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# THE TODOROV MODEL DETERMINED DYNAMICALLY BY COMPARISON WITH THE WHEELER-FEYNMAN CLASSICAL RELATIVISTIC FIELD THEORY THROUGH ORDER $1/c^4$

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## ABSTRACT

The Todorov generalized mass-shell equation

$$(\vec{p}-\vec{V})^2c^2+(m_wc^2+S(\vec{r}))^2-(\epsilon_w-V(\vec{r}))^2=0$$

describes the interaction of two relativistic particles under mutual vector and scalar interaction. It is Lorentz invariant and becomes a simple Schrödinger-like quantum mechanical operator if the relative momentum $\vec{p}$ is replaced by quantum operator $-i\hbar\nabla$. However, the potentials are arbitrary and need to be determined by a comparison with a field theory. In this thesis, only the classical relativistic two body problem is dealt with. The Wheeler-Feynman classical relativistic field theory is used to determine the expressions for the potentials above $(S(\vec{r}), V(\vec{r}), \vec{V})$ through order $1/c^4$, extending earlier work by H.W. Crater and P.Van Alstine done through order $1/c^2$. Then this thesis also shows how the forms of the relativistic reduced mass $m_w$ and energy $\epsilon_w$ of the fictitious particle of relative motion used in the Todorov equation are determined dynamically by field theory. Thus, the Todorov model is determined dynamically with the correct potentials and relativistic reduced mass and energy through order $1/c^4$ and is fully relativistic.

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# CHAPTER 1

## Introduction

In the past few years, several authors have established several two-body models with relativistic kinematics [1][2]. In these models two particles interact through a system of world scalar interaction, which arises from the Lorentz scalar, and the vector interaction, which transforms the time-like components of the Lorentz 4-vector potential. Here it is interesting to investigate the Todorov model [3], one of the two-body models. In this thesis, the relativistically invariant Einstein mass-shell relation (i.e., $m^2c^4+\vec{p}^2c^2 = \epsilon^2$, where $m$ is the rest mass of particle, $\vec{p}$ is the momentum of the particle and $\epsilon$ is the energy of the particle) is used to describe two-body problems by generalizing the one-body mass shell relation to a two-body equivalent ( also relativistically invariant ). These generalized mass-shell forms are useful because their quantization leads to Schrödinger-like equations. This generalization leads to the Todorov model. This thesis will give the extension of the earlier work done by H.W. Crater and P.Van Alstine [4], who have expanded the Todorov model through order $1/c^2$ and used the Wheeler-Feynman classical field theory to determine the Todorov model dynamically through order $1/c^2$. The major contribution of this thesis is to determine the Todorov model through order $1/c^4$. In this thesis, the expansion of the Todorov model through order $1/c^4$ will be obtained in chapter two with several arbitrary quantities that will be determined by a field theory. Therefore, chapter three will give the field theory, the Wheeler-Feynman classical field theory, through order $1/c^4$. Chapter four gives a canonical transformation to connect the Todorov expression and the Wheeler-Feynman expression and gives the final Todorov model that has been determined dynamically.

In chapter two this generalization is done in two steps. First, the interactions between two particles are not included. That is a system composed of two free particles. The Einstein mass-shell relation is generalized (in the center of momentum

system (c.m.system) ($\vec{p}_1 + \vec{p}_2 = 0$)) to $m_w^2 c^4 + \vec{p}^2(w)c^2 = \epsilon_w^2$. Since the two particles are relativistic, the quantities $m_w$ and $\epsilon_w$ are assumed to be the relativistic reduced mass and energy of the fictitious particles of relative motion, respectively. They are the functions of the total energy $w$ in c.m. system ), the non-relativistic reduced mass and the total mass. They should have the correct non-relativistic limits. In general, they are dynamically arbitrary in the relativistic version and need to be determined by field theory. Although they are arbitrary, a heuristic construction of $m_w$ for two free particles can be given analogous to the nonrelativistic case. It will be shown in chapter four that this heuristic construction is consistent with that determined by classical field theory. In the two-body mass-shell relation, $\vec{p}^2$ is the square of the relative momentum of two particles (in the c.m. system). It is invariant too. Secondly, when interactions between two particles are considered, the generalized mass shell relation of two free particles will be generalized further by the minimal replacements $m_w \to m_w + S/c^2$, $\vec{p} \to \vec{p} - \vec{V}$, $\epsilon_w \to \epsilon_w - V$, where $S$ is an arbitrary Lorentz scalar field, $V$ and $\vec{V}$ are the arbitrary time-like and space-like components of the Lorentz 4-vector potential, respectively. The quantities $m_w$ and $\epsilon_w$ are still defined as above. Chapter two will give this generalization of the one-body Einstein mass-shell relation

$$(\vec{p} - \vec{V})^2 c^2 + (m_w c^2 + S(\vec{r}))^2 = (\epsilon_w - V(\vec{r}))^2$$

the Todorov generalized mass-shell equation. Chapter two will also give the expressions for the total c.m. energy of the Todorov model through order $1/c^4$. In the Todorov model the potentials $V$ and $S$ are not determined, but are arbitrary functions of $\vec{r}$ only. $\vec{V}$ can be expressed as $\vec{V} = -\vec{p}\Gamma$ and $\Gamma$ is a function of $\vec{r}$ only. The properties of $V, S$ and $\Gamma$ will be obtained from a classical field theory in chapter four.

Chapter three will present a classical relativistic field theory that will be used to examine the Todorov equation and to determine $\epsilon_w, m_w$ and $S, V, \vec{V}$ dynamically. This classical field theory was created by J.A.Wheeler and R.P.Feynman

around 1950 [5][6] to describe a many-particle system under the influence of mutual electromagnetic interactions. It gives a physically reasonable and experimentally satisfactory account of the classical mechanical behavior of a system of point charges in relativistic interaction with one another, and is free of the ambiguities associated with the idea of a particle acting upon itself. In the Wheeler- Feynman theory, the Fokker-Tetrode action is used. In order to determine the potentials in the Todorov model, The Fokker-Tetrode action is expanded also through order $1/c^4$. It will give specific dynamical content to the Todorov model. In chapter four the expression of the Wheeler-Feynman classical field theory is used to examine the Todorov model by connecting the Wheeler-Feynman theory and the Todorov equation by a canonical transformation involving terms through order $1/c^4$. From the results, it can be seen that, through order $1/c^2$ and with no scalar interaction, the Todorov expression recovers the well-known Darwin formula [7] as pointed out by H.W. Crater and P. Van Alstine [8]. They have perform the same canonical transformation through order $1/c^2$. The results of this thesis will also give relativistic corrections to the vector and scalar potentials of the Todorov model beyond the non-relativistic potentials but to order $1/c^4$, and show that the Todorov model with $S = S(\vec{r}), V = V(\vec{r})$ and $\Gamma = \Gamma(\vec{r})$ is dynamically equivalent to the description given by the classical relativistic field theory. In other words, the Todorov model is dynamically determined through order $1/c^4$. Most important, it is relativistically invariant.

Although the Todorov model and the Wheeler-Feynman classical field theory describe the same dynamical system, the Wheeler-Feynman Hamiltonian theory is semi-relativistic and is not Lorentz invariant. It can not be extended to the Schrödinger-like equation when quantized. The Todorov model, however, is full-relativistic and Lorentz invariant. When the relative momentum $\vec{p}$ is replaced by operator $-i\hbar\nabla$, it will be easily generalized to a quantum mechanical equation of the Schrödinger type. This thesis, however, treats only the classical relativistic line and leaves the quantization (correct through order $1/c^4$) for later work.

Because the perturbation calculations through order $1/c^4$ are so complex, MACSYMA is used throughout this work. MACSYMA stands for MAC's SYmbolic MAnipulation system and can be used for performing symbolic as well as numerical mathematical manipulations, such as algebra, calculus and vector analysis [9]. Appendices A,B,C,D give four programs to calculate the expression of the Todorov equation, the expression of the Lagrangian of the Wheeler-Feynman theory, the expression with the velocities and relative position dependence of the Hamiltonian obtained from the Lagrangian, and the expression with the canonical momentum and position dependence of the Hamiltonian respectively. The use of MACSYMA was crucial for the complex perturbation calculations done here.

# CHAPTER 2

## The Todorov Two-Body Model

The Todorov two-body model gives a covariant description of two relativistic particles under mutual vector and scalar interactions. The vector interaction is generated by the time and space like component of a Lorentz four-vector potential. The scalar interaction arises from a Lorentz scalar and appears as a modification of the invariant rest mass.

First, the simple case of a free particle is considered. For a free particle the relativistic Lagrangian is

$$L_{free} = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} = -mc^2/\gamma. \tag{2.1}$$

The relativistic momentum is

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\gamma\vec{v} \tag{2.2}$$

where $\gamma = 1/\sqrt{1-\frac{\vec{v}^2}{c^2}}$.

From the Euler-Lagrange equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0 \tag{2.3}$$

one obtains

$$\frac{d}{dt}(\gamma m\vec{v}) = 0. \tag{2.4}$$

It is obvious that

$$\vec{p} = const. \tag{2.5}$$

Then the free-particle Hamiltonian is

$$H = \vec{v}\cdot\vec{p} - L = \sqrt{\vec{p}^2c^2 + m^2c^4}. \tag{2.6}$$

Since $H$ is not an explicit function of time $t$, $H$ will be equal to total energy of system $E$ and is conserved.

Eq.(2.6) can be rewritten as

$$E^2 = \vec{p}^2c^2 + m^2c^4. \tag{2.7}$$

It can be modified by adding some fields to a free particle system so that the dynamics of a particle moving in the fields can be described. The most familiar case of such a potential, or field , is the one from which one can derive the electromagnetic Lorentz force on a charged particle. It is

$$U = e\phi - \frac{e}{c}\vec{v}\cdot\vec{A}. \tag{2.8}$$

The potential is correct relativistically and gives the relativistic Lagrangian

$$L = -mc^2/\gamma - e\phi + \frac{e}{c}\vec{v}\cdot\vec{A} \tag{2.9}$$

which yields the three-vector canonical momentum

$$\vec{p} = m\gamma\vec{v} - \frac{e}{c}\vec{A}. \tag{2.10}$$

The equation of motion is

$$\frac{d}{dt}\vec{p} = -e\nabla\phi + \frac{e}{c}\vec{v}\cdot\nabla\vec{A}. \tag{2.11}$$

The Hamiltonian

$$H = c\sqrt{(\vec{p} - \frac{e}{c}\vec{A})^2 + m^2c^2} + e\phi \tag{2.12}$$

is a conserved quantity that is just the total energy $E$ if $\vec{A}$ and $\phi$ are time independent.

Then

$$(E - e\phi)^2 = (\vec{p} - \frac{e}{c}\vec{A})^2c^2 + m^2c^4. \tag{2.13}$$

For short one lets $V = e\phi$, $\vec{V} = \frac{e}{c}\vec{A}$. Then eq.(2.13) can be rewritten as

$$(E - V)^2 = (\vec{p} - \vec{V})^2c^2 + m^2c^4. \tag{2.13'}$$

Another important, although unfamiliar, interaction is the Lorentz scalar interaction [10]. It appears as a modification of the invariant rest mass. This modification is associated with a scalar field $S(x)$. Since $m$ and $c$ in the Lagrangian are Lorentz invariant, and $c$ is a universal constant, the only change is that $m$ is replaced by $M = m + \frac{S}{c^2}$. This interaction appears a force similar to the electromagnetic interaction except that there is no polarization. Thus the Lagrangian of the particle subjected simultaneously to two steady fields, a scalar $S(x)$ and a four-vector $(V, \vec{V})$, is

$$L = -(mc^2 + S(x))\gamma - V(x) + \vec{v} \cdot \vec{V} = -Mc^2\gamma - V(x) + \vec{v} \cdot \vec{V} \tag{2.14}$$

where $M = m + \frac{S}{c^2}$. Then the canonical momentum is

$$\vec{p} = M\vec{v}\gamma + \vec{V} \tag{2.15}$$

and by the Euler-Lagrange equations

$$\dot{\vec{p}} = \nabla L. \tag{2.16}$$

The Hamiltonian of the particle is

$$H = \vec{p} \cdot \vec{v} - L = \sqrt{(\vec{p} - \vec{V})^2 c^2 + M^2 c^4} + V(\vec{x}) \tag{2.17}$$

which is a constant of motion $E$ in case of time-independent potentials. Thus

$$(E - V)^2 = M^2 c^4 + (\vec{p} - \vec{V})^2 c^2. \tag{2.18}$$

From those equations it can be seen that if a particle is affected by three fields $S(\vec{x})$ ,$V(\vec{x})$ and $\vec{V}(\vec{x})$, Eq. (2.7) needs to be modified to obtain (2.18). The following correspondences are used to include the vector and scalar interactions

$$\begin{aligned} \vec{p} &\longrightarrow \vec{p} - \vec{V} \\ m &\longrightarrow m + \frac{S}{c^2} \end{aligned} \tag{2.19}$$

$$E \longrightarrow E - V$$

where $V$ is time like component of Lorentz 4-vector potential. The modified equation (2.7) can then be rewritten as

$$(E-V)^2 = (\vec{p}-\vec{V})^2c^2 + (m + \frac{S}{c^2})^2c^4. \tag{2.20}$$

This is just for the one particle case. Next the case of two relativistic particles will be considered.

If the potential between two particles is absent, the total energy $E$ of the system is simply $E_1 + E_2$.

$$E = \sqrt{\vec{p}_1^2c^2 + m_1^2c^4} + \sqrt{\vec{p}_2^2c^2 + m_2^2c^4} \tag{2.21}$$

where $m_1$ and $m_2$ are the particle masses. In center of momentum frame $\vec{p}_1 = -\vec{p}_2 = \vec{p}$

and Eq.(2.21) becomes

$$E = \sqrt{\vec{p}^2c^2 + m_1^2c^4} + \sqrt{\vec{p}^2c^2 + m_2^2c^4} \equiv w \tag{2.22}$$

The quantity $w$ is the total energy in c.m. system.

In the c.m. system $\vec{p}_1 = -\vec{p}_2 = \vec{p}$, and the total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2$ is zero, $\vec{P} = 0$. The square of the four-vector momentum $P_\mu = (iw/c, \vec{P}) = (iw/c, \vec{0})$ leads to

$$P_\mu P_\mu = -\frac{w^2}{c^2} + \vec{P}^2 = -w^2.$$

Since the square of the four-vector momentum is invariant in all Lorentz systems, $w$ must be Lorentz invariant as well.

Using Eq.(2.22) the three-vector momentum $\vec{p}^2$ can be expressed in terms of $w$

$$\vec{p}^2c^2 = \frac{1}{4w^2}(w^4 - 2(m_1^2 + m_2^2)w^2c^4 + (m_1^2 - m_2^2)^2c^8) \equiv b^2(w) \tag{2.23}$$

For the free particle case $\vec{p}^2c^2 \equiv b^2(w)$ where $b^2(w)$ is the invariant defined above.

As it is known, any relativistic dynamics should reduce to nonrelativistic dynamics when the speed of light goes to infinite. Eq.(2.22) thus becomes

$$\begin{aligned} w &= m_1c^2(1+\frac{\vec{p}^2}{m_1^2c^2})^{\frac{1}{2}} + m_2c^2(1+\frac{\vec{p}^2}{m_2^2c^2})^{\frac{1}{2}} \\ &= m_1c^2(1+\frac{\vec{p}^2}{2m_1^2c^2} - \frac{\vec{p}^4}{8m_1^4c^4} + ...) + m_2c^2(1+\frac{\vec{p}^2}{2m_2^2c^2} - \frac{\vec{p}^4}{8m_2^4c^4} + ...) \\ &\overset{c\to\infty}{=} (m_1+m_2)c^2 + \frac{\vec{p}^2}{2\mu} \end{aligned} \tag{2.24}$$

where $\mu = m_1m_2/(m_1+m_2)$. It is what one expects. The nonrelativistic limit is the total rest energy plus relative kinetic energy without potential energy because of no interaction between the two particles.

For the case of only one particle $\vec{p}^2c^2 = \epsilon^2 - m^2c^4$. In analogy to this for the case of two free particles, $\epsilon_w$ and $m_w$ are defined so that in the c.m. system

$$\vec{p}^2c^2 = \epsilon_w^2 - m_w^2c^4 = b^2(w) \tag{2.25}$$

where $b^2(w)$ is defined above. $\epsilon_w$ and $m_w$ are the relativistic energy and reduced mass of the fictitious particle of relative motion. $\epsilon_w$ and $m_w$ are not unique but must have the correct single particle limit, when either $m_1$ or $m_2$ goes to infinite as , e.g.,

$$m_2 \to \infty,\ w \to m_2c^2 + \epsilon_1,\ \epsilon_w \to \epsilon_1,\ m_w \to m_1$$

where $\epsilon_1$ is the c.m. energy of particle 1.

Since two free particles are concerned here, there are no interactions between them, $\epsilon_w$ in the nonrelativistic limit is

$$\epsilon_w^{N.R} = \mu c^2 + \frac{\vec{p}^2}{2m_1} + \frac{\vec{p}^2}{2m_2} = \mu c^2 + \frac{\vec{p}^2}{2\mu}$$

provided that the total energy in c.m. system is

$$w = (m_1+m_2)c^2 + \frac{\vec{p}^2}{2\mu}.$$

$m_w$ becomes

$$m_w^{N.R} = \mu.$$

When the mass of one of the particles becomes infinite, the system becomes a one particle system.

$$m_2 \to \infty,\ w \to m_2c^2 + \epsilon_1,\ \epsilon_w \to \epsilon_1,\ m_w \to m_1$$

where $\epsilon_1$ should be $m_1c^2 + \frac{\vec{p}^2}{2m_1}$. When

$$m_1 \to \infty,\ w \to m_2c^2 + \epsilon_1,\ \epsilon_w \to \epsilon_1,\ m_w \to m_2$$

where $\epsilon_2$ should be $m_2c^2 + \frac{\vec{p}^2}{2m_2}$.

One of the major aims of the thesis is to determine these two quantities $\epsilon_w$ and $m_w$ if one includes interactions.

Now a heuristic construction of $m_w$ for two *free* particles is given as follows. For free particles

$$p_1 = m_1\frac{dx_1}{d\tau_1},\quad p_2 = m_2\frac{dx_2}{d\tau_2} \tag{2.26}$$

where $d\tau_i,\ i = 1,2$ is proper time increment of each particle. Let $d\tau$ be the proper time increment of the c.m. frame, so that

$$\frac{d\tau}{d\tau_1} = \frac{\epsilon_1}{m_1c^2},\quad \frac{d\tau}{d\tau_2} = \frac{\epsilon_2}{m_2c^2} \tag{2.27}$$

where $\epsilon_1$ and $\epsilon_2$ are c.m energies of particle 1 and 2, respectively.

The nonrelativistic relative momentum is

$$\vec{p} = \frac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1 + m_2} = \mu(\vec{v}_1 - \vec{v}_2). \tag{2.28}$$

Then in analogy to this one defines the relativistic relative momentum $p$ and uses (2.26) and (2.27) to obtain

$$p \equiv \frac{\epsilon_2p_1 - \epsilon_1p_2}{w} = \frac{\epsilon_1\epsilon_2}{wc^2}\frac{d(x_1 - x_2)}{d\tau}. \tag{2.29}$$

Now to determine $m_w$, let $d\tau_e$ be a proper time increment of the effective particle of relative motion defined so that

$$d\tau_e \overset{m_2\to\infty}{\longrightarrow} d\tau_1,\quad d\tau_e \overset{m_1\to\infty}{\longrightarrow} d\tau_2.$$

Use of (2.27) shows that $d\tau/d\tau_e = (\epsilon_1\epsilon_2)/(m_1m_2c^4)$ has this proper limiting form for either static limit. Then in analogy with the right hand side of nonrelativistic case one takes

$$p = m_w \frac{d(x_1 - x_2)}{d\tau_e} \tag{2.30}$$

where $m_w$ is considered as the relativistic reduced mass. Form (2.29) and (2.30) the $m_w$ therefore is

$$m_w = \frac{m_1m_2c^2}{w}. \tag{2.31}$$

Since $b^2 = \epsilon_w^2 - m_w^2$, using (2.23) gives

$$\epsilon_w = \frac{w^2 - (m_1^2 + m_2^2)c^4}{2w}. \tag{2.32}$$

When one checks the nonrelativistic limits of $\epsilon_w$ and $m_w$, one uses (2.24) and obtains the required results $\epsilon_w = \mu c^2 + \frac{\vec{p}^2}{2\mu}, m_w = \mu$.

Now the one body equation (2.20) is generalized to an effective one body equation for two bodies written by analogy (in c.m. system) as

$$(\vec{p} - \vec{V})^2c^2 + (m_wc^2 + S)^2 - (\epsilon_w - \mathrm{V})^2 = 0. \tag{2.33}$$

This is the Todorov model. It is a generalization of the effective Einstein mass-energy relation

$$\vec{p}^2c^2 + m_w^2c^4 - \epsilon_w^2 \longrightarrow (\vec{p} - \vec{V})^2c^2 + (m_wc^2 + \mathrm{S})^2 - (\epsilon_w - \mathrm{V})^2 \tag{2.34}$$

with $m_w$ replaced by $m_w + \frac{S}{c^2}$, $\epsilon_w$ replaced by $\epsilon_w - \mathrm{V}$ and $\vec{p}$ replaced by $\vec{p} - \vec{V}$. Note that $S, V, \vec{V}$ are arbitrary except in so far as they are assumed to be functions of $\vec{r}$ only in the c.m. system. For two-body system $\vec{V}$ is due to the relative motion of the two particles. Note that since $\vec{p}$ is the relative momentum, $\vec{V} = -\vec{p}\Gamma(r)$.

The Todorov two-body model obtained here is analogous to Einstein's mass-energy relation modified for two particles. However, in its present form it lacks manifest Lorentz invariance. In order to show that (2.33) is in fact a Lorentz

invariant quantity expressed in the c.m. system, one has to reexpress it in term of 4-vector quantities.

Consider first of all the form when $S, V$ and $\vec{V}$ are zero. One defines a four vector

$$\mathcal{P}_\mu = \frac{\epsilon_w}{c}\hat{P}_\mu + p_\mu \tag{2.35}$$

where $\hat{P}_\mu = P_\mu/w$. Note that in the c.m. system, $p = (\vec{p}, 0)$, since from its definition in (2.29)

$$p_0|_{c.m.} = \frac{\epsilon_2\epsilon_1 - \epsilon_1\epsilon_2}{cw} = 0.$$

As a consequence of this

$$\hat{P}_\mu p_\mu = 0 \tag{2.36}$$

and therefore

$$c^2\mathcal{P}_\mu\mathcal{P}_\mu = -\epsilon_w^2 + p^2c^4. \tag{2.37}$$

Thus the free invariant expression for the Todorov form that reduces to the left side of (2.34) in the c.m. system is

$$\mathcal{P}^2c^2 + m_w^2c^4 = 0. \tag{2.38}$$

When interactions are included, the potential needs to be reexpressed in covariant form. To bring the argument into four vector form, one needs to find a four vector that has zero time component and space component equal to $\vec{r}$ in the c.m. system. For that covariant generalization, one chooses the form

$$x_{\perp\mu} = (\delta_{\mu\nu} + \hat{P}_\mu\hat{P}_\nu)(x_1 - x_2)_\nu \tag{2.39}$$

or

$$x_{\perp\mu} = (x_1 - x_2)_\mu + \hat{P}_\nu \cdot (x_1 - x_2)\hat{P}_\mu.$$

Its time component is zero in the c.m. system (where $\hat{P} = (i/c, 0)$), and its space component is $\vec{r}$. So, interactions are introduced in a Lorentz covariant way by way of the four vector "minimal" substitution

$$\mathcal{P}_\mu = \frac{\epsilon_w}{c}\hat{P}_\mu + p_\mu \to \mathcal{P}_\mu - \mathcal{V}_\mu$$

where $\mathcal{V}_\mu = V(x_\perp)\hat{P}_\mu + \Gamma(x_\perp)p_\mu$, and the mass replacement

$$m_w c^2 \to m_w c^2 + S(x_\perp)$$

The invariant Todorov interacting two-body generalization of the Einstein mass shell condition for one body $c^2 p_\mu p_\mu + m^2 c^4$ is thus

$$c^2(\mathcal{P}_\mu - \mathcal{V}_\mu(x_\perp))(\mathcal{P}_\mu - \mathcal{V}_\mu(x_\perp)) + (m_w c^2 + S(x_\perp))^2 = 0. \qquad (2.40)$$

In order to determine what $\mathcal{V}(x_\perp)$ and $S(x_\perp)$ are, one must develop a relation from this Todorov generalized mass shell expression (2.40) for the total energy in the c.m. system. One does this by first going back to the c.m. form (2.33). One first notes that the covariant form (2.40) can also be written as

$$p^2 c^2 + \Phi(x_\perp^2, p^2) = b^2(w), \qquad (2.41a)$$

where

$$\Phi = 2m_w c^2 S(x_\perp) + 2\epsilon_w V(x_\perp) + S^2(x_\perp) - V(x_\perp)^2 + \vec{p}^2 c^2(2\Gamma(x_\perp) + \Gamma(x_\perp)^2).$$

It is well known how such a generalized classical Schrödinger-like form can be derived from more general principles using Dirac constraint mechanics. The results are the same as given above [11].

In the c.m. system one finds

$$\vec{p}^2 c^2 + \Phi(\vec{r}^2, \vec{p}^2) = b^2(w) \qquad (2.41b)$$

where

$$\Phi = 2m_w c^2 S(\vec{r}) + 2\epsilon_w V(\vec{r}) + \mathrm{S}^2 - \mathrm{V}^2 + \vec{p}^2 c^2(2\Gamma(\vec{r}) + \Gamma^2(\vec{r})). \qquad (2.42)$$

(2.33) can be solved for $w$ using $b^2 = \epsilon_w^2 - m_w^2 c^4$ and (2.23). It leads to

$$w = \sqrt{\vec{p}^2 c^2 + m_1^2 c^4 + \Phi} + \sqrt{\vec{p}^2 c^2 + m_2^2 c^4 + \Phi} \qquad (2.43)$$

where $\Phi$ is given by (2.42).

Eq.(2.43) is used to obtain the total energy $w$ of system in c.m. frame. It is in general an implicit equation since $\Phi$ is also dependent on $w$. An iterative approximation method will be used here.

The relativistic potentials $V$ ,$S$ and $\Gamma$ are assumed to have the forms

$$V = a_0 + \frac{a_1}{c^2} + \frac{a_2}{c^4} + ..... \tag{2.44}$$

$$S = s_0 + \frac{s_1}{c^2} + \frac{s_2}{c^4} + ..... \tag{2.45}$$

$$\Gamma = \frac{\Gamma_1}{c^2} + \frac{\Gamma_2}{c^4} + ..... \tag{2.46}$$

where $a_i$ and $s_i$, $i = 0, 1, 2....$ and $\Gamma_1$, $\Gamma_2$ are independent of $c$. When relativistic dynamics tends to non-relativistic dynamics, $\Phi$ should reduce to

$$\Phi = 2\mu(a_0 + s_0). \tag{2.47}$$

The expansion of Eq.(2.42) leads to

$$\begin{aligned} w &= m_1c^2(1 + \frac{p^2c^2 + \Phi}{m_1^2c^4})^{\frac{1}{2}} + m_2c^2(1 + \frac{p^2c^2 + \Phi}{m_2^2c^4})^{\frac{1}{2}} \\ &= (m_1 + m_2)c^2 + \frac{p^2}{2\mu} + \frac{\Phi}{2\mu c^2} - \frac{p^4}{8c^2}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) \\ &- \frac{p^2\Phi}{4c^4}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) - \frac{\Phi^2}{8c^6}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) + ..... \end{aligned} \tag{2.48}$$

using $w(c^2) = (m_1 + m_2)c^2$ and collecting term through up to $c^0$, one reproduces

$$w = (m_1 + m_2)c^2 + \frac{p^2}{2\mu} + a_0 + s_0 \tag{2.49}$$

It is just non-relativistic limit of the total energy given in Eq.(2.44).

One continues to expand Eq.(2.48) to higher order $1/c^2$ . Now $\Phi$ will be calculated by using Eq.(2.49) and $\Phi$ will be expanded through order $c^0$. Before the expansion, a certain form of $m_w$ and $\epsilon_w$ are chosen. Note that above expression (2.31)-(2.32) for $m_w$ and $\epsilon_w$ were derived without reference to any interactions. It is a aim to see if the interactions will modify those expressions.

One assumes that

$$m_w = \sum_{n=0}^{\infty} \alpha_n \Delta^n, \quad \epsilon_w = \sum_{n=0}^{\infty} \beta_n \Delta^n \tag{2.50}$$

where $\alpha_0 = \beta_0 = \mu$ and $\alpha_n$, $\beta_n$ are function of $(m, \mu)$ alone and $m = m_1 + m_2, \mu = m_1 m_2/(m_1 + m_2)$. and

$$\Delta = \frac{w - mc^2}{mc^2}$$

The Einstein condition $\epsilon_w^2 - m_w^2 c^4 = b^2(w)$ leads to

$$\begin{aligned} \epsilon_w^2 - m_w^2 c^4 &= \sum_{n=1}^{\infty} \sum_{k=0}^{n} (\beta_k \beta_{n-k} - \alpha_k \alpha_{n-k}) \\ &= \frac{w^2}{4} - \frac{(m_1^2 + m_2^2)c^4}{2} + \frac{(m_1^2 - m_2^2)^2 c^8}{4w^2} \end{aligned} \tag{2.51}$$

Writing $w = mc^2(1 + \Delta)$ and comparing coefficients of $\Delta^n$ gives

$$2\mu(\beta_1 - \alpha_1) = 2m_1 m_2, \quad n = 1 \tag{2.52}$$

$$2\mu(\beta_2 - \alpha_2) + \beta_1^2 - \alpha_1^2 = \frac{1}{4}(m^2 + 3(m_1 - m_2)^2), \quad n = 2 \tag{2.53}$$

$$2\mu(\beta_n - \alpha_n) + \sum_{n=1}^{n-1} (\beta_k \beta_{n-k} - \alpha_k \alpha_{n-k}) = (-1)^n (m_1 - m_2)^2 (n+1), n = 3, 4... \tag{2.53}$$

Then combinating $\Phi$ with (2.43)(2.44)(2.45)(2.46) and (2.49)-(2.54) is given by

$$\begin{aligned} \Phi = {} & 2\mu(s_0 + a_0)c^2 + 2\mu(s_1 + a_1) + 2\vec{p}^2 \Gamma_1 + \frac{\alpha_1 p^2 s_0}{m\mu} + \frac{\beta_1 p^2 a_0}{\mu m} \\ & + \frac{2\beta_1 - m}{m} a_0^2 + \frac{2\alpha_1 + m}{m} s_0^2 + \frac{2(\alpha_1 + \beta_1)}{m} a_0 s_0 \end{aligned} \tag{2.55}$$

where $m = m_1 + m_2$.

In general

$$\Phi = \phi_0 c^2 + \phi_1 + \frac{\phi_2}{c^2} + ..... \tag{2.56}$$

When one needs the expansion for $w$ up to order $1/c^2$, $\Phi$ is required only through order $c^0$. Substitute Eq.(2.55) into Eq.(2.48), the total c.m. energy $w$ up to terms

of order $1/c^2$ becomes

$$
\begin{aligned}
w = & mc^2 + \frac{p^2}{2\mu} + a_0 + s_0 \\
& + \frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) \\
& + p^2 a_0(\frac{\beta_1}{2m\mu^2} - \frac{\mu}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) \\
& + p^2 s_0(\frac{\alpha_1}{2m\mu^2} - \frac{\mu}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) \\
& + \frac{\vec{p}^2\Gamma_1}{\mu} \\
& + a_0^2(-\frac{1}{2\mu} + \frac{\beta_1}{m\mu} - \frac{\mu^2}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) \\
& + s_0^2(\frac{1}{2\mu} + \frac{\alpha_1}{m\mu} - \frac{\mu^2}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) \\
& + a_0 s_0(\frac{(\beta_1 + \alpha_1)}{m\mu} - \mu^2(\frac{1}{m_1^3} + \frac{1}{m_2^3})) \\
& + a_1 + s_1]
\end{aligned}
\tag{2.57}
$$

Note that in the expansion (2.57) $a_0$ and $s_0$ do not appear symmetrically. because they transform in different ways. The vector potential $a_0$ is time like component of Lorentz 4-vector potential and the scalar interaction $s_0$ arises from Lorentz scalar. But in $c^0$ expansion they appear symmetrically. Nonrelativistically their transformation property is no consequence.

A MACSYMA program is given in Appendix A to expand Eq.(2.42) with Eqs.(2.56) (2.57) and $\Gamma$ to obtain order $1/c^4$ expression. The results of the program is

$$
\begin{aligned}
w = & mc^2 + \frac{p^2}{2\mu} + a_0 + s_0 \\
& + \frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) \\
& + p^2 a_0(\frac{\beta_1}{2m\mu^2} - \frac{\mu}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) \\
& + p^2 s_0(\frac{\alpha_1}{2m\mu^2} - \frac{\mu}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3}))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\vec{p}^2\Gamma_1}{\mu} \\
&+a_0^2(-\frac{1}{2\mu}+\frac{\beta_1}{m\mu}-\frac{\mu^2}{2}(\frac{1}{m_1^3}+\frac{1}{m_2^3})) \\
&+s_0^2(\frac{1}{2\mu}+\frac{\alpha_1}{m\mu}-\frac{\mu^2}{2}(\frac{1}{m_1^3}+\frac{1}{m_2^3})) \\
&+a_0s_0(\frac{(\beta_1+\alpha_1)}{m\mu}-\mu^2(\frac{1}{m_1^3}+\frac{1}{m_2^3}))+a_1+s_1] \\
&+\frac{1}{c^4}\{\frac{\vec{p}^6}{16}(\frac{1}{m_1^5}+\frac{1}{m_2^5}) \\
&+\vec{p}^4a_0(\frac{15}{8m^2\mu^2}-\frac{15}{8m\mu^3}+\frac{3}{8\mu^4}+\frac{9\beta_1}{8m^2\mu^3}-\frac{3\beta_1}{8m\mu^4}+\frac{\beta_2}{4\mu^3m^2}) \\
&+\vec{p}^4s_0(-\frac{15}{8m^2\mu^2}+\frac{15}{8m\mu^3}-\frac{9\alpha_1}{8m^2\mu^3}+\frac{3\alpha_1}{8m\mu^4}-\frac{3}{8\mu^4}-\frac{\alpha_2}{4m^2\mu^3}) \\
&+\vec{p}^4\Gamma_1(\frac{3}{2m\mu^2}-\frac{1}{2\mu^3}) \\
&+\vec{p}^2a_0^2(\frac{15}{4m^2\mu}-\frac{9}{2m\mu^2}+\frac{4\beta_1}{m^2\mu^2}-\frac{\beta_1}{m\mu^3}+\frac{1}{\mu^3}+\frac{\beta_2}{m^2\mu^2}) \\
&+\vec{p}^2s_0^2(\frac{15}{4m^2\mu}-\frac{3}{m\mu^2}+\frac{\alpha_2}{m^2\mu^2}+\frac{4\alpha_1}{m^2\mu^2}-\frac{3\alpha_1}{2m\mu^3}+\frac{1}{2\mu^3}) \\
&+\vec{p}^2s_0a_0(-\frac{15}{2m^2\mu}+\frac{15}{2m\mu^2}-\frac{\beta_2}{m^2\mu^2}-\frac{4\beta_1}{m^2\mu^2}+\frac{3\beta_1}{2m\mu^3} \\
&\qquad -\frac{\alpha_2}{m^2\mu^2}-\frac{4\alpha_1}{m^2\mu^2}+\frac{\alpha_1}{m\mu^3}-\frac{3}{2\mu^3}) \\
&+\vec{p}^2a_1(\frac{3}{2m\mu}+\frac{\beta_1}{2m\mu^2}-\frac{1}{2\mu^2}) \\
&+\vec{p}^2s_1(\frac{3}{2m\mu}+\frac{\alpha_1}{2m\mu^2}-\frac{1}{2\mu^2}) \\
&+\vec{p}^2\Gamma_1a_0(\frac{3}{m\mu}-\frac{1}{\mu^2}) \\
&+\vec{p}^2\Gamma_1s_0(-\frac{3}{m\mu}+\frac{1}{\mu^2}) \\
&+\vec{p}^2\Gamma_1^2\frac{1}{2\mu} \\
&+\vec{p}^2\Gamma_2\frac{1}{\mu} \\
&+a_0a_1(\frac{\beta_1}{m\mu}-\frac{2}{\mu}+\frac{3}{m})
\end{aligned}
$$

$$
\begin{aligned}
&+ s_0 a_1 (-\frac{\beta_1}{m\mu} + \frac{1}{\mu} - \frac{3}{m}) \\
&+ s_1 a_0 (\frac{\alpha_1}{m\mu} - \frac{1}{\mu} + \frac{3}{m}) \\
&+ s_0 s_1 (-\frac{\alpha_1}{m\mu} - \frac{3}{m}) \\
&+ a_0^2 s_0 (\frac{9}{m\mu} - \frac{2\beta_2}{m^2\mu} - \frac{7\beta_1}{m^2\mu} + \frac{2\beta_1}{m\mu^2} \\
&\quad - \frac{\alpha_2}{m^2\mu} - \frac{7\alpha_1}{2m^2\mu} + \frac{\alpha_1}{m\mu^2} - \frac{15}{2m^2} - \frac{2}{\mu^2}) \\
&+ a_0 s_0^2 (-\frac{6}{m\mu} + \frac{\beta_2}{m^2\mu} + \frac{7\beta_1}{2m^2\mu} - \frac{\beta_1}{m\mu^2} \\
&\quad \frac{\alpha_2}{m^2\mu} + \frac{7\alpha_1}{m^2\mu} + \frac{2\alpha_1}{m\mu^2} - \frac{15}{2m^2} + \frac{1}{\mu^2}) \\
&+ a_0^3 (-\frac{1}{4m\mu} + \frac{\beta_2}{m^2\mu} + \frac{7\beta_1}{2m^2\mu} - \frac{\beta_1}{m\mu^2} + \frac{1}{\mu^2} + \frac{5}{2m^2}) \\
&+ s_0^3 (-\frac{5}{2m^2} - \frac{\alpha_2}{m^2\mu} - \frac{7\alpha_1}{2m^2\mu} + \frac{\alpha_1}{m\mu^2} + \frac{1}{m\mu}) \\
&+ a_2 + s_2 \}
\end{aligned}
\tag{2.58}
$$

This is the total energy in c.m. system in the Todorov model . The aim now becomes two-fold. One needs to determine the non-relativistic potentials $a_0, s_0$ and the relativistic corrections $a_1, s_1, a_2, s_2$ as well as the vector potential $\Gamma$. At the same time one will determine $\alpha_1, \alpha_2, \beta_1, \beta_2$, building up piece by piece the relativistic mass and energy of the fictitious particles of relative motion. In order to do this, one needs to appeal to relativistic classical field theory for dynamical input.

# CHAPTER 3

## Relativistic Classical Field Theory

## For Two Relativistic Charged Particles System

It is the aim in this section to use relativistic classical field theory to derive an $1/c^2$ power expansion for the c.m. energy of two particles under mutual vector and scalar interactions.

One begins by considering the action principle. It states that the motion of a mechanical system is such that in going from a configuration A at time $t_1$ to a configuration B at time $t_2$, the action

$$A = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)dt \tag{3.1}$$

is an extremum. where $L = L(q_i(t), \dot{q}_i(t), t)$, $i = 1, 2....n$, is the Lagrangian of the system.

By considering small variations of coordinates and velocities away from the actual path and requiring $\delta A = 0$, one obtains the Euler-Lagrange equation of motion [12][20]

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0 \tag{3.2}$$

This can be extended to relativistic particle motion. To obtain a relativistic Lagrangian for a particle in external field, one considers the question of the Lorentz transformation properties of the Lagrangian. From the first postulate of special relativity the action integral must be Lorentz scalar because the equations of motion determined by the extremum condition, $\delta A$=0, must be Lorentz covariant. If the particle's proper time $\tau$ is introduced into (3.1), the action integral becomes

$$A = \int_{\tau_1}^{\tau_2} \gamma L d\tau \tag{3.3}$$

where $dt = \gamma d\tau$ and $\gamma = 1/\sqrt{1 - \frac{\vec{v}^2}{c^2}}$. Since proper time is invariant the condition that A also be invariant requires that $\gamma$L be Lorentz invariant.

The Lagrangian for a free particle can be a function of the velocity of the particle and its mass, but cannot depend on its position. The only Lorentz invariant function of the velocity available is $u_\alpha u_\alpha = -c^2$, where $u_\alpha = (i\gamma c, \gamma\vec{v})$ and $\vec{v}$ is three dimension velocity. Thus one concludes that the Lagrangian for a free particle is proportional to $\gamma^{-1}$.

So [12]

$$L_{free} = -mc^2\sqrt{1 - \frac{\vec{v}^2}{c^2}} \tag{3.4}$$

The equation of motion can be obtained by (2):

$$\frac{d}{dt}(\frac{\partial L}{\partial \vec{v}}) = 0. \tag{3.5}$$

This gives

$$\frac{d}{dt}(\frac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}) = \frac{d}{dt}(\gamma m\vec{v}) = 0. \tag{3.6}$$

Let $\vec{p} = \gamma m\vec{v}$ be the relativistic momentum of a particle of rest mass $m$ and velocity $\vec{v}$. For a free particle it is a constant.

The general requirement that $\gamma$L be Lorentz invariant allows one to determine the Lagrangian for a relativistic charged particle in external electromagnetic fields, provided one knows something about the Lagrangian for nonrelativistic motion in static fields. A slowly moving charged particle,(v$\ll$c),is influenced predominantly by the electric field that is derivable from the rotational scalar potential $\Phi$. The interaction potential is $V = e\Phi$. Since the nonrelativistic Lagrangian is $(T - V)$ , the interaction part $L_{int}$ of the relativistic Lagrangian must reduce in the nonrelativistic limit to

$$L_{int} \xrightarrow{v \ll c} L_{int}^{NR} = -e\Phi$$

Then one can construct the interaction term with $\gamma L$ invariant

$$L_{int} = -e\Phi + \frac{e}{c}\vec{v} \cdot \vec{A}$$

When v$\ll$ c, $L_{int} = -e\Phi$. Then the relativistic invariant interaction Lagrangian is

$$\gamma L_{int} = \frac{e}{c} u_\alpha A_\alpha \tag{3.7}$$

where $u_\alpha = (i\gamma c, \gamma\vec{v})$ and $A_\alpha = (i\Phi, \vec{A})$.

Thus, a complete Lagrangian for a charged particle in the external electromagnetic field is

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}. \tag{3.8}$$

The canonical momentum $\vec{P}$ conjugate to the position coordinate $\vec{q}$ is obtained by definition

$$P_i = \frac{\partial L}{\partial \dot{q}_i} = \gamma m\vec{v}_i + \frac{e}{c}A_i = p_i + \frac{e}{c}A_i.$$

Thus

$$\vec{P} = \vec{p} + \frac{e}{c}\vec{A}. \tag{3.9}$$

where $\vec{p} = \gamma m\vec{v}$ and $\vec{P}$ is the three dimension canonical momentum.

The equation of motion is given by (3.2)

$$\frac{d}{dt}\vec{P} = -e\nabla\phi + \frac{e}{c}\vec{v}\cdot\nabla\vec{A} \tag{3.10}$$

If a manifestly covariant description is used, the customary variables $\vec{q}$ and $\vec{v}$ are replaced by 4-vector $x_\alpha$ and $\dot{z}_\alpha$, where $\dot{z} = u_\alpha = dz/d\tau$, and relativistic Lagrangian is

$$\mathcal{L} = -mc\sqrt{-\dot{z}_\alpha\dot{z}_\alpha} + \frac{e}{c}\dot{z}_\alpha A_\alpha \tag{3.11}$$

Note that $\mathcal{L} = \gamma L$.

The equation of motion is given by

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}_\alpha}\right) - \frac{\partial\mathcal{L}}{\partial z_\alpha} = 0 \tag{3.12}$$

where $\dot{z} = \frac{dz}{d\tau}$. This leads to

$$m\frac{d^2x_\alpha}{d\tau^2} = \frac{e}{c}F_{\alpha\beta}\frac{dx_\beta}{d\tau} \tag{3.13}$$

where $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and

$$P_\alpha = \frac{\partial \mathcal{L}}{\partial(\frac{dx_\alpha}{d\tau})} = m\dot{z}_\alpha + \frac{e}{c}A_\alpha \tag{3.14}$$

So, for a charged particle in an external electromagnetic field the form of the Lagrangian suggests that the manifestly covariant form of the action integral is

$$A_{action} = \int_{\tau_1}^{\tau_2} [-mc\sqrt{-\dot{z}^2} + \frac{e}{c}\dot{z}_\alpha A_\alpha]d\tau \tag{3.15}$$

where $\dot{z}_\alpha \dot{z}_\alpha = \dot{z}^2$.

Now one has to introduce the Lagrangian for electromagnetic fields [12]. Here a Lagrangian description of the electromagnetic field in interaction with specified external source of charge and current is examined. The Lagrangian approach to continuous fields closely parallels the techniques used for discrete point particles. The finite number of coordinates $q_i(t)$ and $\dot{q}_i(t)$,i=1,2,..n, are replaced by an infinite number of degrees of freedom. The generalized coordinate $q_i$ is replaced by a continuous field $\phi_k(x)$ with a discrete index, k=1,2,...n, and a continuous index $(x^\alpha)$. The generalized velocity $\dot{q}_i$ is replaced by the 4-vector gradient, $\partial^\beta \phi_k$, The Euler-Lagrangian equations follow from the stationary property of the action integral with respect to variations $\delta\phi_k$ and $\delta(\partial^\beta\phi_k)$ around the physical values. one thus has the following correspondence [12][13]:

$$i \longrightarrow x_\alpha \;\; k$$

$$q_i \longrightarrow \phi_k(x)$$

$$\dot{q}_i \longrightarrow \partial_\alpha \phi_k(x)$$

$$L = \sum L_i(q_i, \dot{q}_i) \longrightarrow \int \mathcal{L}(\phi_k, \partial_\alpha \phi_k) d^3x \tag{3.16}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) = \frac{\partial L}{\partial q_i} \longrightarrow \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k} \tag{3.17}$$

where $k, i = 1, 2, 3.....n$. In formulating the variational principle in the continuous case, it has

$$\delta \int L dt = \delta \int dt \int dx^3 \mathcal{L}(\phi_k, \partial_\alpha \phi_k) = \delta \int \mathcal{L} d^4x \tag{3.18}$$

where $\mathcal{L}$ is Lagrangian density.

The electromagnetic field Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F_{\alpha\beta} - \frac{1}{c} J_\alpha A_\alpha \tag{3.19}$$

where $J_\alpha$ is the current density and $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and is Lorentz invariant and leads to covariant Maxwell equations. The action is

$$A = -\int [\frac{1}{16\pi} F_{\alpha\beta} F_{\alpha\beta} - \frac{1}{c} J_\alpha A_\alpha] d^4x \tag{3.19'}$$

The Lorentz invariant form $-F_{\alpha\beta}F_{\alpha\beta} = 2(\vec{B}^2 - \vec{E}^2)$ is seen as a kinetic energy minus a potential energy when $\vec{B}^2$ and $\vec{E}^2$ are related to $\vec{A}$ and $\phi$.

Then the equations of motion ( field equations ) of the electromagnetic field is obtained from Eq.(3.17) with $\phi_k$ replaced by $A_\mu$,

$$\partial_\alpha F_{\alpha\beta} = -\frac{4\pi}{c} J_\beta \tag{3.20}$$

The equations (3.20) are just the Maxwell equations. They are

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{3.21}$$

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \tag{3.22}$$

Eq.(3.20) is a covariant form of the inhomogeneous Maxwell equations (3.21)-(3.22). The other two equations of Maxwell are

$$\nabla \cdot \vec{B} = 0 \tag{3.23}$$

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \tag{3.24}$$

and are automatically satisfied because of the definition of the four-vector potential $A_\alpha$.

The conservation of the source current density can be obtained by taking the 4-divergence of both side of (3.20), thus

$$\partial_\beta J_\beta = 0 \tag{3.25}$$

Now one should turn to study the dynamics of a charged particle moving in its own electromagnetic field. It can be seen that the external current density $j^\alpha$ does not exist any more since the electromagnetic field is created by the moving charged particle, instead of an external source. In the meantime, however , the moving charged particle has an interaction with the electromagnetic field created by itself. Now $j^\alpha_{ext}$ is regarded as zero. The $j^\alpha_{int}$ is not. It must have some relation with the motion of the charged particle. When a charged particle moving in free region is investigated, classically, it has current density $\vec{J} = e\vec{v}$ and the Lagrangian is $\frac{1}{2}mv^2$. If the equation (3.11) is examined in the relativistic sense, it seems that the first term has a kinetic property and the second term represent the interaction between the current density $e\dot{z}_\mu$ and the vector potential $A^\mu$, which is being created by the charged particle but there is a time delay. So one can postulate the action for the case is

$$A_{action} = -mc\int\sqrt{-\dot{z}^2}d\tau + \frac{e}{c}\int\dot{z}_\mu A_\mu(z)d\tau - \frac{1}{16\pi c}\int F_{\mu\nu}(x)F_{\mu\nu}(x)d^4x \tag{3.26}$$

This action incorporates both the particle and field action (3.11)(3.19). In particular, note that in Eq.(3.26) the second term can be rewritten as

$$\begin{aligned}&\frac{e}{c}\int d^4x\int d\tau\dot{z}_\nu\delta^{(4)}(x-z(\tau))A_\mu(x)\\ &=\frac{1}{c}\int J_\mu(x)A_\mu(x)d^4x.\end{aligned} \tag{3.27}$$

Equation (3.27) indicates clearly that the current density $J_\mu$ is carried by a charged particle with the velocity $\dot{z}^\nu$ at $(\vec{r},t)$.

When the least action principle $\delta A = 0$ is applied, the equation of motion of the particle is given by (3.13) and the equation of field is obtained by Eq.(3.17) and leads to

$$\partial\mu F_{\mu\nu}(x) = -4\pi J_\nu(x) \tag{3.28}$$

where

$$J_\nu(x) = e\int d\tau \dot{z}_\nu \delta^{(4)}(x - z(\tau)) \tag{3.29}$$

and

$$\partial_\nu J_\nu = 0. \tag{3.30}$$

In the relativistic case, besides the vector (electromagnetic) interaction, there is a Lorentz scalar interactions also. Unlike the vector field, the scalar field has no polarization degree of freedom. If the particle is affected by a scalar field $S(\vec{x}, t)$, the Lagrangian will be changed by replacing $m$ by $M = m + \frac{S}{c^2}$. Then

$$L = -(mc^2 + gS(\vec{x}, t))\sqrt{1 - \frac{\vec{v}^2}{c^2}}. \tag{3.31}$$

The momentum is

$$\vec{P} = \frac{\partial L}{\partial \vec{v}} = (m + g\frac{S}{c^2})\vec{v}\gamma = M\vec{v}\gamma \tag{3.32}$$

The Lorentz invariant Lagrangian is

$$\mathcal{L} = -(m + g\frac{S}{c^2})c\sqrt{-\dot{z}^2}. \tag{3.33}$$

When there exists a scalar field, the Lagrangian density is

$$\mathcal{L} = -\frac{1}{8\pi c}\partial_\mu S(x)\partial_\mu S(x) + \frac{1}{c}J(x)S(x).$$

The action is

$$A_{action} = -\frac{1}{8\pi c}\int d^4x[\partial_\mu S(x)\partial_\mu S(x) + \frac{1}{c}J(x)S(x)] \tag{3.34}$$

The equation of field is given by (3.17)

$$\partial^2 S(x) = -4\pi J(x). \tag{3.35}$$

In completely analogy to the vector interactions, the action to the scalar interaction

$$A_{action} = -mc\int\sqrt{-\dot{z}^2}d\tau + \frac{g}{c}\int d\tau\sqrt{-\dot{z}^2}S(z) - \frac{1}{8\pi c}\int d^4x\partial_\mu S(x)\partial_\mu S(x) \quad (3.36)$$

where $S(x)$ is the scalar potential. and the equation of field is given by Eq.(3.17) , with $\phi_k$ replaced by $S(\vec{x},t)$,

$$\partial^2 S = -4\pi J \quad (3.37)$$

where now

$$J = g\int d\tau\sqrt{-\dot{z}^2}\delta^{(4)}(x - z(\tau)). \quad (3.38)$$

So the action of a complete system with vector and scalar interactions is

$$A_{action} = -mc\int\sqrt{-\dot{z}^2}d\tau + \frac{e}{c}\int d\tau\dot{z}_\mu A_\mu(z) + \frac{g}{c}\int d\tau\sqrt{-\dot{z}^2}S(z)$$

$$-\frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x)F_{\mu\nu}(x) - \frac{1}{8\pi c}\int d^4x\partial_\mu S(x)\partial_\mu S(x) \quad (3.39)$$

If a system is consist of two relativistic charged particles, the action will be [14]

$$A_{action} = -m_1 c\int\sqrt{-\dot{z_1}^2}d\tau_1 - m_2 c\int\sqrt{-\dot{z_2}^2}d\tau_2$$

$$+\frac{e_1}{c}\int d\tau_1\dot{z}_{1\mu}A_{1\mu}(z_1) + \frac{e_2}{c}\int d\tau_2\dot{z}_{2\mu}A_{2\mu}(z_2)$$

$$+\frac{g_1}{c}\int d\tau_1\sqrt{-\dot{z_1}^2}S_1(z_1) + \frac{g_2}{c}\int d\tau_2\sqrt{-\dot{z_2}^2}S_2(z_2)$$

$$-\frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x)F_{\mu\nu}(x) - \frac{1}{8\pi c}\int d^4x\partial_\mu S(x)\partial_\mu S(x) \quad (3.40)$$

Here $F^{\mu\nu}(x), S(x)$ is related to the total field due to the two moving charged particles themselves not an external sources. Particle and field variation ($\delta A = 0$) lead to

$$(m_i c - \frac{g_i}{c}S_i(z_i))\frac{d}{d\tau}(\frac{\dot{z}_{i\mu}}{\sqrt{-\dot{z_i}^2}}) = \frac{e_i}{c}\dot{z}_{i\nu}F_{\mu\nu}(z_i) \quad (3.41)$$

and

$$\partial\mu F_{\mu\nu}(x) = -4\pi J_\nu(x) \quad (3.42)$$

where

$$J^{\nu}(x) = e_1 \int d\tau_1 \dot{z_1}^{\nu} \delta^{(4)}(x - z_1(\tau_1)) + e_2 \int d\tau_2 \dot{z_2}^{\nu} \delta^{(4)}(x - z_2(\tau_2)) \tag{3.43}$$

and

$$\partial^2 S = -4\pi J \tag{3.44}$$

where

$$J = g_1 \int d\tau_1 \sqrt{-\dot{z_1}^2}\, \delta^{(4)}(x - z_1(\tau_1)) + g_2 \int d\tau_2 \sqrt{-\dot{z_2}^2}\, \delta^{(4)}(x - z_2(\tau_2)) \tag{3.45}$$

To relate the potentials of the previous chapter to the classical field theories presented in this chapter, one introduces the so-called Fokker- Tetrode action [14][15][16]. One gives a heuristic derivation of the Fokker-Tetrode action in which field variables are eliminated by using the above equations of motion and fields.

The terms

$$-\frac{1}{16\pi c} \int d^4x F_{\mu\nu}(x) F_{\mu\nu}(x) - \frac{1}{8\pi c} \int d^4x \partial_{\mu} S(x) \partial_{\mu} S(x)$$

can be written , after integration by parts and using Eq.(3.42)-(3.45) , as

$$-\frac{e_1}{2c} \int d\tau_1 \dot{z}_{1\mu} A_{\mu}(z_1) - \frac{e_2}{2c} \int d\tau_2 \dot{z}_{2\mu} A_{\mu}(z_2)$$

$$-\frac{g_1}{2c} \int d\tau_1 \sqrt{-\dot{z_1}^2}\, S(z_1) - \frac{g_2}{2c} \int d\tau_2 \sqrt{-\dot{z_2}^2}\, S(z_2)$$

Thus

$$\begin{aligned} A_{action} = & -m_1 c \int \sqrt{-\dot{z_1}^2}\, d\tau_1 - m_2 c \int \sqrt{-\dot{z_2}^2}\, d\tau_2 \\ & + \frac{e_1}{2c} \int d\tau_1 \dot{z}_{1\mu} A_{\mu}(z_1) + \frac{e_2}{2c} \int d\tau_2 \dot{z}_{2\mu} A_{\mu}(z_2) \\ & + \frac{g_1}{2c} \int d\tau_1 \sqrt{-\dot{z_1}^2}\, S(z_1) + \frac{g_2}{2c} \int d\tau_2 \sqrt{-\dot{z_2}^2}\, S(z_2). \end{aligned} \tag{3.46}$$

Now working in the Lorentz gauge $\partial_{\mu} A_{\mu} = 0$, the Maxwell equations (3.20) yields

$$\partial^2 A_{\mu}(x) = -4\pi J_{\mu}(x). \tag{3.47}$$

Thus, using the retarded solutions

$$A_\mu(x) = \int d^4y G_R(x-y) J_\mu(y). \tag{3.48}$$

Similarly

$$S(x) = \int d^4y G_R(x-y) J(y). \tag{3.49}$$

where in both case $\partial^2 G_R(x-y) = -4\pi\delta^4(x-y)$ and $G_R$ is the Green function. Thus

$$A_\nu(x) = e_1 \int d\tau_1 \dot{z_1}_\nu G_R(x-z_1) + e_2 \int d\tau_2 \dot{z_2}_\nu G_R(x-z_2) \tag{3.50}$$

and

$$S(x) = g_1 \int d\tau_1 \sqrt{-\dot{z_1}^2} G_R(x-z_1) + g_2 \int d\tau_2 \sqrt{-\dot{z_2}^2} G_R(x-z_2) \tag{3.51}$$

If one substitutes this into the above action and drop infinite self energy terms, the action is

$$\begin{aligned} A_{action} = & -m_1 c \int \sqrt{-\dot{z_1}^2} d\tau_1 - m_2 c \int \sqrt{-\dot{z_2}^2} d\tau_2 \\ & + \frac{e_1 e_2}{c} \int d\tau_1 d\tau_2 \dot{z_1}(\tau_1) \cdot \dot{z_2}(\tau_2) \bar{G}(z_1 - z_2) \\ & + \frac{g_1 g_2}{c} \int d\tau_1 d\tau_2 \sqrt{-\dot{z_1}^2} \sqrt{-\dot{z_2}^2} \bar{G}(z_1 - z_2) \end{aligned} \tag{3.52}$$

where $z_i = z(\tau_i)$ and

$$\bar{G}(z_1 - z_2) = \frac{1}{2}(G_R + G_A) = \frac{1}{2\pi}\delta((z_1 - z_2)^2) \equiv \bar{G}(z_2 - z_1) \tag{3.53}$$

This is the Fokker-Tetrode action with half retarded and half advanced fields [14][17]. It is just the form that Wheeler and Feynman used for their adjunct field formulation of electromagnetism. Note that this involves particle variables only and two proper times. The field have been eliminated.

The aim now is to obtain an expression from this action for the system Hamiltonian for comparison with the Todorov model expression. Unlike the action , this

will not be covariant but rather be developed in a power series expansion in the variable $1/c^2$. One shall accomplish this by relinquishing the manifest covariant forms in favor of a power series expression by changing from the $\tau_i$ to $t_i$ variables. The Green function then allows one to eliminate one of the $t_i$.

$$d\tau_i\sqrt{-\dot{z}_i^2} = dt_i\sqrt{c^2 - \vec{v_i}(t_i)^2} \tag{3.54}$$

$$d\tau_1 d\tau_2 \dot{z}_1\cdot\dot{z}_2 = dt_1 dt_2(\vec{v_1}(t_1)\cdot\vec{v_2}(t_2) - c^2) \tag{3.55}$$

$$\bar{G}(z_1 - z_2) = \frac{1}{4\pi c}\sum_{n=0}^{\infty}\frac{1}{(2n)!}(\frac{r_{12}}{c})^{2n-1}(-\partial_{t_1}\partial_{t_2})^n\delta(t_1 - t_2) \tag{3.56}$$

where $r_{12} = |\vec{r_1}(t_1) - \vec{r_2}(t_2)|$. One can eliminate one of the integrals by multiple integration by parts leaving with

$$A = \int L(\vec{r_i}, \frac{d\vec{r_i}}{dt}, \frac{d^2\vec{r_i}}{dt^2}, ...)dt \tag{3.57}$$

where the Lagrangian $L$ is given by [18]

$$L = -m_1c^2\sqrt{1 - \frac{\vec{v_1^2}(t)}{c^2}} - m_2c^2\sqrt{1 - \frac{\vec{v_2^2}(t)}{c^2}}$$

$$+\sum_{n=0}^{\infty}\frac{(-D_1D_2)^n}{(2n)!c^{2n}}\{[-e_1e_2(1 - \frac{\vec{v_1}(t)\cdot\vec{v_2}(t)}{c^2})$$

$$+g_1g_2\sqrt{1 - \frac{\vec{v_1^2}(t)}{c^2}}\sqrt{1 - \frac{\vec{v_2^2}(t)}{c^2}}]|\vec{r_1}(t) - \vec{r_2}(t)|^{2n-1}\} \tag{3.58}$$

The operators $D_1$ and $D_2$ signify time derivatives of only $\vec{r}_1$ and $\vec{r}_2$ respectively, e.g.

$$D_2r_{12} = \frac{d}{dt}|\vec{r}_1 - \vec{r}_2(t)| = -\frac{\dot{\vec{r}}_2\cdot(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|.}$$

The appearance of all orders of the derivatives at a given instant $t$ replaces the retarded and advanced field effects.

One expands (3.58) through up to $c^0$ terms. It gives

$$L = -m_1c^2(1 - \frac{\vec{v_1^2}}{2c^2} + .....) - m_2c^2(1 - \frac{\vec{v_2^2}}{2c^2} + .....) - \frac{e_1e_2}{r} + \frac{g_1g_2}{r} + ....$$

$$= -(m_1 + m_2)c^2 + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 - \frac{e_1e_2}{r} + \frac{g_1g_2}{r}. \tag{3.59}$$

It is the familiar nonrelativistic results. Since the Lagrangian is not an explicit function of the time, the total energy is conserved. The Hamiltonian of the system can be obtained by

$$H = \vec{p}\cdot\vec{v} - L \tag{3.60}$$

and $\vec{p} = \frac{\partial L}{\partial \vec{v}}$. In the order $c^0$ , the Lagrangian is given by (3.59) then

$$\vec{p}_1 = \frac{\partial L}{\partial \vec{v}_1} = m_1\vec{v}_1, \quad \vec{v}_1 = \frac{\vec{p}_1}{m_1} \tag{3.61}$$

$$\vec{p}_2 = \frac{\partial L}{\partial \vec{v}_2} = m_2\vec{v}_2, \quad \vec{v}_2 = \frac{\vec{p}_2}{m_2} \tag{3.62}$$

then the Hamiltonian is

$$H = \sum_{i=1}^{2} \vec{p}_i\cdot\vec{v}_i - L$$

$$= (m_1 + m_2)c^2 + \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{e_1e_2}{r} - \frac{g_1g_2}{r}. \tag{3.63}$$

It is the nonrelativistic Hamiltonian for two particles with vector and scalar interactions .

The Lagrangian through order $1/c^2$ for two particles interacting by the way of electromagnetic force only was derived first by Darwin [7].

$$L = -(m_1 + m_2)c^2 + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 - \frac{e_1e_2}{r}$$
$$+ \frac{1}{c^2}\{\frac{1}{8}m_1\vec{v}_1^4 + \frac{1}{8}m_2\vec{v}_2^4 + \frac{e_1e_2}{2r}(\vec{v}_1\cdot\vec{v}_2) + \frac{e_1e_2}{2r}(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})\} \tag{3.64}$$

It does not include acceleration terms. In order to verify $L(1/c^2)$, one expands Eq.(3.58) up to $(1/c^2)$ with the vector interaction only, which means $(g_1g_2) = 0$.

$$L = -m_1c^2(1 - \frac{\vec{v}_1^2}{2c^2} + \frac{3\vec{v}_1^4}{8c^4} + .....) - m_2c^2(1 - \frac{\vec{v}_2^2}{2c^2} + \frac{3\vec{v}_2^4}{8c^4} + .....)$$
$$- \frac{e_1e_2}{r} + \frac{e_1e_2\vec{v}_1\cdot\vec{v}_2}{rc^2} + \frac{e_1e_2}{2c^2}D_1D_2r \tag{3.65}$$

Performing $D_1D_2r$, then one gets Eq.(3.64).

When the scalar interaction is included, the expansion up to order $1/c^2$ is

$$\begin{aligned} L = &-(m_1+m_2)c^2 + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 - \frac{e_1e_2}{r} + \frac{g_1g_2}{r} \\ &+ \frac{1}{c^2}\{\frac{1}{8}m_1\vec{v}_1^4 + \frac{1}{8}m_2\vec{v}_2^4 + \frac{e_1e_2}{2r}(\vec{v}_1\cdot\vec{v}_2) + \frac{e_1e_2}{2r}(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}) \\ &- \frac{g_1g_2}{2r}(\vec{v}_1^2+\vec{v}_2^2) - \frac{g_1g_2}{2r}(\vec{v}_1\cdot\vec{v}_2) + \frac{g_1g_2}{2r}(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})\} \end{aligned} \tag{3.66}$$

Note that the vector interaction plays a different role from the scalar interaction. When one lets $g_1g_2 = 0$, (3.66) yields Darwin's result with the vector interaction only since there are not cross terms between the vector and the scalar interactions.

In the Lagrangian up through order $1/c^2$, the interaction part is a function of the velocity and the position but is independent of time, which means that the total energy in c.m. frame is conserved. Then the Hamiltonian of system can still be obtained by (3.60). The result, however, is that interaction is a function of the velocities as well as the positions.

$$\begin{aligned} H(\vec{v},\vec{r}) =&(m_1+m_2)c^2 + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 + \frac{e_1e_2}{r} - \frac{g_1g_2}{r} \\ &+ \frac{1}{c^2}[\frac{3}{8}m_1\vec{v}_1^4 + \frac{3}{8}m_2\vec{v}_2^4 \\ &+ \frac{(\vec{v}_1\cdot\vec{v}_2)e_1e_2}{2r} + \frac{e_1e_2(\hat{r}\cdot\vec{v}_1)(\hat{r}\cdot\vec{v}_2)}{2r} \\ &- \frac{g_1g_2}{2r}(\vec{v}_1^2+\vec{v}_2^2) - \frac{g_1g_2(\hat{r}\cdot\vec{v}_1)(\hat{r}\cdot\vec{v}_2)}{2r} + \frac{(\vec{v}_1\cdot\vec{v}_2)g_1g_2}{2r}. \end{aligned} \tag{3.67}$$

This is the Hamiltonian with $(\vec{v}_1,\vec{v}_2,\vec{r})$. It however is very difficult to find $H$, which should be characterized by the conjugate momentum $(\vec{p}_1,\vec{p}_2)$ and the coordinates $(\vec{r})$, because $\vec{v}_i, i = 1,2$ can not be represented by a simple function of $\vec{p}_i$ . An alternative way has to be used to find $H(\vec{p}_1,\vec{p}_2,\vec{r})$.

One assumes that $H$ can be written as

$$H(\vec{p},\vec{r}) = (m_1+m_2)c^2 + \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{e_1e_2}{r} - \frac{g_1g_2}{r} + \frac{1}{c^2}H_1 \tag{3.68}$$

By the dimensional analysis, $H_1$ should have terms like $\frac{\vec{p}^4}{m^3}$, $\frac{\vec{p}^2e_1e_2}{m^2r}$, $\frac{\vec{p}^2g_1g_2}{m^2r}$ , $\frac{(g_1g_2)^2}{mr^2}$, $\frac{(e_1e_2)^2}{mr^2}$ and $\frac{(e_1e_2)(g_1g_2)}{mr^2}$. So in general, $H_1$ have the form

$$H_1 = h_{10} + \epsilon h_{11} + \delta h_{12} + \epsilon^2 h_{13} + \delta^2 h_{14} + \epsilon\delta h_{15} \tag{3.69}$$

where $\epsilon = \frac{e_1 e_2}{r}$, $\delta = \frac{g_1 g_2}{r}$.

The equation of motion in the Poisson bracket formulation is

$$\vec{v} = \dot{\vec{r}} = [\vec{r}, H] \tag{3.70}$$

where $[A, B]$ is the Poisson bracket. When one applies the Poisson formulation on this two particles system, it leads to

$$\vec{v}_1 = \frac{\vec{p}_1}{m_1} + \frac{1}{c^2}(\frac{\partial h_{10}}{\partial \vec{p}_1} + \epsilon\frac{\partial h_{11}}{\partial \vec{p}_1} + \delta\frac{\partial h_{12}}{\partial \vec{p}_1} + \epsilon^2\frac{\partial h_{13}}{\partial \vec{p}_1} + \delta^2\frac{\partial h_{14}}{\partial \vec{p}_1} + \epsilon\delta\frac{\partial h_{15}}{\partial \vec{p}_1}) + O(\frac{1}{c^4}) \tag{3.71}$$

$$\vec{v}_2 = \frac{\vec{p}_2}{m_2} + \frac{1}{c^2}(\frac{\partial h_{10}}{\partial \vec{p}_2} + \epsilon\frac{\partial h_{11}}{\partial \vec{p}_2} + \delta\frac{\partial h_{12}}{\partial \vec{p}_2} + \epsilon^2\frac{\partial h_{13}}{\partial \vec{p}_2} + \delta^2\frac{\partial h_{14}}{\partial \vec{p}_2} + \epsilon\delta\frac{\partial h_{15}}{\partial \vec{p}_2}) + O(\frac{1}{c^4}) \tag{3.72}$$

Substituting them into (3.67) and collecting terms up to order $1/c^2$ , one has

$$\begin{aligned} H(\vec{p},\vec{r}) = {} & (m_1+m_2)c^2 + \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{e_1e_2}{r} - \frac{g_1g_2}{r} \\ & + \frac{1}{c^2}[\vec{p}_1\cdot\frac{\partial h_{10}}{\partial \vec{p}_1} + \epsilon\vec{p}_1\cdot\frac{\partial h_{11}}{\partial \vec{p}_1} + \delta\vec{p}_1\cdot\frac{\partial h_{12}}{\partial \vec{p}_1} \\ & + \epsilon^2\vec{p}_1\cdot\frac{\partial h_{13}}{\partial \vec{p}_1} + \delta^2\vec{p}_1\cdot\frac{\partial h_{14}}{\partial \vec{p}_1} + \epsilon\delta\vec{p}_1\cdot\frac{\partial h_{15}}{\partial \vec{p}_1} \\ & + \vec{p}_2\cdot\frac{\partial h_{10}}{\partial \vec{p}_2} + \epsilon\vec{p}_2\cdot\frac{\partial h_{11}}{\partial \vec{p}_2} + \delta\vec{p}_2\cdot\frac{\partial h_{12}}{\partial \vec{p}_2} \\ & + \epsilon^2\vec{p}_2\cdot\frac{\partial h_{13}}{\partial \vec{p}_2} + \delta^2\vec{p}_2\cdot\frac{\partial h_{14}}{\partial \vec{p}_2} + \epsilon\delta\vec{p}_2\cdot\frac{\partial h_{15}}{\partial \vec{p}_2} \\ & + \frac{3\vec{p}_1^4}{8m_1^3} + \frac{3\vec{p}_2^4}{8m_2^3} + \frac{e_1e_2\vec{p}_1\cdot\vec{p}_2}{2m_1m_2r} + \frac{e_1e_2(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2r} \\ & - \frac{g_1g_2\vec{p}_1\cdot\vec{p}_2}{2m_1m_2r} - \frac{g_1g_2(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2r} - \frac{g_1g_2}{2r}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2})] \end{aligned} \tag{3.73}$$

Now Eq.(3.73) is set to be equal to Eq.(3.69), since they are the same Hamiltonian but different forms. One selects coefficients of the same power of $\epsilon$ and $\delta$ , solves the equations and obtains $H = H(\vec{p}_1, \vec{p}_2, \vec{r})$,

The $\epsilon^0\delta^0$ terms:

$$\vec{p}_1\cdot\frac{\partial h_{10}}{\partial \vec{p}_1} + \vec{p}_2\cdot\frac{\partial h_{10}}{\partial \vec{p}_2} + \frac{3\vec{p}_1^4}{8m_1^3} + \frac{3\vec{p}_2^4}{8m_2^3} = h_{10}.$$

Solving it, it leads to

$$h_{10} = -\frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_2^4}{8m_2^3}. \tag{3.74}$$

The $\epsilon$ terms:

$$\vec{p}_1 \cdot \frac{\partial h_{11}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{11}}{\partial \vec{p}_2} + \frac{\vec{p}_1 \cdot \vec{p}_2}{2m_1 m_2 r} + \frac{(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1 m_2 r} = h_{11}.$$

The solution is

$$h_{11} = -\frac{\vec{p}_1 \cdot \vec{p}_2}{2m_1 m_2 r} - \frac{(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1 m_2 r}. \tag{3.75}$$

The $\delta$ terms:

$$\vec{p}_1 \cdot \frac{\partial h_{12}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{12}}{\partial \vec{p}_2}$$

$$-\frac{g_1 g_2 \vec{p}_1 \cdot \vec{p}_2}{2m_1 m_2 r} - \frac{g_1 g_2 (\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1 m_2 r} - \frac{g_1 g_2}{2r}\left(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}\right) = h_{12}.$$

The solution to this equation is

$$h_{12} = -\frac{g_1 g_2 \vec{p}_1 \cdot \vec{p}_2}{2m_1 m_2 r} + \frac{g_1 g_2 (\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1 m_2 r} + \frac{g_1 g_2}{2r}\left(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}\right). \tag{3.76}$$

The $\epsilon^2$ terms:

$$\vec{p}_1 \cdot \frac{\partial h_{13}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{13}}{\partial \vec{p}_2} = h_{13}.$$

The solution is

$$h_{13} = 0. \tag{3.77}$$

The $\delta^2$ terms:

$$\vec{p}_1 \cdot \frac{\partial h_{14}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{14}}{\partial \vec{p}_2} = h_{14}.$$

The solution is

$$h_{14} = 0. \tag{3.78}$$

The $\epsilon\delta$ terms:

$$\vec{p}_1 \cdot \frac{\partial h_{15}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{15}}{\partial \vec{p}_2} = h_{15}.$$

The solution is

$$h_{15} = 0. \tag{3.79}$$

So that $h_{13}, h_{14}, h_{15}$ are zero .

Finally, the Hamiltonian of two particles system through up to order $1/c^2$ is

$$\begin{aligned} H(\vec{p},\vec{r}) &= (m_1+m_2)c^2 + \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{e_1e_2}{r} - \frac{g_1g_2}{r} \\ &+ \frac{1}{c^2}[-\frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_1^4}{8m_2^3} - \frac{e_1e_2\vec{p}_1\cdot\vec{p}_2}{2m_1m_2r} - \frac{e_1e_2(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2r} \\ &- \frac{g_1g_2\vec{p}_1\cdot\vec{p}_2}{2m_1m_2r} + \frac{g_1g_2(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2r} + \frac{g_1g_2}{2r}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2})] \end{aligned} \tag{3.80}$$

One continues to expand (3.58) through order $1/c^4$ and consequently the Lagrangian includes acceleration terms to this order. Then the Lagrangian is [ see Appendix B]

$$\begin{aligned} L &= -(m_1+m_2)c^2 + \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 - \frac{e_1e_2}{r} + \frac{g_1g_2}{r} \\ &+ \frac{1}{c^2}\{\frac{1}{8}m_1{v_1}^4 + \frac{1}{8}m_2{v_2}^4 + \frac{e_1e_2}{2r}(\vec{v}_1\cdot\vec{v}_2) + \frac{e_1e_2}{2r}(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}) \\ &- \frac{g_1g_2}{2r}({v_1}^2+{v_2}^2) - \frac{g_1g_2}{2r}(\vec{v}_1\cdot\vec{v}_2) + \frac{g_1g_2}{2r}(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})\} \\ &+ \frac{1}{c^4}\{\frac{1}{16}m_1{v_1}^6 + \frac{1}{16}m_2{v_2}^6 \\ &- \frac{g_1g_2r}{8}[(\vec{a}_1\cdot\vec{a}_2) + (\vec{a}_1\cdot\hat{r})(\vec{a}_2\cdot\hat{r})] - \frac{e_1e_2r}{8}[(\vec{a}_1\cdot\vec{a}_2) + (\vec{a}_1\cdot\hat{r})(\vec{a}_2\cdot\hat{r})] \\ &+ \frac{g_1g_2}{8}(\vec{a}_1\cdot\hat{r})[(v_2^2 - (\vec{v}_2\cdot\hat{r})^2)] - \frac{g_1g_2}{8}(\vec{a}_2\cdot\hat{r})[(v_1^2 - (\vec{v}_1\cdot\hat{r})^2)] \\ &- \frac{e_1e_2}{8}(\vec{a}_1\cdot\hat{r})[(v_2^2 - (\vec{v}_2\cdot\hat{r})^2)] + \frac{e_1e_2}{8}(\vec{a}_2\cdot\hat{r})[(v_1^2 - (\vec{v}_1\cdot\hat{r})^2)] \\ &+ \frac{e_1e_2}{4}[(\vec{a}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r}) - (\vec{a}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\hat{r})] \\ &+ \frac{g_1g_2}{4}[(\vec{a}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r}) - (\vec{a}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\hat{r})] \\ &+ \frac{g_1g_2}{2}[(\vec{a}_2\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r}) - (\vec{a}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r})] \\ &- \frac{g_1g_2}{8r}(v_1^4+v_2^4) - \frac{g_1g_2v_2^2}{4r}(\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) \\ &- \frac{g_1g_2v_1^2}{4r}(\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) + \frac{3g_1g_2v_1^2v_2^2}{8r} - \frac{e_1e_2v_1^2v_2^2}{8r} \\ &- \frac{g_1g_2}{8r}(v_2^2(\vec{v}_1\cdot\hat{r})^2 + v_1^2(\vec{v}_2\cdot\hat{r})^2) + \frac{e_1e_2}{8r}(v_2^2(\vec{v}_1\cdot\hat{r})^2 + v_1^2(\vec{v}_2\cdot\hat{r})^2) \\ &- \frac{g_1g_2}{2r}(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})(\vec{v}_1\cdot\vec{v}_2) + \frac{(g_1g_2)}{8r}(2(\vec{v}_1\cdot\vec{v}_2)^2 - 3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2) \\ &+ \frac{(e_1e_2)}{8r}(2(\vec{v}_1\cdot\vec{v}_2)^2 - 3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2)\} \end{aligned} \tag{3.81}$$

The Lagrangian is the function of velocities $\vec{v}_i$ and accelerations $\vec{a}_i, i = 1, 2$ and relative position $\vec{r}$. It is not an explicit function of time.

Because the Lagrangian has acceleration terms. The equation of motion will be changed. Consider the following action

$$A = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), \ddot{q}_i(t), t)dt \tag{3.82}$$

When one applies $\delta A = 0$, the Euler-Lagrangian equation of motion is

$$\frac{d^2}{dt^2}(\frac{\partial L}{\partial \ddot{q}_i}) - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i} = 0 \tag{3.83}$$

There is one more term than Eq.(3.2), $\frac{d^2}{dt^2}(\frac{\partial L}{\partial \ddot{q}_i})$, which is related to acceleration.

When $L$ is not an explicit function of time $t$, this means that the Hamiltonian is constant and the energy of system is conserved, $H$ can be obtained by [19]

$$H = \vec{v} \cdot \frac{\partial L}{\partial \vec{v}} + \vec{a} \cdot \frac{\partial L}{\partial \vec{a}} - \vec{v} \cdot \frac{d}{dt}\frac{\partial L}{\partial \vec{a}} - L. \tag{3.84}$$

In order to verify it, one assumes that $L = L(\ddot{q}_i(t), \dot{q}_i(t), q_i(t), t)$

$$\frac{dL}{dt} = \frac{\partial L}{\partial \ddot{q}_i}\frac{d\ddot{q}_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial t}.$$

Using equation of motion Eq.(3.83), one has

$$\frac{\partial L}{\partial t} - \frac{d}{dt}(L + \dot{q}_i\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_i} - \ddot{q}_i\frac{\partial L}{\partial \ddot{q}_i} - \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}) = 0 \tag{3.85}$$

Given $\frac{\partial L}{\partial t} = 0$,

$$L + \dot{q}_i\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_i} - \ddot{q}_i\frac{\partial L}{\partial \ddot{q}_i} - \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} = -const.$$

The constant one identifies with the Hamiltonian.

For two particles system Hamiltonian will be

$$H = \sum_{i=1}^{2}(\vec{v}_i \cdot \frac{\partial L}{\partial \vec{v}_i} + \vec{a}_i \cdot \frac{\partial L}{\partial \vec{a}_i} - \vec{v}_i \cdot \frac{d}{dt_i}\frac{\partial L}{\partial \vec{a}_i}) - L. \tag{3.86}$$

The *MACSYMA* system is used to obtain the Hamiltonian [ Appendix C ]. The result of equation (3.86) is a Hamiltonian of function of the relative coordinate

$(\vec{r})$, velocities $(\vec{v}_1, \vec{v}_2)$, accelerations $(\vec{a}_1, \vec{a}_2)$ and jerks $(\vec{j}_1, \vec{j}_2)$. The jerk is defined by $\vec{j} = \frac{d^3\vec{x}}{dt^3}$. The program gives

$$
\begin{aligned}
H(\vec{j},\vec{a},\vec{v},\vec{r}) &= (m_1+m_2)c^2 + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 + \frac{e_1e_2}{r} - \frac{g_1g_2}{r} \\
&+ \frac{1}{c^2}[\frac{3}{8}m_1\vec{v}_1^4 + \frac{3}{8}m_2\vec{v}_2^4 \\
&+ \frac{(\vec{v}_1\cdot\vec{v}_2)e_1e_2}{2r} + \frac{e_1e_2(\hat{r}\cdot\vec{v}_1)(\hat{r}\cdot\vec{v}_2)}{2r} \\
&- \frac{g_1g_2}{2r}(\vec{v}_1^2+\vec{v}_2^2) - \frac{g_1g_2(\hat{r}\cdot\vec{v}_1)(\hat{r}\cdot\vec{v}_2)}{2r} + \frac{(\vec{v}_1\cdot\vec{v}_2)g_1g_2}{2r} \\
&+ \frac{1}{c^4}\{\frac{5m_1\vec{v}_1^6}{16} + \frac{5m_2\vec{v}_2^6}{16} \\
&+ \frac{e_1e_2}{r}[\frac{\vec{v}_1\cdot\vec{v}_2}{8}(\vec{v}_2^2-(\vec{v}_2\cdot\hat{r})^2) + \frac{\vec{v}_1\cdot\vec{v}_2}{8}(\vec{v}_1^2-(\vec{v}_1\cdot\hat{r})^2) \\
&+ \frac{3(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})}{8}(\vec{v}_2^2-(\vec{v}_2\cdot\hat{r})^2) + \frac{3(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})}{8}(\vec{v}_1^2-(\vec{v}_1\cdot\hat{r})^2) \\
&- \frac{\vec{a}_1\cdot\vec{r}}{8}(\vec{v}_2^2-(\vec{v}_2\cdot\hat{r})^2) + \frac{\vec{a}_2\cdot\vec{r}}{8}(\vec{v}_1^2-(\vec{v}_1\cdot\hat{r})^2) \\
&+ \frac{\vec{v}_1\cdot\vec{v}_2}{8}(\vec{a}_1\cdot\vec{r}-\vec{a}_2\cdot\vec{r}) + \frac{3(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})}{8}(\vec{a}_1\cdot\vec{r}-\vec{a}_2\cdot\vec{r}) \\
&+ \frac{5r}{8}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r}) - (\vec{a}_2\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{r}{4}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r}) - (\vec{a}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\hat{r})) \\
&+ \frac{3r}{8}((\vec{a}_2\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r}) - (\vec{a}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r}) - \vec{r}^2(\vec{a}_1\cdot\vec{a}_2)) \\
&+ \frac{\vec{r}^2}{8}(3\vec{j}_1\cdot\vec{v}_2 + 3\vec{j}_2\cdot\vec{v}_1 - (\vec{j}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}) - (\vec{j}_2\cdot\hat{r}))(\vec{v}_1\cdot\hat{r}) \\
&- \frac{\vec{v}_1^2\vec{v}_2^2}{8} + \frac{(\vec{v}_1\cdot\vec{v}_2)^2}{4} + \frac{\vec{v}_1^2(\vec{v}_1\cdot\hat{r})^2}{8} + \frac{\vec{v}_2^2(\vec{v}_2\cdot\hat{r})^2}{8} \\
&- \frac{3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2}{8}] \\
&+ \frac{g_1g_2}{r}[-\frac{3}{8}(\vec{v}_1^4+\vec{v}_2^4) + \frac{3}{8}((\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})^3 + (\vec{v}_1\cdot\hat{r})^3(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{\vec{v}_1^2}{8}(\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) + \frac{\vec{v}_2^2}{8}(\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{3}{8}(\vec{v}_2^2(\vec{v}_1\cdot\hat{r})^2 + \vec{v}_1^2(\vec{v}_2\cdot\hat{r})^2 - (\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})^2 - (\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})^2) \\
&- \frac{3}{8}(\vec{v}_2^2(\vec{a}_1\cdot\vec{r}) - \vec{v}_1^2(\vec{a}_2\cdot\vec{r}))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{8}(\vec{a}_1\cdot\vec{r})(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))-\frac{3}{8}(\vec{a}_2\cdot\vec{r})(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))\\
&-\frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{v}_2\cdot\hat{r})^2-(\vec{a}_2\cdot\vec{r})(\vec{v}_1\cdot\hat{r})^2)\\
&+\frac{3r}{8}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_2\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r}))\\
&+\frac{r}{8}((\vec{a}_2\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r}))\\
&+\frac{r}{4}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})-(\vec{a}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\hat{r}))\\
&-\frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r})+\vec{r}^2(\vec{a}_1\cdot\vec{a}_2))\\
&+\frac{\vec{r}^2}{8}(\vec{j}_1\cdot\vec{v}_2+\vec{j}_2\cdot\vec{v}_1+(\vec{j}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})+(\vec{j}_2\cdot\hat{r}))(\vec{v}_1\cdot\hat{r})\\
&-\frac{\vec{v}_1^2\vec{v}_2^2}{8}+\frac{(\vec{v}_1\cdot\vec{v}_2)^2}{4}++\frac{3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2}{8}]\}. \qquad (3.87)
\end{aligned}
$$

The Hamiltonian is characterized by the conjugate momentum $(\vec{p}_1,\vec{p}_2)$ and coordinates $(\vec{r})$. Using the same technique one has used in order $1/c^2$ calculation, one assumes that in order $1/c^4$ the part of the Hamiltonian has the form

$$
\begin{aligned}
H(1/c^4)=(&h_{20}+\epsilon h_{21}+\delta h_{22}+\epsilon^2 h_{23}+\delta^2 h_{24}\\
&+\epsilon\delta h_{25}+\epsilon^2\delta h_{26}+\epsilon\delta^2 h_{27}+\epsilon^3 h_{28}+\delta^3 h_{29}) \qquad (3.88)
\end{aligned}
$$

where $h_{2i}, i=1,2,...9$ should be functions of $\frac{\vec{p}^6}{m^5}$, $\frac{\vec{p}^4}{m^4}$, $\frac{\vec{p}^2}{m^3}$ and $\frac{1}{m^2}$ . and $\epsilon=\frac{e_1e_2}{r}$, $\delta=\frac{g_1g_2}{r}$ are the same as mentioned before.

The Hamilton equations are

$$
\vec{v}_i=\dot{\vec{r}}_i=[\vec{r}_i,H]=\frac{\partial H}{\partial\vec{p}_i},\quad i=1,2 \qquad (3.89)
$$

$$
\dot{\vec{p}}_i=[\vec{p}_i,H]=-\frac{\partial H}{\partial\vec{r}_i},\quad i=1,2 \qquad (3.90)
$$

One computes $\vec{a}_i$ and $\vec{j}_i$, $i=1,2$, from the Poisson formulas.

$$
\vec{a}_i=\dot{\vec{v}}_i=[\vec{v}_i,H] \qquad (3.91)
$$

$$
\vec{j}_i=\dot{\vec{a}}_i=[\vec{a}_i,H]. \qquad (3.92)
$$

Because $\vec{a}_i, \vec{j}_i, i = 1,2$ appear only in $1/c^4$ terms and $H(c^2)$ terms are constants, the calculation of $\vec{a}_i, \vec{j}_i, i = 1,2$ needs only $c^0$ terms, which is given by parts of (3.63) in the order $c^0$,

$$\vec{a}_i = [\vec{v}_i, H(c^0)] = [\frac{\vec{p}_i}{m_i}, \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} - \frac{e_1 e_2}{r} + \frac{g_1 g_2}{r}] + O(\frac{1}{c^2}) \tag{3.93}$$

$$\vec{j}_i = [\vec{a}_i, H(c^0)] \tag{3.94}$$

By straightforward calculation, One has

$$\vec{a}_1 = \frac{(e_1 e_2 - g_1 g_2)}{m_1} \frac{\hat{r}}{r^2} \tag{3.95}$$

$$\vec{a}_2 = -\frac{(e_1 e_2 - g_1 g_2)}{m_2} \frac{\hat{r}}{r^2} \tag{3.96}$$

$$\vec{j}_1 = \frac{(e_1 e_2 - g_1 g_2)}{m_1 r^3}(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}) - \frac{3(e_1 e_2 - g_1 g_2)}{m_1 r^3}(\frac{(\vec{p}_1 \cdot \hat{r})\hat{r}}{m_1} - \frac{(\vec{p}_2 \cdot \hat{r})\hat{r}}{m_2}) \tag{3.97}$$

$$\vec{j}_2 = -\frac{(e_1 e_2 - g_1 g_2)}{m_2 r^3}(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}) + \frac{3(e_1 e_2 - g_1 g_2)}{m_2 r^3}(\frac{(\vec{p}_1 \cdot \hat{r})\hat{r}}{m_1} - \frac{(\vec{p}_2 \cdot \hat{r})\hat{r}}{m_2}) \tag{3.98}$$

At the order $\frac{1}{c^4}$, the velocities are given by Eq.(3.60)-(3.61),

$$\vec{v}_1 = \frac{\vec{p}_1}{m_1}, \vec{v}_2 = \frac{\vec{p}_2}{m_2} \tag{3.99}$$

One can repeat the process done for obtaining $H(1/c^2)$.

One substitutes Eqs.(3.95)-(3.99) into Eq.(3.87) of $1/c^4$ terms , set this equal to (3.88) and have nine differential equations. The appendix D will give the complete calculations of the Hamiltonian in order $1/c^4$ .

The final Hamiltonian with $(\vec{p}_1, \vec{p}_2, \vec{r})$ terms through order $1/c^4$ is

$$\begin{aligned} H = {} & (m_1 + m_2)c^2 + \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{(e_1 e_2)}{r} - \frac{(g_1 g_2)}{r} \\ & + \frac{1}{c^2}[-\frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_1^4}{8m_2^3} - \frac{(e_1 e_2)\vec{p}_1 \cdot \vec{p}_2}{2m_1 m_2 r} - \frac{(e_1 e_2)(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1 m_2 r} \\ & - \frac{(g_1 g_2)\vec{p}_1 \cdot \vec{p}_2}{2m_1 m_2 r} + \frac{(g_1 g_2)(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1 m_2 r} + \frac{(g_1 g_2)}{2r}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2})] \\ & + \frac{1}{c^4}\{\frac{\vec{p}_1^6}{16m_1^5} + \frac{\vec{p}_1^6}{16m_2^5} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(e_1e_2)}{r}[\frac{5(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2})+\frac{(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2})\\
&-\frac{\vec{p}_1^2(\vec{p}_2\cdot\hat{r})^2}{24m_1^2m_2^2}-\frac{\vec{p}_2^2(\vec{p}_1\cdot\hat{r})^2}{24m_1^2m_2^2}+\frac{\vec{p}_1^2\vec{p}_2^2}{24m_1^2m_2^2}-\frac{(\vec{p}_1\cdot\vec{p}_2)^2}{12m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}\\
&+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{8m_1m_2}(\frac{\vec{p}_1^2+(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{\vec{p}_2^2+(\vec{p}_2\cdot\hat{r})^2}{m_2^2})]\\
&+\frac{(e_1e_2)^2}{r^2}[\frac{5}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2})-\frac{7}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}\\
&+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2})-\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}(\frac{1}{m_1}+\frac{1}{m_2})+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}(\frac{1}{m_1}+\frac{1}{m_2}]\\
&+\frac{(g_1g_2)}{r}[-\frac{3}{4}(\frac{\vec{p}_1^4}{m_1^4}+\frac{\vec{p}_1^4}{m_2^4})+\frac{(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2})\\
&-\frac{(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2})-\frac{5(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2})\\
&-\frac{\vec{p}_1^2(\vec{p}_1\cdot\hat{r})^2+\vec{p}_2^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\vec{p}_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{6m_1^2m_2^2}\\
&+\frac{\vec{p}_1^2\vec{p}_2^2}{24m_1^2m_2^2}-\frac{(\vec{p}_1\cdot\vec{p}_2)^2}{12m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}+\frac{5(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2})]\\
&+\frac{(g_1g_2)^2}{r^2}[-\frac{3}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2})+\frac{5}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2})\\
&+(\frac{\vec{p}_1^4}{2m_1^3}+\frac{\vec{p}_2^4}{2m_2^3})-\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}(\frac{1}{m_1}+\frac{1}{m_2})]\\
&+\frac{(e_1e_2)(g_1g_2)}{r^2}[\frac{1}{4m_1m_2}(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2})+\frac{1}{4m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2})\\
&-\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}(\frac{1}{m_1}+\frac{1}{m_2})]\\
&-\frac{(g_1g_2)^2(e_1e_2)+(g_1g_2)(e_1e_2)^2}{4m_1m_2r^3}\\
&+\frac{(e_1e_2)^3+(g_1g_2)^3}{4m_1m_2r^3}\}. \qquad (3.100)
\end{aligned}
$$

If is is represented in c.m. system, using total mass and reduced mass, one has

$$\vec{p}_1=-\vec{p}_2=\vec{p}$$

$$m_1+m_2=m$$

$$\frac{m_1m_2}{m_1+m_2}=\mu$$

The equation has the form

$$
\begin{aligned}
H(\vec{p},\vec{r}) = {} & mc^2 + \frac{\vec{p}^2}{2\mu} + \frac{(e_1e_2)}{r} - \frac{(g_1g_2)}{r} \\
& + \frac{1}{c^2}[-\frac{p^4}{8}(\frac{1}{m_1^3}+\frac{1}{m_2^3}) + \frac{p^2(e_1e_2)}{2\mu r} + \frac{(\vec{p}\cdot\hat{r})^2(e_1e_2)}{2\mu r} \\
& + \frac{p^2(g_1g_2)}{2\mu r} + \frac{p^2(g_1g_2)}{2r}\frac{m-2\mu}{\mu^2 m} - \frac{(\vec{p}\cdot\hat{r})^2(g_1g_2)}{2\mu r}] \\
& + \frac{1}{c^4}\{\frac{p^6}{16}(\frac{1}{m_1^5}+\frac{1}{m_2^5}) \\
& + \frac{(e_1e_2)}{r}[-\frac{p^4(5m-9\mu)}{24\mu^3m^2} - \frac{p^2(\vec{p}\cdot\hat{r})^2(2m-3\mu)}{12\mu^3m^2} + \frac{(\vec{p}\cdot\hat{r})^4(3\mu-m)}{8\mu^3m^2}] \\
& + \frac{(e_1e_2)^2}{r^2}[\frac{9p^2}{8\mu^2m} - \frac{9(\vec{p}\cdot\hat{r})^2}{8\mu^2m}] \\
& - \frac{(g_1g_2)}{r}[\frac{p^4(9m^2-31\mu m+9\mu^2)}{24\mu^4m^2} + \frac{p^2(\vec{p}\cdot\hat{r})^2(m-3\mu)}{12\mu^3m^2} + \frac{(\vec{p}\cdot\hat{r})^4(3\mu-m)}{8\mu^3m^2}] \\
& + \frac{(g_1g_2)^2}{r^2}[\frac{p^2(4m-15\mu)}{8\mu^3m} + \frac{7(\vec{p}\cdot\hat{r})^2}{8\mu^2m}] \\
& + \frac{(g_1g_2)(e_1e_2)}{r^2}[\frac{3p^2}{4\mu^3m} + \frac{(\vec{p}\cdot\hat{r})^2}{4\mu^2m}] \\
& - \frac{(g_1g_2)^2(e_1e_2)+(e_1e_2)^2(g_1g_2)}{4\mu m r^3} + \frac{(e_1e_2)^3+(g_1g_2)^3}{4\mu m r^3} = w
\end{aligned}
\tag{3.101}
$$

This is the Hamiltonian of two relativistic charged particles in the c.m. system. One will call this the Wheeler-Feynman Hamiltonian. They have both scalar and vector charges. It will be used to determine the dynamical properties of the Todorov model.

# CHAPTER 4

## A Canonical Connection Between

## The Todorov Two-Body Model And The Wheeler-Feynman

## Results Of Two Relativistic Charged Particles System

As is known, if a canonical transformation connects two dynamical systems, the systems are equivalent [20]. Now one tries to find a canonical transformation between the Todorov two-body Model and two relativistic charged particles system that is described by Wheeler-Feynamn classical field theory.

One requires that both theories describe the same dynamics system. It is a common point. They do so, however, in a different fashion. The Todorov expression for $w$ involves a function of the relative momentum $\vec{p}$ and relative position $\vec{r}$ between two particles in c.m.system. The Wheeler-Feynman expression for $w$ has not only these variables but also a projection of $\vec{p}$ on $\vec{r}$. Based on the common point, if the Todorov description is correct, there should be a canonical transformation between them. By the Todorov description one means

$$(\vec{p} - \vec{V})^2 c^2 + (m_w c^2 + S)^2 = (\epsilon_w - V)^2$$

where

$$\vec{V} = -\vec{p}\Gamma(\vec{r})$$

$$S = S(\vec{r})$$

$$V = V(\vec{r})$$

in the c.m. system. If this description is correct, then a canonical transformation would allow one to determine the properties of potentials $a_0, a_1, a_2, s_0, s_1, s_2$ and $\Gamma$, and show that the potentials are a function of the relative position and do not involve the relative momentum $\vec{p}$. One will show that in order $1/c^2$ the Todorov model with $\Gamma = a_1 = s_1 = 0$ agrees dynamically with the Wheeler-Feynman

(WF) expression, but in order $1/c^4$, one needs $\Gamma$ and $a_2, s_2$ in that order. The main proposal however is that one can establish the above form by canonically transforming the WF results in the c.m. system.

The coordinates $Q(q,p)$ and the conjugate momenta $P(q,p)$ are canonical if they have properties

$$[Q,Q] = [P,P] = 0$$

$$[Q,P] = \delta_{ij} \tag{4.1}$$

where $[A,\ B]$ is Poisson bracket, is defined as

$$[A,B]_{p,q} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{4.2}$$

where $A, B$ are the function of $(p,q)$. From the definition (4.2), Eq.(4.1) can be obtained.

In chapter two , one found that the Todorov expression through order $c^0$ is

$$w = (m_1 + m_2)c^2 + \frac{\vec{p}^2}{2\mu} + a_0 + s_0 \tag{4.3}$$

In chapter three, the Wheeler-Feynman expression through order $c^0$ is

$$w = (m_1 + m_2)c^2 + \frac{\vec{p}^2}{2\mu} + \frac{e_1 e_2}{r} - \frac{g_1 g_2}{r} \tag{4.4}$$

It is easy to find a canonical transformation to connect these two descriptions. It is just

$$\vec{r}' = \vec{r}, \quad \vec{p}' = \vec{p}. \tag{4.5}$$

From matching them, one can determine $a_0$ and $s_0$.

$$a_0 = \frac{e_1 e_2}{r}, \qquad s_0 = -\frac{g_1 g_2}{r}. \tag{4.6}$$

Consider the general canonical transformation [21] through order $1/c^2$

$$\vec{r} \longrightarrow \vec{r}' = \vec{r}(1 + \frac{B_1}{mc^2}) + \hat{p}\cdot\vec{r}\hat{p}(\frac{E_1}{mc^2}) \tag{4.7}$$

$$\vec{p} \longrightarrow \vec{p}' = \vec{p}(1 + \frac{G_1}{mc^2}) + \hat{r} \cdot \vec{p}\hat{r}(\frac{K_1}{mc^2}) \tag{4.8}$$

where $B_1, G_1, E_1$ and $K_1$ have units of energy. One assumes that the terms $B_1, E_1, G_1, K_1$ are functions of the three independent scalars $x = \frac{\vec{r}^2}{2}, y = \vec{r} \cdot \vec{p}$, and $z = \frac{\vec{p}^2}{2}$.

First one considers order $1/c^2$ and assumes the absence of scalar interaction for simplicity. Eq.(4.1) applies to (4.7 )-(4.8) to yields the equations for $B_1, G_1, E_1$ and $K_1$ in order $1/c^2$ .

$$B_1 = -G_1 \tag{4.9}$$

$$(1 + y\frac{\partial}{\partial y})E_1 = -2z\frac{\partial}{\partial z}G_1 \tag{4.10}$$

$$(1 + y\frac{\partial}{\partial y})K_1 = 2x\frac{\partial}{\partial x}G_1 \tag{4.11}$$

Since $B_1, G_1, E_1$ and $K_1$ have units of energy, one lets

$$G_1 = -B_1 = \epsilon f_0 \tag{4.12}$$

$$E_1 = \epsilon f_1 \tag{4.13}$$

$$K_1 = \epsilon f_2 \tag{4.14}$$

where $f_0, f_1$ and $f_2$ are dimensionless functions independent of $\epsilon$ to this order , where $\epsilon = \frac{e_1 e_2}{r}$. The only dimensionless combination of the independent $x, y, z$ is $\frac{xz}{y^2}$ or $\frac{y^2}{xz} \equiv \rho$. Hence the partial differential equations (4.11)-(4.12) become

$$f_1 + 2\rho f_1' = 2\rho f_0' \tag{4.15}$$

$$f_2 + 2\rho f_2' = -f_0 - 2\rho f_0' \tag{4.16}$$

where prime means $\frac{\partial}{\partial \rho}$.

Now one performs the canonical transformation Eqs.(4.7)-(4.8) on the field theoretic expression Eq.(3.81) in order $1/c^2$ with no scalar interaction. This yields

$$\begin{aligned} w_{W.F.} = {} & mc^2 + \frac{\vec{p}^2}{2\mu} + \frac{(e_1 e_2)}{r} \\ & + \frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) + \frac{\vec{p}^2(e_1 e_2)}{2m\mu r}(1 + 2f_0 + \frac{\rho}{4} + \frac{\rho f_2}{2}) \\ & + \frac{(e_1 e_2)^2}{mr^2}(f_0 - \frac{\rho f_1}{4})] \end{aligned} \tag{4.17}$$

The Todorov expression Eq.(2.50) without the scalar interaction gives

$$w = mc^2 + \frac{p^2}{2\mu} + a_0 + \frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) + p^2 a_0(\frac{\beta_1}{2m\mu^2} - \frac{\mu}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) + \frac{\vec{p}^2\Gamma_1}{2\mu} + a_0^2(-\frac{1}{2\mu} + \frac{\beta_1}{m\mu} - \frac{\mu^2}{2}(\frac{1}{m_1^3} + \frac{1}{m_2^3})) + a_1] \tag{4.18}$$

where $a_0 = \frac{e_1 e_2}{r}$. Equating the two forms for the c.m. energy $w$ leads to

$$\frac{\vec{p}^2(e_1e_2)}{2m\mu r}(1 + 2f_0 + \frac{\rho}{4} + \frac{\rho f_2}{2} - \frac{\beta_1}{2\mu} + \frac{m - 3\mu}{2\mu}) + \frac{(e_1e_2)^2}{mr^2}(f_0 - \frac{\rho f_1}{4} + \frac{m}{2\mu} - \frac{\beta_1}{\mu} + \frac{m - 3\mu}{2\mu}) = 0 \tag{4.19}$$

Hence one has two relations that must be simultaneously satisfied:

$$1 + 2f_0 + \frac{\rho}{4} + \frac{\rho f_2}{2} - \frac{\beta_1}{\mu} + \frac{m - 3\mu}{\mu} = 0 \tag{4.20}$$

$$f_0 - \frac{\rho f_1}{4} + \frac{m}{2\mu} - \frac{\beta_1}{\mu} + \frac{m - 3\mu}{2\mu} = 0 \tag{4.21}$$

To compare this with (4.15)-(4.16) one differentiates (4.20)-(4.21)

$$2f_0' + \frac{1}{4} + \frac{\rho f_2'}{2} + \frac{f_2}{2} = 0 \tag{4.22}$$

$$f_0' - \frac{\rho f_1'}{4} - \frac{f_1}{4} = 0 \tag{4.23}$$

where $\beta_1 = \beta_1(m, \mu) \neq \beta_1(\rho)$.

Substitution of Eqs.(4.20)-(4.21) into Eqs.(4.15)-(4.16) yields the differential equations:

$$f_1'(2\rho - \frac{\rho^2}{2}) = -f_1(1 - \frac{\rho}{2}) \tag{4.24}$$

$$f_2'(2\rho - \frac{\rho^2}{2}) + f_2(1 - \frac{3\rho}{4}) = \frac{m - \beta_1 - 2\mu}{2\mu} + \frac{3\rho}{8}. \tag{4.25}$$

These possess the solutions

$$f_1 = \frac{c_1}{\sqrt{\rho - \frac{\rho^2}{4}}}, \quad f_2 = \frac{4m - 4\beta_1 - 8\mu + \mu\rho}{2\mu(4 - \rho)} + \frac{c_2}{\sqrt{\rho}(1 - \frac{\rho}{4})} \tag{4.26}$$

These solutions are not compatible with Eqs.(4.20)(4.21) for the same $f_1$ and $\beta_1$ unless $c_1 = c_2 = 0$, Hence

$$f_1 = 0, \quad f_2 = \frac{4m - 4\beta_1 - 8\mu + \mu\rho}{2\mu(4-\rho)} \tag{4.27}$$

Eqs.(4.20)-(4.21) become

$$2f_0 - \frac{\beta_1(6-\rho)}{\mu(4-\rho)} + \frac{\rho}{4} + \frac{4m - 8\mu + \mu\rho}{\mu(8-2\rho)} + \frac{m-2\mu}{\mu} = 0 \tag{4.28}$$

$$f_0 - \frac{\beta_1}{\mu} + \frac{2m-3\mu}{2\mu} = 0 \tag{4.29}$$

There are two equations and two unknowns and $\beta_1$ is not a function of $\rho$. They are solved by

$$f_0 = \frac{1}{2}, \quad \beta_1 = m - \mu. \tag{4.30}$$

From chapter two Eq.(2.43), one can determine more about $m_w$. In particular

$$\alpha_1 = -\mu \tag{4.31}$$

Then Eqs.(4.30)-(4.31) give

$$\epsilon_w = \mu c^2 + (m-\mu)\Delta c^2 + ....., \quad m_w = \mu(1-\Delta)c^2 + ... \tag{4.32}$$

where $\Delta = \frac{w - mc^2}{mc^2}$ and $m = m_1 + m_2, \mu = m_1 m_2/m$, and using (4.7)(4.8)(4.12)-(4.14) and (4.27)-(4.30)

$$\vec{r}' = \vec{r}(1 - \frac{e_1 e_2}{2mc^2 r}) \tag{4.33}$$

$$\vec{p}' = \vec{p}(1 + \frac{e_1 e_2}{2mc^2 r}) - \hat{r}\hat{r}\cdot\vec{p}\frac{e_1 e_2}{2mc^2 r} \tag{4.34}$$

which reproduces Eq.(4.18) from part of the field theoretic expression Eq.(3.101) without scalar interaction.

When one adds scalar interaction, one simply lets $e_1 e_2$ be replaced by $(e_1 e_2 - g_1 g_2)$ and $\alpha_1, \beta_1$ be unchanged.

Note that $\Gamma_1$ and the potentials $a_1$ and $s_1$ are not necessary for a match. This means that $\Gamma_1$ ,$a_1$ and $s_1$ can be zero without affecting anything.

$$\Gamma_1 = 0, \quad a_1 = 0, \quad s_1 = 0. \tag{4.35}$$

Therefore the potentials between two particles have no corrections beyond this nonrelativistic interactions.

Now one can summarize these in order $1/c^2$. The canonical transformation (4.7)-(4.8) can be determined by [4]

$$B_1 = -\frac{e_1e_2 - g_1g_2}{2r} \tag{4.36}$$

$$G_1 = \frac{e_1e_2 - g_1g_2}{2r} \tag{4.37}$$

$$K_1 = -\frac{e_1e_2 - g_1g_2}{2r} \tag{4.38}$$

$$E_1 = 0 \tag{4.39}$$

The potentials $a_1$ and $s_1$ are zero. $m_w$ and $\epsilon_w$ can be written as

$$\epsilon_w = \mu c^2 + (m-\mu)\Delta c^2 + ....., \quad m_w = \mu(1-\Delta) + ..... \tag{4.40}$$

The Todorov's choices are $\epsilon_w = \frac{w^2-(m_1^2+m_2^2)c^4}{w}$ and $m_w = \frac{m_1m_2c^2}{w}$. If one lets $\Delta = \frac{w-mc^2}{mc^2}$, substitute $w = mc^2(1+\Delta)$ into $\epsilon_w$ and $m_w$, then expand them, one gets

$$\epsilon_w = \frac{w^2 - (m_1^2+m_2^2)c^4}{2w} = \mu c^2 + (m-\mu)\Delta c^2 + ....$$

$$m_w = \frac{m_1m_2c^2}{w} = \frac{\mu}{1+\Delta} = \mu - \mu\Delta + ....$$

They are consistent with (4.40). The Todorov's choices are correct through order $1/c^2$.

Now one considers a canonical transformation connecting the two expressions for $w$ in order $1/c^4$. One assumes

$$\vec{r} \longrightarrow \vec{r}' = \vec{r}(1 + \frac{B_1}{mc^2} + \frac{B_2}{m^2c^4}) + \hat{p}\cdot\vec{r}\hat{p}(\frac{E_1}{mc^2} + \frac{E_2}{m^2c^4}) \tag{4.41}$$

$$\vec{p} \longrightarrow \vec{p}' = \vec{p}(1 + \frac{G_1}{mc^2} + \frac{G_2}{m^2c^4}) + \hat{r} \cdot \vec{p}\hat{r}(\frac{K_1}{mc^2} + \frac{K_2}{m^2c^4}) \tag{4.42}$$

where $B_1, G_1, K_1, E_1$ have been found in order $1/c^2$. One applies Eq.(4.1) to (4.41)and (4.42) through order $1/c^4$ with order $1/c^2$ known, using (4.36)-(4.39), one finds

$$F_2 + G_2 = \frac{(e_1e_2 - g_1g_2)^2}{4r^2} \tag{4.43}$$

$$(1 + y\frac{\partial}{\partial y})E_2 = -2z\frac{\partial}{\partial z}G_2 \tag{4.44}$$

$$(1 + y\frac{\partial}{\partial y})K_2 = -2x\frac{\partial}{\partial x}B_2 - \frac{(e_1e_2 - g_1g_2)^2}{4r^2} \tag{4.45}$$

$$2xy\frac{\partial}{\partial x}E_2 = -2yz\frac{\partial}{\partial z}K_2 \tag{4.46}$$

Since $B_2, G_2, E_2$ and $K_2$ have units of energy square, one assumes the general forms

$$G_2 = \epsilon^2 f_1 + \frac{\vec{p}^2\epsilon}{\mu}f_2 + \delta^2 g_1 + \frac{\vec{p}^2\delta}{\mu}g_2 + \epsilon\delta k_1 \tag{4.47}$$

$$B_2 = \epsilon^2 f_3 + \frac{\vec{p}^2\epsilon}{\mu}f_4 + \delta^2 g_3 + \frac{\vec{p}^2\delta}{\mu}g_4 + \epsilon\delta k_2 \tag{4.48}$$

$$K_2 = \epsilon^2 f_5 + \frac{\vec{p}^2\epsilon}{\mu}f_6 + \delta^2 g_5 + \frac{\vec{p}^2\delta}{\mu}g_6 + \epsilon\delta k_3 \tag{4.49}$$

$$E_2 = \epsilon^2 f_7 + \frac{\vec{p}^2\epsilon}{\mu}f_8 + \delta^2 g_7 + \frac{\vec{p}^2\delta}{\mu}g_8 + \epsilon\delta k_4 \tag{4.50}$$

where $G_2, B_2, K_2, E_2$ are function of $\rho$, a dimensionless variable. and $\epsilon = a_0 = \frac{e_1e_2}{r}$ and $\delta = s_0 = \frac{g_1g_2}{r}$. Note that $f_1$ and $f_2$ are different from previous ones in order $1/c^2$.

Substituting Eqs.(4.47)-(4.50) into Eqs.(4.43)-(4.46), and equating the coefficients of $\epsilon^2, \epsilon, \delta^2, \delta, \epsilon\delta$, one has three groups of equation. The first group comes form the coefficients of $\epsilon^2$ and $\delta^2$. one has

$$f_1 + f_3 = \frac{1}{4} \tag{4.51}$$

$$f_5 + 2\rho f_5' = 2\rho f_3' + 2f_3 - \frac{1}{4} \tag{4.52}$$

$$f_7 + 2\rho f_7' = 2\rho f_1' \tag{4.53}$$

$$f_7 + \rho f_7' = -\rho f_5' \tag{4.54}$$

and

$$g_1 + g_3 = \frac{1}{4} \tag{4.55}$$

$$g_5 + 2\rho g_5' = 2\rho g_3' + 2g_3 - \frac{1}{4} \tag{4.56}$$

$$g_7 + 2\rho g_7' = 2\rho g_1' \tag{4.57}$$

$$g_7 + \rho g_7' = -\rho g_5'. \tag{4.58}$$

The second group comes form equaling the coefficients of $\epsilon$ and $\delta$. one has

$$f_2 + f_4 = 0 \tag{4.59}$$

$$f_6 + 2\rho f_6' = 2\rho f_4' + f_4 \tag{4.60}$$

$$f_8 + 2\rho f_8' = 2\rho f_2' - 2f_2 \tag{4.61}$$

$$f_8 + \rho f_8' = 2f_6 - 2\rho f_6' \tag{4.62}$$

and

$$g_2 + g_4 = 0 \tag{4.63}$$

$$g_6 + 2\rho g_6' = 2\rho g_4' + g_4 \tag{4.64}$$

$$g_8 + 2\rho g_8' = 2\rho g_2' - 2g_2 \tag{4.65}$$

$$g_8 + \rho g_8' = 2g_6 - 2\rho g_6'. \tag{4.66}$$

The third group comes from equaling the coefficient of $\epsilon\delta$. one has

$$k_1 + k_2 = -\frac{1}{2} \tag{4.67}$$

$$k_4 + 2\rho k_4' = 2\rho k_1' \tag{4.68}$$

$$k_3 + 2\rho k_3' = 2\rho k_2' + 2k_2 + \frac{1}{2} \tag{4.69}$$

$$k_4 + \rho k_4' = -\rho k_3' \tag{4.70}$$

where prime, which appears in (4.51)-(4.70), means $\frac{\partial}{\partial \rho}$.

One can assume that $f_i, g_i$ and $k_j, i = 1,2....8, j = 1,2,3,4$ have solutions $\sum_{n=0}^{\infty} c_n \rho^n$, where $c_n$ are function of $m, \mu$. However, when one matches the Todorov expression and the field theoretic expression, one can find that $c_n$, $n \geq 2$ will not contribute to the match. Then one can let $f_i, g_i$ and $k_j, i = 1,2....8, j = 1,2,3,4$ have the form $constant_1 + constant_2 \rho$.

So from the first equation group, one assumes that $f_2 = c_1 + c_2\rho, g_2 = d_1 + d_2\rho$,then one has

$$\begin{aligned} f_2 &= c_1 + c_2\rho = -f_4 \\ f_6 &= -c_1 - 2c_2\rho \\ f_8 &= -2c_1 \end{aligned} \tag{4.71}$$

$$\begin{aligned} g_2 &= d_1 + d_2\rho = -g_4 \\ g_6 &= -d_1 - 2d_2\rho \\ g_8 &= -2d_1 \end{aligned} \tag{4.72}$$

From the second equation group, one assumes that $f_1 = c_3 + c_4\rho, g_1 = d_3 + d_4\rho$,then one has

$$\begin{aligned} f_1 &= c_3 + c_4\rho \\ f_3 &= \frac{1}{4} - c_3 - c_4\rho \\ f_5 &= -\frac{4}{3}c_4\rho - 2c_3 + \frac{1}{4} \\ f_7 &= \frac{2}{3}c_4\rho \end{aligned} \tag{4.73}$$

$$g_1 = d_3 + d_4\rho$$

$$g_3 = \frac{1}{4} - d_3 - d_4\rho \tag{4.74}$$

$$g_5 = -\frac{4}{3}d_4\rho - 2d_3 + \frac{1}{4}$$

$$g_7 = \frac{2}{3}d_4\rho$$

From the third equation group, one assumes that $k_1 = e_1 + e_2\rho$,then one has

$$k_1 = e_1 + e_2\rho$$

$$k_2 = -\frac{1}{2} - e_1 - e_2\rho \tag{4.75}$$

$$k_3 = -\frac{1}{2} - 2e_1 - \frac{3}{4}e_2\rho$$

$$k_4 = \frac{2}{3}e_2\rho$$

where $c_i, d_i, e_j,\ \ i = 1,2,3,4\ \ j = 1,2$ are constants.

When one uses transformation (4.41), then

$$\frac{1}{r'} = \frac{1}{r}(1 + \frac{(e_1e_2 - g_1g_2)}{2mrc^2} + \frac{(e_1e_2 - g_1g_2)^2}{4m^2r^2c^4} - \frac{B_2}{m^2c^4} - \frac{E_2(\vec{p}\cdot\hat{r})^2}{\vec{p}^2m^2c^4}) \tag{4.76}$$

In the Wheeler-Feynman expression there are $\vec{p}\cdot\vec{r}$ terms. One uses $(\vec{p}\cdot\vec{r})^2 = \frac{1}{4}\rho\vec{r}^2\vec{p}^2$ . One substitutes it into equation (3.101), which is the Wheeler-Feynman Hamiltonian . Then it becomes

$$\begin{aligned}
H(\vec{p},\vec{r}) = {} & mc^2 + \frac{\vec{p}^2}{2\mu} + \frac{(e_1e_2)}{r} - \frac{(g_1g_2)}{r} \\
& + \frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) + \frac{\vec{p}^2(e_1e_2)}{2\mu r} - \frac{\vec{p}^2(g_1g_2)(m-2\mu)}{2m\mu^2r} + \frac{(e_1e_2 - g_1g_2)^2}{2m}] \\
& + \frac{1}{c^4}\{\frac{\vec{p}^6}{16}(\frac{1}{m_1^5} + \frac{1}{m_2^5}) + \frac{(e_1e_2)}{r}[-\frac{\vec{p}^4(11m-27\mu)}{24\mu^3m^2} \\
& + \frac{\vec{p}^4\rho(m-6\mu)}{48\mu^3m^2} + \frac{\vec{p}^4\rho^2(3\mu-m)}{128\mu^3m^2} - \frac{B_2}{m^2} - \frac{E_2\rho}{4m^2}] \\
& + \frac{(e_1e_2)^2}{r^2}[\frac{\vec{p}^2(9m+7\mu)}{8\mu^2m^2} - \frac{3\vec{p}^2\rho(3m+\mu)}{32\mu^2m^3}] \\
& + \frac{(g_1g_2)}{r}[-\frac{\vec{p}^4(9m^2-37\mu m+27\mu^2)}{24\mu^4m^2} - \frac{\vec{p}^4\rho(m-3\mu)}{24\mu^3m^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\vec{p}^4\rho^2(m-3\mu)}{128\mu^3m^2}+\frac{B_2}{m^2}+\frac{E_2\rho}{4m^2}]\\
&+\frac{(g_1g_2)^2}{r^2}[\frac{\vec{p}^2(7\mu^2-21m\mu+4m^2)}{8\mu^3m^2}+\frac{\vec{p}^2\rho(11m-3\mu)}{32\mu^2m}]\\
&+\frac{(g_1g_2)(e_1e_2)}{r^2}[\frac{\vec{p}^2(6m-7\mu)}{4\mu^2m^2}+\frac{\vec{p}^2\rho(3\mu-m)}{16\mu^2m^2}]\\
&+\frac{K_2\vec{p}^2\rho}{4m^2\mu}+\frac{G_2\vec{p}^2}{m^2\mu}\\
&+\frac{(e_1e_2)^2(g_1g_2)}{r^3}(-\frac{1}{4m\mu}-\frac{3}{4m^2})\\
&+\frac{(g_1g_2)^2(e_1e_2)}{r^3}(-\frac{1}{4m\mu}+\frac{3}{4m^2})\\
&+\frac{(e_1e_2)^3}{r^3}(\frac{1}{4m\mu}+\frac{1}{4m^2})\\
&+\frac{(g_1g_2)^3}{r^3}(\frac{1}{4m\mu}-\frac{1}{4m^2})\}
\end{aligned}
\tag{4.77}
$$

Substituting $B_2, G_2, K_2, E_2$ into Eq.(4.77)

$$
\begin{aligned}
H(\vec{p},\vec{r}) = {} & mc^2+\frac{\vec{p}^2}{2\mu}+\frac{(e_1e_2)}{r}-\frac{(g_1g_2)}{r}\\
&+\frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3}+\frac{1}{m_2^3})+\frac{\vec{p}^2(e_1e_2)}{2\mu r}-\frac{\vec{p}^2(g_1g_2)(m-2\mu)}{2m\mu^2r}+\frac{(e_1e_2-g_1g_2)^2}{2m}]\\
&+\frac{1}{c^4}\{\frac{\vec{p}^6}{16}(\frac{1}{m_1^5}+\frac{1}{m_2^5})\\
&+\frac{\vec{p}^4(e_1e_2)}{r}[-\frac{(11m-27\mu)}{24\mu^3m^2}+\frac{\rho(m-6\mu)}{48\mu^3m^2}+\frac{\rho^2(3\mu-m)}{128\mu^3m^2}+\frac{f_2}{m^2\mu^2}+\frac{f_6\rho}{4m^2\mu^2}]\\
&+\frac{\vec{p}^2(e_1e_2)^2}{r^2}[\frac{(9m+7\mu)}{8\mu^2m^2}-\frac{3\rho(3m+\mu)}{32\mu^2m^3}-\frac{f_8\rho}{4\mu m^2}\\
&-\frac{f_4}{\mu m^2}+\frac{f_5\rho}{4\mu m^2}+\frac{f_1}{\mu m^2}]\\
&+\frac{\vec{p}^4(g_1g_2)}{r}[-\frac{(9m^2-37\mu m+27\mu^2)}{24\mu^4m^2}-\frac{\rho(m-3\mu)}{24\mu^3m^2}\\
&+\frac{\rho^2(m-3\mu)}{128\mu^3m^2}+\frac{g_2}{m^2\mu^2}+\frac{g_6\rho}{4m^2\mu^2}]\\
&+\frac{\vec{p}^2(g_1g_2)^2}{r^2}[\frac{(7\mu^2-21m\mu+4m^2)}{8\mu^3m^2}+\frac{\rho(11m-3\mu)}{32\mu^2m}]\\
&+\frac{g_8\rho}{4\mu m^2}+\frac{g_4}{\mu m^2}+\frac{g_5\rho}{4\mu m^2}+\frac{g_1}{\mu m^2}]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\vec{p}^2(g_1g_2)(e_1e_2)}{r^2}[\frac{(6m-7\mu)}{4\mu^2m^2}+\frac{\rho(3\mu-m)}{16\mu^2m^2}+\frac{k_3\rho}{4\mu m^2}\\
&+\frac{k_1}{\mu m^2}+\frac{f_8\rho}{4\mu m^2}+\frac{f_4}{\mu m^2}-\frac{g_8\rho}{4\mu m^2}-\frac{g_4}{\mu m^2}]\\
&+\frac{(e_1e_2)^2(g_1g_2)}{r^3}(-\frac{1}{4m\mu}-\frac{3}{4m^2}-\frac{k_4\rho}{4m^2}-\frac{k_2}{m^2}+\frac{f_7\rho}{4m^2}+\frac{f_3}{m^2})\\
&+\frac{(g_1g_2)^2(e_1e_2)}{r^3}(-\frac{1}{4m\mu}+\frac{3}{4m^2}+\frac{k_4\rho}{4m^2}+\frac{k_2}{m^2}-\frac{f_7\rho}{4m^2}-\frac{f_3}{m^2})\\
&+\frac{(e_1e_2)^3}{r^3}(\frac{1}{4m\mu}+\frac{1}{4m^2}-\frac{f_7\rho}{4m^2}-\frac{f_3}{m^2})\\
&+\frac{(g_1g_2)^3}{r^3}(\frac{1}{4m\mu}-\frac{1}{4m^2}+\frac{g_7\rho}{4m^2}+\frac{g_3}{m^2})\}
\end{aligned}
\tag{4.78}
$$

After matching the Todorov expression and the Wheeler-Feynman results in order $1/c^2$, one had $\Gamma_1 = a_1 = s_1 = 0$ and $\alpha$, $\beta$ given by (4.30) -(4.31). Therefore the Todorov expression (2.51) simplifies drastically to

$$
\begin{aligned}
w_T = &\, mc^2+\frac{\vec{p}^2}{2\mu}+\frac{(e_1e_2)}{r}-\frac{(g_1g_2)}{r}\\
&+\frac{1}{c^2}[-\frac{\vec{p}^4}{8}(\frac{1}{m_1^3}+\frac{1}{m_2^3})+\frac{\vec{p}^2(e_1e_2)}{2\mu r}-\frac{\vec{p}^2(g_1g_2)(m-2\mu)}{2m\mu^2r}+\frac{(e_1e_2-g_1g_2)^2}{2mr^2}]\\
&+\frac{1}{c^4}\{\frac{\vec{p}^6}{16}(\frac{1}{m_1^5}+\frac{1}{m_2^5})\\
&+\frac{\vec{p}^4(e_1e_2)}{r}(\frac{3}{4\mu^2m^2}-\frac{3}{8\mu^3m}+\frac{\beta_2}{4\mu^3m^2})\\
&+\frac{\vec{p}^4(g_1g_2)}{r}(-\frac{3}{4\mu^2m^2}+\frac{11}{8\mu^3m}-\frac{3}{8\mu^4}-\frac{\beta_2}{4\mu^3m^2})\\
&+\frac{\vec{p}^2(e_1e_2)^2}{r^2}(-\frac{1}{4\mu m^2}+\frac{1}{2m\mu^2}+\frac{\beta_2}{m^2\mu^2})\\
&+\frac{\vec{p}^2(g_1g_2)^2}{r^2}(-\frac{1}{4\mu m^2}-\frac{1}{m\mu^2}+\frac{1}{2\mu^3}+\frac{\beta_2}{m^2\mu^2})\\
&+\frac{\vec{p}^2(g_1g_2)(e_1e_2)}{r^2}(\frac{1}{2\mu m^2}+\frac{1}{2m\mu^2}-\frac{\beta_2}{m^2\mu^2})\\
&+\frac{\vec{p}^2\Gamma_2}{\mu}\\
&+\frac{(e_1e_2)^2(g_1g_2)}{r^3}(\frac{3}{m^2}-\frac{3}{2m\mu}-\frac{3\beta_2}{m^2\mu})\\
&+\frac{(g_1g_2)^2(e_1e_2)}{r^3}(-\frac{3}{m^2}+\frac{3}{2m\mu}+\frac{3\beta_2}{m^2\mu})
\end{aligned}
$$

$$+\frac{(e_1e_2)^3}{r^3}(-\frac{1}{m^2}+\frac{1}{2m\mu}+\frac{\beta_2}{m^2\mu})$$
$$+\frac{(g_1g_2)^3}{r^3}(\frac{1}{m^2}-\frac{1}{2m\mu}-\frac{\beta_2}{m^2\mu})$$
$$+a_2+s_2\} \tag{2.51'}$$

Now one matches terms of order $1/c^4$ in the equations (4.78) and (2.51'), and substitutes $f_i$ and $g_i$, $i = 1,2,3,...8$. The coefficients of the same power $\vec{p}^n(e_1e_2)^m(g_1g_2)^l$,$n = 0,2,4,\quad m,l = 0,1,2,3$ , are required to be equal. The coefficient of the $\frac{\vec{p}^4(e_1e_2)}{r}$ is

$$-\frac{(11m-27\mu)}{24\mu^3m^2}-(\frac{3}{4\mu^2m^2}-\frac{3}{8\mu^3m}+\frac{\beta_2}{4\mu^3m^2})+\frac{c_1}{\mu^2m^2}$$
$$+\frac{\rho}{m^2\mu^2}(\frac{m-6\mu}{48\mu}-\frac{c_1}{4}+c_2)+\frac{\rho^2}{m^2\mu^2}(\frac{3\mu-m}{128\mu}-\frac{c_2}{4})=0 \tag{4.79}$$

Since $\rho$ is a variable, one has the solutions

$$c_2=\frac{3\mu-m}{32\mu}$$
$$c_1=-\frac{3\mu+m}{24\mu}$$
$$\beta_2=\mu-\frac{m}{2}. \tag{4.80}$$

The coefficient of the $\frac{\vec{p}^4(g_1g_2)}{r}$ is

$$-\frac{27\mu^2-37m\mu+9m^2}{24\mu^4m^2}+(\frac{3}{4\mu^2m^2}-\frac{11}{8\mu^3m}+\frac{3}{8\mu^4}+\frac{\beta_2}{4\mu^3m^2})+\frac{d_1}{m^2\mu^2}$$
$$+\frac{\rho}{m^2\mu^2}(\frac{3\mu-m}{24\mu}-\frac{d_1}{4}+d_2)+\frac{\rho^2}{m^2\mu^2}(\frac{m-3\mu}{128\mu}-\frac{d_2}{4})=0 \tag{4.81}$$

The solution is

$$d_2=\frac{m-3\mu}{32\mu}$$
$$d_1=\frac{3\mu-m}{24\mu}$$
$$\beta_2=\mu-\frac{m}{2}. \tag{4.82}$$

Note that one has consistency here as that $\beta_2$ is the same as (4.80).

As one mentioned before that $\Gamma_2$ is the function of $\vec{r}$ only, one will use it for matching. $\Gamma_2$ can be divided into two pieces $\Gamma_{21}\frac{(e_1e_2)^2}{r^2}$ and $\Gamma_{22}\frac{(g_1g_2)^2}{r^2}$ The coefficient of the $\frac{\vec{p}^2(e_1e_2)^2}{r^2}$ is

$$\frac{\Gamma_{21}}{\mu}+\frac{9m+7\mu}{8m^2\mu^2}+\frac{c_1+c_3}{m^2\mu}+(\frac{1}{4\mu m^2}-\frac{1}{2m\mu^2}-\frac{\beta_2}{m^2\mu^2})$$

$$+\frac{\rho}{m^2\mu}(\frac{c_1+2c_2-c_3+2c_4}{\mu}+\frac{\mu-3m}{32\mu})-\frac{c_4\rho^2}{3m^2\mu}=0 \qquad (4.83)$$

Setting $c_4=0$. $c_3$ can be obtained by letting the coefficient of $\rho$ be zero with $c_1, c_2$ and $c_4$ known ,and

$$c_3=-\frac{2m}{3\mu}$$

$$\Gamma_{21}=-\frac{5}{12m\mu}. \qquad (4.84)$$

The coefficient of the $\frac{\vec{p}^2(g_1g_2)^2}{r^2}$ is

$$\frac{\Gamma_{22}}{\mu}+\frac{7\mu^2-21m\mu+4m^2}{8m^2\mu^2}+\frac{d_3-d_1}{m^2\mu}+(\frac{1}{4\mu m^2}+\frac{1}{m\mu^2}-\frac{\beta_2}{m^2\mu^2}-\frac{1}{2\mu^3})$$

$$+\frac{\rho}{m^2\mu}(\frac{2d_4-2d_2-d_3-d_1}{\mu}+\frac{11m-\mu}{32\mu})-\frac{d_4\rho^2}{3m^2\mu}=0 \qquad (4.85)$$

Setting $d_4=0$. $d_3$ can be obtained by letting the coefficient of $\rho$ be zero,and

$$d_3=\frac{2m}{3\mu}$$

$$\Gamma_{22}=\frac{5}{12m\mu}. \qquad (4.86)$$

Substitute $f_4, g_4, f_8, g_8, \beta_2$ into $\frac{\vec{p}^2(e_1e_2)(g_1g_2)}{r^2}$ term.

$$\frac{6m-6\mu}{4m^2\mu^2}+\frac{\rho}{4m^2\mu}(k_3+\frac{1}{2})-\frac{6m-6\mu}{4m^2\mu^2}=0? \qquad (4.87)$$

Setting $k_3=-\frac{1}{2}$. And Eq.(4.87) is surely zero. Then $k_1=0, k_2=-\frac{1}{2}$

Before one matches the next few terms, one has to examine $s_2$ and $a_2$. They should be the function of $r$ and only through $\frac{e_1e_2}{r}$ and $\frac{g_1g_2}{r}$. In the Todorov expansion (2.51') they appear symmetrical . One however lets $a_2 = a_2' + a_2''$ and $s_2 = s_2' + s_2''$. The double prime terms go with the cross-terms of $(e_1e_2)^2g_1g_2$ and $(g_1g_2)^2e_1e_2$ and the single prime terms go with $(e_1e_2)^3$ and $(g_1g_2)^3$ terms. From the next calculations, one finds that one can not match the terms of $(e_1e_2)^2g_1g_2$,$(g_1g_2)^2e_1e_2$,$(e_1e_2)^3$ and $(g_1g_2)^3$ if $a_2, s_2$ are not divided into two parts. The coefficient of the $\frac{(e_1e_2)^2(g_1g_2)}{r^3}$ is

$$(-\frac{1}{4m\mu} - \frac{3}{4m^2} - \frac{k_4\rho}{4m^2} - \frac{k_2}{m^2} + \frac{f_7\rho}{4m^2} + \frac{f_3}{m^2}) - (\frac{3}{m^2} - \frac{3}{2m\mu} - \frac{3\beta_2}{m^2\mu} + a_2'') = 0 \tag{4.88}$$

Letting $k_4 = 0$ and $f_7 = 0$, $k_2, f_3, \beta_2$ are given above. Then

$$a_2'' = \frac{5}{12m\mu}.$$

The coefficient of the $\frac{(g_1g_2)^2(e_1e_2)}{r^3}$ is

$$(-\frac{1}{4m\mu} + \frac{3}{4m^2} + \frac{k_4\rho}{4m^2} + \frac{k_2}{m^2} - \frac{g_7\rho}{4m^2} - \frac{g_3}{m^2}) + (\frac{3}{m^2} - \frac{3}{2m\mu} - \frac{3\beta_2}{m^2\mu} + s_2'') = 0 \tag{4.89}$$

Letting $k_4 = 0$ and $g_7 = 0$, $k_2, g_3, \beta_2$ are known. Then

$$s_2'' = \frac{5}{12m\mu}.$$

The coefficient of the $\frac{(e_1e_2)^3}{r^3}$ is

$$(\frac{1}{4m\mu} + \frac{1}{4m^2} - \frac{f_7\rho}{4m^2} - \frac{f_3}{m^2}) + (\frac{1}{m^2} - \frac{1}{2m\mu} - \frac{\beta_2}{m^2\mu} - a_2') = 0 \tag{4.90}$$

Setting $f_7 = 0$, and $\beta = \mu - \frac{m}{2}$, $f_3$ is given by (4.73) and $c_3, c_4$ are just obtained as above.

Then

$$a_2' = -\frac{5}{12m\mu}.$$

The coefficient of the $\frac{(g_1g_2)^3}{r^3}$ is

$$(\frac{1}{4m\mu}-\frac{1}{4m^2}+\frac{g_7\rho}{4m^2}+\frac{g_3}{m^2})+(-\frac{1}{m^2}+\frac{1}{2m\mu}+\frac{\beta_2}{m^2\mu}+s_2')=0 \qquad (4.91)$$

Setting $g_7=0$, and $\beta=\mu-\frac{m}{2}$, $g_3$ is given by (4.74) and $d_3, d_4$ are given above,

Then

$$s_2'=-\frac{5}{12m\mu}.$$

Hence the Todorov model is

$$(1+\Gamma)^2\vec{p}^2c^2+(m_wc^2+S)^2-(\epsilon_w-A)^2=0 \qquad (4.92)$$

with

$$\begin{aligned}\Gamma=\frac{\Gamma_1}{c^2}+\frac{\Gamma_2}{c^4}&=\Gamma_{21}\frac{(e_1e_2)^2}{r^2c^4}+\Gamma_{22}\frac{(g_1g_2)^2}{r^2c^4}\\&=\frac{5}{12m\mu}\frac{(g_1g_2)^2-(e_1e_2)^2}{r^2c^4}\\V=a_0+\frac{a_1}{c^2}+\frac{a_2}{c^4}&=\frac{e_1e_2}{r}+\frac{a_2}{c^4}\\S=s_0+\frac{s_1}{c^2}+\frac{s_2}{c^4}&=\frac{e_1e_2}{r}+\frac{s_2}{c^4}\end{aligned} \qquad (4.93)$$

with $a_1=0$, and $s_1=0$. $a_2$ and $s_2$ have two choices.

$$(1)\ \ a_2=a_2'+a_2''\ \ and\ \ s_2=s_2'+s_2''$$

or

$$(2)\ \ a_2=a_2'+\frac{1}{2}(a_2''+s_2'')\ \ and\ \ s_2=s_2'+\frac{1}{2}(a_2''+s_2'')$$

How to determine which one is the correct choice depends on next order expansion $O(\frac{1}{c^6})$.

In the first choice,

$$V=\frac{e_1e_2}{r}+\frac{5(e_1e_2)^2(g_1g_2-e_1e_2)}{12m\mu r^3c^4} \qquad (4.94)$$

$$S=-\frac{g_1g_2}{r}-\frac{5(g_1g_2)^2(g_1g_2-e_1e_2)}{12m\mu r^3c^4}. \qquad (4.95)$$

In the second choice,

$$V = \frac{e_1 e_2}{r} - \frac{5(e_1 e_2)}{12m\mu r^3 c^4}((e_1 e_2)^2 - \frac{(g_1 g_2)^2 + (g_1 g_2)(e_1 e_2)}{2}) \tag{4.96}$$

$$S = -\frac{g_1 g_2}{r} - \frac{5(g_1 g_2)}{12m\mu r^3 c^4}((g_1 g_2)^2 - \frac{(e_1 e_2)^2 + (g_1 g_2)(e_1 e_2)}{2}). \tag{4.97}$$

From the above calculations for $\beta_1$ and $\beta_2$, Eq.(2.43) gives

$$m_w = \mu - \mu\Delta + \mu\Delta^2 + .....$$

On the other hand, the Todorov choice , which one derived without interactions, is $m_w = \frac{m_1 m_2 c^2}{w}$. Are these two results consistent ? If one lets $w = mc^2(1+\Delta)$, where $w$ is the total energy in c.m system, then $\Delta = \frac{w - mc^2}{mc^2}$, and substitute it into the Todorov's formula, expand it ($\Delta \ll 1$), one finds

$$m_w = \frac{m_1 m_2 c^2}{w} = \frac{\mu}{1+\Delta} = \mu - \mu\Delta + \mu\Delta^2 + ..... \tag{4.98}$$

which agrees with (2.43). From Eq.(2.43) and $\alpha_1$, $\alpha_2$ one has

$$\epsilon_w = \mu c^2 + (m-\mu)\Delta c^2 + (\mu - \frac{m}{2})\Delta^2 c^2 + ....$$

If one expands the Todorov choice as function of $\Delta$ then

$$\epsilon_w = \frac{w^2 - (m_1^2 + m_2^2)c^4}{2w} = \mu c^2 + (m-\mu)\Delta c^2 + (\mu - \frac{m}{2})\Delta^2 c^2 + .... \tag{4.99}$$

The results here are consistent with the Todorov's choices too. Thus one takes

$$\epsilon_w = \frac{w^2 - (m_1^2 + m_2^2)c^4}{2w}, \quad m_w = \frac{m_1 m_2 c^2}{w}. \tag{4.100}$$

So the Todorov equation (4.92) has the same dynamical properties as the field theory through order $1/c^4$, because there is a canonical transformation connecting them. It is Eqs.(4.41)(4.42) (The general Schwinger transformation [21]) with $B_1, E_1, K_1, G_1$ shown in Eqs.(4.36)-(4.39) and $B_2, G_2, K_2, E_2$ defined as:

$$\begin{aligned} G_2 = &-\frac{2m\epsilon^2}{3\mu} - \frac{\vec{p}^2\epsilon}{\mu}(\frac{m+3\mu}{24\mu}) + \frac{(\vec{p}\cdot\hat{r})^2\epsilon}{\mu}(\frac{3\mu - m}{8\mu}) \\ &+ \frac{2m\delta^2}{3\mu} - \frac{\vec{p}^2\delta}{\mu}(\frac{m-3\mu}{24\mu}) - \frac{(\vec{p}\cdot\hat{r})^2\delta}{\mu}(\frac{3\mu - m}{8\mu}) \end{aligned} \tag{4.101}$$

$$B_2 = (\frac{3\mu + 8m}{12\mu})\epsilon^2 + \frac{\vec{p}^2\epsilon}{\mu}(\frac{m + 3\mu}{24\mu}) - \frac{(\vec{p}\cdot\hat{r})^2\epsilon}{\mu}(\frac{3\mu - m}{8\mu})$$
$$+ (\frac{3\mu - 8m}{12\mu})\delta^2 + \frac{\vec{p}^2\delta}{\mu}(\frac{m - 3\mu}{24\mu}) + \frac{(\vec{p}\cdot\hat{r})^2\delta}{\mu}(\frac{3\mu - m}{8\mu}) - \frac{1}{2}\epsilon\delta \quad (4.102)$$

$$K_2 = (\frac{3\mu + 16m}{12\mu})\epsilon^2 + \frac{\vec{p}^2\epsilon}{\mu}(\frac{m + 3\mu}{24\mu}) - \frac{(\vec{p}\cdot\hat{r})^2\epsilon}{\mu}(\frac{3\mu - m}{4\mu})$$
$$+ (\frac{3\mu - 16m}{12\mu})\delta^2 + \frac{\vec{p}^2\delta}{\mu}(\frac{m - 3\mu}{24\mu}) + \frac{(\vec{p}\cdot\hat{r})^2\delta}{\mu}(\frac{3\mu - m}{4\mu}) - \frac{1}{2}\epsilon\delta \quad (4.103)$$

$$E_2 = \frac{\vec{p}^2\epsilon}{\mu}(\frac{m + 3\mu}{12\mu}) + \frac{\vec{p}^2\delta}{\mu}(\frac{m - 3\mu}{12\mu}) \quad (4.104)$$

# CHAPTER 5

## Summary

In this thesis, the Todorov model is examined in light of classical relativistic field theory. The final c.m. version of the Todorov equation is given as

$$(\vec{p}-\vec{V})^2c^2+(m_wc^2+S)^2=(\epsilon_w-V)^2 \tag{5.1}$$

where

$\vec{V}=-\vec{p}\Gamma(\vec{r})$ is the vector potential.

$S=S(\vec{r})$ is the scalar potential.

$V=V(\vec{r})$ is the time-like component of Lorentz 4-vector potential.

$m_w=\frac{m_1m_2c^2}{w}$ and $\epsilon_w=\frac{w^2-(m_1^2+m_2^2)c^4}{2w}$ is the relativistic reduced mass and energy of the fictitious particles of the relative motion.

The Todorov model also takes the form

$$\vec{p}^2c^2+\Phi=b^2(w) \tag{5.2}$$

where $b^2(w)=\epsilon_w^2-m_w^2c^4$ and $\Phi=2m_wc^2S+2\epsilon_wV+S^2-V^2+\vec{p}^2c^2(2\Gamma+\Gamma^2)$.

To obtain the expressions of $V,S$ and $\Gamma$, one used the classical field theory as input to determine those potentials. First of all, the Todorov expression (2.58) of the Todorov equation (5.1) for c.m. energy $w$ through order $1/c^4$ was obtained. Secondly, one obtained a similar expression of the classical field theory for $w$ in c.m. system (3.101) ,in which comes from expanding the Fokker-Tetrode action (3.52) or (3.58) in the same order $1/c^4$ as one did in the Todorov model. Finally, based on the demand that these two theories describe the same dynamical system, one performed a canonical transformation (4.41)(4.42) to match expressions (2.58) and (3.101). This leads to the following final forms for $V,S$ and $\Gamma$:

$$\Gamma=\frac{5}{12m\mu}\frac{(g_1g_2)^2-(e_1e_2)^2}{r^2c^4}$$

$$V=\frac{e_1e_2}{r}+\frac{5(e_1e_2)^2(g_1g_2-e_1e_2)}{12m\mu r^3c^4} \tag{5.3a}$$

$$S = -\frac{g_1 g_2}{r} - \frac{5(g_1 g_2)^2(g_1 g_2 - e_1 e_2)}{12 m \mu r^3 c^4}.$$

or

$$V = \frac{e_1 e_2}{r} + \frac{5(e_1 e_2)}{12 m \mu r^3 c^4}((e_1 e_2)^2 - \frac{(g_1 g_2)^2 + (g_1 g_2)(e_1 e_2)}{2}) \qquad (5.3b)$$

$$S = -\frac{g_1 g_2}{r} - \frac{5(g_1 g_2)}{12 m \mu r^3 c^4}((g_1 g_2)^2 - \frac{(e_1 e_2)^2 + (g_1 g_2)(e_1 e_2)}{2}).$$

Note that $\Gamma$, $A$, $S$ do not have corrections of order $1/c^2$. The the canonical transformation is given by (4.41)(4.42) with (4.36)-(4.39) and (4.97)-(4.100). Furthermore one found, to the appropriate order in $1/c^2$, that a dynamical determination of $m_w$ and $\epsilon_w$ yield the original Todorov forms.

This is what one obtained for two relativistic spinless charged particle system.

# CHAPTER 6

## Conclusions

The most important results of this thesis are the Todorov model and the corrections to potentials between the particles obtained from classical field theory. The Todorov model generally describes two spinless charged particles interacting mutually through time-like vector, space-like vector and world scalar interactions. In the c.m. system the covariant Todorov equation can be written as (2.33) with $b^2(w)$ given in (2.23) and $\Phi$ in (2.43). $\Phi$ is the effective interactions between these two particles. Even though (2.41b) appears non-relativistic, it is fully relativistic as it is (2.41a) in a special frame (c.m. frame). When the mass of one of particles becomes very heavy, the two-body Todorov equation will reduce to one-body Klein-Gordon (generalized mass-shell) equation with the same non-relativistic limits. The Todorov equation obtained here is completely analogous to the Einstein mass-energy relation, given the relativistic reduced mass and energy of the fictitious particle. From the Todorov equation, the energy for $w$ in the c.m. system (2.58) can be obtained through order $1/c^4$ by iteration. When the non-relativistic energy is known, energy $w$ in order $1/c^4$ can be calculated from $w$ in order $1/c^0$ and $1/c^2$. Todorov's expressions of $m_w$ and $\epsilon_w$ are also dynamically obtained. While the Todorov wave equation is expanded, the potentials are assumed to be a power series of $1/c^2$, the assumptions for $m_w$ and $\epsilon_w$ are also that of a $1/c^2$ power series. In addition to the Todorov model, one expanded the Fokker-Tetrode action through order $1/c^4$, given by (3.101). This classical relativistic field theory describes the same general physical problem as that of the Todorov model, and it gives specific dynamical content to the Todorov model. This theory provides a tool to determine the Todorov model dynamically. One makes a canonical transformation to connect the expressions (2.58) and (3.101). It makes them dynamically equal and gives the results of the unknown invariant potentials $V, S$ and $\vec{V}$. The summary lists the results in (5.3). It can be seen that beyond the non-relativistic potentials, there

are some corrections in order $1/c^4$ (note that there are no corrections in order $1/c^2$). At this order, the potentials $V, S$ and $\vec{V}$ contains both scalar and vector interactions. At the same time, the potential $\Phi$ has more corrections involving $\vec{p}^2$. All the above proves that the Todorov equation can reproduce the correct relativistic dynamics through order $1/c^4$.

As it stands, the Todorov form (5.1) with potentials given in (5.3a,b) displays exact relativistic kinematics (unlike (2.58) which displays the kinematics only approximately). However, the form for the potentials are not in closed forms, appearing as partial series expression. One can use Pade' approximants to resume these series into a closed form. Pade' approximants provides a systematic way of resuming a polynomial approximation of an infinite series into the form of a rational function. The simplest example is the replacement of $1-x$ by $\frac{1}{1+x}$ and is the only one used here as series expression for $S$ and $V$ involve just two terms. From the results (5.3a), one has

$$V = \frac{a_0}{1+\frac{5a_0(a_0-s_0)}{12m\mu c^4}}$$

and

$$S = \frac{s_0}{1+\frac{5a_0(a_0-s_0)}{12m\mu c^4}}. \tag{6.1}$$

From (5.3b), one has

$$V = \frac{a_0}{1+\frac{5(s_0+a_0)(s_0-2a_0)}{24m\mu c^4}}$$

$$S = \frac{s_0}{1+\frac{5(s_0+a_0)(a_0-2s_0)}{24m\mu c^4}} \tag{6.2}$$

where $a_0 = \frac{e_1 e_2}{r}$ and $s_0 = -\frac{g_1 g_2}{r}$. Note the these effective potentials do not blow up at the origin. These may become very generally useful in quantum application of these equations.

After the Todorov equation is determined, the correct expression $w$ through order $1/c^4$ is given by (2.58) with $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, a_0, a_1, a_2, s_0, s_1, s_2, \Gamma_1$ and

$\Gamma_2$ known. It can be converted into the Schrödinger equation, if one lets $\vec{p} \rightarrow (-i\hbar\nabla)$. The wave equation in c.m. system is

$$w\psi = H\psi \tag{6.3}$$

where $\psi$ is the eigenfunction and $H$ is given by right hand side of (2.58). However, this form of the Schrödinger equation is semi-relativistic and very awkward to use because of the many different powers of $\vec{p}^2$ that appeared in (2.58). If one uses instead the Todorov form (2.33) as a quantum operator with $S, V$ and $\vec{V}$ given as above, one obtains a covariant Schrödinger-like form

$$[(\vec{p} - \vec{V})^2 + (m_w + S)^2 - (\epsilon_w - V)^2]\psi = 0 \tag{6.4}$$

that extrapolates the dynamics to all order of $1/c^2$ and agrees with field theory through order $1/c^4$.

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# APPENDICES

# APPENDIX A

## A PROGRAM FOR CALCULATION OF TODOROV EXPRESSION THROUGH ORDER $1/c^4$

Here one gives a MACSYMA program to calculate Todorov expression (2.23) through order $1/c^4$ with order $1/c^2$ matched.

All the assumptions used here are

(1) The relativistic reduced mass

$$m_w = \sum_{n=0}^{\infty} \alpha_n \Delta^n \tag{A.1}$$

In the program $\alpha_n \rightarrow a_n$.

(2) The relativistic energy

$$\epsilon_w = \sum_{n=0}^{\infty} \beta_n \Delta^n \tag{A.2}$$

In the program $\beta_n \rightarrow b_n$.

(3) the potential between two particles

$$\begin{aligned} \Phi &= 2m_w c^2 S + 2\epsilon_w V + S^2 - V^2 + \vec{p}^2 c^2 (2\Gamma + \Gamma^2) \\ &= \phi_0 c^2 + \phi_1 + \frac{\phi_2}{c^2} + .... \end{aligned} \tag{A.3}$$

In the program $\phi_0 \rightarrow t_0$, $\phi_1 \rightarrow t_1$and $\phi_2 \rightarrow t_2$.

(4) The expansion of (2.58) up to order $1/c^2$ is sufficit to reproduce the Wheeler-Feynman results through the same order if

$$a_1 = 0, \quad s_1 = 0, \quad \alpha_0 = \mu, \quad \beta_0 = \mu, \quad \alpha_1 = \mu, \quad \beta_1 = m - \mu$$

. Thus, to that order,

$$\begin{aligned} w = {} & mc^2 + \frac{\vec{p}^2}{2\mu} + a_0 + s_0 \\ & + \frac{1}{c^2}[-\frac{p^4}{8}(\frac{1}{m_1^3} + \frac{1}{m_2^3}) + \frac{\vec{p}^2 a_0}{2\mu} - \frac{\vec{p}^2 s_0 (m - 2\mu)}{2m\mu^2} + \frac{(a_0 + s_0)^2}{2m}] \end{aligned} \tag{A.4}$$

The program will recover the $w(c^2)$, $w(c^0)$ and $w(1/c^2)$. In the program $w(c^2)$ gives $mc^2$, $w(c^0)$ is $w_0$ and $w(1/c^2)$ is $w_1$. Finally the program leads to $w(1/c^4)$ ,which is $w_2$ in the program, with $\beta_1 \rightarrow b_2$ an undetermined variable that will be given after performing a canonical transformation.

The MACSYMA program is

```
/*****************************************************************/
/*                                                               */
/*    The program for calculation of the Todorov expression      */
/*                                                               */
/*     This program name is " cal.tod2 ".  The results will      */
/*        output to file "outsp"                                 */
/*                                                               */
/*****************************************************************/

    /*  writefile( )  defines a output file ( )                  */
    /*  taylor()      means "Taylor expansion"                   */
    /*  subst()       means "substitution    "                   */
    /*  ratsubst()    means "substitution "                      */
    /*  ratsimp()     means "simplification"                     */
    /*  ratcoeff()    means "coefficients of ()"                 */

    writefile(outsp);
    k1:  taylor(m1*c^2*sqrt(1+x1)+m2*c^2*sqrt(1+x2),[x1,x2],0,3)$
    k2:  subst([x1=(p^2*c^2+t)/(m1^2*c^4),x2=(p^2*c^2+t)/(m2^2*c^4)],k1)$
    k3:  ratsubst(t0*c^2+t1+t2/c^2,t,k2)$
    m11: (m-sqrt(m^2-4*u*m))/2$
    m22: (m+sqrt(m^2-4*u*m))/2$
    k3:  subst([m1=m11,m2=m22],k3)$
    k3:  ratsimp(k3)$
    k4:  ratcoeff(k3,c,2);
    w0:  ratcoeff(k3,c,0);
    w1:  ratcoeff(k3,c,-2);
    w2:  ratcoeff(k3,c,-4);

    k0: (p^2/(2*u*m)+(v0+s0)/m);
    k00: ((p^2*v0)/(u*m^2)-(p^2*s0*(m-2*u))/
          (2*u^2*m^2)+(v0+s0)^2/(2*m^2)-p^4*(m-3*u)/(8*u^3*m^2));
    k:  k0/c^2+k00/c^4$
    po: ratsubst(k,k, 2*u*c^2*(v+s)+s^2-v^2+2*k*c^2*(a1*s+b1*v)
           +2*k^2*c^2*(a2*s+b2*v))$
    po:  subst([s=s0+s1/c^2+s2/c^4,v=v0+v1/c^2+v2/c^4],po)$
    t00: ratcoeff(po,c,2);
    t10: ratcoeff(po,c,0);
    t20: ratcoeff(po,c,-2);
    w1: subst([t0=t00,t1=t10],w1);
        ratcoeff(w1,p,4);
        ratcoeff(w1,p,2);
        ratcoeff(w1,p,0);
    w1: subst([a1=-u,b1=m-u],w1)$
        ratcoeff(w1,p,4);
        ratcoeff(w1,p,2);
        ratcoeff(w1,p,0);
```

```
/* COMPUTE 1/C^4 TERMS WITH A VARIABLE B2  */

w2: subst([t0=t00,t1=t10,t2=t20],w2);
w2: expand(%)$
    coeff(w2,p,6);
    coeff(w2,p,4);
    coeff(w2,p,2);
    coeff(w2,p,0);
w2: subst([a1=-u,b1=m-u,a2=b2+m/2,s0=-ss0],w2)$
w2: expand(%)$
    coeff(w2,p,6);
    coeff(w2,p,4);
    coeff(w2,p,2);
    coeff(w2,p,0);
closefile(outsp);
logout();
```

# APPENDIX B

## A PROGRAM FOR CALCULATION OF

## THE LAGRANGIAN THROUGH ORDER $1/c^4$

Here a MACSYMA program is given to calculate the Lagrangian (3.58) through order $1/c^4$. The Lagrangian that is computed is

$$L = -m_1c^2\sqrt{1-\frac{\vec{v}_1^2(t)}{c^2}} - m_2c^2\sqrt{1-\frac{\vec{v}_2^2(t)}{c^2}}$$

$$+\sum_{n=0}^{\infty}\frac{(-D_1D_2)^n}{(2n)!c^{2n}}\{[-e_1e_2(1-\frac{\vec{v_1}(t)\cdot\vec{v_2}(t)}{c^2})$$

$$+g_1g_2\sqrt{1-\frac{\vec{v}_1^2(t)}{c^2}}\sqrt{1-\frac{\vec{v}_2^2(t)}{c^2}}]|\vec{r_1}(t_1)-\vec{r_2}(t_2)|^{2n-1}\} \qquad (B.1)$$

The opeators $D_1$ and $D_2$ signify time derivatives of only $\vec{r}_1$ and $\vec{r}_2$ respectively.

In order to understand the program, one gives some comments of the program. one defines

$$k_1 = -m_1c^2\sqrt{1-\frac{\vec{v}_1^2(t)}{c^2}} - m_2c^2\sqrt{1-\frac{\vec{v}_2^2(t)}{c^2}}$$

$$y_1 = \sqrt{1-\frac{\vec{v}_1^2(t)}{c^2}}\sqrt{1-\frac{\vec{v}_2^2(t)}{c^2}}$$

$$k_2 = -e_1e_2(1-\frac{\vec{v_1}(t)\cdot\vec{v_2}(t)}{c^2}) + g_1g_2\sqrt{1-\frac{\vec{v}_1^2(t)}{c^2}}\sqrt{1-\frac{\vec{v}_2^2(t)}{c^2}}$$

$k_6$ will expand $k_2$ through order $1/c^4$ and gives

$$k_6 = k_3 + \frac{k_4}{c^2} + \frac{k_5}{c^4}$$

Then

$$L_1 = -m_1c^2\sqrt{1-\frac{\vec{v}_1^2(t)}{c^2}} - m_2c^2\sqrt{1-\frac{\vec{v}_2^2(t)}{c^2}} + \frac{k_6}{r}$$

This corresponds to $k_1$ plus $n = 0$ in sum terms.

$k_7$ and $k_8$ are defined as

$$\frac{d}{dt_2}\frac{d}{dt_1}(k_6|\vec{r_1}(t_1)-\vec{r_2}(t_2)|)$$

and are the $n=1$ terms in the sum in $(B.1)$. The result is $L_2$ up to $1/c^4$ order.

$L_3$ is $n=2$ in sum terms, but only one term in $1/c^4$ order.

$$\frac{d^2}{dt_2^2}\frac{d^2}{d_1^2}(|\vec{r_1}(t_1)-\vec{r_2}(t_2)|)^3$$

Finally the Lagrangian is $L=L_1-L_2+L_3$. The output " Lagrangian" gives in Eq.(3.81).

```
/*****************************************************************/
/*                                                               */
/*    The program for calculation of the Lagrangian              */
/*                                                               */
/*     This program name is " lag.in ".  The results will        */
/*        output to file "lagrangian" and                        */
/*        a readable file "datalag".                             */
/*                                                               */
/*****************************************************************/

    /*  load()         loads in the vector package.            */
    /*  declare()      defines variables                       */
    /*  writefile( )   defines a output file ( )               */
    /*  diff()         means "differentiation"                 */
    /*  vectorsimp     means "vector simplification"           */
    /*  ratcoeff()     means "coefficients of ()"              */
    /*  fortran()      means "output in fortran form"          */
    /*  sqrt()         square root of ()                       */

   load(vect);
   declare([m1,m2,s0,g0,x,c],scalar,[a,r,v,j],nonscalar)$
   writefile(lagrange);
        taylor(-m1*c^2*sqrt(1-x1)-m2*c^2*sqrt(1-x2),[x1,x2],0,3);
   k1:  subst([x1=(v(t1).v(t1))/c^2,x2=(v(t2).v(t2))/c^2],%);
   y:   sqrt(r(t1).r(t1)+r(t2).r(t2)-2*r(t1).r(t2));
   y1:  taylor(sqrt(1-x),x,0,3);
   y2:  ratsubst((v(t1).v(t1))/c^2,x,y1);
   y3:  ratsubst((v(t2).v(t2))/c^2,x,y1);
   y1:  ratsimp(y2*y3);
   v10: diff(r(t1),t1,1);
   v20: diff(r(t2),t2,1);
   a10: diff(v(t1),t1,1);
   a20: diff(v(t2),t2,1);
   a11: diff(r(t1),t1,2);
   a21: diff(r(t2),t2,2);
   k2:  -g0*(1-(v(t1).v(t2))/c^2)+s0*y1;
   k3:  ratcoeff(k2,c,0);
   k4:  ratcoeff(k2,c,-2);
   k5:  ratcoeff(k2,c,-4);
```

```
k6:  vectorsimp(k3+k4/c^2+k5/c^4),expandall;
l1:  subst([v(t1)=v1,v(t2)=v2],k1+k6/x);
l1:  vectorsimp(l1),expandall;
k7:  diff((k4/c^2+k3)*y,t1,1);
k8:  diff(k7,t2,1);
l2:  ratsubst(x,y,k8/(2*c^2));
     subst([a10=a1,a20=a2,v10=v1,v20=v2,v(t1)=v1,v(t2)=v2],l2);
     subst(r+r(t2),r(t1),%);
l2:  vectorsimp(%),expandall;
     diff(y^3,t1,2);
     diff(%,t2,2);
     ratsubst(x,y,%);
     subst([v10=v1,v20=v2,a11=a1,a21=a2],%);
     ratsubst(r+r(t2),r(t1),%);
k10: vectorsimp(%),expandall;
l3:  (-g0+s0)*k10/(24*c^4);
l:   l1-l2+l3;
p0:  ratcoeff(l,c,2);
     ratcoeff(l,c,0);
p1:  partfrac(%,x);
     ratcoeff(l,c,-2);
p2:  vectorsimp(%),expandall;
     ratcoeff(l,c,-4);
p3:  vectorsimp(%),expandall;
e1:  ratcoeff(p3,g0,1);
e1:  vectorsimp(e1),expandall;
e2:  ratcoeff(p3,s0,1);
e2:  vectorsimp(e2),expandall;
l:   p0*c^2+p1+p2*(1/c^2)+p3*(1/c^4);
closefile(lagrange);
writefile(datalag);
fortran(l);
closefile(datalag);
logout();
```

## APPENDIX C

## A PROGRAM FOR TRANSFORMING LAGRANGIAN TO HAMILTONIAN H($\vec{v}, \vec{r}$)

Here one gives a MACSYMA program to transform the Lagrangian to the Hamiltonian, using the formula

$$H = \sum_{i=1}^{2} (\vec{v}_i \cdot \frac{\partial L}{\partial \vec{v}_i} + \vec{a}_i \cdot \frac{\partial L}{\partial \vec{a}_i} - \vec{v}_i \cdot \frac{d}{dt_i} \frac{\partial L}{\partial \vec{a}_i}) - L \qquad (C.1)$$

The program is easily understood. Only thing emphasized here is $k_i$, $i = 1, 2, ..., 18$. Because MACSYMA does not recognize $(\vec{a} \cdot \vec{b})$ is a scalar, one has to set it to be a scalar $k_i$ in order to use "DOT SCALAR RULES " in the MACSYMA.

The results of the program gives

$$\begin{aligned}
H(\vec{j}, \vec{a}, \vec{v}, \vec{r}) = {} & (m_1 + m_2)c^2 + \frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 + \frac{e_1 e_2}{r} - \frac{g_1 g_2}{r} \\
& + \frac{1}{c^2} \{ \frac{3}{8} m_1 \vec{v}_1^4 + \frac{3}{8} m_2 \vec{v}_2^4 \\
& + \frac{(\vec{v}_1 \cdot \vec{v}_2) e_1 e_2}{2r} + \frac{e_1 e_2 (\hat{r} \cdot \vec{v}_1)(\hat{r} \cdot \vec{v}_2)}{2r} \\
& - \frac{g_1 g_2}{2r} (\vec{v}_1^2 + \vec{v}_2^2) - \frac{g_1 g_2 (\hat{r} \cdot \vec{v}_1)(\hat{r} \cdot \vec{v}_2)}{2r} + \frac{(\vec{v}_1 \cdot \vec{v}_2) g_1 g_2}{2r} \} \\
& + \frac{1}{c^4} \{ \frac{5 m_1 \vec{v}_1^6}{16} + \frac{5 m_2 \vec{v}_2^6}{16} \\
& + \frac{e_1 e_2}{r} [ \frac{\vec{v}_1 \cdot \vec{v}_2}{8} (\vec{v}_2^2 - (\vec{v}_2 \cdot \hat{r})^2) + \frac{\vec{v}_1 \cdot \vec{v}_2}{8} (\vec{v}_1^2 - (\vec{v}_1 \cdot \hat{r})^2) \\
& + \frac{3(\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r})}{8} (\vec{v}_2^2 - (\vec{v}_2 \cdot \hat{r})^2) + \frac{3(\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r})}{8} (\vec{v}_1^2 - (\vec{v}_1 \cdot \hat{r})^2) \\
& - \frac{\vec{a}_1 \cdot \vec{r}}{8} (\vec{v}_2^2 - (\vec{v}_2 \cdot \hat{r})^2) + \frac{\vec{a}_2 \cdot \vec{r}}{8} (\vec{v}_1^2 - (\vec{v}_1 \cdot \hat{r})^2) \\
& + \frac{\vec{v}_1 \cdot \vec{v}_2}{8} (\vec{a}_1 \cdot \vec{r} - \vec{a}_2 \cdot \vec{r}) + \frac{3(\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r})}{8} (\vec{a}_1 \cdot \vec{r} - \vec{a}_2 \cdot \vec{r}) \\
& + \frac{5r}{8} ((\vec{a}_1 \cdot \vec{v}_2)(\vec{v}_1 \cdot \hat{r}) - (\vec{a}_2 \cdot \vec{v}_1)(\vec{v}_2 \cdot \hat{r})) \\
& + \frac{r}{4} ((\vec{a}_1 \cdot \vec{v}_2)(\vec{v}_2 \cdot \hat{r}) - (\vec{a}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \hat{r}))
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3r}{8}((\vec{a}_2\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r}))\\
&+\frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r})-\vec{r}^2(\vec{a}_1\cdot\vec{a}_2))\\
&+\frac{\vec{r}^2}{8}(3\vec{j}_1\cdot\vec{v}_2+3\vec{j}_2\cdot\vec{v}_1-(\vec{j}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})-(\vec{j}_2\cdot\hat{r}))(\vec{v}_1\cdot\hat{r})\\
&-\frac{\vec{v}_1^2\vec{v}_2^2}{8}+\frac{(\vec{v}_1\cdot\vec{v}_2)^2}{4}+\frac{\vec{v}_1^2(\vec{v}_1\cdot\hat{r})^2}{8}+\frac{\vec{v}_2^2(\vec{v}_2\cdot\hat{r})^2}{8}\\
&-\frac{3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2}{8}]\\
&+\frac{g_1g_2}{r}[-\frac{3}{8}(\vec{v}_1^4+\vec{v}_2^4)+\frac{3}{8}((\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})^3+(\vec{v}_1\cdot\hat{r})^3(\vec{v}_2\cdot\hat{r}))\\
&+\frac{\vec{v}_1^2}{8}(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))+\frac{\vec{v}_2^2}{8}(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))\\
&+\frac{3}{8}(\vec{v}_2^2(\vec{v}_1\cdot\hat{r})^2+\vec{v}_1^2(\vec{v}_2\cdot\hat{r})^2-(\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})^2-(\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})^2)\\
&-\frac{3}{8}(\vec{v}_2^2(\vec{a}_1\cdot\vec{r})-\vec{v}_1^2(\vec{a}_2\cdot\vec{r}))\\
&+\frac{3}{8}(\vec{a}_1\cdot\vec{r})(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))-\frac{3}{8}(\vec{a}_2\cdot\vec{r})(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))\\
&-\frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{v}_2\cdot\hat{r})^2-(\vec{a}_2\cdot\vec{r})(\vec{v}_1\cdot\hat{r})^2)\\
&+\frac{3r}{8}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_2\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r}))\\
&+\frac{r}{8}((\vec{a}_2\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r}))\\
&+\frac{r}{4}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})-(\vec{a}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\hat{r}))\\
&-\frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r})+\vec{r}^2(\vec{a}_1\cdot\vec{a}_2))\\
&+\frac{\vec{r}^2}{8}(\vec{j}_1\cdot\vec{v}_2+\vec{j}_2\cdot\vec{v}_1+(\vec{j}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})+(\vec{j}_2\cdot\hat{r}))(\vec{v}_1\cdot\hat{r})\\
&-\frac{\vec{v}_1^2\vec{v}_2^2}{8}+\frac{(\vec{v}_1\cdot\vec{v}_2)^2}{4}++\frac{3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2}{8}]\}
\end{aligned}
\qquad (C.2)
$$

The program is

```
/*****************************************************************/
/*                                                               */
/*    The program for transforming Lagrangian to Hamiltonian     */
/*                                                               */
/*     This program name is " Ham.in ".  The results will        */
/*        output to file "resh ", meaning "result of             */
/*        Hamiltonian ". It will be the function of the          */
/*        velocity and relative position.                        */
/*                                                               */
/*     The program will read in the result of the program        */
/*        " lag.in "-" resh ", which is convert to the           */
/*        name "lag.res".                                        */
/*                                                               */
/*****************************************************************/

    /*  dotscrules:   factor out scalar terms                 */
    /*  expand        means "expand "                         */
    /*  ratexpand     means "expand"                          */

   declare([m1,m2,g0,s0,c,x,k1,k2,k3,k4,k5,k6,k7,k8,k9,k10,
            k11,k12,k13,k14,k15,k16,k17,k18],scalar,
            [v1,v2,a1,a2,j1,j2,r], nonscalar)$
   writefile(resh);

   batch(lag,res)$
   z0: v1.v1;
   z00:v2.v2;

 /* Calculate v.diff(1,v,1) and a.diff(1,a,1) for both particles    */

   z1:  v1.diff(1,v1,1)+v2.diff(1,v2,1)$
   z1:  expand(%)$
   z2:  a1.diff(1,a1,1)+a2.diff(1,a2,1)$
   z2:  expand(%)$
   z3:  z1+z2$
   z3:  z3,dotscrules: true $

/* Compute a.diff(1,t) terms */

   loadfile(vect)$
   declare([s0,go,c,x],scalar,[v,r,j,a],nonscalar)$
   v10: diff(r(t1),t1,1)$
   v20: diff(r(t2),t2,1)$
   a10: diff(v(t1),t1,1)$
   a20: diff(v(t2),t2,1)$
   a11: diff(r(t1),t1,2)$
   a21: diff(r(t2),t2,2)$
   z4:  diff(1,a1,1)$
   z5:  diff(1,a2,1)$
   z6:  diff(a(t1),t1,1)$
   z7:  diff(a(t2),t2,1)$
   y:    sqrt(r(t1).r(t1)+r(t2).r(t2)-2*r(t1).r(t2))$
   z8:  subst([a1=a(t1),a2=a(t2),v1=v(t1),v2=v(t2),r=r(t1)-r(t2),x=y],z4)$
   z9:  subst([a1=a(t1),a2=a(t2),v1=v(t1),v2=v(t2),r=r(t1)-r(t2),x=y],z5)$
   z10:  diff(z8,t1,1)$
   z11:  diff(z8,t2,1)$
   z12:  subst([z6=j1,z7=j2,a(t1)=a1,a(t2)=a2,v10=v1,v20=v2,a10=a1,
                                  a20=a2,v(t1)=v1,v(t2)=v2],z11+z10)$
        ratsubst(x,y,%)$
```

```
      ratsubst(r+r(t2),r(t1),%)$
  z12:  vectorsimp(%),expandall$
  z13:  diff(z9,t1,1)$
  z14:  diff(z9,t2,1)$
  z15:  subst([z6=j1,z7=j2,a(t1)=a1,a(t2)=a2,v10=v1,v20=v2,a10=a1,
                                  a20=a2,v(t1)=v1,v(t2)=v2],z13+z14)$
        ratsubst(x,y,%)$
        ratsubst(r+r(t2),r(t1),%)$
  z15:  vectorsimp(%),expandall$
  z16:  v1.z12+v2.z15$
  z17:  %,dotdistrib:true$
        expand(%)$

/* z18 is the Hamiltonian with (v,r) but not a nice form     */

  z18: expand(z3-z17-1)$
  z19: subst([r.v1=v1.r,r.v2=v2.r,v2.v1=v1.v2,r.a1=a1.r,r.a2=a2.r],%)$
       subst([a1.a2=a2.a1,v1.a1=a1.v1,v2.a1=a1.v2,v1.a2=a2.v1],%)$
       subst([v2.a2=a2.v2,
              r.j1=j1.r,r.j2=j2.r,v1.j2=j2.v1,v2.j1=j1.v2],%)$
       subst([r.v1=v1.r,r.v2=v2.r,v2.v1=v1.v2,r.a1=a1.r],%)$
       subst([r.a2=a2.r, a1.a2=a2.a1,v1.a1=a1.v1],%)$
       subst([v2.a1=a1.v2,v1.a2=a2.v1,v2.a2=a2.v2],%)$
       subst([v1.r=k1,v2.r=k2,v1.v2=k3,a1.r=k4,a2.r=k5,a2.a1=k6],%)$
       subst([a1.v1=k7,a1.v2=k8,a2.v1=k9,a2.v2=k10,v2.v2=k11],%)$
       subst([v1.v1=k12,j1.r=k13,j2.r=k14,
              j1.v2=k15,j2.v1=k16,z0=k17,z00=k18],%)$
       z20:  %,dotscrules: true$
             expand(%)$

       subst([k1=v1.r,k2=v2.r,k3=v1.v2,k4=a1.r,k5=a2.r,k6=a2.a1],%)$
       subst([k7=a1.v1,k8=a1.v2,k9=a2.v1,k10=a2.v2,k11=v2.v2,k12=v1.v1],%)$
       subst([k13=j1.r,k14=j2.r,k15=j1.v2,k16=j2.v1,
                                  k17=v1.v1,k18=v2.v2],%)$
  z21: expand(%);
       coeff(z21,c,0);
       coeff(z21,c,-2);
  z22: coeff(z21,c,-4);

/* Here z21 is the final results of Hamiltonian       */

  closefile(resh);

/*  For the further calculation of the Hamiltonian,            */
/*  the program outputs the coefficients of the order 1/c^4 .  */

  writefile(hv4);
  fortran(z22);
  closefile(hv4);

  logout();
```

## APPENDIX D

## DERIVATION OF THE HAMILTONIAN WITH H($\vec{p},\vec{r}$)

First,there is a program to calculate the Hamiltonian with $h_{2i}$, $i = 1,2,....9$ From the Poisson bracket formulation

$$\vec{v} = \dot{\vec{r}} = [\vec{r}, H] \tag{D.1}$$

$$\vec{a} = \dot{\vec{v}} = [\vec{v}, H] \tag{D.2}$$

$$\vec{j} = \dot{\vec{a}} = [\vec{a}, H] \tag{D.3}$$

In the chapter 3, it was assumed that

$$H = (m_1 + m_2)c^2 + h_{00} + \frac{1}{c^2}(h_{10} + \epsilon h_{11} + \delta h_{12})$$
$$\frac{1}{c^4}(h_{20} + \epsilon h_{21} + \delta h_{22} + \epsilon^2 h_{23} + \delta^2 h_{24}$$
$$+\epsilon\delta h_{25} + \epsilon^2\delta h_{26} + \epsilon\delta^2 h_{27} + \epsilon^3 h_{28} + \delta^3 h_{29}) \tag{D.4}$$

Then $H$ can be written as

$$H = H_0(c^2) + H_1(c^0) + H_2(1/c^2) + H_3(1/c^4) \tag{D.5}$$

where

$$H_0(c^2) = (m_1 + m_2)c^2 \tag{D.6}$$

$$H_1(c^0) = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{e_1 e_2}{r} - \frac{g_1 g_2}{r} \tag{D.7}$$

$$H_2(1/c^2) = \frac{1}{c^2}[-\frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_1^4}{8m_2^3} - \frac{e_1 e_2 \vec{p}_1\cdot\vec{p}_2}{2m_1 m_2 r} - \frac{e_1 e_2(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1 m_2 r}$$
$$-\frac{g_1 g_2 \vec{p}_1\cdot\vec{p}_2}{2m_1 m_2 r} + \frac{g_1 g_2(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1 m_2 r} + \frac{g_1 g_2}{2r}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2})] \tag{D.8}$$

and

$$H_3(1/c^4) = \frac{1}{c^4}(h_{20} + \epsilon h_{21} + \delta h_{22} + \epsilon^2 h_{23} + \delta^2 h_{24}$$
$$+ \epsilon\delta h_{25} + \epsilon^2\delta h_{26} + \epsilon\delta^2 h_{27} + \epsilon^3 h_{28} + \delta^3 h_{29}) \tag{D.9}$$

From (D.1) one has

$$\vec{v}_1 = [\vec{r}_1, H] = \frac{\partial H}{\partial \vec{p}_1} = \frac{\vec{p}_1}{m_1} + \frac{\partial H_2}{\partial \vec{p}_1} + \frac{\partial H_3}{\partial \vec{p}_1} \qquad (D.10)$$

$$\vec{v}_2 = [\vec{r}_2, H] = \frac{\partial H}{\partial \vec{p}_2} = \frac{\vec{p}_2}{m_2} + \frac{\partial H_2}{\partial \vec{p}_2} + \frac{\partial H_3}{\partial \vec{p}_2} \qquad (D.11)$$

In the program, some terms in (D.10)-(D.11) are set

$$\frac{\vec{d}}{c^2} = \frac{\partial H_2}{\partial \vec{p}_1} \qquad (D.12)$$

$$\frac{\vec{b}}{c^2} = \frac{\partial H_2}{\partial \vec{p}_2} \qquad (D.13)$$

$$\frac{\vec{e}_1}{c^4} = \frac{\partial H_3}{\partial \vec{p}_1} \qquad (D.14)$$

$$\frac{\vec{e}_2}{c^4} = \frac{\partial H_3}{\partial \vec{p}_2} \qquad (D.15)$$

since $H_2$ has being obtained in chapter 4, then

$$\vec{d} = -\frac{\vec{p}_1^6}{2m_1^3} - \frac{e_1e_2\vec{p}_2}{2m_1m_2r} - \frac{g_1g_2\vec{p}_2}{2m_1m_2r} - \frac{e_1e_2\hat{r}\hat{r}\cdot\vec{p}_2}{2m_1m_2r} + \frac{g_1g_2\vec{p}_1}{m_1^2r} + \frac{g_1g_2\hat{r}\hat{r}\cdot\vec{p}_2}{2m_1m_2r} \qquad (D.16)$$

$$\vec{b} = -\frac{\vec{p}_2^6}{2m_2^3} - \frac{e_1e_2\vec{p}_1}{2m_1m_2r} - \frac{g_1g_2\vec{p}_1}{2m_1m_2r} - \frac{e_1e_2\hat{r}\hat{r}\cdot\vec{p}_1}{2m_1m_2r} + \frac{g_1g_2\vec{p}_2}{m_2^2r} + \frac{g_1g_2\hat{r}\hat{r}\cdot\vec{p}_1}{2m_1m_2r} \qquad (D.17)$$

Substituting (D.10)-(D.17) into

$$\begin{aligned} H =& (m_1+m_2)c^2 + \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 + \frac{e_1e_2}{r} - \frac{g_1g_2}{r} \\ &+ \frac{1}{c^2}[\frac{3}{8}m_1\vec{v}_1^4 + \frac{3}{8}m_2\vec{v}_2^4 + \frac{(\vec{v}_1\cdot\vec{v}_2)e_1e_2}{2r} + \frac{e_1e_2(\hat{r}\cdot\vec{v}_1)(\hat{r}\cdot\vec{v}_2)}{2r} \\ &- \frac{g_1g_2}{2r}(\vec{v}_1^2+\vec{v}_2^2) - \frac{g_1g_2(\hat{r}\cdot\vec{v}_1)(\hat{r}\cdot\vec{v}_2)}{2r} + \frac{(\vec{v}_1\cdot\vec{v}_2)g_1g_2}{2r}] \end{aligned} \qquad (D.18)$$

and collecting terms through order $1/c^4$, one calls the result (D.18').

In the Appendix C the Hamiltonian with $(\vec{j}, \vec{a}, \vec{v}, \vec{r})$ has been found in order $1/c^4$. Since one needs to find the Hamiltonian with $(\vec{p}, \vec{r})$ up to order $1/c^4$, one only needs to calculate $\vec{j}, \vec{a}, \vec{v}$ in order $c^0$. They can be obtained by (D.1),(D.2), (D.3)

$$\vec{v}_1 = [\vec{r}, H] = \frac{\partial H}{\partial \vec{p}_1} = \frac{\vec{p}_1}{m_1} + O(1/c^2) \qquad (D.19)$$

$$\vec{v}_2 = [\vec{r}, H] = \frac{\partial H}{\partial \vec{p}_2} = \frac{\vec{p}_2}{m_2} + O(1/c^2) \tag{D.20}$$

$$\vec{a}_1 = [\vec{v}_1, H] = \frac{(e_1 e_2 - g_1 g_2)}{m_1} \frac{\hat{r}}{r^2} + O(1/c^2) \tag{D.21}$$

$$\vec{a}_2 = [\vec{v}_2, H] = -\frac{(e_1 e_2 - g_1 g_2)}{m_2} \frac{\hat{r}}{r^2} + O(1/c^2) \tag{D.22}$$

$$\begin{aligned} \vec{j}_1 = [\vec{a}_1, H] =& \frac{(e_1 e_2 - g_1 g_2)}{m_1 r^3} \left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right) \\ & - \frac{3(e_1 e_2 - g_1 g_2)}{m_1 r^3} \left(\frac{(\vec{p}_1 \cdot \hat{r})\hat{r}}{m_1} - \frac{(\vec{p}_2 \cdot \hat{r})\hat{r}}{m_2}\right) + O(1/c^2) \end{aligned} \tag{D.23}$$

$$\begin{aligned} \vec{j}_2 = [\vec{a}_2, H] =& -\frac{(e_1 e_2 - g_1 g_2)}{m_2 r^3} \left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right) \\ & + \frac{3(e_1 e_2 - g_1 g_2)}{m_2 r^3} \left(\frac{(\vec{p}_1 \cdot \hat{r})\hat{r}}{m_1} - \frac{(\vec{p}_2 \cdot \hat{r})\hat{r}}{m_2}\right) + O(1/c^2) \end{aligned} \tag{D.24}$$

substituting $(D.19) - (D.24)$ into

$$\begin{aligned} H(\vec{v}, \vec{a}, \vec{j}, \vec{r}, 1/c^4) =& \frac{5m_1\vec{v}_1^6}{16} + \frac{5m_2\vec{v}_2^6}{16} \\ & + \frac{e_1 e_2}{r}\Big[\frac{\vec{v}_1 \cdot \vec{v}_2}{8}(\vec{v}_2^2 - (\vec{v}_2 \cdot \hat{r})^2) + \frac{\vec{v}_1 \cdot \vec{v}_2}{8}(\vec{v}_1^2 - (\vec{v}_1 \cdot \hat{r})^2) \\ & + \frac{3(\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r})}{8}(\vec{v}_2^2 - (\vec{v}_2 \cdot \hat{r})^2) + \frac{3(\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r})}{8}(\vec{v}_1^2 - (\vec{v}_1 \cdot \hat{r})^2) \\ & - \frac{\vec{a}_1 \cdot \vec{r}}{8}(\vec{v}_2^2 - (\vec{v}_2 \cdot \hat{r})^2) + \frac{\vec{a}_2 \cdot \vec{r}}{8}(\vec{v}_1^2 - (\vec{v}_1 \cdot \hat{r})^2) \\ & + \frac{\vec{v}_1 \cdot \vec{v}_2}{8}(\vec{a}_1 \cdot \vec{r} - \vec{a}_2 \cdot \vec{r}) + \frac{3(\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r})}{8}(\vec{a}_1 \cdot \vec{r} - \vec{a}_2 \cdot \vec{r}) \\ & + \frac{5r}{8}((\vec{a}_1 \cdot \vec{v}_2)(\vec{v}_1 \cdot \hat{r}) - (\vec{a}_2 \cdot \vec{v}_1)(\vec{v}_2 \cdot \hat{r})) \\ & + \frac{r}{4}((\vec{a}_1 \cdot \vec{v}_2)(\vec{v}_2 \cdot \hat{r}) - (\vec{a}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \hat{r})) \\ & + \frac{3r}{8}((\vec{a}_2 \cdot \vec{v}_2)(\vec{v}_1 \cdot \hat{r}) - (\vec{a}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \hat{r})) \\ & + \frac{1}{8}((\vec{a}_1 \cdot \vec{r})(\vec{a}_2 \cdot \vec{r}) - \vec{r}^2(\vec{a}_1 \cdot \vec{a}_2)) \\ & + \frac{\vec{r}^2}{8}(3\vec{j}_1 \cdot \vec{v}_2 + 3\vec{j}_2 \cdot \vec{v}_1 - (\vec{j}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r}) - (\vec{j}_2 \cdot \hat{r}))(\vec{v}_1 \cdot \hat{r}) \\ & - \frac{\vec{v}_1^2\vec{v}_2^2}{8} + \frac{(\vec{v}_1 \cdot \vec{v}_2)^2}{4} + \frac{\vec{v}_1^2(\vec{v}_1 \cdot \hat{r})^2}{8} + \frac{\vec{v}_2^2(\vec{v}_2 \cdot \hat{r})^2}{8} \end{aligned}$$

$$
\begin{aligned}
&- \frac{3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2}{8}] \\
&+ \frac{g_1g_2}{r}[-\frac{3}{8}(\vec{v}_1^4+\vec{v}_2^4)+\frac{3}{8}((\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})^3+(\vec{v}_1\cdot\hat{r})^3(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{\vec{v}_1^2}{8}(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}))+\frac{\vec{v}_2^2}{8}(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{3}{8}(\vec{v}_2^2(\vec{v}_1\cdot\hat{r})^2+\vec{v}_1^2(\vec{v}_2\cdot\hat{r})^2-(\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})^2-(\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})^2) \\
&- \frac{3}{8}(\vec{v}_2^2(\vec{a}_1\cdot\vec{r})-\vec{v}_1^2(\vec{a}_2\cdot\vec{r})) \\
&+ \frac{3}{8}(\vec{a}_1\cdot\vec{r})(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) \\
&- \frac{3}{8}(\vec{a}_2\cdot\vec{r})(\vec{v}_1\cdot\vec{v}_2-(\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})) \\
&- \frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{v}_2\cdot\hat{r})^2-(\vec{a}_2\cdot\vec{r})(\vec{v}_1\cdot\hat{r})^2) \\
&+ \frac{3r}{8}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_2\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{r}{8}((\vec{a}_2\cdot\vec{v}_2)(\vec{v}_1\cdot\hat{r})-(\vec{a}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\hat{r})) \\
&+ \frac{r}{4}((\vec{a}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\hat{r})-(\vec{a}_2\cdot\vec{v}_1)(\vec{v}_1\cdot\hat{r})) \\
&- \frac{1}{8}((\vec{a}_1\cdot\vec{r})(\vec{a}_2\cdot\vec{r})+\vec{r}^2(\vec{a}_1\cdot\vec{a}_2)) \\
&+ \frac{\vec{r}^2}{8}(\vec{j}_1\cdot\vec{v}_2+\vec{j}_2\cdot\vec{v}_1+(\vec{j}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r})+(\vec{j}_2\cdot\hat{r})(\vec{v}_1\cdot\hat{r})) \\
&- \frac{\vec{v}_1^2\vec{v}_2^2}{8}+\frac{(\vec{v}_1\cdot\vec{v}_2)^2}{4}+\frac{3(\vec{v}_1\cdot\hat{r})^2(\vec{v}_2\cdot\hat{r})^2}{8}] \qquad (D.25)
\end{aligned}
$$

Combination of (D.18') and (D.25) leads to

$$
\begin{aligned}
H(\vec{p},\vec{r}) &= (m_1+m_2)c^2+\frac{\vec{p}_1^2}{2m_1}+\frac{\vec{p}_2^2}{2m_2}+\frac{(e_1e_2)}{r}-\frac{(g_1g_2)}{r} \\
&+ \frac{1}{c^2}\{-\frac{\vec{p}_1^4}{8m_1^3}-\frac{\vec{p}_1^4}{8m_2^3}-\frac{(e_1e_2)\vec{p}_1\cdot\vec{p}_2}{2m_1m_2r}-\frac{(e_1e_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2r} \\
&- \frac{(g_1g_2)\vec{p}_1\cdot\vec{p}_2}{2m_1m_2r}+\frac{(g_1g_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2r}+\frac{(g_1g_2)}{2r}(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2})\} \\
&+ \frac{1}{c^4}\{-\frac{5\vec{p}_1^6}{16m_1^5}-\frac{5\vec{p}_1^6}{16m_2^5} \\
&+ \frac{(e_1e_2)}{r}[-\frac{5(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2})-\frac{(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}) \\
&+ \frac{\vec{p}_1^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}+\frac{\vec{p}_2^2(\vec{p}_1\cdot\hat{r})^2}{8m_1^2m_2^2}-\frac{\vec{p}_1^2\vec{p}_2^2}{8m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\vec{p}_2)^2}{4m_1^2m_2^2}-\frac{3(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{8m_1m_2}\Big(\frac{\vec{p}_1^{\,2}+(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{\vec{p}_2^{\,2}+(\vec{p}_2\cdot\hat{r})^2}{m_2^2}\Big)\Big] \\
&+\frac{(e_1e_2)^2}{r^2}\Big[-\frac{5}{8m_1m_2}\Big(\frac{\vec{p}_1^{\,2}}{m_1}+\frac{\vec{p}_2^{\,2}}{m_2}\Big)+\frac{7}{8m_1m_2}\Big(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\Big) \\
&+\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}\Big(\frac{1}{m_1}-\frac{1}{m_2}\Big)+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\Big(\frac{1}{m_1}+\frac{1}{m_2}\Big)\Big] \\
&+\frac{(g_1g_2)}{r}\Big[\frac{9}{8}\Big(\frac{\vec{p}_1^{\,4}}{m_1^4}+\frac{\vec{p}_1^{\,4}}{m_2^4}\Big)-\frac{3(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}\Big(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}\Big) \\
&-\frac{5(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}\Big(\frac{\vec{p}_1^{\,2}}{m_1^2}+\frac{\vec{p}_2^{\,2}}{m_2^2}\Big)+\frac{5(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{8m_1m_2}\Big(\frac{\vec{p}_1^{\,2}}{m_1^2}+\frac{\vec{p}_2^{\,2}}{m_2^2}\Big) \\
&+\frac{3(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{8m_1m_2}\Big(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}\Big) \\
&+\frac{3\vec{p}_1^{\,2}(\vec{p}_1\cdot\hat{r})^2+3\vec{p}_2^{\,2}(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}-\frac{(\vec{p}_1\cdot\vec{p}_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1^2m_2^2} \\
&-\frac{\vec{p}_1^{\,2}\vec{p}_2^{\,2}}{8m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\vec{p}_2)^2}{4m_1^2m_2^2}+\frac{3(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}\Big] \\
&+\frac{(g_1g_2)^2}{r^2}\Big[\frac{3}{8m_1m_2}\Big(\frac{\vec{p}_1^{\,2}}{m_1}+\frac{\vec{p}_2^{\,2}}{m_2}\Big)-\frac{5}{8m_1m_2}\Big(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\Big) \\
&-\Big(\frac{\vec{p}_1^{\,2}}{2m_1^3}+\frac{\vec{p}_2^{\,2}}{2m_2^3}\Big)+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\Big(\frac{1}{m_1}+\frac{1}{m_2}\Big)\Big] \\
&+\frac{(e_1e_2)(g_1g_2)}{r^2}\Big[-\frac{1}{4m_1m_2}\Big(\frac{\vec{p}_1^{\,2}}{m_1}+\frac{\vec{p}_2^{\,2}}{m_2}\Big)-\frac{1}{4m_1m_2}\Big(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\Big) \\
&+\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}\Big(\frac{1}{m_1}+\frac{1}{m_2}\Big)\Big] \\
&-\frac{(g_1g_2)^2(e_1e_2)+(g_1g_2)(e_1e_2)^2}{4m_1m_2r^3} \\
&+\frac{(e_1e_2)^3+(g_1g_2)^3}{4m_1m_2r^3} \\
&+\vec{p}_1\cdot\vec{e}_1+\vec{p}_2\cdot\vec{e}_2\Big\} \qquad (D.26)
\end{aligned}
$$

(D.26) is the results of the program given here.

To obtain the final Hamiltonian, one can solve the differential equations after (D.26) is set to be equal to $H$ (D.4). From the assumptions, there are nine equations, which correspond to the coefficients of the same $(e_1e_2)^m(g_1g_2)^n$, m,n=0,1,2,3, power.

The equality of the coefficients of $(e_1e_2)^0(g_1g_2)^0$ leads to

$$\vec{p}_1 \cdot \frac{\partial h_{20}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{20}}{\partial \vec{p}_2} - \frac{5\vec{p}_1^6}{16m_1^5} - \frac{5\vec{p}_2^6}{16m_2^5} = h_{20} \tag{D.27}$$

The solution is

$$h_{20} = \frac{\vec{p}_1^6}{16m_1^5} + \frac{\vec{p}_2^6}{16m_2^5} \tag{D.28}$$

The equality of the coefficients of $(e_1e_2)^1(g_1g_2)^0$ leads to

$$\begin{aligned}
&\vec{p}_1 \cdot \frac{\partial h_{21}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{21}}{\partial \vec{p}_2} \\
&[-\frac{5(\vec{p}_1 \cdot \vec{p}_2)}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}) - \frac{(\vec{p}_1 \cdot \vec{p}_2)}{8m_1m_2}(\frac{(\vec{p}_1 \cdot \hat{r})^2}{m_1^2} + \frac{(\vec{p}_2 \cdot \hat{r})^2}{m_2^2}) \\
&+ \frac{\vec{p}_1^2(\vec{p}_2 \cdot \hat{r})^2}{8m_1^2m_2^2} + \frac{\vec{p}_2^2(\vec{p}_1 \cdot \hat{r})^2}{8m_1^2m_2^2} - \frac{\vec{p}_1^2\vec{p}_2^2}{8m_1^2m_2^2} + \frac{(\vec{p}_1 \cdot \vec{p}_2)^2}{4m_1^2m_2^2} - \frac{3(\vec{p}_1 \cdot \hat{r})^2(\vec{p}_2 \cdot \hat{r})^2}{8m_1^2m_2^2} \\
&- \frac{3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{8m_1m_2}(\frac{\vec{p}_1^2 + (\vec{p}_1 \cdot \hat{r})^2}{m_1^2} + \frac{\vec{p}_2^2 + (\vec{p}_2 \cdot \hat{r})^2}{m_2^2})] \\
&= h_{21}
\end{aligned} \tag{D.29}$$

The solution is

$$\begin{aligned}
h_{21} =& \frac{5(\vec{p}_1 \cdot \vec{p}_2)}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}) + \frac{(\vec{p}_1 \cdot \vec{p}_2)}{24m_1m_2}(\frac{(\vec{p}_1 \cdot \hat{r})^2}{m_1^2} + \frac{(\vec{p}_2 \cdot \hat{r})^2}{m_2^2}) \\
&- \frac{\vec{p}_1^2(\vec{p}_2 \cdot \hat{r})^2}{24m_1^2m_2^2} - \frac{\vec{p}_2^2(\vec{p}_1 \cdot \hat{r})^2}{24m_1^2m_2^2} + \frac{\vec{p}_1^2\vec{p}_2^2}{24m_1^2m_2^2} - \frac{(\vec{p}_1 \cdot \vec{p}_2)^2}{12m_1^2m_2^2} + \frac{(\vec{p}_1 \cdot \hat{r})^2(\vec{p}_2 \cdot \hat{r})^2}{8m_1^2m_2^2} \\
&+ \frac{(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{8m_1m_2}(\frac{\vec{p}_1^2 + (\vec{p}_1 \cdot \hat{r})^2}{m_1^2} + \frac{\vec{p}_2^2 + (\vec{p}_2 \cdot \hat{r})^2}{m_2^2}).
\end{aligned} \tag{D.30}$$

The equality of the coefficients of $(e_1e_2)^0(g_1g_2)^1$ leads to

$$\begin{aligned}
&\vec{p}_1 \cdot \frac{\partial h_{22}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{22}}{\partial \vec{p}_2} \\
&+ \frac{9}{8}(\frac{\vec{p}_1^4}{m_1^4} + \frac{\vec{p}_1^4}{m_2^4}) - \frac{3(\vec{p}_1 \cdot \vec{p}_2)}{8m_1m_2}(\frac{(\vec{p}_1 \cdot \hat{r})^2}{m_1^2} + \frac{(\vec{p}_2 \cdot \hat{r})^2}{m_2^2}) - \frac{5(\vec{p}_1 \cdot \vec{p}_2)}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}) \\
&+ \frac{3(\vec{p}_2 \cdot \hat{r})(\vec{p}_1 \cdot \hat{r})}{8m_1m_2}(\frac{(\vec{p}_1 \cdot \hat{r})^2}{m_1^2} + \frac{(\vec{p}_2 \cdot \hat{r})^2}{m_2^2}) + \frac{5(\vec{p}_2 \cdot \hat{r})(\vec{p}_1 \cdot \hat{r})}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}) \\
&+ \frac{3\vec{p}_1^2(\vec{p}_1 \cdot \hat{r})^2 + 3\vec{p}_2^2(\vec{p}_2 \cdot \hat{r})^2}{8m_1^2m_2^2} - \frac{(\vec{p}_1 \cdot \vec{p}_2)(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{2m_1^2m_2^2} \\
&- \frac{\vec{p}_1^2\vec{p}_2^2}{8m_1^2m_2^2} + \frac{(\vec{p}_1 \cdot \vec{p}_2)^2}{4m_1^2m_2^2} + \frac{3(\vec{p}_1 \cdot \hat{r})^2(\vec{p}_2 \cdot \hat{r})^2}{8m_1^2m_2^2} \\
&= h_{22}
\end{aligned} \tag{D.31}$$

The solution is

$$
\begin{aligned}
h_{22} =& -\frac{3}{4}\left(\frac{\vec{p}_1^4}{m_1^4}+\frac{\vec{p}_1^4}{m_2^4}\right)+\frac{(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}\right)+\frac{5(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}\left(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2}\right)\\
&-\frac{(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{8m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}\right)-\frac{5(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{24m_1m_2}\left(\frac{\vec{p}_1^2}{m_1^2}+\frac{\vec{p}_2^2}{m_2^2}\right)\\
&-\frac{\vec{p}_1^2(\vec{p}_1\cdot\hat{r})^2+\vec{p}_2^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\vec{p}_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{6m_1^2m_2^2}\\
&+\frac{\vec{p}_1^2\vec{p}_2^2}{24m_1^2m_2^2}-\frac{(\vec{p}_1\cdot\vec{p}_2)^2}{12m_1^2m_2^2}+\frac{(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2}. \qquad (D.32)
\end{aligned}
$$

The equality of the coefficients of $(e_1e_2)^2(g_1g_2)^0$ leads to

$$
\begin{aligned}
&\vec{p}_1\cdot\frac{\partial h_{23}}{\partial\vec{p}_1}+\vec{p}_2\cdot\frac{\partial h_{23}}{\partial\vec{p}_2}\\
&-\frac{5}{8m_1m_2}\left(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2}\right)+\frac{7}{8m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\right)\\
&+\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\\
&=h_{23} \qquad (D.33)
\end{aligned}
$$

The solution is

$$
\begin{aligned}
h_{23} =& \frac{5}{8m_1m_2}\left(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2}\right)-\frac{7}{8m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}\right.\\
&\left.+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\right)-\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \qquad (D.34)
\end{aligned}
$$

The equality of the coefficients of $(e_1e_2)^0(g_1g_2)^2$ leads to

$$
\begin{aligned}
&\vec{p}_1\cdot\frac{\partial h_{24}}{\partial\vec{p}_1}+\vec{p}_2\cdot\frac{\partial h_{24}}{\partial\vec{p}_2}\\
&+\frac{3}{8m_1m_2}\left(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2}\right)-\frac{5}{8m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\right)\\
&-\left(\frac{\vec{p}_1^2}{2m_1^3}+\frac{\vec{p}_2^2}{2m_2^3}\right)+\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\\
&=h_{24} \qquad (D.35)
\end{aligned}
$$

The solution is

$$
\begin{aligned}
h_{24} =& -\frac{3}{8m_1m_2}\left(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2}\right)+\frac{5}{8m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\right)\\
&+\left(\frac{\vec{p}_1^4}{2m_1^3}+\frac{\vec{p}_2^4}{2m_2^3}\right)-\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right). \qquad (D.36)
\end{aligned}
$$

The equality of the coefficients of $(e_1e_2)^1(g_1g_2)^1$ leads to

$$\begin{aligned}
&\vec{p}_1 \cdot \frac{\partial h_{25}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{25}}{\partial \vec{p}_2} \\
&- \frac{1}{4m_1m_2}\left(\frac{\vec{p}_1^2}{m_1} + \frac{\vec{p}_2^2}{m_2}\right) - \frac{1}{4m_1m_2}\left(\frac{(\vec{p}_1 \cdot \hat{r})^2}{m_1} + \frac{(\vec{p}_2 \cdot \hat{r})^2}{m_2}\right) \\
&+ \frac{(\vec{p}_1 \cdot \vec{p}_2)}{2m_1m_2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \\
&= h_{25}
\end{aligned} \tag{D.37}$$

The solution is

$$\begin{aligned}
h_{25} =& \frac{1}{4m_1m_2}\left(\frac{\vec{p}_1^2}{m_1} + \frac{\vec{p}_2^2}{m_2}\right) + \frac{1}{4m_1m_2}\left(\frac{(\vec{p}_1 \cdot \hat{r})^2}{m_1} + \frac{(\vec{p}_2 \cdot \hat{r})^2}{m_2}\right) \\
&- \frac{(\vec{p}_1 \cdot \vec{p}_2)}{2m_1m_2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).
\end{aligned} \tag{D.38}$$

The equality of the coefficients of $(e_1e_2)^2(g_1g_2)^1$ leads to

$$\vec{p}_1 \cdot \frac{\partial h_{26}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{26}}{\partial \vec{p}_2} - \frac{1}{4m_1m_2r^3} = h_{26} \tag{D.39}$$

The solution is

$$h_{26} = -\frac{1}{4m_1m_2r^3}. \tag{D.40}$$

The equality of the coefficients of $(e_1e_2)^1(g_1g_2)^2$ leads to

$$\vec{p}_1 \cdot \frac{\partial h_{27}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{27}}{\partial \vec{p}_2} - \frac{1}{4m_1m_2r^3} = h_{27} \tag{D.41}$$

The solution is

$$h_{27} = -\frac{1}{4m_1m_2r^3}. \tag{D.42}$$

The equality of the coefficients of $(e_1e_2)^3(g_1g_2)^0$ leads to

$$\vec{p}_1 \cdot \frac{\partial h_{28}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{28}}{\partial \vec{p}_2} + \frac{1}{4m_1m_2r^3} = h_{28} \tag{D.43}$$

The solution is

$$h_{28} = \frac{1}{4m_1m_2r^3}. \tag{D.44}$$

The equality of the coefficients of $(e_1e_2)^0(g_1g_2)^3$ leads to

$$\vec{p}_1 \cdot \frac{\partial h_{29}}{\partial \vec{p}_1} + \vec{p}_2 \cdot \frac{\partial h_{29}}{\partial \vec{p}_2} + \frac{1}{4m_1m_2r^3} = h_{29} \qquad (D.45)$$

The solution is

$$h_{29} = \frac{1}{4m_1m_2r^3}. \qquad (D.46)$$

One can include all the above to give the final Hamiltonian

$$\begin{aligned}
H(\vec{p},\vec{r}) = {} & (m_1+m_2)c^2 + \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{(e_1e_2)}{r} - \frac{(g_1g_2)}{r} \\
& + \frac{1}{c^2}[-\frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_1^4}{8m_2^3} - \frac{(e_1e_2)\vec{p}_1\cdot\vec{p}_2}{2m_1m_2} - \frac{(e_1e_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2} \\
& - \frac{(g_1g_2)\vec{p}_1\cdot\vec{p}_2}{2m_1m_2} + \frac{(g_1g_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{2m_1m_2} + \frac{(g_1g_2)}{2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2})] \\
& + \frac{1}{c^4}\{\frac{\vec{p}_1^6}{16m_1^5} + \frac{\vec{p}_1^6}{16m_2^5} \\
& + \frac{(e_1e_2)}{r}[\frac{5(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}) + \frac{(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2} + \frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}) \\
& - \frac{\vec{p}_1^2(\vec{p}_2\cdot\hat{r})^2}{24m_1^2m_2^2} - \frac{\vec{p}_2^2(\vec{p}_1\cdot\hat{r})^2}{24m_1^2m_2^2} + \frac{\vec{p}_1^2\vec{p}_2^2}{24m_1^2m_2^2} - \frac{(\vec{p}_1\cdot\vec{p}_2)^2}{12m_1^2m_2^2} + \frac{(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2} \\
& + \frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{8m_1m_2}(\frac{\vec{p}_1^2 + (\vec{p}_1\cdot\hat{r})^2}{m_1^2} + \frac{\vec{p}_2^2 + (\vec{p}_2\cdot\hat{r})^2}{m_2^2})] \\
& + \frac{(e_1e_2)^2}{r^2}[\frac{5}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1} + \frac{\vec{p}_2^2}{m_2}) - \frac{7}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1} + \frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}) \\
& - \frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}(\frac{1}{m_1} + \frac{1}{m_2}) + \frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}(\frac{1}{m_1} + \frac{1}{m_2})] \\
& + \frac{(g_1g_2)}{r}[-\frac{3}{4}(\frac{\vec{p}_1^4}{m_1^4} + \frac{\vec{p}_1^4}{m_2^4}) + \frac{(\vec{p}_1\cdot\vec{p}_2)}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2} + \frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}) \\
& - \frac{(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1^2} + \frac{(\vec{p}_2\cdot\hat{r})^2}{m_2^2}) - \frac{5(\vec{p}_2\cdot\hat{r})(\vec{p}_1\cdot\hat{r})}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2}) \\
& - \frac{\vec{p}_1^2(\vec{p}_1\cdot\hat{r})^2 + \vec{p}_2^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2} + \frac{(\vec{p}_1\cdot\vec{p}_2)(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{6m_1^2m_2^2} \\
& + \frac{\vec{p}_1^2\vec{p}_2^2}{24m_1^2m_2^2} - \frac{(\vec{p}_1\cdot\vec{p}_2)^2}{12m_1^2m_2^2} + \frac{(\vec{p}_1\cdot\hat{r})^2(\vec{p}_2\cdot\hat{r})^2}{8m_1^2m_2^2} + \frac{5(\vec{p}_1\cdot\vec{p}_2)}{24m_1m_2}(\frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2})] \\
& + \frac{(g_1g_2)^2}{r^2}[-\frac{3}{8m_1m_2}(\frac{\vec{p}_1^2}{m_1} + \frac{\vec{p}_2^2}{m_2}) + \frac{5}{8m_1m_2}(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1} + \frac{(\vec{p}_2\cdot\hat{r})^2}{m_2})
\end{aligned}$$

$$+\left(\frac{\vec{p}_1^4}{2m_1^3}+\frac{\vec{p}_2^4}{2m_2^3}\right)-\frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{4m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)]$$
$$+\frac{(e_1e_2)(g_1g_2)}{r^2}[\frac{1}{4m_1m_2}\left(\frac{\vec{p}_1^2}{m_1}+\frac{\vec{p}_2^2}{m_2}\right)+\frac{1}{4m_1m_2}\left(\frac{(\vec{p}_1\cdot\hat{r})^2}{m_1}+\frac{(\vec{p}_2\cdot\hat{r})^2}{m_2}\right)$$
$$-\frac{(\vec{p}_1\cdot\vec{p}_2)}{2m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)]$$
$$-\frac{(g_1g_2)^2(e_1e_2)+(g_1g_2)(e_1e_2)^2}{4m_1m_2r^3}$$
$$+\frac{(e_1e_2)^3+(g_1g_2)^3}{4m_1m_2r^3}\} \qquad (D.47)$$

If it is reprensented in c.m. system, using total mass and reduced mass, one has

$$\vec{p}_1=-\vec{p}_2=\vec{p}$$
$$m_1+m_2=m$$
$$\frac{m_1m_2}{m_1+m_2}=\mu$$

The equation has the form

$$H(\vec{p},\vec{r})=mc^2+\frac{\vec{p}^2}{2\mu}+\frac{(e_1e_2)}{r}-\frac{(g_1g_2)}{r}$$
$$+\frac{1}{c^2}[-\frac{p^4}{8}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)+\frac{p^2(e_1e_2)}{2\mu r}+\frac{(\vec{p}\cdot\hat{r})^2(e_1e_2)}{2\mu r}$$
$$+\frac{p^2(g_1g_2)}{2\mu r}+\frac{p^2(g_1g_2)}{2r}\frac{m-2\mu}{\mu^2m}-\frac{(\vec{p}\cdot\hat{r})^2(g_1g_2)}{2\mu r}]$$
$$+\frac{1}{c^4}\{\frac{p^6}{16}\left(\frac{1}{m_1^5}+\frac{1}{m_2^5}\right)+\frac{(e_1e_2)}{r}[-\frac{p^4(5m-9\mu)}{24\mu^3m^2}$$
$$-\frac{p^2(\vec{p}\cdot\hat{r})^2(2m-3\mu)}{12\mu^3m^2}+\frac{(\vec{p}\cdot\hat{r})^2(3\mu-m)}{8\mu^3m^2}]$$
$$+\frac{(e_1e_2)^2}{r^2}[\frac{9p^2}{8\mu^2m}-\frac{9(\vec{p}\cdot\hat{r})^4}{8\mu^2m}]$$
$$-\frac{(g_1g_2)}{r}[\frac{p^4(9m^2-31\mu m+9\mu^2)}{24\mu^4m^2}+\frac{p^2(\vec{p}\cdot\hat{r})^2(m-3\mu)}{12\mu^3m^2}+\frac{(\vec{p}\cdot\hat{r})^4(3\mu-m)}{8\mu^3m^2}]$$
$$+\frac{(g_1g_2)^2}{r^2}[\frac{p^2(4m-15\mu)}{8\mu^3m}+\frac{7(\vec{p}\cdot\hat{r})^2}{8\mu^2m}]$$
$$+\frac{(g_1g_2)(e_1e_2)}{r^2}[\frac{3p^2}{4\mu^3m}+\frac{(\vec{p}\cdot\hat{r})^2}{4\mu^2m}]$$
$$-\frac{(g_1g_2)^2(e_1e_2)+(e_1e_2)^2(g_1g_2)}{4\mu mr^3}+\frac{(e_1e_2)^3+(g_1g_2)^3}{4\mu mr^3}\} \qquad (D.48)$$

This is the Hamiltonian of two relativistic charged particles in the c.m. system.

# VITA

Dujiu Yang was born on December 29, 1963, in Beijing, the People's Republic of China. He attended the school at age of seven. After he graduated from high school, he enrolled Tsinghua University, Beijing, China, in 1981. He received a Bachelor degree of Science in Electron and Information Engineering (equivalent to Electrical Engineering ) in 1986 and then went to the U.S. Having studied in the department of physics at the University of Southern Mississippi for a year, he transfered to the University of Tennessee Space Institute, to pursue a Master of Science Degree in Physics on the field of two-body relativistic field theory. At the same time he worked for Energy Conversion Project as a Graduate Research Assistant.

He has many hobbies, including traveling, classical music, oil painting, history and photography.