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July 29, 1988

**STATISTICAL THEORY FOR
PREDICTING THE FAILURE OF
BRITTLE MATERIALS**

**A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville**

**Si-xing She
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ABSTRACT

Statistical models for predicting failure probability of brittle materials are investigated. A formula is derived from a physical consideration of fracture of microcracks in materials based on the general forms of a fracture criterion and a statistical distribution function incorporating the weakest link principle. The relationships of this model and other statistical models in the literature are discussed; they were found to be equivalent for isotropic materials in which microcracks are randomly distributed in all directions.

The finite element structural analysis program SAP4 is modified; some new subroutines are developed so that the failure analyses can be made using statistical models. A truss element, a triangular element and a quadrilateral element are each modified and several subroutines are developed to calculate the risk of rupture and the failure probability for each element and for the whole structure. The triangular element and quadrilateral elements are capable of doing failure analyses for plane stress, plane strain and axisymmetric problems. Several sample analyses are also provided to verify the program.

The statistical model is also used in a failure analysis of the rounded notch four point bending specimen made of an AISI 1008 steel. The grain boundary carbide particles are considered to be microcracks in the plastic zone near the notch tip. The distribution

function in the statistical theory is derived from the density and size distribution of carbide particles in the steel. The statistical theory for a triaxial stress state is used to predict the failure probability for any given load on the specimen. The failure loads (loads corresponding to 50% failure probability) are calculated for the specimen at different temperatures. The results are compared with experimental data; good agreement is obtained.

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LIST OF SYMBOLS

a	notch length
b	characteristic length along crack front.
B	risk of rupture, or thickness of four point bending specimen
c_1, c_2, c_3	constants used in eq.(4.13)
d_0	diameter of carbides
dA	elemental area in the plastic zone
E	Young's modulus
f	percentage of the carbides participating in fracture process
G_c	critical the coplanar strain-energy release rate
J_{Ic}	critical value of J-integral at fracture
K	fracture criterion parameter
K_{cr}	critical fracture criterion parameter
K_{Ic}	critical value of stress intensity factor at fracture
K_0	material constant in Weibull distribution functions for an arbitrary fracture criterion
K_u	material constant in three-parameter Weibull distribution function for an arbitrary fracture criterion
k	material constant in Weibull distribution function $n(\sigma)$
k_1	material constant in Weibull distribution function $n_1(\sigma)$
k, μ	material constants in eq.(2.9)
$N(\sigma_{cr})$	distribution function in eq. (2.7)
N_0	number of carbides per unit volume

n	work hardening exponent
$n(\sigma)$	Weibull distribution function for a uniaxial stress state
n_1, n_2, n_3	three components of a unit vector n
$n_1(\sigma)$	Weibull distribution function for a multiaxial stress state
M	moment
m	material constant in Weibull distribution function $n(\sigma)$
m_1	material constant in Weibull distribution function $n_1(\sigma)$
P	force
P_f	failure probability
P_s	survival probability
r, θ	polar coordinates
r_0	radius of a penny-shaped microcrack
r_p	extension of plastic zone into the specimen
r_c	distance from notch center, used in eq.(4.13)
s_u, s_0	material constants in Weibull distribution function for a normal tensile stress fracture criterion
T	temperature
v	volume of materials
W	length for describing the geometry of the four point bending specimen (see Figure 4.2)
α	limit of integrals obtained from eq.(3.9) or eq.(3.12)
α, β	characteristic coordinates in slip line region
$\bar{\epsilon}$	equivalent strain
ϵ_0	yield strain
$\bar{\epsilon}_t$	equivalent strain at notch tip

θ, φ	spherical coordinates
θ_s	span of the plastic zone
γ	effective fracture surface energy
ν	Poisson's ratio
Σ	stress state
ρ	root radius of notch
$\bar{\sigma}$	equivalent stress
σ_0	yield stress
σ_1	first principal stress
σ_2	second principal stress
σ_3	third principal stress
σ_f	fracture tension of carbides in eq.(4.1)
σ_{cr}	critical tensile stress
σ_n	normal stress on the surface with normal direction n
σ_{min}	nominal stress defined in eq.(4.2)
τ_n	shear stress on the surface with normal direction n
ψ	distribution function defined in eq.(2.11)
Ω	solid angle
ζ	parameter defining the upper and lower boundaries of plastic region

CHAPTER 1

INTRODUCTION

Brittle materials, such as glass, ceramics and high strength steel at low temperature, exhibit a substantial dispersion in the fracture strength. Two identical structures made of the same material and under the same loading configuration may have considerably different fracture strengths. This is because the failure of brittle materials is caused by the fracture of microscopic cracks which exist in brittle materials.

The effects of microcracks on macroscopic properties of brittle materials can only be described by a statistical theory since the sizes of microcracks are very small and the number of microcracks is very large¹⁻⁴. Thus, for a loaded structure made of a brittle material, the statistical theory can predict the failure probability under the given load. However, it cannot determine a single failure strength of the structure, because the physical basis for failure of brittle materials is the fracture of the microcracks in the materials.

The objective of this work is to further understand the fracture properties of brittle materials by investigating the statistical models for predicting the failure of these materials. The first step is to investigate the physical basis of the statistical theory for predicting the failure of brittle materials; then the methods for applying the the statistical theory for any structure and materials are investigated.

Most brittle materials display an essentially elastic behavior, and they have little or no plastic deformation at fracture. The strength of brittle materials is usually high. There is a large amount of scatter in the fracture strength, however, and the compressive strength is usually much higher than tensile strength. So, failure usually occurs in tension. Brittle materials have low ductility and very low fracture resistance.

The low fracture resistance of brittle materials causes some problems in using them. This can be avoided by using brittle materials in compression, but theories for predicting the strength of brittle materials are needed to design a reliable, efficient and safe structure.

The distribution of microcracks in brittle materials is statistical in nature and has a direct influence on the mechanical properties of the materials. Microcracks in brittle materials can be distributed in the volume or along the surface of the material⁵. They can be randomly oriented or have preferred directions. Fracture of microcracks is mostly caused by a normal tensile stress, but it can also be caused by the remote shear stress around the microcracks. The normal compressive stress which can cause the contact of two microcrack surfaces will also effect the fracture of microcracks. The fracture criteria of microcracks are material dependent.

In the current statistical models¹⁻⁴, structural failures of brittle materials are associated with fractures of microcracks according to the weakest link principle. The weakest link principle

states that the entire structure fails when a single microcrack in the material fails. Application of the weakest link principle to a structure made of brittle material implies that there is no stable microcrack growth in the structure and that any fracture of a microcrack will lead to catastrophic failure of the structure.

Several statistical models have been proposed and applied in predicting the failure of brittle materials¹⁻⁴. Each one has some success in particular applications, but the limitations and the relations between these models are not clear and have not been investigated extensively.

The Weibull model¹ was derived from a uniaxial stress state. It is generally used together with the independent action theory in prediction of fracture of brittle materials under multiaxial stress states⁶. The Weibull theory emphasizes the statistical character of the strength of a material. Two statistical distribution functions were proposed by Weibull which used normal tensile stress as a criterion for fracture of microcracks to evaluate the reliability of a structure subjected to tensile stress. The independent action theory assumes that the three principal stresses act independently to affect the failure probability of the loaded body.

During 1970's, much research was done on this problem in relation to the applications of high strength materials, and some new results were obtained. Batdorf and Crose² developed a statistical model for a multiaxial stress state from physical consideration of the fracture of microcracks. It was also proved that Weibull/independent action theory gives nonconservative results⁷

and the Bardorf and Crose model is more accurate and more realistic than the Weibull/independent action theory⁶. Evans³ conducted a statistical analysis by using the strain energy release rate fracture criterion for penny-shaped cracks in triaxial and biaxial stress states. Alpa⁴ proposed a theory in which a general function with several material-dependent parameters is used as a fracture criterion valid for both tension and compression stress states. However, none of these models which is derived from physical consideration of the fracture of microcracks but they apply to an arbitrary microcrack distribution. In Chapter 2, these models are reviewed and their relations are investigated. A general formula of the failure probability for a triaxial stress state and for an arbitrary distribution of microcracks and an arbitrary fracture criterion is derived.

Application of the statistical theory for predicting the failure probability of a structure requires that the stress distribution in the structure, microcrack distributions in the materials and the fracture criterion of the microcracks are known. The microcrack distribution and the fracture criterion of the microcracks are material properties, and they can be obtained by examining or testing the materials. The stress distribution depends on each particular loaded structure and its loading conditions, and in general, it can be obtained by using finite element methods. The failure probability for the structure is obtained from the volume integration over the entire structure. For a structure subjected to a complex load configuration, the failure probabilities for the whole structure have

to be obtained numerically. Several researchers^{2,8} suggested that the Weibull analysis should be used in conjunction with a finite element program. In Chapter 3, the general procedures for computing the failure probability of a structure using SAP4 and methods for modifying SAP4 are developed. The formulas needed in the program of failure analysis are also derived and several sample problems are solved by this modified SAP4 program to ensure its reliability. The results are identical or very close to the results calculated by hand.

Application of the statistical theory to predict the failure of a particular material requires that the mechanism of the fracture process be well understood so that a fracture criterion for the microcracks can be determined and the application of the weakest link principle can be justified. The process of cleavage fracture of AISI 1008 steel is mainly caused by the cracking of grain boundary carbide particles⁹⁻¹¹. The grain boundary carbide particles initially crack when plastic deformation occurs, and then propagate into the surrounding matrix under a certain stress. The cracked carbide particles are considered to be the major microcracks in the steel, and their size distribution can be found using a microscopic technique called "image analysis"⁹⁻¹¹. The relationship between the diameter of the crack of a carbide particle and the maximum remote tensile stress the crack can sustain without propagating is also found¹². As an example of applying the statistical theory to a particular material, in Chapter 4, the statistical theory is used to predict the failure load of a notched body made of AISI 1008 steel

from its microstructure properties. This prediction is different from the previous research⁹⁻¹¹ in that the statistical model for the triaxial stress state is used instead of the previously used statistical model for a uniaxial state.

CHAPTER 2

STATISTICAL MODELS OF FRACTURE OF BRITTLE MATERIALS

2.1 Literature survey

In 1939, Weibull¹ investigated scatter in the tensile strength of brittle materials and proposed a statistical approach using a distribution function, referred to as the Weibull distribution function, to characterize the statistical distribution of tensile strength in these materials. The statistical model was developed for a uniaxial stress state and was extended to multiaxial stress states without proof. Weibull also applied the weakest link principle in his analysis in which the fracture of brittle materials is related to microcracks, failure of one microcrack causes the failure of the entire structure. Therefore, the strength of brittle materials is controlled by the largest possible microcrack in the materials.

In Weibull theory, the failure probability is

$$P_f = 1 - \exp(-B) \quad (2.1)$$

where B is referred to as the risk of rupture and is calculated from

$$B = \int_v n(\sigma) dv \quad (2.2)$$

where v is the volume of the material and $n(\sigma)$ is referred to as the Weibull distribution function; it is a function of stress components.

For the uniaxial stress state, the form of the Weibull distribution function is

$$n(\sigma) = k(\sigma - s_u)^m \quad (2.3)$$

where k , s_u and m are material constants and must be determined by experiment.

For the multiaxial stress state, Weibull gave the form of the distribution function as

$$n_1(\sigma_1, \sigma_2, \sigma_3) = \frac{k_1}{4\pi} \int_{\Omega} (\sigma_n - s_u)^{m_1} d\Omega \quad (2.4)$$

where σ_1, σ_2 and σ_3 are the three principal stresses; σ_n is a normal stress in the solid angle element $d\Omega$; Ω is a solid angle in which σ_n is a tensile stress; k_1, m_1 are material constants obtained from m, k in eq.(2.3) by arguing that the Weibull distribution function for the multiaxial stress state $n_1(\sigma_1, \sigma_2, \sigma_3)$ should give the same result as the distribution function for uniaxial stress state $n(\sigma)$ when two principal stresses are zero $\sigma_2 = \sigma_3 = 0$. Then the equation $n(\sigma_1) = n_1(\sigma_1, 0, 0)$ should hold. This gives the results

$$m_1 = m \quad (2.5)$$

$$k_1 = \frac{2m+1}{2\pi} k \quad (2.6)$$

The form of the Weibull distribution function is empirical in nature, but its three experimentally determined material parameters k, m and s_u give it great flexibility to fit a variety of experimental results under tensile stresses¹³. Usually, these material parameters are determined from test results of bending or simple tension because the formulas for failure probability P_f can be easily derived for these loading configurations. Subsequently, these parameters are used for failure analysis for structures under more complicated load configurations.

In 1972, Batdorf and Crose² proposed a statistical model for the multiaxial stress state. This model is derived from the consideration of fracture of microcracks existing in brittle materials. The fracture criterion used is maximum tensile stress. With this criterion, the formula for failure probability is

$$P_f = 1 - \exp \left[-v \int_0^{\infty} \frac{\Omega(\Sigma, \sigma_{cr})}{4\pi} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} \right] \quad (2.7)$$

where $N(\sigma_{cr})$ is the number of microcracks per unit volume having a critical tensile stress less than or equal to σ_{cr} ; $\Omega(\Sigma, \sigma_{cr})$ is the solid angle in which normal stress exceeds the value of σ_{cr} ; Σ represents the stress state; and v is a small volume element. This model was extended and reformulated for an arbitrary fracture criterion¹⁴ and for surface distributed microcracks⁵.

In 1977, Evans³ conducted failure analysis for the multiaxial stress state by using the critical coplanar strain energy release rate as a fracture criterion for microcracks.

For the penny-shaped crack

$$\frac{\pi E G_c}{4(1-\nu^2)r_0} = \sigma_n^2 + \frac{4\tau_n^2}{(2-\nu)^2} \quad (2.8)$$

where ν is Poisson's ratio, σ_n is the stress normal to the crack, τ_n is the maximum in-plane shear stress, G_c is the critical value of the coplanar strain-energy release rate and r_0 is the radius of the internal penny-shaped microcrack.

In the analysis, the assumption was made that the three-parameter Weibull distribution function is only valid for the equi-triaxial stress state. The strength ratio of equi-triaxial stress state to equi-biaxial stress state and the strength ratio of equi-biaxial stress state to uniaxial stress state were evaluated. Good agreement with experimental data was obtained.

Matsuo¹⁵ derived the general statistical theory for the multiaxial stress state by assuming that Weibull's three-parameter distribution function is only valid for the equi-triaxial tensile stress state which was proposed by Evans. His formulae can be used in the case that the distribution of microcracks in the materials has some preferred directions. Both effects of internal and surface cracks were included in the theory.

Alpa⁴ proposed a general form of a fracture criterion which he claimed to be valid for both tension and compression stress states; however, this criterion has no physical meaning. For any surface in the material, the critical value of the fracture parameter is a function of both the normal stress and the shear stress in this surface. For the normal tensile stress, the fracture parameter is proportional to $(\sigma^2 + k\tau^2)$, where k is a material parameter. For the normal compressive stress, the fracture parameter is proportional to $(\tau - \mu\sigma)$, where μ is a material parameter. Therefore, the fracture criterion is defined by the function $f(\sigma, \tau)$ as

$$f(\sigma, \tau) = \begin{cases} \sigma^2 + k\tau^2 & \text{when } \sigma > 0 \\ \tau - \mu \sigma & \text{when } \sigma < 0 \end{cases} \quad (2.9)$$

For any σ, τ , the fracture occurs on the material surface when $f(\sigma, \tau)$ is equal to or larger than a critical value.

Rufin et al.⁶ conducted fracture tests on two types of specimens made of the same material. One of the specimen types was a four point bending specimen, and the other was a circular disk mounted on a ring support under a uniform pressure. They evaluated the material parameters in the distribution functions of both the Weibull model and the Batdorf and Crose model and used these two models with a finite element program to predict the failure probability of the circular disks. Comparison of the test results and predicted results for the circular disks indicated that the Batdorf and Crose model is significantly more accurate than the Weibull model for this material and stress state.

2.2 The general formula of failure probability

The fracture of brittle materials is caused by the propagation of microscopic cracks. Because the number of microcracks is so large and the size of microcracks is so small, it is impossible to determine the size and the location of each microcrack and to compute the stress distribution around each microcrack. The existence of microcracks in brittle materials can be characterized by some statistical distribution. Suppose the fracture criterion of microcracks under any stress state is given; then the failure

probability can be calculated from a statistical distribution function which characterizes the fracture strength resulting from the existence of microcracks.

There are three basic problems in the statistical theory for predicting the failure probability for brittle materials. If the solutions for these three problems are known, then the formula for predicting the failure probability can be derived mathematically. One of the problems is determination of the fracture criterion for microcracks in materials. Although many fracture criteria such as K_{Ic} , J_{Ic} have been successfully used for macrocracks in linear elastic materials and elastic-plastic materials¹⁶, it is difficult to apply them to the microcracks if the stress state in the materials is complicated or if the microcracks do not have regular shapes. Another problem is that of characterizing the distribution of microcracks in materials. Since it is usually difficult to determine the distribution of microcracks in materials directly, a certain form is assumed for the distribution function with several material parameters left to be determined from the results of failure tests of the same materials. The last problem is determination of the failure criterion for the entire structure. The commonly used failure criterion for brittle materials is the weakest link principle proposed by Weibull. This implies that the fracture of any microcrack in brittle materials will cause failure of a structure. This is probably true for brittle materials since they have low fracture resistance.

In the following discussion, general forms are assumed for both the fracture criterion of the microcracks and the distribution

function. The formula for predicting failure probability is derived from these general forms and the weakest link principle. This formula is equivalent to the formula given by Batdorf and Crose if microcracks are randomly distributed in the materials, and it is equivalent to the formula derived by Matsou for the Evans model if the three-parameter Weibull distribution function is used for equitriaxial stress state.

Consider a unit volume of material in which stresses are considered constants. In general, the fracture criterion of a crack is a function of both the remote normal stress σ_n and the remote shear stress τ_n , so that

$$K = K(\sigma_n, \tau_n) \quad (2.10)$$

where K represents the fracture criterion. Suppose K_{cr} is the critical value of K ; then the microcrack will fail if $K > K_{cr}$. Let $\psi(\theta, \phi, K_{cr})$ be the distribution function, for which, $[\frac{1}{4\pi}\psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi)]$ is the probability of occurrence of one microcrack which has the normal direction within the solid angle $(\sin\theta d\theta d\phi)$ and has the critical value of the fracture criteria parameter larger than or equal to K_{cr} . Therefore, failure probability of the surface which has a normal direction in the solid angle element $d\Omega = \sin\theta d\theta d\phi$ is

$$P_f(d\Omega) = \frac{1}{4\pi}\psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi) \quad (2.11)$$

The survival probability of this element is

$$P_s = 1 - P_f(d\Omega) = 1 - \frac{1}{4\pi}\psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi) \quad (2.12)$$

The survival probability of a unit volume is the multiplication of the survival probabilities of the small elements in all the

directions. Suppose that Ω_0 represents the entire solid angle from the center of a unit sphere, that is, $\Omega_0 = 4\pi$. Then,

$$P_s = \prod_{d\Omega \in \Omega_0} [1 - P_f(d\Omega)] \quad (2.13)$$

From eq.(2.12), one has

$$\prod_{d\Omega \in \Omega_0} [1 - P_f(d\Omega)] = \prod_{\substack{0 < \theta < \pi \\ 0 < \phi < 2\pi}} \left[1 - \frac{1}{4\pi} \psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi) \right] \quad (2.14)$$

From Taylor expansion of function $\exp(-x)$ when x is small,

$$\exp(-x) = 1 - x + o(x) \quad (2.15)$$

Letting x be $\frac{1}{4\pi} \psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi)$ in eq.(2.15) and noting that $(\sin\theta d\theta d\phi)$ is small, then

$$1 - \frac{1}{4\pi} \psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi) = \exp\left[-\frac{1}{4\pi} \psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi)\right] \quad (2.16)$$

Substituting eq.(2.16) into eq.(2.14), and then into eq.(2.13), one obtains

$$P_s = \prod_{\substack{0 < \theta < \pi \\ 0 < \phi < 2\pi}} \exp\left[-\frac{1}{4\pi} \psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi)\right] \quad (2.17)$$

Changing the multiplication in eq.(2.17) into the summation of the exponents, then one has

$$P_s = \exp\left[\sum_{\substack{0 < \theta < \pi \\ 0 < \phi < 2\pi}} -\frac{1}{4\pi} \psi(\theta, \phi, K_{cr}) (\sin\theta d\theta d\phi)\right] \quad (2.18)$$

In eq.(2.18), supposing $d\theta$ and $d\phi$ are infinitesimal, the summation sign can be changed into integration sign, one has

$$P_s = \exp\left[-\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \psi(\theta, \varphi, K_{cr}) (\sin\theta d\theta d\varphi) \right] \quad (2.19)$$

The right hand side of eq.(2.19) is the survival probability for one crack in the unit volume of materials. If there are N_0 microcracks per unit volume of the material, then from the weakest link principle, the total survival probability is the product of the survival probability of each microcrack. One has

$$P_s = \exp\left[-\frac{N_0}{4\pi} \int_0^\pi \int_0^{2\pi} \psi(\theta, \varphi, K_{cr}) (\sin\theta d\theta d\varphi) \right] \quad (2.20)$$

Using the same argument above, the survival probability for whole structure is

$$P_s = \exp\left\{ -\frac{N_0}{4\pi} \int_V \left[\int_0^\pi \int_0^{2\pi} \psi(\theta, \varphi, K_{cr}) (\sin\theta d\theta d\varphi) \right] dv \right\} \quad (2.21)$$

Finally, from eq.(2.21), the failure probability for the whole structure becomes

$$P_f = 1 - \exp\left\{ -\frac{N_0}{4\pi} \int_V \left[\int_0^\pi \int_0^{2\pi} \psi(\theta, \varphi, K_{cr}) (\sin\theta d\theta d\varphi) \right] dv \right\} \quad (2.22)$$

This is the general formula for predicting the failure probability in the structure in which the stresses are known. In the formula, K_{cr} is a fracture criterion for microcracks with the normal direction specified by θ, φ in the spherical coordinate system under a given stress state. In general, K_{cr} is the function of normal stress, σ_n , and shear stress, τ_n , in the direction specified by spherical coordinates $\theta,$

φ . σ_n and τ_n can be calculated from the general stress state as follows:

$$\sigma_n = n_i \sigma_{ij} n_j \quad (2.23)$$

$$\tau_n = (n_i \sigma_{ij} n_k \sigma_{kj} - \sigma_n^2)^{1/2} \quad (2.24)$$

where σ_{ij} , $i, j = 1, 2, 3$ are nine stress components, n_i , $i=1, 2, 3$ are the three unit vector components specified by θ, φ in the spherical coordinate system. These can be written

$$\begin{cases} n_1 = \sin\theta \cos\varphi \\ n_2 = \sin\theta \sin\varphi \\ n_3 = \cos\theta \end{cases} \quad (2.25)$$

The specific fracture criterion which is used for the microcracks will determine the actual form of K_{cr} . For example, if the maximum tension stress is used as a fracture criterion, then $K_{cr} = \sigma_n$; if the strain energy release rate is used as a fracture criterion, then $K_{cr} = G_c$ and from eq.(2.8)

$$G_c = \frac{4(1-\nu^2)a}{\pi E_c} \left(\sigma_n^2 + \frac{4\tau_n^2}{(2-\nu)^2} \right) \quad (2.26)$$

Except for the hydrostatic stress state in which σ_n is a constant and τ_n is zero, K_{cr} will change with direction because σ_n and τ_n are functions of direction.

The distribution function $\psi(\theta, \varphi, K_{cr})$ reflects the statistical distribution of size and direction of microcracks in the materials. Thus, it is a function only of material properties and should be independent of stress state in the materials. If the microcracks are

randomly distributed in the materials, then the distribution function ψ is not an explicit function of θ and ϕ , but for a specific stress state, since K_{cr} depends on θ and ϕ generally, ψ still changes with direction.

There are two methods that can be used to determine the distribution function. One is to measure microscopically the distributions of microcracks in the materials⁸⁻¹⁰. This method is usually applied when the microcracks have a regular shape and the critical value of the fracture criterion parameter is known for this shape of microcrack. This method allows a direct determination of the distribution function from the microscopic properties of the materials. The distribution function determined through this method should be valid for all stress states. The other method is to do fracture tests under the same loading configuration for the specimens which have identical geometry and are made of the same material⁶. Many specimens must be loaded to failure in order to get the curve of failure probability versus load. By assuming a functional form for the distribution function with some constants as material parameters, the failure probability can be predicted from the statistical model. The material constants should be chosen such that the predicted curve of failure probability versus load best matches the experimental results. Once the distribution function is determined, it can be used in the statistical model to predict failure probability for another structure under a different loading configuration. The commonly used functional forms for the

distribution function are the two-parameter Weibull distribution function which has the form

$$\psi = \left(\frac{K_{cr}}{K_0} \right)^m \quad (2.27)$$

and the three-parameter Weibull distribution function which has the form

$$\psi = \left(\frac{K_{cr} - K_u}{K_0} \right)^m \quad (2.28)$$

where K_{cr} , K_0 and m are materials parameters. ψ can also be assumed to be a polynomial with the constants chosen such that predicted results fits the results of failure tests.

The difference between the two-parameter Weibull distribution function and the three-parameter Weibull distribution function is that for the first one, the failure probability is always greater than zero as long as there is a principal tensile stress in the material. But for the second one, the failure probability will be zero if the maximum tensile stress in the material is less than K_u . The two-parameter Weibull distribution function implies that no matter how low the tensile stress in the materials, there is always a possibility that such a large crack exists that it will fracture under this tensile stress. The three-parameter Weibull distribution function implies that cracks larger than a certain size do not exist, so that the structure will never fail if the maximum tensile stress is less than K_u . Therefore, for the application in which cracks larger than a certain level do not exist and can be excluded from

consideration, the three-parameter Weibull distribution function should be used.

2.3 Discussion

In the previous sections, several statistical models for brittle materials have been reviewed and a general formula for predicting the failure probability has been derived for general forms of the distribution function, ψ , and arbitrary fracture criterion, K_{cr} , using the weakest link principle. In this section, relationships between the statistical models will be discussed.

The Batdorf and Crose model (see eq.(2.7)) was derived physically from a fracture criterion based on normal tensile stress, a distribution function $N(\sigma_{cr})$ and the weakest link principle for isotropic materials in which microcracks are randomly distributed in all directions. The distribution function $N(\sigma_{cr})$ is defined as the density of microcracks having a critical tensile stress less than or equal to σ_{cr} . The general theory developed in section 2.2 can also be applied on the same physical basis. Thus, it should give the same formula for calculating the failure probability if it is applied to isotropic materials and if the normal tensile stress is used as a fracture criterion.

In the theory developed in the previous section, if the fracture criterion of maximum tensile stress is used and material is isotropic, then $K_{cr} = \sigma_{cr}$ and the distribution function ψ is only a function of σ_{cr} , that is, $\psi = \psi(\sigma_{cr})$. From both definitions of distribution functions N and ψ , one has

$$N_0 \psi(\sigma_{cr}) = N(\sigma_{cr}) \quad (2.29)$$

Then, one has

$$\frac{N_0}{4\pi} \iint_{\substack{0 < \theta < \pi \\ 0 < \varphi < 2\pi}} \psi(\sigma_{cr}) (\sin\theta d\theta d\varphi) = \frac{1}{4\pi} \iint_{\substack{0 < \theta < \pi \\ 0 < \varphi < 2\pi}} N(\sigma_{cr}) (\sin\theta d\theta d\varphi) \quad (2.30)$$

Since $N(\sigma_{cr}) = \int_0^{\sigma_{cr}} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr}$, eq.(2.30) can be changed into

$$\frac{N_0}{4\pi} \iint_{\substack{0 < \theta < \pi \\ 0 < \varphi < 2\pi}} \psi(\sigma_{cr}) (\sin\theta d\theta d\varphi) = \frac{1}{4\pi} \iint_{\substack{0 < \theta < \pi \\ 0 < \varphi < 2\pi}} \int_0^{\sigma_{cr}} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} (\sin\theta d\theta d\varphi) \quad (2.31)$$

One should notice the different meaning of σ_{cr} at the different positions in the right side of eq.(2.31). The σ_{cr} as an upper limit of the integral is a function of θ , φ and stress state Σ , that is,

$$\sigma_{cr} = \sigma_{cr}(\theta, \varphi, \Sigma) \quad (2.32)$$

The σ_{cr} inside the integral sign is only an integration variable and does not change with θ and φ .

The order of the integration in the right side of eq.(2.31) can be exchanged. After the change of the order of integration, the limit of integral for θ and φ is a function of σ_{cr} and stress state Σ and is specified by eq.(2.32). Suppose Ω represents the solid angle in which the integration for θ and φ is evaluated, and $d\Omega = \sin\theta d\theta d\varphi$. Then eq.(2.31) becomes

$$\frac{N_0}{4\pi} \int_{0 < \theta < \pi} \int_{0 < \varphi < 2\pi} \psi(\sigma_{cr}) (\sin\theta d\theta d\varphi) = \frac{1}{4\pi} \int_0^\infty \int_{\Omega(\sigma_{cr}, \Sigma)} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\Omega d\sigma_{cr} \quad (2.33)$$

In eq.(2.33), since σ_{cr} in the expression $\frac{dN(\sigma_{cr})}{d\sigma_{cr}}$ is only an integration variable and is independent of θ and φ , one has

$$\int_{\Omega(\sigma_{cr}, \Sigma)} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\Omega d\sigma_{cr} = \Omega(\sigma_{cr}, \Sigma) \frac{dN(\sigma_{cr})}{d\sigma_{cr}} \quad (2.34)$$

Substituting eq.(2.34) into eq.(2.33), one obtains

$$\frac{N_0}{4\pi} \int_{0 < \theta < \pi} \int_{0 < \varphi < 2\pi} \psi(\sigma_{cr}) (\sin\theta d\theta d\varphi) = \frac{1}{4\pi} \int_0^\infty \Omega(\sigma_{cr}, \Sigma) \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} \quad (2.35)$$

By substituting eq.(2.35) into eq.(2.22) and assuming that the volume v is small such that stresses are constant in v , eq.(2.7) is obtained. Therefore, eq.(2.22) is equivalent to eq.(2.7) if it is applied to isotropic materials and if the maximum tensile stress fracture criterion is used.

The statistical model developed by Evans and Matsuo can be used for unisotropic materials in which microcracks are not randomly distributed and can have some preferred directions, but the theory is not deduced physically from fracture of individual microcracks. The formula from their model for predicting the failure probability is identical to eq.(2.21) if the three-parameter

Weibull distribution function is used for the distribution function ψ in eq.(2.21). Thus, the theory developed in section 2.2 is equivalent to the Evans model if a three-parameter Weibull distribution function is used. This also proves that the Batdorf and Crose model and the Evans model are actually equivalent for isotropic materials if same fracture criterion and same form of distribution function are used. In general, for isotropic materials, the models should give the same results when identical forms of the distribution function are used and the constants are evaluated on the same material. Failure to recognize this fact has lead some investigators to try to make unnecessary and useless comparisons between two different models⁶.

CHAPTER 3

MODIFICATION OF A STRUCTURAL ANALYSIS PROGRAM SAP4 BY WEIBULL'S THEORY

3.1 Introduction

In chapter 2, the statistical theory for the fracture of brittle materials is investigated and discussed. It can be seen that the failure probability is dependent upon the stress distribution in a structure and the microcrack distribution function, which is a material property and has to be determined experimentally. For obtaining the failure probability of the whole structure, volume integration is needed.

The purpose of this chapter is to illustrate the modification made in a structural analysis program SAP4 by the statistical theory using a two-parameter Weibull distribution function and normal tensile stress as the fracture criterion. The SAP4 program is modified so that it can do failure analysis from Weibull's theory. The formulations are deduced for isotropic materials and for the two-parameter Weibull distribution function under different stress states. The sample analyses are also performed by the modified SAP4 to verify the program.

Finite element methods have been used for stress analysis for a long time. The stresses in a structure under any loading condition can be obtained accurately if the finite element mesh is fine enough.

In a finite element analysis, a structure is subdivided into many small "elements". The behavior of each element is represented by an element matrix which relates the forces at nodal points of an element to the nodal displacements. The displacements at nodal points are obtained by solving the global equilibrium equation. The stress distribution in each element can be obtained from the nodal displacement through the stress-displacement transformation matrices. The stress distribution in an element obtained from a finite element analysis can be used in the statistical theory for predicting the failure probability of the element from the volume integration over the element. Then the failure probability of the entire structure can be obtained from the weakest link principle, which assumes that failure of any element in the structure will causes the failure of the whole structure.

The risk of rupture, B , originally defined by Weibull (see eq.(2.2)), is used here for triaxial stress state as

$$B = \frac{N_0}{4\pi} \int_v \left[\int_{0 < \theta < \pi} \int_{0 < \varphi < 2\pi} \psi(\theta, \varphi, K_{cr}) (\sin\theta d\theta d\varphi) \right] dv \quad (3.1)$$

B is calculated for each element. The risk of rupture for the whole structure is obtained by adding the risks of rupture of all the elements together. Then, the failure probability is obtained from the relation

$$P_f = 1 - \exp(-B) \quad (3.2)$$

Eq.(3.1) can be simplified for a particular form of distribution function and for a particular fracture criterion used in the statistical model.

For the types of finite element in which the stresses are constant, such as the truss element and the triangular element, the volume integration is not needed. For some types of analysis in which the stress state in the elements is uniaxial or biaxial, such as plane stress analysis, the formulation for calculating the risk of rupture, B , can also be simplified.

For the finite element in which stresses are not constant, Gaussian quadrature must be used to calculate the volume integration. For the biaxial and triaxial stress states, the integration of the distribution function ψ over the surface of the unit sphere must be numerically evaluated by Gaussian quadrature for the variables θ and ϕ in a spherical coordinate system.

A truss element, a triangular element and a quadrilateral element are modified and several subroutines are developed to calculate the risk of rupture and the failure probability for each element and for the whole structure. The triangular element and the quadrilateral element are capable of doing failure analyses for plane stress, plane strain and for axisymmetric cases.

3.2 Formulation of statistical model in the program

In SAP4, the truss element and the two-dimensional finite elements are modified to be capable of doing statistical failure analysis; only a two-parameter Weibull distribution function is

used. The formulas for calculating the failure probability can be simplified due to the simple nature of the two-parameter Weibull distribution function and the specific stress state in the elements. In the truss element, a uniaxial stress state exists; thus, volume integration is not needed. In the triangular element, stresses in each element are constant, a biaxial stress state exists for the plane stress analysis and a triaxial stress state exists for plane strain axisymmetric analysis. In the quadrilateral element, stresses are not constant, and stress state is biaxial for the plane stress problem and is triaxial for the plane strain and axisymmetric problems.

When the two parameter Weibull's distribution function is used in the statistical theory developed in Chapter 2, the distribution function ψ has the form

$$\psi = k \sigma_n^m / N_0 \quad (3.3)$$

where k , m are two material parameters, and σ_n is the normal stress in the n direction specified by $\sigma_1, \sigma_2, \sigma_3$ and θ, ϕ (see Figure 3.1).

$$\sigma_n = (\sigma_1 \cos^2 \phi + \sigma_2 \sin^2 \phi) \sin^2 \theta + \sigma_3 \cos^2 \theta \quad (3.4)$$

From eq.(3.3) and eq.(3.1), one has

$$B = \frac{k}{4\pi} \int_V \left[\int_{0 < \theta < \pi} \int_{0 < \phi < 2\pi} \sigma_n^m (\sin \theta d\theta d\phi) \right] dv \quad (3.5)$$

For the uniaxial stress state, suppose σ_2, σ_3 are zero. Then eq.(3.5) can be further simplified by substituting eq.(3.4) into eq.(3.5) to find

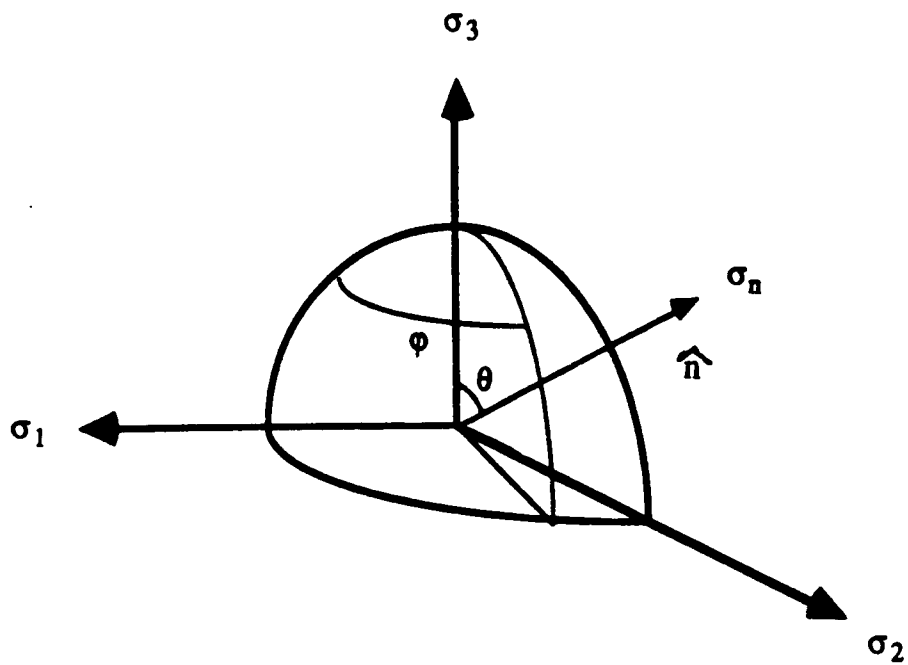


Figure 3.1 Coordinates for Calculation of Normal Stress, σ_n , in the Direction, $\hat{n}$, from the Three Principal Stresses σ_1 , σ_2 and σ_3

$$B = \frac{k}{2m+1} \int_v \sigma_1^m dv \quad (3.6)$$

For each truss element, the stress is constant. From eq.(3.5), the risk of rupture is

$$B = \frac{k}{2m+1} \sigma_1^m v \quad (3.7)$$

where v is the volume of element.

For the biaxial stress state, suppose σ_3 is zero. By substituting eq.(3.4) into eq.(3.5), B is obtained as

$$B = 2k \int_v \left[\int_0^\alpha (\sigma_1 \cos^2 \varphi + \sigma_2 \sin^2 \varphi)^m d\varphi \int_0^{\pi/2} \sin^{2m+1} \theta d\theta \right] dv \quad (3.8)$$

$$\text{where } \alpha = \begin{cases} \tan^{-1} \sqrt{-\frac{\sigma_1}{\sigma_2}} & \text{if } \sigma_1 < 0 \\ \frac{\pi}{2} & \text{if } \sigma_2 \geq 0 \end{cases} \quad (3.9)$$

For calculation of the distribution function in the triaxial stress state, the integration on the unit sphere only needs to be evaluated on the domain in which $\sigma_n > 0$. The triaxial stress state can be classified into three cases: (a) all three principal stresses are not less than zero, (b) only one of the three principal stresses is less than zero, (c) only one of the three principal stresses is greater than zero, (d) all the three principal stresses are less than zero. The formulas for these cases are developed following.

(a) Suppose $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ and $\sigma_3 \geq 0$, then in this case, B can be obtained as

$$B = \frac{2k}{\pi} \int_v \left[\int_0^{\pi/2} \int_0^{\pi/2} \sigma_n^m \sin\theta d\theta d\varphi \right] dv \quad (3.10)$$

where σ_n is calculated from eq.(3.4).

(b) In this case, suppose $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ and $\sigma_3 < 0$, note that

$$\sigma_n \geq 0 \text{ for } 0 \leq \varphi \leq \pi/2, 0 \leq \theta \leq \alpha \quad (3.11)$$

$$\sigma_n \leq 0 \text{ for } 0 \leq \varphi \leq \pi/2, \alpha \leq \theta \leq \pi/2$$

where for a given φ , α is

$$\alpha = \tan^{-1} \sqrt{\frac{\sigma_1 \cos^2 \varphi + \sigma_2 \sin^2 \varphi}{-\sigma_3}} \quad (3.12)$$

B is obtained by

$$B = \frac{2k}{\pi} \int_v \left[\int_0^{\pi/2} \int_0^{\alpha} \sigma_n^m \sin\theta d\theta d\varphi \right] dv \quad (3.13)$$

(c) In this case, suppose $\sigma_1 \leq 0$, $\sigma_2 \leq 0$ and $\sigma_3 > 0$, note that

$$\sigma_n \geq 0 \text{ for } 0 \leq \varphi \leq \pi/2, \alpha \leq \theta \leq \pi/2 \quad (3.14)$$

$$\sigma_n \leq 0 \text{ for } 0 \leq \varphi \leq \pi/2, 0 \leq \theta \leq \alpha$$

where α is also given by eq.(3.12). B can be obtained by

$$B = \frac{2k}{\pi} \int_v \left[\int_{\alpha}^{\pi/2} \int_0^{\pi/2} \sigma_n^m \sin\theta d\theta d\varphi \right] dv \quad (3.15)$$

(d) In this case, since $\sigma_1 \leq 0$, $\sigma_2 \leq 0$ and $\sigma_3 \leq 0$, $\sigma_n \leq 0$. Because the normal tensile stress is used as a fracture criterion, B is zero.

3.3 Description of the program

In order to modify and to extend SAP4 program, it is necessary to understand the organization of the program and the basic function of some subroutines needed for execution. The basic steps for static analysis in the SAP4 program are outlined in this chapter. The structural program SAP4 was developed by Professors K. Bathe, E. L. Wilson and F. E. Peterson in 1974. It includes eight different types of structural element and is capable of doing static and dynamic analyses for these elements. Although SAP4 can analyze very large problems, it is also very flexible and efficient for solving small problems. The subroutines which are actually needed in the execution for a special application can easily be assembled and linked together to form a smaller program. Thus, the SAP4 can be easily used on a personal computer.

SAP4 was also designed to be modified and extended by the users. Additional options may easily be added. The total program is divided into many subroutines which perform different tasks. The main program carries out finite element computations by calling these subroutines. If some part of the program needs to be modified and extended, only the subroutines corresponding to this part need to be changed.

The stress analysis in the structural finite element program SAP4 is accomplished through the following steps for static analysis.

1. The nodal point input data is read and generated.
2. The material properties and element properties are read.

3. The element stiffness matrices are calculated.

4. The stress-displacement transformation matrices are generated.

5. The structural (global) stiffness matrix and load vector are formed.

6. The equilibrium equation is solved.

7. The stresses in elements are calculated and printed.

Each step is accomplished in one or more subroutines. Step 1 is accomplished in subroutine INPUT; step 2 to step 4 are accomplished in subroutines TRUSS, RUSS and CALBAN for truss element, and in subroutine PLANE, ELAW, QUAD, PLNAX, FORMB and CALBAN for the two dimensional elements; step 5 is accomplished in subroutine ADDSTF; step 7 is accomplished in subroutine SOLEQ, SESOL, STRESS and PRINTD; step 6 is accomplished in subroutine TRUSS and STRSC for truss element, and in subroutine PLANE and STRSC for two dimensional elements.

The failure probability and risk of rupture for a structure are calculated from the failure probability and risk of rupture of the elements. The failure probability and risk of rupture of an element are calculated from the material properties, element properties and stresses in the element. If stresses in the element are not constant, Gaussian numerical integration over the element is used in the calculations. If the stresses in an element are constant, the risk of rupture in the element is constant, and risk of rupture of an element is obtained by multiplying the volume of the element with the risk of rupture at any point in the element.

The calculations of failure probability and risk of rupture of a structure in modified SAP4 are accomplished through the following steps.

1. Write material properties and element properties in a data file.
2. Write stresses in elements in a data file.
3. Read the material properties and the element properties written in the data files.
4. Calculate the risk of rupture in an element from stresses and Weibull's material parameters.
5. Calculate the failure probability and risk of rupture of elements from the risk of rupture in the element and the stress distribution. If the stress distribution in the element is not uniform, Gaussian numerical integration over the element is used.
6. Calculate the failure probability and risk of rupture of the structure by applying the weakest link principle.

For the truss element, the length and the area of the cross section of each element are written on data file SMAL13.DAT from subroutine RUSS. The stress of each element is written on the data file SMAL10.DAT from subroutine TRUSS. A new subroutine TF is developed to read the stress, the length and the area of cross section of each element, and then to calculate the risk of rupture and failure probability for each element and for whole structure. Notice that the stress in a truss element is constant, the risk of rupture of an element is equal to the multiplication of the volume of the element and the risk of rupture at a point.

For the triangular element, the area and thickness of each element are written on data file SMAL13.DAT from subroutine PLNAX, and stresses for each element are written on data file SMAL10.DAT. The new subroutines VF2, GAUSS, D3WB are developed to read the area and the thickness of elements and to calculate the risk of rupture and failure probability for each element and for whole structure. The stresses in a triangular element are constant, so risk of rupture of an element is equal to the multiplication of the volume of the element and the risk of rupture at any point in the element.

For the quadrilateral element, the situation is more complicated. The stresses in an element are not constant. The risk of rupture of each element is obtained from two dimensional Gaussian integration over the element. In this program, a 5 by 5 points Gaussian integration is used in each element. Thus, the stresses at these 25 integration points have to be calculated. In the program, the material properties are written in the data file SMAL13.DAT from subroutine PLNAX. The stress-displacement transformation matrices at these 25 points are calculated in subroutine QUAD and are stored in data file SMAL1.DAT in subroutine PLNAX. The stresses at these 25 points are calculated in subroutine PLANE and are written on data file SMAL10.DAT. The values of the Jacobian corresponding to the transformation from a quadrilateral element to the master element at these 25 points are written on data file SMAL10.DAT in subroutine QUAD. Several subroutines VF2, GAUSS, D3WB are developed to read the data

written in data files SMAL10.DAT and SMAL13.DAT and to calculate the risk of rupture and failure probability for each element and for the whole structure.

For the details of the modifications, please see the following: Figure 3.2: The Flowchart of Subroutines in Modified SAP4, Appendix A: Illustration of the Subroutines in Modified SAP4, Appendix B: List of Changes in the Original SAP4 Code and Appendix C: Subroutines Developed in Modified SAP4 Program.

The formats of input data files for the modified SAP4 program are slightly different from the original SAP4 input data files. A few changes are needed, and they are listed as follows:

1. Change value of NDYN to 5 at column 20 - 25 of "master control card" (see reference 17, page II.1).

2. One line should be added for each type of material used in the structure after the end of an original input data file:

Column 1 - 10: material constant k

Column 11 - 20: material constant m (real number).

3. For each quadrilateral element, change the stress evaluation option "n" to 120 at columns 51 - 55 of " element card " (see reference 17, page IV.4.4).

3.4 Sample analyses

Three simple problems are solved by the modified SAP4 program in order to verify the efficiency and reliability of the program. Because the stress distributions in these three problems are very simple, the failure probabilities are also derived in closed

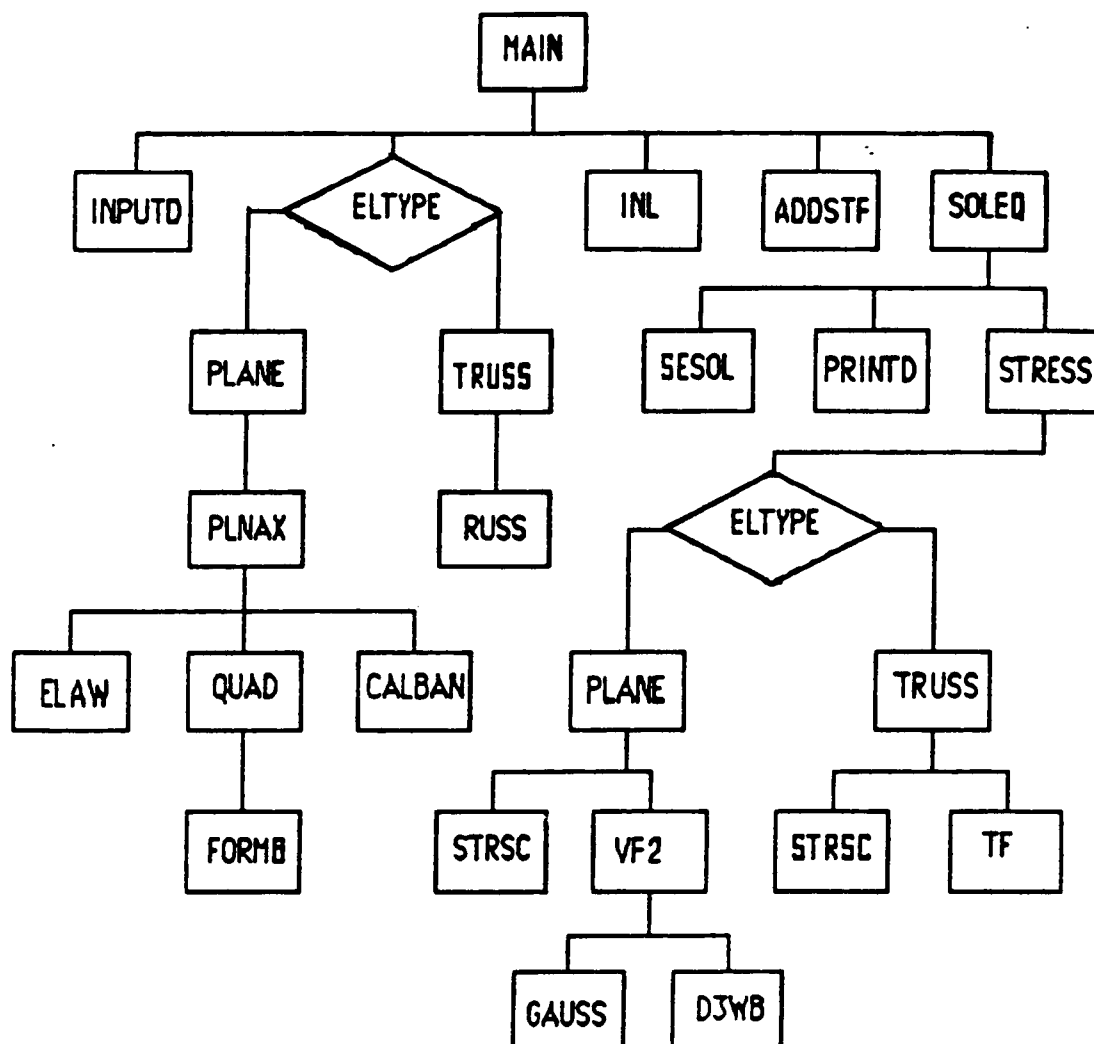


Figure 3.2 Flowchart of Subroutines in Modified SAP4

The subroutines which are modified: PLANE, PLANX, QUAD, TRUSS, RUSS and main program; The subroutines which are developed: D3WB, GAUSS, TF and VT2.

form and their values are calculated by hand. In these sample analyses, the failure probabilities calculated by modified SAP4 program are identical or very close to those calculated from analytical expressions. The good agreement obtained illustrates that the program works properly and can give accurate results.

Following are the three sample problems and their analytical expressions for the failure probabilities. For the details of the input data for these problems, please see Appendix D: Input Data Files for the Sample Problems.

1. Uniaxial tension. For the geometry and loading conditions of the problem, please see Figure 3.3. The expression of failure probability P_f is

$$P_f = 1 - \exp(-k\sigma^m v) \quad (3.14)$$

where v is the volume of the entire loaded body.

For $v = 40 \times 20 \times 1$, $k = 3$, $m = 7$ and $s = 0.4$, from eq.(3.14), P_f is 0.9803987.

In the program, the problem is solved both by using two quadrilateral elements and by using four triangular elements in the plane stress option. The failure probabilities in both cases are identical to the result from eq.(3.14).

2. Three point bending of a cantilever beam. For the geometry and loading condition of the problem, please see Figure 3.4. The left half of the beam is subjected to pure bending. The expression of the risk of rupture for the pure bending part of the beam is

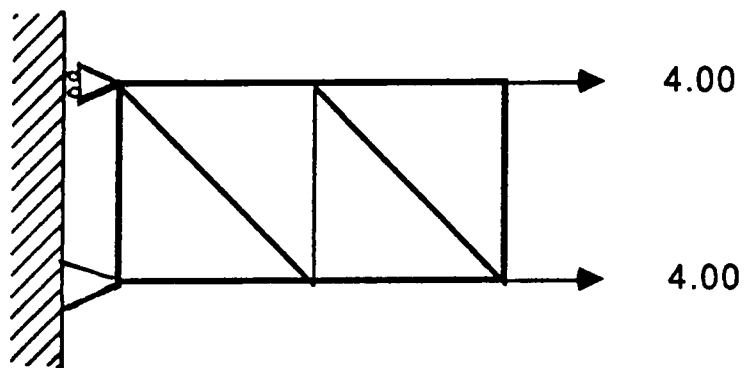
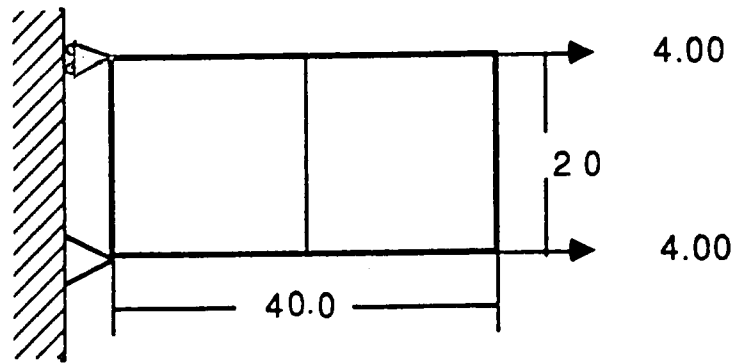


Figure 3.3 Sample Problem 1: Uniaxial Tension

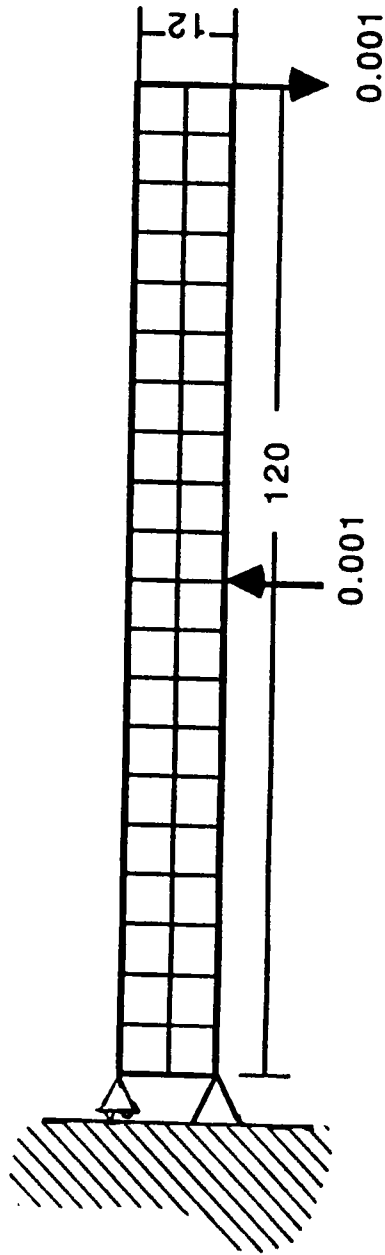


Figure 3.4 Sample Problem: Three Point Bending of a Cantilever Beam

$$\begin{aligned}
B &= -bc \int_0^{d/2} k \left[\frac{\sigma_M y}{d/2} \right]^m dy \\
&= -\frac{k b c d \sigma_M^m}{2(m+1)}
\end{aligned} \tag{3.15}$$

where b is the beam width; d is the beam height; c is the length of a section of the beam under the consideration; σ_M is the maximum tensile stress over the cross section.

For $b = 1$, $c = 6$, $d = 12$, $k = 300$, $\sigma_M = 0.025$ and $m = 2$, from eq.(3.15), B is 2.25.

Forty plane stress quadrilateral elements are used for solving the problem. The the risk of rupture B for the elements in the pure bending part of the beam given by the program are approximately the same to the value from the analytical expression. The differences of B are due to the differences between the stress distributions from finite element computation and from simple beam theory used in eq. (3.15).

3. Axisymmetric tension. For the geometry and loading condition of the problem, please see Figure 3.5. The stress state in the loaded body is an equibiaxial stress state. The analytical expression for B is

$$B = 2kv \int_0^{\pi/2} \int_0^{\pi/2} (\sigma \sin^2 \theta)^m \sin \theta d\theta d\phi \tag{3.16}$$

For $m = 7$, $v = 40^2\pi$, $k = 3$ and $\sigma = 4$, because

$$\int_0^{\pi/2} \cos^{15} \theta d\theta = \frac{14 \times 12 \times \dots \times 2}{15 \times 13 \times \dots \times 3} = 0.3182595 \tag{3.17}$$

B is 0.01439772 from eq.(3.16).

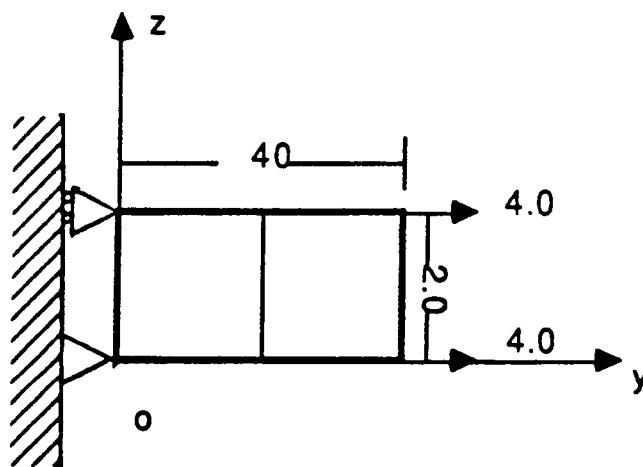
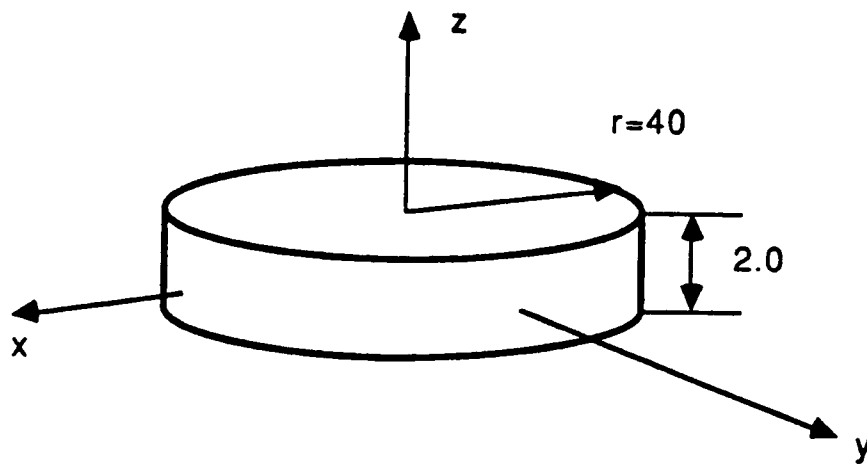


Figure 3.5 Sample Problem: Axisymmetric Tension

Two quadrilateral elements are used in the program for solving this problem, and the value of B computed by the program is identical to the value of B from eq.(3.16).

CHAPTER 4

APPLICATION OF STATISTICAL MODEL FOR PREDICTING THE FAILURE LOAD IN A NOTCHED BODY

4.1 Introduction

The statistical model developed in Chapter 2 can be used for predicting the failure probability for brittle materials if stresses in the materials are known. In this chapter, the theory is used to predict the failure load for a four-point bending specimen of AISI 1008 steel with a rounded notch. The problem was investigated in references 9, 10 and 11, where a quantitative method was developed to calculate the failure probability for a given load P from the microscopic properties of the materials, and the statistical model for a uniaxial stress state was used. In this study, the statistical model for a triaxial stress state is used for the prediction of the failure load. The advantage of using the statistical model for the triaxial stress state is that the percentage of the carbide particles participating in the fracture process f can be automatically determined by the statistical model for triaxial stress state rather than be approximately estimated as is the case when the statistical model for uniaxial stress state is used. Thus, the failure load can be completely determined from the microscopic properties of the steel.

Low strength steel exhibits brittle material properties at low temperature. For the notched specimen under the four point bending, the specimen will break without stable crack growth. This

is called cleavage fracture. The mechanism of cleavage fracture of ferrite steel has been extensively studied by material scientists and the statistical theory are used to predict the failure of the steel¹⁸⁻²³.

The mechanism of the cleavage fracture in the steel is the propagation of the microscopic crack in the materials. In a simple ferrite/carbide steel, the microscopic cracks are initially developed in carbide particles located at the ferrite grain boundary when the plastic deformation occurs in the adjoining ferrite grains. When the stresses in the materials are high enough, the microcracks in the carbide particles will propagate into the adjoining ferrite grains (see Figure 4.1). In the four-point bending specimens of AISI 1008 steel, the propagation of one microcrack of carbide particles into the ferrite grains near the notch tip will cause cleavage fracture of the entire specimen. Thus, the weakest link principle can be applied to predict the material behavior.

The density and the size distribution of the carbide particles in the steel can be determined by a microscopic examination. The local fracture stress perpendicular to the microcrack of a carbide particle is found to be a function of microcrack diameter, d_0 , and other material properties. Therefore, if the stresses in the specimen are known for a given load P , the failure probability can be determined from the distribution of carbide particles in the material.

Since the carbide particles in the steel are considered to become microcracks only when plastic deformation occurs in the

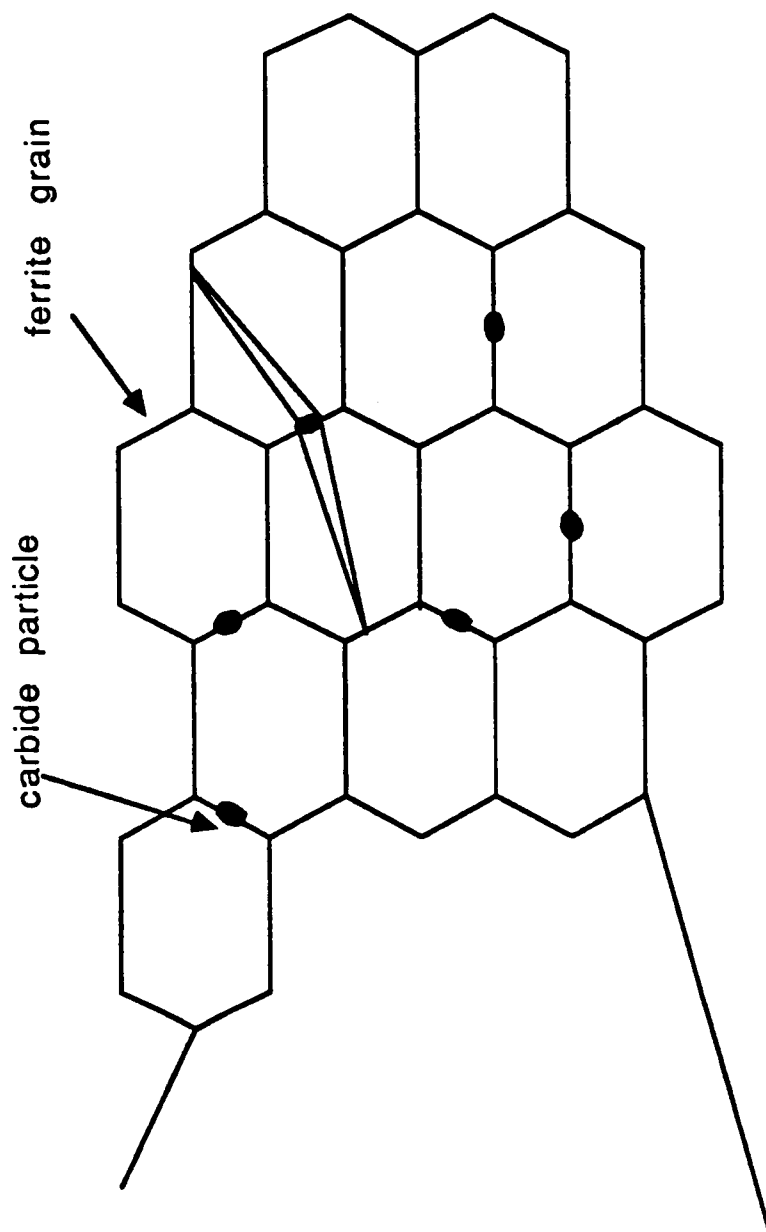


Figure 4.1 Schematic Illustration of Microscopic Structure
and Fracture Mechanism of AISI 1008 Steel

adjoining ferrite grains, only the stresses in the plastic zone need to be known for obtaining the failure probability. In references 9,10 and 11, an approximate method is used to calculate the stresses in the plastic zone at the notch tip for the four point bending specimen. In the method, the plastic zone near the notch tip is determined from plasticity analysis at the notch tip by using the slip line theory of elastic-perfectly plastic materials, and strains near the notch tip are assumed to be related to the external load P in the same empirical function which is obtained through the finite element analysis of the same specimen. Then the stresses in the plastic zone can be obtained through the constitutive equation and the equilibrium equation in the plastic zone.

4.2 Mechanism of the fracture

Smith¹² developed a theoretical model for obtaining the local fracture stress, σ_f , for microcracks which develop from carbide particles. When this model is applied to a penny-shaped microcrack of diameter d_0 , the local fracture tension is found to be

$$\sigma_f = \sqrt{\frac{\pi\gamma E}{(1-\nu^2)d_0}} \quad (4.1)$$

where γ is the effective fracture surface energy of the surrounding materials, E is the Young's modulus, and ν is Possion's ratio. This formula has been used in the statistical model to predict the failure load of the notched body of AISI 1008 steel in references 9, 10 and 11.

4.3 The stress field near the notch tip

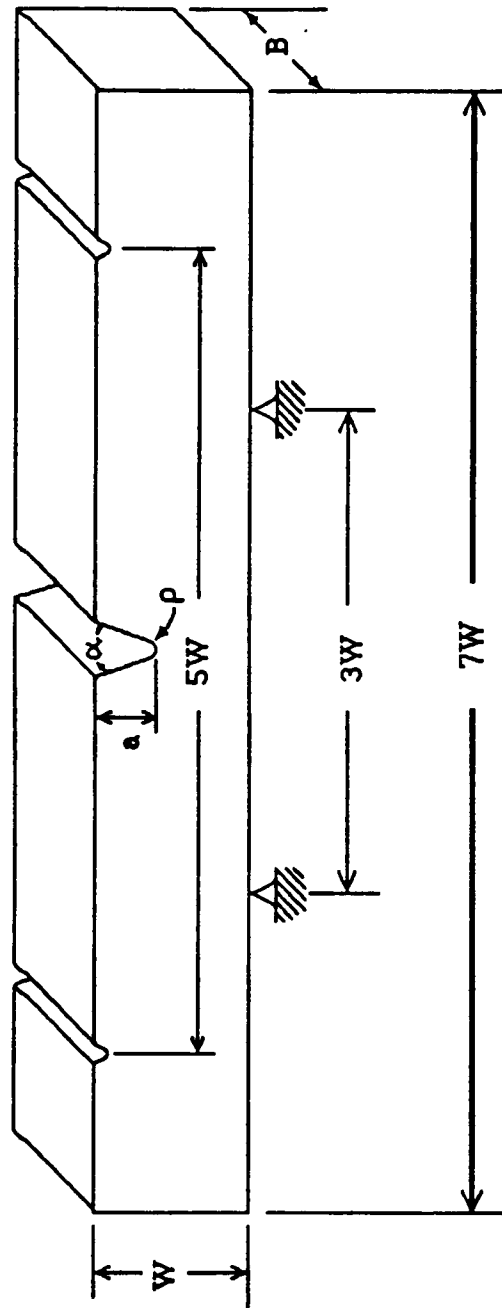
The size and geometry of the specimen is shown in the Figure 4.2(a) and the loading conditions are shown in Figure 4.2(b). The loads P are applied at A, B, and the fixed displacement boundary conditions along vertical direction are applied at C, D. The section of the specimen between C and D is subjected to the pure bending of moment $M = PW$. The notch depth " a " is one-third of the cross-section. The notch is rounded at the tip and has an opening angle of 45° .

As a measure of the applied load, the nominal stress σ_{nom} is defined as:

$$\sigma_{\text{nom}} = \frac{3M}{B(W-a)^2} \quad (4.2)$$

where B is the specimen thickness, " a " is the notch depth and M is the moment in the middle section of the specimen, $M = PW$ for this loading configuration. The nominal stress is related to the bending stress at the notch root by ignoring the stress concentration effect. Thus, the load on the specimen can be described by $\sigma_{\text{nom}} / \sigma_0$, where σ_0 is the yield stress of the material.

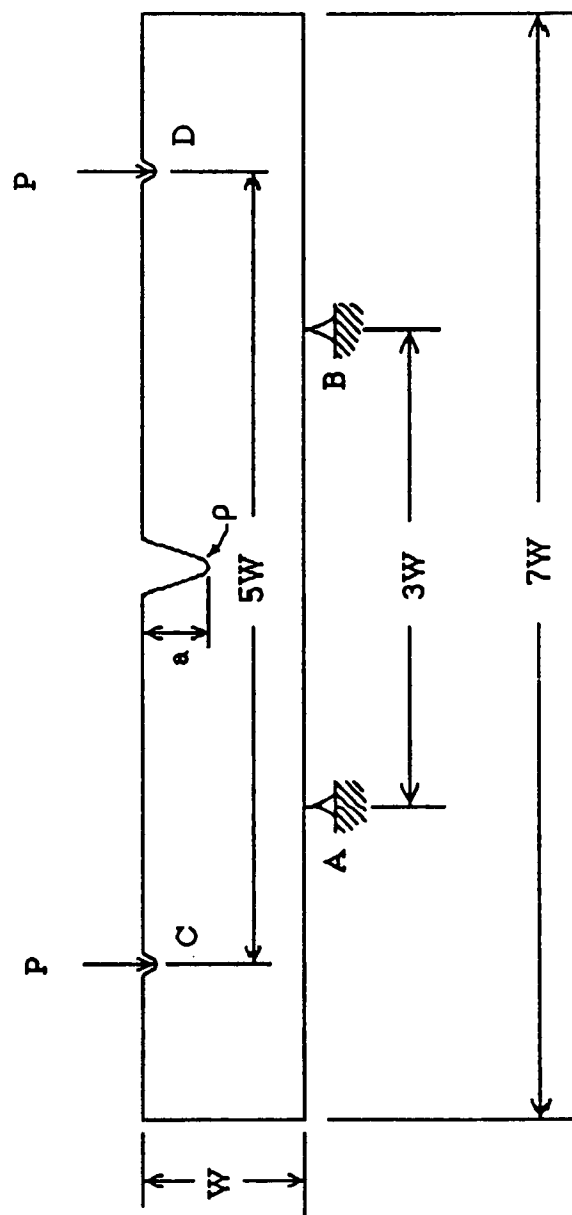
At first, the author tried to use ANSYS (a finite element analysis package) to obtain the stress field in the specimen. Although the final stress field was not obtained due to the extensive plastic deformation near the notch tip and the speed limitation of the computer used, some experience was gained through this attempt. There are two reasons which make the computation of stresses in this specimen very time-consuming.



(a). The Geometry of the Specimen

where $a = 4.23\text{mm}$, $B = 12.7\text{mm}$, $W = 12.7\text{mm}$, $\rho = 0.25\text{mm}$, $\alpha = 45^\circ$

Figure 4.2 The Geometry and Loading Conditions of the Four Point Bending Specimen



(b). The Loading Conditions of the Specimen

Figure 4.2 The Geometry and Loading Conditions of the Four Point Bending Specimen

First, a very fine mesh is needed near the notch tip because the stress gradient is very high there. Thus a large number of elements are needed for this problem. Secondly, the extensive plastic deformation will develop near the notch tip before the applied load reaches the failure load. In order to satisfy the convergence requirement of ANSYS for nonlinear materials (plastic strain increment cannot exceed a certain percent of the elastic strain), many load steps are needed to reach the failure load.

The method from references 9, 10 and 11 is used here to obtain the stress field near the notch tip of the specimen. The method is outlined here.

The slip line field near the rounded notch was obtained in references 9, 10 and 11. The two groups of slip lines are perpendicular to each other and are expressed in the characteristic α , β coordinate system as shown in Figure 4.3. α , β can be related to the cartesian x , y coordinate system as

$$x = \rho \cos (\alpha + \beta) \exp \left(\frac{\pi}{2} + \alpha - \beta \right) \quad (4.3)$$

$$y = \rho \sin (\alpha + \beta) \exp \left(\frac{\pi}{2} + \alpha - \beta \right) \quad (4.4)$$

and related to the polar r , θ coordinate system as

$$r = \rho \exp \left(\frac{\pi}{2} + \alpha - \beta \right) \quad (4.5)$$

$$\theta = \alpha + \beta \quad (4.6)$$

where ρ is the radius of the notch tip. The upper and lower boundaries of the plastic region are $\alpha = -\zeta$ and $\beta = \zeta$ respectively where ζ ranges from $\pi/4$ to $\pi/16$ depending upon the applied load.

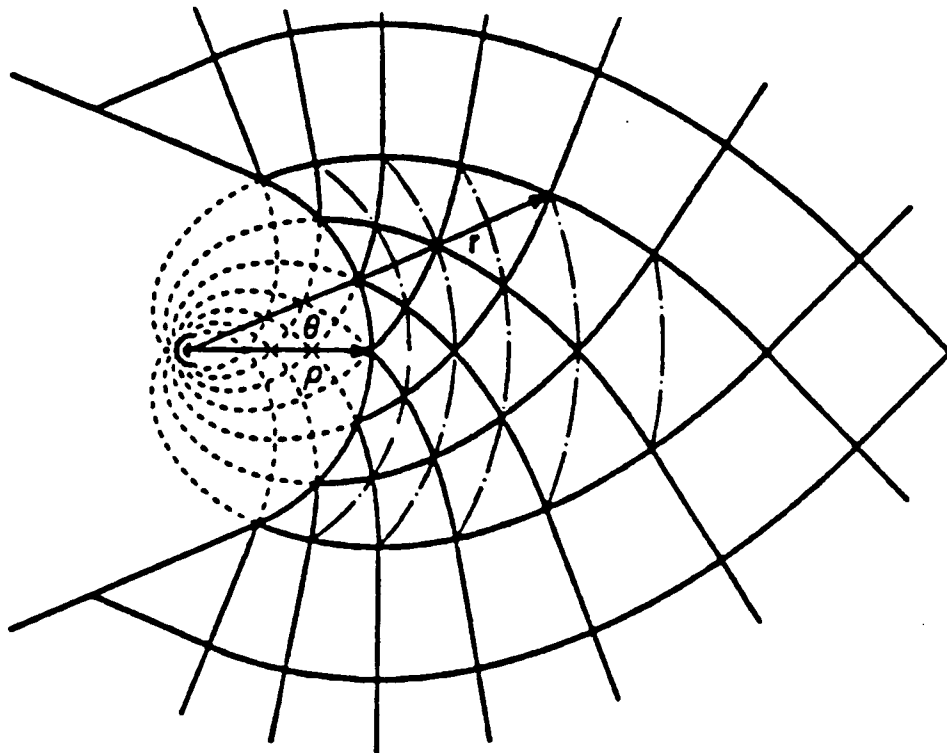


Figure 4.3 Slip Line Field around Notch Tip
(reproduced from reference 11)

The notch has a root radius of ρ and
an opening angle of 45° .

From eq.(4.5), the length of the plastic zone is $r_p - \rho$, where r_p is given by

$$r_p = \rho \exp\left(\frac{\pi}{2} - 2\zeta\right) \quad (4.7)$$

For any $\rho \leq r < r_p$, from eq.(4.5) and eq.(4.6), the span of the plastic zone is

$$\begin{aligned} \theta_s &= 2(\alpha + \beta) \\ &= -2(\alpha - \beta) - 4\zeta \\ &= \pi - 2\ln\frac{r}{\rho} - 4\zeta \end{aligned} \quad (4.8)$$

The three principal stresses within the plastic zone are obtained as

$$\sigma_1 = \frac{2}{\sqrt{3}} \left(\bar{\sigma} + \int_{\rho}^r \frac{\bar{\sigma}}{r} dr \right) \quad (4.9)$$

$$\sigma_2 = \frac{1}{\sqrt{3}} \left(\bar{\sigma} + 2 \int_{\rho}^r \frac{\bar{\sigma}}{r} dr \right) \quad (4.10)$$

$$\sigma_3 = \frac{2}{\sqrt{3}} \int_{\rho}^r \frac{\bar{\sigma}}{r} dr \quad (4.11)$$

where $\bar{\sigma}$ is the equivalent stress which is independent of θ and is a function of r only.

The equivalent stress, $\bar{\sigma}$, is obtained as a function of r from the stress-strain relation for power law hardening materials and the strain field near the notch tip computed by Griffiths and Owen through a finite element analysis²³. For the power hardening materials, one has

$$\frac{\bar{\epsilon}}{\epsilon_0} = \left(\frac{\bar{\sigma}}{\sigma_0} \right)^n \quad (4.12)$$

where σ_0 , ϵ_0 are yield stress and yield strain respectively, and n is the hardening exponent.

The equivalent strain, $\bar{\epsilon}$, is obtained as a function of r within the plastic zone from Griffiths and Owen's finite element analysis²⁴. Their result is plotted in Figure 4.4. The empirical function is

$$\bar{\epsilon} = \bar{\epsilon}_t \exp\left[-c_1 \left(\frac{r}{\rho} - 1\right)\right] \quad \text{for } r < r_c \quad (4.13a)$$

$$\bar{\epsilon} = \bar{\epsilon}_t \exp\left[-c_2 \frac{r}{\rho} + c_3\right] \quad \text{for } r \geq r_c \quad (4.13b)$$

where $\bar{\epsilon}_t$ is the equivalent strain at the notch tip, and is a function of σ_{nom}/σ_0 ; c_1 is a constant determined as 2.25; c_2 is a function of σ_{nom}/σ_0 only; $c_3 = (c_2 - c_1)r_c/\rho + c_1$; and $r_c = 2.3\rho$.

The relation between $\bar{\epsilon}_t$ and σ_{nom}/σ_0 is also obtained from Griffiths and Owen's finite element analysis and is plotted in Figure 4.5.

For an applied load P , the procedures to calculate the stresses near the notch tip are:

1. Calculate the nominal stress σ_{nom} from eq. (4.2).
2. Calculate the equivalent strain at the notch tip $\bar{\epsilon}_t$ from σ_{nom}/σ_0 through Figure 4.5.
3. Calculate the equivalent strain $\bar{\epsilon}$ distribution as a function of r from eq. (4.13).
4. Determine the plastic zone size. Find r_p from the equivalent strain $\bar{\epsilon}$ distribution through the equation $\bar{\epsilon} = \epsilon_0$ when $r = r_p$. Then

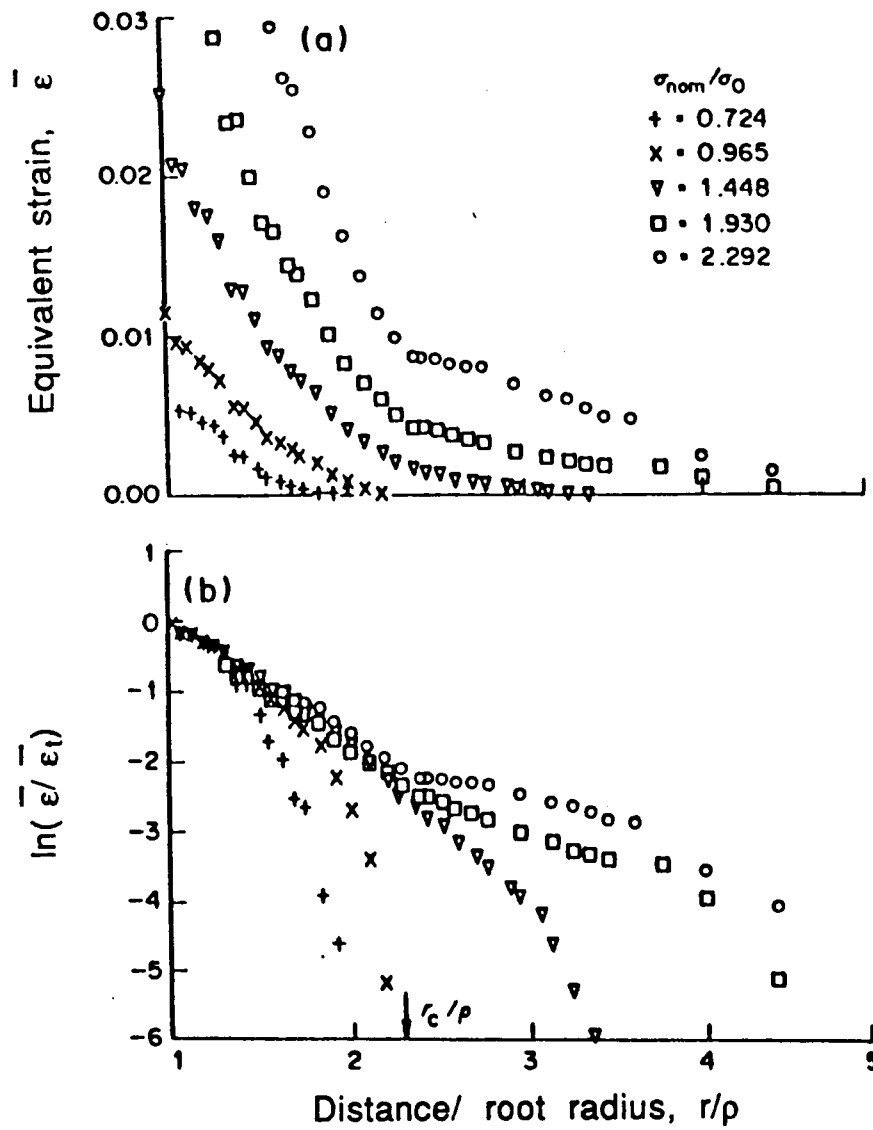


Figure 4.4 Distribution of Equivalent Strain, $\bar{\epsilon}$, as a Function of Normal Distance $\frac{r}{\rho}$ (reproduced from reference 11)

(a). $\bar{\epsilon}$ vs. $\frac{r}{\rho}$ (b). $\ln(\bar{\epsilon}/\bar{\epsilon}_t)$ vs. $\frac{r}{\rho}$

where $\bar{\epsilon}_t$ is the equivalent strain at notch tip.

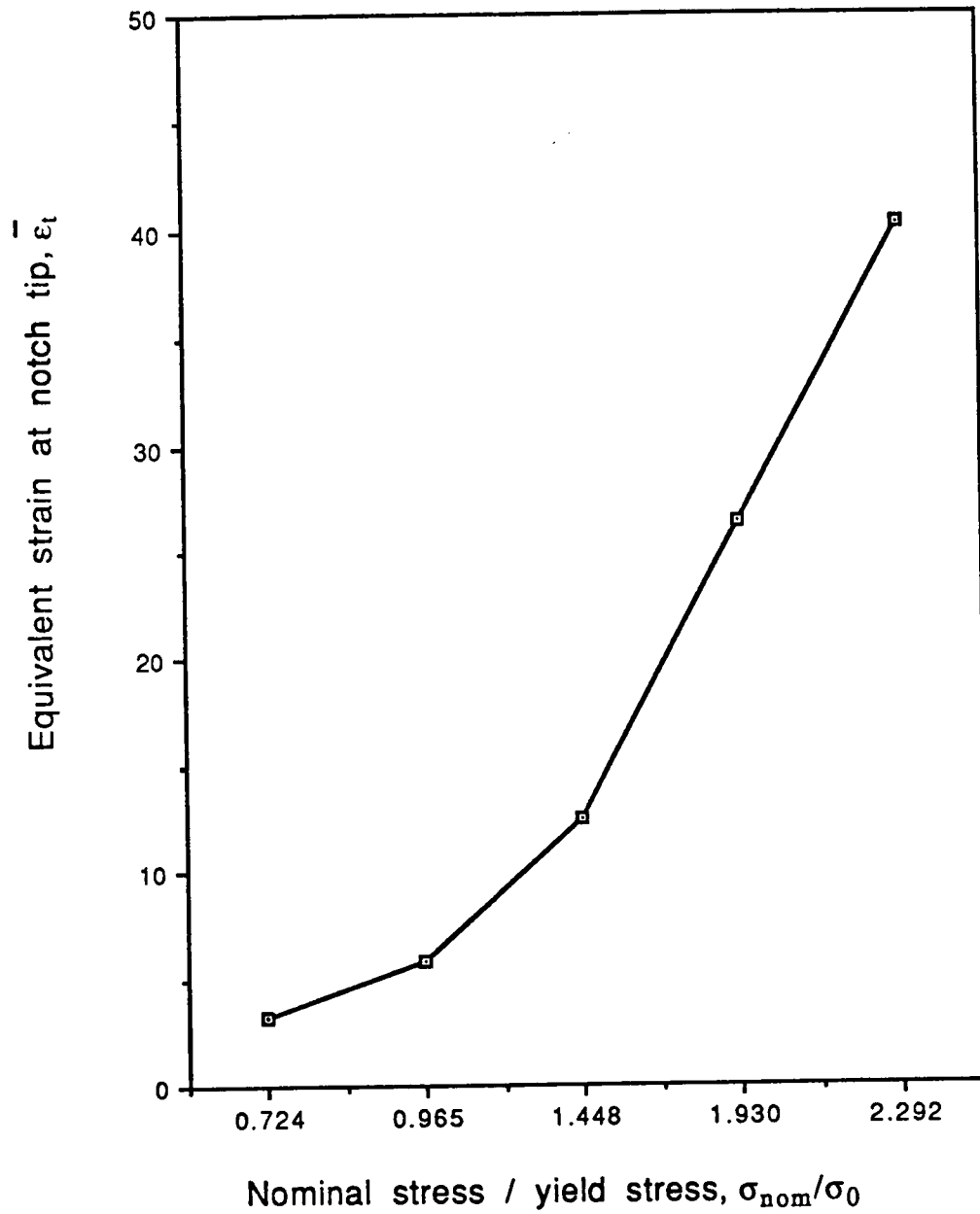


Figure 4.5 Relation between $\bar{\epsilon}_t$ and σ_{nom}/σ_0

where $\bar{\epsilon}_t$ is the equivalent strain at notch tip,
 σ_{nom} is the normal stress and σ_0 is the yield
 stress.

find ζ from eq. (4.7).

5. Calculate the three principal stresses σ_1 , σ_2 and σ_3 within the plastic zone ($\rho \leq r \leq r_p$) from eq.(4.9), eq.(4.10) and eq.(4.11).

4.4 Application of statistical theory

When the stress distribution in the plastic zone and microcrack distribution in the material are known, the statistical model can be easily applied.

The general formula for obtaining the failure probability P_f is

$$P_f = 1 - \exp\left(-b N_0 \int_{\rho}^{r_p} f(\sigma_1, \sigma_2, \sigma_3) dA\right) \quad (4.14)$$

where A is the total area of the plastic zone, and from eq. (4.8), dA is

$$dA = (\pi - 2\ln \frac{r}{\rho} - 4\zeta) dr \quad (4.15)$$

b is the characteristic length which is about 3 times the ferrite grain diameter d_0 ²¹, N_0 is the number of grain boundary carbide particles per unit volume, and $f(\sigma_1, \sigma_2, \sigma_3)$ is Weibull's distribution function for the multiaxial stress state. In this application, $f(\sigma_1, \sigma_2, \sigma_3)$ has the form

$$f(\sigma_1, \sigma_2, \sigma_3) = -\frac{1}{4\pi} \iint_{\substack{0 < \theta < \pi \\ 0 < \phi < 2\pi}} \psi(\sigma_1, \sigma_2, \sigma_3) (\sin\theta d\theta d\phi) \quad (4.16)$$

where $\psi(\sigma_1, \sigma_2, \sigma_3)$ takes the form

$$\psi(\sigma_1, \sigma_2, \sigma_3) = \begin{cases} \left(\frac{\sigma_n - s_u}{s_0} \right)^m & \text{if } \sigma_n > s_u \\ 0 & \text{if } \sigma_n \leq s_u \end{cases} \quad (4.17)$$

where s_u , s_0 , m are the three Weibull distribution function parameters. They are determined from the size distribution of carbide particles and eq. (4.1). σ_n is the normal stress in the direction specified by θ and ϕ in the spherical coordinate system of σ_1, σ_2 and σ_3 .

$$\sigma_n = \sigma_1 \sin^2 \theta \cos^2 \phi + \sigma_2 \sin^2 \theta \sin^2 \phi + \sigma_3 \cos^2 \theta \quad (4.18)$$

If the statistical model for the uniaxial stress state is used as in references 9,10 and 11, then the formula for calculating the failure probability P_f is

$$P_f = 1 - \exp(-bf N_0 \int_p^{r_p} G(\sigma_1) dA) \quad (4.19)$$

$$G(\sigma_1) = \begin{cases} \left(\frac{\sigma_1 - s_u}{s_0} \right)^m & \text{if } \sigma_n > s_u \\ 0 & \text{if } \sigma_n \leq s_u \end{cases} \quad (4.20)$$

Since the microcracks are randomly distributed in the material, not all the microcracks participate in the fracture process. For instance, if σ_2, σ_3 are less than s_u and the normal direction of a microcrack surface is perpendicular to the direction of the first principal stress σ_1 , then this microcrack will not propagate no matter how high σ_1 is. Thus, the parameter f , which is defined as

the percentage of the microcracks participating in the fracture process, has to be introduced in the statistical model for uniaxial stress state in order to match the real situation.

The disadvantage of the statistical model for uniaxial stress state is that it does not accurately include the influence of the second and third principal stresses σ_2, σ_3 on the fracture process. Application of the statistical model for the uniaxial stress state requires that the value of f be evaluated from a separate material toughness test and that f be considered a constant independent of stresses in the material. As can be seen from the statistical model for a triaxial stress state, the percentage of the microcracks participating in the fracture process actually depends upon the local stresses; that is, f is not a material property. If in one small material element the three principal stresses $\sigma_1, \sigma_2, \sigma_3$ are same, then all the microcracks will equally participate in the fracture process and f should be 1. Thus, the value of f determined from one toughness test can only be applied in the statistical model of the same material under the same loading configuration. This makes prediction of the failure load from the statistical model for uniaxial stress state dependent upon the experimental results of the same material under a similar loading condition. Therefore, the experimental verification of the predicted results from the statistical model for uniaxial stress state in this application cannot be justified logically.

Since the statistical theory for multiaxial stress state includes the influence of all three principal stresses, there is no need to

evaluate the percentage of the microcracks participating in the fracture process f from the results of material toughness tests. Thus, prediction of failure load from the statistical theory of multiaxial stress state depends only upon the microscopic properties of the materials such as size of ferrite grains and distribution of carbide particles in the steel, and upon some other material parameters. The experimental results of the toughness test can be used to verify the predicted failure load from this model.

4.5 Results and Discussions

A computer program was developed to compute the failure probability P_f from a given value of load P and temperature, T , as well as to compute the value of load P for a given failure probability P_f and temperature. Because the material properties, namely, the yield stress, σ_0 , change with temperature, the failure probability will change with temperature for a given load P .

All the materials property parameters needed for the calculation in the program are obtained from reference 9. The functional dependency of yield stress, σ_0 , upon temperature is plotted in Figure 4.6. The size distribution of carbide particles in the steel is shown in Figure 4.7. From it, the three Weibull distribution function parameters s_u , s_0 and m are obtained as 1298MPa, 2838MPa and 1.651 respectively. The carbide particle density in AISI 1008 steel is found to be $3.71 \times 10^{-4}/\mu\text{m}^3$. The work hardening exponent is 4.

The load P corresponding to 50%, 1% and 99% of the failure

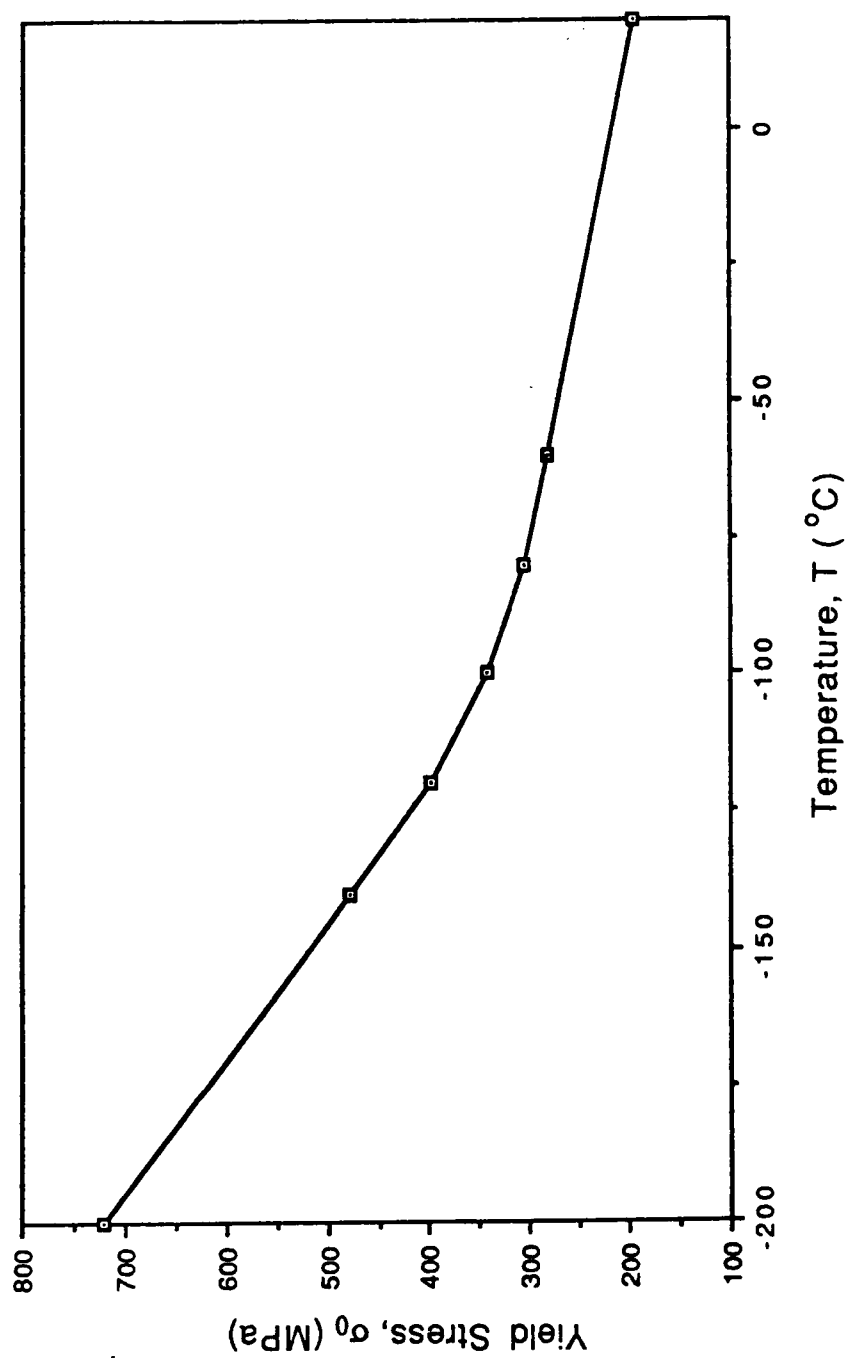


Figure 4.6 The Functional Dependency of Yield Stress, σ_0 , upon Temperature, T

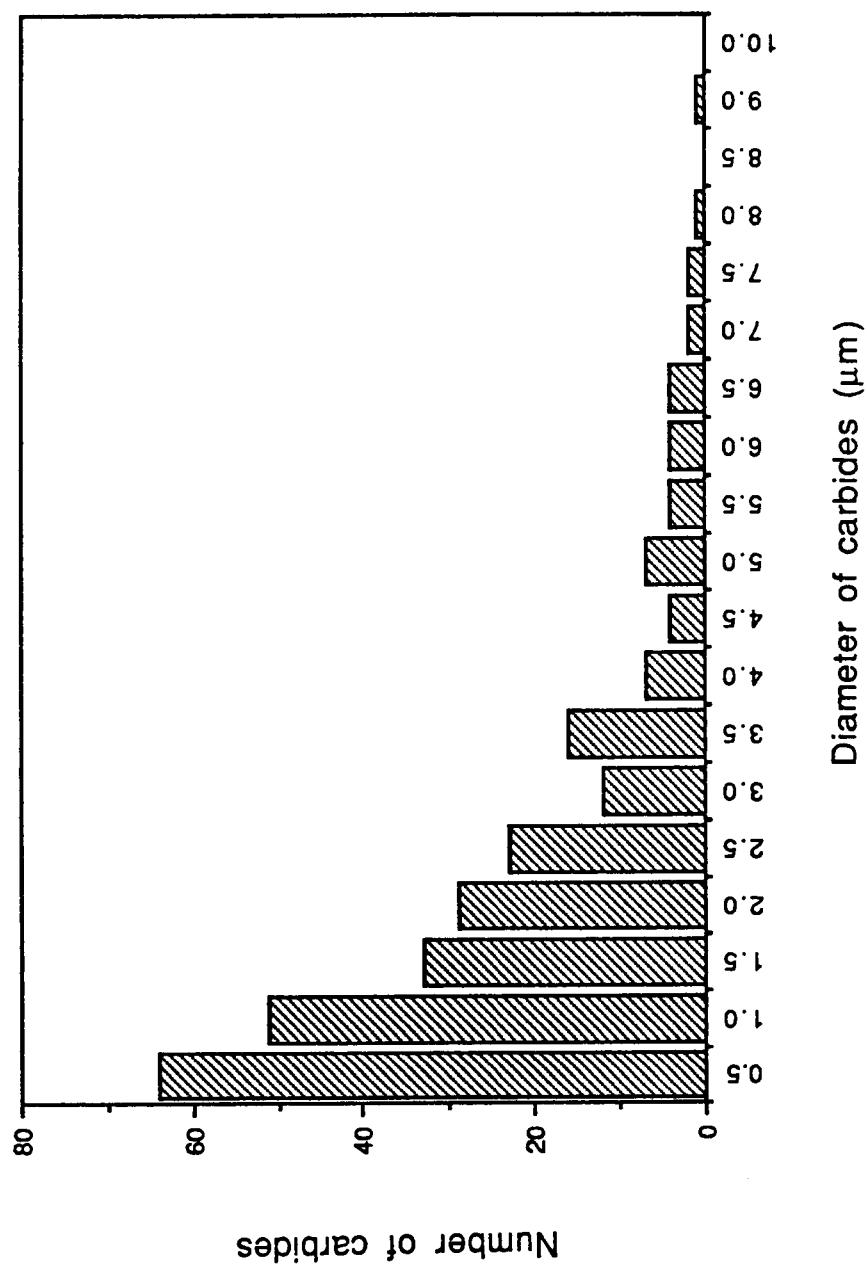


Figure 4.7 Size Distribution of Carbide Particles

probabilities are calculated and are plotted in Figure 4.8 as a functions of temperature. The results from the statistical model for multiaxial stress state are quite close to the previous results from the statistical model for uniaxial stress state with estimated value of the percentage of microcracks participating in the fracture process f as 5%, and both predicted results are in good agreement with results of the experiments in references 9, 10 and 11 at low temperature (see Figure 4.9). The agreement of the two models suggests that $f = 5\%$ is approximately the correct estimation for the percentage of carbide particles near the notch tip eligible for cracking. The good agreement between the experimental results and the results of statistical model of multiaxial stress state at low temperature suggests that this model should be used for the prediction of failure load of brittle materials.

Both predicted failure load and experimental failure load increase when the temperature increases. This is because the yield stress of the steel decreases when temperature increases, consequently the stresses near the notch tip will decrease when the yield stress decreases for a given value of load on the specimen. The lower stresses near the notch tip mean that only the microcracks with the largest size are likely to propagate, and thus, fewer microcracks participate in the fracture process and the specimen is less likely to fail. Therefore, when the temperature increases, the failure load will increase.

The difference between the predicted failure load and the measured failure load becomes greater as the temperature

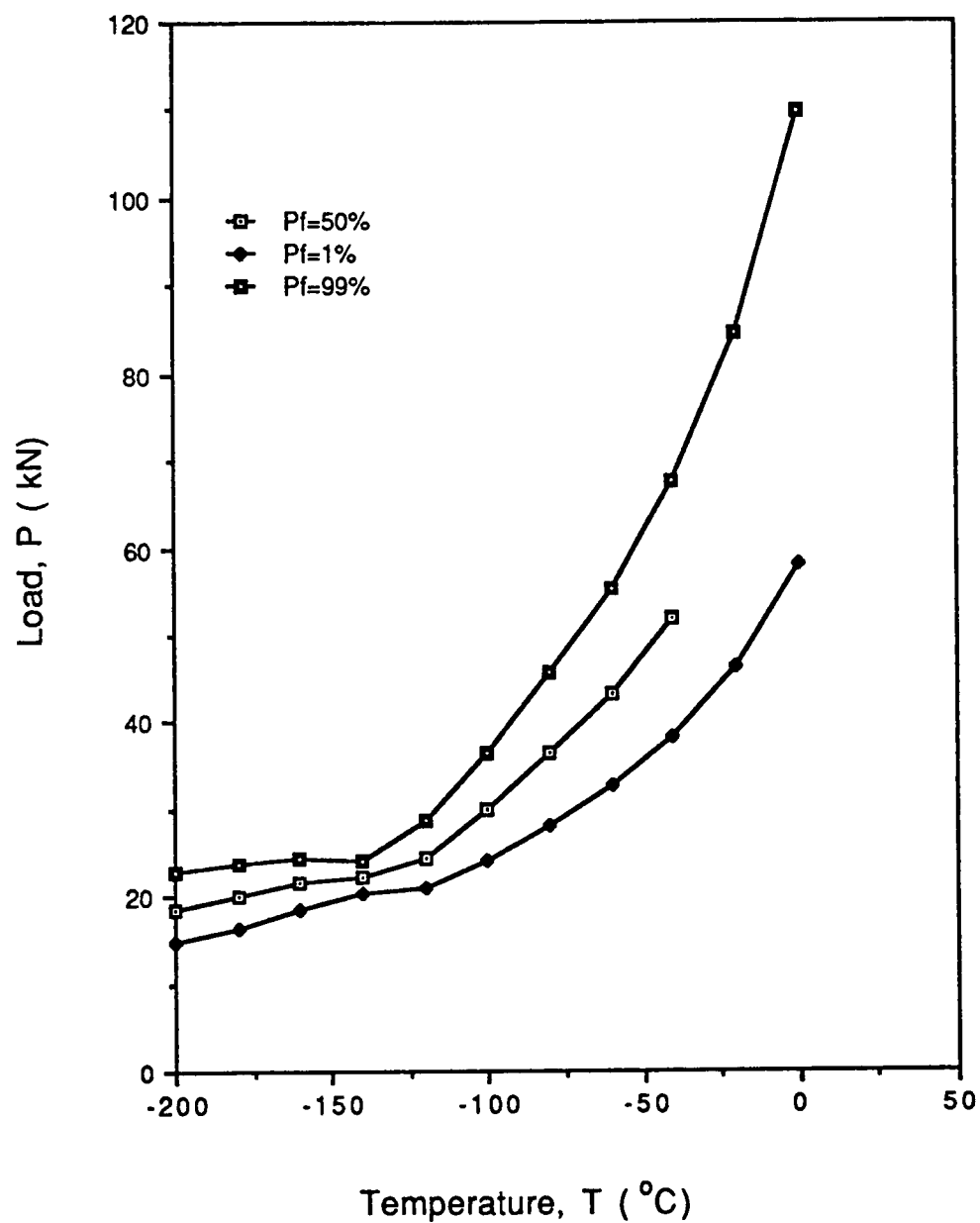


Figure 4.8 Predicted Loads for Failure Probabilities Equal to 1%, 50% and 99% Respectively.

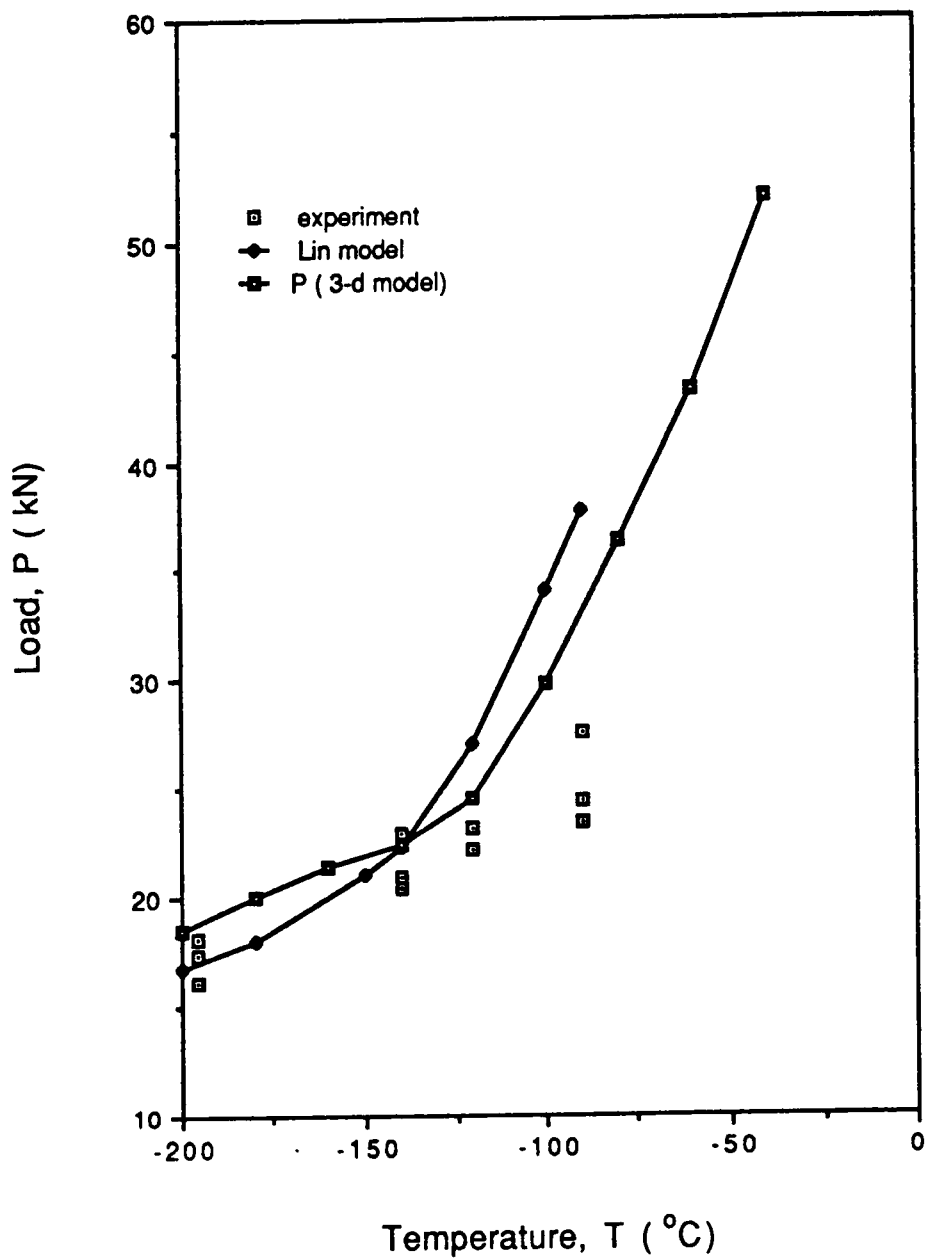


Figure 4.9 Comparison of Predicted Failure Loads with Measured Failure loads

increases. There are two reasons which are probably responsible for the inconsistency of the predicted failure load and the real failure load at the high temperature. One is that the size of the plastic zone near the notch tip, obtained through the method described in section 4.2, becomes a less accurate estimate of the real plastic zone size. Because the plasticity analysis from the slip line theory gives the maximum plastic zone size of $[\exp(\frac{3\pi}{8}) - 1]\rho$, when the temperature increases, the yield stress decreases, and the real plastic zone size may exceed this value. The failure load predicted from a plastic zone smaller than the real plastic zone will give a high value since the failure probability is calculated from the volume integration. The larger the plastic zone is, the more carbide particles will participate in the fracture process, and the more likely the specimen is to crack from the weakest link principle. The second reason is that when the temperature increases, the material becomes more ductile, and the weakest link principle may not be valid any more. That is, the propagation of one carbide particle microcrack into the surrounding grains may not cause catastrophic failure of the whole specimen. Instead, many voids may form and grow around the carbide particles. The growth of these voids may cause the growth of the major crack in the specimen. In this case, the failure load predicted by the weakest link principle should be lower than the real failure load if the real stress distribution near the notch tip is used in the statistical model. Since the predicted failure loads from the model for the four point bend specimen are

higher than the real failure loads obtained from the experiment, the differences are probably mainly caused by the unrealistic prediction of the stress distributions near the notch tip.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The statistical theory for brittle materials includes three important elements: the statistical distributions of microcracks in materials, a fracture criterion of microcrack and the weakest link principle which serves as a failure criterion for the entire loaded body. With arbitrary forms assumed for the distribution function and fracture criterion, the failure probability for a triaxial stress state is obtained mathematically through the weakest link principle. This physically based formula for calculating failure probability is found to be equivalent to the formula given by Batdorf and Crose for isotropic materials and to be equivalent to the formula given by Mastou for the Evans model if a three-parameter Weibull distribution is used.

Two-dimensional finite elements in the structural analysis finite element program SAP4 are modified so that the failure probability can be calculated for a loaded structure. Incorporation of the statistical theory of brittle materials into a finite element program proves to be a very efficient way to apply the statistical model for the structure under an arbitrary loading condition. The volume integration of the entire structure needed in the statistical theory is naturally divided into volume integration over each element. The risk of rupture for each element is calculated by Gaussian quadrature in each element in the same way that the element stiffness matrices are calculated. Thus, the original finite

element program only needs to be slightly modified for statistical failure analysis. The validity of the program was proved by several sample analyses for which analytical expressions of failure probability are available. Good accuracy is obtained for the problems analyzed.

Application of the statistical theory for a triaxial stress state to predict failure loads for four point bending specimens made of AISI 1008 steel also proves to be successful, especially for failure loads at low temperature. The theory is applied to an example where the distribution of microcracks is determined directly by analyzing the microstructure of material under the microscope. Using this theory, failure probability can be predicted directly with no additional information. Therefore, the predicted failure load of this model is totally based on the information from the microstructure of the materials and does not have any empirical fitting constants.

The work conducted for this thesis suggests several areas where further research on statistical prediction for failure of brittle materials is needed. The statistical description of microcracks in materials should be further investigated. Currently, in applying the statistical theory to predicting the failure of brittle materials, the distribution functions corresponding to the microcrack distribution are commonly obtained from failure tests of the same material. In this procedure, the correlation is found between material parameters in the distribution function and failure probability at different loads obtained experimentally for the test specimens. The

microcracks are not treated explicitly in this process. The distribution functions obtained in this way are empirical in nature and their validity can only be justified by prediction of failure for a loaded body having the stress state and loading configuration similar to those of specimen used in the failure test²⁵. For the case in which the stresses are relatively high over a relatively small volume in a loaded body, and the failure probability will be more sensitive to the distribution of relatively small microcracks. If the stresses are relatively uniform over a relatively large volume in a loaded body, the failure probability will be sensitive to the distribution of large cracks. Thus, the distribution function obtained through failure tests can only give an accurate description of microcrack distribution related to the loading configuration of the test specimen. Thus, a method to obtain the distribution function based on the physical distribution of microcracks in materials is more desirable and must be developed. With a physically determined distribution of microcracks, the theory will apply to more general loading configurations.

A fracture criterion for individual microcracks in the materials needs to be developed for cases in which microcracks are subjected to complex remote stresses and do not have a regular geometry. Since the geometry of microcracks is not explicitly treated in many statistical models, this problem has not received much attention. The commonly used fracture criterion of normal tensile stress cannot account for fracture in pure compression. The statistical model using this fracture criterion is not applicable when

the stresses in the loaded body are overwhelmingly compressive. Evans proposed the strain energy release rate as a fracture criterion for penny-shaped microcracks under both remote normal and shear stresses, but the effect of normal compressive stress which will cause the contact of two crack surfaces is not considered in the fracture criterion. Several other fracture criteria applied for both tensile and compressive stress states have been proposed, but their physical basis needs to be further investigated and justified. The fracture mechanics theory for macroscopic cracks can provide a starting point with a theoretical basis for developing more realistic fracture criteria for microcracks, one which will be valid for both tensile and compressive stress states.

A computer program which includes different forms of distribution functions and different forms of fracture criteria of microcracks needs to be developed in order to compare different statistical models as well as to apply the statistical models to the failure analysis of brittle materials. The computer program which computes the failure probability could either be included in the finite element program which computes the stresses in the loaded body, or it could be created separately to process the data of a stress distribution calculated separately by a finite element program. The first way to develop the program requires that the source code of finite element program be available. This program would have a more compact form since the subdomains of volume integrations coincide with finite elements. The second way to develop the program would need more effort since the program

would have to perform the volume integration independently from the finite element program. But the program would be able to process the data calculated by any finite element package.

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APPENDICES

APPENDIX A

ILLUSTRATION OF THE SUBROUTINES IN THE MODIFIED SAP4 PROGRAM

A brief description is given for each subroutine used in modified SAP4. The descriptions for some subroutines may not be comprehensive, but they can help us understand the organization of the program.

ADDSTF: generate structural (global) stiffness matrix.

CALBAN: calculate the band width.

D3WB: calculate the value of the Weibull distribution function n for three-dimensional stress state.

ELAW: compute the constitutive matrix. for two dimensional elements.

ELTYPE: call the element subroutine according to a input data file.

FORMB: compute the stress-displacement transformation matrix at a point in a two dimensional element.

GAUSS: perform 10 points Gaussian numerical integration for a function over a interval.

INPUTD: read and generate nodal point positions.

INL: input nodal loads.

PLANE: subroutine for the two dimensional element.

PLNAX: compute the stiffness matrices for two dimensional element.

PRINTD: write out nodal displacements.

QUAD: compute the stiffness matrix for a two dimensional element.

SESOL: solve the equilibrium equation.

SOLEQ: subroutine for the solution phase.

STRESS: subroutine for computation of stresses.

STRSC: calculate stresses at a point in an element.

TF: subroutine for failure analysis of truss element.

TRUSS: subroutine for truss element.

VF2: subroutine for failure analysis of two dimensional elements.

APPENDIX B

LIST OF CHANGES IN THE ORIGINAL SAP4 PROGRAM

Following are the list of changes made in the original SAP4 program for the modified SAP4 program which can do statistical failure analysis for brittle materials. The changes are made in the main program and subroutines PLANE, PLNAX, QUAD, RUSS, TRUSS. The changes for each subroutine are listed in the above order.

Changes in main program:

```
.....
.....
C
C ONE LINE IS INACTIVE
C   DATA FNAME/' '/
C
C THE FOUR LINES ARE ADDED TO PROMOTE THE INPUT AND
C OUTPUT FILE
      WRITE (*,'(A\)' ) ' INPUT AN INPUT FILE NAME: '
      READ (*,'(BN,A15)' ) FNAME
      WRITE (*,'(A\)' ) ' INPUT AN OUTPUT FILE NAME: '
      READ (*,'(BN,A15)' ) LNAME
C
.....
.....
      OPEN(12,FILE='C:SMAL12.DAT',STATUS='NEW')
C
C ONE LINE IS ADDED TO OPEN A DATA FILE FOR FAILURE ANALYSIS
      OPEN(13,FILE='C:SMAL13.DAT',
          STATUS='NEW',FORM='UNFORMATTED')
C
.....
.....
C
C ONE LINE IS ADDED
C IF NW=1, THE OPTION FOR FAILURE ANALYSIS IS CHOSEN.
C
      NW=1
C
.....
.....
C
```

```

C ONE LINE IS ADDED.
C IF NPAR(10)=1, THE OPTION FOR FAILURE ANALYSIS IS CHOSEN.
C
C     IF (NW.EQ.1) NPAR(10)=1
C
C .....
C .....
C
C THE FOLLOWING LINES ARE NOT NEEDED FOR STATIC AND FAILURE
C ANALYSES
C
C 700 IF (NEQB.LT.NEQ) GOTO 710
C     NIM=3
C     NC=NF+NIM
C     NVM=6
C     NCA=NEQ*MAX0 (MBAND, NC)
C     NTOT=NCA+4*NEQ+2*NVM*NEQ+5*NC
C     IF (NTOT.LE.MTOT) GOTO 720
C 710 NV=MIN0 (2*NF, NF+8)
C     IF (NAD.NE.0) NV=NAD
C     NEQB1=(MTOT-MBAND) / (2*MBAND+1)
C     NEQB2=(MTOT-MBAND-2*NV-NV*(MBAND-2)) / (3*NV+MBAND+1)
C     NEQB3=(MTOT-3*NV*NV-3*NV) / (2*NV+1)
C     NEQB4=(MTOT-6*NV) / (1+MBAND)
C     IF (NEQB1.LT.NEQB) NEQB=NEQB1
C     IF (NEQB2.LT.NEQB) NEQB=NEQB2
C     IF (NEQB3.LT.NEQB) NEQB=NEQB3
C     IF (NEQB4.LT.NEQB) NEQB=NEQB4
C 720 CONTINUE
C     NBLOCK=(NEQ-1) / NEQB+1
C     IF (NEQB.GE.NEQ) NEQB=NEQ
C     KREM=1000
C     NTOT=NBLOCK*NEQB*NF+KREM
C     IF (MTOT.LT.NTOT) GOTO 725
C     GOTO 790
C
C     WRITE (*, 320)
C     GOTO 790
C 730 CONTINUE
C     NN2=NEQ
C     NN3=NEQ
C     NEQB1=(MTOT-NN2-NN3-NEQ-MBAND) / (2*MBAND+1)
C     IF (NEQB1.LT.NEQB) NEQB=NEQB1
C     IF (NEQB2.LT.NEQB) NEQB=NEQB2
C     IF (NEQB.GT.NEQ) NEQB=NEQ
C     NBLOCK=(NEQ-1) / NEQB+1
C     NN2=10
C     NN3=40
C     NN4=6*NUMNP+2*NN2*NEQ
C     IF (NN4.GT.MTOT) GOTO 770
C     GOTO 780
C 770 WRITE (12, 320)

```

```

C      WRITE(*,320)
C 780 NN4=NEQ*2*(NN2+1)+NN2*(1+2*NN3)
C      IF(NN4.GT.MTOT)GOTO 785
C      GOTO 790
C 785 WRITE(12,320)
C      WRITE(*,320)
C      .....
C      .....
C
C THE FOLLOWING LINES ARE NOT NEEDED FOR STATIC AND
C FAILURE ANALYSES
C
C 40 WRITE(*,2140)
C     WRITE(12,2140)
C     T(6)=T(5)
C     CALL SOLEIG
C     CALL TTIME(T(7))
C     T(8)=T(7)
C     T(9)=T(7)
C     T(10)=T(7)
C     WRITE(*,2150)
C     WRITE(12,2150)
C     GOTO 90
C 50 WRITE(*,2160)
C     WRITE(12,2160)
C     T(6)=T(5)
C     IF(NDYN.LT.0)GOTO 52
C     CALL SOLEIG
C     CALL TTIME(T(7))
C     GOTO 54
C 52 DO 53 I=1,6
C 53 T(I+1)=T(I)
C     REWIND 2
C     READ(2)NEQ,NBLOCK,NEQB,MBAND,N1,NF,(AT(I),I=1,NF)
C     REWIND 7
C     IMAX=NEQB*NF
C     READ(7)(A(I),I=1,NF)
C     DO 56 L=1,NBLOCK
C 56 READ(7)(A(I),I=1,IMAX)
C 54 CALL HISTRY
C     CALL TTIME(T(8))
C     T(9)=T(8)
C     T(10)=T(8)
C     WRITE(*,2170)
C     WRITE(12,2170)
C     GOTO 90
C 60 WRITE(*,2180)
C     WRITE(12,2180)
C     T(6)=T(5)
C     IF(NDYN.LT.0)GOTO 62
C     CALL SOLEIG
C     CALL TTIME(T(7))

```

```

C      T(8)=T(7)
C      GOTO 64
C 62 DO 63 I=1,7
C 63 T(I+1)=T(I)
C      REWIND 2
C      REWIND 7
C      IMAX=NEQB*NF
C      READ(7) (A(I), I=1, NF)
C      DO 66 L=1, NBLOCK
C 66 READ(7) (A(I), I=1, IMAX)
C 64 CALL RESPEC
C      CALL TTIME(T(9))
C      T(10)=T(9)
C      WRITE(*, 2190)
C      WRITE(12, 2190)
C      GOTO 90
C 70 WRITE(*, 2200)
C      WRITE(12, 2200)
C      DO 71 I=6, 9
C 71 T(I)=T(5)
C      CALL STEP
C      WRITE(*, 2210)
C      WRITE(12, 2210)
C      CALL TTIME(T(10))
C      TT=0.0
C      DO 95 I=1, 9
C      T(I)=T(I+1)-T(I)
C      TT=TT+T(I)
C 95 CONTINUE
C      WRITE(12, 203) (T(K), K=1, 9), TT
C      WRITE(*, 203) (T(K), K=1, 9), TT
C      WRITE(6, 202) (T(K), K=1, 9), TT
C      .....
C      .....

```

Changes in subroutine ELTYPE

```

C      .....
C      .....
C
C BECAUSE THE ONLY ELEMENT SUBROUTINES OF TRUSS AND PLANE
C ARE MODIFIED AND ARE CAPABLE OF DOING FAILURE ANALYSES,
C ONLY THESE TWO SUBROUTINES ARE LINKED WITH THIS CODE.
C THE CALLINGS OF OTHER SUBROUTINES ARE INACTIVATED
C
C      WRITE(6, 1010)
C      CALL TRUSS
C      GOTO 900
C      WRITE(6, 1020)
C      CALL BEAM
C      GOTO 900

```

```

WRITE(6,1030)
  CALL PLANE
  GOTO 900
WRITE(6,1040)
  CALL PLANE
  GOTO 900
WRITE(6,1050)
C   CALL THREED
  GOTO 900
WRITE(6,1060)
C   CALL SHELL
  GOTO 900
WRITE(6,1070)
C   CALL BOUND
  GOTO 900
WRITE(6,1080)
C   CALL SOL21
  GOTO 900
WRITE(6,1090)
C   CALL PIPE
.....
.....

```

Changes in subroutine PLANE

```

.....
.....
C
C IN THE FOLLOWING TWO LINES, 1960 IS USED INSTEAD OF 240
C IN THE ORIGINAL CODE
  N12=N11+1960
  MM=N12+1960-MTOT
C   N12=N11+240
C   MM=N12+240-MTOT
C
.....
.....
C
C TWO LINES ARE ADDED TO OBTAIN THE ACCURATE VALUE OF PI
  ZERO=0.0
  HPI=DACOS(ZERO)
C
.....
.....
C
C IN THE FOLLOWING LINE, 600 IS CHANGED TO 601
C
  DO 601 KK=1,NS,4
C 510 DO 600 KK=1,NS,4
C
.....

```

```

.....
C
C THREE LINES ARE CHANGED
C
      S5=CC+CR
      S6=CC-CR
      S7=0.0
C      SIG(5)=CC+CR
C      SIG(6)=CC-CR
C      SIG(7)=0.0
C
.....
.....
C
C ONE LINE IS CHANGED
C
      S7=45.0*DATAN2(SIG(4),BB)/HPI
C      SIG(7)=28.648*DATAN2(SIG(4),BB)
C
.....
.....
C
C FOUR LINES ARE ADDED TO WRITE THE STRESSES AT 25
C INTEGRATION POINTS ON DATA FILE SMAL10.DAT
C
      IF(ITAG.EQ.1.AND.NS.LT.20) WRITE(10) MM,L,S5,S6,SIG(3)
      IF(ITAG.LT.6) GO TO 600
      WRITE(10) MM,L,S5,S6,SIG(3)
      IF(ITAG.GT.5) GO TO 601
C
.....
.....
C
C ONE LINE IS ADDED
C
      CONTINUE
C
.....
.....
C
C ONE LINE IS ADDED. WHERE LH IS THE NUMBER OF TOTAL LOAD
C CASES
C
      IF(NPAR(10).EQ.1) CALL VF2(LH,NUME)
C
.....
.....

```

Changes in subroutine PLNAX

.....

```

.....
C
C IN THIS SUBROUTINE, B(20,12), BB(20,12), TI(20,4) ARE
C CHANGED TO B(120,12), BB(120,12), TI(120,4) BECAUSE
C THE STRESSES AT 25 MORE POINTS ARE NEEDED
C
      DIMENSION X(1),Y(1),Z(1),ID(NUMNP,1),
      NTC(1),WT(1),RO(1),WANG(1),
      E(NUMTC,11,1),T(1),B(120,12),BB(120,12)
      COMMON /ELPAR/ NPAR(14),NUMNN,MBAND,NELTYP,
      N1,N2,N3,N4,N5,MTOT,NEQ
      COMMON /EM/ LM(12),S(12,12),P(12,4),XM(12),TI(120,4),
      1IX(4),IE(5),NS,D(4,4),EMUL(4,5),RR(4),ZZ(4),H(6),HS(6),
      2HT(6),HR(6),HZ(6),FAC,XMM,PRESS,EE(10),TTI(4),PP(12,4),
      3THICK,TMP(4),TP(12),ALP(4),IFILL2(4783)
      COMMON /JUNK/ MAT,NT,TEMP,REFT,BETA,
      U(4),V(4),W(4),G(4),IFLL(390)
      COMMON /EXTRA/ MODEX,NT8,IFILL3(14)
C
.....
.....
C
C ONE LINE IS ADDED FOR MONITORING THE PROGRESS OF
C COMPUTATION
C
      WRITE(6,2003) N,IX,MAT,REFT,PRESS,NS,KG,THICK
C
C ONE LINE IS ADDED TO WRITE MATERIAL INFORMATION
C FOR FAILURE ANALYSES
C
      WRITE(13) N,(IX(I),I=1,4),MAT,THICK
C
.....
.....
C
C ONE LINE IS ADDED. NP=1 FOR TRANGULAR ELEMENT
C                               NP=25 FOR QUADRILATERAL ELEMENT
      NP=25
C
.....
.....
C
C TO CALCULATE THE AREA OF TRANGULAR ELEMENT,
C AND TO WRITE IT ON ADTA FILE SMAL10.FOR
C SIX LINES ARE ADDED
C
      IF(IE(3).NE.IE(4)) GO TO 156
      CALL CROSS(RR,ZZ,V)
      AREA=0.5*DABS(V(4))
      NP=1
      WRITE(10) NP
      IF(NP.EQ.1) WRITE(10) AREA

```

```

C
.....
.....
C
C TO CALCULATE THE AREA OF TRANGULAR ELEMENT,
C AND WRITE IT ON DATA FILE SMAL10.DAT
C FIVE LINES ARE ADDED
C
      IF(IE(3).NE.IE(4)) GO TO 164
      NP=1
      AREA=0.5*RR(2)*ZZ(3)
      WRITE(10) NP
      IF(NP.EQ.1) WRITE(10) AREA

```

```

C
.....
.....

```

Changes in subroutine QUAD

```

.....
.....
C
C IN THIS SUBROUTINE, TI(20,4), B(20,12), BB(20,12), SSS(5),
C TTT(5) ARE CHANGED TO TI(120,4), B(120,12), BB(120,12),
C SSS(30), TTT(30) IN ORDER TO CALCULATE STRESSES AT EXTRA 25
C INTEGRATION POINTS.
C
      COMMON /ELPAR/ NPAR(14),NUMNP,MBAND,NELTYP,
      N1,N2,N3,N4,N5,MTOT,NEQ
      COMMON /EM/ LM(12),S(12,12),P(12,4),XM(12),TI(120,4),
      1IX(4),IE(5),NS,D(4,4),EMUL(4,5),RR(4),ZZ(4),H(6),
      2HS(6),HT(6),HR(6),HZ(6),FAC,XMM,PRESS,EE(10),TTI(4),
      3PP(12,4),THICK,TMP(4),TP(12),ALP(4),IFILL2(4783)
      COMMON /JUNK/ MAT,NT,TEMP,REFT,BETA,IFILL1(422)
      DIMENSION B(120,12),BB(120,12)
      DIMENSION SS(2),TT(2),HH(2),SSS(30),TTT(30),
      IVECT(4),JVECT(4),V(4)
C
C 25 MORE VALUES OF SSS & TTT ARE INITIALIZED.
C
      DATA SSS /0.,-1.,1.,0.,0.,
      -0.9061798459,-0.5384693101,0.0,0.5384693101,
      0.9061798459,-0.9061798459,-0.5384693101,0.0,
      0.5384693101,0.9061798459,-0.9061798459,
      -0.5384693101,0.0,0.5384693101,0.9061798459,
      -0.9061798459,-0.5384693101,0.0,0.5384693101,
      0.9061798459,-0.9061798459,-0.5384693101,
      0.0,0.5384693101,0.9061798459/
      DATA TTT /0.,0.,0.,-1.,1.,
      5*-0.9061798459,5*-0.5384693101,
      5*0.,5*0.5384693101,5*0.9061798459/

```

```

C
.....
.....
C
C TWO LINES ARE ADDED TO WRITE THE VALUE OF JACOBIAN
C IN DATA FILE SMAL10.DAT
C
      IF(L.LE.5) GO TO 510
      WRITE(10) FAC
C
.....
.....
C
C THE FOLLOWING SEVEN LINES ARE USED INSTEAD OF ONE LINE
C TO PASS THE CORRECT VALUE TO THE SUBROUTINE VECTER
C
      X1=RR(IV)
      X2=ZZ(IV)
      X3=0.0
      Y1=RR(JV)
      Y2=ZZ(JV)
      Y3=0.0
      CALL VECTOR(V,X1,X2,X3,Y1,Y2,Y3)
C      CALL VECTOR(V,RR(IV),ZZ(IV),0.0,RR(JV),ZZ(JV),0.0)
C
.....
.....

```

Changes in subroutine RUSS

```

.....
.....
C
C THE NUMBER OF DIFFERENT MARERIALS ARE WRITTEN
C ON FILE SMAL13.DAT FOR FAILURE ANALYSES
C
      WRITE(13) NUMMAT
C
.....
.....
C
C ELEMENT NUMBER, LENGTH OF ELEMENT AND MATERIAL TYPE OF
C ELEMENT ARE WRITTEN ON SMAL13.DAT FOR FAILURE ANALYSES
C
      WRITE(13) M,XL,MTYPE
C
.....
.....

```

Changes in subroutine TRUSS

.....
.....

C
C ONE LINE IS CHANGED.
C THE COMMON BLOCK JUNK1 IS USED INSTEAD OF JUNK
C TO MAKE THE PROGRAM WORK
COMMON/JUNK1/LT,LH,L,IPAD,SIG(24)
C COMMON/JUNK/LT,LH,L,IPAD,SIG(20),
C 1 N6,N7,N8,N9,N10,IFILL(381)
C

.....
.....

C
C ONE LINE IS ADDED TO CALL THE SUBROUTINE
C TO DO FAILURE ANALYSIS
C
IF(NPAR(10).EQ.1) CALL TF(LH)
C

.....
.....

APPENDIX C

THE SUBROUTINES DEVELOPED FOR THE MODIFIED SAP4 PROGRAM

```
C D3WB.FOR SI-XING SHE      JULY 10, 1988
      SUBROUTINE D3WB(WM,BP)
      IMPLICIT REAL*8 (A-H,O-Z)
C  DESCRIPTION: THIS SUBROUTINE IS DEVELOPED TO CALCULATE
C  THE VALUE OF DISTRIBUTION FUNCTION AT INTEGRATION
C  POINTS FOR TRIAXIAL STRESS STATE.
C  CALLED BY: VF
C  SUBROUTINE TO CALL: GAUSS
C  DATA INPUT: WM, WB3D (FROM COMMON BLOCK)
C  DATA OUTPUT: BP,
C  VARIABLES: S1,S2,S3, THREE PRINCIPAL STRESSES
C
      COMMON/WB3D/WE(10),X(10),S1,S2,S3,A1,C,HPI
C
C  TO ARRANGE THREE PRINPAL STRESSES
C  SUCH THAT S1 > S2 > S3
C
      IF(S3.LE.S1) GO TO 10
      C=S3
      S3=S2
      S2=S1
      S1=C
10  IF(S3.LE.S2) GO TO 20
      C=S2
      S2=S3
      S3=C
C
20  A=0
      B=HPI
      BP=0.0
      DO 50 I=1,10
      IF(S3.GE.0.0) GO TO 30
      IF(S2.GE.0.0) GO TO 30
      C=S1
      S1=S3
      S3=C
30  C=2.0*HPI*X(I)
      A1=DCOS(C)
      A2=DSIN(C)
      C=(S1*A1*A1+S2*A2*A2)
      IF(S3.GE.0.0.AND.S2.GE.0.0) GO TO 40
      ANG=DSQRT(-C/S3)
      ANG=DATAN(ANG)
```

```

        IF(S2.GT.0.0) B=ANG
        IF(S2.LE.0.0) A=ANG
40    CALL GAUSS(A,B,R,WM,V,3)
50    BP=BP+WE(I)*V
C
    RETURN
    END

C GAUSS.FOR SI-XING SHE      10 JULY 88
    Subroutine gauss(a,b,r,w,v,j)
    implicit real*8(a-h,o-z)
C
C DESCRIPTION: TO PERFORM 10 POINTS GAUSSIAN INTEGRATION
C DATA INPUT: A,B, RANGE OF INTEGRATION
C             J, INDEX OF FUNCTION TO BE INTEGRATED
C             R, A PARAMETER IN THE FUNCTION TO BE INTEGRATED
C DATA OUTPUT: V, THE VALUE OF INTEGRATION
C VARIABLES: WE, INTEGRATION COEFFICIENTS
C
    common/wb3d/we(10),X(10),S1,S2,S3,C,A1,HPI
    v=0.0
    do 10 i=1,10
10    v=v+we(i)*func(i,j,a,b,r,w)
    return
    end
C
    function func(i,j,a,b,r,w)
    implicit real*8 (a-h,o-z)
C
C DESCRIPTION: CALCULATE THE VALUES OF SEVERAL FUNCTIONS
C WHICH NEED TO BE INTEGRATED DATA INPUT:
C     I, THE INDEX OF INTEGRATION POINT ( FROM 1 TO 10 )
C     J, THE INDEX OF FUNCTIONS
C     A,B, THE RANGE OF INTEGRATION
C     R,W, THE PARAMETERS IN THE INTEGRATED FUNCTIONS
C DATA OUTPUT: FUNC
C
    common/wb3d/we(10),x(10),s1,s2,s3,a1,c,hpi
    t=a+(b-a)*x(i)
    if(j.eq.2) func=((dcos(t))**2+r*(dsin(t))**2)**w
1    *(b-a)
    if(j.eq.1) func=((dsin(t))**(2.0*w+1.0))*(b-a)
    if(j.eq.3) then
        b1=dcos(t)
        b2=dsin(t)
        d=c*b1*b1+s3*b2*b2
        func=((c*b1*b1+s3*b2*b2)**w)*b1*(b-a)
    end if
    return
    end

```

```

C TF.FOR SI-XING SHE          JULY 10 88
C   SUBROUTINE TF(LH)
C
C DESCRIPTION: THIS SUBROUTINE IS DEVELOPED AND ADDED TO
C   THE ORIGINAL SAP4 PROGRAM TO DO FAILURE ANALYSES FOR
C   TRUSS ELEMENT.
C CALLED BY: T
C SUBROUTINE TO CALL: NO
C DATA INPUT: READ LENGTH AND AREA OF CROSS SECTION OF EACH
C   ELEMENT
C   FROM DATA FILE SMAL13.DAT
C   READ STRESS OF EACH ELEMENT FROM DATA FILE SMAL10.DAT
C   READ WEIBULL'S MATERIAL CONSTANTS FROM INPUT DATA FILE
C   THE NUMBER OF TOTAL LOAD CASES ARE OBTAINED FROM THE
C   ARGUMENT OF SUBROUTINE
C DATA OUTPUT: WRITE THE RESULTS ON OUTPUT DATA FILE
C VARIABLES: LH, NUMBER OF LOAD CASE;
C           LL, NUMBER OF MATERIAL TYPE;
C           B, RISK OF RUPTURE;
C           S, FAILURE PROBABILITY
C
C   IMPLICIT REAL*8 (A-H,O-Z)
C   COMMON/ELPAR/NPAR(14),NUMNP,MBAND,NELTYP,
1     N1,N2,N3,N4,N5,MTOT,NEQ
C   DIMENSION X(50),Y(50),Z(50),AREA(20),
1     WQ(20),WM(20),BB(20),SS(20)
C   NUME = NPAR(2)
C   WRITE(12,1000)
C   WRITE(*,1000)
C   DO 9 L=1,LH
C     SS(L) = 0.0
9   BB(L) = 0.0
C   REWIND 10
C   REWIND 13
C
C INPUT THE AREA OF CROSS SECTION FOR EACH MATERIAL TYPE
C
C   READ(13) LL
C   DO 20 I1=1,LL
20  READ(13) AREA(I1)
C
C INPUT THE WEIBULL'S CONSTANTS
C
C   DO 80 I1=1,LL
C   READ(5,1010) WQ(I1),WM(I1)
80  WRITE(6,1020) WQ(I1),WM(I1)
C
C   WRITE(6,1030)
C
C INPUT ELEMENT NUMBER, LENGTH AND MATERIAL TYPE
C
30  READ(13) N,XL,MTYPE

```

```

C
C INPUT ELEMENT NUMBER, LOAD CASE NUMBER AND STRESS OF
C ELEMENT
C
  40 READ (10) MM,L,S1
C
  IF (S1.LT.0.0) GO TO 60
  IF (MM.EQ.N) GO TO 50
  WRITE (6,1040) MM,N
  RETURN
C
C CALCULATION OF RISK OF RUPTURE AND FAILURE PROBABILITY
C AND OUTPUT THE RESULTS
C
  50 V = XL*AREA(MTYPE)
  B = V*((S1/WQ(MTYPE))**WM(MTYPE))
  BB(L) = BB(L)+B
  S = 1.0-DEXP(-B)
  WRITE (6,1050) N,B,S
  60 IF (L.NE.LH) GO TO 40
  IF (N.NE.NUME) GO TO 30
  DO 70 L=1,LH
  SS(L) = 1.0-DEXP(-BB(L))
  70 WRITE (6,1060) L,BB(L),SS(L)
  RETURN
  1000 FORMAT(' FRACTURE ANALYSIS STARTED')
  1010 FORMAT(2F10.0)
  1020 FORMAT(/' WK= ',E15.7,' WM= ',E15.7)
  1030 FORMAT(/
  1 ' ELEMENT RISK OF RUPTURE FAILURE PROBABILITY')
  1040 FORMAT(/
  1 '**ERROR.(ELEMENT DATA OUT OF ORDER. MM.NE.N.',
  1 2I5,')')
  1050 FORMAT(1X,I5,2E15.7)
  1060 FORMAT(/' B AND S OF WHOLE STRUCTURE FOR LOAD CASE ',
  1 I3,/,2E15.7)
  END

C VF.FOR SI-XING SHE July 10, 88
  SUBROUTINE VF2(LH,NUME)
  IMPLICIT REAL*8(A-H,O-Z)
C
C DESCRIPTION: THIS SUBROUTINE IS DEVELOPED TO CALCULATE
C THE RISK OF RUPTURE AND FAILURE RPOBABILITY FOR
C TRANGULAR ELEMENT ANG QUADRILATERAL ELEMENT
C CALLED BY: MAIN
C SUBROUTINE TO CALL: WB3D, GAUSS
C DATA INPUT: READ STRESSES AND GEOMETRY INFORMATION OF EACH
C ELENEMT FROM DATA FILE SMAL10.DAT AND SMAL13.DAT
C DATA OUTPUT: WRITE THE RISK OF RUPTURE AND THE FAILURE
C RPOBABILITY ON OUTPUT DATA FILE

```

```

C VARIABLES: B, RISK OF RUPTURE; S, FAILURE PROBABILITY
C   WQ,WM, WAIBULL'S MATERIAL CONSTANTS
C   X1, COODINATES OF GAUSSIAN INTEGRATION POINTS
C   W1, HS, COEFFICIENTS OF GAUSSIAN INTEGRATION
C
COMMON /ELPAR/ NPAR(14),IFILL(10)
COMMON/WB3D/WE(10),X(10),S1,S2,S3,A1,A2,HPI
DIMENSION WQ(20),WM(20),BB(20),SS(20),W1(10),X1(10)
1   ,IX(4),C(20),XJ(100,25),HS(5)
data w1/0.0333356722,0.0747256746,0.1095431813,
*       0.1346333597,0.1477621124,0.1477621124,
*       0.1346333597,0.1095431813,0.0747256746,
*       0.0333356722/
data x1/0.0130467357,0.0674683167,0.1602952159,
*       0.2833023029,0.4255628305,0.5744371695,
*       0.7166976971,0.8397047841,0.9325316833,
*       0.9869532643/
DATA HS /0.2369268851,0.4786286705,
*       0.5688888889,0.4786286705,0.2369268851/

C
C-----PERAPERATION
C
WRITE(12,1000)
WRITE(*,1000)
REWIND 10
REWIND 13
ZERO=0.0
HPI=DACOS(ZERO)
DO 10 I=1,10
WE(I)=W1(I)
10 X(I)=X1(I)

C
C-----INPUT DATA
C
DO 12 K=1,NUME
READ(10) NP
DO 12 J=1,NP
12 READ(10) XJ(K,J)

C
C TO MULTIPLY THE GAUSSIAN INTEGRATION COEFFIENTS
C
IF(NP.EQ.1) GO TO 14
DO 13 I=1,NUME
DO 13 J=1,5
DO 13 K=1,5
J1=(J-1)*5+K
13 XJ(I,J1)=XJ(I,J1)*HS(J)*HS(K)

C
C CALCULATION OF FAILURE PROBABILITY
C
14 DO 80 I1=1,LH
READ(5,1010) WQ(I1),WM(I1)

```

```

      ANG=HPI
      CALL GAUSS(0.0,ANG,0.0,WM(I1),C(I1),1)
80  WRITE(6,1020) WQ(I1),WM(I1)
      WRITE(6,1030)

30  READ(13) N,(IX(I),I=1,4),MAT,THICK

      DO 99 I1=1,LH
      B=0.0

      DO 90 J=1,NP
40  READ (10) MM,L,S1,S2,S3
      IF(S1.LE.0.0.AND.S3.LE.0.0)GO TO 90
      IF(S3.EQ.0.0) GO TO 45
C
C  CALL D3WB TO CALCULATE THE DISTRIBUTION FUNCTION FOR
C  THREE DIMENSIONAL STRESS STATE
C
      CALL D3WB(WM(MAT),BP)
C
      GO TO 55
45  ANG=HPI
      IF(S2.GE.0.0) GO TO 60
      E=DSQRT(-S1/S2)
      ANG=DATAN(E)
60  R=S2/S1
      CALL GAUSS(ZERO,ANG,R,WM(MAT),VI,2)
      BP=VI*C(MAT)*(S1*WM(MAT))*THICK/HPI
55  B=B+BP*XJ(MM,J)*(2.0*WM(MAT)+1.0)*WQ(MAT)
90  CONTINUE

      IF(NPAR(5).EQ.0) B=4.0*HPI*B
      BB(I1)=BB(I1)+B
      S=1.0-DEXP(-B)
      WRITE(*,1050) MM,I1,MAT,B,S
      WRITE(6,1050) MM,I1,MAT,B,S
99  CONTINUE

      IF(MM.NE.NUME) GO TO 30
C
C  TO CALCULATE B & S FOR WHOLE STRUCTURE
C
      DO 70 I=1,LH
      SS(I)=1.0-DEXP(-BB(I))
70  WRITE(6,1060) I,BB(I),SS(I)
C
      RETURN
1000 FORMAT(' FRACTURE ANALYSIS STARTED')
1010 FORMAT(1X,2E15.7)
1020 FORMAT(/'  WK= ',E15.7,'      WM= ',E15.7)
1030 FORMAT(/' ELEMENT  LOAD  MATERIAL ',7X,'B',7X,'S')

```

```

1040 FORMAT(/'**ERROR.(ELEMENT DATA OUT OF ORDER.
MM.NE.N.',
      *      2I5,')')
1050 FORMAT(1X,I5,3X,2(I5,3X),2E15.7)
1060 FORMAT(' LOAD CASE NUMBER:',I3,/,
      *      ' RISK OF THE FRACTURE OF THE WHOLE STRUCTURE:',
      *      E15.7/,
      *      ' PROB0BALITY OF FRACTURE OF THE WHOLE
STRUCTURE:',
      *      E15.7)
      END

```

APPENDIX D

INPUT DATA FILES OF SAMPLE PROBLEMS

1. Input data files for sample problem 1

(a). Two Quadrilateral Elements

```

SIMPLE TENSION ( 2 Quadrilateral Elements )
6  1  1  0  5
1110110
2111110      0.      0.      20.
3100110      20.
4100110      20.      20.
5100110      40.
6100110      40.      20.
3  2  1  1  2
1  1      6.0      7.0      0.
2000000. 2000000. 2000000.      0.0      0.0      0.0
2000000.0

```

```

1  1  3  4  2  1      0.      0. 120  0      1.
2  3  5  6  4  1      0.      0. 120  0      1.
5  1      0.      4.00
6  1      4.00
1  1
3.      7.

```

(b). Four triangular elements

```

Simple Tension ( 4 Triangular elements )
6  1  1  0  5
1110110
2111110      0.      0.      20.
3100110      20.
4100110      20.      20.
5100110      40.
6100110      40.      20.
3  4  1  1  2
1  1      6.0      7.0      0.
2000000. 2000000. 2000000.      0.0      0.0      0.0
2000000.

```

1	1	3	2	2	1	0.	0.	16	0	1.
2	2	3	4	4	1	0.	0.	16	0	1.
3	3	5	4	4	1			16		1.
4	4	5	6	6	1			16		1.
5	1		0.		4.00					
6	1				4.00					
1	1									

3. 7.

2. Input data file for sample problem 2

THREE POINTS BENDING (40 Quadrilateral Elements)

63	1	1	0	5	0	0	0		
1110111				0.			0.	-6.	
2100111				0.			0.	0.	
3111111				0.			0.	6.	
4100111				0.			6.	-6.	
5100111				0.			6.		
6100111				0.			6.	6.	
7100111				0.			12.	-6.	
8100111				0.			12.		
9100111				0.			12.	6.	
10100111				0.			18.	-6.	
11100111				0.			18.		
12100111				0.			18.	6.	
13100111				0.			24.	-6.	
14100111				0.			24.		
15100111				0.			24.	6.	
16100111				0.			30.	-6.	
17100111				0.			30.		
18100111				0.			30.	6.	
19100111				0.			36.	-6.	
20100111				0.			36.		
21100111				0.			36.	6.	
22100111				0.			42.	-6.	
23100111				0.			42.		
24100111				0.			42.	6.	
25100111				0.			48.	-6.	
26100111				0.			48.		
27100111				0.			48.	6.	
28100111				0.			54.	-6.	
29100111				0.			54.		
30100111				0.			54.	6.	
31100111				0.			60.	-6.	
32100111				0.			60.		
33100111				0.			60.	6.	
34100111				0.			66.	-6.	
35100111				0.			66.		

36100111	0.	66.	6.
37100111	0.	72.	-6.
38100111	0.	72.	
39100111	0.	72.	6.
40100111	0.	78.	-6.
41100111	0.	78.	
42100111	0.	78.	6.
43100111	0.	84.	-6.
44100111	0.	84.	
45100111	0.	84.	6.
46100111	0.	90.	-6.
47100111	0.	90.	
48100111	0.	90.	6.
49100111	0.	96.	-6.
50100111	0.	96.	
51100111	0.	96.	6.
52100111	0.	102.	-6.
53100111	0.	102.	
54100111	0.	102.	6.
55100111	0.	108.	-6.
56100111	0.	108.	
57100111	0.	108.	6.
58100111	0.	114.	-6.
59100111	0.	114.	
60100011	0.	114.	6.
61100111	0.	120.	-6.
62100111	0.	120.	
63100111	0.	120.	6.
3 40 1 1 2			
1 1 9. 8. 0.			
2000000. 2000000. 2000000.	0.3	0.3	0.3

769230.77

1	1	4	5	2	1	0.	0.	120	3	1.
20	58	61	62	59	1	0.	0.	120	3	1.
21	2	5	6	3	1	0.	0.	120	3	1.
40	59	62	63	60	1	0.	0.	120	3	1.
31	1		0.		0.	0.0100				
63	1		0.		0.	-0.0100				
1	1									

300.00 2.

3. Input data file for sample problem 3

Axisymmetric tension (2 Quadrilateral Elements)
6 1 1 0 5

1110110								
2111110			0.		0.		20.	
3100110					20.			
4100110					20.		20.	
5100110					40.			
6100110					40.		20.	
4	2	1	1	2				
1	1		0.0		0.0	0.		
2000000.0		2000000.	2000000.	2000000.		0.0	0.0	0.0

1	1	3	4	2	1	0.	0.	120	0	1.
2	3	5	6	4	1	0.	0.	120	0	1.
5	1		0.		4.00					
6	1				4.00					
1	1									
	3.		7.							

VITA

Si-xing She was born on May 2, 1963, in Changchun, Jilin, China. After graduating from Changchun's Seventeenth High School in June of 1981, he entered Beijing University, Beijing, China and received a B. S. degree in Mechanics in June 1985. Study toward an M.S. degree in Engineering Science and Mechanics at the University of Tennessee began in the winter quarter of 1986. When this degree was awarded in August 1988, he was in the Ph.D program of the same department.