ii) $J + R \neq R[X]$;

(iii) $J + R + Rf(X) = R[X]$ *for some* $f(X) \in R[X]$.
If these three conditions hold for T *and* J, *then the conductor* $(R:T) = \{r \in R : rR[X] \subseteq R + J\}$ *is the largest ideal* I_0 *of* R *such that* $I_0[X] \subseteq R + J$.

(2) Given a ring R *and a proper ideal* I *of* R, *there exists an ideal* J *of* $R[X]$ *satisfying conditions (i) - (iii) of (1) such that* $(R:R[X]/J) = I$; *namely, we may let* $J = IX + X^2R[X]$.

Proof. (1) We first prove the "if" part of the first assertion. Let J be an ideal of $R[X]$ satisfying (i) - (iii). The composite map $R \to R[X] \to R[X]/J$ has kernel $J \cap R$ and so, since $J \cap R = 0$, R embeds as a subring of $R[X]/J$. The image of R under the composite map is $(R + J)/J$ and so, since $R + J \neq R[X]$, the ring $R[X]/J$ is a proper extension of R. Given the polynomial $f(X)$ of condition (iii), let t be the canonical image of $f(X)$ in $R[X]/J$. Then $R[X]/J = (R + Rf(X) + J)/J = R + Rt$.

Next we prove the "only if" assertion. Suppose that $T = R + Rt$, for some $t \in T \setminus R$. Clearly, t is integral over R. Let J be the kernel of the surjective R-algebra homomorphism $R[X] \to T$ defined by $X \mapsto t$, so that T is R-algebra isomorphic to $R[X]/J$. Note that since $T = R + Rt$, it follows that $R[X]/J = R + R(X + J)$. We see that $J \cap R = 0$, since R embeds as a subring of $R[X]/J$. Also, $R + J \neq R[X]$, since R is a proper subring of $R[X]/J$. Finally, we verify condition (iii) for J by showing that $J + R + RX = R[X]$. Indeed, for arbitrary $g(X) \in R[X]$, we may write $g(X) + J = a + b(X + J)$ for some $a,b \in R$. Thus, $g(X) \in a + bX + J \subseteq J + R + RX$. This completes the proof of the first assertion.

Now suppose that the ideal J of $R[X]$ satisfies conditions (i) - (iii). An element $r \in R$ is contained in $I := (R:R[X]/J)$ if and only if $r(g(X) + J) \in R + J$ for all $g(X) \in R[X]$ if and only if $rR[X] \subseteq R + J$. It follows that $I = \{r \in R : rR[X] \subseteq R + J\}$. From this description of I, it is clear that $I[X] \subseteq R + J$. On the other hand, if I_0 is an ideal of R such that $I_0[X] \subseteq R + J$, then $rR[X] \subseteq R + J$ for all $r \in I_0$, and so $I_0 \subseteq I$. Thus I is the largest ideal of R satisfying the condition $I[X] \subseteq R + J$.

(2) Let $J = IX + X^2R[X]$. Condition (i) of (1) is obvious. Condition (ii) holds since I is a proper ideal of R. Condition (iii) holds for J by letting $f(X) = X$.

Now let us compute the conductor $(R{:}R[X]/J)$ using the characterization established in (1). Clearly, $I[X] \subseteq R + J$. If I_0 is an ideal of R such that $I_0[X] \subseteq R + J = R + IX + X^2R[X]$, then $I_0 \subseteq I$. Thus $I = (R{:}R[X]/J)$. ∎

(4.15) REMARKS. (a) It was asserted in the second paragraph of this chapter that the conductor of a λ-extension $R \subseteq T$ of rings need not be a prime ideal of R or T. Examples of this may be obtained via (4.13) and (4.14), by choosing a ring R and an ideal I of R such that I is not a prime ideal and R/I is a chained ring. For example, we may take R to be a valuation domain and I any non-prime ideal of R.

Let $R \subseteq T$ be a λ-extension of the kind constructed in the preceding paragraph, i.e., such that $(R{:}T)$ is not a prime ideal of R. By results of Modica [Mo] and Ferrand and Olivier [FO] mentioned earlier, $R \subseteq T$ is not an adjacent extension. If $R \subseteq T$ is obtained by the construction of (4.12) - (4.14), it is easy to see via (4.12) that the number of rings S such that $R \subseteq S \subseteq T$ is the same as the number of ideals A of R such that $I \subseteq A \subseteq R$. By choosing I to be the square of a maximal ideal in the above construction, we thus find a λ-extension $R \subseteq T$, with exactly one ring properly contained between R and T, such that $(R{:}T)$ is not a prime ideal of R.

(b) In Chapter III, the notion of a μ-extension of fields is introduced and it is shown (see (3.15)(2)) that any λ-extension of fields is a μ-extension of fields. The notion of a μ-extension may be extended to extensions of rings as follows. Say that $R \subseteq T$ is a μ-extension if there an element $t \in T \setminus R$ such that R is maximal among the set of subrings of T not containing t. As in (3.17)(3), it can be shown that this condition is equivalent to the condition that there is an intermediate ring R_0 which properly contains R and which is contained in every intermediate ring which properly contains R.

In the more general ring setting, a λ-extension need not be a μ-extension. We may construct examples using (4.12), (4.13) and (4.14). For example, let K be a field, let $R = K[X]$, and let I be the zero ideal of R. By (4.13) and (4.14), there is a λ-extension T of R such that $(R{:}T) = I = 0$. Since R has no minimal non-zero ideal, it follows from (4.12) that there is no minimal ring among the rings between R and T which properly contain R. Hence, by the remark in the preceding paragraph, T is not a μ-extension of R.

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