To the Graduate Council:

I am submitting herewith a thesis written by Christos N. Mazis entitled "Error Estimates for Finite Difference Approximations to the Wave Equation." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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ERROR ESTIMATES FOR FINITE DIFFERENCE APPROXIMATIONS TO THE WAVE EQUATION

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ABSTRACT

In this thesis we consider the approximation of the solution of the wave equation in two dimensional rectangles by three finite difference methods, two of which are of second order of accuracy and one of fourth order of accuracy. We prove, using energy methods, stability and convergence of the difference schemes in the discrete ℓ_2-norm under suitable restrictions on their Curant number.

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CHAPTER I

INTRODUCTION

We consider the wave equation in two dimensions

$$u_{tt} = c^2[u_{xx} + u_{yy}] \,, \tag{1.1}$$

and we consider approximating $u = u(x, y, t)$ in $[0, 1] \times [0, 1] \times [0, T]$ satisfying (1.1), the homogeneous, Dirichlet boundary conditions

$$u(0, y, t) = u(x, 0, t) = u(1, y, t) = u(x, 1, t) = 0$$

for all $x \in [0, 1]$, $y \in [0, 1]$, $t \in [0, T]$ and the initial conditions

$$u(x, y, 0) = f(x, y) \,, \quad u_t(x, y, 0) = g(x, y)$$

$$(x, y) \in [0, 1] \times [0, 1]$$

where f and g are given smooth functions defined on $[0, 1] \times [0, 1]$, and $c > 0$ is a constant.

We choose positive integers J and N and define constant step sizes h and k by $h = \frac{1}{J + 1}$ and $k = \frac{T}{N}$ for the space and time discretization, respectively.

We partition $[0, 1] \times [0, 1] \times [0, T]$ by introducing the mesh points

$$(x_i, y_j, t^n) \ , \ 0 \le i \ , \ j \le J + 1 \ , \ 0 \le n \le N \ ,$$

$$x_i = ih \qquad i = 0, 1, \ldots, J + 1 \ ,$$

$$y_j = jh \qquad j = 0, 1, \ldots, J + 1 \ , \text{ and}$$

$$t^n = nk \qquad n = 0, 1, 2, \ldots, N \ .$$

We let, if u is the solution of (1.1),

$$u_{ij}^n = u(x_i, y_j, t^n) \ .$$

We seek $\{U_{ij}^n\}$, $0 \le n \le N$, $0 \le i \ , \ j \le J + 1$, as approximations to the values of the exact solution u_{ij}^n , $0 \le n \le N$, $0 \le i \ , \ j \le J + 1$. We will compute the U_{ij}^n as solutions of the difference schemes.

In Chapter II, we consider one-parameter family of generally implicit schemes

$$(1.2) \qquad \delta_t^2 U_{ij}^n = r^2[\alpha \diamond U_{ij}^{n+1} + (1 - 2\alpha) \diamond U_{ij}^n + \alpha \diamond U_{ij}^{n-1}] \ , \ \alpha \ge 0 \ ,$$

where

$$\diamond U_{ij} = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij} \ ,$$

and

$$\delta_t^2 U_{ij}^n = U_{ij}^{n+1} - 2U_{ij}^n + U_{ij}^{n-1} \quad , \quad r = \frac{ck}{h} \quad .$$

The scheme (1.2) has, in general, a formal accuracy of $O(h^2 + k^2)$, and this is due to Von Neumann [24]. Of course, $\alpha = 0$ yields the classical, explicit, Courant-Friedrichs-Lewy scheme [9]. We equip the scheme with standard boundary and initial conditions and prove, using energy methods, that for $\alpha \geq \frac{1}{4}$ the scheme is unconditionally stable and has a global error of $O(k^2 + h^2)$ in the discrete ℓ_2-norm. If $\alpha < \frac{1}{4}$, the scheme is stable only if $r \leq [2(1 - 4\alpha)]^{-1/2}$, i.e. $r \leq \frac{1}{\sqrt{2}}$ if $\alpha = 0$. Thus, we recover for the case of the initial-boundary-value problem on the unit square the results that Von Neumann L^2-stability analysis gives for the pure initial value problem on $\mathbb{R}^2$.

In Chapter III, we consider the application of an alternating direction method (ADI) to the solution of (1.1) in rectangles (we consider the unit square again). Lees [22] noticed that the Von Neumann scheme (1.2) can be perturbed to

$$(1.3) \qquad \delta_t^2 U_{ij}^n = r^2 \diamond [\alpha U_{ij}^{n+1} + (1 - 2\alpha) U_{ij}^n + U_{ij}^{n-1}] - r^4 \alpha^2 \delta_x^2 \delta_y^2 (\delta_t^2 U_{ij}^n) \ ,$$

where

$$\delta_x^2 U_{ij} = U_{i+1,j} - 2U_{ij} + U_{i-1,j} \ ,$$

$$\delta_y^2 U_{ij} = U_{i,j+1} - 2U_{ij} + U_{i,j-1} \ ,$$

without loss of accuracy of stability. The advantage of (1.3) is that

it is equivalent to a two-step scheme (see Chapter III), each of which requires the solution of one tridiagonal system of difference equations, in contrast with the implicit scheme (1.2) whose associated matrix is not tridiagonal. We prove that (1.3) has accuracy $O(h^2 + k^2)$ and is also unconditionally stable if $\alpha > \frac{1}{4}$ in the discrete ℓ_2-norm. We note that Lees [22] has also given a proof of stability and convergence for this method, but we have not used his technique of error estimation. Instead, we extend our energy techniques of Chapter II to the ADI case.

In Chapter IV, we consider the scheme

$$(1.4) \qquad [(1 - \rho)\diamond - 12\rho](U_{ij}^{n+1} + U_{ij}^{n-1}) + [2\square + (6 + 2\rho) + 24\rho]U_{ij}^{n} = 0 ,$$

where the operators $\square$, $\diamond$ are given by

$$\square u_{ij} = u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i+1,j+1} - 4u_{ij} ,$$

$$\diamond u_{ij} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} ,$$

and

$$\rho = \frac{1}{r^2} .$$

This scheme is one of the class of implicit, fourth-order scheme due to Collatz [8] which uses only 5 points at the advanced time level $n + 1$. We supplement (1.4) with the appropriate boundary and initial conditions and prove that (1.4) is (discrete ℓ_2) stable provided

$r \leq \frac{\sqrt{2}}{3}$ and of $O(h^4 + k^4)$. We are not aware of any proof of stability or convergence for such fourth-order schemes in the case of the initial-boundary value problem. For more details on the derivation of these and other schemes for the wave equation, we refer the reader to Lecture 3, "Sound Waves" [5].

Typically, each of the chapters that follow consists of three parts. In the first part, we prove the existence-uniqueness of the solutions of the difference schemes and specify the required initial and boundary conditions. In the second part, we study the consistency, and in the third part, we prove the convergence (and hence stability) of the various difference schemes in the discrete ℓ_2-sense.

CHAPTER II

SECOND ORDER ACCURATE DIFFERENCE APPROXIMATIONS TO THE WAVE EQUATION ON THE UNIT SQUARE

We consider approximating $u = u(x, y, t)$ in $[0, 1] \times [0, 1] \times [0, T]$ satisfying the wave equation

$$u_{tt} = c^2[u_{xx} + u_{yy}] , \tag{2.1}$$

$$(x, y, t) \in (0, 1) \times (0, 1) \times (0, T] ,$$

and the boundary conditions

$$u(0, y, t) = u(x, 0, t) = u(1, y, t) = u(x, 1, t) = 0 , \tag{2.2}$$

for all $x \in [0, 1]$, $y \in [0, 1]$, $t \in [0, T]$, and the initial conditions

$$u(x, y, 0) = f(x, y) , \; u_t(x, y, 0) = g(x, y) , \tag{2.3}$$

$$(x, y) \in [0, 1] \times [0, 1] ,$$

where f and g are given smooth functions defined on $[0, 1] \times [0, 1]$ and where $c > 0$ is a constant. For consistency we assume that f and g are zero on the perimeter of the unit square. We shall approximate the solution of (2.1) - (2.3) by using a one-parameter family of implicit

difference schemes due to Von Neumann, [24]. Using the notation introduced in Chapter I, we seek U_{ij}^n , $0 \le n \le N$, $0 \le i$, $j \le J + 1$ satisfying

$$\delta_t^2 U_{ij}^n = r^2[\alpha \diamond U_{ij}^{n+1} + (1 - 2\alpha) \diamond U_{ij}^n + \alpha \diamond U_{ij}^{n-1}], \quad \alpha \ge 0 , \tag{2.4}$$

$$1 \le i , j \le J , 1 \le n \le N - 1$$

where

$$\delta_t^2 U_{ij}^n = U_{ij}^{n+1} - 2U_{ij}^n + U_{ij}^{n-1} ,$$

$$0 \le i , j \le J + 1 , 1 \le n \le N - 1 ,$$

and

$$\diamond U_{ij} = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij} ,$$

$$1 \le i , j \le J ,$$

and $r = \frac{ck}{h}$ as usual.

The difference scheme (2.4) is supplemented with the boundary conditions

$$U_{0,j}^n = U_{i,0}^n = U_{J+1,j}^n = U_{i,J+1}^n = 0 , \tag{2.5}$$

$$0 \le i , j \le J + 1 , 0 \le n \le N .$$

We compute the initial conditions u^0_{ij} , u^1_{ij} as follows: first, we take

$$u^0_{ij} = f_{ij} = f(x_i, y_i) \ , \tag{2.6}$$

$$0 \le i \ , \ j \le J + 1 \ .$$

To motivate the choice for u^1_{ij} , approximating u^1_{ij} by a Taylor expansion, we have for $1 \le i \ , \ j \le J$,

$$u^1_{ij} = u(x_i, y_j, k) = u(x_i, y_j, 0) + Ku_t(x_i, y_j, 0)$$

$$+ \frac{K^2}{2} u_{tt}(x_i, y_j, 0) + \frac{K^3}{6} u_{ttt}(x_i, y_j, \theta_{ij}) \ ,$$

for some θ_{ij} such that $0 \le \theta_{ij} \le k$. Using (2.1) and (2.3), we obtain

$$u^1_{ij} = f_{ij} + kg_{ij} + \frac{c^2k^2}{2} [\Delta f]_{ij} + \frac{k^3}{6} u_{tt}(x_i, y_j, \theta_{ij}) \ , \tag{2.7}$$

$$0 \le \theta_{ij} \le k \ .$$

Hence, U^1_{ij} may be computed as

$$\begin{cases} U^1_{ij} = f_{ij} + kg_{ij} + \dfrac{c^2k^2}{2} (\Delta f)_{ij} \ , & 1 \le i,j \le J \\ \\ U^1_{ij} = 0 & \text{if } i \text{ or } j \text{ equals } 0 \text{ or } J + 1 \ , \end{cases} \tag{2.8}$$

provided the Laplacian Δf of f is known. Alternatively, if Δf is not known, we may approximate $(\Delta f)_{ij}$ by $\diamond f_{ij}/h^2$ and have the alternative formula

$$(2.8') \quad \begin{cases} U^1_{ij} = f_{ij} + kg_{ij} + \frac{r^2}{2} \diamond f_{ij}, \ 1 \le i, j \le J \ , \\ U^1_{ij} = 0 \qquad\qquad , \text{ if } i \text{ or } j \text{ equals } 0 \text{ or } J+1 \ . \end{cases}$$

We note that as far as the accuracy of the approximations (2.8) and (2.8') is concerned, we have the following result:

Lemma 2.1.

If U^1_{ij} is computed by (2.8), we have that

$$(2.9) \quad \max_{1\le i,j\le J} |U^1_{ij} - U^1_{ij}| \le \frac{k^3}{6} \max |u_{ttt}(x, y, t)| \ ,$$

$$0 \le x, y \le 1$$

$$0 \le t \le k \qquad .$$

If U^1_{ij} is computed by (2.8'), we have that

$$(2.9') \quad \max_{1\le i,j\le J} |U^1_{ij} - U^1_{ij}| \le \frac{k^3}{6} \max_{x,y,t} |\frac{\partial^3 u}{\partial t^3}| + \frac{c^2k^2h^2}{12} \max_{0\le x,y\le 1} \{|\frac{\partial^4 f}{\partial x^4}|, |\frac{\partial^4 f}{\partial y^4}|\}.$$

Proof. (2.9) follows immediately by subtracting (2.8) from (2.7). To prove (2.9') we have for $1 \le i, j \le J$

$$u^1_{ij} = f_{ij} + kg_{ij} + \frac{c^2k^2}{2}(\Delta f)_{ij} + \frac{k^3}{6} u_{ttt}(x_i, y_j, \theta_{ij}) ,$$

$$0 \leq \theta_{ij} \leq k .$$

In order to estimate the difference $(\Delta f)_{ij} - \diamond f_{ij}/h^2$, we observe that

$$f(x_{i+1}, y_j) = f(x_i, y_j) + h f_x(x_i, y_j) + \frac{h^2}{2} f_{xx}(x_i, y_j)$$

$$+ \frac{h^3}{6} f_{xxx}(x_i, y_j) + \frac{h^4}{24} f_{xxxx}(\theta'_{ij}, y_j) ,$$

$$x_i \leq \theta'_{ij} \leq x_{i+1} .$$

Also, we have

$$f(x_{i-1}, y_j) = f(x_i, y_j) - hf_x(x_i, y_j) + \frac{h^2}{2} f_{xx}(x_i, y_j) -$$

$$- \frac{h^3}{6} f_{xxx}(x_i, y_j) + \frac{h^4}{24} f_{xxxx}(\theta''_{ij}, y_j) ,$$

$$x_{i-1} \leq \theta''_{ij} \leq x_i .$$

From the above we obtain

$$f_{xx}(x_i, y_j) = (f_{i+1,\, j} - 2f_{ij} + f_{i-1,j})/h^2 -$$

$$- \frac{h^2}{24}[f_{xxxx}(\theta'_{ij}, y_j) + f_{xxxx}(\theta''_{ij}, y_j)] .$$

Similarly,

$$f_{yy}(x_i, y_j) = (f_{i,j+1} - 2f_{ij} + f_{i,j-1})/h^2 - \frac{h^2}{24}[f_{yyyy}(x_i, \tilde{\theta}_{ij}) + f_{yyyy}(x_i, \tilde{\tilde{\theta}}_{ij})] ,$$

where

$$y_j \leq \tilde{\theta}_{ij} \leq y_{j+1}, \; y_{j-1} \leq \tilde{\tilde{\theta}}_{ij} \leq y_j .$$

Thus,

$$\max_{1\leq i,j\leq J} |(\Delta f)_{ij} - \diamond f_{ij}/h^2| \leq \frac{h^2}{6} \max_{0\leq x\leq y\leq 1} \{|f_{xxxx}|, |f_{yyyy}|\} . \tag{2.10}$$

From (2.10) we obtain (2.9'). ■

We can now easily see that (with initial conditions computed by (2.6) and (2.8) or (2.8') and taking into account the boundary conditions (2.5)) the scheme (2.4) has a unique solution U^n_{ij} for each n, $2 \leq n \leq N$. Note first that the case $\alpha = 0$ gives the classical, explicit, Courant-Friedrichs-Lewy [9] scheme. All other values of $\alpha > 0$ yield implicit schemes. In each case $\{U^n_{ij}\}$, $1 \leq i, j \leq J$ may be found as the solution of a $J^2 \times J^2$ system of linear equations. Specifically, we order $\{U^n_{ij}\}$ as a vector U^n_h in $\mathbb{R}^{J^2}$, by setting $U^n_{h,1} = U^n_{1,1}$, $U^n_{h,2} = U^n_{2,1}, \ldots, U^n_{h,J} = U^n_{J,1}$, $U^n_{h,J+1} = U^n_{1,2}, \ldots,$ $U^n_{h,J^2} = U^n_{J,J}$, $2 \leq n \leq N - 1$. Then, the scheme (2.4) is equivalent to

the linear system

$$A_h U_h^n = F_h^n ,$$

where F_h^n is a J^2-vector that depends on the previous values U_h^{n-1}, U_h^{n-2} and where A_h is a constant, block tridiagonal $J^2 \times J^2$ matrix of the form

$$(2.11) \qquad A_h = \begin{bmatrix} C & B & 0 \\ B & \ddots & B \\ 0 & B & C \end{bmatrix} ,$$

where C is the $J \times J$ tridiagonal matrix

$$C = \begin{bmatrix} 1 + 4\alpha r^2 & -\alpha r^2 & 0 \\ -\alpha r^2 & \ddots & \alpha r^2 \\ 0 & \alpha r^2 & 1 + 4\alpha r^2 \end{bmatrix}$$

and B is the $J \times J$ diagonal matrix

$$B = -\alpha r^2 I$$

where I is the $J \times J$ identity. We now have

Lemma 2.2.

A_h is symmetric positive definite for all $\alpha > 0$.

Proof. Since $1 + 4r^2\alpha > 4r^2\alpha$ for each $\alpha \geq 0$, we conclude that A_h is strictly diagonally dominant. Therefore, A_h is symmetric positive definite being, of course, symmetric and having positive diagonal. ■

Hence, from Lemma 2.2 we conclude that the solution U_h^n exists uniquely for $n = 2, \ldots, N$.

We now turn to obtain an error estimate for the difference scheme (2.4). As a first step, we address the question of its consistency. We have

Lemma 2.3.

Let u be the solution of (2.1). Then, there exists a constant C, independent of k, h, and u such that

$$\max_{\substack{1\leq i,\, j\leq J \\ 1\leq n\leq N-1}} |\delta_t^2 u_{ij}^n - r^2[\alpha\diamond u_j^{n+1} + (1 - 2\alpha)\, u_{ij}^n + \alpha\diamond u_{ij}^{n-1}]| \leq \tag{2.12}$$

$$\leq Ck^2(k^2 + h^2) \max_{(x,y,t)} \left(\left|\frac{\partial^4 u}{\partial x^4}\right|, \left|\frac{\partial^4 u}{\partial y^4}\right|, \left|\frac{\partial^4 u}{\partial t^4}\right|, \left|\frac{\partial^6 u}{\partial t^6}\right|\right).$$

Proof. By Taylor expansion we obtain that

$$\delta_t^2 u_{ij}^n = k^2 \frac{\partial^2 u}{\partial t^2}\Big|_{ij}^n + \frac{k^4}{24}\left[\frac{\partial^4 u}{\partial t^4}(x_i, y_j, \theta) + \frac{\partial^4 u}{\partial t^4}(x_i, y_j, \theta')\right], \tag{2.13}$$

where θ, θ' are some points (depending on x_i, y_j) in the intervals $[t^n, t^{n+1}]$ and $[t^{n-1}, t^n]$, respectively. On the other hand, we obtain, as in Lemma 2.11, that

$$(2.14)\qquad \diamond u_{ij}^n = h^2\Delta\ u|_{ij}^n + \frac{h^4}{24}\left[\frac{\partial^4 u}{\partial x^4}(\xi_{ij}, y_j, t^n) + \frac{\partial^4 u}{\partial x^4}(\xi'_{ij}, y_j, t^n)\right.$$

$$\left. + \frac{\partial^4 u}{\partial y^4}(x_i, \sigma_{ij}, t^n) + \frac{\partial^4 u}{\partial y^4}(x_i, \sigma'_{ij}, t^n)\right] ,$$

where ξ_{ij}, ξ'_{ij}, σ_{ij}, σ'_{ij} are points in the intervals $[x_{i-1}, x_i]$, $[x_i, x_{i+1}]$, $[y_{j-1}, y_j]$, $[y_j, y_{j+1}]$, respectively.

Now, we have that

$$|\delta_t^2 u_{ij}^n - r^2(\alpha\, u_{ij}^{n+1} + (1 - 2\alpha)\diamond u_{ij}^n + \alpha\diamond u_{ij}^{n-1})| \leq$$

$$(2.15)\qquad \leq |\alpha(\delta_t^2 u_{ij}^{n+1} - r^2\diamond u_{ij}^{n+1}) + (1 - 2\alpha)(\delta_t^2 u_{ij}^n - r^2\diamond u_{ij}^n)$$

$$+ \alpha(\delta_t^2 u_{ij}^{n-1} - r^2\diamond u_{ij}^{n-1})| + \alpha|\delta_t^2 u_{ij}^{n+1} - 2\delta_t^2 u_{ij}^n + \delta_t^2 u_{ij}^{n-1}| .$$

Now, (2.13) and (2.14) applied for n, $n - 1$, $n + 1$ give (in view of (2.1)) that

$$(2.16)\qquad |\delta_t^2 u_{ij}^m - r^2\diamond u_{ij}^m| \leq \frac{k^4}{12}\max_{x,y,t}|u_{tttt}| + \frac{c^2k^2h^2}{6}\cdot$$

$$\max_{x,y,t}\{|u_{xxxx}|, |u_{yyyy}|\} ,$$

where $m = n - 1, n, n + 1$.

Moreover, (2.13) gives that

$$(2.17)\qquad |\delta_t^2 u_{ij}^{n+1} - 2\delta_t^2 u_{ij}^n + \delta_t^2 u_{ij}^{n-1}| \leq k^2|(u_{tt}|_{ij}^{n+1} - 2u_{tt}|_{ij}^n + u_{tt}|_{ij}^{n-1})| +$$

$$+ \frac{k^4}{4} \max_{x,y,t} |u_{tttt}| .$$

Since, now, in analogy with (2.13)

$$|\delta_t^2 u_{tt}|_{ij}^n| = |(u_{tt}|_{ij}^{n+1} - 2u_{tt}|_{ij}^n + u_{tt}|_{ij}^{n-1})|$$

$$\leq k^2 |u_{tttt}|_{ij}^n| + \frac{k^4}{12} \max_{x,y,t} |\frac{\partial^6 u}{\partial t^6}| ,$$

we conclude by (2.17) that

$$(2.18) \qquad |\delta_t^2 u_{ij}^{n+1} - 2\delta_t^2 u_{ij}^n + \delta_t^2 u_{ij}^{n-1}| \leq \frac{5k^4}{4} \max_{x,y,t} |\frac{\partial^4 u}{\partial t^4}| + \frac{k^6}{12} \max_{x,y,t} |\frac{\partial^6 u}{\partial t^6}| .$$

Therefore, (2.18) and (2.16) give (for $k \leq 1$) (2.12). ■

In preparation of our convergence theorem, let us first introduce some notation. For a mesh function $v = v_{ij}$, $0 \leq i$, $j \leq J$, we define

$$\delta_x^2 v_{ij} = v_{i+1,j} - 2v_{ij} + v_{i-1,j} ,$$

$$\delta_y^2 v_{ij} = v_{i,j+1} - 2v_{ij} + v_{i,j-1} , \quad 1 \leq i , j \leq J .$$

Hence, we have that

$$\diamond = \delta_x^2 + \delta_y^2 .$$

For mesh functions $v = v_{ij}$, $w = w_{ij}$, $0 \leq i , j \leq J + 1$, define the

bilinear form $a_h(v, w)$ by

$$(2.19) \qquad a_h(v, w) = -h^2 \sum_{i,j=1}^{J} (\diamond v_{ij}) v_{ij} .$$

Let S_h be the J^2-dimensional space of mesh functions $v = v_{ij}$, $0 \le i, j \le J + 1$ which are zero on the boundary of the unit square, i.e. such that $v_{0,j} = v_{J+1,j} = v_{i,0} = v_{i,J+1} = 0$, $0 \le i, j \le J + 1$. Then, we have:

Lemma 2.4.

Let $v = v_{ij}$ and $w = w_{ij}$ be mesh function in S_h. Then,

$$(2.20) \qquad a_h(v, w) = a_h(w, v) ,$$

$$(2.21) \qquad a_h(v, v) \ge 0 \text{ and if } a_h(v, v) = 0 \text{ then}$$

$$v_{ij} = 0 \text{ for all } i, j ,$$

$$(2.22) \qquad a_h(v, w) \le (a_h(v, v))^{1/2} \cdot (a_h(w, w))^{1/2} .$$

Moreover,

$$(2.23) \qquad a_h(v, v) \le 8 \|v\|_h^2$$

where

$$(2.24) \qquad \|v\|_h \underset{\text{def}}{=} h\Big(\sum_{i,j=1}^{J} v_{ij}^2\Big)^{1/2} ,$$

i.e. $\|v\|_h$ is the discrete ℓ_2-norm for functions on S_h .

Proof. We first prove some preliminary results. Consider, for the moment, one-dimensional mesh functions v_i and w_i , $0 \le i \le J+1$, such that $v_0 = w_0 = v_{J+1} = w_{j+1} = 0$. We will show that for such mesh functions

$$(i) \quad \sum_{i=1}^{J} (\delta_x^2 v_i) w_i = \sum_{i=1}^{J} (\delta_x^2 w_i) v_i ,$$

$$(ii) \quad -\sum_{i=1}^{J} (\delta_x^2 v_i) v_i \ge 0 \quad \text{and if}$$

$$-\sum_{i=1}^{J} (\delta_x^2 v_i) v_i = 0 \quad \text{then}$$

$$v_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, J ,$$

$$(iii) \quad \Big|-\sum_{i=1}^{J} (\delta_x^2 v_i) w_i\Big| \le \Big[-\sum_{i=1}^{J} (\delta_x^2 v_i) v_i\Big]^{1/2} \cdot \Big[-\sum_{i=1}^{J} (\delta_x^2 w_i) w_i\Big]^{1/2} ,$$

$$(iv) \quad \Big[-\sum_{i=1}^{J} (\delta_x^2 v_i) v_i\Big]^{1/2} \le 2\Big(\sum_{i=1}^{J} v_i^2\Big)^{1/2} .$$

In order to prove (i), we observe that

$$\sum_{i=1}^{J} (\delta_x^2 v_i) w_i = \sum_{i=1}^{J} (v_{i+1} w_i - 2 v_i w_i + v_{i-1} w_i) .$$

On the other hand,

$$\sum_{i=1}^{J} (\delta_x^2 w_i) v_i = \sum_{i=1}^{J} (w_{i+1} - 2w_i + w_{i-1}) v_i =$$

$$= \sum_{i=1}^{J} (w_{i+1} v_i - z\, w_i v_i + w_{i-1} v_i) .$$

We easily see that in view of $v_0 = w_0 = v_{J+1} = w_{J+1} = 0$, the above have identical right-hand sides and hence (i) holds.

Now, in order to prove (ii), we observe that

$$-\sum_{i=1}^{J} (\delta_x^2 v_i) v_i = -\sum_{i=1}^{J} (v_{i+1} - 2v_i + v_{i-1}) v_i =$$

$$= -\sum_{i=1}^{J} (v_{i+1} v_i - 2v_i^2 + v_{i-1} v_i)$$

$$= -(v_2 v_1 + v_3 v_2 + \ldots + v_J v_{J-1}) - (v_1 v_2 + v_2 v_3 + \ldots + v_{J-1} v_J) +$$

$$+ 2[v_1^2 + v_2^2 + \ldots + v_J^2] =$$

$$= 2[v_1^2 + v_2^2 + \ldots + v_J^2 - v_1 v_2 - v_2 v_3 - \ldots - v_{J-1} v_J] =$$

$$= v_1^2 + (v_1 - v_2)^2 + \ldots + (v_{J-1} - v_J)^2 + v_J^2 .$$

Hence, (ii) follows immediately. Moreover, if $\sum_{i=1}^{J} (\delta_x^2 v_i) v_i = 0$, it follows from the above that $v_1 = 0$, $v_J = 0$ and $v_i = v_{i+1}$, $1 \le i \le J - 1$, i.e. that $v_i = 0$, for all i.

We see that (i) and (ii) imply that $\sum_{i=1}^{J} (\delta_x^2 v_i) w_i$ is an inner product for one-dimensional mesh function which are zero on the boundary. We then recognize (iii) as the Cauchy-Schwartz inequality for this inner product. Finally, (iv) follows immediately from the computation used to prove (ii), since $2v_i v_j \le v_i^2 + v_j^2$.

We are ready to show the validity of (2.20) - (2.24). (2.20) follows from

$$a_h(v, w) = -h^2 \sum_{i,j=1}^{J} (\diamond v_{ij}) w_{ij} =$$

$$= -h^2 \sum_{i,j=1}^{J} (\delta_x^2 v_{ij}) w_{ij} - h^2 \sum_{i,j=1}^{J} (\delta_y^2 v_{ij}) \cdot w_{ij} =$$

$$= -h^2 \sum_{i,j=1}^{J} (\delta_x^2 w_{ij}) v_{ij} - h^2 \sum_{i,j=1}^{J} (\delta_y^2 w_{ij}) v_{ij} =$$

$$= -h^2 \sum_{i,j=1}^{J} (\diamond w_{ij}) v_{ij} = a_h(w, v) .$$

The result of (i) was used here since, for each fixed i, $w_{i,0} = w_{i,J+1} = v_{i,0} = v_{i,J+1} = 0$ and for each fixed j, $w_{0,j} = w_{J+1,j} = v_{0,j} = v_{J+1,j} = 0$. Now, for $v \in S_h$,

$$a_h(v, v) = -h^2 \sum_{i,j=1}^{J} (\diamond v_{ij}) v_{ij} =$$

$$= -h^2 \sum_{i,j=1}^{J} (\delta_x^2 v_{ij}) v_{ij} - h^2 \sum_{i,j=1}^{J} (\delta_y^2 v_{ij}) v_{ij} \geq 0 .$$

Where again (ii) is used fixing j, i respectively and then summing. Assume now that $a_h(v, v) = 0$, i.e. that

$$-h^2 \sum_{i,j=1}^{J} (\delta_x^2 v_{ij}) v_{ij} - h^2 \sum_{i,j=1}^{J} (\delta_y^2 v_{ij}) v_{ij} = 0 ,$$

i.e. that

$$\sum_{j=1}^{J}\Big(-\sum_{i=1}^{J}(\delta_x^2 v_{ij})v_{ij}\Big) + \sum_{i=1}^{J}\Big(-\sum_{j=1}^{J}(\delta_y^2 v_{ij})v_{ij}\Big) = 0 .$$

Now, by (ii) , $-\sum_{i=1}^{J}(\delta_x^2 v_{ij})v_{ij} \geq 0$ for each j , and $-\sum_{j=1}^{J}(\delta_y^2 v_{ij})v_{ij} \geq 0$ for each i . Hence, the above yields e.g. $\sum_{i=1}^{J}(\delta_x^2 v_{ij})v_{ij} = 0$ for $1 \leq j \leq J$, from which, using (ii) again, we see that $v_{ij} = 0$, $1 \leq i, j \leq J$, and the proof of (2.21) is complete.

Now, (2.20) and (2.21) imply that $a_h(\cdot, \cdot)$ is an inner product on S_h . Hence, (2.22) is just the Cauchy-Schwartz inequality.

Finally, observe that the relation (iv) implies

$$-h^2 \sum_{i,j=1}^{J}(\delta_x^2 v_{ij})v_{ij} \leq 4h^2 \sum_{i,j=1}^{J} v_{ij}^2 ,$$

and also

$$-h^2 \sum_{i,j=1}^{J}(\delta_y^2 v_{ij})v_{ij} \leq 4h^2 \sum_{i,j=1}^{J} v_{ij}^2 ,$$

from which (2.23) follows by addition. The proof of Lemma 2.4 is now complete. ■

We now proceed to the error estimate. For concreteness, we suppose that the initial conditions of the scheme (2.4) are computed by (2.8').

Theorem 2.1

Let u be the solution of (2.1). Suppose that $\{U_{ij}^n\}$, $0 \leq n \leq N$, $0 \leq i, j \leq J$ is the solution of the difference scheme (2.4) with the

initial conditions (2.8'). If U_h^n, u^n are mesh functions U_{ij}^n, u_{ij}^n respectively, then for every $\alpha \geq 0$ there exists a constant $C = C(\alpha, r, T)$, such that if $r = \frac{ck}{h} < r_0(\alpha)$, then

$$\max_{1\leq n\leq N} \|U_h^n - u^n\|_h \leq C(\alpha, r, T) \cdot (h^2 + k^2) \cdot \max_{x,y,t} \left[\left|\frac{\partial^4 u}{\partial x^4}\right|, \left|\frac{\partial^4 u}{\partial y^4}\right|, \left|\frac{\partial^4 u}{\partial t^4}\right|, \left|\frac{\partial^6 u}{\partial t^6}\right|\right] \tag{2.25}$$

where

$$r_0(\alpha) = \begin{cases} +\infty & , \text{ if } \alpha \geq \frac{1}{4}, \\ \dfrac{1}{\sqrt{2(1 - 4\alpha)}} & , \text{ if } 0 \leq \alpha < \frac{1}{4} \end{cases} .$$

Proof. Let $e_{ij}^n = U_{ij}^n - u_{ij}^n$. Then we have, for $1 \leq i, j \leq J$, $1 \leq n \leq N - 1$

$$\begin{aligned} \delta_t^2 e_{ij}^n &= \delta_t^2 U_{ij}^n - \delta_t^2 u_{ij}^n \\ &= r^2\alpha\diamond U_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond U_{ij}^n + r^2\alpha\diamond U_{ij}^{n-1} - \delta_t^2 u_{ij}^n + \\ &\quad + r^2\alpha\diamond u_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond u_{ij}^n + r^2\alpha\diamond u_{ij}^{n-1} - \\ &\quad - r^2\alpha\diamond u_{ij}^{n+1} - r^2(1 - 2\alpha)\diamond u_{ij}^n - r^2\alpha\diamond u_{ij}^{n-1} = \\ &= r^2\alpha\diamond e_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond e_{ij}^n + r^2\alpha\diamond e_{ij}^{n-1} + E_{ij}^n \end{aligned} \tag{2.26}$$

where

$$E_{ij}^n = r^2\alpha\diamond u_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond u_{ij}^n + r^2\alpha\diamond u_{ij}^{n-1} - \delta_t^2 u_{ij}^n ,$$

and

$$\max_{\substack{1\le n\le N-1\\ 1\le i,j\le J}} |E_{ij}^n| \le ck^2(k^2 + h^2) \max_{x,y,t} \{|\frac{\partial^4 u}{\partial x^4}|, |\frac{\partial^4 u}{\partial y^4}|, |\frac{\partial^4 u}{\partial t^4}|, |\frac{\partial^6 u}{\partial t^6}|\} . \tag{2.27}$$

Let

$$C_1(u) = C \max_{x,y,t} \{|\frac{\partial^4 u}{\partial x^4}|, |\frac{\partial^4 u}{\partial y^4}|, |\frac{\partial^4 u}{\partial t^4}|, |\frac{\partial^6 u}{\partial t^6}|\} .$$

Now, (2.26) implies that

$$h^2 \sum_{i,j=1}^{J} \delta_t^2 e_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) = h^2 r^2 \alpha \sum_{i,j=1}^{J} \diamond e_{ij}^{n+1} (e_{ij}^{n+1} - e_{ij}^{n-1}) +$$

$$+ r^2(1 - 2\alpha)\, h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) +$$

$$+ r^2\alpha h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^{n-1} (e_{ij}^{n+1} - e_{ij}^{n-1}) + h^2 \sum_{i,j=1}^{J} E_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) ,$$

$$1 \le n \le N - 1 .$$

Hence, recalling the definition of $\|\cdot\|_h$ and $a_h(\cdot, \cdot)$ we obtain

$$\|e^{n+1} - e^n\|_h^2 - \|e^n - e^{n-1}\|_h^2 =$$

$$= -r^2\alpha a_h(e^{n+1}, e^{n+1} - e^{n-1}) - r^2(1 - 2\alpha) \cdot$$

$$\cdot a_h(e^n, e^{n+1} - e^{n-1}) - r^2\alpha a_h(e^{n-1}, e^{n+1} - e^{n-1}) + \tag{2.28}$$

$$+ h^2 \sum_{i,j=1}^{J} E_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) ,$$

$$1 \leq n \leq N - 1 .$$

The RHS of (2.28) can be rewritten as:

$$- \frac{r^2}{2} (1 - 2\alpha) [a_h(e^{n+1}, e^{n+1}) - a_h(e^{n-1}, e^{n-1})] +$$

$$+ \frac{r^2}{2} (1 - 2\alpha)[a_h(e^{n+1} - e^n, e^{n+1} - e^n) -$$

$$- a_h(e^n - e^{n-1}, e^n - e^{n-1})] - r^2\alpha[a_h(e^{n+1}, e^{n+1}) -$$

$$- a_h(e^{n-1}, e^{n-1})] = - \frac{r^2}{2} [a_h(e^{n+1}, e^{n+1}) - a_h(e^{n-1}, e^{n-1})] +$$

$$+ \frac{r^2}{2} (1 - 2\alpha)[a_h(e^{n+1} - e^n, e^{n+1} - e^n) -$$

$$- a_h(e^n - e^{n-1}, e^n - e^{n-1})] .$$

Thus (2.28) becomes:

$$\|e^{n+1} - e^n\|_h^2 - \|e^n - e^{n-1}\|_h^2 + \frac{r^2}{2}[a_h(e^{n+1}, e^{n+1}) - a_h(e^{n-1}, e^{n-1})]$$

$$- \frac{r^2}{2}(1 - 2\alpha)[a_h(e^{n+1} - e^n, e^{n+1} - e^n) -$$

$$- a_h(e^n - e^{n-1}, e^n - e^{n-1})] = h^2 \sum_{i,j=1}^{J} E_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) ,$$

$$1 \leq n \leq N - 1 .$$

Summing up both sides of the above with respect to n from $n = 1$ to $n = n' - 1$ and changing the index n' back to n we get (noting that $e^0 = 0$) ,

$$\|e^n - e^{n-1}\|_h^2 + \frac{r^2}{2}[a_h(e^n, e^n) + a_h(e^{n-1}, e^{n-1})]$$

$$- \frac{r^2}{2}(1 - 2\alpha)\, a_h(e^n - e^{n-1}, e^n - e^{n-1}) =$$

$$= \|e^1\|_h^2 + \alpha r^2\, a_h(e^1, e^1) + h^2 \sum_{\ell=1}^{n-1} \sum_{i,j=1}^{J} E_{ij}^{\ell} (e_{ij}^{\ell+1} - e_{ij}^{\ell-1}) ,$$

$$1 \leq n \leq N .$$

Since $a_h(e^1, e^1) \leq 8\|e^1\|_h^2$, and

$$h^2 \sum_{\ell=1}^{n-1} \sum_{i,j=1}^{J} E_{ij}^{\ell} (e_{ij}^{\ell+1} - e_{ij}^{\ell-1}) \leq \sum_{\ell=1}^{n-1} \|E^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h ,$$

(Cauchy-Schwartz inequality for $\|\cdot\|_h$) (2.29) becomes:

$$\|e^n - e^{n-1}\|_h^2 + \frac{r^4}{4} a_h(e^n + e^{n-1},\ e^n + e^{n-1})$$

$$+ \frac{r^2}{4}(4\alpha - 1)a_h(e^n - e^{n-1},\ e^n - e^{n-1})$$

(2.30)

$$\leq (1 + 8\alpha r^2)\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|E\|_h \|e^{\ell+1} - e^{\ell-1}\|_h ,$$

$$1 \leq n \leq N .$$

Now we distinguish two cases depending on the sign of $4\alpha - 1$ in (2.30).

(i) Let $\alpha \geq \frac{1}{4}$.

Then $4\alpha - 1 \geq 0$ and (2.30) implies

$$\|e^n - e^{n-1}\|_h^2 \leq (1 + 8\alpha r^2)\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|E^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h . \tag{2.31}$$

Using the Cauchy-Schwartz inequality we obtain

$$\sum_{\ell=1}^{n-1} \|E^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \leq [\sum_{\ell=1}^{n-1} \|E^\ell\|_h^2]^{1/2} \cdot [\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2]^{1/2}. \tag{2.32}$$

We put

$$E = [\sum_{\ell=1}^{N-1} \|E^\ell\|_h^2]^{1/2} . \tag{2.33}$$

Now, we have by the triangle inequality that

$$\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2 \leq \sum_{\ell=1}^{n-1} (\|e^{\ell+1} - e^{\ell}\|_h + \|e^{\ell} - e^{\ell-1}\|_h)^2$$

$$\leq 2\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell}\|_h^2 + 2\sum_{\ell=1}^{n-1} \|e^{\ell} - e^{\ell-1}\|_h^2 \leq 4\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2 .$$

Hence,

$$(2.34) \qquad [\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2]^{1/2} \leq 2[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_j^2]^{1/2} .$$

From (2.32) - (2.34) we obtain, using the inequality $2ab \leq \frac{a^2}{A} + Ab^2$ for real a, b and any $A > 0$ that

$$\sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \leq 2E[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2]^{1/2}$$

$$\leq \frac{E^2}{A} + A[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2]$$

$$\leq \frac{E^2}{A} + A\, N \max_{1\leq n\leq N} \|e^n - e^{n-1}\|_h^2 .$$

Choosing $A = \frac{1}{2N}$, we obtain

$$(2.35) \qquad \sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \leq 2NE^2 + \frac{1}{2} \max_{1\leq n\leq N} \|e^n - e^{n-1}\|_h^2 .$$

From (2.31), (2.35) we deduce that

$$\max_{1\leq n\leq N} \|e^n - e^{n-1}\|_h^2 \leq (1 + 8\alpha r^2)\|e^1\|_h^2 + 2NE^2$$

$$+ \frac{1}{2} \max_{1\leq n\leq N} \|e^n - e^{n-1}\|_h^2 ,$$

or that

$$(2.36)\qquad \max_{1\le n\le N} \|e^n - e^{n-1}\|_h^2 \le 2(1 + 8\alpha r^2)\, \|e^1\|_h^2 + 4NE^2 .$$

From (2.27), since $NK = T$ we obtain

$$(2.37)\qquad NE^2 \le N \sum_{\ell=1}^{N-1} C_1^2(u)k^4(k^2 + h^2)^2 \le N^2\, C_1^2(u)\, k^4(k^2 + h^2)^2$$

$$= T^2C_1^2(u)\, K^2(k^2 + h^2)^2 .$$

Using the inequality $\sqrt{a^2 + b^2} \le a + b$ for $a, b \ge 0$, we obtain from (2.36) and (2.37) that

$$(2.38)\qquad \max_{1\le n\le N} \|e^n - e^{n-1}\|_h \le \sqrt{2(1 + 8\alpha r^2)}\|e^1\|_h + 2TC_1(u)\, K(k^2 + h^2) = R.$$

But since $\|e^n\|_h \le \|e^n - e^{n-1}\|_h + \|e^{n-1}\|_h$, $e_h^0 = 0$, we obtain by (2.38) that

$$(2.39)\qquad \max_{1\le n\le N} \|e^n\|_h \le [\max_{1\le n\le N} \|e^n - e^{n-1}\|_h] \cdot N$$

$$\le [\sqrt{2(1 + 8\, r^2)}\|e^1\|_h + 2k(k^2 + h^2)\, C_1(u)T] \cdot N .$$

If we take in to account (2.9'), (2.39) implies the result of Theorem 2.1. ■

(ii) If $0 \le \alpha < \frac{1}{4}$, then $1 - 4\alpha > 0$ and (2.30) gives, since $a_h(u, u) \ge 0$

$$\|e^n - e^{n-1}\|_h^2 - \frac{r^2}{h}(1 - 4\alpha)\, a_h(e^n - e^{n-1}, e^n - e^{n-1}) \tag{2.40}$$

$$\le (1 + 8\alpha r^2)\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|E^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h ,$$

$$1 \le n \le N .$$

Using $a_h(u, u) \le 8\|v\|_h^2$ we obtain

$$[1 - 2r^2(1 - 4\alpha)]\|e^n - e^{n-1}\|_h^2 \tag{2.41}$$

$$\le (1 + 8\alpha r^2)\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|E^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h .$$

If $r < r_0(\alpha) = [2(1 - 4\alpha)]^{-1/2}$ then $[1 - 2r^2(1 - 4\alpha)] > 0$. Thus, if we let

$$[1 - 2r^2(1 - 4\alpha)] = [C(r, \alpha)]^{-1} ,$$

(2.41) becomes

$$\|e^n - e^{n-1}\|_h^2 \le C(r, \alpha)(1 + 8\alpha r^2)\|e^1\|_h^2 \tag{2.42}$$

$$+ C(r, \alpha) \sum_{\ell=1}^{n-1} \|E^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h ,$$

which is the analog of (2.31). The rest of the proof follows exactly as before with

$$E = C(r, \alpha) \left[\sum_{\ell=1}^{N-1} \|E^{\ell}\|_h^2 \right]^{1/2} ,$$

and therefore we come to the conclusion that for $0 \leq \alpha < \frac{1}{4}$, the scheme (2.4) is stable and convergent if $r < r_0(\alpha) = [2(1 - 4\alpha)]^{-1/2}$.

For the special case $\alpha = 0$ we capture the classical Courant-Friedrichs-Lewy stability condition (in two dimensions)

$$r \leq \frac{1}{\sqrt{2}} . \qquad \blacksquare$$

CHAPTER III

AN ALTERNATING DIRECTION METHOD FOR THE SOLUTION OF THE WAVE EQUATION IN RECTANGLES

In this chapter, we consider again the approximation of the initial and boundary value problem (2.1) - (2.3). We shall approximate the solution of (2.1) - (2.3) starting again from the one-parameter family of (generally) implicit difference scheme due to Von Neumann, which we considered in Chapter II. We shall perturb this scheme to obtain a more computationally efficient method. Using the notation introduced in Chapter I, we seek U_{ij}^n , $0 \le n \le N$, $0 \le i , j \le J + 1$ satisfying

$$(3.1) \qquad \delta_t^2 U_{ij}^n = r^2[\alpha\diamond U_{ij}^{n+1} + (1 - 2\alpha) U_{ij}^n + \alpha\diamond U_{ij}^{n-1}]$$

$$- r^4\alpha^2 \delta_x^2 \delta_y^2(\delta_t^2 U_{ij}^n) ,$$

$$1 \le i,j \le J , r = \frac{ck}{h} , 1 \le n \le N - 1,$$

where as usual

$$\delta_t^2 U_{ij}^n = U_{ij}^{n+1} - 2U_{ij}^n + U_{ij}^{n-1} ,$$

$$0 \le i,j \le J + 1 , 1 \le n \le N - 1 ,$$

$$\diamond U_{ij} = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij} ,$$

$$1 \leq i,j \leq J \ .$$

Also,

$$\delta_x^2 U_{ij} = U_{i+1,j} - 2U_{ij} + U_{i-1,j} \ ,$$

$$\delta_y^2 U_{ij} = U_{i,j+1} - 2U_{ij} + U_{i,j-1} \ ,$$

$$1 \leq i,j \leq J \ ,$$

and

$$\alpha \geq 0$$

is a parameter. Again the difference scheme (3.1) is supplemented with the boundary conditions

$$(3.2) \qquad U_{0,j}^n = U_{i,0}^n = U_{i,J+1}^n = U_{J+1,j}^n = 0 \ ,$$

where

$$0 \leq i,j \leq J + 1 \quad \text{and} \quad 0 \leq n \leq N \ .$$

The initial conditions U_{ij}^0 and U_{ij}^1 are given by (2.6) and (2.8) and (2.8') .

As it was pointed out in the introduction, the scheme (3.1) has the advantage that it is equivalent to a two-step scheme, each step of which requires the solution of one tridiagonal system of equations in the x-, resp., y-, direction; according e.g. to Fairweather and Mitchell [16], we may rewrite (3.1) in two steps as:

$$(3.3)\qquad \tilde{U}^{n+1}_{ij} - 2U^{n}_{ij} + U^{n-1}_{ij} = r^2\,\delta_x^2[\alpha\,\tilde{U}^{n+1}_{ij} + (1 - 2\alpha)\,U^{n}_{ij} + \alpha U^{n-1}_{ij}] + r^2\,\delta_y^2\,U^{n}_{ij}\,,$$

$$(3.4)\qquad U^{n+1}_{ij} = \tilde{U}^{n+1}_{ij} + \alpha r^2\,\delta_y^2(U^{n+1}_{ij} - 2U^{n}_{ij} + U^{n-1}_{ij})\,.$$

We put $\tilde{U}^{n+1}_{ij} = U^{n+1}_{ij} = 0$ on the boundary of the unit square. Hence $\{\tilde{U}^{n+1}_{ij}\}$, $\{U^{n+1}_{ij}\}$, $1 \le i, j \le J$ may be found as the solution of two $J^2 \times J^2$ systems of linear equations. Specifically, we order $\{\tilde{U}^{n+1}_{ij}\}$ and $\{U^{n+1}_{ij}\}$ as vectors $\tilde{U}^{n+1}_{h}$ and U^{n+1}_{h} in $\mathbb{R}^{J^2}$ by setting:

$$\tilde{U}^{n}_{h,1} = \tilde{U}^{n}_{1.1}\,,\quad \tilde{U}^{n}_{h,2} = \tilde{U}^{n}_{2,1}\,,\ \dots\,,\ \tilde{U}^{n}_{h,J} = \tilde{U}^{n}_{J,1}\,,$$

$$\tilde{U}^{n}_{h,J+1} = \tilde{U}^{n}_{1,2}\,,\ \dots\,,\ \tilde{U}^{n}_{h,J^2} = \tilde{U}^{n}_{J,J}\,,\ \text{and}$$

$$U^{n}_{h,1} = U^{n}_{1,1}\,,\ U^{n}_{h,2} = U^{n}_{1,2}\,,\ \dots\,,\ U^{n}_{h,J} = U^{n}_{1,J}\,,$$

$$U^{n}_{h,J+1} = U^{n}_{2,1}\,,\ \dots\,,\ U^{n}_{h,J^2} = U^{n}_{J,J}\,.$$

Clearly, the schemes (3.3) and (3.4) are equivalent for each n , $2 \leq n \leq N$ to the linear systems

$$A_h \tilde{U}_h^n = F_h \ , \tag{3.5}$$

$$A_h U_h^n = \tilde{F}_h \ , \tag{3.6}$$

where F_h and $\tilde{F}_h$ are J^2-vectors depending upon known nodal values. A_h is a constant block diagonal $J^2 \times J^2$ of the form

$$A_h = \begin{bmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{bmatrix} \ ,$$

where the matrix C is given by:

$$C = \begin{bmatrix} (1 + 2\alpha r^2) & -\alpha r^2 & 0 \\ -\alpha r^2 & \ddots & -\alpha r^2 \\ 0 & -\alpha r^2 & (1 + 2\alpha r^2) \end{bmatrix} \ .$$

Lemma 3.1

A_h is symmetric positive definite.

Proof. Since $\alpha \geq 0$ and $1 + 2\alpha r^2 > 2\alpha r^2$ for every α , we conclude that A_h is strictly diagonal dominant. Therefore A_h is symmetric positive definite being, of course, symmetric and having positive diagonal. ■

Hence, from Lemma 3.1, we obtain that the solutions $\tilde{U}_h^n$ and U_h^n exist uniquely for $n = 2, 3, \ldots, N$.

We now proceed to obtain an error estimate for the difference scheme (3.1). In order to do that, as a first step we are going to show its consistency. We have

Lemma 3.2

Let u be the solution of (2.1) - (2.3). Then there exists a constant C, independent of K, h, and u such that

$$(3.7)\qquad \max_{\substack{1\le i,j\le J\\ 1\le n\le N-1}} \left| \delta_t^2 u_{ij}^n - r^2[\alpha\diamond u_{ij}^{n+1} + (1-2\alpha)\diamond u_{ij}^n + \alpha\diamond u_{ij}^{n-1}] + r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n)\right|$$

$$\le C\,k^2(k^2+h^2)\max_{x,y,t}\Big[\Big|\frac{\partial^4 u}{\partial x^4}\Big|, \Big|\frac{\partial^4 u}{\partial y^4}\Big|, \Big|\frac{\partial^4 u}{\partial t^4}\Big|, \Big|\frac{\partial^6 u}{\partial t^6}\Big|,$$

$$\Big|\frac{\partial^6 u}{\partial t^2\partial x^2\partial y^2}\Big|, \Big|\frac{\partial^8 u}{\partial t^2\partial y^2\partial x^4}\Big|, \Big|\frac{\partial^8 u}{\partial t^2\partial x^2\partial y^4}\Big|, \Big|\frac{\partial^{10} u}{\partial t^2\partial x^4\partial y^4}\Big|,$$

$$\Big|\frac{\partial^8 u}{\partial t^4\partial x^2\partial y^2}\Big|, \Big|\frac{\partial^{10} u}{\partial t^4\partial x^2\partial y^4}\Big|, \Big|\frac{\partial^{10} u}{\partial t^4\partial x^4\partial y^2}\Big|, \Big|\frac{\partial^{12} u}{\partial t^4\partial x^4\partial y^4}\Big|\Big].$$

Proof. For convenience we let

$$C^*(u) = \max_{x,y,t}\Big[\Big|\frac{\partial^4 u}{\partial x^4}\Big|, \ldots, \Big|\frac{\partial^{12} u}{\partial t^4\partial x^4\partial y^4}\Big|\Big].$$

We observe that

$$(3.8)\qquad |\delta_t^2 u_{ij}^n - r^2\diamond[\alpha u_{ij}^{n+1} + (1 - 2\alpha)u_{ij}^n + \alpha u_{ij}^{n-1}] + r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n)|$$

$$\leq |\delta_t^2 u_{ij}^n - r^2\diamond[\alpha u_{ij}^{n+1} + (1 - 2\alpha)u_{ij}^n + \alpha u_{ij}^{n-1}]|$$

$$+ r^4\alpha^2\,\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n) \quad .$$

Also we have

$$\delta_t^2 u_{ij}^n = k^2 \frac{\partial^2 u}{\partial t^2}(x_i, y_j, t^n)$$

$$+ \frac{k^4}{24}[\frac{\partial^4 u}{\partial t^4}(x_i, y_j, \theta_1) + \frac{\partial^4 u}{\partial t^4}(x_i, y_j, \theta_2)] ,$$

where $\theta_1 = \theta_1(i, j, n)$, $\theta_2 = \theta_2(i, j, n)$, and $t^n \leq \theta_1 \leq t^{n+1}$, $t^{n-1} \leq \theta_2 \leq t^n$,

$$\delta_y^2 u_{ij}^n = h^2 \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t^n) + \frac{h^4}{24}[\frac{\partial^4 u}{\partial y^4}(x_i, \theta_1', t^n)$$

$$+ \frac{\partial^4 u}{\partial y^4}(x_i, \theta_2', t^n)] ,$$

where again $y_j \leq \theta_1' \leq y_{j+1}$, $y_{j-1} \leq \theta_2' \leq y_j$,

$$\delta_x^2 u_{ij}^n = h^2 \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t^n) + \frac{h^4}{24}[\frac{\partial^4 u}{\partial x^4}(\theta_1'', y_j, t^n)$$

$$+ \frac{\partial^4 u}{\partial x^4}(\theta_2'', y_j, t^n)] ,$$

where $x_i \leq \theta_1'' \leq x_{i+1}$, $x_{i-1} \leq \theta_2'' \leq x_i$. Hence, we obtain that

$$(3.9)\qquad \delta_x^2\delta_y^2(\delta_t^2\delta_{ij}^n) = \delta_x^2\delta_y^2[k^2\frac{\partial^2 u}{\partial t^2}(x_i, y_j, t^n) + \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4}(x_i, y_j, \theta_1)$$

$$+ \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4}(x_i, y_j, \theta_2)]$$

$$= \delta_x^2[k^2h^2\frac{\partial^4 u}{\partial t^2\partial y^2}(x_i, y_j, t^n) + \frac{k^2h^4}{24}\frac{\partial^6 u}{\partial t^2\partial y^4}(x_i, \xi_1, t^n)$$

$$+ \frac{k^2h^4}{24}\frac{\partial^6 u}{\partial t^2\partial y^4}(x_i, \xi_2, t^n) + \frac{k^4h^2}{24}\frac{\partial^6 u}{\partial t^4\partial y^2}(x_i, y_j, \theta_1) +$$

$$+ \frac{k^4h^4}{(24)^2}\frac{\partial^8 u}{\partial t^4\partial y^4}(x_i, \xi_3, \theta_1) + \frac{k^4h^4}{(24)^2}\frac{\partial^8 u}{\partial t^4\partial y^4}(x_i, \xi_4, \theta_1)$$

$$+ \frac{k^4}{24}h^2\frac{\partial^6 u}{\partial t^4\partial y^2}(x_i, y_j, \theta_2) + \frac{k^4h^4}{(24)^2}\frac{\partial^8 u}{\partial t^4\partial y^4}(x_i, \xi_2, \theta_2)$$

$$+ \frac{k^4h^4}{(24)^2}\frac{\partial^8 u}{\partial t^4\partial y^4}(x_i, \xi_6, \theta_2)] ,$$

where $t^n \le \theta_1 = \theta_1(i, j, k) \le t^{n+1}$, $t^{n-1} \le \theta_2 = \theta_2(i, j, k) \le t^n$,

$$y_j \le \xi_1, \xi_2, \xi_3 \le y_{j+1} \ , \quad y_{j-1} \le \xi_2, \xi_4, \xi_6 \le y_j \ .$$

If we now apply the operator δ_x^2 to the RHS of (3.9), we obtain that

$$(3.10)\qquad |\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n)| \le ck^2h^4 \max_{x,y,t}[|\frac{\partial^6 u}{\partial t^2\partial x^2\partial y^2}|, |\frac{\partial^8 u}{\partial t^2\partial y^2\partial x^4}|, |\frac{\partial^8 u}{\partial t^2\partial y^4\partial x^2}|,$$

$$|\frac{\partial^{10} u}{\partial t^2\partial y^4\partial x^4}|, |\frac{\partial^8 u}{\partial t^4\partial x^2\partial y^2}|, |\frac{\partial^{10} u}{\partial t^4\partial y^2\partial x^4}|,$$

$$\left|\frac{\partial^{10} u}{\partial t^4 \partial y^4 \partial x^2}\right|, \left|\frac{\partial^{12} u}{\partial t^4 \partial y^4 \partial x^4}\right|] ,$$

where C is a constant independent of h, k, and u.

(3.10) implies that

$$(3.11) \quad r^4\alpha^2 |\delta_x^2 \delta_y^2 (\delta_t^2 u_{ij}^n)| \le Ck^6 \max_{x,y,t} [\left|\frac{\partial^6 u}{\partial t^2 \partial x^2 \partial y^2}\right|, \left|\frac{\partial^8 u}{\partial t^2 \partial x^4 \partial y^2}\right|,$$

$$\left|\frac{\partial^8 u}{\partial t^2 \partial x^2 \partial y^4}\right|, \left|\frac{\partial^{10} u}{\partial t^2 \partial x^4 \partial y^4}\right|, \left|\frac{\partial^8 u}{\partial t^4 \partial x^2 \partial y^2}\right|, \left|\frac{\partial^{10} u}{\partial t^4 \partial y^4 \partial x^2}\right|,$$

$$\left|\frac{\partial^{10} u}{\partial t^4 \partial y^2 \partial x^4}\right|, \left|\frac{\partial^{12} u}{\partial t^4 \partial x^4 \partial y^4}\right|] ,$$

$$1 \le i, j \le J \quad , \quad 1 \le n \le N - 1 .$$

Thus, (3.8), (3.11) and Lemma (2.3) imply the result of Lemma (3.2). ■

We are now ready to proceed to the error estimate. For concreteness, we suppose that the initial conditions U_{ij}^1 are computed by (2.8') .

In preparation of our convergence theorem, we define for mesh functions $u, w \in S_h$ (for definition of S_h , see page 16) the bilinear form $\beta_h(v, w)$ by:

$$\beta_h(v, w) = -h^2 \sum_{i,j=1}^{J} (\delta_x^2 \delta_y^2 u_{ij}) w_{ij} .$$

Then we have

Lemma 3.3

Let $v = v_{ij}$ and $w = w_{ij}$ be mesh functions in S_h. Then, with β_h as above

$$\beta_h(v, w) = \beta_h(w, v) , \tag{3.12}$$

$$\beta_h(v, v) \leq 0 . \tag{3.13}$$

Proof. In order to show (3.12) we observe that

$$\begin{aligned}
\beta_h(v, w) &= -h^2 \sum_{i,j=1}^{J} (\delta_x^2 \delta_y^2 u_{ij}) w_{ij} \\
&= -h^2 \sum_{i,j=1}^{J} [(v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1}) \\
&\qquad - 2(v_{i+1,j} - 2v_{ij} + v_{i-1,j}) \\
&\qquad + (v_{i+1,j-1} - 2v_{i,j-1} + v_{i-1,j-1})] w_{ij} = \\
&= -h^2 \sum_{i,j=1}^{J} [v_{i+1,j+1} w_{ij} - 2v_{i,j+1} w_{ij} + v_{i-1,j+1} w_{ij} \\
&\qquad - 2v_{i+1,j} w_{ij} + 4v_{ij} w_{ij} - 2v_{i-1,j} w_{ij} \\
&\qquad + v_{i+1,j-1} w_{ij} - 2v_{i,j-1} w_{ij} + v_{i-1,j-1} w_{ij}] .
\end{aligned} \tag{3.14}$$

Similarly,

$$(3.15)\qquad \beta_h(w, v) = -h^2 \sum_{i,j=1}^{J} (\delta_x^2\delta_y^2 w_{ij})v_{ij}$$

$$= -h^2 \sum_{i,j=1}^{J} [w_{i+1,j+1} v_{ij} - 2w_{i,j+1} v_{ij} + w_{i-1,j+1} v_{ij}$$

$$-2w_{i+1,j} v_{ij} + 4w_{ij} v_{ij} - 2w_{i-1,j} v_{ij}$$

$$+ w_{i+1,j-1} v_{ij} - 2w_{i,j-1} v_{ij} + w_{i-1,j-1} v_{ij}] .$$

It is easy to see, considering the boundary conditions, that the right hand sides of (3.14) and (3.15) are equal. Thus, for all $v, w \in S_h$

$$\beta_h(v, w) = \beta_h(w, v) .$$

In order to show (3.13) we have

$$(3.16)\qquad \beta_h(v, v) = -h^2 \sum_{i,j=1}^{J} (\delta_x^2\delta_y^2 v_{ij})v_{ij}$$

$$= -h^2 \sum_{i=1}^{J} \delta_x^2 \Big(\sum_{j=1}^{J} \delta_y^2 v_{ij} \Big) v_{ij}$$

$$= -h^2 \sum_{i=1}^{J} \delta_x^2 \sum_{j=1}^{J} (v_{i,j+1} - 2v_{ij} + v_{i,j-1})v_{ij} =$$

$$= -h^2 \sum_{i=1}^{J} \delta_x^2 c_i ,$$

where $C_i = \sum_{j=1}^{J} (v_{i,j+1} - 2v_{ij} + v_{i,j-1})v_{ij}$, $i = 1, 2, \ldots, J + 1$.

From (ii) in Lemma (2.4), we conclude

$$C_i < 0 \ , \ 1 \le i \le J \ , \ C_0 = C_{j+1} = 0 \ .$$

Hence, (3.16) becomes

$$\beta_h(v, v) = -h^2 \sum_{i=1}^{J} (C_{i+1} - 2C_i + C_{i-1})$$

$$= -h^2[C_2 + C_3 + \ldots + C_J] + 2h^2[C_1 + C_2 + \ldots + C_J]$$

$$- h^2[C_1 + C_2 + \ldots + C_{J-1}] = h^2(C_1 + C_j) < 0 \ .$$

Thus, the proof of (3.13) is complete. ■

Theorem 3.1

Let u be the solution of (2.1) - (2.3). Suppose that $\{U_{ij}^n\}$, $0 \le n \le N$, $0 \le i, j \le J$ is the solution of the difference scheme (3.1) with the initial conditions (2.8'). If U_h^n, u^n are mesh functions U_{ij}^n, u_{ij}^n respectively, then for every $\alpha \ge 0$ there exists a constant $C = C(\alpha, r, T)$, such that if $r = \frac{ck}{h} < r_0(\alpha)$, then

$$(3.17) \qquad \max_n \| U_h^n - u^n \|_h \le C(\alpha, r, T)(h^2 + k^2)C^*(u) \ ,$$

where

$$r_0(\alpha) = \begin{cases} +\infty \ , & \text{if } \alpha \ge \frac{1}{4} \\ \dfrac{1}{\sqrt{2(1 - 4\alpha)}} \ , & \text{if } 0 \le \alpha < \frac{1}{4} \ , \end{cases}$$

the norm $\|\cdot\|_h$ was defined by (2.24), and $C^*(u)$ was defined in Lemma 3.2.

Proof. For $1 \le i, j \le J$, if $e_{ij}^n = U_{ij}^n - u_{ij}^n$ we have

$$e_{ij}^{n+1} - 2e_{ij}^n + e_{ij}^{n-1} = (U_{ij}^{n+1} - 2U_{ij}^n + U_{ij}^{n-1}) - (u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1})$$

$$= r^2\alpha\diamond U_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond U_{ij}^n + r^2\alpha\diamond U_{ij}^{n-1} - r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 U_{ij}^n) -$$

$$- (u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}) - r^2\alpha\diamond u_{ij}^{n+1} - r^2(1 - 2\alpha)\diamond u_{ij}^n$$

$$- r^2\alpha\diamond u_{ij}^{n-1} + r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n) + r^2\alpha\diamond u_{ij}^{n+1}$$

$$+ r^2(1 - 2\alpha)\diamond u_{ij}^n + r^2\alpha\diamond u_{ij}^{n-1} - r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n) =$$

$$= r^2\alpha\diamond e_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond e_{ij}^n + r^2\alpha\diamond e_{ij}^{n-1} - r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 e_{ij}^n) + \varepsilon_{ij}^n ,$$

where

$$\varepsilon_{ij}^n = r^2\alpha\diamond u_{ij}^{n+1} + r^2(1 - 2\alpha)\diamond u_{ij}^n + r^2\alpha\diamond u_{ij}^{n-1}$$

$$- r^4\alpha^2\delta_x^2\delta_y^2(\delta_t^2 u_{ij}^n) - \delta_t^2 u_{ij}^n ,$$

$$1 \le i,j \le J , \quad 1 \le n \le N ,$$

and furthermore

$$\max_{\substack{1\le i,j\le J\\ \bar{1}\le n\le \bar{N}}} |\varepsilon_{ij}^n| \le C\, k^2(k^2 + h^2)\, C^*(u) ,$$

The constant C is independent of h, k, and u. From the above we obtain for $1 \le n \le N - 1$

$$(3.18)\quad h^2 \sum_{i,j=1}^{J} (e_{ij}^{n+1} - 2e_{ij}^n + e_{ij}^{n-1})(e_{ij}^{n+1} - e_{ij}^{n-1}) =$$

$$= r^2\alpha h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^{n+1}\,(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ r^2(1 - 2\alpha)\, h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ r^2\alpha h^2 \sum_{i,j=1}^{J} e_{ij}^{n-1}(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$- r^4\alpha^2 h^2 \sum_{i,j=1}^{J} \delta_x^2\delta_y^2(\delta_t^2\, e_{ij}^n)(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) .$$

(3.18) implies that

$$\|e^{n+1} - e^n\|_h^2 - \|e^n - e^{n-1}\|_h^2 = -r^2\alpha a_h(e^{n+1},\, e^{n+1} - e^{n-1})$$

$$- r^2(1 - 2\alpha)a_h(e^n,\, e^{n+1} - e^{n-1}) - r^2\alpha a_h(e^{n-1},\, e^{n+1} - e^{n-1})$$

$$- r^4\alpha^2 h^2 \sum_{i,j=1}^{J} \delta_x^2\delta_y^2(\delta_t^2\, e_{ij}^n)(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) ,$$

where a_h has been defined by (2.19). We observe that

$$-r^4\alpha^2h^2 \sum_{i,j=1}^{J} \delta_x^2\delta_y^2(e_{ij}^{n+1} - 2e_{ij}^n + e_{ij}^{n-1})(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$= r^4\alpha^2\beta_h(e^{n+1} - 2e^n + e^{n-1}, e^{n+1} - e^{n-1}) =$$

$$= r^4\alpha^2[\beta_h(e^{n+1} - e^n, e^{n+1} - e^n) + \beta_h(e^{n+1} - e^n, e^n - e^{n-1})$$

$$- \beta_h(e^n - e^{n-1}, e^{n+1} - e^n) - \beta_h(e^n - e^{n-1}, e^n - e^{n-1})]$$

$$= r^4\alpha^2[\beta_h(e^{n+1} - e^n, e^{n+1} - e^n) - \beta_h(e^n - e^{n-1}, e^n - e^{n-1})] ,$$

where the symmetry of β_h was used. Thus, (3.19) becomes

$$\|e^{n+1} - e^n\|_h^2 - \|e^n - e^{n-1}\|_h^2 + \frac{r^2}{2}[a_h(e^{n+1}, e^{n+1}) - a_h(e^{n-1}, e^{n-1})]$$

$$- \frac{r^2}{2}(1 - 2\alpha)[a_h(e^{n+1} - e^n , e^{n+1} - e^n)$$

$$-a_h(e^n - e^{n-1}, e^n - e^{n-1})]$$

$$= r^4\alpha^2[\beta_h(e^{n+1} - e^n, e^{n+1} - e^n) - \beta_h(e^n - e^{n-1}, e^n - e^{n-1})]$$

$$+ h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^n(e_{ij}^{n+1} - e_{ij}^{n-1}) .$$

Summing both sides of the above from $n = 1$ to $n = n' + 1$ and putting $n = n'$ again, we obtain

$$\|e^n - e^{n-1}\|_h^2 + \frac{r^4}{4} a_h(e^n + e^{n-1}, e^n - e^{n-1})$$

$$+ \frac{r^2}{4} (4\alpha - 1)\, a_h(e^n - e^{n-1}, e^n - e^{n-1})$$

$$\leq (1 + 8\alpha r^2)\|e^1\|_h^2 + r^4\alpha^2\beta_h(e^n - e^{n-1}, e^n - e^{n-1})$$

$$- r^4\alpha^2\, \beta_h(e^1, e^1) + \sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \, \|e^{\ell+1} - e^{\ell-1}\|_h \, .$$

Taking into account the fact that

$$\frac{r^4}{4} a_h(e^n + e^{n-1}, e^n + e^{n-1}) \geq 0 \, ,$$

$$r^4\alpha^2\beta_h(e^n - e^{n-1}, e^n - e^{n-1}) \leq 0 \, ,$$

we obtain

$$\|e^n - e^{n-1}\|_h^2 + \frac{r^2}{4}(4\alpha - 1)a_h(e^n - e^{n-1}, e^n - e^{n-1}) \tag{3.21}$$

$$\leq (1 + 8\alpha r^2)\|e^1\|_h^2 - r^4\alpha^2\beta_h(e^1, e^1) + \sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \, \|e^{\ell+1} - e^{\ell-1}\|_h \, .$$

Now, we observe that

$$-r^4\alpha^2\beta_h(e^1, e^1) = -r^4\alpha^2h^2 \sum_{i=1}^{J} (e^1_{i+1,J} - 2e^1_{1,J} + e^1_{i-1,J})e^1_{i,J}$$

$$- r^4\alpha^2h^2 \sum_{i=1}^{J} (e_{i+1,1} - 2e_{i,1} + e_{i-1,1})e^1_{i,1}$$

$$\leq -r^4\alpha^2h^2 \sum_{i,j=1}^{J} (e^1_{i+1,j} - 2e^1_{ij} + e^1_{i-1,j})e^1_{ij} \, .$$

Similarly, we obtain

$$-r^4\alpha^2\beta_h(e^1, e^1) \le -r^4\alpha^2h^2 \sum_{i,j=1}^{J} (e^1_{i,j+1} - 2e^1_{ij} + e^1_{i,j-1})e^1_{ij} .$$

From the above we conclude

$$-2r^4\alpha^2\beta_h(e^1, e^1) \le -r^4\alpha^2h^2 \sum_{i,j=1}^{J} (\diamond e^1_{ij})e^1_{ij} = r^4\alpha^2 a_h(e^1, e^1) .$$

Hence, by (2.23)

$$-r^4\alpha^2\beta_h(e^1, e^1) \le 4r^4\alpha^2\|e^1\|_h^2 .$$

Therefore, (3.21) becomes

$$(3.22) \qquad \|e^n - e^{n-1}\|_h^2 + \frac{r^2}{4}(4\alpha - 1)a_h(e^n - e^{n-1}, e^n - e^{n-1})$$

$$\le (1 + 8\alpha r^2 + 4r^4\alpha^2)\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \|e^{\ell+1} - e^{\ell-1}\|_h .$$

Let $\alpha \ge \frac{1}{4}$. Then

$$(3.23) \qquad \|e^n - e^{n-1}\|_h^2 \le (1 + 8\alpha r^2 + 4r^4\alpha^2)\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \|e^{\ell+1} - e^{\ell-1}\|_h .$$

Using the Cauchy-Schwartz inequality, we obtain

$$(3.24) \qquad \sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \le [\sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h^2]^{1/2}$$

$$\cdot [\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2]^{1/2} .$$

We put

$$E = \Big[\sum_{\ell=1}^{N-1} \|\varepsilon^{\ell}\|_h^2\Big]^{1/2} .$$

Also, we have that

$$\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2 \le \sum_{\ell=1}^{n-1} [\|e^{\ell+1} - e^{\ell}\|_h + \|e^{\ell} - e^{\ell-1}\|_h]^2$$

$$\le 2 \sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell}\|_h^2 + 2 \sum_{\ell=1}^{n-1} \|e^{\ell} - e^{\ell-1}\|_h^2 \le 4 \sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2 .$$

Hence,

$$\Big[\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|^{1/2} \le 2\Big[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2\Big]^{1/2} .$$

Thus, (3.24) becomes

$$\sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \le 2E\Big[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2\Big]^{1/2}$$

$$\le \frac{E^2}{A} + A \sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2 \le \frac{E^2}{A} + AN \max_{1\le n\le N} \|e^{n} - e^{n-1}\|_h^2$$

for any $A > 0$. Choosing $A = \frac{1}{2N}$, we obtain

$$(3.25)\qquad \sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \le 2NE^2 + \frac{1}{2} \max_{1\le n\le N} \|e^n - e^{n-1}\|_h^2 .$$

Hence, in conclusion (3.23) becomes

$$(3.26)\qquad \max_{1\le n\le N} \|e^n - e^{n-1}\|_h^2 \le 2(1 + 8\alpha r^2 + 4r^4\alpha^2)\|e^1\|_h^2 + 4NE^2 .$$

We notice

$$NE^2 \le CN \sum_{\ell=1}^{N-1} C^{*2}(u)\, k^4(k^2 + h^2)^2,$$

where C is a constant independent of h, k, and u.

Thus,

$$NE^2 \le CN^2\, C^{*2}(u)\, k^4(k^2 + h^2)^2 = CT^2k^2C^{*2}(u)\, k^2(k^2 + h^2)^2 .$$

Using $\sqrt{a^2 + b^2} \le a + b$ for $a, b \ge 0$, we get

$$(3.27)\qquad \max_{1\le n\le N} \|e^n - e^{n-1}\|_h \le \sqrt{2(1 + 8\alpha r^2 + 4r^4\alpha^2)}\,\|e^1\|_h + 2CTk(k^2 + h^2)C^*(u) = R ..$$

Since $\|e^n\|_h \le \|e^n - e^{n-1}\|_h + \|e^{n-1}\|_h$ and $\|e^0\|_h = 0$, we obtain

$$(3.28)\qquad \max_{1\le n\le N} \|e^n\|_h \le [\max_{1\le n\le N} \|e^n - e^{n-1}\|_h] \cdot N \le RN .$$

If we take into account the fact that $KN = T$ and (2.9'), we obtain finally the result of Theorem 3.1.

Following exactly the same procedure as we did in the proof of Theorem 2.1, we can easily obtain that if $0 \leq \alpha < \frac{1}{4}$ we have again convergence only if $r < \frac{1}{\sqrt{2(1 - 4\alpha)}}$. ■

Hence, we have proved that if $\alpha \geq \frac{1}{4}$ the difference scheme (3.1) is unconditionally stable and has an accuracy of $O(h^2 + k^2)$ in the discrete ℓ_2-norm. If $0 \leq \alpha < \frac{1}{4}$ we obtain stability only if $r < r_0(\alpha)$ with the same accuracy of $O(h^2 + k^2)$.

CHAPTER IV

A FOURTH-ORDER ACCURATE DIFFERENCE APPROXIMATION TO THE WAVE EQUATION ON THE UNIT SQUARE

We consider again the problem stated in Chapter II by (2.1), (2.2), and (2.3). In this chapter, we shall approximate the solution of (2.1) - (2.3) by using a particular fourth-order in space and time implicit difference scheme proposed by Collatz [8]. Using the notation introduced in Chapter I, we seek U_{ij}^n , $0 \le n \le N$, $0 \le i, j \le J + 1$ satisfying

$$(4.1) \qquad \delta_t^2 U_{ij}^n = \frac{(1 - \rho)}{12\rho} \diamond (U_{ij}^{n+1} + U_{ij}^{n-1}) + \frac{(10 + 2\rho)}{12\rho} U_{ij}^n + \frac{2}{12\rho} \delta_x^2 \delta_y^2 U_{ij}^n ,$$

$$1 \le i, j \le J , \quad 1 \le n \le N - 1$$

where, as usual,

$$\delta_t^2 U_{ij}^n = U_{ij}^{n+1} - 2U_{ij}^n + U_{ij}^{n-1} ,$$

$$\diamond U_{ij}^n = U_{i+1,j}^n + U_{i-1,j}^n + U_{i,j+1}^n + U_{i,j-1}^N - 4 U_{ij}^n ,$$

$$1 \le i, j \le J \quad \text{and} \quad \rho = \frac{1}{r^2} .$$

The difference scheme (4.1) is supplemented with the boundary conditions

$$(4.2) \qquad U_{0,j}^n = U_{i,0}^n = U_{i,J+1}^n = U_{J+1,j}^n = 0 , \quad 0 \le i , j \le J + 1 , \; 0 \le n \le N.$$

We compute the initial conditions U^0_{ij} and U^1_{ij} as follows:

Anticipating that the scheme (4.1) will be of 4^{th} order accuracy, we note that Taylor's theorem gives

$$(*) \qquad u^1_{ij} = u^0_{ij} + ku_t\Big|^0_{ij} + \frac{k^2}{2} u_{tt}\Big|^0_{ij} + \frac{k^3}{6} u_{ttt}\Big|^0_{ij} + \frac{k^4}{24} u_{tttt}\Big|^0_{ij} +$$

$$+ \frac{k^5}{5!} \frac{\partial^5 u}{\partial t^5} (x_i, y_i, \theta_{ij}) ,$$

where $0 \leq \theta_{ij} \leq k$.

Now, using the wave equation

$$u^0_{tt} = c^2(u^0_{xx} + u^0_{yy}) = c^2(f_{xx} + f_{yy}) ,$$

$$u^0_{ttt} = c^2(u^0_{txx} + u^0_{tyy}) = c^2(g_{xx} + g_{yy}) ,$$

$$u^0_{tttt} = c^2(u^0_{xxtt} + u^0_{yytt}) = c^2(u^0_{ttxx} + u^0_{ttyy})$$

$$= c^4(u^0_{xxxx} + u^0_{xxyy} + u^0_{xxyy} + u^0_{yyyy})$$

$$= c^4(f_{xxxx} + 2f_{xxyy} + f_{yyyy}) .$$

For simplicity, we assume that the functions f_{xx}, f_{yy}, g_{xx}, g_{yy}, f_{xxxx}, f_{yyyy}, f_{xxyy} are available. Hence we may compute the initial conditions U^0_{ij}, U^1_{ij} of the scheme (4.1) as follows:

$$
(4.3)\qquad \begin{cases} U_{ij}^0 = f_{ij} \ , \ 1 \le i, j \le J \\ \\ U_{ij}^1 = f_{ij} + kg_{ij} + \dfrac{k^2c^2}{2}(f_{xx} + f_{yy})\big|_{ij} + \dfrac{k^3c^2}{6}(g_{xx} + g_{yy})\big|_{ij} \\ \qquad\qquad + \dfrac{k^4c^4}{24}(f_{xxxx} + 2f_{xxyy} + f_{yyyy})\big|_{ij} \\ \\ U_{ij}^{\alpha} = 0 \ , \text{ on the boundary of the unit square, } \ \alpha = 0, 1 \ . \end{cases}
$$

It follows immediately by (*) that

$$
(**)\qquad \max_{i,j} |u_{ij}^1 - U_{ij}^1| \le \frac{k^5}{120} \max_{x,y,t} \left|\frac{\partial^5 u}{\partial t^5}\right| \ .
$$

$\{U_{ij}^n\}$ $1 \le i, j \le J$ may be found as the solution of a $J^2 \times J^2$ system of linear equations. Specifically, we order $\{U_{ij}^n\}$ as vector U_j^n in $\mathbb{R}^{J^2}$ by setting $U_{h,1}^n = U_{1,1}^n$, $U_{h,2}^n = U_{2,1}^n, \ldots, U_{h,J}^n = U_{J,1}^n$, $U_{h,J+1}^n = U_{1,2}^n$, $\ldots$, $U_{h,J^2}^n = U_{J,J}^n$. Clearly the scheme (4.1) is equivalent for each n , $n = 2$, $J - 1$ to the linear system

$$
A_h \, U_h^n = F_h^n \ ,
$$

where F_h^n is a J^2-vector depending upon the previous values U_h^{n-1}, U_h^{n-2}, and A_h is the block tridiagonal matrix

$$A_h = \begin{bmatrix} C & B & 0 \\ B & & B \\ 0 & B & C \end{bmatrix}$$

where if we let $d = 1 + \frac{1 - \rho}{3\rho}$ and $s = \frac{1 - \rho}{12\rho}$ then the $J \times J$ matrices C, B are given by

$$C = \begin{bmatrix} d & -s & 0 \\ -s & & -s \\ 0 & -s & d \end{bmatrix}$$

$$B = \begin{bmatrix} -s & & \\ & 0 & \\ & 0 & \\ & & -s \end{bmatrix} .$$

Lemma 4.1.

A_h is symmetric positive definite.

Proof. We observe that for all $\rho > 0$

$$|1 + \frac{1 - \rho}{3\rho}| > |\frac{1 - \rho}{12\rho}| \cdot 4 \quad .$$

Thus, A_h is strictly diagonally dominant. Therefore, A_h is symmetric positive definite being symmetric and having positive diagonal. ■

Lemma 4.2.

Let u be the solution of (2.1) - (2.3). Then, there exists a constant C, independent of k, h, and u such that

$$(4.4)\qquad \max_{\substack{1\le i,j\le J\\ 1\le n\le N-1}} \Big|\delta_t^2 u_{ij}^n - \frac{(1-\rho)}{12\rho}\diamond(u_{ij}^{n+1}+u_{ij}^{n-1}) - \frac{(10+2\rho)}{12\rho}\diamond u_{ij}^n - \frac{2}{12\rho}\delta_x^2\delta_y^2 u_{ij}^n\Big| \le Ck^2(k^4+h^4)\quad C_4(u)\ ,$$

where

$$C_4(u) = \max_{x,y,t}\Big[\Big|\frac{\partial^6 u}{\partial t^6}\Big|,\ \Big|\frac{\partial^6 u}{\partial x^6}\Big|,\ \Big|\frac{\partial^6 u}{\partial y^6}\Big|,\ \Big|\frac{\partial^{\alpha+\beta} u}{\partial t^\alpha \partial x^\beta}\Big|,\ \Big|\frac{\partial^{\alpha+\beta} u}{\partial t^\alpha \partial y^\beta}\Big|,\ \Big|\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial y^\beta}\Big|\Big].$$

$$\alpha,\ \beta = 1,\ldots,6 \quad\text{and}\quad \alpha+\beta = 6\ .$$

Proof. We have that

$$u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1} - \frac{(r^2-1)}{12}\big[u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1} - 4u_{ij}^{n+1}\big]$$

$$- \frac{(r^2-1)}{12}\big[u_{i+1,j}^{n-1} + u_{i-1,j}^{n-1} + u_{i,j+1}^{n-1} + u_{i,j-1}^{n-1} - 4u_{ij}^{n-1}\big]$$

$$- \frac{(10r^2+2)}{12}\big[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{ij}^{n}\big]$$

$$- \frac{r^2}{6}\big[u_{i+1,j+1}^{n} + u_{i-1,j+1}^{n} + u_{i+1,j-1}^{n} + u_{i-1,j-1}^{n}$$

$$- 2u^n_{i,j+1} - 2u^n_{i+1,j} - 2u^n_{i-1,j} - 2u^n_{i,j-1} + 4u^n_{ij}] =$$

$$= (u^n_{ij} + k\frac{\partial}{\partial t}u^n_{ij} + \frac{k^2}{2}\frac{\partial^2}{\partial t^2}u^n_{ij} + \frac{k^3}{6}\frac{\partial^3}{\partial t^3}u^n_{ij} + \frac{k^4}{24}\frac{\partial^4}{\partial t^4}u^n_{ij}$$

$$+ \frac{k^5}{5!}\frac{\partial^5}{\partial t^5}u^n_{ij} + \frac{k^6}{6!}\frac{\partial^6}{\partial t^6}u(x_i, y_j, \theta_1))$$

$$- 2u^n_{ij} + (u^n_{ij} - k\frac{\partial}{\partial t}u^n_{ij} + \frac{k^2}{2}\frac{\partial^2}{\partial t^2}u^n_{ij}$$

$$- \frac{k^3}{6}\frac{\partial^3}{\partial t^3}u^n_{ij} + \frac{k^4}{24}\frac{\partial^4}{\partial t^4}u^n_{ij} - \frac{k^5}{5!}\frac{\partial^5}{\partial t^5}u^n_{ij} + \frac{k^6}{6!}\frac{\partial^6}{\partial t^6}u(x_i, y_j, \theta_2))$$

$$- \frac{(r^2-1)}{12}[u^n_{ij} + (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})u^n_{ij} + (\frac{h^2}{2}\frac{\partial^2}{\partial x^2}u^n_{ij} + \frac{k^2}{2}\frac{\partial^2}{\partial t^2}u^n_{ij}$$

$$+ hk\frac{\partial^2}{\partial x\partial t}u^n_{ij}) + \frac{1}{3!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})^3 u^n_{ij} + (\frac{h^4}{24}\frac{\partial^4}{\partial x^4}u^n_{ij} + \frac{k^4}{24}\frac{\partial^4}{\partial t^4}u^n_{ij}$$

$$+ \frac{hk^3}{6}\frac{\partial^4}{\partial t^3\partial x}u^n_{ij} + \frac{h^2k^2}{4}\frac{\partial^4}{\partial t^2\partial x^2}u^n_{ij} + \frac{h^3k}{6}\frac{\partial^4}{\partial x^3\partial t}u^n_{ij})$$

$$+ \frac{1}{5!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})^5 u^n_{ij} + \frac{1}{6!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})^6 u(\xi_1, y_j, \theta_3)$$

$$+ u^n_{ij} + (-h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})u^n_{ij} + (\frac{h^2}{2}\frac{\partial^2}{\partial x^2}u^n_{ij} + \frac{k^2}{2}\frac{\partial^2}{\partial t^2}u^n_{ij}$$

$$- hk\frac{\partial^2}{\partial x\partial t}u^n_{ij}) + \frac{1}{3!}(-h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})^3 u^n_{ij} + (\frac{h^4}{24}\frac{\partial^4}{\partial x^4}u^n_{ij}$$

$$+ \frac{k^4}{24}\frac{\partial^4}{\partial t^4}u^n_{ij} - \frac{hk^3}{6}\frac{\partial^4}{\partial x\partial t^3}u^n_{ij} + \frac{h^2k^2}{4}\frac{\partial^4}{\partial x^2\partial t^2}u^n_{ij} -$$

$$- \frac{h^3k}{6}\frac{\partial^4}{\partial x^3\partial t}u^n_{ij}) + \frac{1}{5!}(-h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})^5 u^n_{ij}$$

$$+ \frac{1}{6!}\left(-h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t}\right)^6 u(\xi_2, y_j, \theta_4)$$

$$+ u_{ij}^n + \left(h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right) u_{ij}^n + \left(\frac{h^2}{2}\frac{\partial^2}{\partial y^2} u_{ij}^n + \frac{k^2}{2}\frac{\partial^2}{\partial t^2} u_{ij}^n\right.$$

$$\left. + hk\frac{\partial^2}{\partial y \partial t} u_{ij}^n\right) + \frac{1}{3!}\left(h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right)^3 + \left(\frac{h^4}{24}\frac{\partial^4}{\partial y^4} u_{ij}^n\right.$$

$$+ \frac{k^4}{24}\frac{\partial^4}{\partial t^4} u_{ij}^n + \frac{h^3 k}{6}\frac{\partial^4}{\partial y^3 \partial t} u_{ij}^n + \frac{hk^3}{6}\frac{\partial^4}{\partial y \partial t^3} u_{ij}^n +$$

$$\left. + \frac{h^2 k^2}{4}\frac{\partial^4}{\partial y^2 \partial t^2} u_{ij}^n\right) + \frac{1}{5!}\left(h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right)^5 u_{ij}^n$$

$$+ \frac{1}{6!}\left(h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right)^6 u(x_i, \xi_3, \theta_5)$$

$$+ u_{ij}^n - h\frac{\partial}{\partial y} u_{ij}^n + k\frac{\partial}{\partial t} u_{ij}^n + \left(\frac{h^2}{2}\frac{\partial^2}{\partial y^2} u_{ij}^n\right.$$

$$\left. + \frac{k^2}{2}\frac{\partial^2}{\partial t^2} u_{ij}^n - hk\frac{\partial^2}{\partial y \partial t} u_{ij}^n\right) =$$

$$+ \frac{1}{3!}\left(-h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right)^3 u_{ij}^n + \left(\frac{h^4}{24}\frac{\partial^4}{\partial y^4} u_{ij}^n + \frac{k^4}{24}\frac{\partial^4}{\partial t^4} u_{ij}^n\right.$$

$$\left. - \frac{hk^3}{6}\frac{\partial^4}{\partial y \partial t^3} u_{ij}^n + \frac{h^2 k^2}{4}\frac{\partial^4}{\partial y^2 \partial t^2} u_{ij}^n - \frac{h^3 k}{6}\frac{\partial^4}{\partial y^3 \partial t} u_{ij}^n\right)$$

$$+ \frac{1}{5!}\left(-h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right)^5 u_{ij}^n + \frac{1}{6!}\left(-h\frac{\partial}{\partial y} + k\frac{\partial}{\partial t}\right)^6 u(x_i, \xi_4, \theta_6)$$

$$-4u_{ij}^n - 4\,k\frac{\partial}{\partial t} u_{ij}^n - 4\,\frac{k^2}{2}\frac{\partial^2}{\partial t^2} u_{ij}^n - 4\,\frac{k^3}{6}\frac{\partial^3}{\partial t^3} u_{ij}^n - 4\,\frac{k^4}{24}\frac{\partial^4}{\partial t^4} u_{ij}^n$$

$$- 4\,\frac{k^5}{5!}\frac{\partial^5}{\partial t^5} u_{ij}^n - 4\,\frac{k^6}{6!}\frac{\partial^6}{\partial t^6} u(x_i, y_j, \theta_7)$$

$$+ u_{ij}^n + \left(h\frac{\partial}{\partial x} - k\frac{\partial}{\partial t}\right) u_{ij}^n + \left(\frac{h^2}{2}\frac{\partial^2}{\partial x^2} u_{ij}^n + \frac{k^2}{2}\frac{\partial^2}{\partial t^2} u_{ij}^n\right.$$

$$- h\,k\,\frac{\partial^2}{\partial x \partial t}\,u_{ij}^n) +$$

$$+ \frac{1}{3!}\left(h\,\frac{\partial}{\partial x} - k\,\frac{\partial}{\partial t}\right) u_{ij}^n + \left(\frac{h^4}{24}\,\frac{\partial^4}{\partial x^4}\,u_{ij}^n + \frac{k^4}{24}\,\frac{\partial^4}{\partial t^4}\,u_{ij}^n\right.$$

$$\left. - \frac{h^3 k}{6}\,\frac{\partial^4}{\partial x^3 \partial t}\,u_{ij}^n + \frac{h^2 k^2}{4}\,\frac{\partial^4}{\partial x^2 \partial t^2}\,u_{ij}^n - \frac{h k^3}{6}\,\frac{\partial^4}{\partial x \partial t^3}\,u_{ij}^n\right)$$

$$+ \frac{1}{5!}\left(h\,\frac{\partial}{\partial x} - k\,\frac{\partial}{\partial t}\right)^5 u_{ij}^n + \frac{1}{6!}\left(h\,\frac{\partial}{\partial x} - k\,\frac{\partial}{\partial t}\right)^6 u(\xi_5, y_j, \theta_8)$$

$$+ u_{ij}^n - h\,\frac{\partial}{\partial x}\,u_{ij}^n - k\,\frac{\partial}{\partial t}\,u_{ij}^n + \left(\frac{h^2}{2}\,\frac{\partial^2}{\partial x^2}\,u_{ij}^n\right.$$

$$\left. + \frac{k^2}{2}\,\frac{\partial^2}{\partial t^2}\,u_{ij}^n + hk\,\frac{\partial^2}{\partial x \partial t}\,u_{ij}^n\right) + \frac{1}{3!}\left(-h\,\frac{\partial}{\partial x} - k\,\frac{\partial}{\partial t}\right)^3 u_{ij}^n$$

$$+ \left(\frac{h^4}{24}\,\frac{\partial^4}{\partial x^4}\,u_{ij}^n + \frac{k^4}{24}\,\frac{\partial^4}{\partial t^4}\,u_{ij}^n + \frac{h^3 k}{6}\,\frac{\partial^4}{\partial x^3 \partial t}\,u_{ij}^n +\right.$$

$$\left. + \frac{h^2 k^2}{4}\,\frac{\partial^4}{\partial x^2 \partial t^2}\,u_{ij}^n + \frac{h k^3}{6}\,\frac{\partial^4}{\partial x \partial t^3}\,u_{ij}^n\right) + \frac{1}{5!}\left(-h\frac{\partial}{\partial x} - k\,\frac{\partial}{\partial t}\right)^5 u_{ij}^n$$

$$+ \frac{1}{6!}\left(-h\,\frac{\partial}{\partial x} - k\,\frac{\partial}{\partial t}\right)^6 u(\xi_6, y_j, \theta_9)$$

$$+ u_{ij}^n + \left(h\,\frac{\partial}{\partial y} - k\,\frac{\partial}{\partial t}\right)u_{ij}^n + \left(\frac{h^2}{2}\,\frac{\partial^2}{\partial y^2}\,u_{ij}^n + \frac{k^2}{2}\,\frac{\partial^2}{\partial t^2}\,u_{ij}^n\right.$$

$$\left. - hk\,\frac{\partial^2}{\partial y \partial t}\,u_{ij}^n\right) + \frac{1}{3!}\left(h\,\frac{\partial}{\partial y} - k\,\frac{\partial}{\partial t}\right)^3 u_{ij}^n + \left(\frac{h^4}{24}\,\frac{\partial^4}{\partial y^4}\,u_{ij}^n\right.$$

$$\left. + \frac{k^4}{24}\,\frac{\partial^4}{\partial t^4}\,u_{ij}^n - \frac{h^3 k}{6}\,\frac{\partial^4}{\partial y^3 \partial t}\,u_{ij}^n + \frac{h^2 k^2}{4}\,\frac{\partial^4}{\partial y^2 \partial t^2}\,u_{ij}^n\right.$$

$$\left. - \frac{h k^3}{6}\,\frac{\partial^4}{\partial y \partial t^3}\,u_{ij}^n\right) + \frac{1}{5!}\left(h\frac{\partial}{\partial y} - k\frac{\partial}{\partial t}\right)^5 u_{ij}^n +$$

$$+ \frac{1}{6!}\left(h\frac{\partial}{\partial y} - k\frac{\partial}{\partial t}\right)^6 u(x_i, \xi_7, \theta_{10}) +$$

$$+ u_{ij}^n - h\frac{\partial}{\partial y} u_{ij}^n - k\frac{\partial}{\partial t} u_{ij}^n + (\frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n + \frac{k^2}{2} \frac{\partial^2}{\partial t^2} u_{ij}^n$$

$$+ hk \frac{\partial^2}{\partial y \partial t} u_{ij}^n) + \frac{1}{3!} (-h\frac{\partial}{\partial y} - k\frac{\partial}{\partial t})^3 u_{ij}^n$$

$$+ (\frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n + \frac{k^4}{24} \frac{\partial^4}{\partial t^4} u_{ij}^n + \frac{h^3 k}{6} \frac{\partial^4}{\partial y^3 \partial t} u_{ij}^n$$

$$+ \frac{h^2 k^2}{4} \frac{\partial^4}{\partial y^2 \partial t^2} u_{ij}^n + \frac{hk^3}{6} \frac{\partial^4}{\partial y \partial t^3} u_{ij}^n) + \frac{1}{5!}(-h\frac{\partial}{\partial y} - k\frac{\partial}{\partial t})^5 u_{ij}^n$$

$$+ \frac{1}{6!}(-h\frac{\partial}{\partial y} - k\frac{\partial}{\partial t})^6 u(x_i, \xi_8, \theta_{11})$$

$$- 4 u_{ij}^n + 4 k\frac{\partial}{\partial t} u_{ij}^n - 4 \frac{k^2}{2} \frac{\partial^2}{\partial t^2} u_{ij}^n - \frac{4k^3}{6} \frac{\partial^3}{\partial t^3} u_{ij}^n - \frac{4k^4}{24} \frac{\partial^4}{\partial t^4} u_{ij}^n$$

$$+ 4 \frac{k^5}{5!} \frac{\partial^5}{\partial t^5} u_{ij}^n - 4 \frac{k^6}{6!} \frac{\partial^6}{\partial t^6} (x_i, y_j, \theta_{12})] -$$

$$- \frac{(10r^2 + 2)}{12} [(u_{ij}^n + h \frac{\partial}{\partial x} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n + \frac{h^3}{6} \frac{\partial^3}{\partial x^3} u_{ij}^n$$

$$+ \frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^5}{5!} \frac{\partial^5}{\partial x^5} u_{ij}^n + \frac{h^6}{6!} \frac{\partial^6}{\partial x^6} u(\xi_9, y_j, t^n))$$

$$+ (u_{ij}^n - h \frac{\partial}{\partial x} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n$$

$$- \frac{h^3}{6} \frac{\partial^3}{\partial x^3} u_{ij}^n + \frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n - \frac{h^5}{5!} \frac{\partial^5}{\partial x^5} u_{ij}^n$$

$$+ \frac{h^6}{6!} \frac{\partial^6}{\partial x^6} u(\xi_{10}, y_j, t^n))$$

$$+ (u_{ij}^n + h \frac{\partial}{\partial y} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n + \frac{h^3}{6} \frac{\partial^3}{\partial y^3} u_{ij}^n$$

$$+ \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n + \frac{h^5}{5!} \frac{\partial^5}{\partial y^5} u_{ij}^n$$

$$+ \frac{h^6}{6} \frac{\partial^6}{\partial y^6} u(x_i, \xi_{11}, t^n)) + (u_{ij}^n - h \frac{\partial}{\partial y} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n$$

$$- \frac{h^3}{6} \frac{\partial^3}{\partial y^3} u_{ij}^n +$$

$$+ \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n - \frac{h^5}{5!} \frac{\partial^5}{\partial y^5} u_{ij}^n + \frac{h^6}{6!} \frac{\partial^6}{\partial y^6} u(x_i, \xi_{12}, t^n))$$

$$- 4 u_{ij}^n]$$

$$- \frac{r^2}{6}[u_{ij}^n + (h\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}) u_{ij}^n + (\frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n$$

$$+ h^2 \frac{\partial^2}{\partial x \partial y} u_{ij}^n)$$

$$+ \frac{1}{3!}(h\frac{\partial}{\partial x} + h\frac{\partial}{\partial y})^3 u_{ij}^n + (\frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n +$$

$$+ \frac{h^4}{6} \frac{\partial^4}{\partial x^3 \partial y} u_{ij}^n$$

$$+ \frac{h^4}{4} \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n + \frac{h^4}{6} \frac{\partial^4}{\partial x \partial y^3} u_{ij}^n) + \frac{1}{5!}(h \frac{\partial}{\partial x} + h \frac{\partial}{\partial y})^5 u_{ij}^n$$

$$+ \frac{1}{6!}(h \frac{\partial}{\partial x} + h \frac{\partial}{\partial y})^6 u(\xi_{13}, \xi_{14}, t^n)$$

$$+ u_{ij}^n - h \frac{\partial}{\partial x} u_{ij}^n + h \frac{\partial}{\partial y} u_{ij}^n + (\frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n$$

$$- h^2 \frac{\partial^2}{\partial x \partial y} u_{ij}^n)$$

$$+ \frac{1}{3!}(-h \frac{\partial}{\partial x} + h \frac{\partial}{\partial y})^3 u_{ij}^n + (\frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n$$

$$+ \frac{h^4}{4} \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n$$

$$- \frac{h^4}{6} \frac{\partial^4}{\partial x^3 \partial y} u_{ij}^n - \frac{h^4}{6} \frac{\partial^4}{\partial x \partial y^3} u_{ij}^n) + \frac{1}{5!}(-h \frac{\partial}{\partial x} + h \frac{\partial}{\partial y})^5 u_{ij}^n$$

$$+ \frac{1}{6!} (-h \frac{\partial}{\partial x} + h \frac{\partial}{\partial y})^6 u(\xi_{15}, \xi_{16}, t^n)$$

$$+ u_{ij}^n + (h \frac{\partial}{\partial x} - h \frac{\partial}{\partial y}) u_{ij}^n + (\frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n + \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n$$

$$- h^2 \frac{\partial^2}{\partial x \partial y} u_{ij}^n)$$

$$+ \frac{1}{3!}(h \frac{\partial}{\partial x} - h \frac{\partial}{\partial y})^3 u_{ij}^n + (\frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n$$

$$- \frac{h^4}{6} \frac{\partial^4}{\partial x \partial y^3} u_{ij}^n$$

$$+ \frac{h^4}{4} \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n - \frac{h^4}{6} \frac{\partial^4}{\partial x^3 \partial y} u_{ij}^n) + \frac{1}{5!}(h \frac{\partial}{\partial x} - h \frac{\partial}{\partial y})^5 u_{ij}^n$$

$$+ \frac{1}{6!}(h \frac{\partial}{\partial x} - h \frac{\partial}{\partial y})^6 u(\xi_{17}, \xi_{18}, t^n)$$

$$+ u_{ij}^n - h \frac{\partial}{\partial x} u_{ij}^n - h \frac{\partial}{\partial y} u_{ij}^n + (\frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n$$

$$+ \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n + h^2 \frac{\partial^2}{\partial x \partial y} u_{ij}^n)$$

$$+ \frac{1}{3!}(-h \frac{\partial}{\partial x} - h \frac{\partial}{\partial y})^3 u_{ij}^n + (\frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n$$

$$+ \frac{h^4}{6} \frac{\partial^4}{\partial x^3 \partial y} u_{ij}^n$$

$$+ \frac{h^4}{6} \frac{\partial^4}{\partial x \partial y^3} u_{ij}^n + \frac{h^4}{4} \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n) + \frac{1}{5!}(-h\frac{\partial}{\partial x} - h\frac{\partial}{\partial y})^5 u_{ij}^n$$

$$+ \frac{1}{6!} (-h\frac{\partial}{\partial x} - h \frac{\partial}{\partial y})^6 u(\xi_{19}, \xi_{20}, t^n)$$

$$- 2 u_{ij}^n - 2h \frac{\partial}{\partial y} u_{ij}^n - 2 \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n - 2 \frac{h^3}{6} \frac{\partial^3}{\partial y^3} u_{ij}^n$$

$$- 2 \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n$$

$$- 2 \frac{h^5}{5!} \frac{\partial^5}{\partial y^5} u_{ij}^n - 2 \frac{h^6}{6!} \frac{\partial^6}{\partial y^6} (x_i, \xi_{21}, t^n)$$

$$- 2 u_{ij}^n - 2h \frac{\partial}{\partial x} u_{ij}^n - 2 \frac{h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n - 2 \frac{h^3}{6} \frac{\partial^3}{\partial x^3} u_{ij}^n$$

$$- 2 \frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n$$

$$- 2 \frac{h^5}{5!} \frac{\partial^5}{\partial x^5} u_{ij}^n - 2 \frac{h^6}{6!} \frac{\partial^6}{\partial x^6} u(\xi_{22}, y_j, t^n)$$

$$- 2 u_{ij}^n + 2h \frac{\partial}{\partial x} u_{ij}^n - \frac{2h^2}{2} \frac{\partial^2}{\partial x^2} u_{ij}^n + 2 \frac{h^3}{6} \frac{\partial^3}{\partial x^3} u_{ij}^n$$

$$- 2 \frac{h^4}{24} \frac{\partial^4}{\partial x^4} u_{ij}^n$$

$$+ 2 \frac{h^5}{5!} \frac{\partial^5}{\partial x^5} u_{ij}^n - 2 \frac{h^6}{6!} \frac{\partial^6}{\partial x^6} u(\xi_{23}, y_j, t^n)$$

$$- 2u_{ij}^n + 2h \frac{\partial}{\partial y} u_{ij}^n - 2 \frac{h^2}{2} \frac{\partial^2}{\partial y^2} u_{ij}^n + 2 \frac{h^3}{6!} \frac{\partial^3}{\partial y^3} u_{ij}^n$$

$$- 2 \frac{h^4}{24} \frac{\partial^4}{\partial y^4} u_{ij}^n$$

$$+ 2 \frac{h^5}{5!} \frac{\partial^5}{\partial y^5} u_{ij}^n - 2 \frac{h^6}{6!} \frac{\partial^6}{\partial y^6} u(x_i, \xi_{24}, t^n) + 4u_{ij}^n] \quad .$$

After the proper cancellations, the RHS of (4.5) becomes

$$\text{RHS} = k^2 \frac{\partial^2}{\partial t^2} u_{ij}^n + 2 \frac{k^4}{24} \frac{\partial^4}{\partial t^4} u_{ij}^n - \frac{(r^2 - 1)}{12} [2h^2 \frac{\partial^2}{\partial x^2} u_{ij}^n$$

$$+ 2h^2 \frac{\partial^2}{\partial y^2} u_{ij}^n$$

$$+ h^2k^2 \frac{\partial^4}{\partial t^2 \partial x^2} u_{ij}^n + h^2k^2 \frac{\partial^4}{\partial t^2 \partial y^2} u_{ij}^n + \frac{h^4}{6} \frac{\partial^4}{\partial x^4} u_{ij}^n$$

$$+ \frac{h^4}{6} \frac{\partial^4}{\partial y^4} u_{ij}^n]$$

$$- \frac{(10r^2 + 2)}{12} [h^2 \frac{\partial^2}{\partial x^2} u_{ij}^n + h^2 \frac{\partial^2}{\partial y^2} u_{ij}^n + \frac{h^4}{12} \frac{\partial^4}{\partial x^4} u_{ij}^n$$

$$+ \frac{h^4}{12} \frac{\partial^4}{\partial x^4} u_{ij}^n]$$

$$- \frac{r^2}{6} h^4 \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n + (\text{sixth order terms})$$

$$= k^2 \frac{\partial^2}{\partial t^2} u_{ij}^n + \frac{2k^4}{24} \frac{\partial^4}{\partial t^4} u_{ij}^n - \frac{c^2k^2}{12h^2}[2h^2 \frac{\partial^2}{\partial x^2} u_{ij}^n$$

$$+ 2h^2 \frac{\partial^2}{\partial y^2} u_{ij}^n$$

$$+ h^2k^2 \frac{\partial^4}{\partial t^2 \partial x^2} u_{ij}^n + h^2k^2 \frac{\partial^4}{\partial t^2 \partial y^2} u_{ij}^n + \frac{h^4}{6} \frac{\partial^4}{\partial x^4} u_{ij}^n$$

$$+ \frac{h^4}{6} \frac{\partial^4}{\partial y^4} u_{ij}^n]$$

$$+ \frac{1}{12}[2h^2 \frac{\partial^2}{\partial x^2} u_{ij}^n + 2h^2 \frac{\partial^2}{\partial y^2} u_{ij}^n + h^2k^2 \frac{\partial^4}{\partial t^2 \partial x^2} u_{ij}^n$$

$$+ h^2k^2 \frac{\partial^4}{\partial t^2 \partial y^2} u_{ij}^n$$

$$+ \frac{h^4}{6} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{6} \frac{\partial^4}{\partial y^4} u_{ij}^n]$$

$$- \frac{10c^2k^2}{12h^2} [h^2 \frac{\partial^2}{\partial x^2} u_{ij}^n + h^2 \frac{\partial^2}{\partial y^2} u_{ij}^n + \frac{h^4}{12} \frac{\partial^4}{\partial x^4} u_{ij}^n$$

$$+ \frac{h^4}{12} \frac{\partial^4}{\partial y^4} u_{ij}^n]$$

$$- \frac{1}{6} [h^2 \frac{\partial^2}{\partial x^2} u_{ij}^n + h^2 \frac{\partial^2}{\partial y^2} u_{ij}^n + \frac{h^4}{12} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{12} \frac{\partial^4}{\partial y^4} u_{ij}^n]$$

$$- \frac{ck^2}{6h^2} h^4 \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n + (\text{sixth order terms})$$

$$= \frac{-c^2k^2}{12h^2} [\frac{h^4}{6} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{6} \frac{\partial^4}{\partial y^4} u_{ij}^n] + \frac{1}{12}[h^2k^2 \frac{\partial^4}{\partial t^2 \partial x^2} u_{ij}^n$$

$$+ h^2k^2 \frac{\partial^4}{\partial t^2 \partial y^2} u_{ij}^n]$$

$$- \frac{10c^2k^2}{12c^2} [\frac{h^4}{12} \frac{\partial^4}{\partial x^4} u_{ij}^n + \frac{h^4}{12} \frac{\partial^4}{\partial y^4} u_{ij}^n] - \frac{c^2k^2}{6h^2} h^4 \frac{\partial^4}{\partial x^2 \partial y^2} u_{ij}^n$$

$$+ (\text{sixth order terms}) = (\text{sixth order terms}) .$$

If we consider that u is the solution of (2.1) - (2.3), we come to the conclusion of Lemma (4.2). ■

We now proceed to obtain an error estimate.

Theorem 4.1

Let u be the solution of the wave equation. Suppose that $\{U_{ij}^n\}$, $0 \le n \le N$, $0 \le i, j \le J$ is the solution (4.1) with the initial conditions (4.3). Then if U_k^n, u^n are the mesh functions U_{ij}^n, u_{ij}^n, respectively, and if $r = \frac{ck}{h} \le r_0 < \frac{\sqrt{2}}{3}$, there exists a constant $C = C(r, T)$ such that

$$\max_{0 \le n \le N} \|U_h^n - u^n\|_h \le C(r, T)(k^4 + h^4)\, C_4(u) ,$$

where

$$C_4(u) = \max_{x,y,t} \{ \left|\frac{\partial^6 u}{\partial t^6}\right|, \left|\frac{\partial^6 u}{\partial x^6}\right|, \left|\frac{\partial^6 u}{\partial y^6}\right|, \left|\frac{\partial^6 u}{\partial t^\alpha \partial x^\beta}\right|, \left|\frac{\partial^6 u}{\partial t^\alpha \partial y^\beta}\right|,$$

$$\left|\frac{\partial^6 u}{\partial x^\alpha \partial y^\beta}\right|_{\alpha + \beta = 6} \quad \alpha, \beta = 1, 2, \ldots, 6 \} .$$

Proof. We have that

$$(4.6) \qquad e_{ij}^{n+1} - 2e_{ij}^n + e_{ij}^{n-1} = (U_{ij}^{n+1} - 2\,U_{ij}^n + U_{ij}^{n-1})$$

$$- (u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1})$$

$$= \frac{(1 - \rho)}{12\rho} \diamond U_{ij}^{n+1} + \frac{(1 - \rho)}{12\rho} \diamond U_{ij}^{n-1} + \frac{(10 + 2\rho)}{12\rho} \diamond U_{ij}^n$$

$$+ \frac{2}{12\rho} \delta_x^2 \delta_y^2\, U_{ij}^n + \frac{2}{12\rho} \delta_x^2 \delta_y^2\, u_{ij}^n$$

$$- \delta_t^2 u_{ij}^n - \frac{(1-\rho)}{12\rho} \diamond u_{ij}^{n+1} - \frac{(1-\rho)}{12\rho} \diamond u_{ij}^{n-1} - \frac{(10+2\rho)}{12\rho} u_{ij}^n -$$

$$- \frac{2}{12\rho} \delta_x^2 \delta_y^2 u_{ij}^n + \frac{(1-\rho)}{12\rho} \diamond u_{ij}^{n+1} + \frac{(1-\rho)}{12\rho} \diamond u_{ij}^{n-1} + \frac{(10+2\rho)}{12\rho} \diamond u_{ij}^n +$$

$$= \frac{(1-\rho)}{12\rho} \diamond e_{ij}^{n+1} + \frac{(1-\rho)}{12\rho} \diamond e_{ij}^{n-1} + \frac{(10+2\rho)}{12\rho} \diamond e_{ij}^n$$

$$+ \frac{2}{12\rho} \delta_x^2 \delta_y^2 e_{ij}^n + \varepsilon_{ij}^n ,$$

where

$$\varepsilon_{ij}^n = \frac{(1-\rho)}{12\rho} \diamond u_{ij}^{n+1} + \frac{(1-\rho)}{12\rho} \diamond u_{ij}^{n-1} + \frac{(10+2\rho)}{12\rho} \diamond u_{ij}^n$$

$$+ \frac{2}{12\rho} \delta_x^2 \delta_y^2 u_{ij}^n - \delta_t^2 u_{ij}^n ,$$

and furthermore, by Lemma (4.2)

$$(4.7) \qquad \max_{\substack{1 \le i, j \le J \\ 1 \le n \le N-1}} |\varepsilon_{ij}^n| \le ck^2(k^4 + h^4) \cdot C_4(u) .$$

From (4.6), we get that

$$h^2 \sum_{i,j=1}^{J} (e_{ij}^{n+1} - 2e_{ij}^n + e_{ij}^{n-1})(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$= \frac{(1-\rho)}{12\rho} h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^{n+1}(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ \frac{(1-\rho)}{12\rho} h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^{n-1}(e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ \frac{(10 + 2\rho)}{12\rho} h^2 \sum_{i,j=1}^{J} \diamond e_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) +$$

$$+ \frac{h^2}{6\rho} \sum_{i,j=1}^{J} \delta_x^2 \delta_y^2 e_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1})$$

$$+ h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^n (e_{ij}^{n+1} - e_j^{n-1}) .$$

From the above we obtain

$$\|e^{n+1} - e^n\|_h^2 - \|e^n - e^{n-1}\|_h^2 = - \frac{(1 - \rho)}{12\rho} a_h(e^{n+1}, e^{n+1} - e^{n-1})$$

$$- \frac{(1 - \rho)}{12\rho} a_h(e^{n-1}, e^{n+1} - e^{n-1}) - \frac{(10 + 2\rho)}{12\rho} a_h(e^n, e^{n+1} - e^{n-1})$$

$$+ \frac{h^2}{6\rho} \sum_{i,j=1}^{J} \delta_x^2 \delta_y^2 e_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) + h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) ,$$

or

$$\|e^{n+1} - e^n\|_h^2 - \|e^n - e^{n-1}\|_h^2 + \frac{(1 - \rho)}{12\rho} [a_h(e^{n+1}, e^{n+1})$$

$$- a_h(e^{n-1}, e^{n-1})]$$

$$+ \frac{(10 + 2\rho)}{24\rho} [a_h(e^{n+1}, e^{n+1}) - a_h(e^{n-1}, e^{n-1})]$$

$$- \frac{(10 + 2\rho)}{24\rho} [a_h(e^{n+1} - e^n, e^{n+1} - e^n) - a_h(e^n - e^{n-1}, e^n - e^{n-1})]$$

$$= - \frac{1}{6\rho} \beta_h(e^n, e^{n+1} - e^{n-1}) + h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^n (e_{ij}^{n+1} - e_{ij}^{n-1}) .$$

Summing up from $n = 1$ to $n = n - 1$, we obtain

$$(4.8)\qquad \|e^n - e^{n-1}\|_h^2 + \frac{1}{2\rho}[a_h(e^n, e^n) + a_h(e^{n-1}, e^{n-1})]$$

$$- \frac{(10 + 2\rho)}{24\rho} a_h(e^n - e^{n-1}, e^n - e^{n-1}) =$$

$$= \frac{(1 + \rho)}{12\rho} a_h(e^1, e^1) - \frac{1}{6\rho} \beta_h(e^n, e^{n-1})$$

$$+ h^2 \sum_{\ell=1}^{n-1} \sum_{i,j=1}^{J} \varepsilon_{ij}^{\ell}(e_{ij}^{\ell+1} - e_{ij}^{\ell-1}) .$$

We also have that for $v \in S_h$

$$-\beta_h(v, v) = -h^2 \sum_{i=1}^{J} (v_{i+1,J} - 2v_{iJ} + v_{i-1,J})v_{iJ}$$

$$- h^2 \sum_{i=1}^{J} (v_{i+1,1} - 2v_{i1} + v_{i-1,1})v_{i1}$$

$$\leq -h^2 \sum_{i,j=1}^{J} (v_{i+1,j} - 2v_{ij} + v_{i-1,j})v_{ij} .$$

Similarly,

$$-\beta_h(v, v) \leq -h^2 \sum_{i,j=1}^{J} (v_{i,j+1} - 2v_{ij} + v_{i,j-1})v_{ij} .$$

Thus,

$$(4.9)\qquad 2\beta_h(v, v) \leq -h^2 \sum_{i,j=1}^{J} (\diamond v_{ij})v_{ij} = a_h(v, v) \leq 8\|v\|_h^2 .$$

or

$$-\beta(v, v) \leq 4\|v\|_h^2 .$$

Furthermore,

$$\frac{1}{6\rho}\,\beta_h(e^n,\, e^{n-1}) = \frac{1}{12\rho}\,\beta_h(e^n,\, e^n) + \frac{1}{12\rho}\,\beta_h(e^{n-1},\, e^{n-1})$$

$$- \frac{1}{12\rho}\,\beta_h(e^n - e^{n-1},\, e^n - e^{n-1})\ .$$

Using (4.9) and taking into consideration the fact that

$$- \frac{1}{12\rho}\,\beta_h\,(e^n - e^{n-1},\, e^n - e^{n-1}) \geq 0\ ,$$

(4.8) can be written

$$\|e^n - e^{n-1}\|_h^2 + \frac{11}{24\rho}\,[a_h(e^n,\, e^n) + a_h(e^{n-1},\, e^{n-1}) \tag{4.10}$$

$$- a_h(e^n - e^{n-1},\, e^n - e^{n-1})]$$

$$+ \frac{(1 - 2\rho)}{24\rho}\,a_h[e^n - e^{n-1},\, e^n - e^{n-1}) \leq 8\,\frac{(1 - \rho)}{12\rho}\,\|e^1\|_h^2$$

$$+ h^2 \sum_{i,j=1}^{J} \varepsilon_{ij}^{\ell}\,(e_{ij}^{\ell+1} - e_{ij}^{\ell-1})\ .$$

Also,

$$\frac{11}{24\rho}\,[a_h(e^n,\, e^n) + a_h(e^{n-1},\, e^{n-1}) - a_h(e^n - e^{n-1},\, e^n - e^{n-1})] \tag{4.11}$$

$$= \frac{11}{12\rho}[a_h(e^{n-1},\, e^{n-1}) + a_h(e^{n-1},\, e^n - e^{n-1})]\ .$$

Now, we have

$$|a_h(e^{n-1}, e^n - e^{n-1})| \le [a_h(e^{n-1}, e^{n-1})]^{1/2}$$

$$\cdot [a_h(e^n - e^{n-1}, e^n - e^{n-1})]^{1/2}$$

$$\le [a_h(e^{n-1}, e^{n-1})]^{1/2} \cdot 2\sqrt{2}\|e^n - e^{n-1}\|_h$$

$$\le a_h(e^{n-1}, e^{n-1}) + 2\|e^n - e^{n-1}\|_h^2 \quad .$$

Thus, we obtain

$$\frac{11}{12\rho} a_h(e^{n-1}, e^n - e^{n-1}) \ge -\frac{11}{12\rho} a_h(e^{n-1}, e^{n-1}) - \frac{11}{6\rho}\|e^n - e^{n-1}\|_h^2 \quad .$$

Also, since $1 - 2\rho < 0$ if $r^2 < 2$ we obtain

$$\frac{(1 - 2\rho)}{24\rho} a_h(e^n - e^{n-1}, e^n - e^{n-1}) \ge \frac{(1 - 2\rho)}{3\rho} \|e^n - e^{n-1}\|_h^2 \quad .$$

From the above, (4.10) becomes

$$(1 - \frac{11}{6\rho} + \frac{1 - 2\rho}{3\rho})\|e^n - e^{n-1}\|_h^2 \le \frac{8(1 + \rho)}{12\rho} \|e^1\|_h^2$$

$$+ \sum_{\ell=1}^{n-1} \|\varepsilon^1\|_h \|e^{\ell+1} - e^{\ell-1}\|_h \quad ,$$

since,

$$h^2 \sum_{\ell=1}^{n-1} \sum_{i,j=1}^{J} \varepsilon_{ij}^{\ell}(e_{ij}^{\ell+1} - e_{ij}^{\ell-1}) \le \sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \quad .$$

If $r^2 < \frac{2}{9}$, then $1 - \frac{11}{6\rho} + \frac{1 - 2\rho}{3\rho} > 0$. Then, for $r \le r_0 < \frac{\sqrt{2}}{3}$, if we let $(1 - \frac{11}{6\rho} + \frac{1 - 2\rho}{3\rho}) = c(r)^{-1}$, we obtain

$$(4.12) \qquad \|e^n - e^{n-1}\|_h^2 \le C(r)\,\frac{8(1-\rho)}{12\rho}\,\|e^1\|_h^2 + \sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h .$$

Using the Cauchy-Schwartz inequality, we have

$$\sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \le \Big[\sum_{\ell=1}^{n-1} \|\varepsilon^\ell\|_h^2\Big]^{1/2} \cdot \Big[\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2\Big]^{1/2} .$$

We let

$$E = \Big[\sum_{\ell=1}^{N-1} \|\varepsilon^\ell\|_h^2\Big]^{1/2} .$$

Also,

$$\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2 \le \sum_{\ell=1}^{n-1} \big[\|e^{\ell+1} - e^{\ell}\|_h + \|e^{\ell} - e^{\ell-1}\|_h\big]^2$$

$$\le 2\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell}\|_h^2 + 2\sum_{\ell=1}^{n-1} \|e^{\ell} - e^{\ell-1}\|_h^2 \le 4\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2 .$$

Hence,

$$(4.13) \qquad \Big[\sum_{\ell=1}^{n-1} \|e^{\ell+1} - e^{\ell-1}\|_h^2\Big]^{1/2} \le 2\Big[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2\Big]^{1/2} .$$

Thus,

$$\sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \leq 2E\Big[\sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2\Big]^{1/2}$$

$$\leq \frac{E^2}{A} + A \sum_{\ell=1}^{n} \|e^{\ell} - e^{\ell-1}\|_h^2 \leq \frac{E^2}{A} + A \cdot N \max_{1\leq n\leq N} \|e^{n} - e^{n-1}\|_h^2 .$$

(NB. This holds for any $A > 0$: $2ab \leq \frac{a^2}{A} + Ab^2$) .

Choosing $A = \frac{1}{2N}$, we obtain

$$(4.14) \qquad \sum_{\ell=1}^{n-1} \|\varepsilon^{\ell}\|_h \cdot \|e^{\ell+1} - e^{\ell-1}\|_h \leq 2NE^2 + \frac{1}{2} \max_{1\leq n\leq N} \|e^{n} - e^{n-1}\|_h^2 .$$

From the above, (4.12) becomes

$$(4.15) \qquad \max_{1\leq n\leq N} \|e^{n} - e^{n-1}\|_h^2 \leq 16\, C(r)\, \frac{(1-\rho)}{12\rho} \|e^{1}\|_h^2 + 4NE^2 .$$

Furthermore,

$$NE^2 \leq N \sum_{\ell=1}^{N-1} C_4^2(u)\, k^4(k^4 + h^4)^2 \leq N^2\, C_4^2(u)\, k^4(k^4 + h^4)^2$$

$$= T^2\, C_4^2(u)\, k^2(k^4 + h^4)^2 .$$

Using $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$, we obtain

$$\max_{1\leq n\leq N} \|e^{n} - e^{n-1}\|_h \leq 4\sqrt{C(r)\frac{(1-\rho)}{12\rho}} \cdot \|e^{1}\|_h$$

$$+ 2T\, C_4(u)\, k(k^4 + h^4) \equiv R .$$

Since $\|e^n\|_h \leq \|e^n - e^{n-1}\|_h + \|e^{n-1}\|_h$ and $\|e^0\|_h = 0$, we obtain

$$\max_{1\leq n\leq N} \|e^n\|_h \leq [\max_{1\leq n\leq N} \|e^n - e^{n-1}\|_h] \cdot N \leq RN$$

If we take under consideration the fact $KN = T$, we obtain the result of Theorem 4.1. ■

We note that the Von Neumann stability analysis for the pure initial value problem for the scheme (4.1) gives the stability bound $r \leq \frac{1}{\sqrt{2}} \simeq .707$ [5]. Moreover in his Master's thesis, K. Dickens [K.E. Dickens, "A Comparison of finite difference schemes applied to the wave equation", M.S. Thesis, Mathematics Department, The University of Tennessee, Knoxville, 1979] has computed with (4.1) using zero boundary conditions on the unit square for $r = \frac{1}{\sqrt{2}}$ and found that the scheme was stable. Hence it is clear that our bound $r \leq \frac{\sqrt{2}}{3} \simeq .471$ is pessimistic. We conjecture that $r \leq \frac{1}{\sqrt{2}}$ is the correct bound, but we were unable to obtain it by an energy proof of the type shown here.

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