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Date July 11, 1991

THE BOUNDARY ELEMENT METHOD FOR THE TWO-DIMENSIONAL EXTERIOR STOKES PROBLEM

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DEDICATION

This thesis is dedicated to my wonderful and patient family, Jon, Jamie, and Jennifer.

ABSTRACT

This research applied the boundary element method to the two-dimensional exterior Stokes problem. Numerical experiments were carried out for the specific case where the domain is the exterior of a circle. Vector fields were sketched with these numerical results.

Numerical test computations indicate that the boundary element method is useful for computational hydrodynamics.

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1. INTRODUCTION

The boundary integral approach has a long tradition of being used to prove existence and uniqueness of solutions of boundary value problems. The numerical analogue, called the boundary element method (BEM), has become very popular for the numerical treatment of engineering problems for the last ten to fifteen years. Boundary element methods have been applied to a large variety of problems arising in the engineering sciences, such as problems of potential flow, elasticity, electromagnetic and acoustic theories (e.g. [5], [18] and the references therein). Most research so far has been carried out in the framework of the h-version, in which accuracy is achieved by decreasing the mesh size of elements on the boundary while keeping the degree of piecewise polynomials fixed (usually at a low level, $p = 1$ or 2). For this method several detailed results, including asymptotic rates of convergence for both first-kind and second-kind integral equations are known (see [7], [18]). The basic ideas of the above convergence proofs can also be used to analyze the recently introduced p- and h-p-versions for the BEM (see [2], [16], [17], [19]). In the p-version, a fixed mesh with constant h is used and accuracy is achieved by increasing the degree p of the polynomials used. The h-p-version combines the two approaches.

In particular, boundary element methods have been successfully applied to viscous flow problems in the last few years ([8], [9], [11], [12], [20]). The main advantages of boundary element methods generally are:

1. The unknowns are located on the boundary only, hence we are led to algebraic systems with nonsparse, but relatively small, matrices.

2. Exterior problems do not cause essential disadvantages.

3. Boundary data are involved in a natural way, and the methods often immediately given the physically important information about stresses or displacements on the boundary.

Conversely, the disadvantage of treating nonhomogeneities should be noted. Apart form the general advantages of such methods over domain-type ones, we point out that the BEM applied to viscous flows yields approximate flow fields which automatically satisfy the incompressibility constraint $\operatorname{div} u = 0$. The purpose of this thesis is to discuss the BEM for the two dimensional exterior Stokes problem, which is a mathematical model of the steady-state, slow, viscous, incompressible fluid flow past an obstacle.

The thesis is organized as follows: in Section 2, the problem is stated and we introduce the hydrodynamical potentials. In Section 3, we introduce the weak formulation of our problem, which is approximated by the BEM in Section 4. This section also gives the error estimates for the solution of the discrete problems. In Section 5, we discuss a specific situation with a circular obstacle. Finally, in Section 6 we give some instructive results of numerical tests which essentially confirms the theoretical statements on this boundary element method.

2. THE STOKES PROBLEM

Let Ω be a domain exterior to a smooth, simple closed curve γ in the plane $\mathbb{R}^2$. The two-dimensional exterior Stokes problem is the boundary value problem of determining u and p from the Stokes equations,

$$\left.\begin{aligned} &\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} = \frac{\partial p}{\partial x_1} , \\ &\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} = \frac{\partial p}{\partial x_2} , \\ &\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 , \end{aligned}\right\} \quad \forall (x_1, x_2) \in \Omega \tag{1}$$

satisfying the boundary condition

$$\left.\begin{aligned} &u_1 = f_1 , \\ &u_2 = f_2 , \end{aligned}\right\} \quad \forall (x_1, x_2) \in \gamma \tag{2}$$

and the conditions at infinity

$$u = O(1) \quad p = o(1) \text{ as } |x| \to \infty , \tag{3}$$

where $u = (u_1, u_2)$ is the velocity vector and p is the pressure.

The above mathematical model describes the steady-state, slow, viscous, incompressible fluid flow past an obstacle. In order to define the hydrodynamical potentials, we need the fundamental tensor pair $E(x - y) = (E_{ij}(x - y))$, $e(x - y) = (e_j(x - y))$ of the Stokes equation (see [4]):

$$E_{ij}(x - y) = \frac{1}{4\pi}\left[\delta_{ij} \ln \frac{1}{|x - y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2}\right],$$

$$e_j(x - y) = \frac{1}{2\pi} \frac{\partial}{\partial x_j} \ln|x - y| . \tag{4}$$

For fixed j, each column vector $E_{ij}(x - y)$ is a solution of (1) with corresponding pressure $e_j(x - y)$, and exhibits a fundamental singularity at $x = y$. Now the potentials of a single-layer with density $a = (a_1, a_2)$ for the velocity and pressure are defined by the integrals:

$$U(x; a) = \int_\gamma E(x - y)\, a(y)\, ds_y, \text{ i.e.,}$$

$$U_1(x; a) = \int_\gamma [E_{11}(x - y)a_1(y) + E_{12}(x - y)\, a_2(y)]\, ds_y ,$$

$$U_2(x; a) = \int_\gamma [E_{21}(x - y)a_1(y) + E_{22}(x - y)\, a_2(y)]\, ds_y ,$$

and (5)

$$P(x; a) = \int_\gamma (e(x - y), a(y))\, ds_y, \text{ i.e. },$$

$$P(x; a) = \int_\gamma [e_1(x-y)\, a_1(y) + e_2(x-y)\, a_2(y)]\, ds_y \ ,$$

where ds_y denotes the arc length element at the integration point y. By the potential of a double-layer with density $b = (b_1, b_2)$, we mean the integrals:

$$\begin{aligned} W(x; b) &= -\int_\gamma (T'_y E(x-y)\, n(y))^t\, b(y)\, ds_y \ , \\ \Pi(x; b) &= 2 \operatorname{div} \int_\gamma (e(x-y), b(y))\, n(y)\, ds_y \ . \end{aligned} \tag{6}$$

Here n is the interior (with respect to Ω) normal vector to γ and $(T'_y E(x-y)\, n(y))^t$ denotes the transpose of $(T'_y E_1(x-y)\, n(y), T'_y E_2(x-y)\, n(y))$ and $T'_y E_j(x-y)$ is the tensor defined by

$$(T'_y E_j)_{ik} = e_j\, \delta_{ik} + \left(\frac{\partial E_{ij}}{\partial y_k} + \frac{\partial E_{kj}}{\partial y_i}\right) .$$

The subscript y on $T'_y E_j(x-y)$ indicates the differentiation in $(T' E_j)_{ik}$ is carried out with respect to y. It can be verified that $(T_x E_j(x-y))_{ik} = -(T'_y E_j(x-y))_{ik}$.

Let us note that the stress tensor

$$(Tu)_{ij} = -p\, \delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \ i,j = 1, 2$$

replaces the normal derivative in the potential theory (e.g. the exterior Dirichlet problem for a Laplace equation) and the tensor E replaces the scalar fundamental theory (see [14]) .

If we substitute the explicit expressions for E and e from (4) into (5) and (6), we get

$$U(x; a) = \frac{1}{4\pi} \int_{\gamma} [a(y) \ln \frac{1}{|x - y|} + \frac{(x - y, a(y))}{|x - y|^2} (x - y)] \, ds_y \ ,$$

$$W(x; b) = \frac{-1}{\pi} \int_{\gamma} \frac{(x - y, b(y))(x - y, n(y))}{|x - y|^4} (x - y) \, ds_y \ , \tag{7}$$

for the velocity and

$$P(x; a) = \frac{1}{2\pi} \int_{\gamma} \frac{(x - y, a(y))}{|x - y|^2} \, ds_y \ ,$$

$$\Pi(x; b) = \frac{1}{\pi} \operatorname{div} \int_{\gamma} \frac{(x - y, b(y))}{|x - y|^2} n(y) \, ds_y \ , \tag{8}$$

for the pressure. Now, we can construct the solution of the Stokes problem by using

(a) a single-layer potential in some quotient spaces ([8], [20]) ,

(b) a double-layer potential, which leads to a system of boundary equations of the second kind where eigensolutions arise,

(c) a combination of a single- and double-layer potentials ([9], [11], [12]) .

We will construct the solution of the Stokes problem (1-3) by using a combination of double- and single-layer potentials. For the velocity we seek a solution in the form

$$u(x) = W(x; a) - \eta \, V(x; a; \alpha) \qquad x \in \Omega \, , \tag{9}$$

where $a \in C(\gamma)$ is the unknown density function to be determined from the integral equation below. The constants η and α are real parameters such that

$$\eta > 0 \text{ and } \alpha \neq 0 .$$

The suggestion on how to select α and η will be discussed in Section 5. $V(x; a; \alpha)$ is the potential defined by

$$V(x; a; \alpha) = U(x; a - \frac{1}{\ell} \int_\gamma a(y)\, ds_y) + \frac{\alpha}{4\pi} \int_\gamma a(y)\, ds_y \quad ,$$

with $\ell = \int_\gamma ds_y$.

For the pressure, we take the representation

$$p(x) = \Pi(x; a) - \eta\, Q(x; a; \alpha) \quad x \in \Omega \quad , \tag{10}$$

where

$$Q(x; a; \alpha) = P(x; a - \frac{1}{\ell} \int_\gamma a(y)\, ds_y) .$$

The functions u and p in (9) and (10) satisfy the Stokes equation (1) since all the functions U and P, W and Π satisfy (1). Moreover, u and p have the correct behavior at infinity [12].

Next, we have to determine the unknown density function $a = (a_1, a_2)$ by imposing the boundary condition (2). By using the standard jump relation [14], we see that the vector u and the scalar p defined by (9) and (10) will indeed solve the exterior Stokes problem (1-3), provided the density $a = (a_1, a_2)$ is a solution of the integral equation

$$a + Ka + \eta\, Sa = -2f\,, \tag{11}$$

$x \in \gamma$ and $K, S\colon C(\gamma) \to C(\gamma)$ are integral operators defined by

$$\begin{aligned}(Ka)(x) &= \frac{2}{\pi}\int_\gamma \frac{(x-y,\, a(y))(x-y,\, n(y))}{|x-y|^4}\,(x-y)\, ds_y\,,\\[2ex] (Sa)(x) &= L\Big(a - \frac{1}{\ell}\int_\gamma a(y)\, ds_y\Big)(x) + \frac{\alpha}{2\pi}\int_\gamma a(y)\, ds_y\end{aligned} \tag{12}$$

where

$$(L\tilde{a})(x) = \frac{1}{2\pi}\int_\gamma \Big[\tilde{a}(y)\ln\frac{1}{|x-y|} + \frac{(x-y,\, \tilde{a}(y))}{|x-y|^2}\,(x-y)\Big]\, ds_y\,.$$

The existence and uniqueness of the solution to the problem (1-3) is given by the following theorem [12].

<u>Theorem 1</u>: For any continuous vector function f there exists a unique solution pair u, p to the exterior Stokes problem (1-3) and the solution has the form (9), (10), with a satisfying (11).

Remark: If, as in the standard approach to the exterior Dirichlet problem, we seek a solution in the form of the potential of a double-layer only, then such a solution exists only for boundary values f(x) which satisfy the condition $\int_{\gamma} (f, n)\, ds = 0$.

3. WEAK FORMULATION

We will now construct approximate solutions by applying Galerkin's method in the appropriate Sobolev spaces.

First, we need some notation concerning Sobolev spaces, that is, $H^s(\gamma)$, s real.

For $s = 0$, we have the L_2-norm and the L_2-inner product (see Appendix A).

For $s = 1$, $H^1(\gamma)$ is a space of functions such that

$$||f||_{0,\gamma} = \int_\gamma f^2(x)\, dx < +\infty$$

and

$$||f'||_{0,\gamma} = \int_\gamma (f')^2\, (x)\, dx < +\infty$$

and the norm is defined by

$$||f||_{H^1(\gamma)} = ||f||_{1,\gamma} = \sqrt{||f||^2_{0,\gamma} + ||f'||^2_{0,\gamma}} \quad .$$

If we are in $H^2(\gamma)$, then we include the condition that

$$||f''||_{0,\gamma} = \int_\gamma (f'')^2(x)\, dx < +\infty .$$

This can be extended to $H^s(\gamma)$, where s is a real number by using the so-called interpolation s spaces [1].

Next, we will consider the vector-valued space of functions $f = (f_1, f_2)$, denoted by $f \in H^s(\gamma)$ if $f_i \in H^s(\gamma)$, $i = 1, 2$. The corresponding inner-product for f and g is defined by

$$(f, g)_s = (f_1, g_1)_s + (f_2, g_2)_s$$

and similarly for the norm.

Letting $\mathfrak{L}a = (I + K + \eta S)a$, where I is the identity operator, we introduce the bilinear form

$$\mathfrak{B}(a, b)_0 = (\mathfrak{L}a, b)_0 \tag{13}$$

and we look for $a \in L_2(\gamma)$ (where $L_2(\gamma) = H^0(\gamma)$) such that

$$\mathfrak{B}(a, b) = -2(f, b)_0 \qquad \forall\ b \in L_2(\gamma). \tag{14}$$

The basic properties of the integral operator (11) and the bilinear form (13) in the Sobolev spaces can be summarized in the following two lemmas.

<u>Lemma 1</u>: For smooth γ, the operator from (11),

$$\mathfrak{L}a = (I + K + \eta S)a,$$

defines a continuous mapping: $H^s(\gamma) \to H^s(\gamma)$, $s \in \mathbb{R}$. Moreover, the Gårding inequality

$$|\mathfrak{B}(a, a)| \geq \|a\|_0^2 - |\mathcal{K}(a, a)| \tag{15}$$

holds for all $a \in L_2(\gamma)$, where $\mathcal{K}(a, a)$ is a compact bilinear form on $L_2(\gamma) \times L_2(\gamma)$.

Lemma 2: For smooth γ, there exists constants $c_1, c_2 > 0$ depending on γ, such that

$$c_1\|a\|_s \leq \|\mathfrak{L}a\|_s \leq c_2\|a\|_s \tag{16}$$

for all $a \in H^s(\gamma)$, $s \in \mathbb{R}$.

The proof of Lemma 1 can be found in [13]. We note that S is an integral operator with a logarithmetic kernel, and hence the operator S maps $H^s(\gamma) \to H^{s+1}(\gamma)$ continuously. If $\gamma \in C^{r+2}$, r is an integer, the integral operator K defines compact mappings $H^s(\gamma) \to H^{s+r}(\gamma)$, $H^s(\gamma) \to C^{s+r}(\gamma)$ for $|s| \leq r + 2$; see [18]. Hence K: $H^s(\gamma) \to H^s(\gamma)$ is compact for any $s \in \mathbb{R}$ for sufficiently smooth γ. Lemma 2 simply means that the operator $\mathfrak{L}$ is continuous [12].

4. BOUNDARY ELEMENT DISCRETIZATION

Let S_N denote a finite dimensional subspace of $L_2(\gamma)$. We replace (14) by the following approximate formulation: find $a_N \in S_N$ such that

$$\mathcal{B}(a_N, b_N) = -2(f, b_N)_0 \qquad \forall\ b_N \in S_N . \tag{17}$$

The key to the error analysis is in the following Lemma (analogue of Cea's lemma in the finite element method; see [10]) .

<u>Lemma 3</u>: Let H be a Hilbert space (see Appendix A) with dual H' and let A be injective and continuous from H into H' satisfying a Gårding inequality. Let u denote the solution of

(i) $\quad Au = F$

where $F \in H'$, and let $u_N \in S_N \subset H$ denote the solution of the Galerkin equations.

(ii) $\langle Au_N, v_N\rangle = \langle F, v_N\rangle \qquad \forall\ v_N \in S_N$,

where $\langle\cdot, \cdot\rangle$ denotes the duality pair (see Appendix A) . Furthermore, assume that for any $\phi \in H$ there exists $\phi_N \in S_N$ with

$$\phi = \lim_{N\to\infty} \phi_N \text{ in } H .$$

Then for N large enough, the Galerkin equations (ii) are uniquely solvable and the estimate

$$\|u - u_N\| \leq c \inf_{v_N \in S_N} \|u - v_N\|$$

holds for a constant c independent of u, u_N, and N, where $\|\cdot\|$ denotes the norm in H.

We have three ways to choose S_N with algebraic or trigonometric polynomials. They are the h-version, the p-version, or the h-p-version. For the h-version, the degree of the polynomials is fixed and the accuracy is achieved by refining the partition. For the p-version, the partition is fixed and the accuracy is achieved by increasing the degree of p. For the h-p-version, the partition is refined and the degree of the polynomial is increased simultaneously.

Let the set of points on γ, $t_0 < t_1 < \dots < t_{n-1}$, $t_0 = t_{n_h - 1}$, $t_0 = t_{n_h}$, define partition π_h of γ into the subarcs $\gamma_{h,j} = (t_{j-1}, t_j)$ and $\bar{\gamma} = \bigcup_{j=1}^{n_h} \bar{\gamma}_{h,j}$. We assume that π_h is quasi-uniform, i.e., if h_j is the length of the subarc γ_{hj}, and $h = \max_{j=1,\dots,n_h} h_j$, then there exists a constant τ independent of h such that $\frac{h}{h_j} \leq \tau$ for all $\gamma_{h,j} \in \pi_h$. For any interval I, $\mathfrak{T}_p(I)$ denotes the set of all trigonometric polynomials of degree $\leq p$ on I and $\mathfrak{P}_p(I)$ denotes the set of all algebraic polynomials of degree $\leq p$ on I. So, for $p \geq 0$, $S_{h,p}(\gamma)$ denotes the set of all continuous functions $v_{h,p}$ such that $v_{h,p}|_{\gamma_{h,j}} \in \mathfrak{T}_p(\gamma_{h,j})$ or $\mathfrak{P}_p(\gamma_{h,j})$.

In order to derive the rate of convergence for the h-p-version of the boundary element method we need the following lemma.

Lemma 4([3]): Let $a \in H^s(\gamma)$, $s \geq 0$. Then there exists $a_{h,p} \in S_{h,p}$ such that

$$\|a - a_{h,p}\|_0 \leq ch^{\mu} p^{-s} \|a\|_s ,$$

$\mu = \min(p + 1, s)$ and c is independent of a, h, p .

The h-p-version of the boundary element method consists of finding $a_{h,p} \in S_{h,p}(\gamma)$ such that

$$\mathfrak{B}(a_{h,p}, b_{h,p}) = -2(f, b_{h,p})_0 \qquad \forall b_{h,p} \in S_{h,p}(\gamma) . \tag{18}$$

Now, combining Lemmas 3 and 4, we have the following result:

Theorem 2: Let $a \in H^s(\gamma)$ be the solution of problem (14) and $a_{h,p}$ be the solution to the h-p-version of the boundary element method (18). Then

$$\|a - a_{h,p}\|_0 \leq ch^{\mu} p^{-s} \|a\|_s ,$$

$\mu = \min(p + 1, s)$, c does not depend on a, h, and p .

For many practical problems, the integrals appearing in (18) cannot be computed exactly. Instead, they are approximated through the process of numerical integration. This yields the following fully discrete problem:

find $\bar{a}_{h,p} \in S_{h,p}(\gamma)$ such that

$$\mathcal{B}_{h,p}(\bar{a}_{h,p}, b_{h,p}) = -2(f, b_{h,p})_0 \qquad \forall\, b_{h,p} \in S_{h,p}(\gamma)\,, \tag{20}$$

where $B_{h,p}$ is the discrete bilinear form with numerical quadratures.

For some special curves γ (as, for example, circles or ellipses), it is possible to remove the logarithmic singularity in the integral equation (11). In such situations, the estimates presented in [6], Chapt. 9, for the case of trigonometric polynomials $\mathcal{T}_p(\gamma_{h,j})$ combined with (19) give the following error estimate

$$||a - \bar{a}_{h,p}||_0 \le c h^{\mu} p^{-s} ||a||_s\,, \tag{21}$$

provided that the numerical quadratures used are of the degree $2p$. A similar result in the case of algebraic polynomials $\mathcal{P}_p(\gamma_{h,j})$ can be obtained by means of Theorem 5.1 of [4] .

In the general case when logarithmic singularities occur in (11), we need to apply special Gauss formulas for integrals of the type

$$\int_0^1 \frac{1}{r}\, dr\,, \text{ (see, e.g. [15])} .$$

5A. Problem formulation. In the integral equation (11) as well as (9) and (10), there are two real parameters η and α. Our only restrictions on these parameters are $\eta > 0$ and $\alpha \neq 0$. One may choose η and α to minimize the condition numbers for the corresponding integral operators as well as for the appropriate discrete system.

Let us consider the special case where Ω is the exterior of the circle γ.

Let γ be the circle of radius r, such that γ: $x(t) = (r \cos t, r \sin t)$, $0 \leq t \leq 2\pi$.

Now we form the weak formulation. Find $a = (a_1, a_2)$ such that

$$\int_\gamma (a(x) + Ka(x) + \eta\, Sa(x), v(x))\, d\gamma_x =$$

$$= -2 \int_\gamma (f(x), v(x))\, d\gamma_x \quad \text{for all } v(x) \in C(\gamma) \times C(\gamma). \tag{22}$$

We choose the parameterization:

$$\begin{array}{ll} x: x_1(t) = r \cos t & y: y_1(s) = r \cos s \\ \quad x_2(t) = r \sin t & \quad y_2(s) = r \sin s \\ \quad t \in [0, 2\pi] & \quad s \in [0, 2\pi]\,. \end{array}$$

Then

(a) $n(y)$: $n_1(s) = \cos s \qquad s \in [0, 2\pi]$

$$n_2(s) = \sin s$$

(b) $ds_y = \sqrt{y_1^2(s) + y_2^2(s)}\, ds = r\, ds$

(c) $\ell = 2\pi r$

(d) $|x - y| = 2r|\sin(\frac{t - s}{2})|$

(e) $|x - y|^2 = 4r^2 \sin^2 (\frac{t - s}{2})$

(f) $\ln \frac{1}{|x - y|} = \ln \frac{1}{2r|\sin(\frac{t-s}{2})|} = -\ln 2r|\sin (\frac{t - s}{2})| = -\frac{1}{2} \ln(4r^2 \sin^2(\frac{t - s}{2}))$

We now derive the two components of $(Ka)(t)$. A straightforward but tedious computation gives:

$$(Ka)_1(t) = -\frac{1}{4\pi}\int_0^{2\pi} \{a_1(s)\, 4 \sin^2(\tfrac{t + s}{2}) + a_2(s)(-2 \sin(t + s))\}\, ds \ . \qquad (23)$$

Similarly, for $(Ka)_2\, (t)$, we have,

$$(Ka)_2\, (t) = -\frac{1}{4\pi} \int_0^{2\pi} \{a_1(s)\, (-2 \sin(t + s)) + a_2(s)\, 4 \cos^2 (\tfrac{t + s}{2})\}\, ds\ . \ (24)$$

Next we construct $(Sa)(t)$. A tedious computation yields

$$(Sa)_1\, (t) = \frac{r}{2\pi} \int_0^{2\pi} a_1(s)[-\tfrac{1}{2} \ln(4r^2 \sin^2(\tfrac{t - s}{2})) + \sin^2 (\tfrac{t + s}{2})]ds$$
$$(25)$$
$$- \frac{r}{4\pi} \int_0^{2\pi} a_2(s) \sin(t + s)\, ds + [\frac{r \ln r}{2\pi} - \frac{r}{4\pi} + \frac{\alpha r}{2\pi}] \int_0^{2\pi} a_1(s)\, ds$$

and

$$(Sa)_2(t) = \frac{-r}{4\pi}\int_0^{2\pi} a_1(s)\sin(t+s)ds - \frac{r}{4\pi}\int_0^{2\pi} a_2(s)\ln(4\sin^2(\frac{t-s}{2}))ds$$

$$+\frac{r}{2\pi}\int_0^{2\pi} a_2(s)\cos^2(\frac{t+s}{2})ds + \frac{r}{2\pi}(\alpha-\frac{1}{2})\int_0^{2\pi} a_2(s)\,ds \qquad (26)$$

Note that we have removed the logarithmic singularities in S . From the integral equation (22), we obtain the following system of equations:

$$\int_0^{2\pi} a_1(t)\,v(t)\,dt - \frac{1}{\pi}(1-\frac{r\eta}{2})\int_0^{2\pi}\int_0^{2\pi} a_1(s)\sin^2(\frac{t+s}{2})\,v(t)\,ds\,dt$$

$$-\frac{r\eta}{4\pi}\int_0^{2\pi}\int_0^{2\pi} a_1(s)\ln(4\sin^2(\frac{t-s}{2}))\,v(t)\,ds\,dt$$

$$+\frac{1}{2\pi}(1-\frac{r\eta}{2})\int_0^{2\pi}\int_0^{2\pi} a_2(s)\sin(t+s)\,v(t)\,ds\,dt \qquad (27)$$

$$+\frac{r\eta}{2\pi}(\alpha-\frac{1}{2})(\int_0^{2\pi} a_1(s)\,ds)(\int_0^{2\pi} v(t)\,dt)$$

$$= -2\int_0^{2\pi} f_1(t)\,v(t)\,dt$$

and

$$\int_0^{2\pi} a_2(t)\,v(t)\,dt - \frac{1}{\pi}(1-\frac{r\eta}{2})\int_0^{2\pi}\int_0^{2\pi} a_2(s)\cos^2(\frac{t+s}{2})\,v(t)\,ds\,dt$$

$$- \frac{r\eta}{4\pi} \int_0^{2\pi} \int_0^{2\pi} a_2(s) \ln(4 \sin^2(\frac{t-s}{2}))\, v(t)\, ds\, dt$$

$$+ \frac{1}{2\pi} (1 - \frac{r\eta}{2}) \int_0^{2\pi} \int_0^{2\pi} a_1(s) \sin(t+s)\, v(t)\, ds\, dt$$

$$+ \frac{r\eta}{2\pi} (\alpha - \frac{1}{2}) (\int_0^{2\pi} a_2(s)\, ds)(\int_0^{2\pi} v(t)\, dt)$$

$$= -2 \int_0^{2\pi} f_2(t)\, v(t)\, dt$$

for all continuous $v: [0, 2\pi] \to \mathbb{R}$.

In [12] it is indicated how to choose the parameters α and η. Namely, if we want to select α and η in such a way that the condition number of the operator in (27)-(28) is as small as possible, then the optimal set of (α, η) is described in Figure 1.

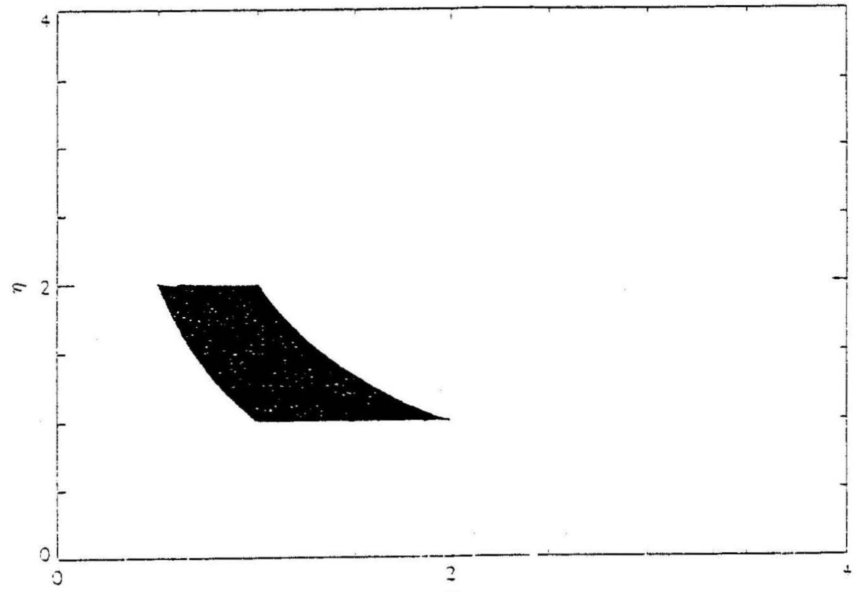

FIGURE 1

OPTIMAL SET OF (α, γ)

5B. Finite Dimensional Discretization. Let $m \subset C^0(0, 2\pi)$, $\dim S_m = m + 1$ so $S_m = \mathrm{span}\{\phi_0, \phi_1, \phi_2, \ldots, \phi_m\}$. Let us consider $\tilde{a} = (\tilde{a}_1(t), \tilde{a}_2(t)) \in S_m \times S_m$ such that $\tilde{a} = (\tilde{a}_1(t) = \sum_{j=0}^{m} \tilde{a}_{1j}\, \phi_j(t), \tilde{a}_2 = \sum_{j=0}^{m} \tilde{a}_{2j}\, \phi_j(t))$. Now equations (18) become

$$\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{1j}\, \phi_j(t)\, \phi_i(t)dt - \frac{1}{\pi}(1 - \frac{r\eta}{2}) \int_0^{2\pi}\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{1j}(s) \sin^2(\frac{t+s}{2})\, \phi_i(t)\, ds\, dt$$

$$- \frac{r\eta}{4\pi} \int_0^{2\pi}\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{1j}(s)\, \phi_j(s) \ln(4 \sin^2(\frac{t-s}{2}))\, \phi_i(t)\, ds\, dt$$

$$+ \frac{1}{2\pi}(1 - \frac{r\eta}{2}) \int_0^{2\pi}\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{2j}(s)\, \phi_j(s) \sin(t+s)\, \phi_i(t)\, ds\, dt$$

$$+ \frac{r\eta}{2\pi}(\alpha - \frac{1}{2})(\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{ij}(s)\, \phi_j(s)ds)(\int_0^{2\pi} \phi_i(t)\, dt)$$

$$= -2 \int_0^{2\pi} f_1(t)\, \phi_i(t)\, dt \qquad (29)$$

and

$$\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{2j}\, \phi_j(t)\, \phi_i(t)dt - \frac{1}{\pi}(1 - \frac{r\eta}{2}) \int_0^{2\pi}\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{2j}(s)\, \phi_j(s) \cos^2(\frac{t+s}{2})\, \phi_i(t)\, ds\, dt$$

$$- \frac{r\eta}{4\pi} \int_0^{2\pi}\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{2j}(s)\, \phi_j(s) \ln(4 \sin^2(\frac{t-s}{2}))\, \phi_i(t)\, ds\, dt \qquad (30)$$

$$+ \frac{1}{2\pi}(1 - \frac{r\eta}{2}) \int_0^{2\pi}\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{1j}(s)\, \phi_j(s) \sin(t+s)\, \phi_i(t)\, ds\, dt$$

$$+ \frac{r\eta}{2\pi}\left(\alpha - \frac{1}{2}\right)\left(\int_0^{2\pi} \sum_{j=0}^{m} \tilde{a}_{2j}(s)\, \phi_j(s) ds\right)\left(\int_0^{2\pi} \phi_i(t)\, dt\right)$$

$$= -2 \int_0^{2\pi} f_2(t)\, \phi_i(t)\, dt$$

for all $i = 0, 1, 2, \ldots, m$. This yields the following linear system:

$$\left[\begin{array}{c:c} K + SR + SU + SV & SQ \\ \hdashline SQ & K + SP + SU + SV \end{array}\right] \begin{bmatrix} \tilde{a}_{10} \\ \vdots \\ \tilde{a}_{1m} \\ \hline \tilde{a}_{20} \\ \vdots \\ \tilde{a}_{2m} \end{bmatrix} = \begin{bmatrix} F_{10} \\ \vdots \\ F_{1m} \\ \hline F_{20} \\ \vdots \\ F_{2m} \end{bmatrix}$$

where

(a) $K(0{:}\ m, 0{:}m)$ $\qquad k_{ij} = \int_0^{2\pi} \phi_i(t)\, \phi_j(t)\, dt$

(b) $R(0{:}m, 0{:}m)$ $\qquad r_{ij} = -\frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \phi_i(t)\, \phi_j(s) \sin^2\left(\frac{t+s}{2}\right) ds\, dt$

and $SR = \left(1 - \frac{r\eta}{2}\right) \times R$

(c) $Q(0{:}m, 0{:}m)$ $\qquad q_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \phi_i(t)\, \phi_j(s) \sin(t + s)\, ds\, dt$

and $SQ = \left(1 - \frac{r\eta}{2}\right) \times Q$

(d) P(0:m, 0:m)

$$p_{ij} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \phi_i(t)\, \phi_j(s) \cos^2\left(\frac{t+s}{2}\right) ds\, dt$$

$$\text{and } SP = \left(1 - \frac{r\eta}{2}\right) \times P$$

(e) SU(0:m, 0:m)

$$su_{ij} = -\frac{r\eta}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \phi_j(s) \ln\left(4\sin^2\left(\frac{t-s}{2}\right)\right) \phi_i(t)\, ds\, dt$$

(f) SV(0:m, 0:m)

$$sv_{ij} = \frac{r\eta}{2\pi}\left(\alpha - \frac{1}{2}\right)\left(\int_0^{2\pi} \phi_j(s)\, ds\right)\left(\int_0^{2\pi} \phi_i(t)\, dt\right)$$

5C. Discretization by Trigonometric Polynomials

Let us consider a finite dimensional space S_m spanned by the following $m = 2n + 1$ trigonometric polynomials:

$$
\begin{aligned}
&\phi_0(t) = 1 \\
&\phi_1(t) = \cos t \\
&\phi_2(t) = \cos 2t \\
&\quad \vdots \\
&\phi_n(t) = \cos nt \\
&\phi_{n+1}(t) = \sin t \\
&\phi_{n+2}(t) = \sin 2t \\
&\quad \vdots \\
&\phi_m(t) = \sin nt\,, \qquad m = 2n\,,
\end{aligned}
\tag{32}
$$

which are mutually orthogonal, i.e.

$$\int_0^{2\pi} \phi_i(t)\, \phi_j(t)\, dt = 0\,, \qquad i \neq j\,.$$

Thus

(a) $K(0{:}m,\ 0{:}m)$ is such that

$$k_{ij} = \begin{cases} 2\pi & \text{if } i = j = 0 \\ \pi & \text{if } i = j > 0 \\ 0 & \text{if } i \neq j \end{cases}$$

(b) $R(0:m, 0:m)$

$$r_{00} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \sin^2(\frac{t+s}{2})\, ds\, dt$$

$$r_{0i} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(is) \sin^2(\frac{t+s}{2})\, ds\, dt \quad \text{for } i = 1,\ldots, n$$

$$r_{0i} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(i-n)s] \sin^2(\frac{t+s}{2})\, ds\, dt \quad \text{for } i = n+1,\ldots, m$$

$$r_{i0} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(it) \sin^2(\frac{t+s}{2})\, ds\, dt \quad \text{for } i = 1,\ldots, n$$

$$r_{i0} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(i-n)s] \sin^2(\frac{t+s}{2})\, ds\, dt \quad \text{for } i = n+1,\ldots, m$$

$$r_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(jt) \cos(ls) \sin^2(\frac{t+s}{2})\, ds\, dt \quad \text{for } j = 1,\ldots, n \text{ and } l = 1,\ldots, n$$

$$r_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(jt)\ [\sin(l-n)s] \sin^2(\frac{t+s}{2})\, ds\, dt$$

for $j = 1,\ldots, n$ and $l = n+1,\ldots, m$

$$r_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(j-n)t] \cos(ls)\ \sin^2(\frac{t+s}{2})\, ds\, dt$$

for $j = n+1, \ldots, m$

$$r_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(j-n)t]\,[\sin(l-n)s]\,\sin^2\!\left(\frac{t+s}{2}\right) ds\, dt$$

for $j = n+1,\dots, m$ and $l = n+1,\dots, m$

(c) $Q(0{:}m, 0{:}m)$

$$q_{00} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sin(t+s)\, ds\, dt$$

$$q_{0i} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(is)\sin(t+s)\, ds\, dt \quad \text{for } i = 1,\dots, n$$

$$q_{0i} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(i-n)s]\sin(t+s)\, ds\, dt \quad \text{for } i = n+1,\dots, m$$

$$q_{i0} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(it)\sin(t+s)\, ds\, dt \quad \text{for } i = 1,\dots, n$$

$$q_{i0} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(i-n)s]\sin(t+s)\, ds\, dt \quad \text{for } i = n+1,\dots, m$$

$$q_{jl} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(jt)\cos(ls)\sin(t+s)\, ds\, dt \quad \text{for } j = 1,\dots, n \text{ and } l = 1,\dots, n$$

$$q_{jl} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(jt)\,[\sin(l-n)s]\sin(t+s)\, ds\, dt$$

for $j = 1,\dots, n$ and $l = n+1,\dots, m$

$$q_{jl} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin (j - n)t] \cos (ls) \sin(t + s)\, ds\, dt$$

for $j = n + 1,\dots, m$ and $l = 1,\dots, n$

$$q_{jl} = +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(j - n)t] [\sin(l - n)s] \sin(t + s)\, ds\, dt$$

for $j = n + 1,\dots, m$ and $l = n + 1,\dots, m$

(d) P(0:m, 0:m)

$$p_{00} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos^2\left(\frac{t + s}{2}\right) ds\, dt$$

$$p_{0i} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos (is) \cos^2\left(\frac{t + s}{2}\right) ds\, dt \quad \text{for } i = 1,\dots, n$$

$$p_{0i} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(i - n)s] \cos^2\left(\frac{t + s}{2}\right) ds\, dt \quad \text{for } i = n + 1,\dots, m$$

$$p_{i0} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos (it) \cos^2\left(\frac{t + s}{2}\right) ds\, dt \quad \text{for } i = 1,\dots, n$$

$$p_{i0} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(i - n)s] \cos^2\left(\frac{t + s}{2}\right) ds\, dt \quad \text{for } i = n + 1,\dots, m$$

$$p_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(jt) \cos(ls) \cos^2\left(\frac{t + s}{2}\right) ds\, dt \quad \text{for } j = 1,\dots, n \text{ and } l = 1,\dots, n$$

$$p_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} \cos(jt)\ [\sin(l-n)s] \cos^2\left(\frac{t+s}{2}\right) ds\, dt$$

for $j = 1,\ldots, n$ and $l = n+1,\ldots, n$

$$p_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(j-n)t] \cos(l-s) \cos^2\left(\frac{t+s}{2}\right) ds\, dt$$

for $j = n+1,\ldots, m$ and $l = 1,\ldots, m$

$$p_{jl} = -\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} [\sin(j-n)t] \cos(ls) \cos^2\left(\frac{t+s}{2}\right) ds\, dt$$

for $j = n+1,\ldots, m$ and $l = n+1,\ldots, m$

(e) SU(0:m, 0:m)

$$su_{ij} = \begin{cases} \dfrac{r\eta\pi}{i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(f) SV(0:m, 0:m)

$$SV_{ij} = \begin{cases} 2\pi r\gamma\left(\alpha - \dfrac{1}{2}\right) & \text{if } i = j = 0 \\ 0 & \text{if } i \neq j = 0 \end{cases}$$

6A. Rate of Convergence

The goal of our first numerical test is to compare the theoretical rate of convergence given by (21) with that obtained in numerical computations. To accomplish this, we select

i) $f^{(1)}(t) = (t(t - 2\pi), \cos t)$, $t \in [0, 2\pi]$,

and

ii) $f^{(2)}(t) = (t^2(t - 2\pi)^2 , 1 + \cos t + \sin t)$, $t \in [0, 2\pi]$,

as the given vector field of f. Note that $f^{(1)} \in H^{3/2-\varepsilon}(\gamma)$ and $f^{(2)} \in H^{7/2-\varepsilon}(\gamma)$ for any $\varepsilon > 0$ which by Lemma 2 yields $a^{(1)} \in H^{3/2-\varepsilon}(\gamma)$ and $a^{(2)} \in H^{7/2-\varepsilon}(\gamma)$, respectively. In our computations we used only one interval (the p-version with $m = 2n$ trigonometric polynomials), hence, according to the estimate (21), the rates of convergence should be

i) $n^{-3/2+\varepsilon}$

and

ii) $n^{-7/2+\varepsilon}$.

The results of our computations presented in Figure 2 display very good agreement with the theoretical rates. For the numerical data, see Appendix B .

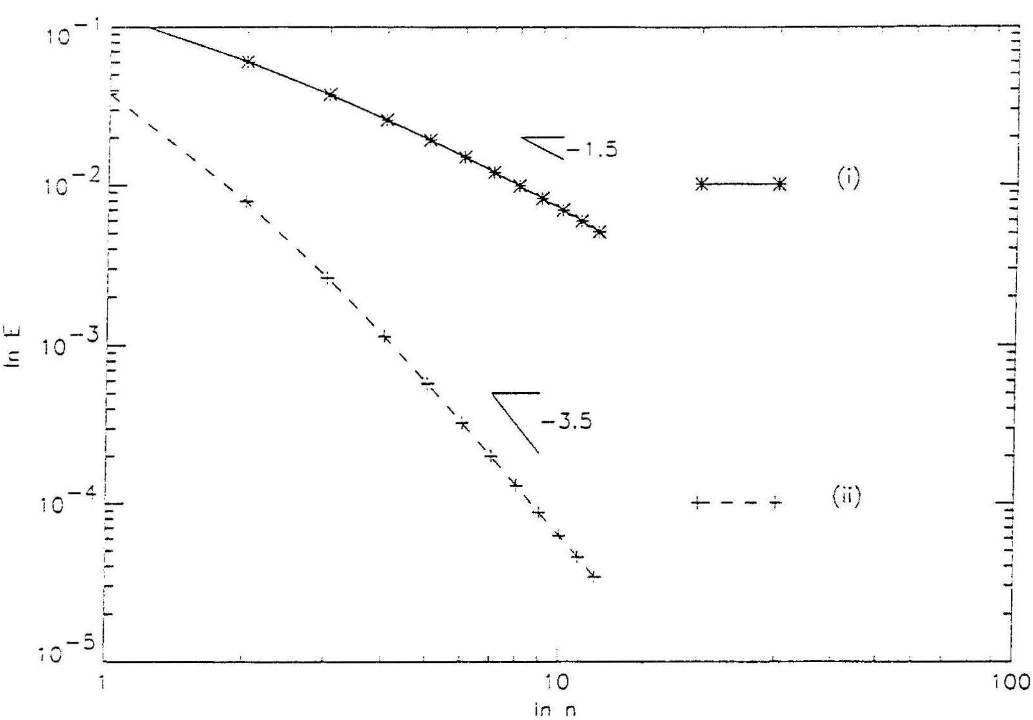

FIGURE 2

RATE OF CONVERGENCE

6B. Approximation and Visualization of the Velocity Field

For this display we select

i) $f^{(0)}(t) = (-\sin t, \cos t)$, $t \in [0, 2\pi]$,

and

ii) $f^{(1)}(t) = (t(t - 2\pi), \cos t)$, $t \in [0, 2\pi]$,

as the boundary fields. The computed velocity fields are presented in Figure 3 for $f^{(0)}$ and in Figure 4 for $f^{(1)}$. For the numerical data, see Appendix B.

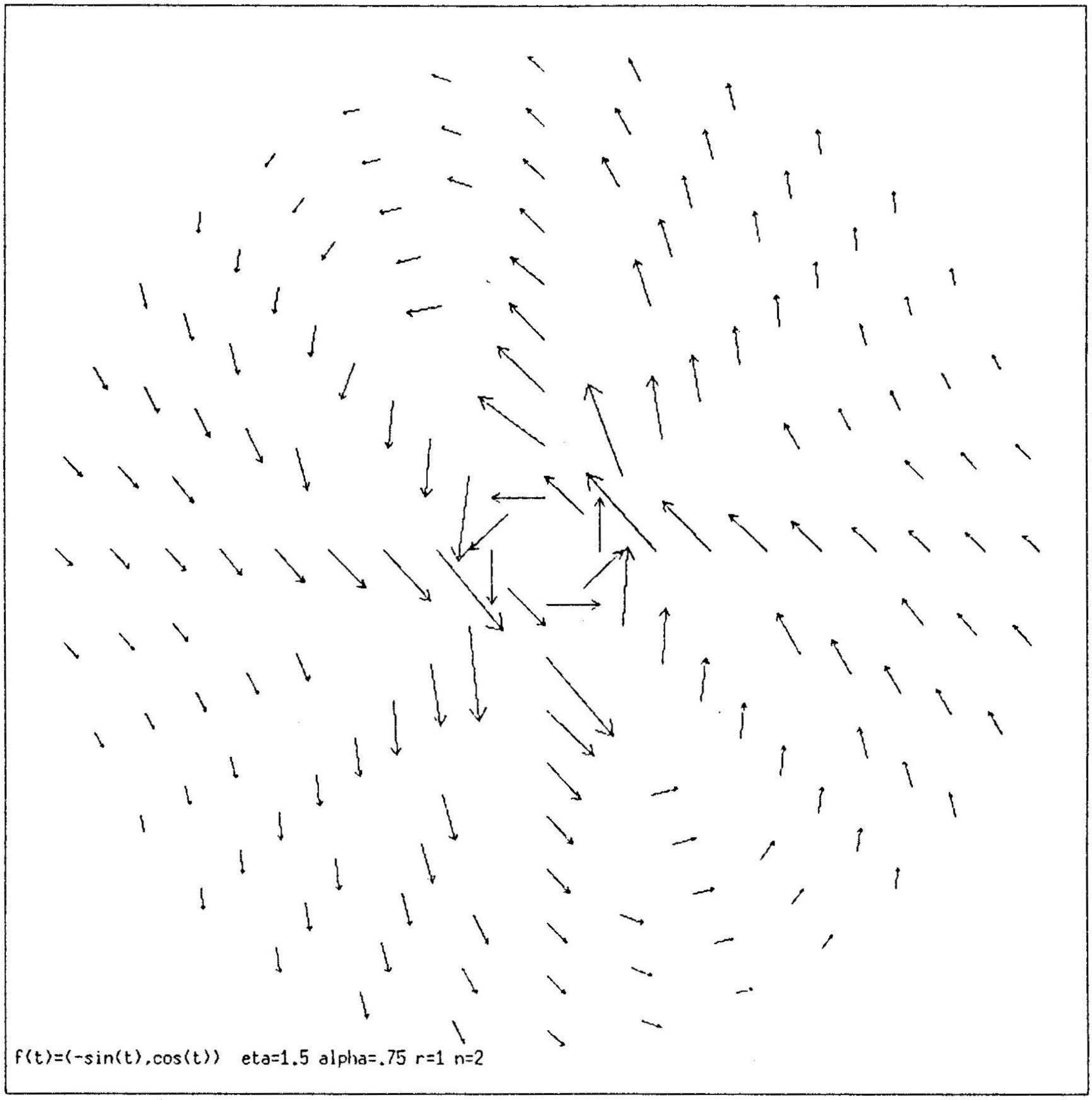

FIGURE 3. VELOCITY FIELD FOR $f(t) = (-\sin t , \cos t)$

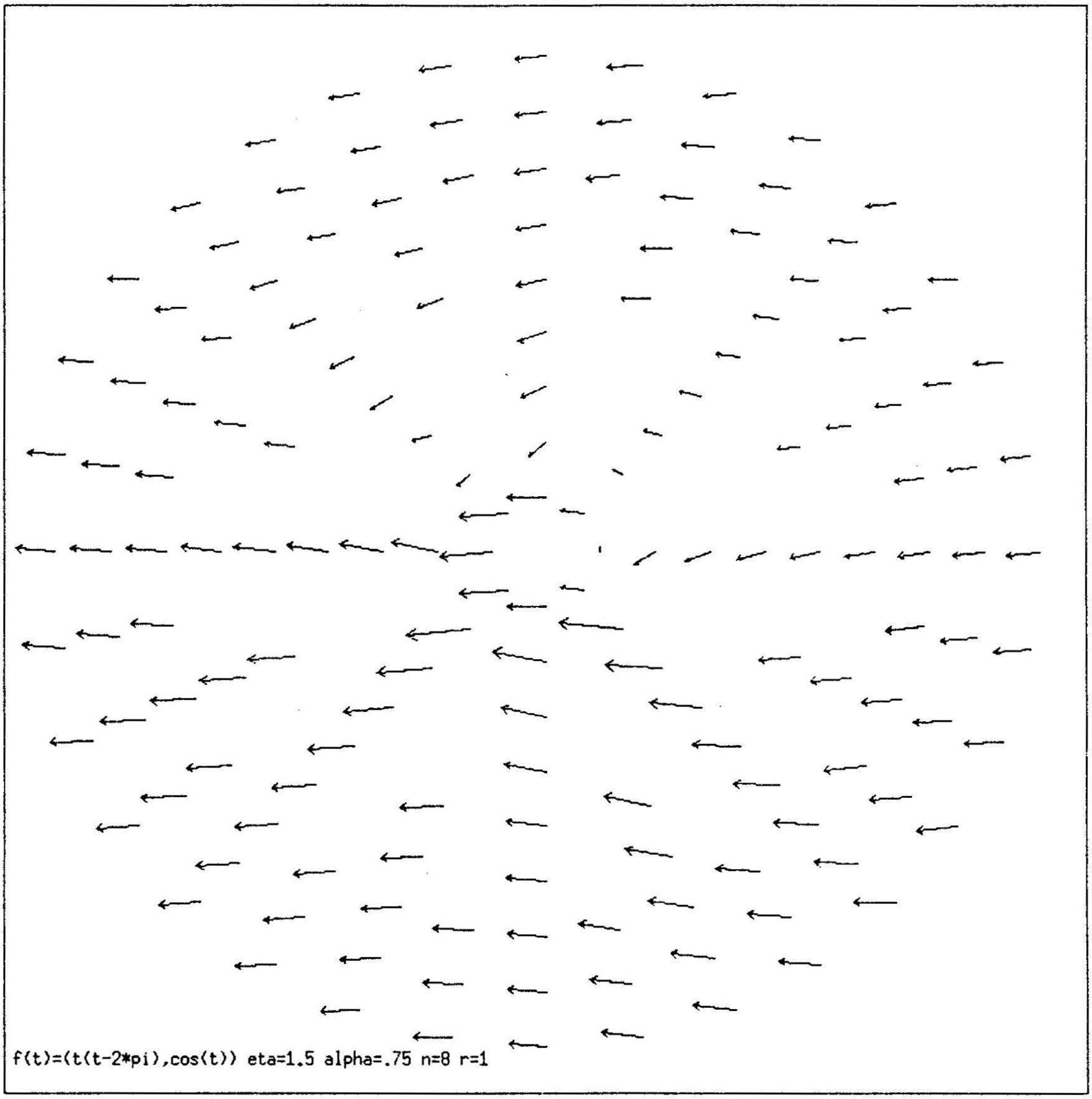

FIGURE 4. VELOCITY FIELD FOR $f(t) = (t(t - 2\pi), \cos t)$

7. CONCLUSION

The theoretical analysis and numerical test computations indicate that the boundary element method is an important tool of computational hydrodynamics. These results seem to be promising for further developments of boundary elements methods to more practical applications as, for example, piecewise smooth boundary γ or piecewise smooth boundary field f.

Some numerical tests carried out without theoretical analysis are very interesting. We selected, for example, a discontinuous boundary field f and we obtained a computational rate of convergence close to $p^{-1/2}$, which by (21) is optimal for the solution $a \in H^{1/2}(\gamma)$.

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BIBLIOGRAPHY

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APPENDICES

APPENDIX A

LEBESGUE INTEGRALS

Since two integrable functions which differ on a set of zero measure have the same integral, they belong to the same equivalence class and can be identified by the Lebesgue integration theorem.

Let (a, b) be a bounded interval of $\mathbb{R}$. Then $L_2(a, b)$ is the space of measurable functions $u: (a, b) \to \mathbb{R}$ such that

$$\int_a^b |u(x)|^2 \, dx < +\infty$$

and the norm is given by:

$$||u||_{L_2(a,\, b)} = \left(\int_a^b |u(x)|^2 \, dx\right)^{1/2} .$$

The inner-product is given by:

$$(u, v) = \int_a^b u(x)\, v(x) \, dx .$$

The space $L_2(a, b)$ is a Hilbert space.

Let X be a real vector space. An inner product on X is a function $X \times X \to \mathbb{R}$, denoted by (u, v), which satisfies the following properties:

(i) $(u, v) = (v, u)$ for all $u, v \in X$;

(ii) $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$ for all $\alpha, \beta \in \mathbb{R}$ and all $u, v, w \in X$;

(iii) $(u, u) \geq 0$ for all $u \in X$;

(iv) $(u, u) = 0 \Rightarrow u = 0$.

The inner product (u, v) defines a norm on X by the relationship

$$||u|| = (u, u)^{1/2} \text{ for all } u \in X.$$

A Cauchy sequence in X is a sequence $\{u_k | k = 0, 1, ...\}$ of elements of X which satisfy the following property:

> for each positive number $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) > 0$ such that the distance $||u_k - u_m||$ between any 2 elements of the sequence is smaller than ε provided both k and m are larger than $N(\varepsilon)$.

A sequence in X is said to converge to an element $u \in X$ if the distance $||u_k - u||$ tends to 0 as k tends to ∞.

A Hilbert space is a vector space equipped with an inner product for which all the Cauchy sequences are convergent.

Let X be a Hilbert space. A linear form $F: X \to \mathbb{R}$ is said to be continuous if there exists a constant $c > 0$ such that

$$|F(u)| \leq c||u|| \text{ for all } u \in X .$$

The set of all linear continuous forms on X is a vector space. We can define a norm on this space by setting

$$||F|| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{F(u)}{||u||} .$$

The vector space of all linear continuous forms on X is called the dual space of X, and is denoted by X'.

The bilinear form from $X' \times X$ into $\mathbb{R}$ defined by

$$\langle F, u \rangle = F(u)$$

is called the duality pairing between X and X'. For more details, see [1] .

APPENDIX B

R A T E   O F   C O N V E R G E N C E

i) f(t) = (t*(t-2*pi) , cost)

polynomial degree n	relative error
1	0.1208422444
2	0.0603559800
3	0.0373399098
4	0.0258649413
5	0.0191950420
6	0.0149123279
7	0.0119620301
8	0.0098192639
9	0.0081966628
10	0.0069249854
11	0.0058985558
12	0.0050480413

ii) f(t) = (t*t*(t-2*pi)*(t-2*pi) , 1+cost+sint)

polynomial degree n	relative error
1	0.0379040357
2	0.0079160563
3	0.0026418505
4	0.0011386031
5	0.0005760443
6	0.0003251776
7	0.0001987457
8	0.0001290089
9	0.0000877832
10	0.0000620366
11	0.0000452245
12	0.0000338298

TABLE 1

SUPPORTING DATA FOR FIGURE 2

Table 2. Supporting Data For Figure 3

f(t)=(-sin(t),cos(t))

rr= 1.00 eta= 1.50 alpha= 0.75

aa(j,l)
0.711718E-17 0.634876E+00 -.502277E-18 0.665686E+01 -.717041E-16
-.185093E-15 -.665686E+01 -.854466E-16 -.634876E+00 -.259428E-16

R is equal to rr+ih
t is equal 0,h,2h,...,2*pi

R	t	u(x1)	u(x2)
0.100000E+01	0.000000E+00	0.000000E+00	0.100000E+01
0.100000E+01	0.785398E+00	-.431795E+01	0.707107E+00
0.100000E+01	0.157080E+01	-.740220E+01	0.612303E-16
0.100000E+01	0.235619E+01	-.925275E+01	-.707107E+00
0.100000E+01	0.314159E+01	-.986960E+01	-.100000E+01
0.100000E+01	0.392699E+01	-.925275E+01	-.707107E+00
0.100000E+01	0.471239E+01	-.740220E+01	-.183691E-15
0.100000E+01	0.549779E+01	-.431795E+01	0.707107E+00
0.200000E+01	0.000000E+00	-.125109E+01	0.144982E+01
0.200000E+01	0.785398E+00	-.611377E+00	0.175483E+01
0.200000E+01	0.157080E+01	-.122606E+01	0.900422E+00
0.200000E+01	0.235619E+01	-.210579E+00	-.160462E+01
0.200000E+01	0.314159E+01	0.119519E+01	-.150845E+01
0.200000E+01	0.392699E+01	0.166609E+00	-.181760E+01
0.200000E+01	0.471239E+01	0.122606E+01	-.149218E+01
0.200000E+01	0.549779E+01	0.840287E-01	0.148443E+01
0.300000E+01	0.000000E+00	-.866938E+00	0.950046E+00
0.300000E+01	0.785398E+00	-.178038E+00	0.118018E+01
0.300000E+01	0.157080E+01	-.871466E+00	0.857101E+00
0.300000E+01	0.235619E+01	-.106182E+00	-.111145E+01
0.300000E+01	0.314159E+01	0.860825E+00	-.953286E+00
0.300000E+01	0.392699E+01	0.177180E+00	-.118306E+01
0.300000E+01	0.471239E+01	0.867831E+00	-.861100E+00
0.300000E+01	0.549779E+01	0.581217E-01	0.105398E+01
0.400000E+01	0.000000E+00	-.662240E+00	0.701239E+00
0.400000E+01	0.785398E+00	-.148717E+00	0.882782E+00
0.400000E+01	0.157080E+01	-.666000E+00	0.654590E+00
0.400000E+01	0.235619E+01	-.815513E-01	-.910108E+00
0.400000E+01	0.314159E+01	0.662238E+00	-.701243E+00
0.400000E+01	0.392699E+01	0.578357E-01	-.104221E+01
0.400000E+01	0.471239E+01	0.626918E+00	-.709499E+00
0.400000E+01	0.549779E+01	0.706338E-01	0.853068E+00
0.500000E+01	0.000000E+00	-.536148E+00	0.555856E+00
0.500000E+01	0.392699E+00	-.276056E+00	0.492892E+00
0.500000E+01	0.785398E+00	-.850321E-01	0.706830E+00
0.500000E+01	0.117810E+01	-.261595E+00	0.847198E+00
0.500000E+01	0.157080E+01	-.643788E+00	0.533235E+00
0.500000E+01	0.196350E+01	-.659589E+00	-.143196E+00
0.500000E+01	0.235619E+01	-.249373E+00	-.697723E+00
0.500000E+01	0.274889E+01	0.224305E+00	-.788675E+00
0.500000E+01	0.314159E+01	0.535943E+00	-.613722E+00
0.500000E+01	0.353429E+01	0.227536E+00	-.526315E+00
0.500000E+01	0.392699E+01	0.812887E-01	-.730881E+00
0.500000E+01	0.431969E+01	0.224267E+00	-.863413E+00
0.500000E+01	0.471239E+01	0.495911E+00	-.540877E+00
0.500000E+01	0.510509E+01	0.500935E+00	0.142191E+00
0.500000E+01	0.549779E+01	0.608514E-01	0.691932E+00
0.500000E+01	0.589049E+01	-.431722E+00	0.768794E+00
0.600000E+01	0.000000E+00	-.449508E+00	0.460939E+00
0.600000E+01	0.392699E+00	-.228681E+00	0.407386E+00
0.600000E+01	0.785398E+00	-.644971E-01	0.589045E+00
0.600000E+01	0.117810E+01	-.202588E+00	0.710272E+00
0.600000E+01	0.157080E+01	-.474824E+00	0.448527E+00
0.600000E+01	0.196350E+01	-.471613E+00	-.120494E+00
0.600000E+01	0.235619E+01	-.108176E+00	-.581044E+00
0.600000E+01	0.274889E+01	0.299298E+00	-.646443E+00
0.600000E+01	0.314159E+01	0.386004E+00	-.475407E+00
0.600000E+01	0.353429E+01	0.218223E+00	-.414793E+00
0.600000E+01	0.392699E+01	0.624702E-01	-.594265E+00
0.600000E+01	0.431969E+01	0.188883E+00	-.713519E+00
0.600000E+01	0.471239E+01	0.428679E+00	-.450087E+00
0.600000E+01	0.510509E+01	0.428018E+00	0.120133E+00
0.600000E+01	0.549779E+01	0.536141E-01	0.580190E+00
0.600000E+01	0.589049E+01	-.362269E+00	0.642349E+00
0.700000E+01	0.000000E+00	-.386676E+00	0.393915E+00
0.700000E+01	0.196350E+00	-.307670E+00	0.341748E+00
0.700000E+01	0.392699E+00	-.195513E+00	0.347425E+00
0.700000E+01	0.589049E+00	-.978041E-01	0.410590E+00
0.700000E+01	0.785398E+00	-.532807E-01	0.504798E+00
0.700000E+01	0.981748E+00	-.819992E-01	0.586669E+00
0.700000E+01	0.117810E+01	-.170652E+00	0.610728E+00
0.700000E+01	0.137445E+01	-.293067E+00	0.546084E+00
0.700000E+01	0.157080E+01	-.393764E+00	0.386335E+00
0.700000E+01	0.176715E+01	-.431451E+00	0.154807E+00
0.700000E+01	0.196350E+01	-.386024E+00	-.103724E+00

R	t	u(x1)	u(x2)
0.700000E+01	0.215984E+01	-.241026E+00	-.337581E+00
0.700000E+01	0.235619E+01	-.662518E-01	-.499327E+00
0.700000E+01	0.255254E+01	0.150152E+00	-.570195E+00
0.700000E+01	0.274889E+01	0.288698E+00	-.552550E+00
0.700000E+01	0.294524E+01	0.374342E+00	-.479612E+00
0.700000E+01	0.314159E+01	0.362980E+00	-.398505E+00
0.700000E+01	0.333794E+01	0.260923E+00	-.343502E+00
0.700000E+01	0.353429E+01	0.192697E+00	-.349547E+00
0.700000E+01	0.373064E+01	0.728882E-01	-.410872E+00
0.700000E+01	0.392699E+01	0.515621E-01	-.506254E+00
0.700000E+01	0.412334E+01	0.862124E-01	-.586796E+00
0.700000E+01	0.431969E+01	0.164588E+00	-.611547E+00
0.700000E+01	0.451604E+01	0.279814E+00	-.546478E+00
0.700000E+01	0.471239E+01	0.376566E+00	-.386749E+00
0.700000E+01	0.490874E+01	0.423179E+00	-.155416E+00
0.700000E+01	0.510509E+01	0.371711E+00	0.103522E+00
0.700000E+01	0.530144E+01	0.238948E+00	0.336484E+00
0.700000E+01	0.549779E+01	0.470525E-01	0.499168E+00
0.700000E+01	0.569414E+01	-.152769E+00	0.568877E+00
0.700000E+01	0.589049E+01	-.311958E+00	0.551465E+00
0.700000E+01	0.608684E+01	-.393748E+00	0.478136E+00
0.800000E+01	0.000000E+00	-.339137E+00	0.344005E+00
0.800000E+01	0.196350E+00	-.269462E+00	0.298016E+00
0.800000E+01	0.392699E+00	-.170755E+00	0.302967E+00
0.800000E+01	0.589049E+00	-.845837E-01	0.358600E+00
0.800000E+01	0.785398E+00	-.456160E-01	0.441615E+00
0.800000E+01	0.981748E+00	-.702003E-01	0.513839E+00
0.800000E+01	0.117810E+01	-.148531E+00	0.535400E+00
0.800000E+01	0.137445E+01	-.254478E+00	0.478900E+00
0.800000E+01	0.157080E+01	-.341394E+00	0.339011E+00
0.800000E+01	0.176715E+01	-.376053E+00	0.135989E+00
0.800000E+01	0.196350E+01	-.332819E+00	-.909992E-01
0.800000E+01	0.215984E+01	-.211170E+00	-.295643E+00
0.800000E+01	0.235619E+01	-.493874E-01	-.437869E+00
0.800000E+01	0.255254E+01	0.133429E+00	-.499178E+00
0.800000E+01	0.274889E+01	0.263906E+00	-.483379E+00
0.800000E+01	0.294524E+01	0.337879E+00	-.418801E+00
0.800000E+01	0.314159E+01	0.328969E+00	-.345735E+00
0.800000E+01	0.333794E+01	0.249700E+00	-.298612E+00
0.800000E+01	0.353429E+01	0.169881E+00	-.303720E+00
0.800000E+01	0.373064E+01	0.745295E-01	-.358671E+00
0.800000E+01	0.392699E+01	0.443801E-01	-.442103E+00
0.800000E+01	0.412334E+01	0.642500E-01	-.513920E+00
0.800000E+01	0.431969E+01	0.145482E+00	-.535644E+00
0.800000E+01	0.451604E+01	0.249646E+00	-.479052E+00
0.800000E+01	0.471239E+01	0.334107E+00	-.339145E+00
0.800000E+01	0.490874E+01	0.372307E+00	-.136189E+00
0.800000E+01	0.510509E+01	0.327478E+00	0.908890E-01
0.800000E+01	0.530144E+01	0.209659E+00	0.295281E+00
0.800000E+01	0.549779E+01	0.415810E-01	0.437834E+00
0.800000E+01	0.569414E+01	-.134157E+00	0.498705E+00
0.800000E+01	0.589049E+01	-.273848E+00	0.483032E+00
0.800000E+01	0.608684E+01	-.345558E+00	0.418263E+00
0.900000E+01	0.000000E+00	-.301947E+00	0.305371E+00
0.900000E+01	0.196350E+00	-.239701E+00	0.264286E+00
0.900000E+01	0.392699E+00	-.151577E+00	0.268704E+00
0.900000E+01	0.589049E+00	-.746285E-01	0.318349E+00
0.900000E+01	0.785398E+00	-.399041E-01	0.392489E+00
0.900000E+01	0.981748E+00	-.615871E-01	0.457056E+00
0.900000E+01	0.117810E+01	-.131770E+00	0.476505E+00
0.900000E+01	0.137445E+01	-.225347E+00	0.426329E+00
0.900000E+01	0.157080E+01	-.302758E+00	0.301897E+00
0.900000E+01	0.176715E+01	-.333640E+00	0.121124E+00
0.900000E+01	0.196350E+01	-.294400E+00	-.810414E-01
0.900000E+01	0.215984E+01	-.187779E+00	-.263118E+00
0.900000E+01	0.235619E+01	-.406746E-01	-.389841E+00
0.900000E+01	0.255254E+01	0.119315E+00	-.444056E+00
0.900000E+01	0.274889E+01	0.239227E+00	-.429794E+00
0.900000E+01	0.294524E+01	0.304362E+00	-.371947E+00
0.900000E+01	0.314159E+01	0.297076E+00	-.306114E+00
0.900000E+01	0.333794E+01	0.230293E+00	-.264517E+00
0.900000E+01	0.353429E+01	0.151281E+00	-.268996E+00
0.900000E+01	0.373064E+01	0.700552E-01	-.318369E+00
0.900000E+01	0.392699E+01	0.390841E-01	-.392677E+00
0.900000E+01	0.412334E+01	0.590842E-01	-.457102E+00
0.900000E+01	0.431969E+01	0.130037E+00	-.476587E+00
0.900000E+01	0.451604E+01	0.223402E+00	-.426399E+00
0.900000E+01	0.471239E+01	0.299343E+00	-.301948E+00
0.900000E+01	0.490874E+01	0.331671E+00	-.121183E+00
0.900000E+01	0.510509E+01	0.292206E+00	0.809825E-01
0.900000E+01	0.530144E+01	0.186767E+00	0.262981E+00
0.900000E+01	0.549779E+01	0.371259E-01	0.389833E+00
0.900000E+01	0.569414E+01	-.119531E+00	0.443863E+00
0.900000E+01	0.589049E+01	-.243985E+00	0.429666E+00
0.900000E+01	0.608684E+01	-.307793E+00	0.371724E+00

Table 3. Supporting Data For Figure 4

f(t)=(t(t-2*pi),cos(t))

aa(j,l)

0.734964E+02 -.204372E+02 -.592931E+01 -.268567E+01 -.152528E+01
-.981869E+00 -.684514E+00 -.504313E+00 -.386926E+00 -.218384E+01
0.176780E-15 0.358001E-11 -.240132E-11 0.303809E-11 -.241874E-15
0.170660E-14 -.100791E-14 -.190550E-15 -.612969E+01 -.223418E-15
-.276840E-14 0.408938E-14 -.472390E-14 0.210723E-12 -.362606E-11
-.262069E-15 0.410723E+00 -.132044E-16 -.179026E-11 0.120072E-11
-.151865E-11 0.465323E-16 -.791053E-15 0.582991E-15

R is equal to rr+ih
t is equal 0,h,2h,....,2*pi

R	t	u(x1)	u(x2)
0.100000E+01	0.000000E+00	0.000000E+00	0.100000E+01
0.100000E+01	0.785398E+00	-.431795E+01	0.707107E+00
0.100000E+01	0.157080E+01	-.740220E+01	0.612303E-16
0.100000E+01	0.235619E+01	-.925275E+01	-.707107E+00
0.100000E+01	0.314159E+01	-.986960E+01	-.100000E+01
0.100000E+01	0.392699E+01	-.925275E+01	-.707107E+00
0.100000E+01	0.471239E+01	-.740220E+01	-.183691E-15
0.100000E+01	0.549779E+01	-.431795E+01	0.707107E+00
0.200000E+01	0.000000E+00	-.364048E+01	-.234305E+01
0.200000E+01	0.785398E+00	-.175943E+01	0.991818E+00
0.200000E+01	0.157080E+01	-.334612E+01	-.250348E+01
0.200000E+01	0.235619E+01	-.226619E+01	-.208854E+01
0.200000E+01	0.314159E+01	-.851682E+01	0.188721E+01
0.200000E+01	0.392699E+01	-.116804E+02	-.131437E+01
0.200000E+01	0.471239E+01	-.996290E+01	0.183339E+01
0.200000E+01	0.549779E+01	-.118706E+02	0.126073E+01
0.300000E+01	0.000000E+00	-.462266E+01	-.173464E+01
0.300000E+01	0.785398E+00	-.311928E+01	0.987140E+00
0.300000E+01	0.157080E+01	-.463095E+01	-.182320E+01
0.300000E+01	0.235619E+01	-.344402E+01	-.938750E+00
0.300000E+01	0.314159E+01	-.806189E+01	0.150747E+01
0.300000E+01	0.392699E+01	-.101027E+02	-.780349E+00
0.300000E+01	0.471239E+01	-.859803E+01	0.155569E+01
0.300000E+01	0.549779E+01	-.102883E+02	0.517995E+00
0.400000E+01	0.000000E+00	-.527712E+01	-.133827E+01
0.400000E+01	0.785398E+00	-.393993E+01	0.787802E+00
0.400000E+01	0.157080E+01	-.516514E+01	-.166975E+01
0.400000E+01	0.235619E+01	-.410149E+01	-.237420E+01
0.400000E+01	0.314159E+01	-.775819E+01	0.120405E+01
0.400000E+01	0.392699E+01	-.926019E+01	-.891255E+00
0.400000E+01	0.471239E+01	-.802686E+01	0.122426E+01
0.400000E+01	0.549779E+01	-.935206E+01	0.709818E+00
0.500000E+01	0.000000E+00	-.543563E+01	-.108129E+01
0.500000E+01	0.392699E+00	-.410409E+01	-.166469E+00
0.500000E+01	0.785398E+00	-.444949E+01	0.646134E+00
0.500000E+01	0.117810E+01	-.534952E+01	0.111597E+00
0.500000E+01	0.157080E+01	-.546902E+01	-.123873E+01
0.500000E+01	0.196350E+01	-.479331E+01	-.150414E+01
0.500000E+01	0.235619E+01	-.454684E+01	-.186395E+01
0.500000E+01	0.274889E+01	-.584059E+01	0.495302E+00
0.500000E+01	0.314159E+01	-.773218E+01	0.993294E+00
0.500000E+01	0.353429E+01	-.884937E+01	-.587048E+00
0.500000E+01	0.392699E+01	-.873632E+01	-.741492E+00
0.500000E+01	0.431969E+01	-.790588E+01	-.299312E+00
0.500000E+01	0.471239E+01	-.770479E+01	0.897948E+00
0.500000E+01	0.510509E+01	-.842575E+01	0.158522E+01
0.500000E+01	0.549779E+01	-.879588E+01	0.381026E+00
0.500000E+01	0.589049E+01	-.761990E+01	-.753673E+00
0.600000E+01	0.000000E+00	-.567004E+01	-.904570E+00
0.600000E+01	0.392699E+00	-.452250E+01	-.107591E+00
0.600000E+01	0.785398E+00	-.479626E+01	0.544578E+00
0.600000E+01	0.117810E+01	-.555758E+01	0.107794E+00
0.600000E+01	0.157080E+01	-.566357E+01	-.100615E+01
0.600000E+01	0.196350E+01	-.508438E+01	-.127090E+01
0.600000E+01	0.235619E+01	-.486118E+01	-.157163E+01
0.600000E+01	0.274889E+01	-.578349E+01	0.516423E+00
0.600000E+01	0.314159E+01	-.740688E+01	0.842628E+00
0.600000E+01	0.353429E+01	-.849621E+01	-.535203E+00
0.600000E+01	0.392699E+01	-.838171E+01	-.683526E+00
0.600000E+01	0.431969E+01	-.766822E+01	-.273928E+00
0.600000E+01	0.471239E+01	-.756529E+01	0.779929E+00
0.600000E+01	0.510509E+01	-.841192E+01	0.131516E+01
0.600000E+01	0.549779E+01	-.842432E+01	0.221198E+00
0.600000E+01	0.589049E+01	-.744486E+01	-.630904E+00
0.700000E+01	0.000000E+00	-.584967E+01	-.776471E+00
0.700000E+01	0.196350E+00	-.515463E+01	-.774525E+00
0.700000E+01	0.392699E+00	-.482079E+01	-.128584E+00
0.700000E+01	0.589049E+00	-.480789E+01	-.265079E+00
0.700000E+01	0.785398E+00	-.504684E+01	0.469336E+00
0.700000E+01	0.981748E+00	-.539721E+01	0.404601E+00
0.700000E+01	0.117810E+01	-.570455E+01	0.100567E+00
0.700000E+01	0.137445E+01	-.613165E+01	-.591978E+00
0.700000E+01	0.157080E+01	-.590944E+01	-.847428E+00
0.700000E+01	0.176715E+01	-.557794E+01	-.106427E+01
0.700000E+01	0.196350E+01	-.529432E+01	-.109895E+01
0.700000E+01	0.215984E+01	-.506961E+01	-.868780E+00
0.700000E+01	0.235619E+01	-.509305E+01	-.135175E+01
0.700000E+01	0.255254E+01	-.536542E+01	-.188432E+00
0.700000E+01	0.274889E+01	-.595305E+01	0.488866E+00
0.700000E+01	0.294524E+01	-.674352E+01	0.468729E+00
0.700000E+01	0.314159E+01	-.729831E+01	0.730584E+00
0.700000E+01	0.333794E+01	-.788750E+01	0.498971E+00
0.700000E+01	0.353429E+01	-.823707E+01	-.466291E+00
0.700000E+01	0.373064E+01	-.830368E+01	-.545445E+00
0.700000E+01	0.392699E+01	-.812656E+01	-.634943E+00
0.700000E+01	0.412334E+01	-.781301E+01	-.559763E+00
0.700000E+01	0.431969E+01	-.750374E+01	-.249333E+00
0.700000E+01	0.451604E+01	-.750994E+01	0.211327E+00
0.700000E+01	0.471239E+01	-.736551E+01	0.681804E+00
0.700000E+01	0.490874E+01	-.765394E+01	0.105100E+01
0.700000E+01	0.510509E+01	-.816546E+01	0.110900E+01
0.700000E+01	0.530144E+01	-.813176E+01	0.784536E+00
0.700000E+01	0.549779E+01	-.815876E+01	0.130701E+00
0.700000E+01	0.569414E+01	-.788767E+01	-.835109E+00
0.700000E+01	0.589049E+01	-.731972E+01	-.541488E+00
0.700000E+01	0.608684E+01	-.686165E+01	-.801676E+00
0.800000E+01	0.000000E+00	-.597961E+01	-.679674E+00
0.800000E+01	0.196350E+00	-.533759E+01	-.606369E+00
0.800000E+01	0.392699E+00	-.504394E+01	-.146096E+00
0.800000E+01	0.589049E+00	-.502959E+01	-.237890E+00
0.800000E+01	0.785398E+00	-.523610E+01	0.411762E+00
0.800000E+01	0.981748E+00	-.554275E+01	0.357885E+00
0.800000E+01	0.117810E+01	-.581415E+01	0.929619E-01
0.800000E+01	0.137445E+01	-.617339E+01	-.505884E+00
0.800000E+01	0.157080E+01	-.595187E+01	-.731364E+00
0.800000E+01	0.176715E+01	-.597682E+01	-.934310E+00
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0.800000E+01	0.333794E+01	-.773329E+01	0.438309E+00
0.800000E+01	0.353429E+01	-.803913E+01	-.407730E+00
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0.900000E+01	0.000000E+00	-.607551E+01	-.604089E+00
0.900000E+01	0.196350E+00	-.547911E+01	-.494788E+00
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0.900000E+01	0.176715E+01	-.603920E+01	-.832384E+00
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0.900000E+01	0.215984E+01	-.541226E+01	-.684705E+00
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0.900000E+01	0.314159E+01	-.714832E+01	0.576060E+00
0.900000E+01	0.333794E+01	-.761137E+01	0.390654E+00
0.900000E+01	0.353429E+01	-.788313E+01	-.358358E+00
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0.900000E+01	0.392699E+01	-.778462E+01	-.549037E+00
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0.900000E+01	0.608684E+01	-.671893E+01	-.622554E+00

VITA

Suzan Jacqueline (Jackie) Kearney was born in Hickory, North Carolina on June 21, 1966. She attended various schools in North Carolina and Kentucky before her family settled into Jefferson County, Tennessee. She graduated from Jefferson County High School in Dandridge, Tennessee in 1984. She enrolled in Carson-Newman College in Jefferson City, Tennesee in August 1985. She received a Bachelor's of Arts in Mathematics from Carson-Newman in May 1989. The following fall, she began graduate school at The University of Tennessee, Knoxville, and earned a Master's of Science in Mathematics in August 1991.

She has accepted a teaching position at Arizona Western College in Yuma, Arizona.