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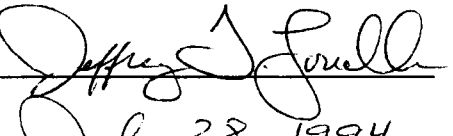
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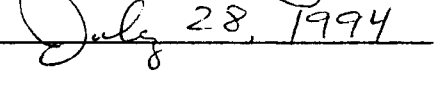
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SURVIVAL AND EXTINCTION: A BRIEF DISCUSSION OF BRANCHING PROCESSES

A Thesis

Presented for the

Master of Science

Degree

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Jeffrey T. Louallen

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ABSTRACT

The theory of branching processes is a subfield of probability which tries to answer questions pertaining to the growth and decay of a general population. The purpose of this thesis is fourfold. First, a brief summary of the general theory of the branching process will be given. Second, some examples of branching processes will be given along with their applications. Third, the process of primary interest, the Galton/Watson branching process, will be given a thorough treatment. Fourth and finally, some simulations based on the discussion of the Galton/Watson model will be given, demonstrating the applicability of the developed theory to reality.

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CHAPTER I

INTRODUCTION

The theory of branching processes is a subfield of probability which tries to answer questions pertaining to the growth and decay of a general population. Origins of this field can be traced back to an article by the Reverend. H.W. Watson and Francis Galton in 1874 entitled "On the Probability of the Extinction of Families" [4]. In this paper Reverend Watson expressed concern over "The decay of families of men who occupied conspicuous positions...," observing that many once common surnames were becoming scarce. At times the article is somewhat amusing such as the assertion of one common belief that this waning was a result of a "diminution in fertility." However, the basic assumptions made in order to attempt to study the phenomena mathematically were the starting point for serious treatment of the general theory.

The purpose of this thesis is fourfold. First, a brief summary of the general theory of the branching process will be given. Second, some examples of branching processes will be given along with their applications. Third the process of primary interest, the Galton/Watson branching process, will be given a thorough treatment. Fourth and finally, some simulations based on the discussion of the Galton/Watson model will be given.

CHAPTER II

GENERAL THEORY OF BRANCHING PROCESSES

The author wishes to make note from the onset that this chapter is not meant to be a thorough treatment of the general theory of this field. It is, however, simply meant to be a summarization of the general theory, based primarily on the text *The Theory of Branching Processes* by Theodore E. Harris, thorough enough to see the origins for conclusions to be drawn in the more simplified treatments to follow. With this in mind, we now begin.

In order to state pertinent conclusions about the general theory, it is first necessary to state a few definitions.

Definition 2.1. The *type* of an object of a particular population is a point x contained in R^d where each of the d components of x measures a distinct characteristic of the object in question. The set of all such types will be denoted by X and is normally restricted to some part of R^d.

Assuming that the number of types of a particular population is finite, the following definition can be stated:

Definition 2.2. A *point distribution* Z on X, $Z = (x_1, n_1; x_2, n_2; \ldots;$ $x_k, n_k)$, is a finite set of distinct points $x_1, x_2, \ldots, x_k$ in X, with a positive integer

"weight" n_i attached to x_i, $i = 1, \ldots, k$; k may be any nonnegative integer, including 0, which corresponds to the null point-distribution. The order in which the pairs x_i, n_i appear is immaterial [4].

Definition 2.3. Associated with any point distribution Z is a *set function* $\tilde{Z}$ defined for every subset A of X as follows:

$$\tilde{Z}(A) = \sum_{x_i \in A} n_i \quad .$$

If Z is null, $\tilde{Z}(A) = 0$ [4].

So, in essence, the set function associated with a point distribution counts the number of elements of the population whose type is contained in the set of interest.

Combining Definitions 2.2 and 2.3, it can be demonstrated that, given a point distribution Z and the corresponding set function $\tilde{Z}$, the following properties hold when $A, A_1, A_2, \ldots$ are subsets of X:

(a) $\tilde{Z}$ is a set function taking on only nonnegative integer values;

(b) if $A_1, A_2, \ldots, A_n$ are disjoint sets, then

$$\tilde{Z}(A_1 \cup A_2 \cup \cdots \cup A_n) = \tilde{Z}(A_1) + \tilde{Z}(A_2) + \cdots + \tilde{Z}(A_n) \ ;$$

(c) if $A_1 \supset A_2 \supset \ldots$ and $\cap_i A_i = 0$, then $\tilde{Z}(A_i)$ is 0 for all sufficiently large i.

Using these properties, it can be shown that there is a one-to-one correspondence between point distributions and set functions. Consequently, for simplicity, the symbol Z will be used in reference to both.

With these foundations, we are now ready to proceed with some initial discussion concerning the general branching process. The progress of a general population can be described as a sequence of point distributions given by $Z_0, Z_1, \ldots,$ where Z_n gives the complete information about the population concerning the total numbers of each of the finite possible types. An example would be a sequence which measures the number of people in a population with a particular color of hair. The number of objects in the n^{th} generation of the population with types contained in some subset A of X will be given by $Z_n(A)$. The following two assumptions will be made: (a) an object's type is enough information so that, coupled with the point distribution Z_n, information concerning earlier generations does not affect the ability to predict the future and (b) interaction between elements of the population does not affect procreation. Note that assumption (a) means that the sequence $\{Z_n\}$ is a Markov process whose states are point distributions. Assumption (b) is the distinction which makes the process a branching process [4].

Before going any further, the notion of probabilities for point distributions needs to be considered. It is necessary to define probabilities such as the following:

$$Prob\{Z(A_1) = r_1, Z(A_2) = r_2, \ldots, Z(A_n) = r_n\}$$

where $r_1, \ldots, r_n$ are nonnegative integers. Assume that $p(A_1, \ldots, A_n; r_1, \ldots, r_n)$ defines a set of functions defined for any $n \geq 1$, any n sets $A_1, \ldots, A_n$, and any $r_1, \ldots, r_n$, $r_i \geq 0$ $\forall i$. For any such set of functions p, the following conditions

must hold:

Conditions 2.1. *Assume n is a positive integer, $A_1, A_2, \ldots$ are subsets of X, and $r_1, \ldots, r_n$ are nonnegative integers.*

(a) $p(A_1, A_2, \ldots, A_n; r_1, r_2, \ldots, r_n)$ is a probability distribution on n-tuples of nonnegative integers $r_1, r_2, \ldots, r_n$. If the A's and the r's are altered with the same permutation, p is not changed. Therefore,

$$p(A_1, A_2; r_1, r_2) = p(A_2, A_1; r_2, r_1) \ .$$

(b) The functions p are consistent. For instance,

$$\sum_{r_2=0}^{\infty} p(A_1, A_2; r_1, r_2) = p(A_1; r_1)$$

(c) if $A_1, A_2, \ldots, A_n$ are disjoint sets and $A = A_1 \cup A_2 \cup \cdots \cup A_n$, then $p(A, A_1, \ldots, A_n; r, r_1, \ldots, r_n) = 0$ unless $r = r_1 + r_2 + \cdots + r_n$, and then $p(A, A_1, \ldots, A_n; r_1 + r_2 + \cdots + r_n, r_1, \ldots, r_n) = p(A_1, A_2, \ldots, A_n; r_1, r_2, \ldots, r_n)$.

(d) If $A_1 \supset A_2 \supset \ldots$ and $\cap_i A_i = 0$, then $\lim_{i \to \infty} p(A_i; 0) = 1$.

Note that condition (c) is motivated by (b) of the properties of a set function associated with a point distribution, and that condition (d) corresponds to property (c) of that list.

Now, under certain conditions, it can be proven that given a set of functions p which satisfy Conditions 2.1, there is a unique probability measure P such that

$$P\{Z(A_1) = r_1, \ldots, Z(A_n) = r_n\} = p(A_1, \ldots, A_n; r_1, \ldots, r_n)$$

where $r_i \geq 0$ $\forall i$[4]. Consider a nonnegative function $s(x)$ defined on X. Clearly $e^{-\int s\, dZ}$ is a random variable. Note that since Z is finite, this integral can be defined as a summation. This will be the case for all integrals to appear in this paper. Define Φ by the following:

$$\Phi(s) = E\{e^{-\int s\, dZ}\}$$

Also clear is that $\Phi(s)$ is both positive and not greater than 1 when $s \geq 0$. Φ is referred to as the *moment generating functional* (MGF) corresponding to the probability measure P. Note that if s is taken to be the indicator function of a set A in X and t is nonnegative, $\Phi(ts) = E\{e^{-tZ(A)}\}$, the ordinary moment generating function of $Z(A)$, and therefore the moments of $Z(A)$ can be obtained by differentiating $\Phi(ts)$ at $t = 0$.

Returning to the general theory, let us consider an individual of type x of some general population. Assume that we can define a set of probabilities p_x such that $p_x(A_1, \ldots, A_n; r_1, \ldots, r_n)$ represents the probability that such an individual has r_1 children with types contained in $A_1, \ldots,$ and r_n with types contained in A_n. We know that this family defines a unique probability distribution P_x. For convenience, assume further that, in reference to the sequence of point distributions which represent our population's future, the initial point distribution Z_0 is nonrandom. The symbols P_x and E_x will represent the probabilities and expectations, respectively, of the given process, if the original population consists of one object of type x. In other words $Z_0 = (x, 1)$. Finally, the MGF of Z_n will be

denoted by $\Phi_x^{(n)}(s) = E_x\{e^{-\int s\ dZ_n}\}$, and $\Phi_x^{(0)}(s) = e^{-s(x)}$.

The remainder of this chapter will be devoted to stating some valuable general conclusions based on our discussion thus far. To begin with, a conclusion involving a recurrence relation for the moment generating functionals discussed above is given.

Theorem 2.1. *If* $\omega = (x_1, n_1; \ldots; x_k, n_k)$, *then*

$$E\{e^{-\int s\ dZ_{n+m}} | Z_n = \omega\} = \left[\Phi_{x_1}^{(m)}(s)\right]^{n_1} \cdots \left[\Phi_{x_k}^{(m)}(s)\right]^{n_k}$$

$$= e^{\int_X log\left[\Phi_x^{(m)}(s)\right]\ d\omega(x)}, \quad n, m = 0, 1, \ldots$$

Furthermore,

$$\Phi_x^{(m+n)}(s) = \Phi_x^{(n)}\big(-log\big[\Phi_{\cdot}^{(m)}(s)\big]\big), \quad n, m = 0, 1, \ldots$$

where $\Phi_{\cdot}^{(m)}(s) = \left[\Phi_{x_1}^{(m)}(s)\right]^{n_1} \cdots \left[\Phi_{x_k}^{(m)}(s)\right]^{n_k}$ *[4].*

The usefulness of this theorem will be demonstrated later when trying to find generating functions for the single-type Galton/Watson model. It will allow us to find a recurrence relationship for the generating functions which can in turn be used to find the individual functions. Knowledge of the generating functions allows one to gain a great deal of useful information which will characterize the population. More will be said about this in the following chapter.

Another important piece of information to have about a branching process is knowledge about the first moments. Recall from the earlier discussion that if s is the indicator function for a subset A of X, then $\Phi_x^{(1)}(ts)$ is the ordinary mgf of

the random variable $Z_1(A)$. Therefore

$$E_x\{Z_1(A)\} = -\frac{\partial \Phi_x^{(1)}(ts)}{\partial t}\bigg|_{t=0}$$

Definition 2.4. We will define $M(x, A) = E_x\{Z_1(A)\}$ [4].

Using Definition 2.4, we can get the following recurrence relationship which will eventually be used to obtain first moments of future generations. Note that for the remainder of this chapter, it will be assumed that $M(x, X)$ is a bounded function of x.

Definition 2.5. Let $M_1(x, A) = M(x, A)$, and let

$$M_{n+1}(x, A) = \int_X M_n(y, A)\, dM(x, y) = \int_X M(y, A)\, dM_n(x, y), \quad n = 1, 2, \ldots$$

We also put $M_0(x, A) = 1$ if x is in A and 0 otherwise.

Note also that the condition concerning the boundedness of $M(x, X)$ assures the boundedness for $M_n(x, X)$ $\forall n \geq 1$.

We are now ready to state the following theorem:

Theorem 2.2. *The functions M_n of Definitions 2.4 and 2.5 are the first moments:* $E_x\{Z_n(A)\} = M_n(x, A), \quad n = 1, 2, \ldots$ *Furthermore,*

$$E\{Z_{n+m}(A)|Z_n = \omega\} = \int M_m(y, A)\, d\omega(y), \quad m, n = 0, 1, \ldots [4]$$

Again, this is quite a strong conclusion. This relationship between the expected values for ensuing generations will afford much in the way of determining the survival for a population.

We now want to consider the existence of eigenfunctions for M. Though the purpose may at first seem elusive, the importance will soon be demonstrated. As before, we must first state some necessary definitions and conditions.

Definition 2.6. Let V be a measure on X such that $0 < V(X) < \infty$.

Definition 2.7. A function f on X is *uniformly positive* if there is a positive constant c such that $f(x) \geq c$ for each x.

Definition 2.8. M_n has a *density* if there is a nonnegative function $m_n(x, y)$ such that for each $x \in X$

$$M_n(x, A) = \int_A m_n(x, y)\, dV(y).$$

The next two definitions deal with eigenfunctions and eigenvalues of M.

Definition 2.9. A function μ, not identically 0, is a *right eigenfunction* of M associated with the eigenvalue ϱ if for each x we have

$$\varrho\mu(x) = \int m(x, y)\mu(y)\, dV(y).$$

Similarly ν is a *left eigenfunction* associated with ϱ if it is not identically 0 and if for each y we have

$$\varrho\nu(x) = \int \nu(x)m(x, y)\, dV(x).$$

Finally, we need the following condition before stating the next theorem.

Condition 2.2. *$M(x, X)$ is a bounded function of x. M has a density m and there is an integer n_0 such that the density $m_{n_0}(x, y)$ is a uniformly positive bounded function, $0 < a \leq m_{n_0}(x, y) \leq b < \infty$.*

With these definitions is mind let us consider the following theorem:

Theorem 2.3. *If Condition 2.2 is satisfied then M has a positive eigenvalue ϱ, larger in magnitude than any other eigenvalue, and corresponding to right and left eigenfunctions μ and ν that are bounded and uniformly positive. Moreover, μ and ν are the only nonnegative right and left eigenfunctions (bounded or not). Furthermore, if we normalize μ and ν so that $\int \mu(x)\nu(x)\, dV(x) = 1$, which will henceforth be assumed, then*

$$m_n(x, y) = \varrho^n \mu(x)\nu(x)\big[1 + O(\Delta^n)\big], \qquad 0 < \Delta < 1,\ n \to \infty\ ,$$

where the bound Δ can be taken independently of x and y.

As stated above, this may at first seem like a merely cerebral conclusion, however, it turns out that this ϱ is uniquely pivotal in determining whether a population will become extinct. Again more will be said about this in the following chapter.

We now come to what are possibly the most important theorems in the theory of branching processes. As stated in the introductory chapter, the main question which this field tries to answer is that of the survival of a population or family. Though clearly these first three theorems are important, they are preliminary conclusions for the more important theorems to follow. The remaining theorems of this chapter have much to say in regards to answering this ultimate question.

Once more, we must begin with a definition.

Definition 2.10. Let $q_n(x) = P_x\{Z_n(X) = 0\}$, $n \geq 0$. Let $q(x) = \lim_{n\to\infty} q_n(x) = P\{Z_n(X) = 0 \text{ for some } n\}$.

We refer to q as the extinction probability. Consider now the following theorem:

Theorem 2.4. *Suppose that Condition 2.2 holds, and that there is a positive integer N such that $q_N(x) > 0$ for each $x \in X$. Then for each $x \in X$ and each $K > 0$ we have*

$$P_x\{0 < Z_n(X) \leq K \text{ infinitely often}\} = 0.$$

The importance of this theorem is that it demonstrates the transience of any branching process. In other words, the population either becomes extinct or grows without bound.

The final three theorems demonstrate, as promised, the pivotal significance of the eigenvalue ϱ.

Theorem 2.5. *Suppose Condition 2.2 holds, $\varrho \leq 1$, and there is a positive integer N such that for each x*

$$q_N(x) > 0$$

Then $q(x) \equiv 1$.

In other words, assuming that we have a positive and finite measure, knowledge of the value of ϱ can tell us immediately if the population will succumb to extinction.

Our final definition will be utilized in the closing theorem of this chapter.

Definition 2.11. Let $M_1^{(2)}(x, A, B) = E_x\{Z_1(A)Z_1(B)\}$. Then define $M_n^{(2)}(x, A, B) = E_x\{Z_n(A)Z_n(B)\}$ $n = 0, 1 \ldots$ and $\upsilon(x, A, B) = M_1^{(2)}(x, A, B) - M(x, A)M(x, B)$.

The next theorem shows the asymptotic nature of certain branching processes.

Theorem 2.6. *Suppose that Condition 2.2 holds, the second moment* $E_x\{Z_1(X)\}^2$ *is a bounded function of* x, *and* $\varrho > 1$. *For each subset* A *of* X *put* $W_n(A) = Z_n(A)/\varrho^n$, $n \geq 1$. *Then for each* x *we have*

$$P_x\{\lim_{n\to\infty} W_n(A) = W(A)\} = 1$$

where $W(A)$ *is a random variable such that*

$$E_x\{W(A)\} = \mu(x) \int_A \nu(y)\, dV(y)$$

$$E_x\{W(A)\} = U(x)\left(\int_A \nu(y)\, dV(y) \right)^2,$$

where $U(x) =$

$$\left(\mu(x)\right)^2 + \sum_{k=1}^{\infty} \varrho^{-2k} \int_X \left\{ \int_X \int_X \mu(y_1)\mu(y_2)\, d_{y_1 y_2} \upsilon(\zeta, y_1, y_2) \right\} dM_{k-1}(x, \zeta).$$

Also, the correlation between $W(A)$ *and* $W(B)$ *is* 1; *i.e., the relation*

$$W(B) = \frac{\int_B \nu(y)\, dV(y)}{\int_A \nu(y)\, dV(y)} W(A)$$

holds with probability 1.

In other words, not only does this theorem prove the asymptotic behavior of $\{Z_n\}$, but more specifically, it gives an expression for the first and second moments of this limiting random variable W. We also have the following corollary as a result of Theorem 2.6:

Corollary 2.1. *If $\varrho > 1$ and the conditions of Theorem 2.6 hold, then the probability of extinction $q(x)$ is, for each x, less than 1.*

With this our knowledge of the future of the population has increased in that we know that if $\varrho > 1$, there is a positive probability of unbounded growth.

The next and final theorem for the general branching process gives even stronger results for the case of $\varrho > 1$.

Theorem 2.7. *Suppose that the conditions of Theorem 2.6 hold and that there is a positive integer N such that $q_N(x) > 0$ for each x. Then (a) q is uniformly positive and uniformly less than 1; (b) q is the only function that is positive, uniformly less than 1, and satisfies the functional equation*

$$q(x) = \Phi_x^{(1)}(-logq) \ ;$$

and (c) if $q^{(0)}$ is any function that is positive and uniformly less than 1, and if we define the sequence $q^{(1)}, q^{(2)}, \ldots$ by

$$q^{(n+1)}(x) = \Phi_x^{(1)}(-logq^n) = \Phi_x^{(n+1)}(-logq^{(0)}) \ ,$$

then $\lim_{n\to\infty} q^{(n)}(x) = q(x)$, $x \in X$.

As with Theorem 2.6, the usefulness of the conclusions given here will be more readily demonstrated in the next chapter. This concludes the discussion of the general theory of branching processes.

CHAPTER III

EXAMPLES OF BRANCHING PROCESSES

In the previous chapter, we gave the basics which define a branching process. As demonstrated, many results can be obtained from the basic assumptions set forth in the general theory. Often, however, it is useful to adapt general theory to a more specific situation in order to obtain stronger results for the given situation. Now that the preliminary foundation has been laid, it seems prudent to consider a few such examples of specific types of processes.

Immigration.

In the last chapter, several of the theorems' conclusions were dependent upon the value of the eigenvalue ϱ. In general, a process is referred to as *critical* if $\varrho = 1$. Similarly, a process is referred to as *supercritical* if $\varrho \geq 1$ and *subcritical* if $\varrho \leq 1$. Consider the subcritical case. From Theorem 2.5 we see right away that the population will die out with certainty. However, an interesting point to ponder is the possibility of immigration. Consider η to be a point distribution, where $\eta(n)$ is the number of immigrants of each of a finite number of types which

join the population in the n^{th} generation. Adding this assumption to the basic branching process allows for further conclusions to be drawn about such a population; most importantly, conditions can be stated under which the population would not in fact die out. [5]

Multiple Ancestry.

Recall again from the previous chapter, that in a general branching process, it is assumed that $Z_0 = (x, 1)$. In other words, the population begins initially with one ancestor of type x. Another interesting twist on the general theory is to disregard this assumption. The question to be answered now is that of the survival of a population which begins with an initial distribution $Z_0 = (x_1, r_1; \ldots; x_n, r_n)$. Again this variation on the general process affords some new, more specific conclusions for such a process. [5]

Continuous Time Process.

Up to this point it has been assumed that the process of consideration is based on a discrete time parameter n. In other words each generation lasted one unit of time. A logical progression from this point would be to consider a process which is dependent upon a continuous time parameter. So the lifetimes for each member of the population is now a random variable. For example, if Z is a point distribution,

then $Z(t)$ is the distribution for the finite types, where $t \in \mathtt{R}$. Generally, when the lifetimes are independent, exponentially distributed random variables, the process is referred to as a *continuous-time Markov branching process.* In the general case however, the process is referred to as *age dependent.* [1] Again, the study of such processes will lead to specialized results.

CHAPTER IV

THE SINGLE-TYPE GALTON/WATSON PROCESS

Upon inspection, one would soon find that to treat any of the examples given in the previous chapter with proper mathematical rigor would be quite an undertaking, one of a much higher caliber than is intended here. For the purpose of this paper, we will consider what is referred to as the Galton/Watson process. At first glance, it will seem a trivial model; however, with respect to mathematical rigor, it will be shown to be nothing of the sort.

A prudent first step is to describe the process mathematically. Now, consider a sequence of point distributions $Z_0, Z_1, \ldots$ for a population of only one type. In this case, Z_n simply counts the number of elements of the population alive during the n^{th} generation. Clearly the lifespan of each element is one time unit. Clearly the basic assumptions associated with a general process will hold. Therefore, the Z_n's form a Markov process, since the states are nonnegative integers, and procreation is independent of the presence of other members of the population. Also, it is assumed that $Z_0 = 1$. Our probability measure will be denoted by P and the probability distribution for Z_1 is determined by $P(Z_1 = k) = p_k$, $k = 0, 1, 2, \ldots$, and $\sum p_k = 1$. The quantity p_k can also be interpreted as the probability that an element of any generation has k children in the ensuing generation. Finally, the

following two additional assumptions will be made. First, $p_k \leq 1$ $k = 0, 1, \ldots$ and $p_0 + p_1 < 1$. Second, $E(Z_1) = \sum_{k=0}^{\infty} kp_k < \infty$. The process has now been defined completely. We can now begin applying the general results of Chapter I. Note that unless otherwise stated, the proofs, at least in sketch form, are taken from [4].

Throughout this section, the usefulness of the probability generation function will be exploited. It is defined as follows:

$$f(s) = \sum_{k=0}^{\infty} p_k s^k \qquad |s| \leq 1$$

Using Theorem 2.1, we have the following recurrence relationship between the moment generating functions of two generations:

$$\phi_{n+m}(s) = \phi_m\big(-ln\big(\phi_n(s)\big)\big) \tag{4.1}$$

If we replace s by e^{-s} we can define $\phi(s) = f(e^{-s})$ and $\phi_n(s) = f_n(e^{-s})$. Using these relationships together with (4.1), we get the following relationships.

$$\begin{aligned} f_{n+1}(s) &= f[f_n(s)] \\ &= f_n[f(s)] \\ f_{n+m}(s) &= f_n[f_m(s)] \end{aligned} \tag{4.2}$$

This lead us to an analog for Theorem 2.1.

Theorem 4.1. *The generating function of Z_n is the n^{th} iterate of $f_n(s)$.*

Proof. Let $f_{(n)}(s)$ be the generating function of Z_n, $n = 0, 1, 2, \ldots$. Conditioning on $Z_n = k$, we can see that the distribution of Z_{n+1} is the sum of k independent random variables, each of which is distributed identically as Z_1. Then the

generating function of $Z_{n+1}|Z_n = k$ is given by:

$$\begin{aligned} E(s^{Z_{n+1}}|Z_n = k) &= E(s^{Z_1{}^1+\cdots+Z_1{}^k}) \\ &= E(s^{Z_1{}^1}\cdots s^{Z_1{}^K}) \\ &= E(s^{Z_1{}^1})\cdots E(s^{Z_1{}^k}) \\ &= (f(s))^k \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad f_{(n+1)}(s) &= E(s^{Z_{n+1}}) \\ &= \sum_{k=0}^{\infty} E(s^{Z_{n+1}}|Z_n = k)P(Z_n = k) \\ &= \sum_{k=0}^{\infty}(f(s))^k \ P(Z_n = k) \\ &= f_{(n)}(f(s)) \end{aligned}$$

Now, using induction, it will be shown that $f_n(s) = f_{(n)}(s)$. Clearly $f_{(0)}(s) = f_0(s)$ by their respective definitions. Now, assume $f_{(n)}(s) = f_n(s)$. Then

$$\begin{aligned} f_{(n+1)}(s) &= f_{(n)}(f(s)) \\ &= f_n(f(s)) \\ &= f_{n+1}(s) \quad \blacksquare \end{aligned}$$

Theoretically, this theorem would enable us to find the probability distribution for Z_n. However, according to Harris, an explicit form is rare. The primary importance of this theorem lies in implementation in the proofs which remain.

With this in mind, let us consider the moments of Z_n. First, let $m = E(Z_1)$ and $\sigma^2 = Var(Z_1) = E[Z_1^2] - \left[E(Z_1)\right]^2$. Using these definitions, we can easily obtain the mean and variance of any generation. However, before stating and proving the next theorem, it is first necessary to prove some preliminary assertions.

Assertion A. *Given the generating function, $f_X(s)$, where X is some non-negative integer-valued random variable, $\lim_{s\to 1^-} f_X^{(n)}(s)$ exists and is finite iff $E[X^n] < \infty$, where $f_X^{(n)}(s)$ is the n^{th} derivative of $f_X(s)$.*

Proof. "$\Rightarrow$"

Assume $\lim_{s\to 1^-} f_X^{(n)}(s) = L_n < \infty$. We want to show that $E[X^n] = \sum_{k=1}^{\infty} k^n p_k$ converges to a finite sum. We know that since $k^n p_k \geq 0$ the conclusion is true iff $\exists C_n \ni \sum_{k=1}^{N} k^n p_k \leq C_n \forall N$. We want to find C_n. Consider the n^{th} derivative of $f_X(s)$, $|s| < 1$,

$$f_X^{(n)}(s) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)s^{k-n}p_k$$

Define d_n such that the following relationship holds:

$$k(k-1)\cdots(k-n+1) \geq d_n k^n$$

$$\iff \quad \frac{k}{k}\frac{(k-1)}{k}\cdots\frac{(k-n+1)}{k} \geq d_n$$

The following is also true:

$$\begin{aligned}\frac{k}{k}\frac{(k-1)}{k}\cdots\frac{(k-n+1)}{k} &= 1\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{(n-1)}{k}\right)\\ &\geq 1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{(n-1)}{n}\right)\\ &= \frac{n}{n}\frac{(n-1)}{n}\cdots\frac{1}{n}\\ &= \frac{n!}{n^n}\end{aligned}$$

$$\Rightarrow \quad d_n = \tfrac{n!}{n^n}$$

$$\Rightarrow \qquad f_X^{(n)}(s) \geq \frac{n!}{n^n} \sum_{k=n}^{\infty} k^n s^{k-n} p_k$$

$$\Leftrightarrow \qquad f_X^{(n)}(s) \geq \frac{n!}{n^n} \sum_{k=n}^{N} k^n s^{k-n} p_k \ ,$$

since $k^n s^{k-n} p_k \geq 0$ and for $N \geq n$.

$$\Leftrightarrow \qquad \frac{n^n}{n!} f_X^{(n)}(s) + \sum_{k=1}^{n} k^n s^{k-n} p_k \geq \sum_{k=1}^{N} k^n s^{k-n} p_k$$

$$\Leftrightarrow \qquad \lim_{s \to 1^-} \left[\frac{n^n}{n!} f_X^{(n)}(s) + \sum_{k=1}^{n} k^n s^{k-n} p_k \right] \geq \lim_{s \to 1^-} \sum_{k=1}^{N} k^n s^{k-n} p_k$$

$$\Leftrightarrow \qquad \frac{n^n}{n!} L_n + \sum_{k=1}^{n} k^n p_k \geq \sum_{k=1}^{N} k^n p_k \ ,$$

since the summations are polynomials, which are continuous, in s. Let $C_n = \frac{n^n}{n!} L_n + \sum_{k=1}^{n} k^n p_k < \infty$. Then $C_n \geq \sum_{k=1}^{N} k^n s^{k-n} p_k$, $\forall N$.

$$\Rightarrow \qquad \sum_{k=1}^{\infty} k^n s^{k-n} p_k = E[X^n] < \infty$$

"$\Leftarrow$" Assume $E[X^n] < \infty$. If a real-valued function is bounded and increasing, over some finite interval, then clearly the function must increase to a finite limit for each element of the domain. Consider $f_X^{(n)}(s)$.

$$f_X^{(n)}(s) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) s^{k-n} p_k$$

For any n, clearly $f_X^{(n)}(s)$ is increasing on $(0,1)$ since $k(k-1)\cdots(k-n+1)p_k \geq 0$, $|s| < 1$, and $k - n \geq 0$. If it can be shown that the function of interest is also

bounded, independent of s, the proof will be finished. Consider the following:

$$\begin{aligned} f_X^{(n)}(s) &= \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)s^{k-n}p_k \\ |s|<1 \rightarrow \qquad &\leq \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)p_k \\ &\leq \sum_{k=n}^{\infty} k^n p_k \\ &\leq \sum_{k=1}^{\infty} k^n p_k \\ &= E[X^n] < \infty \quad \blacksquare \end{aligned}$$

Assertion B. *Given a non-negative integer-valued random variable, X, and its generating function, $f_X(s) = \sum_{k=0}^{\infty} p_k s^k$ for $|s| \leq 1$, the following are true: (1) $\lim_{s\to 1^-} f_X'(s) = E[X]$, and (2) $\lim_{s\to 1^-} f_X''(s) = E[X^2] - E[X]$, where $p_k = P(X = k)$, assuming that the limits exist.*

Proof. (1) Consider the following:

$$\begin{aligned} |f_X'(s) - E[X]| &= \left| \sum_{k=1}^{\infty} k p_k s^{k-1} - \sum_{k=1}^{\infty} k p_k \right| \\ &= \left| \sum_{k=1}^{\infty} k p_k (s^{k-1} - 1) \right| = \spadesuit \end{aligned}$$

If it can be shown that this difference goes to zero as s goes to 1 from the left, the desired result will be achieved. Recall that if $\sum_{k=1}^{\infty} a_n$ converges, then $\forall\, \varepsilon > 0 \quad \exists\, N$ such that $\sum_{k=N+1}^{\infty} a_n < \frac{\varepsilon}{2}$. Choose $\varepsilon > 0$. For appropriately large N, the following is true:

$$\begin{aligned} \Rightarrow \qquad \spadesuit &\leq \left| \sum_{k=1}^{N} k p_k (s^{k-1} - 1) \right| + \left| \sum_{k=N+1}^{\infty} k p_k (s^{k-1} - 1) \right| \\ &< \left| \sum_{k=1}^{N} k p_k (s^{k-1} - 1) \right| + \frac{\varepsilon}{2} \quad , \end{aligned}$$

since $\left| \sum_{k=N+1}^{\infty} kp_k(s^{k-1}-1) \right| \leq \sum_{k=N+1}^{\infty} kp_k|s^{k-1}-1| \leq \sum_{k=N+1}^{\infty} kp_k \leq \frac{\varepsilon}{2}$,
a result of the fact that $E[X] < \infty$.

$$\Leftrightarrow \quad \lim_{s\to 1^-} \left| \sum_{k=1}^{\infty} kp_k(s^{k-1}-1) \right|$$
$$\leq \lim_{s\to 1^-} \left(\left| \sum_{k=1}^{N} kp_k(s^{k-1}-1) \right| + \frac{\varepsilon}{2} \right) = 0 + \frac{\varepsilon}{2} < \varepsilon$$

The limit exists since the summation is a polynomial in s. Since ε is arbitrary, we have the desired result.

(2) Consider the following:

$$\left| f_X''(s) - E[X^2] + E[X] \right|$$
$$= \left| \sum_{k=2}^{\infty} k(k-1)p_k s^{k-2} - \sum_{k=1}^{\infty} k^2 p_k + \sum_{k=1}^{\infty} kp_k \right|$$

$$= \left| \sum_{k=2}^{\infty} k^2 p_k(s^{k-2}-1) + \sum_{k=2}^{\infty} kp_k(1-s^{k-2}) \right|$$

$$\leq \left| \sum_{k=2}^{N_1} k^2 p_k(s^{k-2}-1) + \sum_{k=2}^{N_2} kp_k(1-s^{k-2}) \right|$$
$$+ \left| \sum_{k=N_1+1}^{\infty} k^2 p_k(s^{k-2}-1) + \sum_{k=N_2+1}^{\infty} kp_k(1-s^{k-2}) \right|$$

$$< \left| \sum_{k=1}^{N_1} k^2 p_k(s^{k-2}-1) + \sum_{k=1}^{N_2} kp_k(1-s^{k-2}) \right| + \varepsilon \ ,$$

since $|s^{k-2}-1| \leq 1$ and since $\forall\ \varepsilon > 0$, $\exists\ N_1$,N_2 such that $\sum_{k=N_1+1}^{\infty} k^2 p_k < \frac{\varepsilon}{2}$

and $\sum_{k=N_2+1}^{\infty} kp_k < \frac{\varepsilon}{2}$. Therefore,

$$\lim_{s\to 1^-} \left| f_X''(s) - E[X^2] + E[X] \right|$$
$$\leq \lim_{s\to 1^-} \left| \sum_{k=1}^{N_1} k^2 p_k (s^{k-2} - 1) + \sum_{k=1}^{N_2} kp_k (1 - s^{k-2}) \right| + \varepsilon = \varepsilon$$

As before the limit exists since the summation is a polynomial, which is continuous in s. Since ε is arbitrary, we have the desired result. ■

We now state the next theorem which is an analog to Theorem 2.2.

Theorem 4.2. *The expected value $E[Z_n]$ is m^n, $n = 0, 1, 2, ...$*

If $\sigma^2 = Var[Z_n] < \infty$, then

$$Var[Z_n] = \begin{cases} \dfrac{\sigma^2 m^n (m^n - m)}{m^2 - m}, & if\ m \neq 1 \\ \\ \sigma^2 n, & if\ m = 1 \end{cases}$$

Proof. First we need to show that $E[Z_n] = m^n$. We will achieve this by looking at the generating function for Z_n and deriving a recurrence relationship. This will then be used with induction to obtain the desired result.

Consider the generating function for Z_{n+1}.

$$f_{n+1}(s) = \sum_{k=0}^{\infty} s^k P(Z_{n+1} = k)$$

$$\longrightarrow \qquad f'_{n+1}(s) = \sum_{k=1}^{\infty} k s^{k-1} P(Z_{n+1} = k)$$

For notational purposes, in the remainder of this thesis, $f_n'(1) = \lim_{s \to 1^-} f_n'(s)$. If it can be shown that $f_{n+1}'(1) < \infty$, then by Assertion B $f_{n+1}'(1) = E[Z_{n+1}]$. Consider the following recurrence relationship based on the definition $f_{n+1}(s) = f\big(f_n(s)\big)$.

$$f_{n+1}'(s) = f'\big(f_n(s)\big) f_n'(s) \qquad (a)$$

Since it is assumed that $E[Z_1] < \infty$, this recurrence relationship coupled with Assertion A assures that $f_n'(1) < \infty$ for all n. Therefore, $f_{n+1}'(1) = E[Z_{n+1}]$. By (a) above, the following must also be true.

$$\begin{aligned} E[Z_{n+1}] &= f'\big(f_n(1)\big) E[Z_n] \\ &= f'(1) E[Z_n] \\ &= m E[Z_n] \end{aligned}$$

Claim 4.2.1. $E[Z_{n+1}] = m^{n+1}$.

Proof. This will be proven by induction. First, clearly $E[Z_1] = m$ by assumption. Now, assume $E[Z_n] = m^n$. Then $E[Z_{n+1}] = mE[Z_n] = mm^n = m^{n+1}$.

Now it remains to be shown that the variance for each generation is given by

$$Var[Z_n] = \begin{cases} \dfrac{\sigma^2 m^n (m^n - m)}{m^2 - m}, & if\ m \neq 1 \\ \\ \sigma^2 n, & if\ m = 1 \end{cases}$$

To achieve this, we will employ the same methods as above. Consider the second derivative of the generating function of Z_{n+1}.

$$f''_{n+1}(s) = f'\big(f_n(s)\big)f''_n(s) + \big[f'_n(s)\big]^2 f''\big(f_n(s)\big)$$

hence

$$\begin{aligned} f'_{n+1}(1) &= f'(1)f''_n(1) + m^{2n}f''(1) \\ &= mf''_n(1) + m^{2n}(\sigma^2 + m^2 - m) \end{aligned}$$

As before, if it can be shown that $f''_n(1) < \infty$, Assertion B will give that

$$f''_{n+1}(1) = E[Z^2_{n+1}] - E[Z_{n+1}].$$

We are given that $Var[Z_1] < \infty$ which implies that the second moment of Z_1 is finite and therefore, by Assertion A, that $f''(1) < \infty$. Again, using the given recurrence relationship assures that $f''_n(1) < \infty$ for all n. The following is therefore true.

$$f''_{n+1}(1) = E[Z^2_{n+1}] - E[Z_{n+1}] = E[Z^2_{n+1}] - m^{n+1} \qquad (b)$$

Claim 4.2.2. $f''_{n+1}(1) = (\sigma^2 - m + m^2)m^n \sum_{k=0}^{n} m^k$

Proof. This will be proven with induction also. First, using (b) above shows the statement to be true for $n = 0$. Now assume that

$$f''_{n+1}(1) = (\sigma^2 - m + m^2)m^n \sum_{k=0}^{n} m^k.$$

The following is therefore true.

$$\begin{aligned} f''_{n+2}(1) &= mf''_{n+1}(1) + m^{2(n+1)}(\sigma^2 + m^2 - m) \\ &= m\left[(\sigma^2 - m + m^2)m^n \sum_{k=0}^{n} m^k\right] + m^{2(n+1)}(\sigma^2 + m^2 - m) \\ &= (\sigma^2 - m + m^2)m^{n+1}\left(\sum_{k=0}^{n} m^k + m^{n+1}\right) \\ &= (\sigma^2 - m + m^2)m^{n+1} \sum_{k=0}^{n+1} m^k \end{aligned}$$

The variance of the $(n+1)^{st}$ is as given below.

$$\begin{aligned} Var[Z_{n+1}] &= E[Z_{n+1}^2] - E[Z_{n+1}]^2 \\ &= E[Z_{n+1}^2] - m^{2(n+1)} \\ &= f''_{n+1}(1) + m^{n+1} - m^{2(n+1)} \\ &= (\sigma^2 - m + m^2)m^n \sum_{k=0}^{n} m^k + m^{n+1} - m^{2(n+1)} \end{aligned}$$

Where the last equality holds as a result of claim 4.2.2, and the next to last equality holds as a result of (b) above. Using the fact that

$$\sum_{k=0}^{n} m^k = \begin{cases} \dfrac{m^{n+1} - 1}{m - 1}\,, & if\ m \neq 1 \\ \\ n + 1\,, & if\ m = 1 \end{cases}$$

coupled with the equation derived for $Var[Z_{n+1}]$ will give the final result. ■

We now turn to a theorem which gives an answer to the question of survival of the population. Let us first define more mathematically what it means for a population to become extinct.

Definition 4.1. *Extinction* refers to the event that the sequence $\{Z_n\}$ is composed of zeros for all but a finite number of terms.

Also, it will be helpful to restate Definition 2.10 with respect to our current process.

Definition 4.2. Define q to be the probability of extinction. In other words,

$$q = P(Z_n \to 0) = \lim_{n\to\infty} f_n(0) \tag{4.3}$$

Now consider the following theorem which is a result of Theorems 2.5, 2.7, and Corollary 2.1.

Theorem 4.3. *If $m = E[Z_1] \leq 1$, then the extinction probability, $\lim_{n\to\infty} f_n(0)$, denoted by q, is 1. If $m > 1$, q is the unique non-negative solution less than 1 of the equation $s = f(s)$.*

Proof. In order to prove the theorem, it is necessary to first prove two claims about the sequence $\{f_n(0)\}$.

Claim 4.3.1. $f_n(0) < 1,\ n = 0, 1, 2, ...$

Proof. This will be demonstrated with induction. First, clearly $f_0(0) = 0 < 1$. Next, assume $f_n(0) < 1$.

$$\begin{aligned} f_{n+1}(0) = f\big(f_n(0)\big) &= p_0 + \sum_{k=1}^{\infty} \big(f_n(0)\big)^k p_k \\ &< p_0 + \sum_{k=1}^{\infty} p_k = 1 \end{aligned}$$

Claim 4.3.2. $f_n(0) \leq f_{n+1}(0) \ \ \forall n.$

Proof. This will also be shown with induction. First, it is obvious that $f_1(0) = p_0 \geq 0 = f_0(0)$. Now, assume that $f_{n+1}(0) \geq f_n(0)$.

$$f_{n+2}(0) = f\big(f_{n+1}(0)\big) = p_0 + f_{n+1}(0)p_1 + {f^2}_{n+1}(0)p_2 + \cdots$$

$$\geq p_0 + f_n(0)p_1 + {f^2}_n(0)p_2 + \cdots$$

$$= f\big(f_n(0)\big) = f_{n+1}(0)$$

Let q be as denoted above. This limit exists since $\{f_n(0)\}_{n\geq 0}$ is an increasing sequence, claim 4.3.2, and it is bounded above by 1, claim 4.3.1. Also note that, since $f_{n+1}(0) = f\big(f_n(0)\big)$, $\lim_{n\to\infty} f_{n+1}(0) = \lim_{n\to\infty} f\big(f_n(0)\big)$, and therefore

$$q = f(q) \quad 0 \leq q \leq 1 \ , \qquad\qquad (*)$$

since f is a continuous function. The proof will be completed by considering two cases: $m \leq 1$ and $m > 1$.

Case 1. $m \leq 1$.

Since it is true that $f'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}$ it must also be true that $f'(s)$ is strictly increasing and therefore $f'(s) < f'(1) = E[Z_1] \leq 1$, when $0 \leq s < 1$. Also, for f, the Mean Value Theorem gives the following.

$$\exists\, c \in (s,1) \ \ni \quad \frac{f(1) - f(s)}{1 - s} = f'(c)$$

$$\longleftrightarrow \qquad f(1) - f(s) = f'(c)(1 - s)$$

$$\longleftrightarrow \qquad 1 - f(s) < 1 - s$$

$$\longleftrightarrow \qquad f(s) > s$$

since $f'(c) < 1$ and $f(1) = 1$. Therefore $(*)$ has no solution for $0 \leq s < 1$, and $q = 1$.

Case 2. $m > 1$

Since $m = f'(1) > 1$, there exists an $\varepsilon > 0$ such that $f'(1-\varepsilon) > 1$ by the continuity of f'. Choose $s \in (1-\varepsilon, 1)$ and note that $f'(s) > 1$ (f' is increasing). Again the MVT gives the following.

$$\exists\, c \in (s,1) \ \ni \quad \frac{f(1)-f(s)}{1-s} = f'(c) > 1$$

$$\longleftrightarrow \qquad f(1)-f(s) = f'(c)(1-s)$$

$$\longleftrightarrow \qquad 1-f(s) > 1-s$$

$$\longleftrightarrow \qquad f(s) < s \qquad\qquad (\diamondsuit)$$

Now, it is also true that $f(0) > 0$ and $f(1) = 1$. It must therefore be true that there exists $p \in [0,1)$ such that $f(p) = p$.

Claim 4.3.3. $\exists!\ p \in [0,1) \ \ni \ f(p) = p$.

Proof. Assume there are two solutions, a and b. Define $g(s) = f(s) - s$.

$$\longrightarrow \qquad g(a) = g(b) = g(1) = 0$$

Rolle's Theorem then gives the following.

$$\exists\, c \in (a,b),\ d \in (b,1) \ \ni g'(c) = 0 = g'(d)$$

$$\longrightarrow \qquad f'(c) - 1 = 0 = f'(d) - 1$$

$$\longleftrightarrow \qquad f'(c) = 1 = f'(d)$$

This implies that $f'(s)$ is not strictly increasing, a contradiction. Therefore, the solution must be unique, $0 \leq s < 1$.

Claim 4.3.4. $q = p$

Proof. Assume that $q = 1$ and choose $\varepsilon > 0$. This implies that there is an $n \geq 0$ such that $f_n(0) \in (1-\varepsilon, 1)$. This in turn implies that $f_{n+1}(0) = f\big(f_n(0)\big) < f_n(0)$ by $(\Diamond)$ above. This violates claim 4.3.2 above.

Therefore q is the unique nonnegative solution less than 1 of $f(s) = s$. ■

Note that the previous theorem was divided into cases based on the value of m. Recall from Chapter I that most of the theorems were conditional based on the value of the eigenvalue ϱ. In this simple case, the matrix M from Chapter I is simply the $1x1$ matrix given by $M = m = E(Z_1)$ which has m as its only eigenvalue. It should therefore come as no surprise that the remainder of the theorems in this chapter will pivot on the value of m.

We now have some results concerning extinction. However, what are some other possible fates for the population? The next two theorems address this question.

Theorem 4.4. *If* $m > 0$, $\lim_{n\to\infty} P(Z_n = k) = 0$, *for* $k = 1, 2, \ldots$. *Moreover,* $Z_n \to \infty$ *with probability* $1 - q$, *and* $Z_n \to 0$ *with probability* q.

Proof. In order to prove that the above limit is equal to 0, it will be demonstrated that each state k, $k = 1, 2, \ldots$, is transient. In other words, it will be shown that

$P\{Z_n = k \text{ for infinitely many values of n}\} = 0$. Define $R_k = P\{Z_{n+j} = k \text{ for some } j \geq 1 | Z_n = k\}$. By definition, if each state k is transient, then $R_k < 1$, $k = 1, 2, \ldots$ The proof will be given in two cases.

Case 4.4.1. Assume $p_0 = 0$.

Proof. $R_k = P\{Z_{n+j} = k \text{ for some } j \geq 1 | Z_n = k\}$

$= \sum_{i=1}^{\infty} P\{Z_{n+i} = k, Z_{n+i-1} \neq k, \ldots, Z_{n+1} \neq k | Z_n = k\}$

$= P\{Z_{n+1} = k | Z_n = k\} + P\{Z_{n+2} = k, Z_{n+1} \neq k | Z_n = k\} + \ldots$

$$+P\{Z_{n+l} = k, Z_{n+l-1} \neq k, \ldots, Z_{n+1} \neq k | Z_n = k\} + \ldots$$

$= p_1^k + \frac{P\{Z_{n+2}=k, Z_{n+1}\neq k, Z_n=k\}}{P\{Z_n=k\}} + \ldots$

$$+\frac{P\{Z_{n+l}=k, Z_{n+l-1}\neq k, \ldots, Z_{n+1}\neq k, Z_n=k\}}{P\{Z_n=k\}} + \ldots$$

$= p_1^k + \frac{P\{Z_{n+2}=k, Z_{n+1}>k, Z_n=k\}}{P\{Z_n=k\}} + \ldots$

$$+\frac{P\{Z_{n+l}=k, Z_{n+l-1}>k, \ldots, Z_{n+1}>k, Z_n=k\}}{P\{Z_n=k\}} + \ldots$$

$= p_1^k + 0 + \cdots + 0 + \cdots = p_1^k < 1$

Case 4.4.2. Assume $p_0 > 0$

Proof. Define $A_k = \{Z_{n+j} = k \text{ for some } j \geq 1, Z_n = k\}$ and $B_k = \{Z_{n+1} \neq 0, Z_n = k\}$. Note that $A_k \subset B_k$. This implies the following.

$$R_k = \frac{P(A_k)}{P(Z_n = k)} \leq \frac{P(B_k)}{P(Z_n = k)}$$

$$\longleftrightarrow \qquad R_k \leq 1 - p_0^k < 1$$

Since Z_n therefore does not take on the same positive value infinitely often, $\lim_{n\to\infty} P(Z_n = k) = 0$, for $k = 1, 2, \ldots$ and, moreover, $Z_n \to \infty$ or $Z_n \to 0$.

Theorem 4.3 already states that the probability of the latter is q. Therefore the probability of the former must be $1 - q$. ■

This demonstrates the fact that a Galton/Watson process is quite unstable. Note that this theorem is actually an application of Theorem 2.4.

Now, consider the following question: What can we say about the probability of extinction of the population given that it has not died out in the n^{th} generation? In other words, what is $P[\text{population becomes extinct eventually } | Z_n > 0]$? Further what is the limiting value of this probability as $n \to \infty$? The answer is as follows.

Theorem 4.5. *The sequence* $c_n = P[\text{population becomes extinct eventually } | Z_n > 0]$ *decreases to* 0 *as* $n \to \infty$.

Proof. We will prove this theorem using properties of conditional expectation.

$$c_n = P[\text{population becomes extinct eventually } | Z_n > 0]$$

$$= \frac{P[\text{population becomes extinct eventually }, Z_n > 0]}{P[Z_n > 0]}$$

$$= \frac{\sum_{k=1}^{\infty} P[\text{population becomes extinct eventually } | Z_n = k] P[Z_n = k]}{P[Z_n > 0]}$$

$$= \frac{\sum_{k=1}^{\infty} q^k P[Z_n = k]}{P[Z_n > 0]} = \frac{\sum_{k=1}^{\infty} q^k P[Z_n = k]}{\sum_{k=1}^{\infty} P[Z_n = k]}$$

$$= \frac{\sum_{k=1}^{\infty} q^k P[Z_n = k]}{1 - f_n(0)} = \frac{f_n(q) - P(Z_n = 0)}{1 - f_n(0)}$$

$$= \frac{q - f_n(0)}{1 - f_n(0)} = 1 - \frac{1 - q}{1 - f_n(0)}$$

decreases to 0 since $1 - f_n(0)$ decreases to $1 - q$. ■

It should be stated here that this theorem was not taken from [4] but rather is the result of a prompt from the author's advisor. The result means that the chances of the survival of the population increases with time.

In the final theorem to be considered we will draw upon Theorems 4.3 and 4.4. From Theorem 4.3, we know that if the expected value of the first generation is greater than 1, there is a positive probability that our population will survive indefinitely. Coupling this with Theorem 4.4, we can further state that the population will increase indefinitely. A natural expectation would be that the size of the population should grow geometrically. We make the following definition.

Definition 4.3. Define $W_n = Z_n/m^n$.

The final theorem may now be stated.

Theorem 4.6. *If $m > 1$ and $E(Z_1^2) < \infty$, then the random variables $W_n = \frac{Z_n}{m^n}$ converge with probability 1 and in mean square to a random variable W, and*

$$E(W) = 1 \qquad Var(W) = \frac{Var(Z_1)}{m^2 - m} > 0 \tag{4.4}$$

Proof. First we must prove a claim which will aid in the main proof of the theorem.

Claim 4.6.1. $E[X|Y] = E\big[E[X|Y,Z] \mid Y\big]$, where X, Y, Z are positive, integer-valued random variables.

Proof. Define $E[X|Y] := h_1(Y)$. Then $h_1(y) = E[X|Y = y] = \sum_x xP(X = x|Y = y)$. Also, define $V = E[X|Y,Z]$ and $V := h_2(Y,Z)$. Then $h_2(y,z) = E\big[X|\, Y = y, Z = z\big] = \sum_x xP(X = x|\, Y = y, Z = z)$. Finally, define $E[V|Y] := h_3(Y)$. This implies the following:

$$h_3(y) = E\big[h_2(Y,Z)|\, Y = y\big]$$

$$\text{substitution principle} \rightarrow = E\big[h_2(y,Z)|\, Y = y\big]$$

$$\text{law of unconscious statistician} \rightarrow = \sum_z h_2(y,z)P(Z = z|Y = y)$$

We want to show that $h_1(y) = h_3(y)$. We have that

$$\begin{aligned}
h_3(y) &= \sum_z h_2(y,z) P(Z=z|Y=y) \\
&= \sum_z h_2(y,z) \frac{P(Z=z, Y=y)}{P(Y=y)} \\
&= \sum_z \sum_x x P(X=x|Y=y, Z=z) \frac{P(Z=z, Y=y)}{P(Y=y)} \\
&= \sum_z \sum_x x \frac{P(X=x, Y=y, Z=z)}{P(Y=y, Z=z)} \frac{P(Z=z, Y=y)}{P(Y=y)} \\
&= \sum_z \sum_x x \frac{P(X=x, Y=y, Z=z)}{P(Y=y)} \\
&= \sum_z \sum_x x P(X=x, Z=z|Y=y) \\
&= \sum_x x \sum_z P(X=x, Z=z|Y=y) \\
&= \sum_x x P(X=x|Y=y) = E[X|Y=y] = h_1(y) \ .
\end{aligned}$$

One final preliminary to the main proof is to derive a helpful recurrence relationship. Consider $E[Z_{n+1}|Z_n = k]$.

$$E[Z_{n+1}|Z_n = k] := E[Z_1^1 + \cdots + Z_1^k] = kE[Z_1] = km = Z_n m \tag{4.5}$$

Now consider $E[Z_{n+j}|Z_n = k]$.

Claim 4.6.1 $\rightarrow E[Z_{n+j}|Z_n] = E\{\ E[Z_{n+j}|\ Z_{n+j-1}, \ldots, Z_n]\big|\ Z_n\}$

Markov process $\rightarrow$ $= E\{\ E[Z_{n+j}|\ Z_{n+j-1}]\ |\ Z_n\}$

(4.5) $\rightarrow$ $= E[mZ_{n+j-1}\ |\ Z_n]$

Claim 4.6.2. $E[Z_{n+j}|Z_n] = m^j Z_n$.

Proof. This will be demonstrated with induction. First, for $j = 1$ see (4.5). Next, assume $E[Z_{n+j}|Z_n] = m^j Z_n$.

$$E[Z_{n+j+1}|Z_n] = E[mZ_{n+j}|Z_n] = mE[Z_{n+j}|Z_n] = mm^j Z_n = m^{j+1} Z_n$$

Now we can proceed with the main proof. If the left side of Claim 4.6.2 is divided by m^{n+j}, the result will be the following:

$$\frac{E[Z_{n+j}|Z_n]}{m^{n+j}} = E\left[\frac{Z_{n+j}}{m^{n+j}}\middle|Z_n\right]$$
$$= E[W_{n+j}|Z_n]$$

Consider the following two definitions.

$$g_1(Z_n) := E[W_{n+j}|Z_n]$$

$$g_2(Z_n) := E[W_{n+j}|\frac{Z_n}{m^n}] := E[W_{n+j}|W_n]$$

Therefore

$$g_1(k) = E[W_{n+1}|Z_n = k]$$

$$= \sum_N NP[W_{n+j} = N|Z_n = k]$$

$$= \sum_N NP\left[W_{n+j} = N|\frac{Z_n}{m^n} = \frac{k}{m^n}\right]$$

$$= \sum_N NP\left[W_{n+j} = N|W_n = \frac{k}{m^n}\right]$$

$$= E\left[W_{n+j}|W_n = \frac{k}{m^n}\right] = g_2\left(\frac{k}{m^n}\right)$$

$$\Rightarrow E[W_{n+j}|Z_n] = g_1(Z_n) = g_2\left(\frac{Z_n}{m^n}\right) = h_2(W_n)$$

where the first and third equalities are by definition and the second equality is a result of the work above. Therefore, $E[W_{n+j}|Z_n] := E[W_{n+j}|W_n]$.

Now, dividing the right side of Claim 4.6.2 by m^{n+j} we achieve the following.

$$\frac{m^j Z_n}{m^{n+j}} = Z_n/m^n = W_n$$

Therefore

$$E[W_{n+j}|W_n] = W_n \tag{4.6}$$

Using Theorem 4.2, $E(W_n^2)$ can be found. First, note the following:

$$E(W_n^2) = E\left[\frac{Z_n^2}{m^{2n}}\right] = \frac{1}{m^{2n}}E(Z_n^2)$$

According to Theorem 4.2

$$\begin{aligned} Var(Z_n) &= E[Z_n^2] - [E(Z_n)]^2 \\ &= E[Z_n^2] - m^{2n} \\ &= \frac{\sigma^2 m^n(m^n - 1)}{m^2 - m} \end{aligned}$$

since $m \neq 1$.

$$\begin{aligned} &\leftrightarrow \quad E[Z_n^2] = \frac{\sigma^2 m^n(m^n - 1)}{m^2 - m} + m^{2n} \\ &\leftrightarrow \quad E[W_n^2] = \frac{\sigma^2(1 - m^{-n})}{m^2 - m} + 1 \end{aligned} \tag{4.7}$$

Now consider $Var(W_n)$.

$$Var(W_n) = E[(W_{n+k} - W_n)^2] = E(W_{n+k}^2) - 2E(W_{n+k}W_n) + E(W_n^2) = *$$

Now,

$$\begin{aligned} E(W_{n+k}W_n) &= E\big[E(W_{n+k}W_n|W_n)\big] \\ &= E\big[W_n E(W_{n+k}|W_n)\big] \\ (4.6) \rightarrow \quad &= E(W_n W_n) = E(W_n^2) \end{aligned}$$

Therefore

$$\begin{aligned} * &= E(W_{n+k}^2) - E(W_n^2) \\ (4.7) \rightarrow \quad &= \frac{\sigma^2(1 - m^{-n+k})}{m^2 - m} - \frac{\sigma^2(1 - m^{-n})}{m^2 - m} \\ &= \frac{\sigma^2(1 - m^{-k})}{(m^2 - m)m^n} \\ &\leq \frac{\sigma^2}{(m^2 - m)m^n} = \frac{c}{m^n} \end{aligned}$$

where $c = \frac{\sigma^2}{m^2-m}$. By the Cauchy condition for convergence in L^2, this implies that the W_n's converge in the mean square. [2] Let call the random variable to which the W_n's converge W. It remains to be shown that $W_n \to W$ with probability 1. Consider the following:

$$\begin{aligned}\lim_{k\to\infty} E\big[(W_{n+k} - W_n)^2\big] &= E\big[(W - W_n)^2\big] \\ &= \lim_{k\to\infty} \frac{\sigma^2(1-m^{-k})}{(m^2-m)m^n} \\ &= \frac{\sigma^2}{m^2-m}\, m^{-n} \le c\, m^{-n} \qquad \forall n \\ \to \sum_{n=0}^{\infty} E\big[(W - W_n)^2\big] &\le \sum_{n=0}^{\infty} c\, m^{-n} < \infty \end{aligned} \tag{4.8}$$

since $m > 1$.

Now, define $Y_n = (W - W_n)^2$ and $\Phi(x) = |x|$. For all $\varepsilon > 0$ Chebyshev's inequality gives that

$$P(Y_n \ge \varepsilon) \le \frac{E(|Y_n|)}{|\varepsilon|} = \frac{E(Y_n)}{\varepsilon}$$

since $\varepsilon > 0$, $Y_n \ge 0$, and $E(|Y_n|) = E(Y_n) < \infty$ as a result of (4.8) above. [2] Finally, consider $\sum_{n=0}^{\infty} P(Y_n \ge \varepsilon)$.

$$\sum_{n=0}^{\infty} P(Y_n \ge \varepsilon) \le \sum_{n=0}^{\infty} \frac{E(Y_n)}{\varepsilon} = 1/\varepsilon \sum_{n=0}^{\infty} E(Y_n) < \infty$$

From here the Borel-Cantelli lemmas give that $P(Y_n \ge \varepsilon,\ \text{i.o}) = 0$ and therefore $(W - W_n)^2 \to 0 \ \to\ W_n \to W$ with probability 1, or almost surely. [2]

To finish the proof, it is only necessary to apply the limit as $n \to \infty$ to the formulas for the mean and variance of W_n.

$$E(W) = \lim_{n\to\infty} E(W_n) = \lim_{n\to\infty} E(Z_n/m^n) = \lim_{n\to\infty} \frac{E(Z_n)}{E(Z_n)} = 1$$

$$\begin{aligned} Var(W) = \lim_{n\to\infty} Var(W_n) &= \lim_{n\to\infty} \left(1 + \frac{\sigma^2(1-m^{-n})}{m^2-m} - 1\right) \\ &= \lim_{n\to\infty} \frac{\sigma^2(1-m^{-n})}{m^2-m} \\ &= \frac{\sigma^2}{m^2-m} = \frac{Var(Z_1)}{m^2-m} > 0 \quad \blacksquare \end{aligned}$$

This concludes the treatment of the single-type Galton/Watson model. As has been demonstrated, even with a simple model such this many important and interesting results can be obtained. Even more interesting, and applicable, is what is known as the multi-type Galton/Watson model. This is a generalization of the basic model to allow for the possibility of differing types within the same population. The results follow quite logically from the work already set forth and the interested reader can refer to [4] for explicit statements of these results. The next section will be devoted to summarizing some simulations of the growth of specific types of populations; the purpose of these simulations is to demonstrate that the result just obtained in this section are not merely theoretical but are a true description of reality.

CHAPTER V

APPLICATIONS OF THE GALTON/WATSON PROCESS

Though the theory given up to this point is quite interesting mathematically speaking, there are those who believe theory in the absence of eventual applicability is vanity of vanities. So, lest the theory of branching processes be declared vanity, it will now be demonstrated how this field can be applied to realistic situations. Although the specific examples given herein may arguably be a bit idealistically stated, there are numerous examples of the application of branching processes to absolutely realistic situations which unfortunately could themselves be the topic of a master's thesis.

Three situations will be investigated briefly. First, a most simplistic model in which each member of the population either dies or reproduces by splitting into two new members. Second, an example which models animal gene reproduction is given. And finally, a multitype process is given which models the occurrence of mutant subunits of genes.

Death or Bifurcation.

The first model to be investigated is that of a population whose members at the end of each generation either die or split into two new members of the population. The main purpose of this particular model is not to show applicability, but to demonstrate the accuracy of some of the conclusions stated previously by theorems. In this as with the other two models, all computations were completed by the Maple software package.

First we should determine the form of the generating functions for this model. By simply substituting the proper values for the p_k's in the formula for the generating function, the following result is obtained:

$$f(s) = p_0 + p_2 s^2$$

$$= 1 - p_2 + p_2 s^2$$

where p_2 is the probability of bifurcation. Next it seems prudent to answer the all important question: "What is the probability of extinction?" Again, applying the formula for q, the answer was found; a graph of q versus p_2 is given in figure 1. There are a couple of interesting things to note from this graph. First, the extinction probability is 1 for values of p_2 less than 0.5. This is a result of the fact that it is precisely at this value of p_2 where the expected number of offspring is equal to 1. Secondly, note that it appears that this probability decreases exponentially

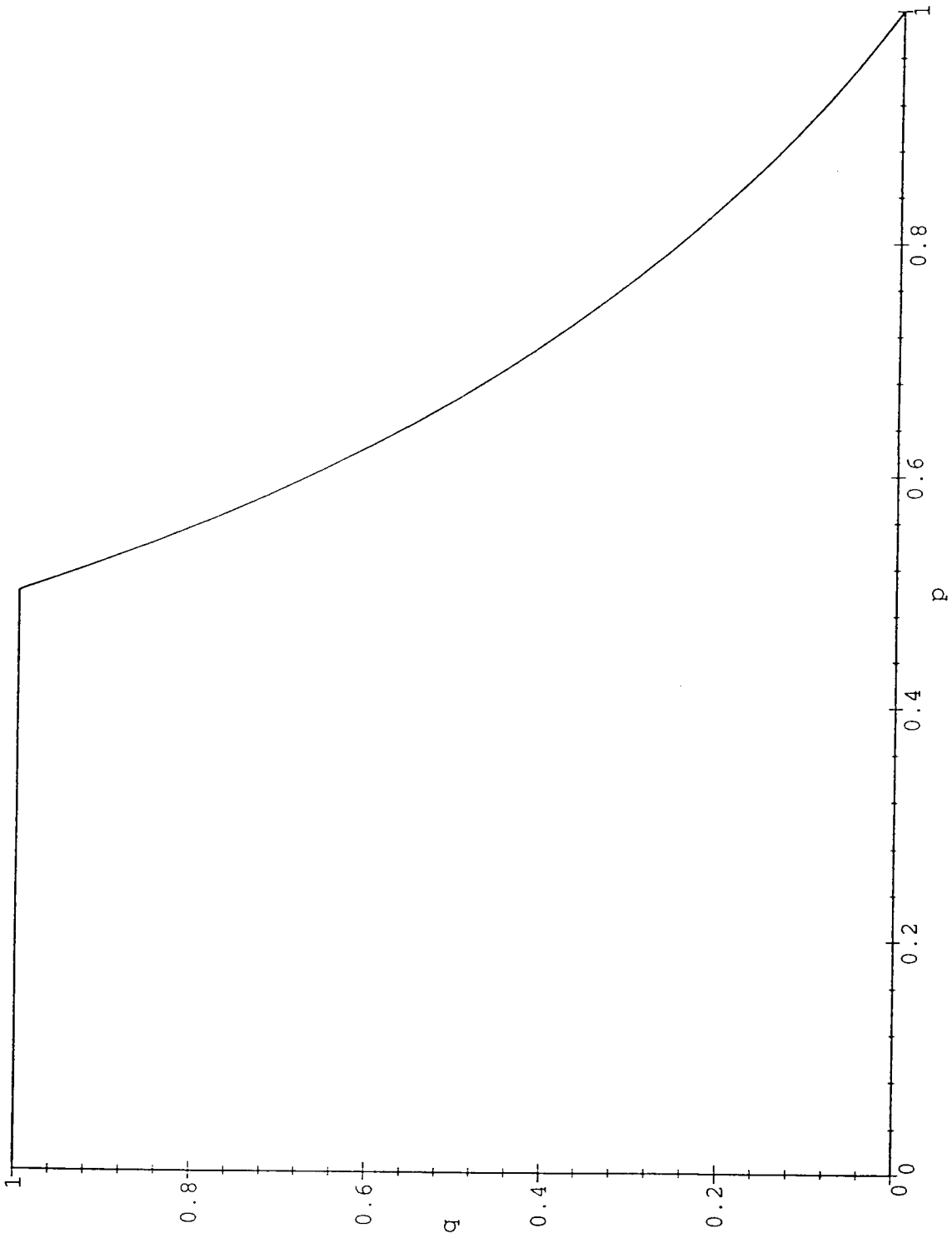


Figure 1 q vs. p

for values of p_2 greater than 0.5. This fact is further supported by observation of figures 2 through 5. For relatively small differences in values of p_2, p in the figures, the population grows indefinite with much greater frequency. Of course the population still has a positive probability of becoming extinct. Finally, another point illustrated by this model is that of the fact that populations which meet the stated criteria will either eventually become extinct of grow without bound, exponentially in fact. Note the figures 2 through 5 are actually the graphs of the natural log of the population sizes. Clearly the graphs are becoming linear as the "years" go by. This behavior is to be expected of exponential growth. Table 1 gives a comparison of the actual versus observed slopes for each of the simulations.

Table 1 Theoretical vs. Observed Slopes

$P(Z_1 = 2)$	Observed	Theoretical
0.5500	0.091953274	0.095310179
0.5505	0.092657409	0.096218858
0.5510	0.093230927	0.097126711
0.5515	0.094150754	0.098033740

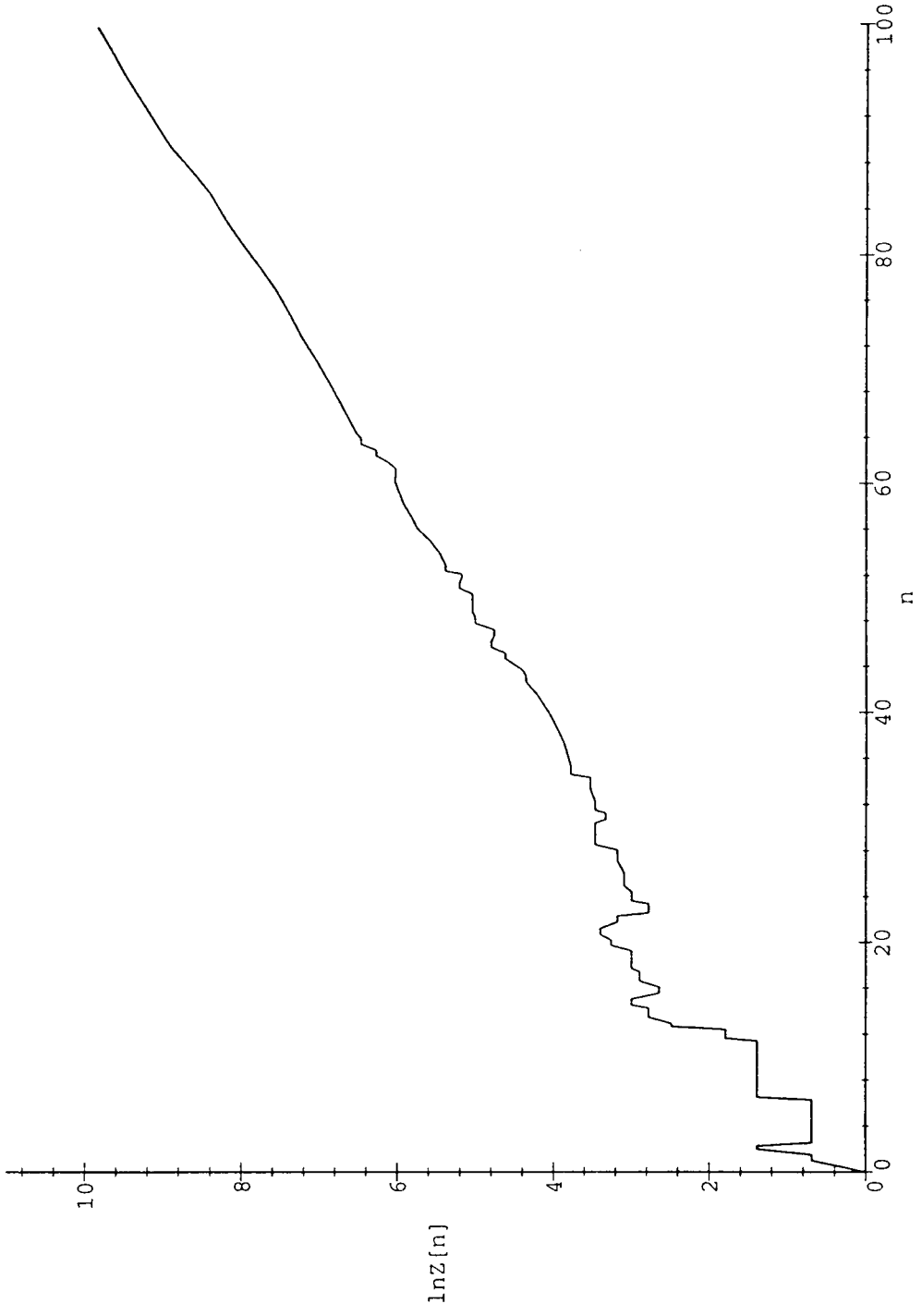


Figure 2 $p_2 = 0.55$

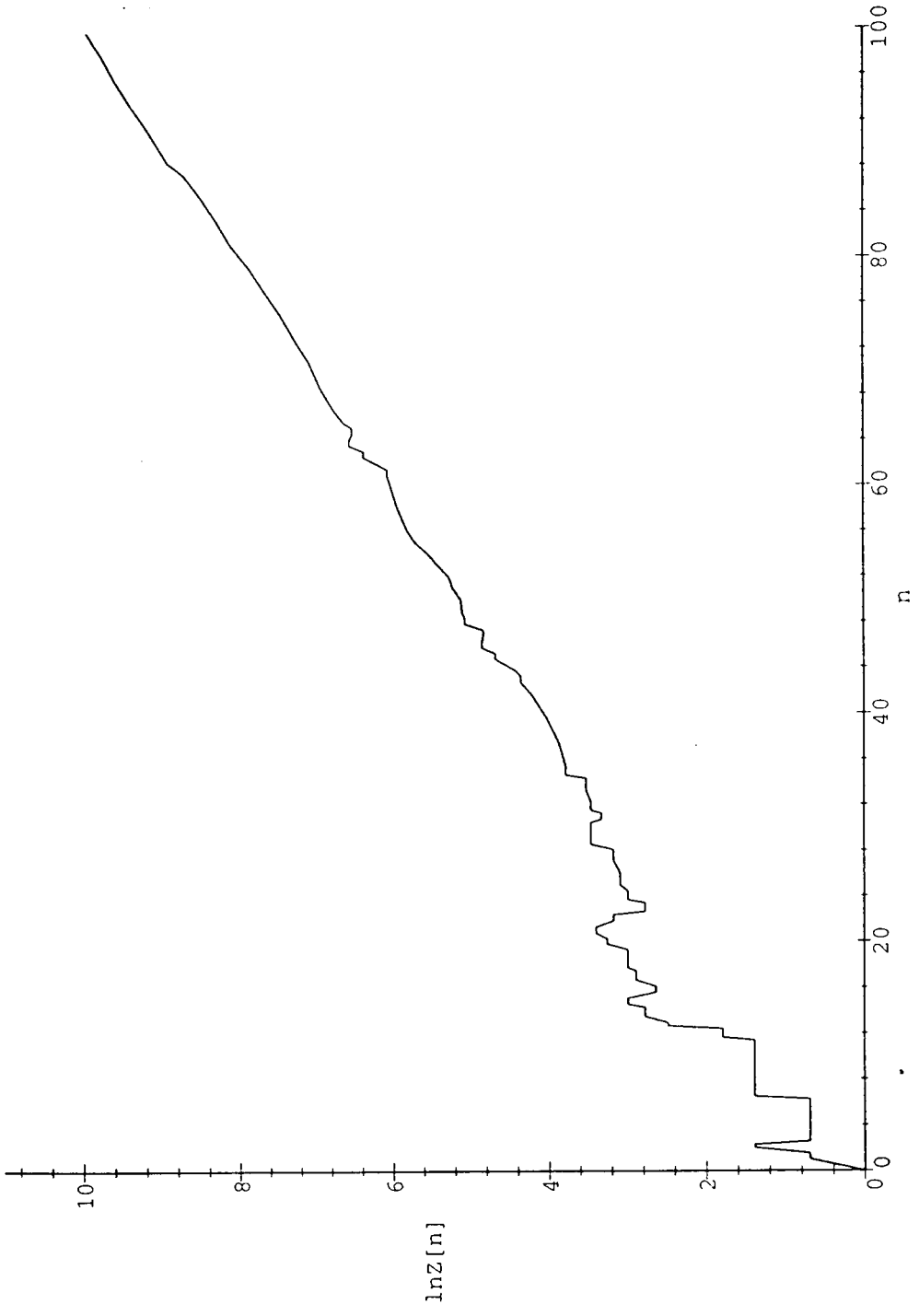


Figure 3 $p_2 = 0.5505$

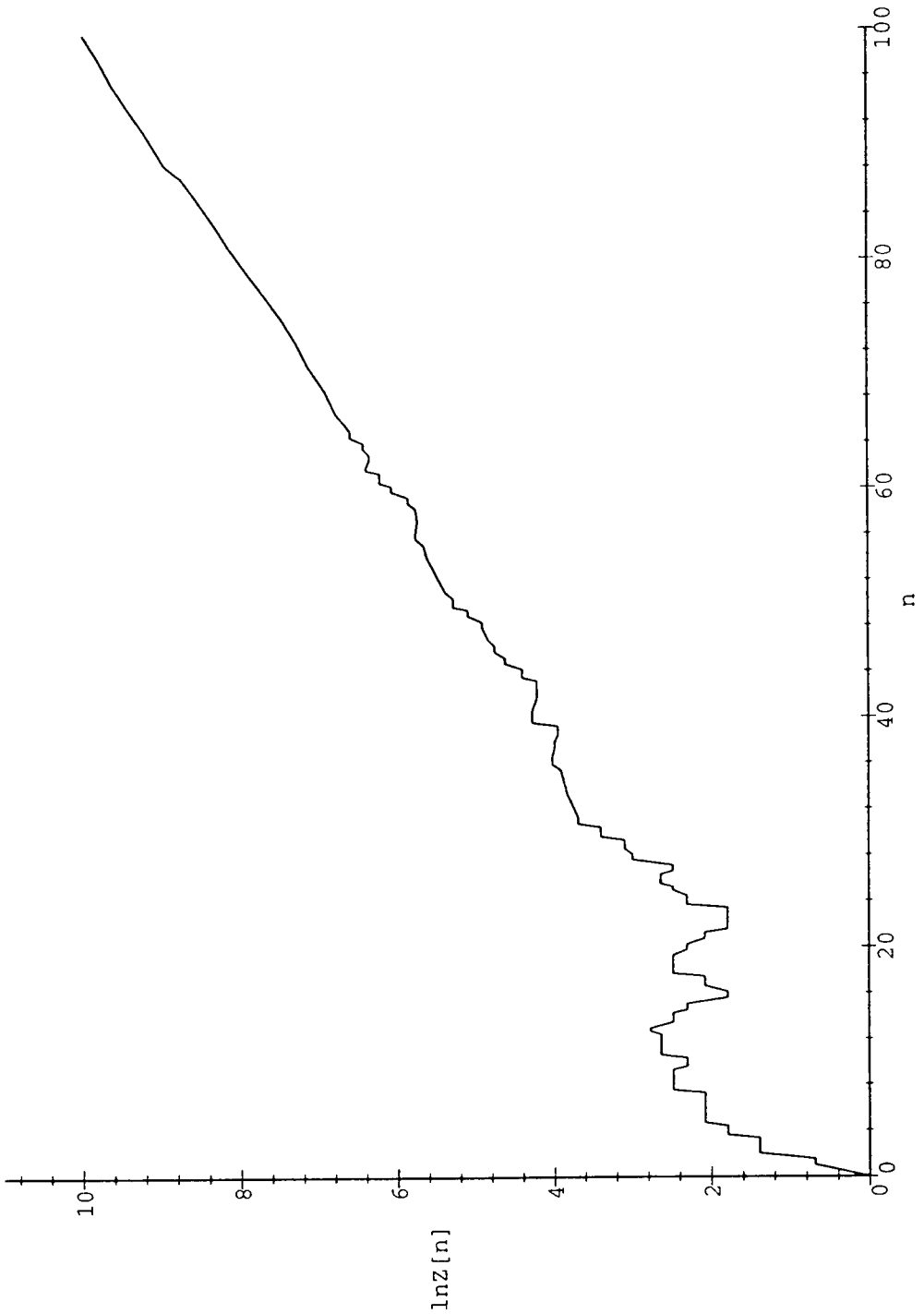


Figure 4 $p_2 = 0.551$

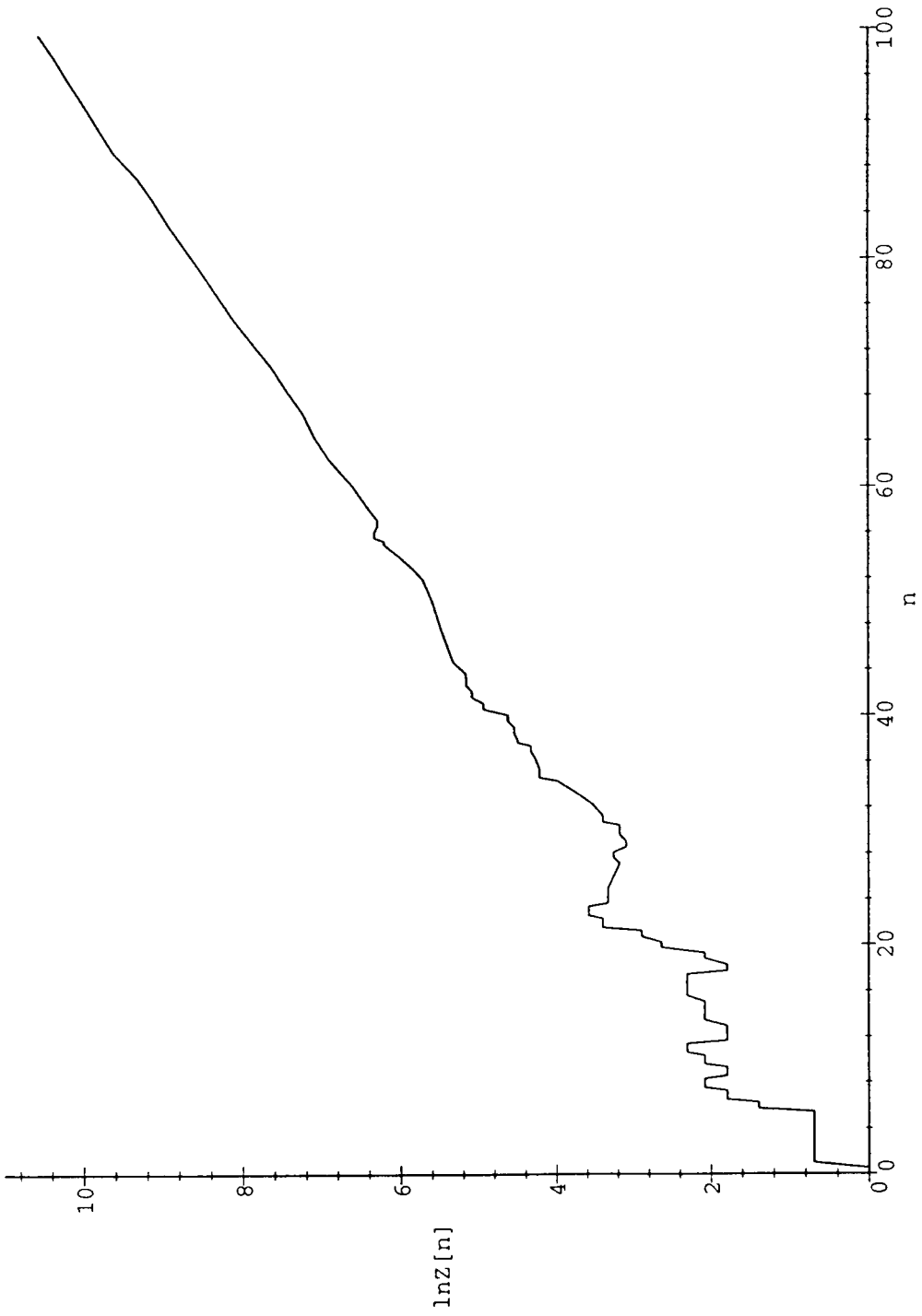


Figure 5 $p_2 = 0.5515$

Animal Reproduction.

The second model to be investigated is one of animal reproduction. As suggested by Harris, if one has the situation wherein some animal produces a large number of offspring of which few survive to adulthood, it would make sense that using the Poisson distribution would give realistic results [4]. Again a first step is to see what the form of the generating function should be. Applying the given formula for the generating function yields the following result:

$$f(s) = e^{-\lambda}e^{\lambda s}$$

where λ is of course the parameter associated with the Poisson distribution.

Upon trying to solve $f(s) = s$ in order to determine a function for q based on λ, one would soon note that in fact we are dealing with a transcendental function. To some that term carries great meaning; however for the author, it simply means that a closed form is impossible. So, the next best thing would be to look at some specific cases for λ. Figures 6 through 8 show the intersecting points of $f(s)$ and s and state the corresponding value of q. An interesting thing to note is that for very small changes in the value of λ, which as a reminder is also the expected value of our distribution, result in comparably large decreases in the corresponding value of q. As suggested by Harris, however, a realistic situation would be to observe

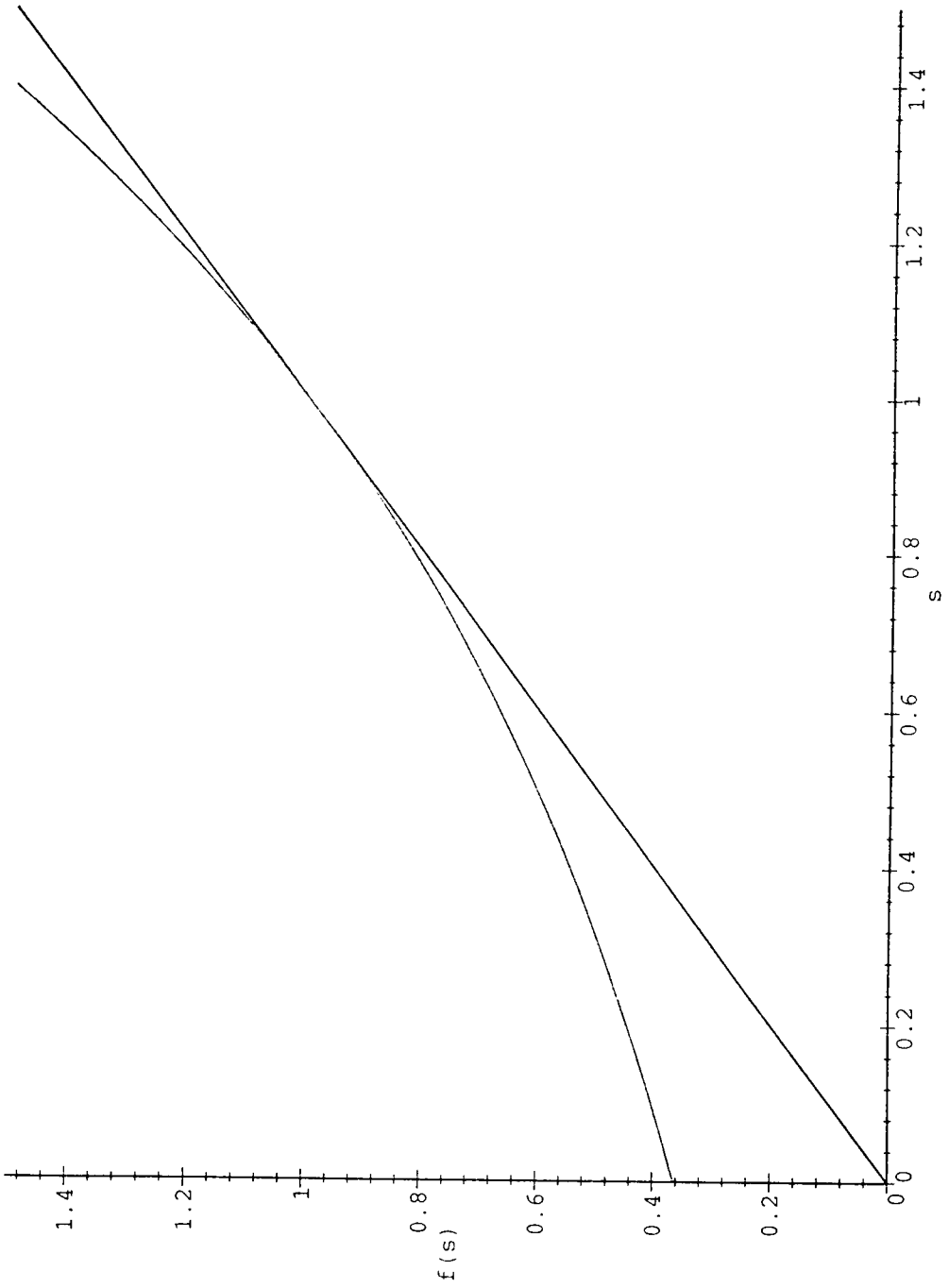


Figure 6 $q = 0.9803$ $\lambda = 1.01$

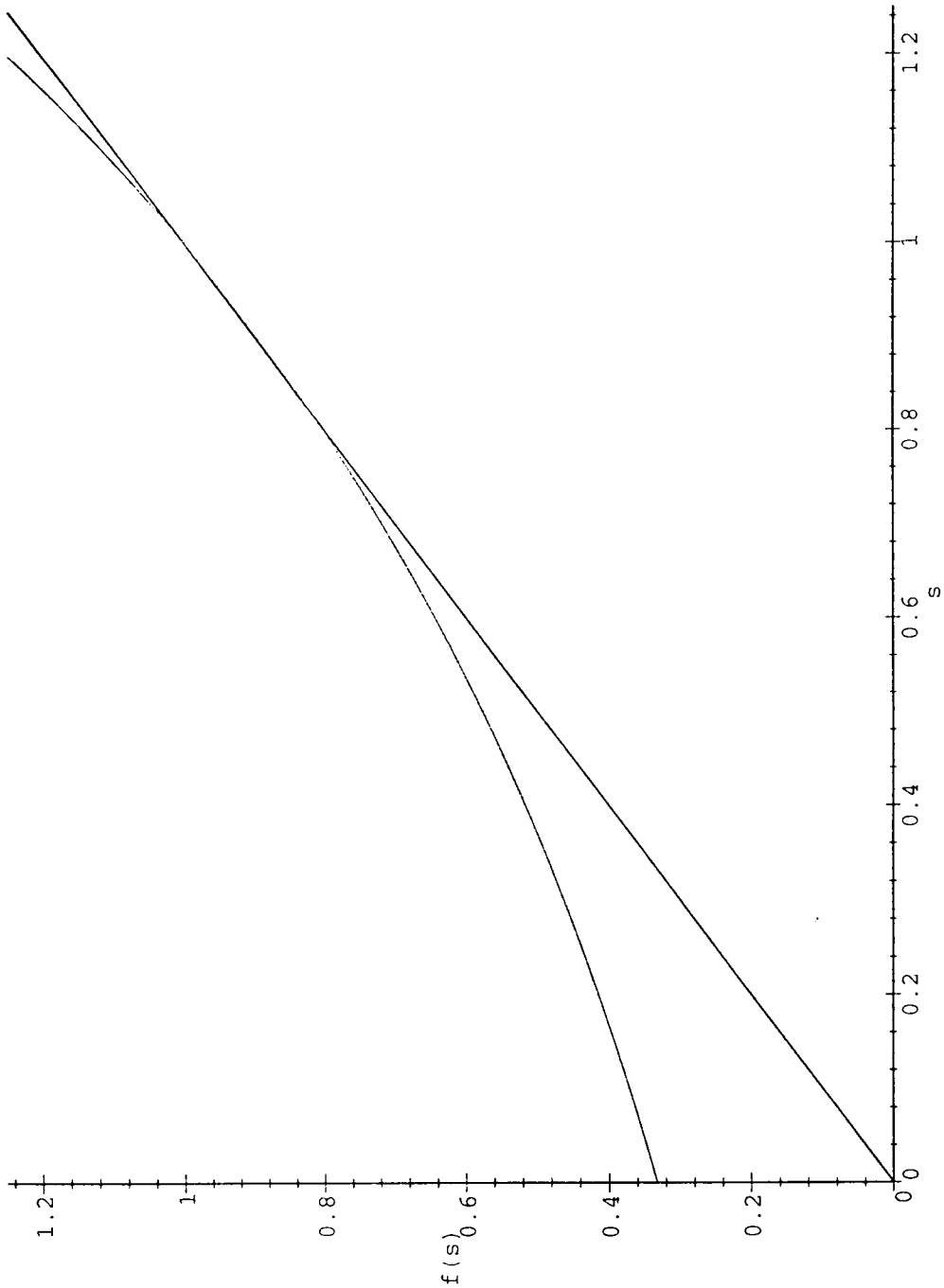


Figure 7 $q = 0.8239$ $\lambda = 1.1$

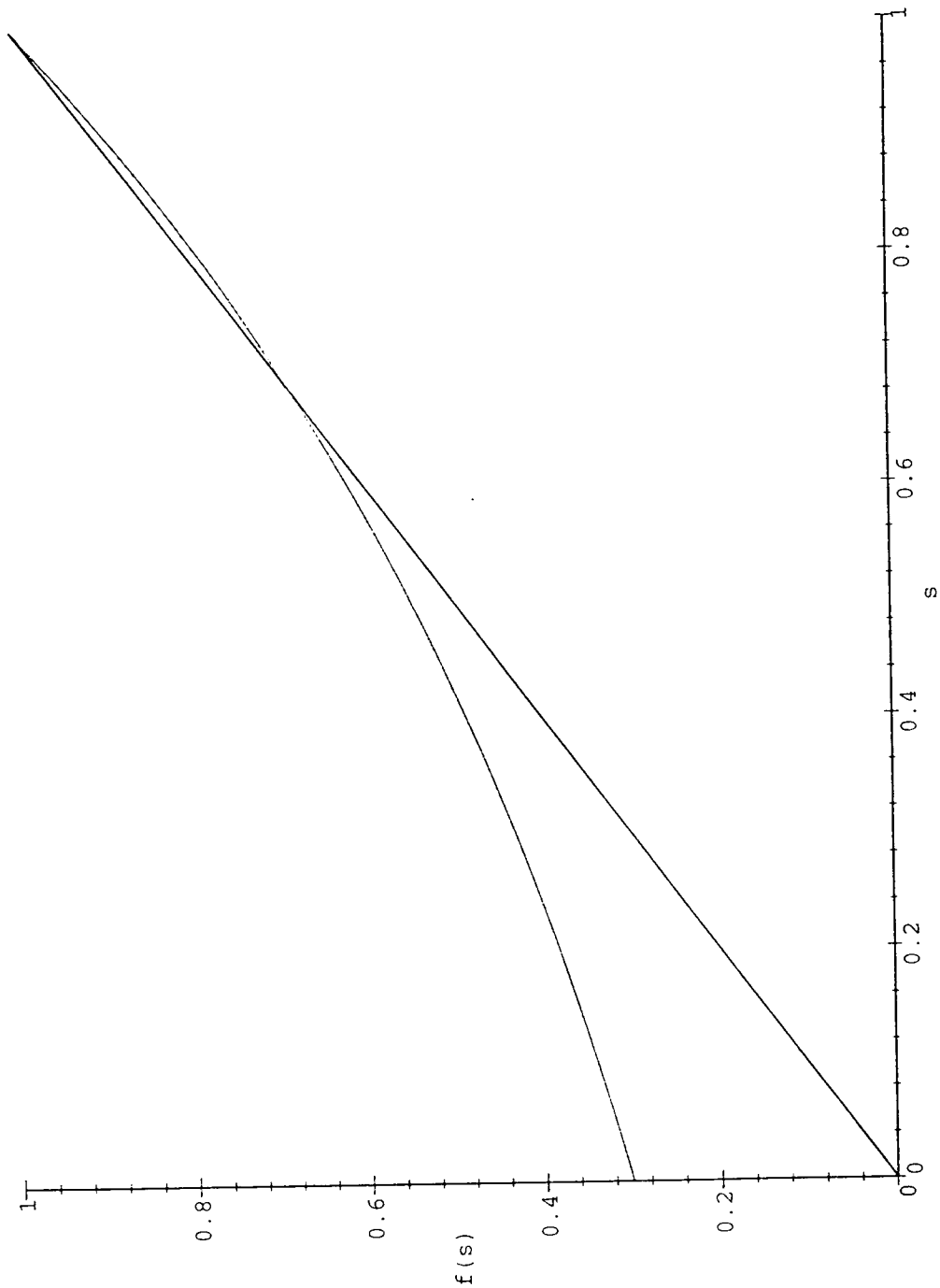


Figure 8 $q = 0.6863$ $\lambda = 1.2$

values of λ slightly above 1. Upon completion of the simulation program for this model, the author noted what should have been an obvious conclusion: the chances for indefinite survival are slim and none. Out of over fifty trials, the longest the population survived was to the fourteenth generation. Figure 9 shows the progression of those fourteen generations.

Mutant Genes.

The final model to be investigated in this thesis is one of mutant gene growth and distribution. Again, this is a model which is suggested by Harris in his section on multitype Galton/Watson processes. In this model the following assumptions are made. First it is assumed that genes are actually made up of subunits. Some of these subunits may be of mutant form. A gene reproduces by splitting. Just before splitting, however, each subunit is doubled, mutant and nonmutant alike. However, the ratio of mutant to nonmutant subunits in the "children" is not necessarily that of the parent. So, assume that a gene has k subunits. The type of a gene is given by the number of mutant subunits which it has. Assume that the environment allows doubling at each generation end, although more often it assumed that only one of the children survives. Finally, types 0 and k will be considered dead since these types can only reproduce their own types. For this

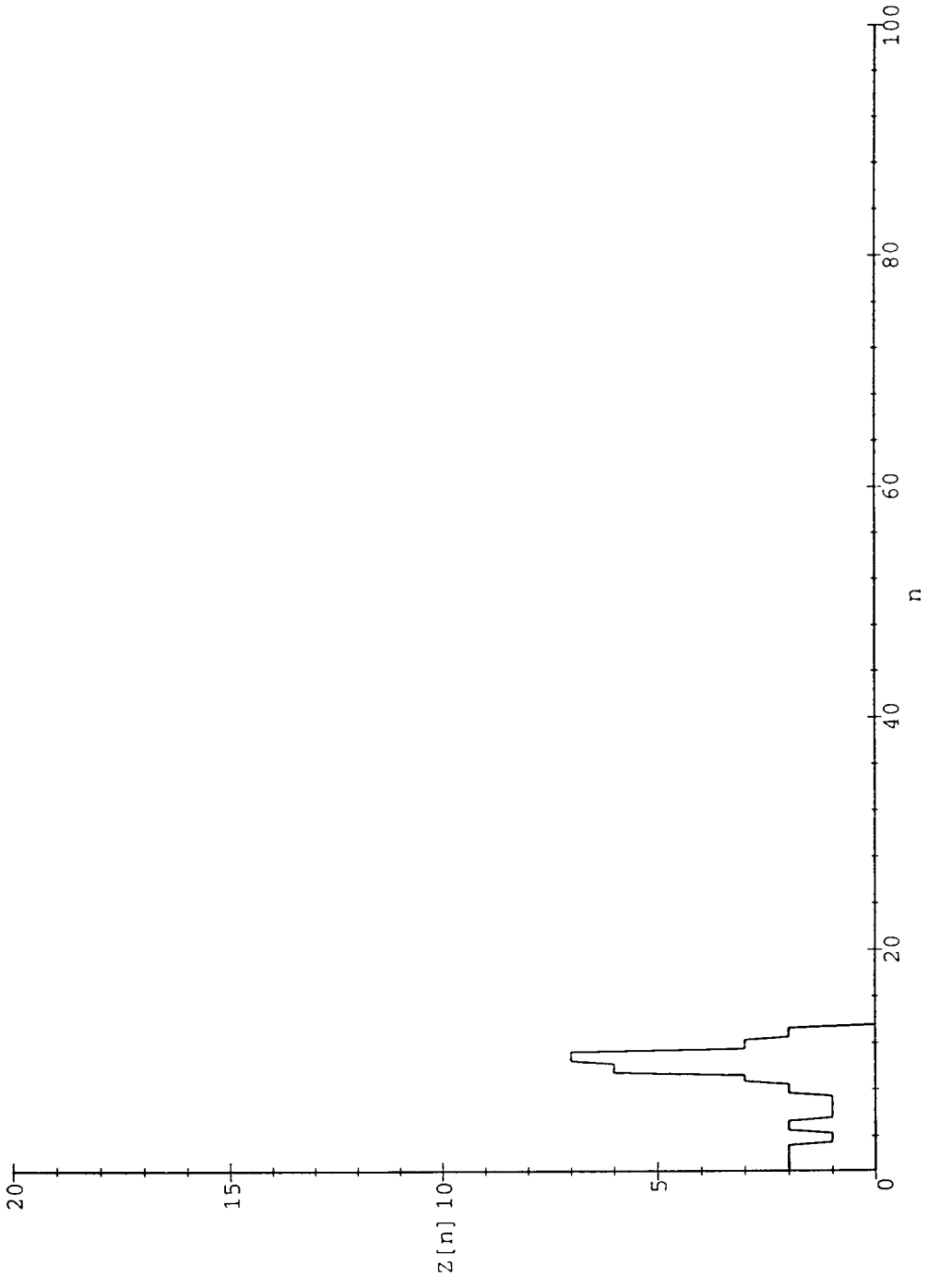


Figure 9 $\lambda = 1.01$

situation, it seems quite natural to use the hypergeometric distribution. With this in mind, assuming that a parent gene is of type i, $0 \leq i \leq k$, each child has the following probability of being of type j, $0 \leq j \leq k$,

$$\frac{\binom{2i}{j}\binom{2k-2i}{k-j}}{\binom{2k}{k}}$$

Let us consider the case when $k = 4$. Since we are only interested in three states, we need only look at the three generating functions associated with these states. As given by Harris and verified by the author, the generating functions are as follows:

$$f^1(s_1, s_2, s_3) = \frac{4}{7}s_1^2 + \frac{3}{7}s_2$$

$$f^2(s_1, s_2, s_3) = \frac{1}{35} + \frac{16}{35}s_1 s_2 + \frac{18}{35}s_2^2$$

$$f^3(s_1, s_2, s_3) = \frac{3}{7}s_2 + \frac{4}{7}s_3^2$$

Also easily verifiable is that the expectation matrix is given by

$$M = \begin{pmatrix} \frac{8}{7} & \frac{3}{7} & 0 \\ \frac{16}{35} & \frac{36}{35} & \frac{16}{35} \\ 0 & \frac{3}{7} & \frac{8}{7} \end{pmatrix}$$

Recall that in the multitype case, as with the general case, there is an eigenvalue, ϱ, which plays a similar role to that of m in the singletype case in determining

what the extinction vector will be. Again as given by Harris, that eigenvalue is in this case $\varrho = \frac{12}{7}$. Since ϱ is clearly greater than one, the conclusion is that the extinction matrix $\mathbf{q}$ will be less than one. In other words, the probabilities that the individual types will survive indefinitely are positive. In this particular simulation, the author arbitrarily chose type 2 to be the originator of the population. Figures 10 through 12 show the progression of the populations of each of the three types of interest. Noting that the graph is actually a step wise connection of actual data points, one can clearly see that, as should be expected, the natural log of the population sizes are becoming linear in only 25 generations.

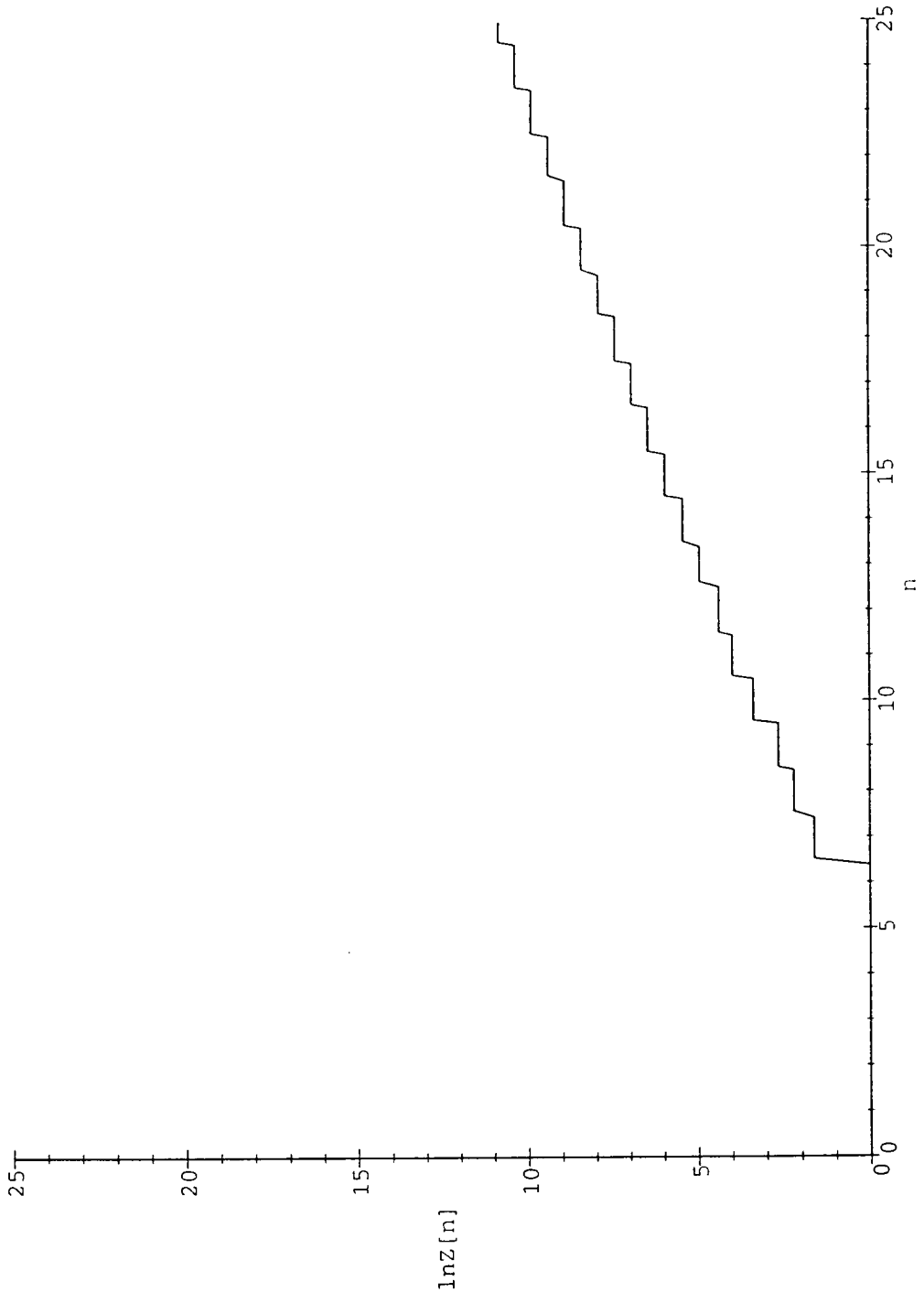


Figure 10 Type 1 Initial Type 2

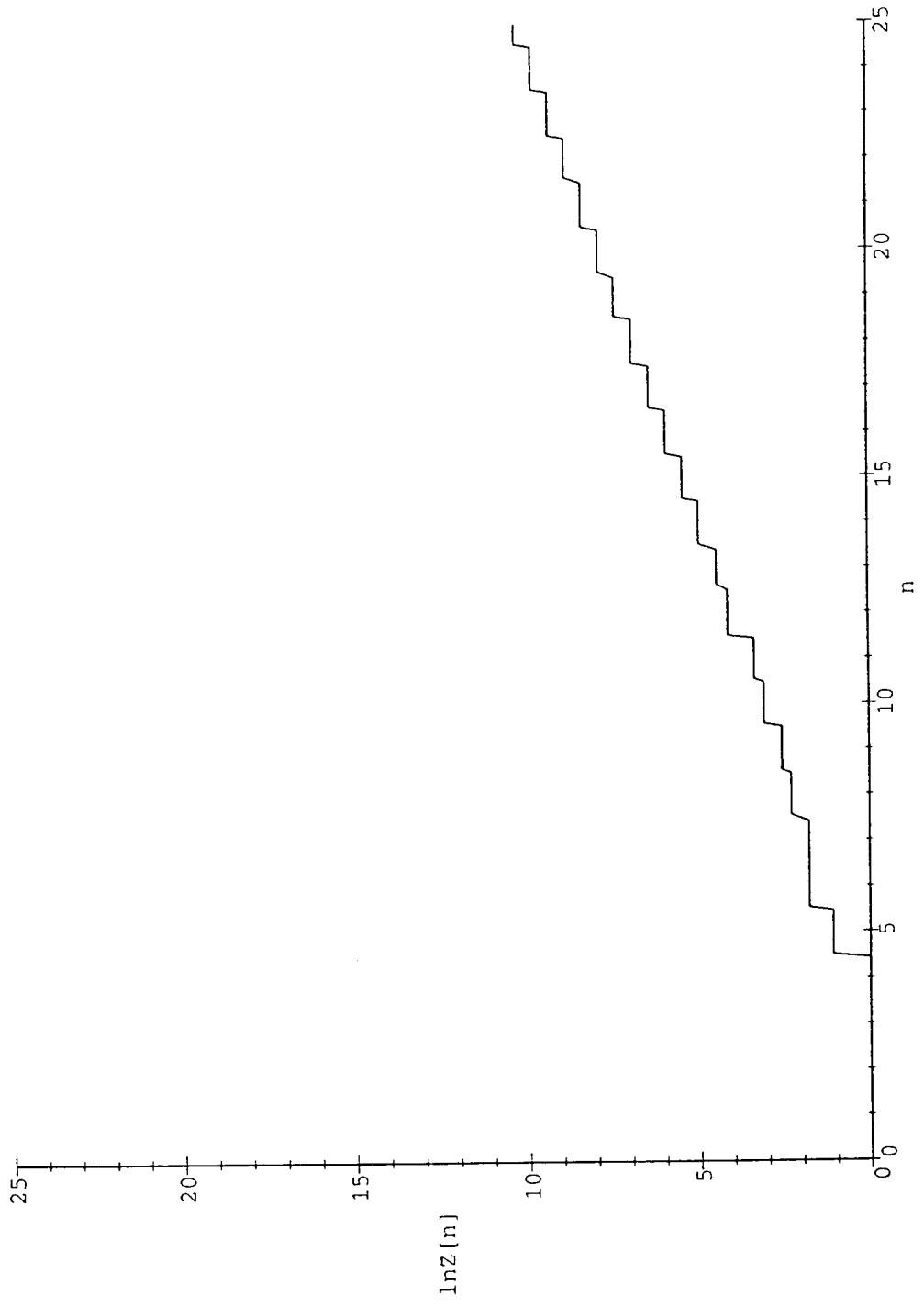


Figure 11 Type 2 Initial Type 2

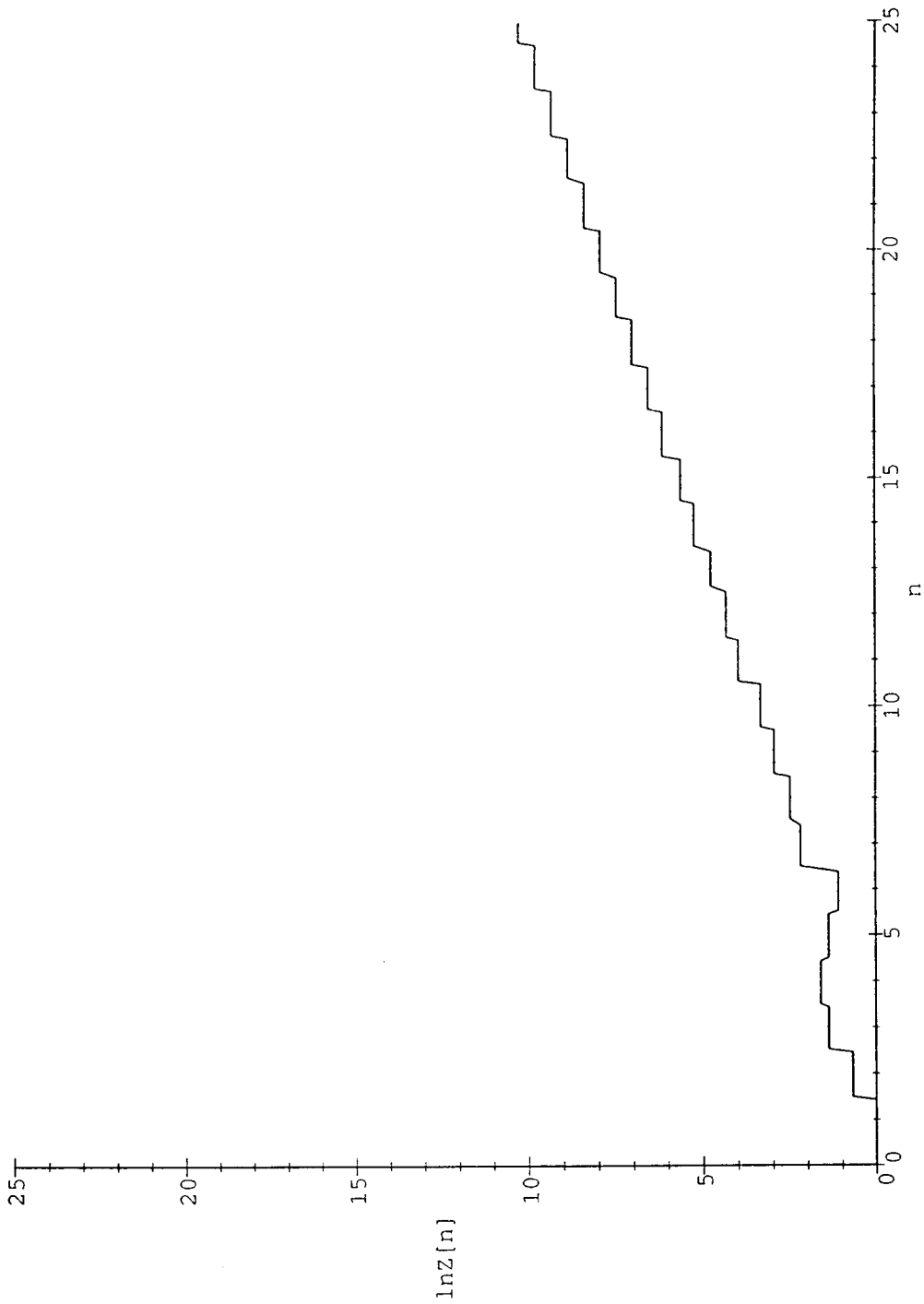


Figure 12 Type 3 Initial Type 2

BIBLIOGRAPHY

1. K.B. Athreya and P.E. Ney, *Branching Processes*, Springer-Verlag, 1972.

2. Kai Lai Chung, *A Course in Probability Theory, 2nd Ed.*, Academic Press, Inc., 1974.

3. Robert T. Goettge, *Limit Theorems for the Supercritical Galton/Watson Process in Varying Environments*, Mathematical Biosciences **28** (1975), 171–190.

4. Theodore E. Harris, *The Theory of Branching Processes*, Springer-Verlag, 1963.

5. P. Jagers, *Branching Processes with Biological Applications*, John Wiley & Sons, 1975.

6. Norman Kaplan, Aidan Sudbury, and Trygve S. Nilsen, *A Branching Process with Disasters*, J. Appl. Prob. **12** (1975), 47–59.

7. Catherine A. Macken and Alan S. Perelson, *Branching Processes Applied to Cell Surface Aggregation Phenomena*, Springer-Verlag, 1985.

8. Charles J. Mode, *Multitype Branching Processes*, American Elsevier, 1971.

9. Sheldon M. Ross, *Introduction to Probability Models, 4th Ed.*, Academic Press, Inc., 1989.

10. Rev. H.W. Watson and Francis Galton, *On the Probability of the Extinction of Families*, Royal J. Anthropol. Inst. Great Britain and Ireland **4**, 138–144.

APPENDIX

SIMULATION PROGRAMS

Death or Bifurcation.

```
Simulate:=proc(p);

with(linalg);

A:=array(0..100); Z:=array(0..100);

A[0]:=0; Z[0]:=0; k:=0;

X:=random(0..100000);

while k≤99 and 0<A[k] do

   ## simulate size of generations 1 through 100 ##

   A[k+1]:=0;

   for n from 1 to A[k] do

                rn:=X();

                    if rn≤p*100000 then A[k+1]:=A[k+1]+2; fi; od;

   Z[k+1]:=A[k+1]; k:=k+1; print(k,Z[k]);

od;

   ## if population dies, assign all future generations to 0 ##

for kn from k+1 to 100 do

   Z[kn]:=0; od;
```

```
g(x):=log(Z[round(x)]);

plot(g(x),x=0..100,ln(Z[n])=0..1);

end;
```

Animal Reproduction.

```
Simulate:=proc(lambda);

with(linalg);

with(stats);

A:=array(0..100); Z:=array(0..100);

A[0]:=0; Z[0]:=0; k:=0;

X:=random[poisson[lambda]]('generator');

while k≤99 and 0<A[k] do

   ## simulate size of generations 1 through 100 ##

   A[k+1]:=0;

   for n from 1 to A[k] do

                rn:=X(); A[k+1]:=A[k+1]+rn; od;

   Z[k+1]:=A[k+1]; k:=k+1; print(k,Z[k]);

od;

   ## if population dies, assign all future generations to 0 ##

for kn from k+1 to 100 do

   Z[kn]:=0; od;

g(x):=Z[round(x)];
```

```
plot(g(x),x=0..100);

end;
```

Mutant Genes.

```
Simulate:=proc(k,s);

with(linalg); with(stats);

A:=array(0..25,1..3); Z:=array(0..25,1..3);

p:=array(0..4); P:=array(0..4);

readlib(unassign);

   ## initialize the original generation ##

for n from 1 to d-1 do

   if n=s then Z[0,n]:=1; else Z[0,n]:=0; fi; od;

   ## initialize index and rand num gen ##

   nn:=0; X:=rand(0..100000);

   ## begin simulation ##

while nnle24 do ## initialize temp counter ##

   for c from 1 to k-1 do

      A[nn+1,c]:=0; od;

   for i from 1 to k-1 do

   ## determine the prob. for each division ##

      for m from 0 to k do

         p[m]:=binomial(2*i,m)*binomial(2*k-2*i,k-m)/binomial(2*k,k):
```

```
        od;
        for m from 0 to k do
            P[m]:=sum(p[j],j=0..m): od:
    ##determine contrib. to next generation from ith comp ##
        if Z[nn,i]>0 then
            for a from 1 to Z[nn,i] do
             rn:=X();
             if rn≤P[0]*100000 then A[nn+1,i]:=A[nn+1,i]+1;
             else for aa from 1 to k-1 do
              if P[aa-1]*100000<rn and rn≤P[aa]*100000
              then A[nn+1,aa]:=A[nn+1,aa]+1;
                if (2*i-aa)>0 and (2*i-aa)<4 then
                A[nn+1,2*i-aa]:=A[nn+1,2*i-aa]+1; fi;
        fi; od; fi; od; fi; od;
    unassign('p'); unassign('P');
    for c from 1 to k-1 do
        Z[nn+1,c]:=A[nn+1,c]; print(nn+1,c,Z[nn+1,c]); od;
    nn:=nn+1;
od;
    ## get graphs of the growth of each type ##
for y from 1 to k-1 do
```

```
for x from 0 to 25 do
   if Z[x,y]=0 then Z[x,y]:=1; fi; od; od;
unassign('n');
   g[1](n):=log(Z[round(n),1]);
   g[2](n):=log(Z[round(n),2]);
   g[3](n):=log(Z[round(n),3]);
   plot(g[1](n));
   plot(g[2](n));
   plot(g[3](n));
end;
```

VITA

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