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To the Graduate Council:

I am submitting herewith a dissertation written by George F. Hampton entitled "Green's Relations and Dimension in Abstract Semi-groups." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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GREEN'S RELATIONS AND DIMENSION IN ABSTRACT SEMIGROUPS

A Dissertation

Presented to

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In Partial Fulfillment

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INTRODUCTION

This thesis originated in an effort to find an efficient algorithm for the construction of finite inverse semigroups of small order. At one stage in trying to devise such a scheme, an attempt was made to construct an inverse semigroup by adjoining two non-idempotent elements to a semilattice in such a way that each of them would be $\mathcal{D}$ -equivalent to a pair of distinct $\mathcal{D}$ -equivalent idempotents. It was noticed that such adjunction yielded an inverse semigroup only when the elements of the pair were incomparable in the partial ordering of the semilattice, and only when, for each positive integer n , either both or neither of the elements of the pair had an n -chain of idempotents descending from it. Two theorems on inverse semigroups emerged from this observation; they were subsequently generalized to regular semigroups, and finally to arbitrary semigroups, and in this form they appear herein as Lemma 1.2 and Theorem 1.4.

It also appeared that, in a program of constructing inverse semigroups from a given semilattice of idempotents, the familiar notion of "height" or "dimension" in a lower semilattice could be useful. A first step in this direction was to try to extend the notion of dimension to non-idempotent elements, and a tentative definition of dimension for such elements led to the conclusion that all elements of a regular $\mathcal{D}$ -class had the same dimension. Subsequently the definition was refined and modified in various ways, and, in the form in which it appears herein, allows us to state that in any semigroup all elements of a $\mathcal{J}$ -class have the same dimension. The dimension function thus defined has the property (a strong form of subadditivity) that the dimension of a product is less than or

equal to the dimension of each factor.

Although the problem with which we began is still far from solved, and indeed appears much more difficult than was thought at first, we have been led to study in a more general setting the relationship between the well-known equivalence relations of Green and the partly ordered set of all idempotent elements in an arbitrary semigroup. The results of this investigation are presented in Chapter I of this dissertation. It was found that the results set forth in that chapter were intimately related to the problem of finding a reasonable (and potentially useful) extension of the dimension concept to the non-idempotent elements of a semigroup, and Chapter II is devoted to that problem.

Many unsolved problems in this domain will occur to the reader. One of the most interesting, perhaps, concerns the possibility of defining dimension in a semiring in such a way that the dimension of the product of two elements is the sum of their dimensions. Theorem 1.16 suggests both the possibility of investigating the property of local minimality in other equivalence classes, and also of studying the stronger property of global minimality, defined in the obvious way.

The fundamental semigroup theory on which all our work is based is the work of many authors and is widely scattered in the periodical literature. Fortunately most of it is collected in the recent treatise [3] of Clifford and Preston, where precise and detailed references to the original papers may be found. For the convenience of readers of this thesis, the general theory that we use has been condensed and presented without proofs in an appendix hereto, the contents of which may all be found in the works referred to in our Bibliography.

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CHAPTER I

GREEN'S RELATIONS AND THE PARTIAL ORDERING OF IDEMPOTENTS

The theorems in this chapter fall into two clusters, one finding its origin in Lemma 1.2 and the other in Lemma 1.12. Because it may not always be obvious that semigroups having the properties laid down in our hypotheses actually exist, we intersperse among our propositions examples to show that they are not vacuous. We begin with such an example, which illustrates the situation contemplated in our basic Lemma 1.2 and in Corollary 1.3 and Theorem 1.4.

Example 1.1. The non-commutative band S having the multiplication table

	e	f	g	h
e	e	f	g	h
f	e	f	g	h
g	g	h	g	h
h	g	h	g	h

One sees at once from the table that $R(e) = S = R(f)$, $R(g) = \{g, h\} = R(h)$, and $J(e) = S \neq \{g, h\} = J(g)$. It follows that $D_e = D_f$, $D_g = D_h$, and $J_e \neq J_g$. Also $g < e$ and $h < f$.

Lemma 1.2. If e and f are $\mathcal{D}$ -equivalent idempotent elements of a semigroup, and if g is an idempotent element strictly under e , then some idempotent element h , $\mathcal{D}$ -equivalent to g , is strictly under f .

Proof. Let S be a semigroup, let e, f, g , be idempotent elements of S , and assume $e \mathcal{D} f$ and $g < e$. Since $D_e = D_f$ there exist mutually inverse elements a and a' such that $aa' = e$, $a'a = f$, $af = afa' = a'$ and $R_e = R_a$. Having assumed $g < e$ we have $g = ge = eg$. Now we shall show that $a'ga$ will serve as the required h , first observing

that $g = eg = aa'g \in Sa'g = L(a'g)$, whence $L(g) \subseteq L(a'g)$. But also $a'g \in Sg = L(g)$, so that $L(a'g) \subseteq L(g)$. Therefore $L(a'g) = L(g)$, and consequently $L_{a'g} = L_g$. It follows that $g = ge \in L_g R_e = L_{a'g} R_a$, from which we conclude that $L_{a'g} R_a \subseteq D_g$ and therefore $a'ga \in D_g$. Having shown that $a'ga$ is $\mathcal{D}$ -equivalent to g , we now show it to be idempotent:

$$\begin{aligned} (a'ga)(a'ga) &= (a')g(aa')ga = (fa')g(e)ga = fa'g(eg)a \\ &= ga'gga = (fa')ga = a'ga. \end{aligned}$$

Furthermore, the last of these equalities shows that $f(a'ga) = a'ga$ and we have also $(a'ga)f = a'g(af) = a'ga$, so that $a'ga \leq f$. Finally,

$$afa' = aa' = e \neq g = eg = ege = (aa')g(aa') = a(a'ga)a',$$

so that $f \neq a'ga$ and therefore $a'ga < f$.

Corollary 1.3. Either all the idempotent elements in a $\mathcal{D}$ -class of a semigroup are primitive or else none is primitive.

Proof. Let e and f be $\mathcal{D}$ -equivalent idempotents in a semigroup S , and let e be primitive. If no idempotent in S is strictly under f then f is also primitive; suppose, on the contrary that $g^2 = g$ and $g < f$. By Lemma 1.2 there is an idempotent $h \in D_g$ such that $h < e$. But e is primitive; hence S must have a zero element 0 , and $h = 0$. Therefore $g \in D_g = D_h = D_0 = \{0\}$, whence $g = 0$ and f is primitive.

We remark in passing that Corollary 1.3 holds also with "primitive" replaced by "minimal". For if a semigroup S has no zero element then the two terms are synonymous for S , while if S has a zero element 0 then 0 is the only minimal element of S and (a fact we have used in the foregoing proof) is the only element in its $\mathcal{D}$ -class. This remark does not establish the similar-sounding proposition in Corollary 1.7 below, although

the latter could just as well be exhibited as a direct consequence of Lemma 1.2.

Theorem 1.4. Let e and f be $\mathcal{D}$ -equivalent idempotent elements of a semigroup S , and $e = e_0 > e_1 > \dots > e_m$ a chain of idempotent elements of S . There exists a chain $f = f_0 > f_1 > \dots > f_m$ of idempotent elements of S such that e_i and f_i are $\mathcal{D}$ -equivalent for each $i = 0, 1, \dots, m$.

Proof. The theorem is clearly true for $m = 0$; suppose it true for $m = j$, j an arbitrary positive integer, and let $e = e_0 > e_1 > \dots > e_j > e_{j+1}$ be a chain of idempotents in S . By our inductive assumption, there is a chain $f = f_0 > f_1 > \dots > f_j$ of idempotents such that $e_i \mathcal{D} f_i$ for each $i = 0, 1, \dots, j$. Since $e_j \mathcal{D} f_j$ and $e_j > e_{j+1}$, Lemma 1.2 guarantees the existence of an idempotent f_{j+1} , $\mathcal{D}$ -equivalent to e_{j+1} , such that $f_j > f_{j+1}$. The theorem therefore holds for $m = j + 1$, and our induction is complete.

Our next result is a quick consequence of Lemma 1.2, but we prepare for it by giving examples. The bicyclic semigroup illustrates Theorem 1.6, but in a dull way because it is bisimple. More interesting is

Example 1.5. R. H. Bruck has given [2, p. 48] a construction for embedding an arbitrary semigroup in a simple semigroup with identity. To obtain an example of a non-bisimple semigroup containing comparable $\mathcal{D}$ -equivalent idempotent elements, we apply Bruck's procedure (formulated in a slightly different way) to the semigroup $S = \{e, f\}$ in which e is identity element and f is zero element. Let N be the set of all non-negative integers, and turn $N \times S \times N$ into a semigroup by defining

$$(a,u,b)(c,v,d) = (a + c - \min\{b,c\}, w, b + d - \min\{b,c\}),$$

where $w = uv$ if $b = c$, $w = u$ if $b > c$, $w = v$ if $b < c$.

Bruck proves that this definition yields a simple semigroup with $(0,e,0)$ as identity element. As A. H. Clifford has pointed out (personal communication to D. D. Miller) the new semigroup has the same $\mathcal{D}$ -structure as S for arbitrary S ; more precisely, $(a,u,b) \mathcal{D} (c,v,d)$ if and only if $u \mathcal{D} v$. It is easy to see that in the general case the idempotents in $N \times S \times N$ are precisely the elements (a,u,a) such that $u^2 = u$. One readily verifies that if (a,u,a) and (b,v,b) are idempotent then $(a,u,a) \leq (b,v,b)$ if and only if either $a > b$ or else $a = b$ and $u \leq v$. In our case ($S = \{e,f\}$), all elements of the form (a,u,a) are idempotent and we abbreviate such an idempotent by u_a . The two $\mathcal{D}$ -classes are $\{(a,e,b)\}$ and $\{(a,f,b)\}$. There are infinitely many infinite descending chains of idempotents, both chains within a $\mathcal{D}$ -class and chains having infinitely many elements in each $\mathcal{D}$ -class; for instance, $e_0 > e_1 > e_2 > \dots$ and $e_0 > f_0 > e_1 > f_1 > \dots$, the latter being the chain of all idempotents in the semigroup.

Theorem 1.6. If a $\mathcal{D}$ -class D in a semigroup contains two comparable idempotent elements then every idempotent in D is the greatest element of an infinite chain of idempotents in D .

Proof. Let e and g be idempotents lying in the same $\mathcal{D}$ -class D of a semigroup, $e > g$, and let f be any idempotent in D . By Lemma 1.2, there is an idempotent $h \in D$ such that $f > h$. Repetition of the argument yields an infinite chain descending from f .

Corollary 1.7. If a regular $\mathcal{D}$ -class in a semigroup is finite then no two of its idempotent elements are comparable.

Corollary 1.8. Let E_D be the set of all idempotent elements of a regular $\mathcal{D}$ -class D in a semigroup. Either each element of E_D is minimal in E_D or else none is.

Proof. Let $e, f \in E_D$ and suppose e is not minimal in E_D , i.e., $e > g$ for some $g \in E_D$. By Theorem 1.6, f is the greatest element of some chain of idempotents in E_D having positive length, and so is not minimal in E_D .

Definition 1.9. If ρ is an equivalence relation defined in a semigroup, we may call an idempotent element ρ -minimal if it is minimal in the set of all idempotents in its ρ -class. Let us call a ρ -class locally minimal if it contains a ρ -minimal idempotent. Then Corollary 1.8 asserts that a $\mathcal{D}$ -class is locally minimal if and only if it is regular and all its idempotents are $\mathcal{D}$ -minimal. If a semigroup is 0-bisimple (i.e., if it has only one non-zero $\mathcal{D}$ -class) then every $\mathcal{D}$ -minimal idempotent is either 0 or primitive; hence a completely 0-simple semigroup is just a 0-bisimple semigroup whose (two) $\mathcal{D}$ -classes are locally minimal.

One might also define ρ -least idempotents in the obvious way, but when $\rho = \mathcal{D}$ the concept is sterile; for, by Theorem 1.6, a $\mathcal{D}$ -least idempotent is the only idempotent in its $\mathcal{D}$ -class, from which it follows that the $\mathcal{D}$ -class is an $\mathcal{H}$ -class containing an idempotent and so is a group having the given idempotent as its identity element.

From the elementary observation that the product of a pair of commuting idempotents is an idempotent under each of them, it follows at once

that if a member e of a semigroup I of idempotents is minimal in I , and commutes with every element of I , then e is the least member of I ; for $ef \leq e$ and the minimality of e yield $fe = ef = e$, i.e., $e \leq f$, for any $f \in I$. In particular, if the idempotent elements of a semigroup commute then either there is a least idempotent or no minimal idempotent. The remarks in this and the preceding two paragraphs may be summarized in

Theorem 1.10. Let D be a $\mathcal{D}$ -class in a semigroup S . The following statements are equivalent:

- (1) D is a subgroup of S ;
- (2) D contains an idempotent that is least among the idempotents in D ;
- (3) D is locally minimal and is a subsemigroup of S with commuting idempotents.

At one time in the course of this study we thought that commuting idempotents would play a significant role. This has not turned out to be the case, but we record here one of our early results which may be of some interest because it extends slightly the familiar theorem that a regular semigroup in which idempotents commute is an inverse semigroup (i.e., each element has a unique inverse). We shall say that a subsemigroup T of a semigroup S is an inverse subsemigroup of S if T itself is an inverse semigroup. (In [3] the term "inverse subsemigroup" is defined only when S itself is inverse; clearly our definition agrees with that of Clifford and Preston in that case.)

Theorem 1.11. Let S be a semigroup (containing at least one idempotent element) in which idempotents commute. The set T of all regular elements of S is the greatest inverse subsemigroup of S .

Proof. It is immediate that $T \neq \square$, for any idempotent element is regular. Let $a, b \in T$, and let a' and b' be inverses in S (hence in T) of a and b , respectively. Then $a = aa'a$, $b = bb'b$, and the elements $a'a$ and bb' are idempotent. Since the idempotents commute,

$$ab = (aa'a)(bb'b) = a(a'a)(bb')b = a(bb')(a'a)b = (ab)(b'a')(ab),$$

whence ab is regular and T is a subsemigroup of S . Since $a' \in T$, T is a regular semigroup, whence, because its idempotent elements commute, it is an inverse semigroup and so is an inverse subsemigroup of S . Now if V is any inverse subsemigroup of S then each of its elements is regular in V and therefore in S ; hence $V \subseteq T$, i.e., T is the greatest inverse subsemigroup of S .

We turn now to the second cluster of theorems in this chapter, beginning with the repulsive but useful

Lemma 1.12. Let e and f be idempotent elements of a semigroup S and let s and t be elements of S such that $set = f$. Then $etfs$, $tfse$, and $etfse$ are idempotent elements of D_f such that $etfs \mathcal{R} etfse$, $tfse \mathcal{L} etfse$, and $etfse \leq e$.

Proof. We show first that $etfse$ is an idempotent element of D_f . Since

$$(fse)(etf)(fse) = f[s(ee)t](ff)se = f(set)fse = fffse = fse,$$

and since

$$(etf)(fse)(etf) = et(ff)[s(ee)t]f = etf(set)f = etfff = etf ,$$

we see that fse and etf are mutually inverse elements of S . Therefore the elements $(etf)(fse) = et(ff)se = etfse$ and $(fse)(etf)$

$= fs(ee)tf = f(set)f = fff = f$ are idempotents belonging to the same

$\mathcal{D}$ -class. To show that $etfs$ and $tfse$ are idempotents, we observe that

$$tfsetfs = tf[(set)f]_s = tfs ,$$

whence

$$(etfs)(etfs) = e(tfsetfs) = etfs$$

and

$$(tfse)(tfse) = (tfsetfs)e = tfse .$$

We show next that $etfs \mathcal{R} etfse$, omitting the parallel proof that $tfse \mathcal{L} etfse$; from these facts it follows at once that $etfs$ and $tfse$ belong to D_f . Having shown that $etfs$ and $etfse$ are idempotent, we know that $\mathcal{R}(etfs) = etfs \cdot S$ and $\mathcal{R}(etfse) = etfse \cdot S$. Now $etfse \cdot S \subseteq etfs \cdot S$; and if $x \in S$ then $etfsx = (etfs)^2x = etfse(tfxx) \in etfse \cdot S$, whence $etfs \cdot S \subseteq etfse \cdot S$. Therefore $etfs \cdot S = etfse \cdot S$, i.e., $etfs \mathcal{R} etfse$. Finally, $(etfse)e = etfse = e(etfse)$, so that $etfse \leq e$.

Theorem 1.13. Let A and B be $\mathcal{D}$ -classes in a semigroup S .

The following statements are equivalent:

- (1) there are regular elements a and b of S such that
 $a \in A$, $b \in B$, and $b \in J(a)$;
- (2) A contains at least one idempotent element, and if e is
any idempotent in A then some idempotent in B is under e ;
- (3) some idempotent in B is under some idempotent in A .

Proof. To show that (1) implies (2), let a and b be regular elements lying in A and B , respectively, such that $b \in J(a)$. Then A and B are regular, so that each contains at least one idempotent; let $e^2 = e \in A$, $f^2 = f \in B$. Since f and b are $\mathcal{D}$ -equivalent they are $\mathcal{J}$ -equivalent, so that $J(f) = J(b)$. Since $b \in J(a)$, $J(b) \subseteq J(a)$. Since a and e are $\mathcal{D}$ -equivalent they are $\mathcal{J}$ -equivalent, so that $J(a) = J(e)$. We now have $f \in J(f) = J(b) \subseteq J(a) = J(e) = SeS$, the last equality holding because e is idempotent. Hence, for some $s, t \in S$, $f = set$. The hypotheses of Lemma 1.12 being fulfilled we conclude that $(etsfe)^2 = etsfe \in D_f = B$ and $etsfe \leq e$, so that (2) holds.

That (2) implies (3) is immediate. To prove that (3) implies (1), let $b^2 = b \in B$, $a^2 = a \in A$, $b \leq a$. Then a and b are regular; and $b = ba = ba^2 = baa \in SaS = J(a)$.

It is true generally that $\mathcal{D} \subseteq \mathcal{J}$, so that every $\mathcal{J}$ -class is a union of $\mathcal{D}$ -classes. Only in special (but important) cases, however, such as that of the full transformation semigroup on a set, is it true [3, Theorem 2.9, p. 52] that $\mathcal{D} = \mathcal{J}$ and consequently every $\mathcal{D}$ -class is a $\mathcal{J}$ -class. We are about to present a theorem asserting that $\mathcal{D}$ -classes having the additional property of local minimality (and therefore, in particular, regular) are always $\mathcal{J}$ -classes, but first we give an example in which a $\mathcal{J}$ -class contains both regular and irregular $\mathcal{D}$ -classes.

Example 1.14. Let $S = \{1, x, e\}$ be the three-element semigroup in which 1 is identity element, e is zero element, and $x^2 = e$. (The associativity is evident from the fact that this semigroup may be regarded as obtained by adjoining an identity element to the null semigroup of order 2.) It

is readily seen that in S we have $D_1 = J_1 = \{1\}$, $D_x = J_x = \{x\}$, and $D_e = J_e = \{e\}$. But now we embed S in the simple semigroup $T = N \times S \times N$ by the procedure explained in Example 1.5 (p. 3). Since elements (a,u,b) and (c,v,d) of T are $\mathcal{D}$ -equivalent if and only if $u \mathcal{D} v$ in S , the $\mathcal{D}$ -classes of T are the sets $\{(a,1,b) : a,b \in N\}$, $\{(a,x,b) : a,b \in N\}$, and $\{(a,e,b) : a,b \in N\}$. Since an element (a,u,b) of T is idempotent only if u is an idempotent in S , and $1^2 = 1$, $x^2 \neq x$, $e^2 = e$, we see that the second of these $\mathcal{D}$ -classes in T is irregular and the other two are regular. But, T being simple, T itself is its only $\mathcal{J}$ -class.

We shall see in the next chapter that if a $\mathcal{J}$ -class contains a "finite-dimensional" element then it cannot contain both regular and irregular $\mathcal{D}$ -classes. For the present we shall show, in the form of a lemma that will be useful to us in proving our next theorem, that even though a $\mathcal{J}$ -class may contain both regular and irregular elements nonetheless the presence of a regular element forces upon all elements of the $\mathcal{J}$ -class a property enjoyed by regular elements.

Lemma 1.15. If an element x of a semigroup S is $\mathcal{J}$ -equivalent to a regular element then $J(x) = SxS$.

Proof. Let x be an element and y a regular element of a semigroup S , and assume $x \mathcal{J} y$. Since y is regular, some idempotent is $\mathcal{D}$ -equivalent to y ; let $e^2 = e \in D_y$. Since $\mathcal{D}$ -equivalent elements are $\mathcal{J}$ -equivalent we have $J_e = J_y = J_x$, whence $e \in J_x = S^1 x S^1$. There exist, then, elements $s_1, s_2 \in S^1$ such that $e = s_1 x s_2$. Let $t_1 = e s_1$ and $t_2 = s_2 e$, and note that $t_1 \in S$ even if $s_1 = 1 \notin S$

($i = 1, 2$). Now we have

$$t_1 x t_2 = (e s_1) x (s_2 e) = e (s_1 x s_2) e = e e e = e.$$

But $J(x) = J(e)$ because $J_x = J_e$, and therefore

$$J(x) = SeS = S(t_1 x t_2)S = (S t_1) x (t_2 S) \subseteq S x S \subseteq J(x),$$

and the desired equality follows.

Theorem 1.16. Every locally minimal $\mathcal{D}$ -class in a semigroup is a $\mathcal{J}$ -class.

Proof. Let D be a locally minimal $\mathcal{D}$ -class in a semigroup S , and let e be an idempotent element minimal in the set of all idempotents in D . Let $x \in J_e$. Then, e being regular, Lemma 1.15 assures us that $J(x) = SxS$. We have, therefore, both

$$e \in J(e) = J(x) = SxS,$$

from which we conclude that $e = s_1 x t_1$ for some $s_1, t_1 \in S$, and

$$x \in J(x) = J(e) = SeS,$$

from which we conclude that $x = s_2 e t_2$ for some $s_2, t_2 \in S$. Now

$(s_1 s_2) e (t_2 t_1) = s_1 (s_2 e t_2) t_1 = s_1 x t_1 = e$, so that if in the statement of Lemma 1.12 we set $s = s_1 s_2$, $t = t_2 t_1$, and $e = f$ then the hypotheses of that lemma are verified. The conclusion then is that

$$e \geq e t f s e = e t_2 t_1 e s_1 s_2 e = (e t_2 t_1 e s_1 s_2 e)^2 \in D_e.$$

Since e is minimal among the idempotents in D_e , it follows that

$$e t_2 t_1 e s_1 s_2 e = e. \text{ Now}$$

$$\begin{aligned}
x(t_1 e s_1 s_2 e t_2 t_1 e s_1) x &= (s_2 e t_2) t_1 e s_1 s_2 e e t_2 t_1 e s_1 (s_2 e t_2) \\
&= s_2 (e t_2 t_1 e s_1 s_2 e) (e t_2 t_1 e s_1 s_2 e) t_2 \\
&= s_2 e e t_2 = s_2 e t_2 = x .
\end{aligned}$$

Therefore x is regular.

We are now in a position to use Theorem 1.13, statement (1) therein being verified if we set $a = e$, $b = x$, $A = D_e$, and $B = D_x$. Statement (2) then asserts that some idempotent $g \in D_x$ is under e . Now $g \not\leq x$ because $g \mathcal{D} x$; hence $e \in J(e) = J(x) = J(g)$.

Again we use Theorem 1.13, this time setting $a = g$, $b = e$, $A = D_g$, $B = D_e$. Statement (1) of the theorem again being verified, we conclude from statement (2) that some idempotent $h \in D_e$ is under g . Now we have $h \leq g \leq e$ and $h, e \in D_e = D$. But e is minimal in D , so that $h = g = e$ and $x \in D_x = D_g = D_e$. We have shown that $x \mathcal{D} e$ for an arbitrary element x of J_e , whence $D = D_e = J_e$.

We have just seen, in the first part of the proof of Theorem 1.16, that if an idempotent element e is minimal among the idempotents of its $\mathcal{D}$ -class then every element of its $\mathcal{J}$ -class J_e is regular. It is by no means the case, however, that the ideal $J(e)$ must be regular, even if we suppose e to be minimal in the whole set E of idempotents in S . A simple example is the infinite cyclic semigroup with identity element adjoined, which is isomorphic to the additive semigroup N of all non-negative integers; for the only idempotent element of N is 0 , but $J(0) = N$ and all non-zero elements of N are irregular. In our next theorem we shall assume regularity for the principal ideal generated by a minimal idempotent, and get correspondingly stronger conclusions than that

of Theorem 1.16. The new assumption is far from being too strong, however, for it holds in every completely simple semigroup.

Theorem 1.17. Let f be a minimal idempotent element of a semi-
group S , and assume that the principal ideal $J(f)$ is regular. Then

- (1) $J(f) = J_f = D_f$,
- (2) $J(f)$ is the kernel of S ,
- (3) each idempotent $e \in S$ is over some idempotent $h_e \in J(f)$,
- (4) an idempotent belongs to $J(f)$ if and only if it is minimal
in S ,
- (5) either f is a zero element of S or $J(f)$ is a completely
simple subsemigroup of S ,
- (6) no two distinct idempotents in $J(f)$ commute.

Proof of (1). Since f is a minimal idempotent, D_f is a fortiori locally minimal and we have $J_f = D_f$ by Theorem 1.16. And always $J_f \subseteq J(f)$. It suffices, then, to prove $J(f) \subseteq D_f$. Let $b \in J(f)$, and in statement (1) of Theorem 1.13 set $a = f$; then $A = D_f$, $B = D_b$, and (1) is verified; by statement (2), therefore, there is an idempotent $h \in D_b$ such that $h \leq f$, and since f is minimal it follows that $h = f$. Therefore $b \in D_b = D_h = D_f$, whence $J(f) \subseteq D_f$.

Proof of (2). Let M be an arbitrary ideal in S . Now that we know that $J_f = J(f)$, and the product of two-sided ideals is always contained in their intersection, we have $M \cdot J(f) \subseteq M \cap J(f) = M \cap J_f$ and $M \cap J_f$ is therefore non-empty. Let $x \in M \cap J_f$. Since $x \in J_f$, $J(x) = J(f)$. Since $x \in M$, $J(x) \subseteq M$. Hence $J(f) \subseteq M$, so that $J(f)$ is the kernel of S .

Proof of (3). Let e be any idempotent element of S . By (1) above we have $ef = eff \in SfS = J(f) = D_f$, whence ef is a regular element of S . Again we apply Theorem 1.13, setting $a = e$ and $b = ef$ in the first statement therein, so that $A = D_e$, $B = D_{ef}$, and, since $ef = eef \in SeS = J(e)$, that first statement is verified. From the second statement of Theorem 1.13 we conclude that some idempotent in D_{ef} is under e , and the observation that $D_{ef} = D_f = J(f)$ completes the proof.

Proof of (4). Let e be a minimal idempotent element of S . By what we have just proved, there is an idempotent $h \in J(f)$ such that $h \leq e$. Therefore $e = h \in J(f)$. Conversely, let g be any idempotent in $J(f)$ and suppose g is not minimal in S . Then $g > g'$ for some idempotent $g' \in S$. But $g \in J(f) = D_f$ by (1) above, so that $g \mathcal{L} f$. By Lemma 1.2, there is an idempotent $f' < f$, contrary to the hypothesis that f is minimal. We conclude that if $g^2 = g \in J(f)$ then g is a minimal idempotent.

Proof of (5). Assume that f is not a zero element of S . We show first that it cannot be a zero element of $J(f)$; for if it were, and if $y \in S$, then

$$fy = f \cdot f y \in f \cdot SfS = f \cdot J(f) = f$$

and similarly

$$yf = y f f \cdot f \in SfS \cdot f = J(f) \cdot f = f,$$

whence the assumption that f is not a zero element of S is contradicted. Now f , being minimal in S , is a fortiori minimal in $J(f)$, and, not being a zero of $J(f)$, is a primitive idempotent of the semigroup $J(f)$. But we know from (2) above that $J(f)$ is the kernel of S , and hence is

a simple semigroup. Therefore $J(f)$ is completely simple.

Proof of (6). Let s and t be idempotents in $J(f)$ such that $st = ts$. Then, as observed earlier, st is idempotent and is under both s and t . By (4) above, s and t are minimal in S , and therefore $s = st = t$.

Closely related both to Theorem 1.16 and to Theorem 1.19, but apparently not an immediate consequence of either, is the final result in this chapter:

Theorem 1.18. Let e be a primitive idempotent in a semigroup S , and let b be a regular element of S lying in $J(e)$. Then either b is a zero element of S or $b \notin e$.

Proof. Once more we use Theorem 1.13, setting $a = e$ in the first statement therein. Then $A = D_e$, $B = D_b$, and $b \in J(e)$, so that (1) is verified. From statement (2) we infer the existence of an idempotent $f \in D_b$ such that $f \leq e$. Suppose now that $b \notin D_e$. Then $f \in D_b \neq D_e$, whence $f \neq e$ and so $f < e$. Since e is primitive, f is then a zero element of S . But then $b \in D_f = \{f\}$, whence $b = f$. We have shown that b is a zero element of S unless $b \notin e$.

Corollary 1.19. In a semigroup S without zero, the principal ideal generated by a minimal idempotent contains exactly one regular $\mathcal{D}$ -class.

Remark 1.20. For semigroups without zero, Corollary 1.19 yields the first conclusion in Theorem 1.17.

CHAPTER II

DIMENSION

The function δ that we are about to define on an arbitrary semigroup will be called "dimension" because on the semigroup of all subspaces of a finite-dimensional vector space, under intersection, it is just the ordinary (projective) dimension. It is one of several closely similar functions that we have investigated, and its peculiarities arise from an effort to find a function which, while enjoying the property claimed for δ in our last theorem, will at the same time "separate" the elements of the semigroup as much as possible.

How the dimension of an element x is to be defined will depend on the relationship of x to the set E of all idempotents of S , and we divide the definition into three mutually exclusive and exhaustive cases. In the first case, $\delta(x)$ will be a positive integer, or a positive integer plus $\frac{1}{2}$, or else a symbol ∞ concerning which we specify only that (by definition) it is greater than every positive integer. All elements falling under the second case are assigned to the dimension $\frac{3}{4}$; any number strictly between $\frac{1}{2}$ and 1 would do as well. The range of δ in the third case is a certain set of rational numbers in the interval $[0, \frac{1}{2}]$, specifically $\frac{1}{2}$ and the fractions $\frac{n}{2(n+2)}$ ($n = 0, 1, 2, \dots$).

Definition 2.1. The dimension $\delta(x)$ of an element x of a semigroup S is defined as follows.

Case I: $J(x)$ contains at least one idempotent.

(a) If the set of lengths of finite chains of idempotents descending from idempotents in $J(x)$ is bounded, and the length of the

longest such chain is $m(x)$, then $\delta(x) = 1 + m(x)$ if x is regular and $\delta(x) = \frac{3}{2} + m(x)$ if x is irregular.

(b) If the set of lengths of finite chains of idempotents descending from x is unbounded, then $\delta(x) = \infty$.

Case II: for every idempotent f , $f \notin J(x)$ and $x \notin J(f)$. We define $\delta(x) = \frac{3}{4}$.

Case III: $J(x)$ contains no idempotent, but $x \in J(f)$ for at least one idempotent f .

(a) If, among all finite chains of idempotents descending from idempotents f such that $x \in J(f)$, there is at least one maximal chain, and if $s(x)$ is the length of the shortest such maximal chain, then

$$\delta(x) = \frac{1}{2} + \frac{1}{2 + s(x)}.$$

(b) If there is no such maximal chain (i.e., if every finite chain of idempotents that descends from an idempotent f such that $x \in J(f)$ is a proper subchain of some such finite chain) then $\delta(x) = \frac{1}{2}$.

Example 2.2. We remark here that under this definition the finite-dimensional regular elements of a semigroup are precisely the elements having positive integral dimension. Any element of the bicyclic semigroup is infinite-dimensional. Each element of the infinite cyclic semigroup Z has dimension $\frac{3}{4}$. Adjoining an identity element 1 to the infinite cyclic semigroup, we obtain a semigroup Z^1 in which $\delta(1) = 1$ and $\delta(x) = 0$ for all $x \in Z$. If we then adjoin to Z^1 a new identity element e , we have a semigroup in which $\delta(e) = 2$, $\delta(1) = 1$, and $\delta(x) = \frac{1}{6}$ for all $x \in Z$.

Clearly the "natural" part of our definition of dimension is Case I, and for large classes of semigroups (e.g., all regular semigroups, all periodic [in particular, all finite] semigroups) only this part applies. Our first theorem in this chapter has the restriction to Case I as part of the hypothesis.

Theorem 2.3. Let P be a principal ideal of a semigroup S , and assume that P has a generator x such that $1 \leq \delta(x) < \infty$. Then each idempotent in P is minimal among the idempotents in its $\mathcal{J}$ -class, which coincides with its $\mathcal{D}$ -class.

Proof. Since $\delta(x) \geq 1$, there is at least one idempotent in P ; let e be any such idempotent. Since $\delta(x) < \infty$, there is a positive integer m such that no m -chain of idempotents descends from e . Hence, by Theorem 1.6, no two idempotents in D_e are comparable, whence D_e is locally minimal. By Theorem 1.16, $D_e = J_e$, so that e is minimal in J_e , as we wished to show. We next obtain the result promised on page 10.

Theorem 2.4. The $\mathcal{D}$ -class of a finite-dimensional regular element of a semigroup coincides with its $\mathcal{J}$ -class, and every element thereof is regular.

Proof. Let x be a finite-dimensional regular element of a semigroup S . Then D_x is regular, whence there is an idempotent $e \in D_x \subseteq J_x \subseteq J(x)$. Since $e \in J(x)$, $1 \leq \delta(x) < \infty$. By Theorem 2.3, $D_x = J_x$.

Theorem 2.5. In any semigroup, all elements of a $\mathcal{J}$ -class have the same dimension.

Proof. Let J be a $\mathcal{J}$ -class in a semigroup S , and let $x, y \in J$, $P = J(x) = J(y)$, and let U be the set of all idempotents in P . Assume first that $U \neq \emptyset$, so that Case I of the definition of δ applies. If $\delta(x)$ is infinite then so is $\delta(y)$ since $J(x) = J(y)$. If $\delta(x)$ is finite then, by Theorem 2.4, either every element of J is regular, in which case $\delta(x) = n = \delta(y)$ for some positive integer n , or else every element of J is irregular, in which case $\delta(x) = n + \frac{1}{2} = \delta(y)$ for some positive integer n .

Assume now that $U = \emptyset$, so that either Case II or Case III of the definition of δ applies. Since x and y generate the same principal ideal of S , for an arbitrary two-sided ideal M it is true that either both x and y belong to M or else neither belongs to M . In particular, for any idempotent f , either $x, y \in J(f)$ or $x, y \notin J(f)$. Hence $\delta(x) = \delta(y)$.

Theorem 2.6. In a semigroup S , let x be a regular element of finite dimension n and let e be any idempotent in the principal ideal generated by x . In order that there exist an $(n-1)$ -chain of idempotents descending from e , it is necessary and sufficient that e be $\mathcal{J}$ -equivalent to x .

Proof of necessity. Assume that there is an $(n-1)$ -chain of idempotents descending from e . Since e and x are regular, and $e \in J(x)$, the first of the three equivalent statements in Theorem 1.13 is verified. By the third statement in that theorem, then, there exist idempotents g and h such that $g \in D_x$, $h \in D_e$, and $g \geq h$. Since $h \mathcal{J} e$, and since by hypothesis there is an $(n-1)$ -chain of idem-

potents descending from e , we know from Theorem 1.4 that there is also such an $(n-1)$ chain descending from h . Consequently if $g > h$ then there is an n -chain of idempotents descending from g . But, since $g \in D_x \subseteq J_x \subseteq J(x)$, Case I of the definition of δ applies, and we conclude that if $g > h$ then $\delta(x) \geq n+1$, contrary to hypothesis. Therefore both $g = h \in D_e$ and $g \in D_x$, whence $D_e = D_x$.

Proof of sufficiency. Conversely, assume that $e \not\geq x$. Since $\delta(x) = n$, there is at least one idempotent in $J(x)$ from which an $(n-1)$ chain of idempotents descends; let f be any such idempotent. Since f and x are regular, and $f \in J(x)$, Theorem 1.13 yields an idempotent $g \in D_f$ such that $g \leq e$. Since $f \not\geq g$, there is an $(n-1)$ -chain descending from g . Hence if $g < e$ there is an n -chain of idempotents descending from $e \in D_e = D_x \subseteq J(x)$, contrary to the hypothesis $\delta(x)=n$. Therefore $g = e$, and there is an $(n-1)$ -chain descending from e as predicted.

Corollary 2.7. Let x be a finite-dimensional regular element of a semigroup, and let e be any idempotent in $J(x)$ from which there descends an idempotent chain of greatest length. Then $D_e = D_x = J_x = J_e$.

Proof. Theorems 2.4 and 2.6.

Theorem 2.8. No element of a principal ideal of a semigroup is of dimension greater than the dimension of the generators of the ideal.

Proof. Let P be a principal ideal of a semigroup S . By Theorem 2.5, all generators of P have the same dimension; let x be any such generator, and let $y \in P$. Then $J(y) \subseteq P$. If $\delta(x) = \infty$ then $\delta(y) \leq \delta(x)$ by definition. If $\delta(x) = n + \frac{1}{2}$ for some positive integer n then, since $J(y) \subseteq P = J(x)$, $\delta(y) \leq \delta(x)$. Assume now that $\delta(x)$

is a positive integer n . If $J(y)$ contains no idempotent element then $\delta(y) \leq \frac{3}{4} \leq \delta(x)$. On the other hand, still assuming $\delta(x) = n$, assume also that $J(y)$ contains at least one idempotent. Since $J(y) \subseteq J(x)$ we know at least that $\delta(y) \leq n + \frac{1}{2}$. Suppose that $\delta(y) = n + \frac{1}{2}$. Then there is an $(n-1)$ -chain of idempotents descending from some idempotent $e \in J(y) \subseteq P$, whence, by Corollary 2.7, $D_e = D_x = J_x = J_e$ and therefore $J(x) = J(e)$. Since $e \in J(y)$ we have $J(e) \subseteq J(y) \subseteq J(x) = J(e)$, so that $J(y) = J(x)$ and therefore $J_x = J_y$. But now Theorem 2.5 asserts that $\delta(y) = \delta(x) = n$, contrary to the assumption $\delta(y) = n + \frac{1}{2}$. Therefore $\delta(y) \leq \delta(x)$.

Having disposed of the case $\delta(x) \geq 1$, we now assume $\delta(x) < 1$, so that $J(y)$ (and a fortiori, P) contains no idempotent, and both $\delta(y) \leq \frac{3}{4}$ and $\delta(x) \leq \frac{3}{4}$. If $\delta(x) = \frac{3}{4}$ then $\delta(y) \leq \frac{3}{4} = \delta(x)$, and we are through. If $\delta(x) < \frac{3}{4}$, then there is an idempotent f such that $x \in J(f)$, whence $y \in J(y) \subseteq J(x) \subseteq J(f)$, $0 \leq \delta(y) \leq \frac{1}{2}$, and $0 \leq \delta(x) \leq \frac{1}{2}$. If $\delta(x) = \frac{1}{2}$ then $\delta(y) \leq \frac{1}{2} = \delta(x)$, and we are through. Assume then, that $\delta(x) < \frac{1}{2}$. Then

$$\delta(x) = \frac{1}{2} - \frac{1}{2+m}$$

for a certain non-negative integer m , and there is an idempotent $g \in S$ such that some maximal m -chain of idempotents descends from g and such that $x \in J(g)$, whence $y \in J(y) \subseteq J(x) \subseteq J(g)$. Thus

$$\delta(y) = \frac{1}{2} - \frac{1}{2+j}$$

for a certain non-negative integer j . Furthermore, since $y \in J(x) \subseteq J(h)$ for all idempotents h such that $x \in J(h)$, the chains descending from idempotents h such that $x \in J(h)$ are a subset of the set of chains descending from idempotents g such that $y \in J(g)$, whence we see that $j \leq m$. Therefore $\delta(y) \leq \delta(x)$.

From Theorem 2.8 we obtain immediately the strong "subadditivity" of our dimension function:

Theorem 2.9. For any elements x and y of a semigroup,

$$\delta(xy) \leq \min \{ \delta(x), \delta(y) \} .$$

Proof. Since $xy \in J(x)J(y) \subseteq J(x) \cap J(y)$, it follows from Theorem 2.8 that $\delta(xy) \leq \delta(x)$ and $\delta(xy) \leq \delta(y)$.

Corollary 2.10. Let S be a semigroup and r a non-negative real number or ∞ . The set of all elements of dimension less than r , and the set of all elements of dimension less than or equal to r , are ideals of S .

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BIBLIOGRAPHY

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APPENDIX

APPENDIX

A binary operation on a set S makes correspond to $u, v \in S$ a unique element $w \in S$, denoted multiplicatively by $uv = w$. The binary operation is associative provided that $a(bc) = (ab)c$ whenever $a, b, c \in S$. A semigroup is a set on which an associative, binary operation is defined. A subsemigroup of a semigroup is a subset T of a semigroup such that $a, b \in T$ implies that $ab \in T$. For the remainder of the appendix, we shall use S to denote a semigroup, and $\square$ to denote the empty set.

The elements $a, b \in S$ commute if $ab = ba$. If $f \in S$ and $ff = f$, then f is an idempotent element belonging to S . If $e \in T \subseteq S$ then e is called a left [right] identity element of T provided that $ex = x$ [$xe = x$] for all $x \in T$. If $xe = ex = x$ for all $x \in T$ then e is called a two-sided identity element, or simply an identity element, of T . The element z is a zero of S if $zx = xz = z$ whenever $x \in S$. The semigroup S can contain no more than one identity and no more than one zero. A semigroup consisting only of idempotent elements is called a band, and a commutative band is a semilattice.

A subsemigroup G of S is a subgroup of S if G is a group, i.e., if there exists an element $e \in G$ which is an identity element of G and if, for each $g \in G$, there exists a unique $g' \in G$ such that $gg' = g'g = e$.

The element $y \in S$ is regular if there exists $y' \in S$ such that $yy'y = y$; furthermore, if also $y'yy' = y'$ then y and y' are mutually inverse elements of S and y' is an inverse of y . An element which is not a regular element is an irregular element. Every regular element of S

has at least one inverse. If every element of S is regular, then S is a regular semigroup. If S is regular and each element of S has only one inverse, then S is an inverse semigroup. The semigroup S is an inverse semigroup if and only if S is regular and the idempotents of S commute. If S is an inverse semigroup, then S is a group if and only if S contains only one idempotent. If the semigroup S is a group, and if $x, x' \in S$, then x and x' are mutually inverse elements of S if and only if $x'x = xx' = e$, where e is the identity element of the group.

Let E be the set of all idempotents of S . We introduce a partial order (i.e., a reflexive, antisymmetric, transitive binary relation) on E by defining $e \leq f$ to mean $ef = fe = e$. If $e \leq f$ we say that e is under f and f is over e . By $e < f$, we mean that $e \leq f$ and $e \neq f$, and we say that e is strictly under f (f strictly over e). By $f \geq e$ and $f > e$, we mean $e \leq f$ and $e < f$, respectively. An idempotent is called primitive if it is not a zero and is strictly over no non-zero element. If either $e \leq f$ or $f \leq e$, then e and f are comparable. A subset of E in which every two elements are comparable is a chain. A finite chain $e_0 > e_1 > \dots > e_m$ is said to be of length m , or to be an m -chain descending from e_0 . A chain C is maximal among chains having a property π if no chain having property π properly contains C .

As is customary in discussions of partial order, we distinguish between the notion's "minimal" and "least" (and similarly between "maximal" and "greatest"). An element e of a subset T of a partly ordered set

is said to be minimal in T if for no $t \in T$ is $t < e$. On the other hand, e is a least element of T (necessarily unique) if $e \leq t$ for all $t \in T$. Similar definitions are made for maximal and greatest elements. The notion of maximal chain defined above is a special case: in the set of chains having property π , partly ordered by inclusion, C is a maximal element.

If $\emptyset \neq M \subseteq S$ and $MS \subseteq M$ [$SM \subseteq M$] then M is a right [left] ideal of S . If M is a left and a right ideal, then M is a two-sided ideal, or ideal, of S . If M_1 and M_2 are ideals then $M_1 M_2$ and $M_1 \cap M_2$ are ideals such that $M_1 M_2 \subseteq M_1 \cap M_2$. The semigroup S itself is an ideal of S , and if S has no other ideal it is called a simple semigroup. For any $a \in S$, $J(a)$ denotes the intersection of all ideals containing a , $L(a)$ the intersection of all left ideals containing a , and $R(a)$ the intersection of all right ideals containing a ; $J(a)$ is an ideal, $L(a)$ a left ideal, and $R(a)$ a right ideal, and these are called the principal two-sided, left, and right ideals generated by a . Now $J(a) = \{a\} \cup Sa \cup aS \cup SaS$, $L(a) = \{a\} \cup Sa$, and $R(a) = \{a\} \cup aS$. If a is regular in S , then $J(a) = SaS$, $R(a) = aS$, and $L(a) = Sa$.

An ideal is called minimal if it does not contain properly an ideal of S . There can be at most one minimal ideal of S , and if such exists it is the intersection of all two-sided ideals of S and is called the kernel of S . If S contains a kernel K , then K is a simple subsemigroup of S . The semigroup S is completely simple if it is simple and contains a primitive idempotent. Every idempotent

in a completely simple semigroup is primitive.

Let $a, b \in S$. If $L(a) = L(b)$, $R(a) = R(b)$, or $J(a) = J(b)$, we say that a and b are $\mathcal{L}$ -equivalent, $\mathcal{R}$ -equivalent, or $\mathcal{J}$ -equivalent, respectively, and we write $a \mathcal{L} b$, $a \mathcal{R} b$, and $a \mathcal{J} b$, respectively. In fact, these definitions each define an equivalence (i.e., a reflexive, symmetric, and transitive) relation on the set S . By L_a , R_a , and J_a , we mean the set of all elements each of which is $\mathcal{L}$ -, $\mathcal{R}$ -, or $\mathcal{J}$ -equivalent to a , respectively. If there exists $x \in S$ such that $a \mathcal{L} x$ and $x \mathcal{R} b$, we say that a and b are $\mathcal{D}$ -equivalent, denoted by $a \mathcal{D} b$, and we denote by D_a the set of all elements each of which is $\mathcal{D}$ -equivalent to a . If both $a \mathcal{L} b$ and $a \mathcal{R} b$, we say that a and b are $\mathcal{H}$ -equivalent, written $a \mathcal{H} b$, and we denote by H_a the set of all elements of S each of which is $\mathcal{H}$ -equivalent to a . The relations $\mathcal{D}$ and $\mathcal{H}$ are also equivalence relations. These five equivalences are referred to as Green's relations, having been introduced by J. A. Green in [4].

If $a \in S$, then $a \in H_a \subseteq L_a[R_a] \subseteq D_a \subseteq J_a \subseteq J(a)$. Also, $L_a \subseteq L(a) \subseteq J(a)$ and $R_a \subseteq R(a) \subseteq J(a)$. Now $J(a)$ is a subset of any ideal containing a , $R(a)$ is a subset of any right ideal containing a , and $L(a)$ is a subset of any left ideal containing a .

The set product LR of any $\mathcal{L}$ -class $L \subseteq S$ and $\mathcal{R}$ -class $R \subseteq S$ is always contained in a single $\mathcal{D}$ -class. If a $\mathcal{D}$ -class D of a semigroup S contains a regular element, then every element of D is regular, and D itself is said to be regular. If D is regular, then every $\mathcal{L}$ -class and every $\mathcal{R}$ -class contained in D contains at least one idempotent. If S itself is a $\mathcal{D}$ -class then it is called a bisimple

semigroup, and is necessarily simple.

If a and a' are inverse elements of a semigroup S , then $e = aa'$ and $f = a'a$ are idempotents such that $ea = af = a$ and $a'e = fa' = a'$. Hence $e \in R_a \cap L_{a'}$ and $f \in R_{a'} \cap L_a$. The elements a, a', e, f all belong to the same $\mathcal{D}$ -class of S . Any idempotent element g of S is a right identity element of L_g and a left identity element of R_g . Any $\mathcal{H}$ -class containing an idempotent is a subgroup of S .

Let a be a regular element of S . Every inverse of a belongs to D_a . If $b \in S$, then H_b contains an inverse of a if and only if both of the $\mathcal{H}$ -classes $R_a \cap L_b$ and $R_b \cap L_a$ contain idempotents.

The null semigroup of order 2 is the semigroup $\{x, z\}$ such that $xx = xz = zx = zz = z$.

Let S be a semigroup. Let 1 be an element not belonging to S and define $s*t = st$ if $s, t \in S$, and $1*s = s*1 = s$ if $s \in S \cup \{1\}$; then $S \cup \{1\}$ is a semigroup under the operation $*$. We use the symbol S^1 to denote the semigroup S if S has an identity and the semigroup $S \cup \{1\}$ if S has no identity.

Let N be the set of non-negative integers. If $(a, b), (c, d) \in N \times N$, define $(a, b)(c, d) = (a + c - \min\{b, c\}, b + d - \min\{b, c\})$; then $N \times N$ is a semigroup, called the bicyclic semigroup. The bicyclic semigroup is inverse and bisimple, but not completely simple.