

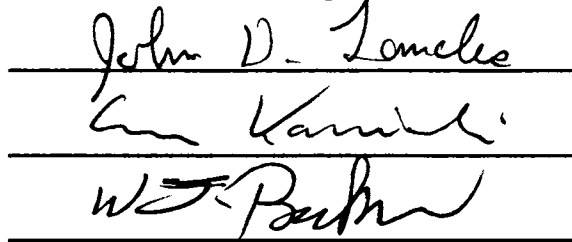
To the Graduate Council:

I am submitting herewith a dissertation written by Julian J. Raphael entitled "Residual Stress, Shakedown and Failure in Carburized Hoisting Load Chain." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mechanical Engineering.



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We have read this dissertation
and recommend its acceptance



Accepted for the Council:



Associate Vice Chancellor and
Dean of The Graduate School

RESIDUAL STRESS, SHAKEDOWN AND FAILURE IN
CARBURIZED HOISTING LOAD CHAIN

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Julian J. Raphael
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ABSTRACT

Electric chain hoists lift their loads by passing carburized chain over a rotating load sprocket. A link passing over the load sprocket experiences relative motion with respect to adjacent links, giving rise to link/link interface frictional tractions that do not occur in those links not passing over the load sprocket. This study, largely conducted with finite element analysis, has focused on the contact stresses in both instances. Yield was assumed to obey the von Mises criterion, an associated flow rule employed and material hardening was taken to be isotropic. Factors complicating the analysis are multiple element groups representing the hardened outer case and the tough core, nonlinear material behavior and the contact itself.

Validation of the finite element model was achieved with a convergence study and by comparing classical elasticity solutions to results obtained from analysis of linearly elastic materials in normal contact. Results of the frictionless problem with multiple nonlinear element groups show the peak von Mises stress to occur along the load line beneath the surface in the carburized case. However, yield was only observed in the core. By comparison, when friction was introduced into the problem the peak von Mises stress was found at the surface in front of the load line within the contact patch. Yield occurred in both the case and the core.

However, merely exceeding the yield strength after one load application does not mean the design is poor or the chain is unusable. That issue can be resolved by examining the stress states after several load/unload cycles to explore shakedown. It was found that in both frictionless and tractive contact a state of elastic shakedown was achieved after one load cycle. In both instances this was attributed to the development of a system of protective residual stresses which arose in the vicinity of the case/core interface.

Superposition integrals which allow calculation of all contact stress components in response to both normal and tangential tractions, given those tractions and the contact patch dimensions, have been developed from the work of Boussinesq and Cerruti. Agreement of the integrals with Sackfield's solution was quite good. A correction was also made in the analytical expression for τ_{xy} in response to a concentrated normal force.

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1 INTRODUCTION

PROBLEM DEFINITION

Electric chain hoists are often called upon to operate at or near their rated capacities at high duty cycles. A critical hoist component is its load chain. It is frequently manufactured from a carburizing, or case hardening, grade of nickel alloy steel which, after thermal treatment, possesses a thin hard outer case and a tough inner core. The function of the case is to provide a wear resistant surface, while the core not only carries most of the applied load but also supports the case. If the case is too soft, excessive wear occurs; while if it is too hard, fatigue fractures occur prematurely because of its brittleness. On the other hand, if the core is too soft, failure of the case can be expected as a result of excessive deformation in the core. Although material composition and hardness values vary somewhat from country to country, typical values across the industry are, for the case, Knoop 690, and for the core, Knoop 400, corresponding to ultimate tensile strengths in excess of 300 ksi for the case and 200 ksi for the core.

As a lifting medium, hoisting chain is normally subjected to two types of loading, denoted as types I and II. Both generate complex stress distributions not only in the vicinity of link/link contact but also throughout the remainder of the link. Type I loading occurs as chain is drawn over the hoist load sprocket, or liftwheel (pocket wheel), and two adjacent links experience relative motion consisting of both rolling and sliding. Out-of-plane bending and frictional tractions characterize type I loading. On the other hand, type II loading is seen in those links which do not pass over the liftwheel. Link-to-link load transmission occurs in the absence of relative motion. No out-of-plane bending occurs, and since the links possess identical elastic constants, no frictional forces exist between the links which essentially operate as two-force members.

Chain Failure and Wear

Hoisting load chain is manufactured from wire of circular cross section. During fabrication each interconnecting link is formed and welded into an oval shape with toroidal ends. Thermal treatment consists of carburization, hardening and tempering. The final process calls for each chain segment to be pitched, i.e., stretched in the heat treated condition to establish a uniform pitch so the chain will track properly in the load sprocket. An ancillary benefit of the stretching operation is an improvement in fatigue performance which occurs since the applied force serves as a proof load.

Contact between adjacent links is made in the saddle shaped, or inter link, area of each end. Fatigue fractures of properly loaded and maintained chain seldom occur since stresses remote from the contact area are normally sufficiently below the endurance limit. Chain failure normally arises from excessive wear in the inter link area in those links which are subjected to mode I loading. The increase in inside length, or pitch, results in poor kinematic action between the chain and load sprocket, preventing proper seating of the chain in a pocket.

Wear has been classified (Rabinowicz, 1995) into the following four categories: adhesive, abrasive, corrosive and surface fatigue.

Adhesive Wear

When two bodies experience sliding contact, adhesive wear can be expected. Its origins are found in strong adhesive forces generated when atoms are sufficiently close. Rabinowicz describes adhesive wear as a process in which particles tear off one surface and subsequently adhere to the second surface. The resulting wear particle may become detached from the second surface and then reattach to the first or become part of the wear debris

as a loose particle. Hutchings (1992) describes several proposed mechanisms by which this wear particle can be generated; all are plasticity dominated.

Abrasive Wear

Abrasive wear is seen when when a hard, rough surface slides over a soft surface and hard surface asperities plow a series of grooves in the softer one. This type of wear is also encountered when hard particles are embedded in a soft matrix and the composite slides over a soft surface.

Corrosive Wear

Corrosive wear results when sliding occurs in a corrosive environment. The oxide film that normally forms on metallic surfaces and which offers at least partial protection against corrosion wears as a result of the sliding. As new oxide layers form and are, in turn, worn away, the corrosive attack continues.

Surface Fatigue Wear

Surface fatigue arises as the result of cyclic sliding and/or rolling motion between two surfaces. Under some conditions a subsurface crack may be nucleated; under others a surface crack may appear. It is the mechanics leading to surface fatigue wear in carburized hoisting load chain that constitute the focus of this investigation.

Scope

This study is focused on the displacement and stress distributions in the vicinity of contact between two smooth adjacent links. First, the elastic frictionless problem is examined, for which analytical solutions exist if the problem can be considered Hertzian

and if both the size and shape of the contact patch as well as the surface tractions are known. Second, the carburized case and core are modeled as nonlinear materials and comparisons drawn to the linear frictionless results. Third, residual stresses and their effects on shakedown are studied. Fourth, the effects of friction for nonlinear material models are examined, with the principal goal of demonstrating a decrease in the depth of maximum shear (or von Mises) stress in an attempt to help explain the pitting phenomenon often attributed to maximum von Mises or sub-surface shear stress. This analysis was carried out by applying a presumed traction to one link rather than by using contact elements, since their adequacy for three dimensional combined sliding and rolling is unclear and the computational cost is excessive for the benefit to be derived.

Additional goals of this work are validation of the superposition integral approach to calculating linear subsurface stresses and displacements as well as the development of some bounds for the use of Hertzian theory relative to the extents and curvatures of the contacting bodies.

ASSESSMENT OF ORIGINALITY

Contact problems have been extensively studied and numerical methods have enhanced our ability to examine increasingly complex problems. In recent years some work has been done on indentation and sliding on layered media. Much of this work examined somewhat artificial problems – artificial in the sense that application and extension of the results to a specific problem may not be immediately obvious. Much of the recent work has been two dimensional. Typically, geometry in these problems has consisted of a large radius spherical indenter sliding over a flat surface. Material models typically, though not always, exhibited perfectly plastic post yield behavior.

A specific three dimensional contact geometry is examined herein under conditions which would seemingly violate the size and curvature constraints of Hertzian theory. Also employed are multiple isotropic-hardening material models in a layered medium.

This work and its methodologies should have relevance for those wishing to investigate surface endurance of hoisting load chain or those studying contacting toroids in general. The results should also be of interest to those who design, specify or manufacture carburized hoisting load chain and are concerned with its surface strength.

2 LITERATURE REVIEW

HERTZ THEORY

Much of the following discussion is derived from Johnson, 1985 and Boresi et al, 1978. Assume that initial contact between two bodies (the subscripts 1 and 2 refer to bodies one and two, respectively, throughout this discussion) occurs at a point which is the origin of a coordinate system whose x-y plane is tangent to both surfaces and which is pictured in figure 1. Note that all tables and figures appear in the appendices. The z-axis is then directed into the lower body. Each body is assumed to be smooth on both a micro and a macro scale. Smoothness on the micro scale assures that there will be no surface irregularities or asperities to interrupt contact or to concentrate stress. On the other hand, smoothness on the macro scale allows each surface to be represented as an ellipsoid of the form

$$z_1 = A_1x^2 + B_1y^2 + C_1xy \quad [1]$$

With a judicious choice of x-y axis orientation the constant C_1 can be made zero by writing the equation as

$$z_1 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A_1 & \frac{C_1}{2} \\ \frac{C_1}{2} & B_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and letting $\begin{bmatrix} x_1 & y_1 \end{bmatrix}^T = \mathbf{K} \begin{bmatrix} x & y \end{bmatrix}^T$ where $\mathbf{K}$ is the matrix of eigenvectors of the center matrix and the superscript T indicates the transpose operation. This eliminates C_1 and allows the expression for z_1 to be rewritten as follows:

$$z_1 = \frac{1}{2R_1'}x_1^2 + \frac{1}{2R_1''}y_1^2$$

where R_1' and R_1'' are the principal radii of curvature of the surface at the origin. Since the z -axis is directed into the the lower body the corresponding expression for z_2 is given by

$$z_2 = - \left(\frac{1}{2R_2'} x_2^2 + \frac{1}{2R_2''} y_2^2 \right)$$

Separation between the surfaces is given as $h = z_1 - z_2$. There is an equation analogous to [1] but expressing the profile for surface two. Hence, the separation, h , can be expressed in a similar manner as:

$$h = \frac{1}{2R_1'} x_1^2 + \frac{1}{2R_1''} y_1^2 + \frac{1}{2R_2'} x_2^2 + \frac{1}{2R_2''} y_2^2$$

In the most general case the x_1 - y_1 axes and x_2 - y_2 axes will not be aligned, though they will be in the plane $z=0$. The following transformation gives x_2 - y_2 as functions of x_1 - y_1 .

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

where α is the angle of rotation between the two sets of axes. The separation can then be written as $h = A'x_1^2 + 2H'x_1y_1 + B'y_1^2$. Now the same eigenvalue diagonalization procedure can be applied to eliminate H' . The results are given by:

$$h = Ax^2 + By^2 \tag{2a}$$

$$B = \frac{1}{4} \left(\frac{1}{R_1'} + \frac{1}{R_1''} + \frac{1}{R_2'} + \frac{1}{R_2''} \right) + J \tag{2b}$$

$$A = \frac{1}{4} \left(\frac{1}{R_1'} + \frac{1}{R_1''} + \frac{1}{R_2'} + \frac{1}{R_2''} \right) - J \tag{2c}$$

$$J = \frac{1}{4} \sqrt{\left[\left(\frac{1}{R_1''} - \frac{1}{R_1'} \right) + \left(\frac{1}{R_2''} - \frac{1}{R_2'} \right) \right]^2 - 4 \left(\frac{1}{R_1''} - \frac{1}{R_1'} \right) \left(\frac{1}{R_2''} - \frac{1}{R_2'} \right) \sin^2 \alpha} \tag{2d}$$

Prior to load application equation [2a] represents the distance between two surface points whose x and y coordinates are equal. During deformation a remote point in body one approaches the origin, parallel to the z -axis, an amount equal to δ_1 . Similarly a remote point in body two, possessing the same x and y coordinates as the remote point in body one, approaches parallel to the z -axis by an amount δ_2 . Resulting from the loading, the surface of each body is deformed an amount equal to $\bar{u}_{z1}$ and $\bar{u}_{z2}$ (the bar over a variable indicates the variable's value at the surface – Johnson's notation). If points S_1 and S_2 are coincident after deformation then

$$\bar{u}_{z1} + \bar{u}_{z2} + h = \delta_1 + \delta_2$$

If we set $\delta = \delta_1 + \delta_2$ then using equations [2] the elastic displacements are written as

$$\bar{u}_{z1} + \bar{u}_{z2} = \delta - Ax^2 - By^2 \quad [3]$$

if S_1 and S_2 are within the contact zone and

$$\bar{u}_{z1} + \bar{u}_{z2} > \delta - Ax^2 - By^2 \quad [4]$$

if S_1 and S_2 lie outside the contact zone.

The elasticity problem is to find a surface pressure distribution such that equation [3] is satisfied within the contact patch and equation [4] is satisfied outside it. Hertz recognized that equation [3] has the same form as the Newtonian potential equation for the attraction of a constant density ellipsoidal mass on a unit mass located remotely. The Newtonian potential function satisfies equations of the same form as those necessary for an elasticity solution. The solution for the principal stresses along the principal axis (z -axis) were left in the form of Newtonian potentials. One of the principal results held that the surface

pressure distribution is semi-ellipsoidal and the shape of the contact patch is elliptical. The surface pressure distribution and peak normal pressure are thus given as

$$p(x, y) = p_0 \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad [5a]$$

$$p_0 = \frac{3P}{2\pi ab} \quad [5b]$$

where $p(x, y)$ is the pressure distribution, p_0 is the peak pressure, P is the applied force, a and b are the semi-major and semi-minor axes of the contact ellipse, respectively.

Becker (1999) has summarized the assumptions normally made in Hertzian contact problems:

- The bodies are homogeneous and isotropic. Stress and strain are linearly related and deformations, strains and rotations are 'small'.
- The surfaces which come into contact are smooth and, in the vicinity of contact, can be well approximated by a quadric, i.e., similar to equation [1].
- Contact is frictionless.
- Each of the bodies in contact can be treated as an elastic half-space. This implies that the contact patch is small in relation to other significant dimensions and permits the full body of elasticity theory to be used.
- After deformation dimensions of the contact ellipse, i.e., its semi-major and semi-minor axes, are small in comparison to the principal radii of curvature.

OTHER MECHANICS RESEARCHERS

Thomas and Hoersch (1930) were able to build on Hertz's original work by recasting the stresses along the principal axis in terms of standard elliptic integrals. They also devel-

oped an algorithm for calculating the contact ellipse shape, (b/a) , which, as it turns out, is only a function of geometry. Once the loading and materials' elastic properties are known, then the semi-major and semi-minor axes can be determined. Due to its importance the algorithm is presented below.

$$\frac{B}{A} = \frac{\left(\frac{1}{k^2}\right)E(k') - K(k')}{K(k') - E(k')}$$

$$k' = \sqrt{1 - k^2} \quad k = \frac{b}{a}$$

$$b = \sqrt[3]{\frac{3kE(k')}{2\pi}(P\Delta)}$$

$$M = \frac{2k^2}{k'^2E(k')}$$

$$\Delta = \frac{1}{A + B} \left(\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)$$

$$n = \sqrt{\frac{k^2 + k^2\left(\frac{z}{b}\right)^2}{1 + k^2\left(\frac{z}{b}\right)^2}}$$

$$\Omega_x = -\frac{1-n}{2} + k\frac{z}{b}[F(\phi, k') - H(\phi, k')]$$

$$\Omega_x' = -\frac{n}{k^2} + 1 + k\frac{z}{b}\left[\frac{1}{k^2}H(\phi, k') - F(\phi, k')\right]$$

$$\Omega_y = \frac{1}{2n} + \frac{1}{2} - \frac{n}{k^2} + k\frac{z}{b}\left[\frac{1}{k^2}H(\phi, k') - F(\phi, k')\right]$$

$$\Omega_y' = -1 + n + k\frac{z}{b}[F(\phi, k') - H(\phi, k')]$$

$$\sigma_{xx} = [M(\Omega_x + \nu\Omega_x')] \frac{b}{\Delta}$$

$$\sigma_{yy} = [M(\Omega_y + \nu\Omega_y')] \frac{b}{\Delta}$$

$$\sigma_{zz} = [M(\Omega_y + \nu\Omega_y')] \frac{b}{\Delta}$$

where A and B are defined in equations [2b] and [2c]. E and ν are Young's modulus and Poisson's ratio respectively. As before, a and b are the semi-major and semi-minor axes of the contact ellipse, respectively. The elliptic integrals are defined below.

$$F(\phi, k') = \int_0^\phi \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}$$

$$H(\phi, k') = \int_0^\phi \sqrt{1 - k'^2 \sin^2 \theta} d\theta$$

$$K(k') = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}$$

$$E(k') = \int_0^{\frac{\pi}{2}} \sqrt{1 - k'^2 \sin^2 \theta} d\theta$$

$$\phi = \cot^{-1} k \left(\frac{z}{b} \right)$$

Smith and Liu (1953) investigated plane stress and plane strain problems under combined normal and tangential loading. They assumed that the frictional traction, $q(x,y)$, could be represented as $q(x,y) = fp(x,y)$ where f was the coefficient of friction. They found that contrary to the case of normal loading alone in which the greatest shear stress occurs beneath the surface, that the effect of friction is to move the point of maximum shear stress closer to the surface, and when $f = 1/3$ the peak stress is found at the surface. Poritsky's work (1950) confirmed the results of Smith and Liu.

Kral and Komvopoulos (1996a) investigated the problem of a spherical indenter sliding along an elastic-plastic layered medium with finite element analysis. Large variations in the layers' elastic moduli were considered as was heavy loading, with peak loads being 100 and 200 times that required for initial yield of the substrate. Differences in the nor-

mal contact stress distribution between their study and the Hertzian problem were noted. The normal pressure distribution was found to be flatter and peak values were found near the edge of contact rather than along the normal load line.

THE ELASTICITY APPROACH

Several elasticity solutions to the problem of a concentrated force acting on an elastic half-space were developed in the late 19th century. These solutions prove useful in the displacement and stress analysis of contacting bodies, as they form the basis for superposition integrals.

The Solutions of Flamant, Boussinesq and Cerruti

One method of calculating elastic contact stresses is superposition. In 1892 Flamant developed a solution for a point load normal to the surface of a two dimensional half space by consideration of an Airy stress function, χ , of the form $\chi = Cr\theta\sin(\theta)$. The stress components in cylindrical coordinates were shown to be

$$\sigma_r = \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} = -\frac{2P}{\pi} \frac{\cos(\theta)}{r}$$

$$\sigma_\theta = \frac{\partial^2 \chi}{\partial r^2} = 0$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial \theta} \right) = 0$$

When transformed from polar into Cartesian planar coordinates with the x-axis directed to the right and the z-axis directed into the body below, the stress components are found to be

$$\sigma_x = -\frac{2P}{\pi} \frac{x^2 z}{(x^2 + z^2)^2}$$

$$\sigma_z = -\frac{2P}{\pi} \frac{z^3}{(x^2 + z^2)^2}$$

$$\tau_{zx} = -\frac{2P}{\pi} \frac{xz^2}{(x^2 + z^2)^2}$$

The three dimensional problem is solved in a similar manner to the two dimensional case. Love (1927) and Sokolnikoff (1956) reproduce the Boussinesq solution for the displacement field of an elastic half space in response to a normal point load applied at the origin. Displacements are given as

$$u_x = \frac{P}{4\pi\mu} \frac{xz}{\varrho^3} - \frac{P}{4\pi(\lambda + \mu)} \frac{x}{\varrho(\varrho + z)} \quad [6a]$$

$$u_y = \frac{P}{4\pi\mu} \frac{yz}{\varrho^3} - \frac{P}{4\pi(\lambda + \mu)} \frac{y}{\varrho(\varrho + z)} \quad [6b]$$

$$u_z = \frac{P}{4\pi\mu} \frac{z^2}{\varrho^3} + \frac{P(\lambda + 2\mu)}{4\pi\mu(\lambda + \mu)} \frac{1}{\varrho} \quad [6c]$$

where λ and μ are the Lamé constants and $\varrho^2 = x^2 + y^2 + z^2$. Since Young's modulus and Poisson's ratio are a frequently used set of elastic constants, λ and μ may be expressed in terms of them.

$$\lambda = \frac{Ev}{(1 + \nu)(1 - 2\nu)} \quad \mu = \frac{E}{2(1 + \nu)}$$

where E is Young's modulus and ν is Poisson's ratio.

The strains are determined by differentiating equations [6]. Stresses are then found from the strains to be

$$\sigma_{ij} = \lambda \varepsilon_{ii} \delta_{ij} + 2\mu \varepsilon_{ij}$$

where δ_{ij} is the Kronecker delta, $\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, $i, j = 1, 2, 3$, and indicial notation is used. Johnson (1985) and Hills, et al. (1993) calculate the stress components to be

$$\sigma_x = \frac{P}{2\pi} \left[\left(\frac{1-2\nu}{r^2} \right) \left\{ \left(1 - \frac{z}{Q} \right) \frac{x^2 - y^2}{r^2} + \frac{zy^2}{Q^3} \right\} - \frac{3zx^2}{Q^5} \right] \quad [7a]$$

$$\sigma_y = \frac{P}{2\pi} \left[\left(\frac{1-2\nu}{r^2} \right) \left\{ \left(1 - \frac{z}{Q} \right) \frac{y^2 - x^2}{r^2} + \frac{zx^2}{Q^3} \right\} - \frac{3zy^2}{Q^5} \right] \quad [7b]$$

$$\sigma_z = - \frac{3P}{2\pi} \frac{z^3}{Q^5} \quad [7c]$$

$$\tau_{xy} = \frac{P}{2\pi} \left[\left(\frac{1-2\nu}{r^2} \right) \left\{ \left(1 - \frac{z}{Q} \right) \frac{xy}{r^2} - \frac{xyz}{Q^3} \right\} - \frac{3xyz}{Q^5} \right] \quad [7d]$$

$$\tau_{xz} = - \frac{3P}{2\pi} \frac{xz^2}{Q^5} \quad [7e]$$

$$\tau_{yz} = - \frac{3P}{2\pi} \frac{yz^2}{Q^5} \quad [7f]$$

where $r^2 = x^2 + y^2$.

In addition to the normal loading considered above, frictional forces play an important role in contact mechanics. Hence, it is equally desirable to express the displacement and stress responses to tangential loading in general three dimensional contact problems. If the tangential surface traction is given by $q(x, y)$ it is common to let $q(x, y) = fp(x, y)$, where f is the coefficient of friction. For the purposes of this discussion we assume that sliding occurs only in the x -direction.

In 1886 Cerutti developed an elasticity solution to the problem of a tangential point force acting on an elastic half space. If that force, denoted by Q_x , operates in the x -direction then the resulting displacements are found to be

$$u_x = \frac{Q_x}{4\pi\mu} \left\{ \frac{1}{\varrho} + \frac{x^2}{\varrho^3} + (1 - 2\nu) \left[\frac{1}{\varrho + z} - \frac{x^2}{\varrho(\varrho + z)^2} \right] \right\} \quad [8a]$$

$$u_y = \frac{Q_x}{4\pi\mu} \left\{ \frac{xy}{\varrho^3} - (1 - 2\nu) \frac{xy}{\varrho(\varrho + z)^2} \right\} \quad [8b]$$

$$u_z = \frac{Q_x}{4\pi\mu} \left\{ \frac{xz}{\varrho^3} - (1 - 2\nu) \frac{x}{\varrho(\varrho + z)} \right\} \quad [8c]$$

Stress results are presented by Johnson in a slightly different form than that previously used for the response to a normal point force. They are

$$\frac{2\pi\sigma_x}{Q_x} = -\frac{3x^3}{\varrho^5} + (1 - 2\nu) \left\{ \frac{x}{\varrho^3} - \frac{3x}{\varrho(\varrho + z)^2} + \frac{x^3}{\varrho^3(\varrho + z)^2} + \frac{2x^3}{\varrho^2(\varrho + z)^3} \right\} \quad [9a]$$

$$\frac{2\pi\sigma_y}{Q_x} = -\frac{3xy^2}{\varrho^5} + (1 - 2\nu) \left\{ \frac{x}{\varrho^3} - \frac{x}{\varrho(\varrho + z)^2} + \frac{xy^2}{\varrho^3(\varrho + z)^2} + \frac{2xy^2}{\varrho^2(\varrho + z)^3} \right\} \quad [9b]$$

$$\frac{2\pi\sigma_z}{Q_x} = -\frac{3xz^2}{\varrho^5} \quad [9c]$$

$$\frac{2\pi\tau_{xy}}{Q_x} = -\frac{3x^2y}{\varrho^5} + (1 - 2\nu) \left\{ -\frac{y}{\varrho(\varrho + z)^2} + \frac{x^2y}{\varrho^3(\varrho + z)^2} + \frac{2x^2y}{\varrho^2(\varrho + z)^3} \right\} \quad [9d]$$

$$\frac{2\pi\tau_{yz}}{Q_x} = -\frac{3xyz}{\varrho^5} \quad [9e]$$

$$\frac{2\pi\tau_{zx}}{Q_x} = -\frac{3x^2z}{\varrho^5} \quad [9f]$$

Superposition Integrals

Since point loads only exist in the abstract, equations [6] through [9] are perhaps best used to calculate elastic displacements and stresses by superposition. Since integration must be performed over the contact ellipse, its extent as well as the pressure distribution must be known a priori. If contact can be considered Hertzian then the method of Thomas

and Hoersch may be used to determine the dimensions of the contact ellipse and surface pressure distribution. In general the integrals cannot be evaluated in closed form and a numerical scheme must be employed.

In forming the superposition integrals it is necessary to replace the differential force dP with $p(\xi, \eta)d\xi d\eta$, and the coordinates x, y with $(x - \xi), (y - \eta)$ respectively, where ξ, η are the dummy integration variables used to denote surface coordinates ($z = 0$) of the differential force $dp(\xi, \eta) d\xi d\eta$ relative to the fixed x - y - z coordinate system, where

$$p(\xi, \eta) = p_0 \sqrt{1 - \left(\frac{\xi}{a}\right)^2 - \left(\frac{\eta}{b}\right)^2}$$

Since the z -coordinate is common to both x - y and ξ - η it is unmodified. A sample integral is presented to illustrate the process.

$$\sigma_{xx} = \frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{(1-2\nu)}{r'^2} \left\{ \left(1 - \frac{z}{\varrho'}\right) \frac{x'^2 - y'^2}{r'^2} + \frac{zy'^2}{\varrho'^3} \right\} - \frac{3zx'^2}{\varrho'^5} \right] d\xi d\eta$$

where $r'^2 = (x - \xi)^2 + (y - \eta)^2$, $x' = x - \xi$, $y' = y - \eta$, $\varrho'^2 = (x - \xi)^2 + (y - \eta)^2 + z^2$, and integration is carried out over the contact patch. It will be seen that agreement between these integrals and the Sackfield, et al. (1983) solution is quite good for the responses to both normal and tangential loading.

It should be noted that when equations [6] were differentiated to obtain the strains and the stresses subsequently obtained from equations [7] an expression for τ_{xy} was derived which differs from that of both Johnson (1985) and Hills, et al. (1993). This new expression agrees quite well with the Sackfield (1983) solution. Thus, τ_{xy} in response to normal loading is given below.

$$\tau_{xy} = \frac{P}{2\pi} \left[(1 - 2\nu) \frac{xy(2q + z)}{q^3(q + z)^2} - \frac{3xyz}{q^5} \right] \quad [10]$$

Sackfield and Hills Solution

Much work has been done in the field of contact mechanics to develop the displacement and stress fields in general three dimensional linear contact problems of bodies with non-conforming or counterformal surfaces. In 1983 Sackfield and Hills developed a potential theory-based method for calculating the contact stress field in response to both normal and tangential semi-ellipsoidal surface pressure distributions in a semi-infinite body. Application of the algorithm requires a priori knowledge of the size and shape of the contact patch. Outputs are normalized with respect to peak surface pressures, but in order that the results be useful, the peak pressures must also be known.

The algorithm for determining the stress state in response to normal loading inside the contact ellipse is presented below. As before, a and b are the semi-major and semi-minor axes of the contact ellipse respectively, P is the applied force, p_0 is the peak normal pressure and is numerically equal to $3P/(2\pi ab)$, k is the ratio b/a , $k' = (1-k^2)^{1/2}$, x, y, z are the coordinates of the point of interest and $x^*, y^*,$ and z^* are the same coordinates normalized to the semi-major axis, so that, e.g., $y^* = y/a$. Intermediate values are defined as necessary. γ is the largest root of

$$1 - \frac{x^{*2}}{a^2 + \gamma} - \frac{y^{*2}}{b^2 + \gamma} - \frac{z^{*2}}{\gamma} = 0$$

$$s = \sqrt{\frac{Y}{a^2}}$$

$$H = \left(\frac{x^*}{1 + s^2} \right)^2 + \left(\frac{y^*}{k^2 + s^2} \right)^2 + \left(\frac{z^*}{s^2} \right)^2$$

$$L = \frac{z^*}{s^3 H \sqrt{(1 + s^2)(k^2 + s^2)}}$$

This is sufficient to calculate σ_{zz} , τ_{xz} and τ_{yz} which are given by

$$\sigma_{zz} = - \frac{p_0 k z^{*2} L}{s^2}$$

$$\tau_{xz} = - \frac{p_0 k x^* z^* L}{1 + s^2}$$

$$\tau_{yz} = - \frac{p_0 k y^* z^* L}{k^2 + s^2}$$

If p is the smallest root of

$$1 - \frac{x^{*2}}{1 - p^2} - \frac{y^{*2}}{k^2 - p^2} = 0$$

then the remaining stress components can be found from

$$\alpha = \sqrt{\frac{(s^2 + p^2)(k^2 - p^2)^2}{(k^2 - p^2)(s^2 + k^2) + k'^2 y^{*2}}}$$

$$X = \frac{x^* k'}{1 - p^2}$$

$$Y = \frac{y^* k'}{k^2 - p^2}$$

$$\theta_1 = \tan^{-1} \left[\frac{Y(\alpha - 1)}{1 + Y^2 \alpha} \right]$$

$$\theta_2 = \tanh^{-1} \left[\frac{X(\alpha - 1)}{1 - X^2 \alpha} \right]$$

$$\chi_{12} = \frac{2\pi p_0 k}{k'^3} (x^* \theta_1 - y^* \theta_2)$$

$$\chi_{22} = -2\pi p_0 k \left[z * I_2 + \frac{1}{k'^3} \left(y * \theta_1 + x * \theta_2 + k' - \frac{z * k'}{s} \sqrt{\frac{s^2 + 1}{s^2 + k^2}} \right) \right]$$

$$\chi_{11} = -2\pi p_0 k \left[z * I_1 - \frac{1}{k'^3} \left(y * \theta_1 + x * \theta_2 + k' - \frac{z * k'}{s} \sqrt{\frac{s^2 + k^2}{s^2 + 1}} \right) \right]$$

$$\phi_3 = -2\pi p_0 k z * I_3$$

$$\phi_{12} = \frac{2\pi p_0 k x * y * L s^2}{z * (1 + s^2)(k^2 + s^2)}$$

$$\phi_{22} = -\frac{2\pi p_0 k}{z *} \left[z * I_2 - \left(\frac{y * s}{k^2 + s^2} \right)^2 L \right]$$

$$\phi_{11} = -\frac{2\pi p_0 k}{z *} \left[z * I_1 - \left(\frac{x * s}{1 + s^2} \right)^2 L \right]$$

The quantities I_1 , I_2 and I_3 above are the integrals defined as

$$I_1 = \int_s^\infty \frac{dw}{(1 + w^2)^{\frac{3}{2}}(k^2 + w^2)^{\frac{1}{2}}} \quad I_2 = \int_s^\infty \frac{dw}{(1 + w^2)^{\frac{1}{2}}(k^2 + w^2)^{\frac{3}{2}}} \quad I_3 = \int_s^\infty \frac{dw}{w^2(1 + w^2)^{\frac{1}{2}}(k^2 + w^2)^{\frac{1}{2}}}$$

Thomas and Hoersch (1930) showed that these integrals can be represented as functions of standard elliptic integrals as follows

$$I_1 = \frac{1}{k'^2} [F(\phi, k') - E(\phi, k')]$$

$$I_2 = \frac{1}{k'^2} \left[\left(\frac{1}{k^2} \right) E(\phi, k') - F(\phi, k') \right] - \frac{\tau}{k^2 k'^2 s} + \frac{1}{k'^2 s \tau}$$

$$I_3 = \frac{\tau}{k^2 s} - \frac{E(\phi, k')}{k^2}$$

$$\text{where} \quad \tau = \sqrt{\frac{k^2 + s^2}{1 + s^2}}$$

The solution for the remaining stress components is shown to be

$$\sigma_{xx} = p_0 \left[\frac{\nu \phi_3}{\pi p_0} - \frac{1 - 2\nu \chi_{11}}{2\pi p_0} - \frac{1}{2\pi} \left(\frac{z^* \phi_{11}}{p_0} \right) \right]$$

$$\sigma_{yy} = p_0 \left[\frac{\nu \phi_3}{\pi p_0} - \frac{1 - 2\nu \chi_{22}}{2\pi p_0} - \frac{1}{2\pi} \left(\frac{z^* \phi_{22}}{p_0} \right) \right]$$

$$\tau_{xy} = - p_0 \left[\frac{1 - 2\nu \chi_{12}}{2\pi p_0} + \frac{1}{2\pi} \left(\frac{z^* \phi_{12}}{p_0} \right) \right]$$

3 RESIDUAL STRESSES AND SHAKEDOWN

Under repeated contact loading many surface and subsurface layers show evidence of plastic deformation. Typical rolling elements are railway rails and bearings of both the ball and roller type. Surface fatigue wear as described by Rabinowicz is perhaps the single most common failure mode experienced by such elements.

There is an increasing body of evidence that residual stresses initiated by plastic deformation play a major role in surface crack propagation (Ponter et al, 1984). After the first application of a contact load which causes plastic deformation, three effects, each tending to oppose plastic strain accumulation, occur. The first is the introduction of residual stresses into the component which has been so stressed. The second is strain hardening of the subject material. The third occurs as the initial plastic deformation increases the area of contact, thereby reducing the peak pressure and thus the contact stresses in subsequent load applications. All three effects contribute to shakedown. However, only the first of these will be examined.

Melan's theorem, also known as the statical theory, discussed separately herein, provides a lower (safe) bound for the true shakedown limit while Koiter's kinematical theory, provides an upper bound. The kinematical theory, which will not be discussed further and is included only for completeness, states (Johnson, 1985) that shakedown will not occur if "any kinematically admissible cycle of plastic strain can be found in which the work done by the external load exceeds the internal plastic work."

The concept of shakedown is perhaps best understood by reviewing the following discussion taken from Johnson's (1985) analysis of rolling contact. For analytical purposes it is assumed that material behavior is linear up to the yield strengths Y in tension or com-

pression and k_s in simple shear. Post yield behavior is assumed to be perfectly plastic. Motion is confined to the x -direction. The distribution of pressure normal to the surface is assumed to be given by the standard Hertz equations and is independent of the tangential loading.

THE ONSET OF YIELD

Yield in free rolling is identical to yield in frictionless normal contact. It first appears below the surface at a depth determined by the yield criterion used and by geometry factors and occurs when the maximum contact pressure, p_0 , is approximately $1.6Y$.

In tractive rolling, friction at the surface influences the point where yielding initiates. If complete slip occurs then $Q_x = fP$ and the point of initial yielding approaches the surface as the coefficient of friction approaches about $1/3$.

However, if $Q < fP$ then micro-slip occurs only over the rear portion of the contact patch. If the traction coefficient, defined as the ratio Q_x/P , is held constant, variations in f cause the micro-slip zone to change in size and the distribution of tangential traction to vary, with peak values becoming larger as f increases. An ancillary though not unexpected result of increasing friction is that the contact pressure required to cause initial yielding decreases.

REPEATED ROLLING - SHAKEDOWN

Most contact applications involve repeated applications of the service load. If the components are originally unstressed and if the stress state is purely elastic then upon unloading there will be no residual stresses. However, if upon the first load application the elastic limit is exceeded then after load removal some residual stresses will be present. After

the second load application the material is subjected to the combination of both residual stresses and load induced stresses. Not infrequently the residual stresses are protective in the sense that they act to oppose the load induced stresses and hence make yielding less likely. Indeed it is possible that after several load/unload cycles the residual stresses are of such values that when they are added to the load induced stresses the material remains elastic. When this happens shakedown is said to have occurred.

To illustrate the process, the case of a cylinder freely rolling over an elastic half space is considered. The assumption of uniform plane deformation, in which $u_z = 0$ and both u_x and u_y are independent of z , is made. According to Malvern (1969) plane deformation is approximated by a prismatic body with boundary conditions on its lateral surfaces which are independent of z . An elastic half space is given as an example that meets this criterion.

Johnson has invoked Melan's theorem which states that "if any time-independent distribution of residual stresses can be found which, together with the elastic stresses due to the load, constitute a system of stresses within the elastic limit, then the system will shakedown." Conversely, if no such residual stresses can be found, then the system will not shakedown and plastic deformation will occur with each subsequent load application. This condition is known as ratcheting. In what follows, we discuss Johnson's application of Melan's theorem under plane rolling conditions to derive the shakedown limit.

A standard right hand coordinate system in which the z -axis is directed into the elastic half space and the x -axis directed to the right parallel to the motion is chosen. In this case the assumption of uniform plane deformation eliminates both $(\tau_{xy})_r$ and $(\tau_{yz})_r$ since u_x is independent of y , u_z is identically zero and u_y is independent of both x and z . The sub-

script r indicates a residual value. Since the deformation is considered uniform the remaining stress components must be independent of both x and y. Since the residual stresses must be in equilibrium with surface free of traction, $(\sigma_z)_r$ and $(\tau_{zx})_r$ must also be zero. Thus the only possible system of residual stresses must therefore be:

$$\begin{aligned}(\sigma_x)_r &= f_1(z) & (\sigma_y)_r &= f_2(z) \\ (\sigma_z)_r &= (\tau_{xy})_r = (\tau_{yz})_r = (\tau_{zx})_r = 0\end{aligned}$$

Due to the combination of residual stresses and service stresses, the principal stresses are found to be:

$$\begin{aligned}\sigma_1 &= \frac{1}{2}[\sigma_x + (\sigma_x)_r + \sigma_z] + \frac{1}{2}\sqrt{[\sigma_x + (\sigma_x)_r - \sigma_z]^2 + 4\tau_{zx}^2} \\ \sigma_2 &= \frac{1}{2}[\sigma_x + (\sigma_x)_r + \sigma_z] - \frac{1}{2}\sqrt{[\sigma_x + (\sigma_x)_r - \sigma_z]^2 + 4\tau_{zx}^2} \\ \sigma_3 &= \nu\{\sigma_x + \sigma_z\} + (\sigma_y)_r\end{aligned}$$

Melan's theorem allows the residual stresses at any depth to take on any value to avoid yield. $(\sigma_y)_r$ can be chosen so that σ_3 becomes the intermediate principal stress. Considering a Tresca material yield can thus be prevented if

$$\frac{1}{4}(\sigma_1 - \sigma_2)^2 \leq k_s^2$$

In terms of stress components the yield criterion is

$$\frac{1}{4}[\sigma_x + (\sigma_x)_r - \sigma_z]^2 + \tau_{zx}^2 \leq k_s^2$$

The inequality cannot be satisfied if $\tau_{zx} > k_s$. However, it can be satisfied if $\tau_{zx} = k_s$. In this case the only residual stress component $(\sigma_x)_r$ can be chosen such that $(\sigma_x)_r = \sigma_z - \sigma_x$.

An important feature of Melan's theorem is that it is unnecessary to find the precise system of residual stresses that develops. Elastic shakedown will occur if any set of self-equilibrating residual stresses can be found that meets Melan's criterion.

McEwen (1949) derived the following closed form expressions for the stress components at a general point in frictionless line contact such as occurs in free rolling.

$$\sigma_x = -\frac{P_0}{a} \left\{ m \left(1 + \frac{z^2 + n^2}{m^2 + n^2} \right) - 2z \right\}$$

$$\sigma_z = -\frac{P_0}{a} m \left(1 - \frac{z^2 + n^2}{m^2 + n^2} \right)$$

$$\tau_{zx} = -\frac{P_0}{a} n \left(\frac{m^2 - z^2}{m^2 + n^2} \right)$$

$$m^2 = \frac{1}{2} \left[\sqrt{(a^2 - x^2 + z^2) + 4x^2z^2} + (a^2 - x^2 + z^2) \right]$$

$$n^2 = \frac{1}{2} \left[\sqrt{(a^2 - x^2 + z^2) + 4x^2z^2} - (a^2 - x^2 + z^2) \right]$$

In these equations the variable 'a' represents the semi-width of contact while all others retain their usual interpretations. Johnson has found the maximum value of τ_{zx} , which occurs at $x = .87a$ and $z = .50a$, to be $.25p_0$. Thus elastic shakedown can be expected as long as $p_0 \leq 4.0k_s$.

Identical results are found when the von Mises yield criterion is used. However, application of von Mises yield criterion predicts initial yielding whenever $p_0 \geq 3.1k_s$. Constructing the ratio of the shakedown limit load to the limit load for first yield it is seen that

$$\frac{P_{sh}}{P_y} = \frac{(P_0)_{sh}^2}{(P_0)_y^2} = \left(\frac{4.0}{3.1} \right)^2 = 1.665$$

since for line contact $P \propto p_0^2$. Thus the applied load must be increased 66.5% over that necessary for first yield before the shakedown limit is exceeded. It is worth noting that initial yielding occurs along the load line at a point where no shear stresses are present. Thus, shakedown is governed by the stress state at a point which is not the first to yield.

Three dimensional shakedown analysis of frictionless contact is more difficult since all six components of residual stress are generally present and plane deformation conditions are absent. Johnson considers the plane of symmetry ($y=0$) of a sphere freely rolling over an elastic-plastic half space. Symmetry considerations require $(\tau_{xy})_r = (\tau_{yz})_r = 0$. However, $(\tau_{zx})_r$ can be present. Johnson and Jeffries (1963) suggested that shakedown in point contact such as would occur with a sphere freely rolling over an elastic-plastic half space would be controlled by $(\tau_{zx})_{\max}$ in the plane of symmetry ($y=0$). Fessler and Ollerton (1957) determined that the maximum orthogonal shear stress lay in the plane $y=0$, so the conclusion seems reasonable. That suggestion was later reiterated by Johnson (1985). For a circular contact patch of radius 'a', Hills and Ashelby(1981) found that τ_{zx} takes on its extreme values of $\pm .214p_0$ at $(x,z) = (\pm .848a, .351a)$. For a Tresca or von Mises material the elastic shakedown limit is given by $p_0 = 4.675k_s$.

Initial yield, however, occurs along the load line where no shear stresses are present. The value of p_0 for first yield is $2.8k_s$. Since for three dimensional contact the applied force is proportional to the cube of p_0 , the ratio of the shakedown limit load to the load necessary for initial yield is significantly greater than that in line contact. This is due to the greater amount of material under stress in line contact. Thus

$$\frac{P_{sh}}{P_y} = \left(\frac{4.67k_s}{2.8k_s} \right)^3 = 4.65$$

Johnson (1985) discussed qualitatively the mechanism of elastic shakedown in free rolling of a point contact and the reason why shakedown is governed by the maximum orthogonal shear stress τ_{zx} . Initial yielding occurs at a point directly beneath the load. The principal stresses are all compressive, but the largest is perpendicular to the surface. Hence, expansion occurs parallel to the surface. However, since each element at that depth is similarly deformed, a residual compressive stress develops along a line parallel to the direction of rolling. After the development of these residual stresses compression of the surface stops. The orthogonal shear stress τ_{zx} cannot be mitigated by the introduction of a constant residual stress, $(\tau_{zx})_r$, because its extreme values alternate on either side of the load line. Johnson concluded that the orthogonal shear stress determines the shakedown limit and the accompanying plastic deformation which follows each load application.

When the kinematical theorem was applied to the case of a cylinder freely rolling over a half-space the identical result was obtained as that when the statical theory was used. The result that $\tau_{zx} = .25p_0$ is thus exact since it is both an upper and a lower bound.

Hills and Ashelby (1981) examined shakedown in an elliptical contact patch with friction acting in the x -direction. Finite element analyses performed by Rydholm and Fredriksson (1978) suggest a shakedown limit of $p_0/k_s = f$.

4 METHOD OF APPROACH

LINEAR PROBLEM

A. A. Becker (1999) described the conditions found in the classic Hertzian contact problem. If these requirements are met, then at least the linearly elastic stress state in response to normal loading can be calculated using either the superposition integrals or the Sackfield solution. However, it is unclear that hoisting load chain geometry conforms to either the size or curvature requirements stipulated in the Hertz problem. That is yet to be determined.

It was noted earlier that at least some of the chain links experience relative motion composed of both rolling and sliding. The presence of friction can be inferred, thus surely violating one of the conditions of Hertzian contact. If thermoelastic phenomena due to frictional heating are ignored, we must still contend with the effects, if any, of the tangential load on the size and shape of the contact patch.

Johnson (1985) addresses this issue with the following discussion. He notes that $\bar{u}_z$ in response to a tangential load is proportional to $(1 - 2\nu)/\mu$ and that the tangential tractions acting on each surface are equal in magnitude but opposite in sign. Therefore the normal displacements are proportional to the values of $(1 - 2\nu)/\mu$ in each body and are opposite in direction. Thus

$$\frac{\mu_1}{1 - 2\nu_1} \bar{u}_{z1}(x, y) = - \frac{\mu_2}{1 - 2\nu_2} \bar{u}_{z2}(x, y)$$

If the bodies have identical elastic constants, as load chain does, then the normal displacements must be equal and opposite. The shape and size of the contact patch is then determined by the contour of the surfaces and the normal force. Johnson notes further

that if the bodies possess different elastic constants, then the tangential load does indeed have an effect on the contact patch, although for moderate values of f the effect is small.

It is reasonable to assume that, given Hertzian contact, the stress state induced by the addition of a frictional traction will be the superposition of the responses to normal loading and tangential loading.

The foregoing discussion is predicated upon the normal loading, acting alone, producing a Hertzian surface pressure. However, due to the large curvature (small radius of curvature) present in chain contact problems it is necessary to establish this result. There are no closed form solutions to the problem, so verification is made with the finite element method.

It is a matter of common experience among finite element analysts that acceptance of results is frequently a matter of judgement, more so if there is no known solution against which a comparison can be made. For contacting chain links this is the case.

Contact analysis of normal loading using linear material properties has three goals. The first is to establish the size and shape of the contact patch. The second is to determine the surface pressure distribution. It would be desirable if the surface pressure distribution and contact patch can be adequately modeled by Hertzian theory, since this would allow the elastic stresses to be calculated directly. Linear analysis could show whether yielding occurs by selection and evaluation of a yield criterion. If yielding is not present then there would be no need to perform analysis with nonlinear materials. The third is to establish a mesh which provides an acceptable solution time so that once adequate linear results will have been obtained, subsequent analyses with nonlinear materials employing the same mesh will be credible.

FINITE ELEMENT ANALYSIS CONVERGENCE STUDY

When no theoretical result is available against which finite element results can be compared, a convergence study is frequently performed to assess suitability and adequacy of the analysis. In this case u_z^* and σ_z at the point of load application ($x=0.0$ inches, $y=0.0$ inches, $z=0.0$ inches) and directly beneath the load at coordinates $x=0.0$ inches, $y=0.0$ inches, $z=0.156$ inches were chosen as appropriate. The volume of that element which contains $(x,y,z)=(0,0,0)$ was selected as the independent variable. It should be noted that the Hertz problem presumes a semi-infinite body, with $u_z|_{z=\infty} = 0$. u_z^* is the load line compression that occurs, i.e., $u_z^* = u_z - u_z|_{z=OD}$, where OD is the diameter of the wire from which the chain link is made.

NONLINEAR PROBLEM

Given that analyses with nonlinear materials are necessary, the same mesh used to validate linear results was employed. Two element groups, differing only in material properties, for each link were developed. The carburized case was modeled with bilinear material properties, .020 inches thick, with yield strength and tangent modulus of 300 ksi and 2,000 ksi respectively. Young's modulus and Poisson's ratio are assumed to be 28,000 ksi and .28 respectively.

It should be noted that case thickness is defined as that depth at which hardness is HRC 50, which corresponds to a yield strength of about 250 ksi. Since the yield strength of material at the surface is about 300 ksi, some hardness gradient exists in a real material. In addition, the peak hardness may not occur at the surface, but may actually lie slightly below it due to material or process variables, such as decarburization in the parent material or excessive delay in material transport to the quench medium from the furnace. A better material profile could possibly be developed to model these variations.

The core was modeled as a Ramberg–Osgood material with a yield strength of 185 ksi and α of 1.5 and hardening exponent of 7.

The materials were assumed to exhibit von Mises yield behavior, with isotropic hardening and an associated flow rule which assumes plastic flow to occur incompressibly ($\nu = .5$). Total Lagrangian formulation and full integration were employed. Node-to-surface gap elements were selected to model the contact.

Since the material models can accommodate inelastic behavior and the problem runs in a nonlinear module, residual stresses were determined by use of the time curve. Force control was selected as preferable to either displacement or arc length control.

FRICTIONAL MODEL

As noted earlier it is desired to demonstrate a decrease in the depth of maximum subsurface shear stress with friction present. Since the size and shape of the contact area will have been determined and the pressure distribution calculated, it is then a simple, though tedious, task to apply surface tractions, to each element face in the contact patch. Johnson has demonstrated that for materials with identical elastic constants and subjected to a normal load neither the contact patch shape nor surface pressure distribution is affected by the presence of a frictional traction. Assuming that to be true, it is unnecessary to use contact elements in the finite element formulation; nor is it required that the nonlinear module be used for analysis of the linear material. However, the presence of friction requires a one quarter model rather than a one eighth model since one plane of symmetry is lost.

As long as the friction coefficient is modeled parametrically, then analysis can be easily performed for several coefficients of friction. However, COSMOS/M is unable to model

three dimensional contact/friction problems with combined rolling and sliding. At least for the linear problem then, combined normal and tangential tractions can be examined. It is possible to model nonlinear material behavior by applying one load which contains both the normal and tangential traction. However, the ability to make a parametric model is somewhat compromised.

RESIDUAL STRESS AND SHAKEDOWN

In nonlinear finite element problems, loads are applied incrementally through the use of time curves in which the applied traction is the ordinate and pseudo-time is the abscissa. Assuming an initial stress-free state a final state is determined when the load reaches 1 (100% of the applied load) and pseudo-time reaches 1 (an arbitrary user-assigned value). Thus at a pseudo-time of 1 all displacements and stresses are known.

Since the yield locus is retained it is also possible to subsequently reduce the load to any value (in this case zero) by modifying the time curve so that at pseudo-time equal to 2 the load is zero, i.e., the time curve is triangular. The resulting displacements and stresses then are all residual values.

Then by setting the load to 1 at pseudo-time equal to 3 we have effectively reapplied the load beginning with an initial state whose displacements and stresses are residual values. We thus have a mechanism for evaluating shakedown. It is a simple matter to compare the stress states after the first load application to that after the second. If the stresses do not increase after the second load application we can conclude that the system has shaken down. If the loads are not excessive shakedown frequently occurs after a few load applications. In many instances the stresses after the second load application are actually less than those after the first. Of course not all systems that shake down will do so after

the first load cycle, but it is common for shakedown to occur after only three or four cycles.

5 PITTING

Quite apart from shakedown, pitting failures often result when a preexisting surface crack propagates as the result of repeated rolling/sliding contact under heavy load. Perhaps the first serious work on the subject was conducted by Stewart Way (1934) at Westinghouse. There were four principal conclusions:

- Pitting will not occur without lubricant
- Pits develop from cracks that start at the surface
- Pitting can be prevented by using oil of sufficiently high viscosity
- Hard and smooth surfaces better resist pitting than soft and rough surfaces

The theory was proposed that pitting originates from small surface cracks which fill with oil and exert a hydraulic pressure on a crack's side walls. However, Way was careful to avoid saying that oil definitely causes pitting in rolling/sliding contact, although the implications in his paper are compelling. His theory does seem to explain the observed phenomena.

Recent work in the field has examined the pit formation process under the combined action of Hertzian and hydraulic loading in combined rolling and sliding contact. Murakami, Sakae and Ichimaru (1994) found that arrow-headed cracks first propagate in shear then in tension (Mode I) due to the hydraulic action of oil on the crack surfaces. They also found that the shape of the arrow-headed crack provides an estimate of the operating service loads. Hsia and Xu (1996) and later Xu and Hsia (1997) developed a mathematical framework for surface crack propagation under the combined action of Hertzian and hydraulic loadings. They created the concept of a characteristic penetration time which is useful in establishing whether or not the hydraulic loading has a meaningful effect on

propagation of the surface crack by determining whether or not pressure variations at the crack mouth are transmitted sufficiently deep into the crack. Their work, though, was limited to linearly elastic materials.

Mura and Nakasone (1990) proposed a theory of fatigue crack initiation which relies on a critical value of the Gibbs free energy being attained relative to the number of cycles. At this point initiation is said to have occurred. This theory was used by Glodez and Flasket in concert with T. Hellen's (1975) method of virtual crack extensions to predict pitting resistance and crack propagation in well-lubricated spur gears experiencing combined rolling and sliding. In their model the total number of cycles to pit formation is the sum of initiation cycles and propagation cycles. The number of propagation cycles is the sum of short crack propagation and long crack propagation cycles. Short crack propagation employed a modified Bilby, Cottrell and Swinden model while long crack propagation was taken to follow the Paris law. Their model is only applicable when the material under consideration lies below the surface.

Cheng et al (1994) extended the notion of dislocation pileup to model crack initiation under contact fatigue. They explain by appropriate consideration of boundary conditions that surface crack initiation requires fewer load cycles than does subsurface initiation.

In a series of experiments with case hardened nickel-chromium steel rollers under rolling and combined rolling and sliding conditions, Fujita and Yoshida (1979b) found that there exists an optimum case depth for surface durability that depends on the ratio of maximum orthogonal shear stress to Vickers hardness. They present methods for determining optimum case depths as well as p_0 vs N , where N is the number of cycles to failure. Although not surprising, they did conclude that the permissible load per width in-

creases with increasing relative radii of curvature, defined as $\frac{1}{2}(AB)^{-\frac{1}{2}}$, where A and B are defined in equations [2]. Not unexpectedly they also found that fatigue performance was degraded in the presence of sliding.

6 RESULTS

Results are comprised of two basic components. First are the superposition integrals in response to both normal and tangential loading. Second are finite element analysis results for:

- Frictionless contact of chain links – linear material
- Frictionless contact of chain links – nonlinear case and core
- Frictionless contact of chain links – residual stress
- Frictionless contact of chain links – shakedown analysis
- Chain link with friction – nonlinear case and core
- Chain link with friction – residual stress
- Chain link with friction – shakedown analysis

Validation of the twelve superposition integrals was achieved by favorable comparisons with the Sackfield (1983) solution. A FORTRAN program, developed by Hills et al in 1985, is available in ESDU publication number 85007 from:

Engineering Sciences Data Unit

251 – 259 Regent Street

London W1R 7AD

United Kingdom

Validation of finite element analysis for a linear material was achieved by demonstrating agreement between those results and both the contact patch and surface pressure distribution in a contact problem.

VALIDATION OF SUPERPOSITION INTEGRALS

As previously noted if material behavior is linear and contact is Hertzian, then the stress state can be calculated for general three dimensional contact using the Sackfield algorithm or the superposition integrals which are presented below. The size and shape of a contact ellipse were arbitrarily chosen. Normal and tangential surface tractions were specified and a point was selected at random so that the stress state could be calculated with both methods using MATHCAD. The environment variable, TOL, which establishes convergence criteria, was set to .0001 in each case. Input variables were:

Length of semi-major axis of contact ellipse (inches), a:	0.043
Length of semi-minor axis of contact ellipse (inches), b:	0.038
Applied force (pounds), P:	2000
Poisson's ratio, ν :	.28
x coordinate (inches)	0.03225
y coordinate (inches)	0.03225
z coordinate (inches)	0.027329

A comparison of results for the stress response to normal loading for each method is found in table 1.

Had the superposition integral for τ_{xy} been formed using the the Johnson/Hills expression, its value would have been -36,947 psi, an error in excess of 40 percent. A comparison of results for tangential loading, assuming that all variables are the same as for normal loading and that the coefficient of friction is equal to .30 is shown in table 2.

Equations [11] are the the superposition integrals used for calculating the stress state beneath an elliptical contact path of known size in response to a known semi-ellipsoidal

normal pressure distribution. Equations [12] are the superposition integrals used for calculating the stress state beneath the same contact patch in response to a surface tangential traction, acting in the x-direction, presumed to be everywhere proportional to the normal pressure.

$$\sigma_{xx} = \frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{(1-2\nu)}{r'^2} \left\{ \left(1 - \frac{z}{\varrho'}\right) \frac{x'^2 - y'^2}{r'^2} + \frac{zy'^2}{\varrho'^3} \right\} - \frac{3zx'^2}{\varrho'^5} \right] d\xi d\eta \quad [11a]$$

$$\sigma_{yy} = \frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{(1-2\nu)}{r'^2} \left\{ \left(1 - \frac{z}{\varrho'}\right) \frac{y'^2 - x'^2}{r'^2} + \frac{zx'^2}{\varrho'^3} \right\} - \frac{3zy'^2}{\varrho'^5} \right] d\xi d\eta \quad [11b]$$

$$\sigma_{zz} = -\frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} \frac{3p(\xi, \eta)z^3}{\varrho'^5} d\xi d\eta \quad [11c]$$

$$\tau_{xy} = \frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{(1-2\nu)x'y'(2\varrho' + z)}{\varrho'^3(\varrho' + z)^2} - \frac{3x'y'z}{\varrho'^5} \right] d\xi d\eta \quad [11d]$$

$$\tau_{yz} = -\frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} 3p(\xi, \eta) \frac{y'z^2}{\varrho'^5} d\xi d\eta \quad [11e]$$

$$\tau_{xz} = -\frac{1}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} 3p(\xi, \eta) \frac{x'z^2}{\varrho'^5} d\xi d\eta \quad [11f]$$

Superposition integrals which calculate the stress response to tangential loading are formed in a similar manner. In the development of these equations it is assumed that the tangential traction is applied in the x-direction, that the tangential traction has no effect on the size or shape of the contact patch and that the magnitude of the tangential traction

is at all points proportional to the normal traction which is assumed to be Hertzian. The constant of proportionality is f , the coefficient of friction.

$$\sigma_{xx} = \frac{f}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[-\frac{3x'^3}{\rho'^5} + (1-2\nu) \left\{ \frac{x'}{\rho'^3} - \frac{3x'}{\rho'(\rho'+z)^2} + \frac{x'^3}{\rho'^3(\rho'+z)^2} + \frac{2x'^3}{\rho'^2(\rho'+z)^3} \right\} \right] d\xi d\eta \quad [12a]$$

$$\sigma_{yy} = \frac{f}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[-\frac{3x'y'^2}{\rho'^5} + (1-2\nu) \left\{ \frac{x'}{\rho'^3} - \frac{x'}{\rho'(\rho'+z)^2} + \frac{x'y'^2}{\rho'^3(\rho'+z)^2} + \frac{2x'y'^2}{\rho'^2(\rho'+z)^3} \right\} \right] d\xi d\eta \quad [12b]$$

$$\sigma_{zz} = -\frac{f}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \frac{3x'z'^2}{\rho'^5} d\xi d\eta \quad [12c]$$

$$\tau_{xy} = \frac{f}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[-\frac{3x'^2y'}{\rho'^5} + (1-2\nu) \left\{ -\frac{y'}{\rho'(\rho'+z)^2} + \frac{x'^2y'}{\rho'^3(\rho'+z)^2} + \frac{2x'^2y'}{\rho'^2(\rho'+z)^3} \right\} \right] d\xi d\eta \quad [12d]$$

$$\tau_{yz} = -\frac{f}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \frac{3x'y'z}{\rho'^5} d\xi d\eta \quad [12e]$$

$$\tau_{xz} = -\frac{f}{2\pi} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \frac{3x'z'^2}{\rho'^5} d\xi d\eta \quad [12f]$$

For materials which do not violate either Hertzian assumptions and/or the above restrictions the stress state at any point below the surface may then be determined by superposing equations [11] and [12]. If sliding is not in the x -direction then the tangential surface traction should be resolved into x and y components and a third set of superposition integrals written to accommodate stresses caused by that portion of the tangential traction in the y -direction.

VALIDATION OF FINITE ELEMENT MODEL - LINEAR MATERIAL

COSMOS/M is a displacement-based, general-purpose finite element code. Three dimensional contact problems are frequently best modeled in commercially available soft-

ware with eight-noded rather than twenty-noded isoparametric elements. The reason for this is that when equivalent nodal forces are found using twenty-noded brick elements the midside surface forces are tensile. Although elements with twenty seven nodes have been developed to obviate this problem, they are unavailable in most codes.

Contact was defined as occurring between a fully constrained target body (zero displacement on the three planes of symmetry) and a partially constrained source body which was given an initial separation of .001 inches from the target. Appropriate loading representing a one ton weight was applied to the source. To assure a non-singular global coefficient matrix, a stiffness of two thousand pounds per inch was added to the source in its direction of rigid body motion, so that approximately two pounds of force were necessary to make initial contact.

Dimensions of the target elements in the vicinity of contact were .0038 inches by .0041 inches on the surface. Beneath the contact area and for a depth of .020 inches element thickness was .0025 inches. Below that element size was generally larger and less regular in shape. At points far removed from the contact zone stress gradients are less severe so larger elements can be used with little penalty.

Node-to-surface contact was defined with single-noded gap elements on the source and contact surfaces on the target. Contact problems are inherently nonlinear since force and deflection are generally not proportional. So, although material behavior is linear the geometric nonlinearity requires solution in the nonlinear module of COSMOS/M.

It was planned that the same mesh would be used for analysis of both linear and nonlinear materials. Since material properties are significantly different for the case and core, it was necessary to define two different element groups. However, for linear material anal-

ysis both element groups were given identical properties. For the alloy under examination Young's modulus and Poisson's ratio were set to 28,000 ksi and 0.28 respectively.

The geometry of contacting chain links does not seem close to that originally envisaged by Hertz. It would appear then that no analytical model exists against which finite element results could be compared with some expectation of agreement. Remarkably, this is not the case.

Selected Stress Results

By searching the results data base for closed gap elements it was found that contact was a circle of radius .0417 inches. Theory predicts a circular contact patch of radius .0425 inches. The normal pressure distribution within the contact patch appears to be reasonably modeled by Hertzian theory. Table 3 compares Hertzian and FEA normal pressure distributions at the surface along the x-axis. Data points along the axis corresponds to the nodal spacing. Relative error was calculated from

$$\text{Relative Error} = \frac{|(\sigma_{zz})_{\text{FEA}} - (\sigma_{zz})_{\text{Hertz}}|}{|(\sigma_{zz})_{\text{Hertz}}|} 100$$

As table 3 illustrates, frictionless chain contact results in a surface normal pressure distribution which can be approximated by Hertzian theory. Tables 4 and 5 compare values of σ_{zz} , σ_{xx} and σ_{yy} along the z-axis for the Hertzian problem to those of chain contact. Since the contact patch for the particular case under study is circular σ_{xx} and σ_{yy} would be equal in purely Hertzian contact. Hence, σ_{xx} and σ_{yy} are compared against the same value in table 5. As is evident, agreement of the subsurface stresses σ_{xx} and σ_{yy} with Hertzian theory is not quite as good as it is for σ_{zz} .

The peak orthogonal shear stress, τ_{yz} , was located .0118 inches below the surface at a y-coordinate of -.0359 inches. Its value was 127,031 psi. The calculated value at this point is 112 ksi.

Selected Displacement Results

As before superposition integrals can be developed from the displacement responses to a normal load. For completeness the displacement superposition integrals in response to normal loading are provided.

$$u_x = \frac{1}{4\pi\mu} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{(x-\xi)z}{\rho'^3} - (1-2\nu) \frac{(x-\xi)}{\rho'(\rho'+z)} \right] d\xi d\eta \quad [13a]$$

$$u_y = \frac{1}{4\pi\mu} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{(y-\eta)z}{\rho'^3} - (1-2\nu) \frac{(y-\eta)}{\rho'(\rho'+z)} \right] d\xi d\eta \quad [13b]$$

$$u_z = \frac{1}{4\pi\mu} \int_{-b}^b \int_{-\frac{a}{b}\sqrt{b^2-\eta^2}}^{\frac{a}{b}\sqrt{b^2-\eta^2}} p(\xi, \eta) \left[\frac{z^2}{\rho'^3} + \frac{2(1-\nu)}{\rho'} \right] d\xi d\eta \quad [13c]$$

Johnson (1985) shows that the surface displacement field for a hemispherical normal pressure distribution is radial. Displacements within the contact patch are given by

$$\bar{u}_r(r) = -\frac{(1-2\nu)(1+\nu)p_0 a^2}{3E} \frac{1}{r} \left[1 - \left(1 - \frac{r^2}{a^2} \right)^{\frac{3}{2}} \right]$$

where $\bar{u}_r(r)$ is the radial surface displacement, E is Young's modulus, r is the radial position and a is the radius of the contact circle. Comparisons of $\bar{u}_z$ along the x-axis and u_z along the z-axis are made in tables 6 and 7 respectively.

Although a large difference exists between the theoretical and FEA results, the displacement gradients $\frac{\partial \bar{u}_z}{\partial x}$ and $\frac{\partial u_z}{\partial z}$ along the x-axis and z-axis respectively are virtually identical

to linear theory. The discrepancies are solely attributable to differences in boundary conditions between the classical Hertz problem and the chain contact problem. In the Hertz problem a semi-infinite body is assumed and the z -displacement beneath contact at the boundary is set to zero. No supporting structures are considered except the contacting bodies themselves. Displacements are calculated relative to a coordinate system whose origin is in a plane tangent to both surfaces when contact is first made and is located at that point. In the chain contact problem a plane of symmetry exists in the z -direction above contact. The target body support structure is such that deformation of points remote from the contact precludes a direct comparison of displacement results in the z -direction. The slender diameter chain is able to stretch, accounting for the entire difference. Clearly, the almost constant difference in u_z means that ϵ_{zz} for both cases is nearly identical. It is immediately evident that ϵ_{zz} for chain contact can be well modeled as Hertzian. This is undoubtedly the principal reason that σ_{zz} along the z -axis is nearly Hertzian. Graphical results are presented to support the argument.

Several observations are apparent from an examination of the surface displacements. First, $\bar{u}_z$ in the contact patch is circular as predicted by Hertzian theory. Second, neither $\bar{u}_x$ along the x -axis nor $\bar{u}_y$ along the y -axis conforms to Hertzian displacements to the same degree as does $\bar{u}_z$, although agreement near the load line is not significantly in error. Consequently, it should be anticipated that $\bar{\epsilon}_{xx}$ and $\bar{\epsilon}_{yy}$ and hence $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ may not agree with the linear theory as well as does $\bar{\sigma}_{zz}$. This is in fact the case, although again the discrepancies are not excessive.

Convergence Results

As expected, it was seen that improved results are achieved as element size decreased from 8.297×10^{-5} to 4.050×10^{-8} in³. For example, at the point of initial contact the coars-

est mesh estimated σ_{zz} as -149,045 psi, while the most refined mesh produced an estimate of -468,312 psi. The theoretical (Hertzian) value is -419,944 psi. At the center of the cross section the coarsest mesh estimated σ_{zz} as -14,292 psi and the finest mesh estimated the value as -16,492 psi. The theoretical value is -18,710 psi. Results of the convergence study are found in tables 8 and 9. Further mesh refinement was not feasible due to limitations in both hardware and software.

RESULTS FROM FRICTIONLESS NONLINEAR MATERIAL ANALYSIS

No theory exists that provides a benchmark against which nonlinear results can be compared. Moreover, since the geometry of chain contact is so different from that of the 'standard' Hertzian problem, it is indeed difficult to validate the finite element solution. Good agreement between some linear theory and linear finite element analysis results can help support a particular solution's credibility, but ultimately, assessing the validity of the nonlinear finite element results is a matter of judgement.

The peak orthogonal shear stress, τ_{yz} , was found at a depth below the surface of .00939 inches, a value .0025 shallower than it was when only linear material was considered, at a y-coordinate of -.03551 inches. Its value was 125,521 psi.

Table 10 compares von Mises stress along the z-axis for the frictionless linear material and nonlinear material problems. Von Mises stress for the nonlinear material is lower than that found in the linear material analysis. A discontinuity across the case/core interface is also observed in the nonlinear material case. Since the case and core were both modeled as von Mises isotropic materials, the discontinuity can be explained by the lower yield strength of the core.

Values for σ_{xx} , σ_{yy} and σ_{zz} along the z-axis are presented in table 11. Discontinuities in both σ_{xx} and σ_{yy} across the case/core interface were found. No tabular comparison to linear results is made.

FRICTIONLESS CONTACT MODEL - RESIDUAL STRESS & SHAKEDOWN

In contact problems residual stresses can be determined simply by removing the load incrementally after the load has been applied once, given that nonlinear material models have been used and at least some yield will have occurred. This is possible since the displacements and material state are retained after the first load application. They serve as initial conditions for load removal which is treated as a separate problem.

After load removal the peak value of von Mises stress in the case was 20,796 psi, located along the load line and below the surface a distance of .0175 inches (called point A in this discussion). The peak value in the core was found similarly along the load line at the case/core interface, a distance of .020 inches below the surface (called point B in this discussion). However, at these two points the residual normal stresses are all tensile, in marked contrast to the stress state found after loading.

σ_{xx} , σ_{yy} , and σ_{zz} at point A were found to be 43,419, 45,381, and 20,796 psi respectively. At point B σ_{xx} , σ_{yy} , and σ_{zz} were seen to be 7,093, 3,392, and 23,233 psi respectively.

Upon reapplication of the load von Mises stress was found to be 293,453 psi at point A and 257,721 psi at point B. These values are slightly lower than the corresponding results after the first load application, so it can be concluded that the system shakes down after the first load cycle.

CHAIN LINK WITH FRICTION - INITIAL LOAD APPLICATION

The addition of a tangential traction resulting from the introduction of friction into the problem has two detrimental effects. Not only has the peak von Mises stress moved to the surface but also its magnitude has increased. It took on a value of 324,308 psi in the case at the surface at an x-coordinate of .00774 inches. Its maximum value in the core was found to be 267,812 psi at the case/core interface at an x-coordinate of .00673 inches. These two points will be designated for the purposes of this discussion as C and D respectively. It was observed that von Mises stress was nearly constant over much of the case/core interface and that point D lay almost directly beneath point C along a radial line through the center of the wire.

CHAIN LINK WITH FRICTION - RESIDUAL STRESS & SHAKEDOWN

Upon load removal von Mises stress at points C and D were found to be 22,110 and 92,217 psi respectively. After reapplication of the load von Mises stress at C and D were 324,306 and 267,812 psi respectively. It is noteworthy that after a second load application the maximum von Mises stress in the case was slightly lower (2 psi) than it was after the first loading. Similarly, the peak von Mises stress in the core reached the same value that it did after the first load. This suggests that even with a frictional load shakedown of the system occurs after one load cycle given that material properties and friction characteristics are as stipulated.

7 DISCUSSION OF RESULTS

SUPERPOSITION INTEGRALS

Good agreement has been demonstrated between the superposition integrals and the Sackfield solution for both normal and tangential loadings, providing that contact is Hertzian and material behavior is linear. The Sackfield solution assumes an elliptical contact patch as well as semi-ellipsoidal loading, be it normal or tangential. A priori knowledge of both the contact area and surface tractions is required.

While inputs to the superposition integral solution are also the contact patch dimensions and the pressure loading, there are no requirements that contact be Hertzian or that the surface tractions be semi-ellipsoidal. It is only required that their functional forms be specified and that material behavior be linear. So, in one sense the superposition integrals provide a more general solution to three dimensional linear contact problems.

Given equivalent convergence criteria, the Sackfield algorithm is much faster than the superposition integral solution. However, in a general purpose mathematical program such as MATHCAD, it may be a less cumbersome to code the superposition integral solution.

A final result associated with the superposition integrals is development of the correct expression for τ_{xy} (see equation [10]). It is not known if others have uncovered the discrepancy before, and if so, this paper will hopefully confirm their work.

LINEAR MATERIAL ANALYSIS - FRICTIONLESS

An important motive for examining the linear material problem was to generate numerical results which would compare favorably to a linearized theory. In this way some confi-

dence in subsequent nonlinear material analyses could be developed. It was certainly known at the outset that significant differences existed in geometry and constraints between the chain contact problem and Hertz's problem, so any agreement between them might be noteworthy.

An examination of the graphical results for u_z along the z -axis suggests that ϵ_{zz} for chain contact is almost identical to that expected for Hertzian contact. Since the support mechanisms are quite different for each problem, it is not unreasonable to conclude that ϵ_{zz} is independent of the support mechanism and method of load application. Similar results were seen when $\bar{u}_z$ was examined. Normal surface displacements follow a very nearly circular pattern as predicted by Hertzian theory. It is likely the case, then, that the Hertzian restrictions on size and curvature are not too severe, since the chain diameter is less than four times the contact patch diameter.

The case is somewhat different when $\bar{u}_x$ and $\bar{u}_y$ are compared to the linear theory. Hertzian theory predicts a radially symmetric surface displacement distribution within the contact patch given by

$$\bar{u}_r = - \frac{(1 - 2\nu)(1 + \nu)a^2}{3E} \frac{1}{r} \left[1 - \left(1 - \frac{r^2}{a^2} \right)^{\frac{3}{2}} \right]$$

where $\bar{u}_r$ is the radial surface displacement and r is the radial position at the surface.

Since $0 \leq r \leq a$ over the contact area, $\bar{u}_r$ is never positive. Thus the material in contact is always being drawn toward the center of contact. This behavior is observed in the finite element analysis results, even though small discrepancies exist in the magnitudes. Beginning at the center of contact and proceeding along the negative y -axis (see figure 4) good agreement between $\bar{u}_y$ and Hertzian theory is seen for the first .025 inches. At that point

theory predicts the displacement plot to flatten out. However, finite element analysis predicts a continuation of the almost linear relationship within the contact patch, indicating ϵ_{yy} is nearly constant along the y-axis. Similar results are seen for $\bar{u}_x$ in figure 5. Finite element analysis predicts slightly greater displacements than would be expected from the linear theory. This may be caused by lesser constraint along the x-axis due to curvature effects in the x-z plane. Similarly, it is seen that finite element analysis results for $\bar{u}_y$ are, in absolute magnitude, slightly less than theoretical predictions for much of the contact area. This may be attributable to somewhat greater constraint due to negative curvature in the y-z plane.

Observed differences in displacements between Hertzian theory and finite element analysis results are likely the result of significant differences in geometry rather than the result of an inappropriate mesh, especially given the agreement in u_z .

A comparison of the σ_{zz} finite element analysis results to Hertzian theory along the three axes suggests that in linear problems of this type contact gap elements may not always be necessary. The gap element approach predicted a contact patch radius of .0417 inches in the x-direction and .0415 inches in the y-direction, an error of 2 percent when compared to the Thomas and Hoersch solution and a difference in circularity of only .2%. Moreover, the resulting normal stress distribution is not unlike theoretical predictions, differing generally by less than 10 percent. Thus, it may be possible to obtain adequate results to a linear chain contact problem merely by applying an appropriate pressure distribution to a contact area of proper size rather than solving a contact problem in the non-linear module, a process that is far more computationally intensive. Agreement of finite element analysis results with linear theory for σ_{zz} along the z-axis, i.e., below the surface along the load line, is reasonable for a distance somewhat greater than the contact patch

radius. Beyond that point the mesh is coarse and the stress has largely decreased, so results were not examined. Furthermore, material failures seldom originate below that point. Much of the discrepancy appears to begin with the peak value at the origin which is 11 percent larger than predictions.

Comparisons of σ_{zz} against linear theory were also made at the surface. Agreement was reasonably good along the x-axis, but along the y-axis the finite element approximation to σ_{zz} is greater by about 10 percent than Hertzian theory. Since finite element codes calculate stresses from strains according to $\sigma_{ij} = \lambda \epsilon_{ii} \delta_{ij} + 2\mu \epsilon_{ij}$, the generalized Hooke's law, discrepancies in σ_{zz} are likely the result in differences already noted in ϵ_{xx} and ϵ_{yy} since $\bar{\epsilon}_{zz}$ and ϵ_{zz} along the z-axis agree well with theoretical predictions.

In a circular contact area subjected to Hertzian loading the surface normal stresses, $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{\theta\theta}$, are found to be compressive and equal to

$$\bar{\sigma}_r = \bar{\sigma}_{\theta\theta} = -\frac{1}{2}(1 + 2\nu)p(r)$$

where $p(r)$ is the Hertzian normal surface pressure distribution. Along the x-axis in circular contact σ_{xx} is σ_{rr} and σ_{yy} is $\sigma_{\theta\theta}$ and along the y-axis σ_{yy} is σ_{rr} and σ_{xx} is $\sigma_{\theta\theta}$. Results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ along both x and y axes are tabulated in tables 12 and 13. It is seen that along the x-axis within the contact patch both $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are slightly greater than predicted. Again it must be remembered that geometric differences between Hertzian contact and chain contact are considerable. However, along the y-axis $\bar{\sigma}_{yy}$ closely approximates the theoretical solution, while $\bar{\sigma}_{xx}$, although larger than predicted, very nearly parallels the Hertzian stress distribution. Additionally, the $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ stress contours are smoother than those along the x-axis, despite a coarser finite element mesh. One possible explanation for this lies in the smaller absolute curvature along the y-axis.

Peak values of the shear stress components were examined in orthogonal planes. Comparisons of the linear material finite element analysis results were made to corresponding values calculated from the superposition integrals. The point at which examination occurred was chosen to be the same point at which the peak finite element analysis value was found. Results are presented in table 14. Differences between the theoretical solution and the finite element solution vary between 10.1 percent and 21.8 percent.

Considered by Fessler and Ollerton to play an important role in failure of rolling elements, the orthogonal shear stresses, τ_{xz} and τ_{yz} , occur in planes perpendicular to the z -axis and to the contact plane (Fessler and Ollerton, 1957). If rolling occurs in the x -direction then the maximum orthogonal shearing stress is defined as τ_{xz} and identified as τ_0 . Even though τ_{xz} may be smaller than τ_{yz} , τ_{xz} is still called the maximum orthogonal shear stress because it reverses sign when rolling reverses whereas τ_{yz} does not. Since τ_0 is an odd function in the rolling direction, its range is $2\tau_0$. Fessler and Ollerton derived closed form expressions for τ_{xz} and τ_{yz} .

$$\tau_{xz} = -\frac{3PQ}{2\pi a^2}$$

$$\tau_{yz} = -\frac{3PR}{2\pi b^2}$$

where Q and R are given by

$$Q = \frac{\left(\frac{y}{b}\right)\left(\frac{z}{b}\right)^2 \left[\left(1 + \frac{c^2}{a^2}\right) \frac{c^2}{a^2} \right]^{-\frac{3}{2}} \left(k^2 + \frac{c^2}{a^2}\right)^{-\frac{1}{2}}}{\left[\left(\frac{ax}{a^2+c^2}\right)^2 + \left(\frac{ay}{b^2+c^2}\right)^2 + \left(\frac{az}{c^2}\right)^2 \right]}$$

$$R = \frac{\left(\frac{y}{b}\right)\left(\frac{z}{b}\right)^2 \left[\left(1 - \frac{c^2}{b^2}\right) \frac{c^2}{b^2} \right]^{-\frac{3}{2}} \left(\frac{1}{k^2} + \frac{c^2}{b^2}\right)^{-\frac{1}{2}}}{\left[\left(\frac{bx}{a^2+c^2}\right)^2 + \left(\frac{by}{b^2+c^2}\right)^2 + \left(\frac{bz}{c^2}\right)^2 \right]}$$

in which c^2 is the positive root of

$$\frac{x^2}{a^2 + c^2} + \frac{y^2}{b^2 + c^2} + \frac{z^2}{c^2} = 1$$

Since maximum values of the orthogonal shear stresses occur in the planes of symmetry, i.e. those planes in which $x = 0$ and $y = 0$, they are found by evaluating $Q|_{y=0}$ and $R|_{x=0}$.

NONLINEAR MATERIAL ANALYSIS - FRICTIONLESS

Recent finite element analysis work in this field deals with elastic-perfectly plastic materials. While this approach to the material provides much insight into contact problems, many materials exhibit post-yield hardening behavior. In recognition of this the material models described earlier were selected as being representative of real materials.

Selected Displacement Results

Linear finite element analysis showed the normal surface displacement, $\bar{u}_z$, to be circular with $\bar{\epsilon}_{zz}$ and ϵ_{zz} along the z -axis near the surface in good agreement with elastic theory. Nonlinear frictionless analysis shows that while the contact patch increases in size by about 5 percent the circular nature of $\bar{u}_z$ is preserved. However, ϵ_{zz} along the z -axis is only slightly different than that predicted by linear theory.

Due to the difference in elastic properties between the case and core it might be expected that some discontinuity might exist in u_z . None was observed. Although the reason for this remains unclear, a finer finite element mesh might uncover a discontinuity which eludes the present analysis.

Selected Stress Results Along the Z-Axis

Pronounced changes in both σ_{xx} and σ_{yy} were observed across the case/core interface along the z -axis. However, no significant change in stress gradient was seen in σ_{zz} along

the z-axis. As might be expected a large gradient across the interface was also observed in von Mises stress. Changes across the interface in von Mises stress as well as in both σ_{xx} and σ_{yy} are attributable to the lower yield strength of the core material and the difference in Poisson's ratio, which is 0.28 in the case, but very nearly 0.50 in the core material near the interface since yielding is assumed to occur incompressibly.

Noteworthy is the fact that initial yielding probably would not occur in the case but would in the core. Peak von Mises stress in the case was 294 ksi while the yield strength is 300 ksi. However, the maximum stress intensity (defined as twice the maximum shear stress), found at the same point, was 305 ksi. Yield strength in the core is 175 ksi. Von Mises stress falls to this level at a depth below the surface of 0.062 inches. Thus, yielding can certainly be expected to have taken place after the first load application, but whether further yielding on subsequent loadings is still unclear.

Selected Stress Results at the Surface in the Contact Zone

According to linear theory $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are equal at all points along the x- and y- axes within the contact zone. For the linear problem along the x-axis $\bar{\sigma}_{xx}$ is seen to be somewhat greater than theory near the center of contact, but very nearly identical to expected results at the limit of contact. The opposite condition is observed in the behavior of $\bar{\sigma}_{yy}$ along the x-axis. In this case a somewhat flatter distribution in $\bar{\sigma}_{yy}$ is seen near the center of contact.

For the nonlinear material analysis, results along the x-axis for both $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are very similar to those obtained from the linear material analysis. This could be anticipated since it is known that yielding in the case probably does not occur.

Along the negative y-axis (hereinafter called the y-axis) results for both $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are qualitatively different than those along the x-axis. First, the curve shapes for both $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ are much smoother. Again there are very few noteworthy differences between corresponding stress component results in the linear and nonlinear material analyses. $\bar{\sigma}_{yy}$ along the y-axis is very nearly identical to linear theory predicts. Values for the nonlinear analysis are very slightly smaller than those for the linear analysis. As was the case along the x-axis, $\bar{\sigma}_{xx}$ was also seen to be greater than linear theory predicts along the y-axis. It was noted that the shape and magnitude of $\bar{\sigma}_{xx}$ along the x-axis were quite similar to $\bar{\sigma}_{xx}$ along the y-axis. However, the same cannot be said for $\bar{\sigma}_{yy}$.

NONLINEAR MATERIAL ANALYSIS - WITH FRICTION

Introduction of a frictional traction into the problem has several mechanistic effects which can now be enumerated:

- The maximum value of von Mises stress is increased. Concomitant increases are also seen in the constituent stress components.
- The location of the maximum von Mises stress has changed from a point directly beneath the load to a point on the surface which is no longer on the load line.
- A large area of tensile stress develops at the surface in the wake of the traction.

The implications for initiation and propagation of surface cracks are obvious. Since it is known that fewer load cycles to crack initiation at the surface are required compared to those necessary for subsurface initiation, it is fair to say that the larger surface stresses will lead to earlier failure. This is especially true in a material which may contain superfi-

cial defects that could be subjected to mode I loading. A result of this is the possibility of lubricant penetration of a short surface defect. S. Way (1936) showed that pitting is greatly facilitated in the presence of a lubricant. The mechanism for crack propagation is quite clear if the lubricant fully penetrates the crack. Due to the lubricant's incompressibility, significant mode I loading occurs. Crack extension is the logical consequence.

Thus, the frictional traction leads to reduced surface life. In a series of experiments Sheehan and Howes (1973) found that they were able to generate surface-initiated pits in carburized discs of good surface finish ($10\text{--}15\text{ }\mu\text{in}$) subjected to tractive rolling under Hertzian normal stresses in the vicinity of 325 ksi. Fatigue lives in the vicinity of 10×10^6 cycles in the presence of a small frictional traction ($f=.075$) were observed, whereas in the absence of friction (rolling contact) no pits were observed after 30×10^6 cycles, even after the peak normal stress was raised to 550 ksi. Although they discounted the effect of frictional loading on pitting resistance they were unable to satisfactorily explain the result.

8 CONCLUDING REMARKS

LINEAR MATERIAL ANALYSIS

Analysis of chain contact using linear material properties had three goals which have been largely achieved. First was determining both size and shape of the contact area. Second was developing the surface pressure distribution over the contact patch. Third was establishment of an optimal mesh. However, if the results are to be believed then they must somehow be validated. Since there is no known solution to the exact problem and mesh density can not be further increased due to computational constraints it is necessary to compare analysis results to solutions of the Hertz problem, bearing in mind that both geometry and boundary conditions in this problem differ significantly.

Review of the linear material analysis results suggests that the problem formulation is adequate for the purposes at hand. Despite the obvious differences in geometry and boundary conditions close agreement between FEA results and linear theory in the following program outputs is compelling.

- Displacement gradient $\frac{\partial \Pi_z}{\partial u_x}$ at the surface
- Displacement gradient $\frac{\partial u_z}{\partial u_z}$ along the z-axis
- Contact was circular as predicted
- Radius of contact circle was within 2 percent of theory
- $\bar{\sigma}_{zz}$ along the x-axis
- $\bar{\sigma}_{yy}$ along the y-axis
- σ_{xx} and σ_{yy} along the z-axis

NONLINEAR MATERIAL ANALYSIS

Interesting features of the nonlinear material analysis results are the discontinuities seen in σ_{xx} and σ_{yy} cross the case/core boundary, an effect likely resulting from differing elastic properties.

However, σ_{zz} is not similarly discontinuous. For both the linear and nonlinear material analyses nearly linear stress gradients are observed in σ_{zz} along the z-axis. Quantitatively, these stress gradients are similar.

In the nonlinear material analysis the orthogonal shear stress, τ_{yz} , was found to be 125 ksi, a value within two percent of that found in the linear material analysis. The peak value occurred below the surface at the same y-coordinate at which it was found in the linear material analysis. However, it was found at a shallower depth than it was in the linear material analysis.

SUPERPOSITION INTEGRALS

Agreement of the superposition integrals with the Sackfield solution is quite good. Based upon that agreement it is not unreasonable to conclude that they can be useful in the study of linear nonconformal contacts. No reference was found prior to 1985 containing an expression for the τ_{xy} response to a concentrated normal force on an elastic half-space. It is indeed hoped that some will benefit from the correction that was made herein to this expression (see equation [10]).

RESIDUAL STRESS & SHAKEDOWN

Residual stress and shakedown analyses were completed for the frictionless problem using the contact algorithm. Two considerations prevented using a similar approach for the

frictional problem. The first is that one plane of symmetry is lost when friction is introduced, thereby making the problem too large. The second is that simulating chain articulation is beyond the software's capability. Thus the problem only became tractable when the surface tractions were simulated. An initial stress free state was assumed since it could not be determined a priori.

The first load application in the frictionless problem caused plastic deformation in the core while the case remained elastic. The state of stress can be described as 'compressive', i.e., the principal stresses are all negative in the vicinity of contact. Upon load removal, however, it is seen that residual tensile stresses develop in a relatively large volume of material near the case/core interface.

Similar results were obtained when friction was included in the problem. It was seen that a system of protective residual stresses developed in the vicinity of the case/core interface that retard further plastic deformation and incremental collapse even though large service stresses were present in both the case and core.

These results are all consistent with observed behavior of hoisting load chain in service. Those links subjected to mode II loading, i.e., they do not pass over the load sprocket show no tendency to pit and virtually no evidence of wear, while links suffering mode I loading do pit and wear out after several hundred thousand load cycles.

SUMMARY

This work has examined the contact problem in hoisting load chain with a focus on the structural mechanics of idealized contact in strain-hardening materials that were treated as a layered medium. Initially linear materials were modeled so that finite element results

could be compared against Hertzian theory. When favorable results in this analysis were achieved, sufficient confidence in the finite element model was gained so that the nonlinear materials could be studied.

Finite element evaluation of nonlinear behavior was limited to the frictionless contact problem and a frictional problem in which the contact loading was applied element-by-element over the area found in the frictionless problem. In the absence of friction, yield was observed only in the core whereas in its presence yield was seen in both the case and the core.

Residual stresses were found in both cases by incrementally relaxing the applied loads. Similar results were found in both problems in that a self-equilibrating system of stresses forms which opposes further yield upon reapplication of the load.

Shakedown was found to occur in both cases after the first load cycle, since peak values of von Mises stress in both the case and the core were lower after a second load application than they were after the first. Shakedown was attributed to the relatively large amount of material near the case/core interface in which the residual stresses were of opposite sign to the service stresses. Secondary factors are likely to have been the increase in yield strength of the plastically deformed zone as well as the increased contact area.

Since pitting is observed only in those links which undergo mode I loading, it can be concluded that limiting initial surface defect size to the maximum extent practicable would improve surface endurance.

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APPENDICES

APPENDIX A
TABLES

Table 1. Comparison of results of Sackfield solution to superposition solution for the contact stress state in response to normal loading.

Stress Component	Sackfield Algorithm (psi)	Superposition Integral (psi)
σ_{xx}	-42,775	-42,775
σ_{yy}	-47,185	-47,185
σ_{zz}	-97,116	-97,115
τ_{xy}	-26,260	-26,260
τ_{yz}	-64,389	-64,388
τ_{xz}	-57,342	-57,343

Table 2. Comparison of results of Sackfield solution to superposition solution for the contact stress state in response to tangential loading.

Stress Component	Sackfield Algorithm (psi)	Superposition Integral (psi)
σ_{xx}	-20,831	-20,837
σ_{yy}	-5,896	-5,897
σ_{zz}	-17,202	-17,204
τ_{xy}	-14,012	-14,012
τ_{yz}	-11,405	-11,407
τ_{xz}	-17,534	-17,533

Table 3. Comparison of Hertzian theory to finite element analysis results for $\bar{\sigma}_{zz}$ along the x-axis.

Position Along x-Axis (in)	σ_{zz} – Hertzian Theory (psi)	σ_{zz} – Linear Mate- rial FEA (psi)	Relative Error (%)
.00000	-529178	-588668	11.2
.00387	-526974	-520513	1.2
.00773	-518163	-521274	.6
.01158	-509121	-506985	.4
.01542	-493090	-494664	.3
.01924	-471812	-475416	.8
.02304	-444611	-467450	5.1
.02682	-410405	-455368	11.0
.03058	-367357	-379426	3.3
.03431	-311971	-324524	4.0
.03803	-235911	-266406	12.9
.04171	-100207	-101854	1.6

Table 4. Comparison of Hertzian theory to finite element analysis results for σ_{zz} along the z-axis.

Position Along z-Axis (in)	σ_{zz} - Hertzian Theory (psi)	σ_{zz} - Linear Mate- rial FEA (psi)	Relative Error (%)
.0000	-529178	-588668	11.2
.0025	-527266	-570527	8.2
.0050	-521863	-540550	3.6
.0075	-513101	-520276	1.4
.0100	-501306	-503820	.5
.0125	-486937	-487477	.1
.0150	-470445	-470365	.0
.0175	-452340	-452250	.0
.0200	-433107	-431772	.3
.0235	-405132	-406245	.3
.0270	-376872	-369753	1.9
.0340	-322522	-340040	5.4
.0410	-273948	-278861	3.8
.0480	-232411	-225216	3.1

Table 5. Comparison of Hertzian theory to finite element analysis results for σ_{xx} and σ_{yy} along the z-axis.

Position Along z-Axis (in)	σ_{xx} & σ_{yy} - Hertzian Theory (psi)	σ_{xx} - Linear Material FEA (psi)	σ_{yy} - Linear Material FEA (psi)	Relative Error in σ_{xx} (%)	Relative Error in σ_{yy} (%)
.0000	-410189	-473896	-435678	15.5	6.2
.0025	-353343	-398081	-362684	12.7	2.6
.0050	-300439	-343968	-305618	14.5	1.7
.0075	-253774	-295883	-258527	16.6	1.9
.0100	-213027	-253361	-219230	18.9	2.9
.0125	-177784	-215577	-185932	21.3	4.6
.0150	-147569	-182278	-157774	23.5	6.9
.0175	-121867	-153126	-134002	25.7	10.0
.0200	-100157	-127202	-113439	27.0	13.3
.0235	-75504	-97692	-91387	29.4	21.0
.0270	-56387	-73468	-71075	30.3	26.1
.0340	-30418	-52411	-56434	72.3	85.5
.0410	-15324	-22640	-35713	47.7	133.1
.0480	-6657	-12306	-23668	84.9	255.5

Table 6. Comparison of Hertzian theory to finite element analysis results for $\bar{u}_z$ along the x-axis.

Distance at Surface Along X-Axis (inches)	$\bar{u}_z$ – Linear Theory (inches)	$\bar{u}_z$ – Linear FEA (inches)	Difference Be- tween Linear Theory and FEA Results (inches)
.00000000	.00108328	.00157882	.0004955
.00387356	.00107926	.00156574	.0004865
.00773559	.00106562	.00155026	.0004846
.01158450	.00104326	.00152688	.0004836
.01541870	.00101209	.00149625	.0004842
.01923650	.00097245	.00145805	.0004856
.02303650	.00092425	.00141372	.0004895
.02681690	.00086774	.00135940	.0004917
.03057620	.00080277	.00128897	.0004862
.03431280	.00073013	.00121482	.0004847
.03802510	.00064926	.00113475	.0004855
.04171140	.00056127	.00104490	.0004836
.04537030	.00048897	.00097566	.0004867
.04900000	.00044084	.00093192	.0004911

Table 7. Comparison of Hertzian theory to finite element analysis results for u_z along the z-axis.

Distance Below Surface Along Z-Axis (inches)	u_z - Linear Theory (inches)	u_z - Linear FEA (inches)	Difference Between Linear Theory and FEA Results (inches)
.0000	.001082127	.00157882	.0004967
.0025	.001058344	.00154934	.0004910
.0050	.001029811	.00151974	.0004899
.0075	.000999556	.00148972	.0004902
.0100	.000968107	.00145905	.0004909
.0125	.000936023	.00142793	.0004919
.0150	.000903705	.00139666	.0004930
.0175	.000871560	.00136551	.0004940
.0200	.000839828	.00133477	.0004949
.0235	.000796630	.00129275	.0004961
.0270	.000755299	.00125259	.0004973
.0340	.000679320	.00117973	.0005004
.0410	.000612770	.00111402	.0005013
.0480	.000555229	.00105757	.0005023

Table 8: Convergence results for u_z^* and σ_{zz} at $(x,y,z)=(0,0,0)$ inches.

Element Volume (in ³)	u_z^* (in)	σ_{zz} (psi)
8.297×10^{-5}	.000334202	-149045
1.496×10^{-5}	.000413217	-350518
4.050×10^{-8}	.000620205	-468312

Table 9: Convergence results for u_z^* and σ_{zz} at $(x,y,z)=(0,0,0.156)$ inches.

Element Volume (in ³)	u_z^* (in)	σ_{zz} (psi)
8.297×10^{-5}	.000050599	-14292
1.496×10^{-5}	.000045418	-14663
4.050×10^{-8}	.000041578	-16492

Table 10. Comparison of linear finite element analysis results to nonlinear finite element analysis results for von Mises stress along the z-axis.

Distance Below Surface Along Z-Axis (inches)	Von Mises Stress for Frictionless Linear Material Problem (psi)	Von Mises Stress for Frictionless Nonlinear Material Problem (psi)
.0000	140972	78382
.0025	194281	108382
.0050	219123	148244
.0075	245765	184468
.0100	269677	217492
.0125	288429	247958
.0150	301698	278327
.0175	309789	293508
.0200	312377	257773
.0235	312474	220060
.0270	298325	219374
.0340	286871	218171
.0410	250917	207346
.0480	208690	191691

Table 11. σ_{xx} , σ_{yy} and σ_{zz} for nonlinear frictionless contact along the z-axis.

Distance Below Surface Along Z- Axis (inches)	σ_{xx} for Frictionless Nonlinear Material Problem (psi)	σ_{yy} for Frictionless Nonlinear Material Problem (psi)	σ_{zz} for Frictionless Nonlinear Material Problem (psi)
.0000	-474383	-424481	-510347
.0025	-408276	-359873	-481805
.0050	-345591	-296819	-462404
.0075	-290607	-242578	-445674
.0100	-239631	-194118	-430168
.0125	-190724	-149428	-414871
.0150	-142620	-107369	-401138
.0175	-106200	-80932	-385811
.0200	-117111	-103614	-367424
.0235	-137294	-131077	-353691
.0270	-114422	-111026	-331505
.0340	-89514	-94839	-309429
.0410	-47139	-58424	-259022
.0480	-20099	-31141	-216154

Table 12. Comparison of Hertzian theory to finite element analysis results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ along the x-axis.

Position Along x-Axis (in)	σ_{xx} and σ_{yy} - Hertzian Theory (psi)	σ_{xx} - Linear Material FEA (psi)	σ_{yy} - Linear Material FEA (psi)
.0000000	-412758	-473896	-435678
.0038735	-411039	-461325	-413642
.0077356	-404167	-446742	-410872
.0115845	-397114	-439192	-406113
.0154187	-384610	-426306	-400602
.0192365	-368013	-412750	-393089
.0230365	-346796	-397081	-387750
.0268169	-320115	-370123	-378206
.0305762	-286538	-322897	-345296
.0343128	-243337	-269144	-315432
.0380251	-184010	-186258	-273715
.0417114	-78161	-67888	-189347

Table 13. Comparison of Hertzian theory to finite element analysis results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ along the negative y-axis.

Position Along y-Axis (in)	σ_{xx} and σ_{yy} - Hertzian Theory (psi)	σ_{xx} - Linear Material FEA (psi)	σ_{yy} - Linear Material FEA (psi)
.0000000	-412758	-473896	-435678
.0041885	-410747	-465176	-426044
.0083749	-404659	-459404	-413009
.0125574	-394316	-449048	-402811
.0167339	-379390	-434159	-387871
.0209024	-359344	-413746	-367535
.0250610	-333295	-387411	-340381
.0292077	-299739	-349965	-307869
.0333404	-255819	-314803	-255565
.0374573	-194759	-238693	-174818
.0415564	-85781	-135627	-50241

Table 14. Comparison of finite element analysis maximum shear stress results to superposition solution in response to normal loading.

Stress Component	x-coordinate (inches)	y-coordinate (inches)	z-coordinate (inches)	Finite Element Analysis Result (psi)	Superposition Integral Solution (psi)
τ_{xy}	.0305762	.0338658	.0000000	49995	64779
τ_{yz}	.0000000	.0359452	.0118552	127031	108013
τ_{xz}	.0383692	.0000000	.0177244	-97186	-108119

APPENDIX B

FIGURES

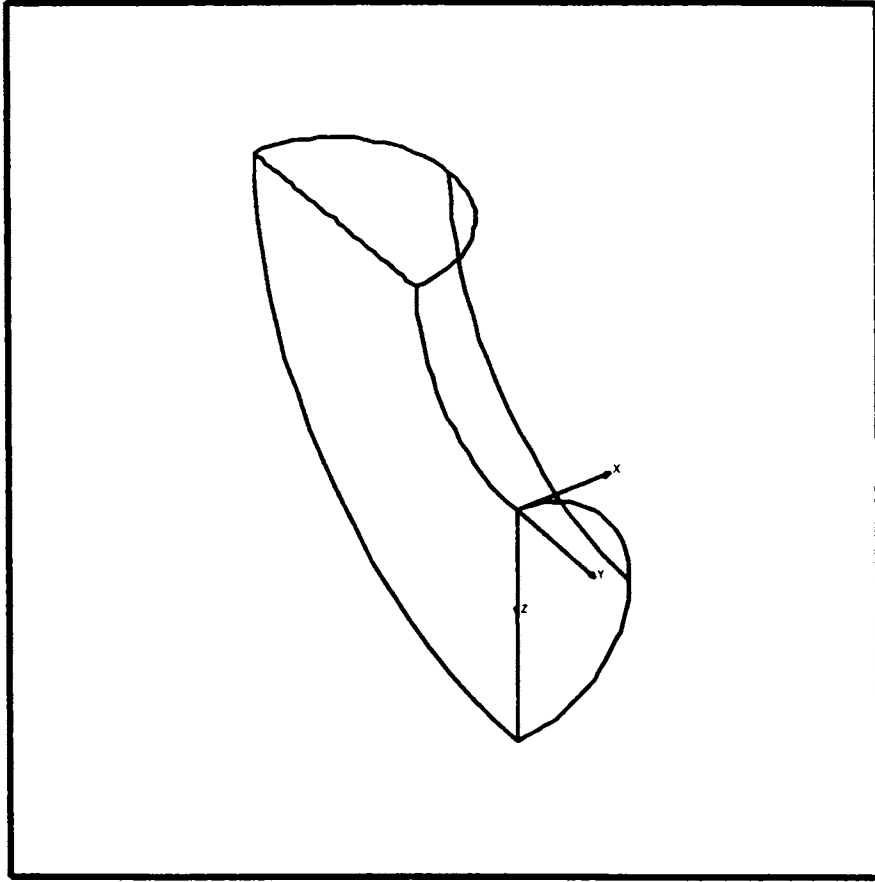


Figure 1: Coordinate system used in contact mechanics problems.

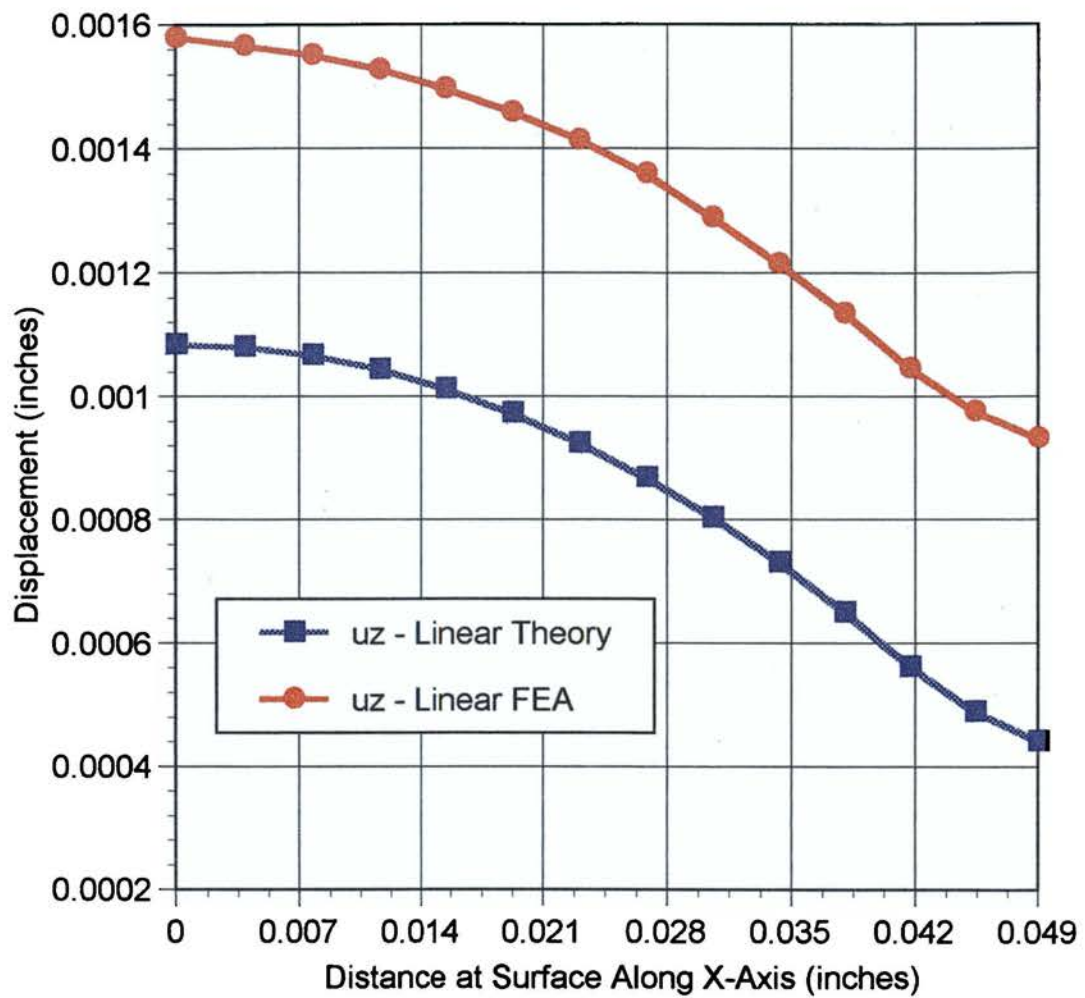


Figure 2: Comparison of linear theory to linear FEA results for $\bar{u}_z$ along the x-axis.

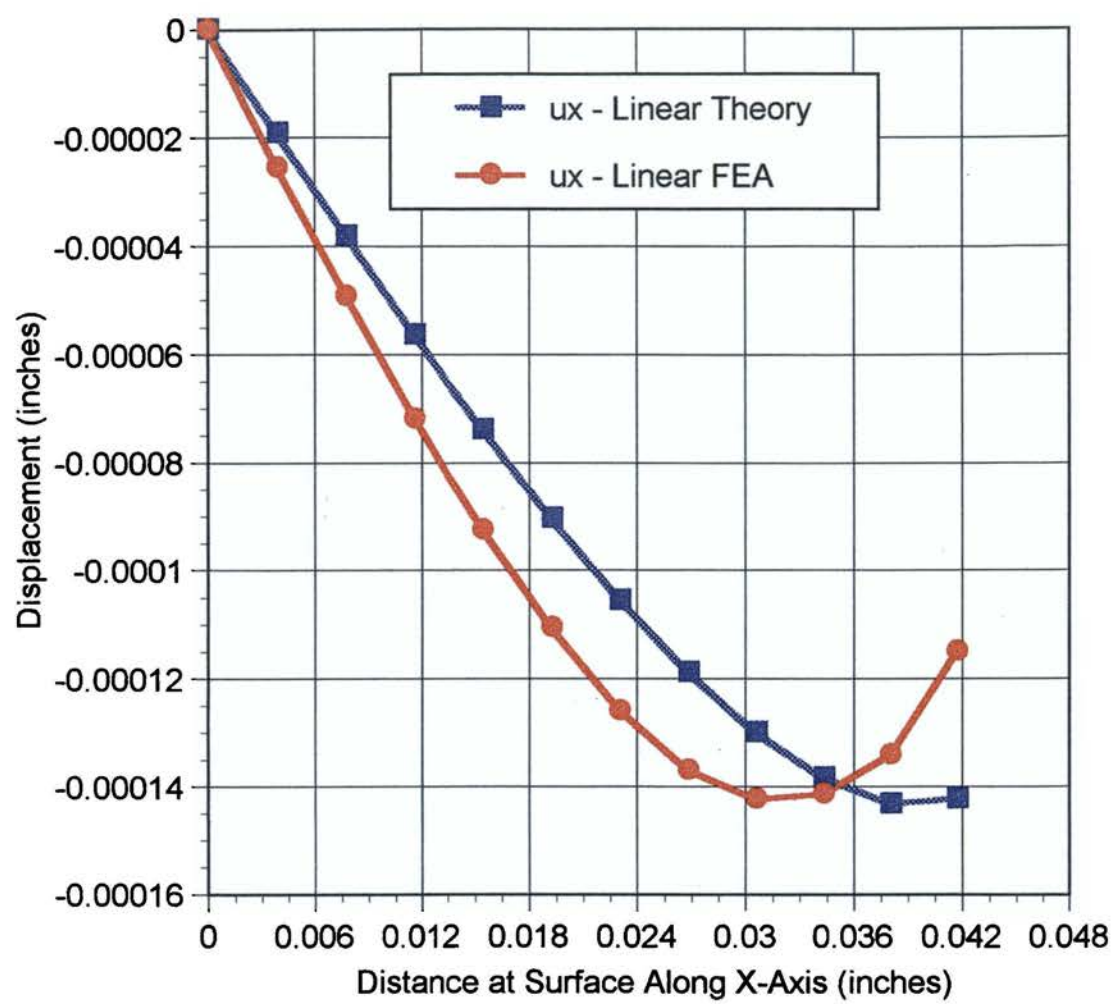


Figure 3: Comparison of linear theory to linear FEA results for $\bar{u}_x$ along the x-axis.

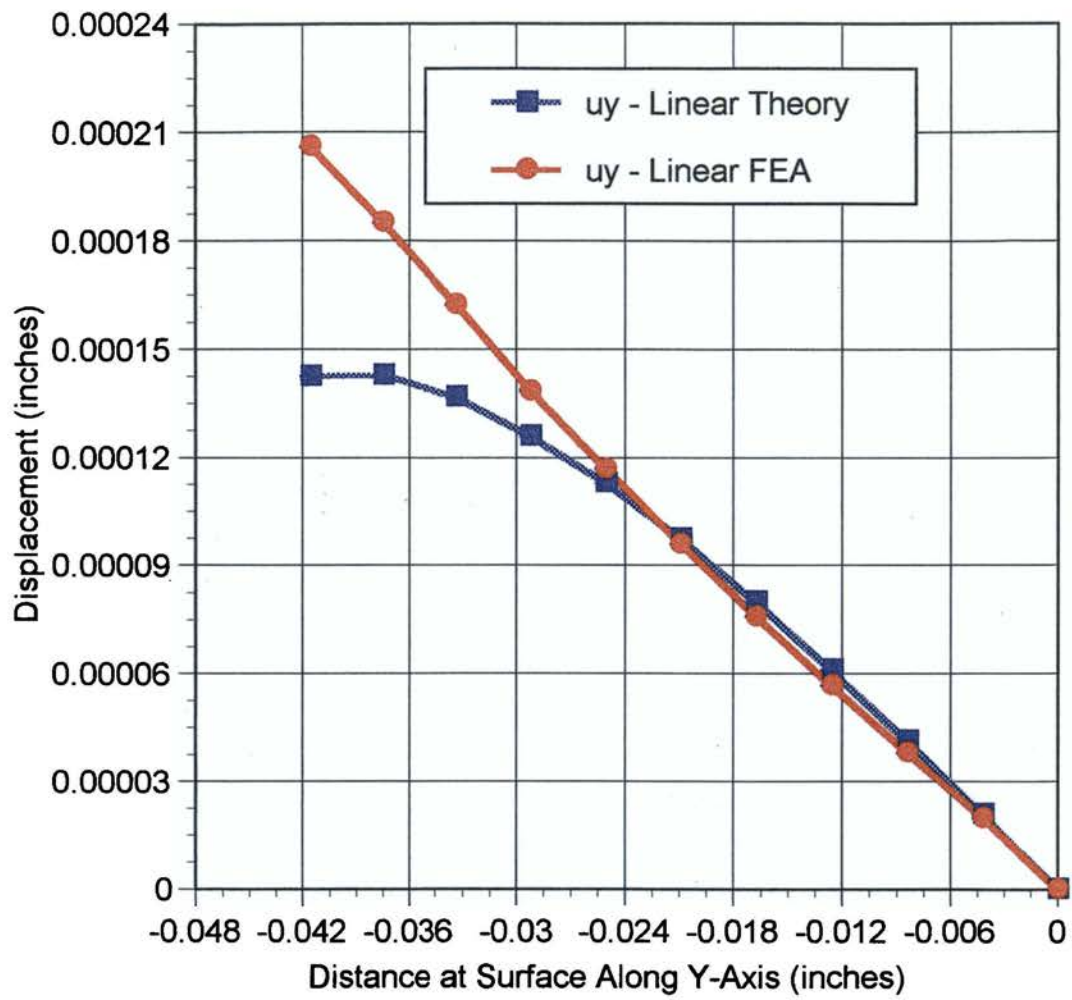


Figure 4: Comparison of linear theory to linear FEA results for $\bar{u}_y$ along the y-axis.

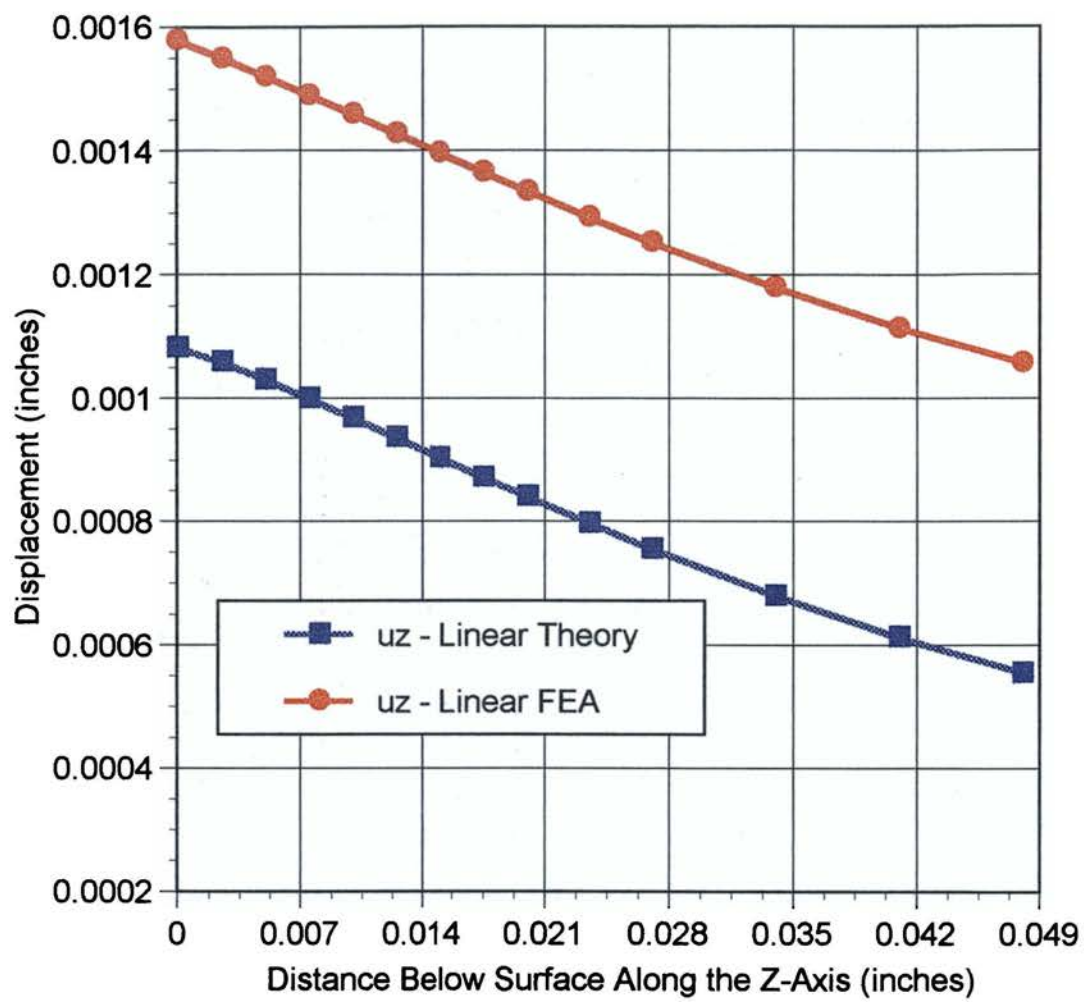


Figure 5: Comparison of linear theory to linear and frictionless nonlinear FEA results for u_z along the z-axis.

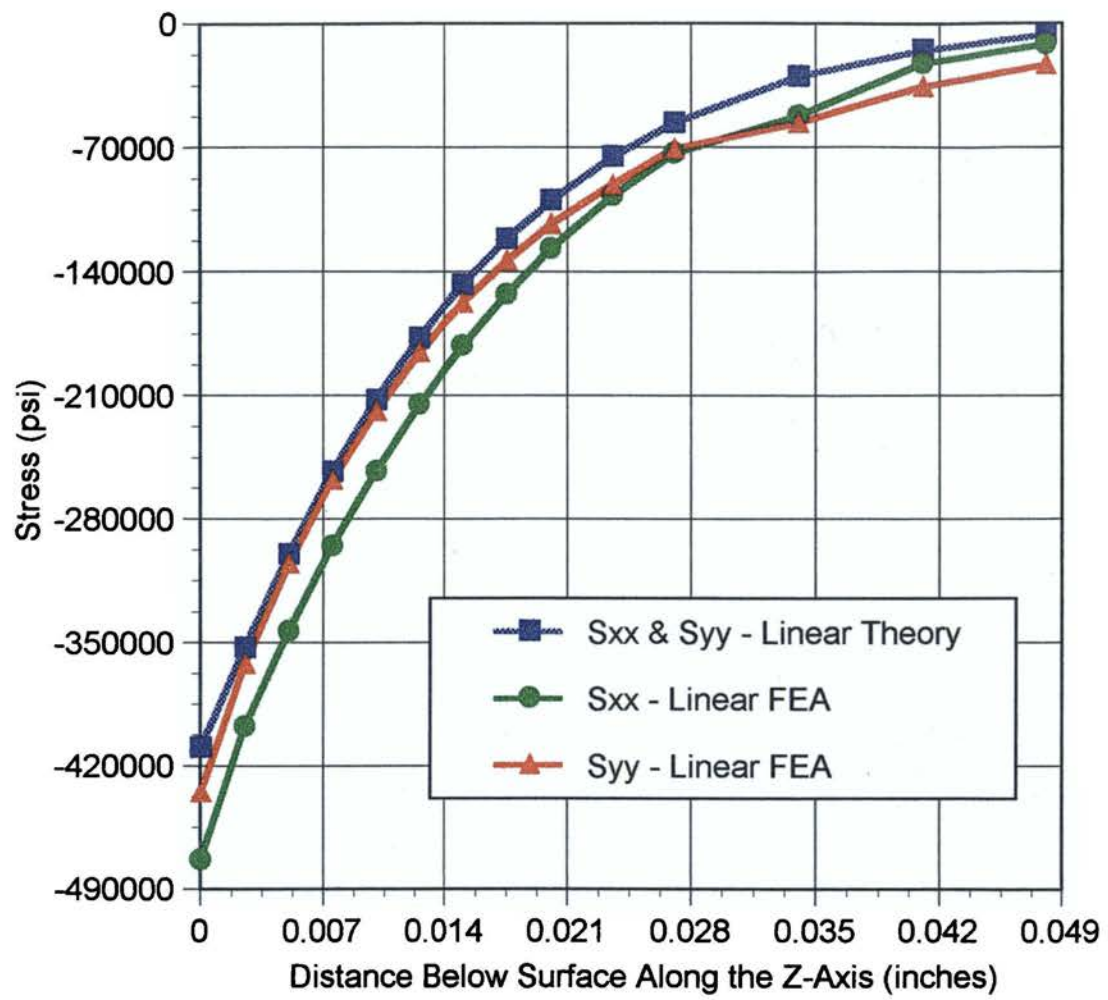


Figure 6: Comparison of linear theory to linear FEA results for σ_{xx} and σ_{yy} along the z-axis.

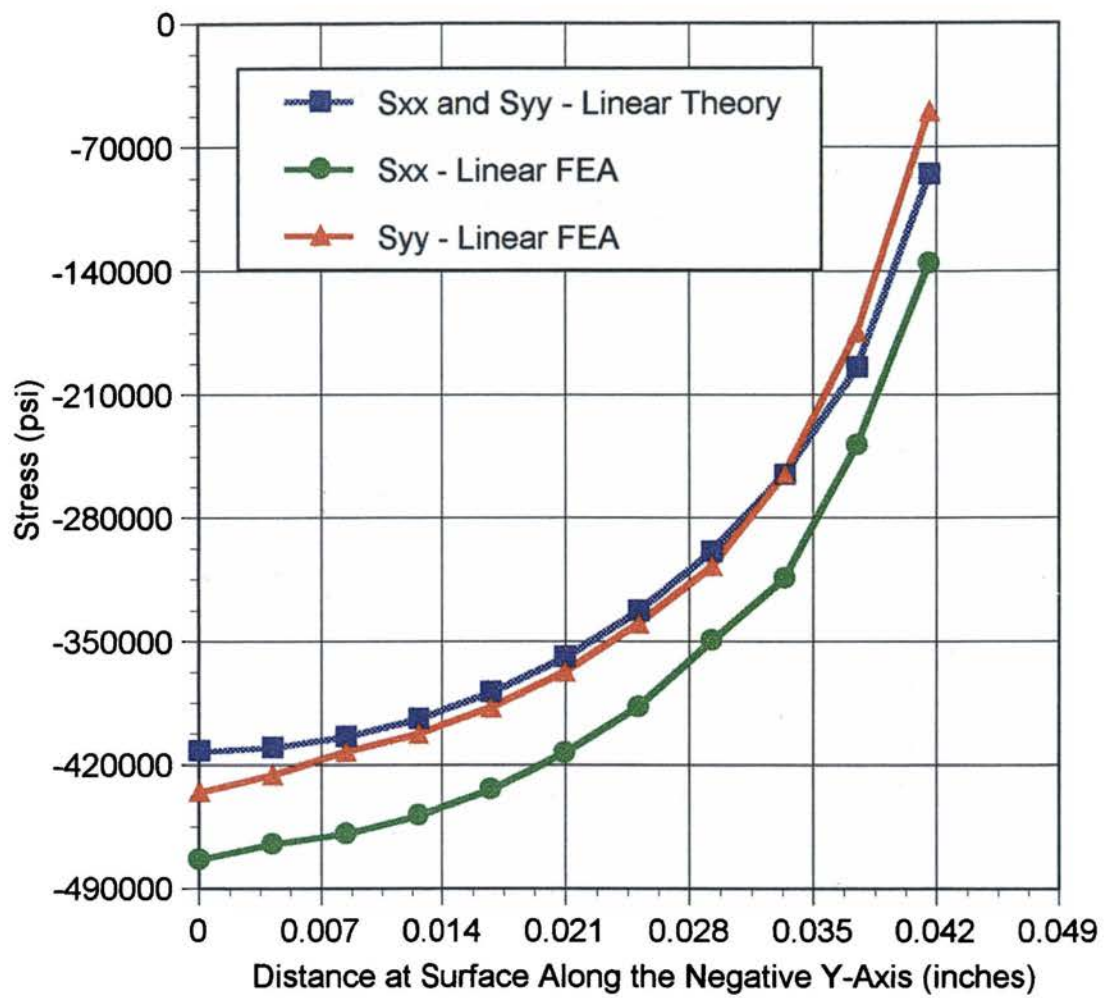


Figure 7: Comparison of linear theory to linear FEA results for $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yy}$ along the y-axis.

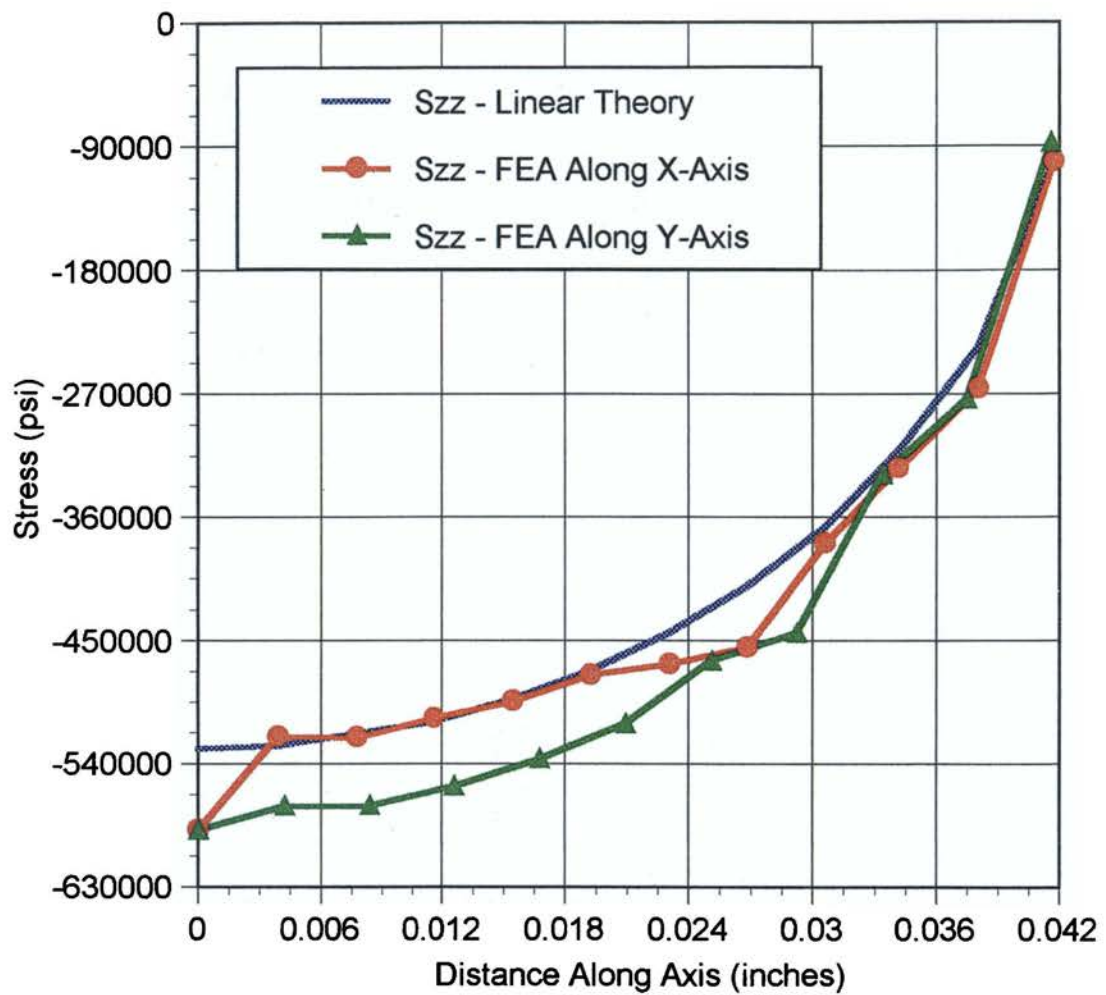


Figure 8: Comparison of linear theory to linear FEA results for $\bar{\sigma}_{zz}$ along the x- and y- axes.

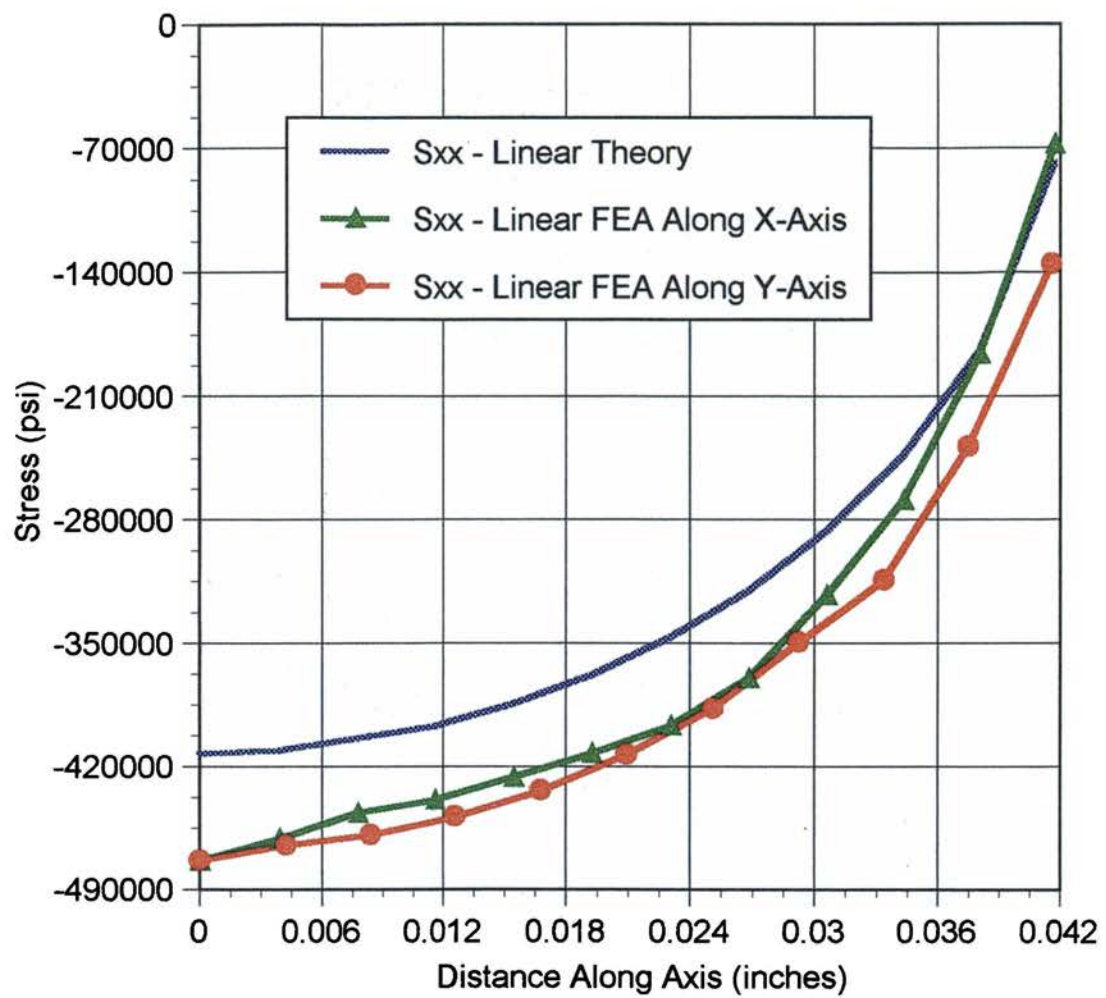


Figure 9: Comparison of linear theory to linear FEA results for $\bar{\sigma}_{xx}$ along the x- and y- axes.

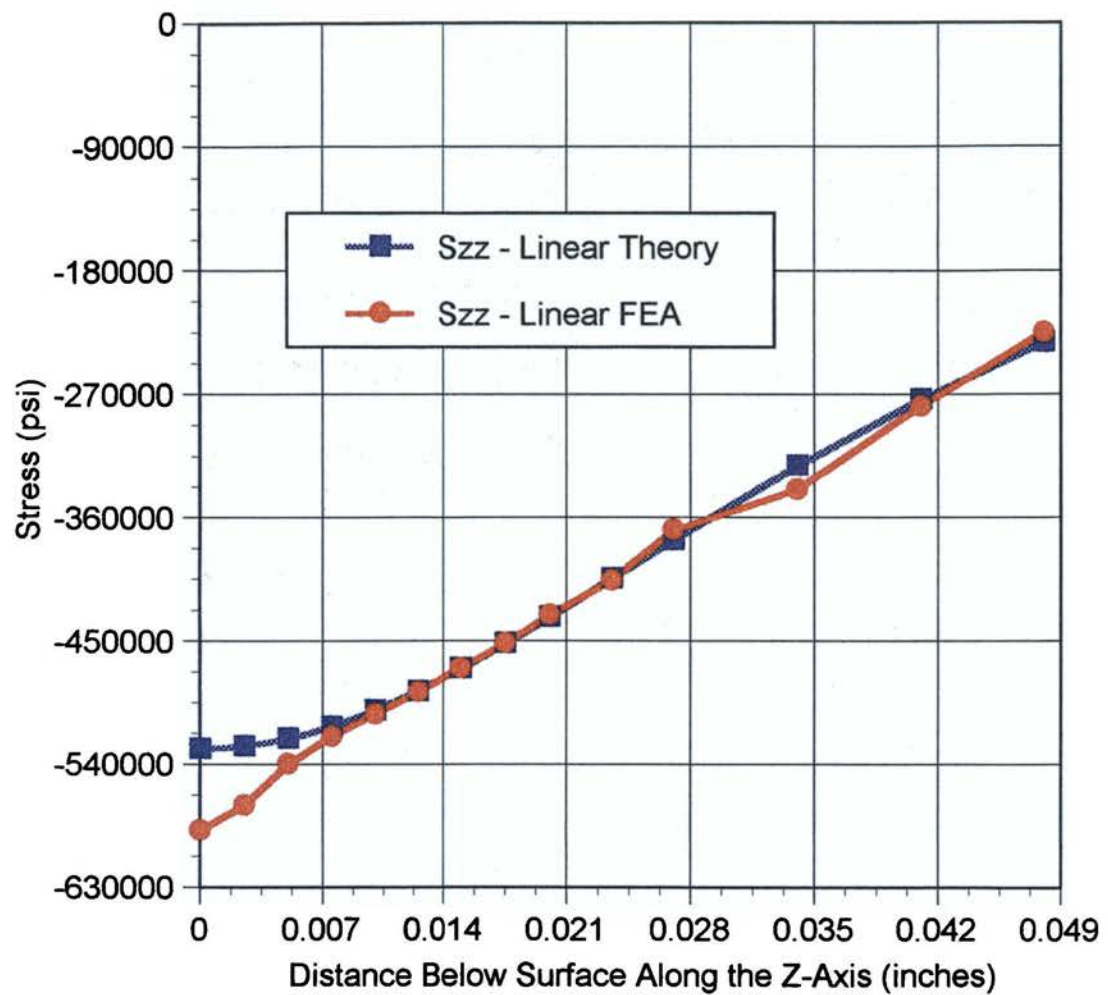


Figure 10: Comparison of linear theory to linear FEA results for σ_{zz} along the z-axis.

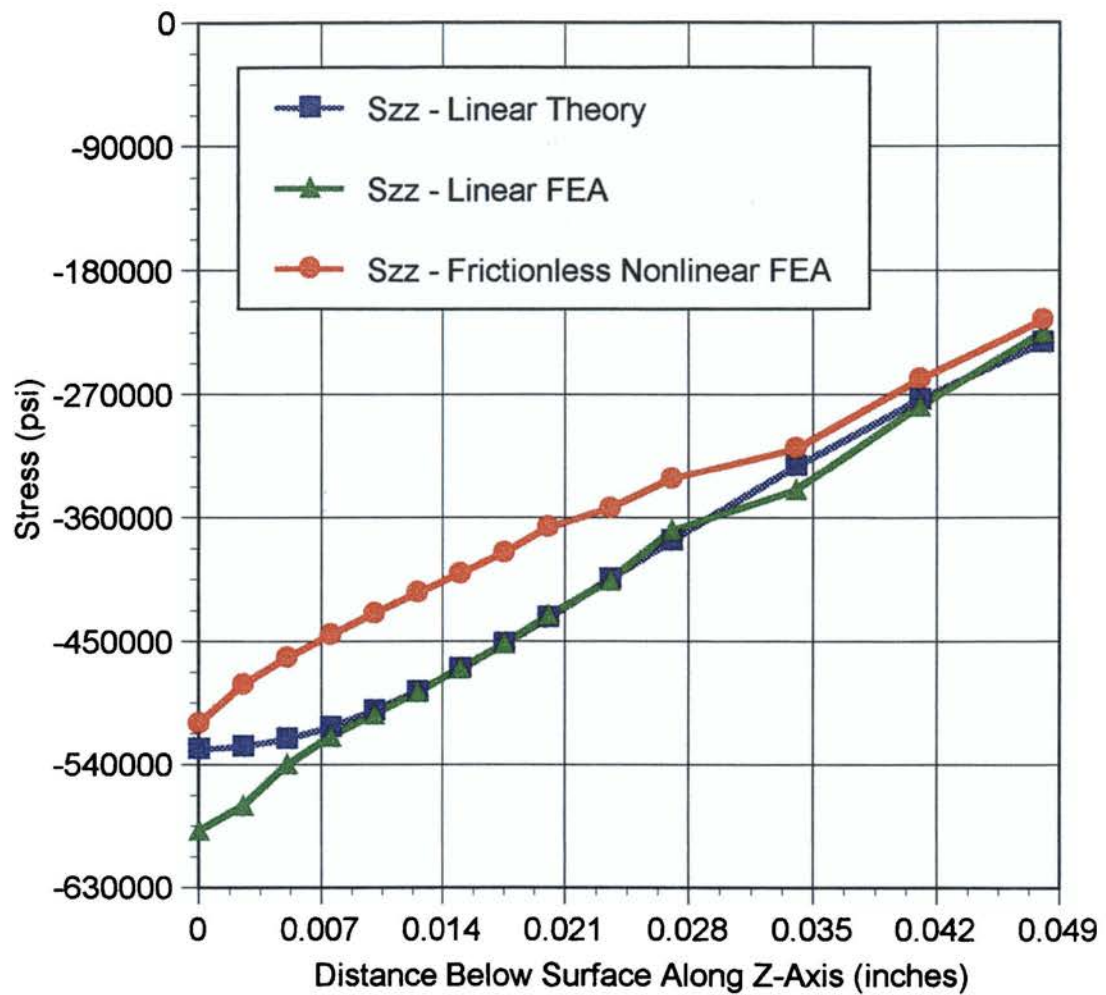


Figure 11: Comparison of linear theory to linear and frictionless nonlinear FEA results for σ_{zz} along the z-axis.

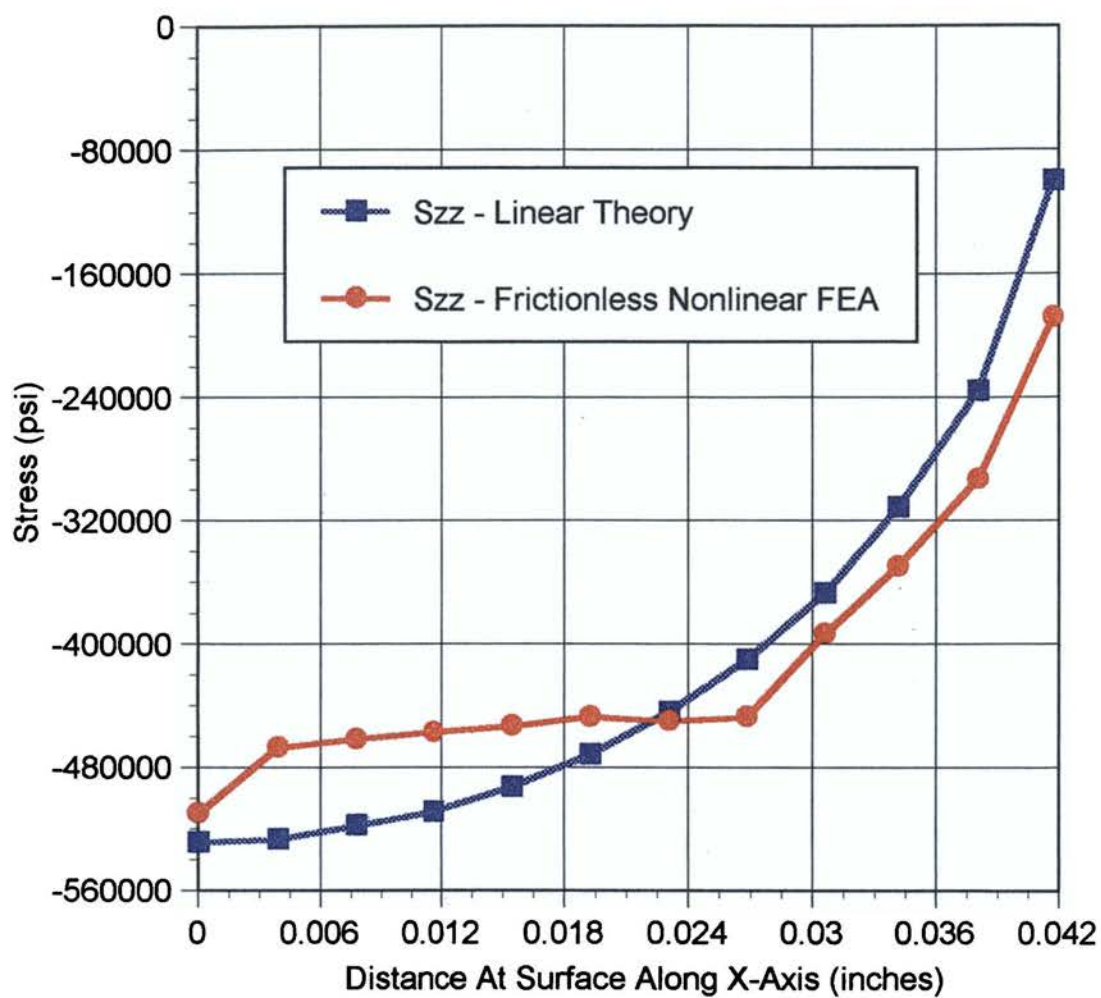


Figure 12: Comparison of linear theory to frictionless nonlinear FEA results for σ_{zz} along the x-axis.

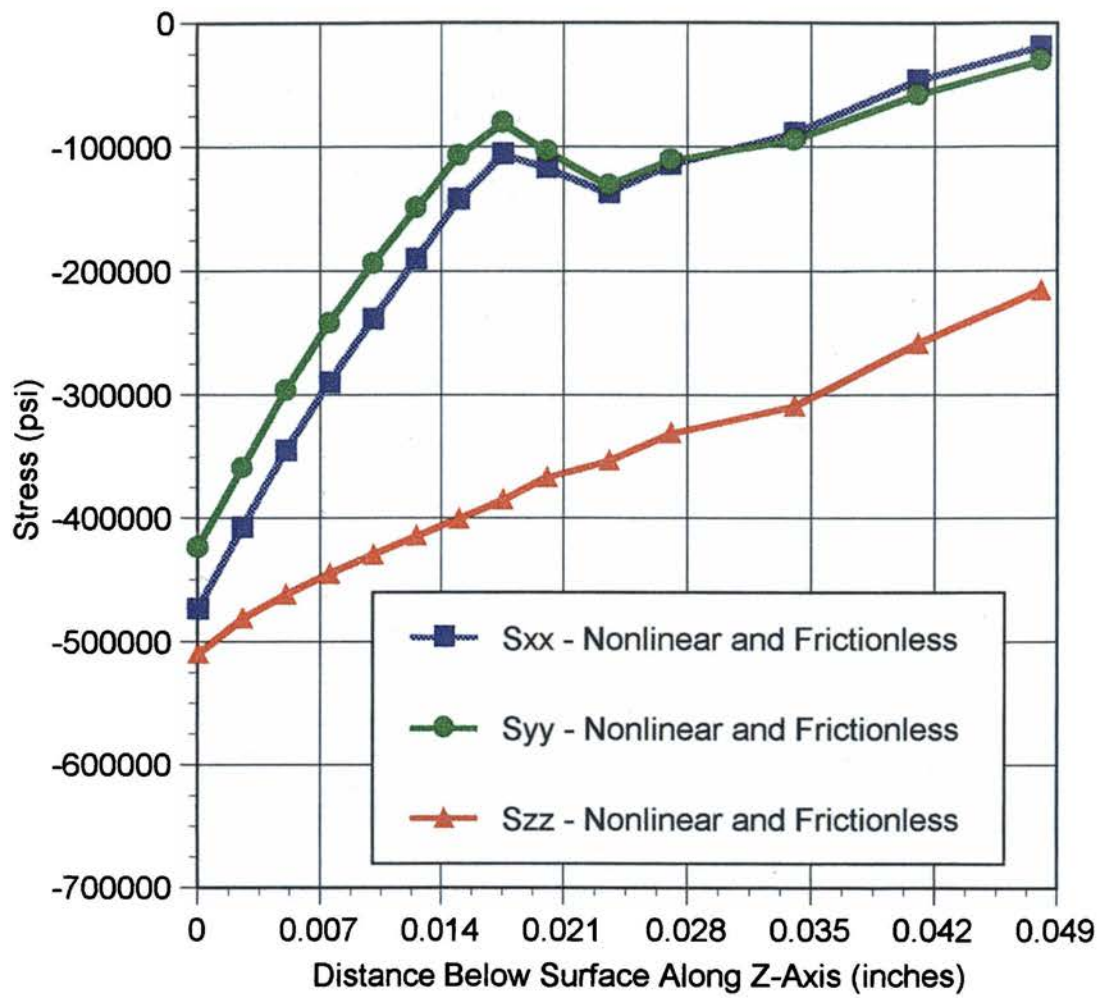


Figure 13: Comparison of frictionless nonlinear FEA results for σ_{xx} , σ_{yy} and σ_{zz} along the z-axis.

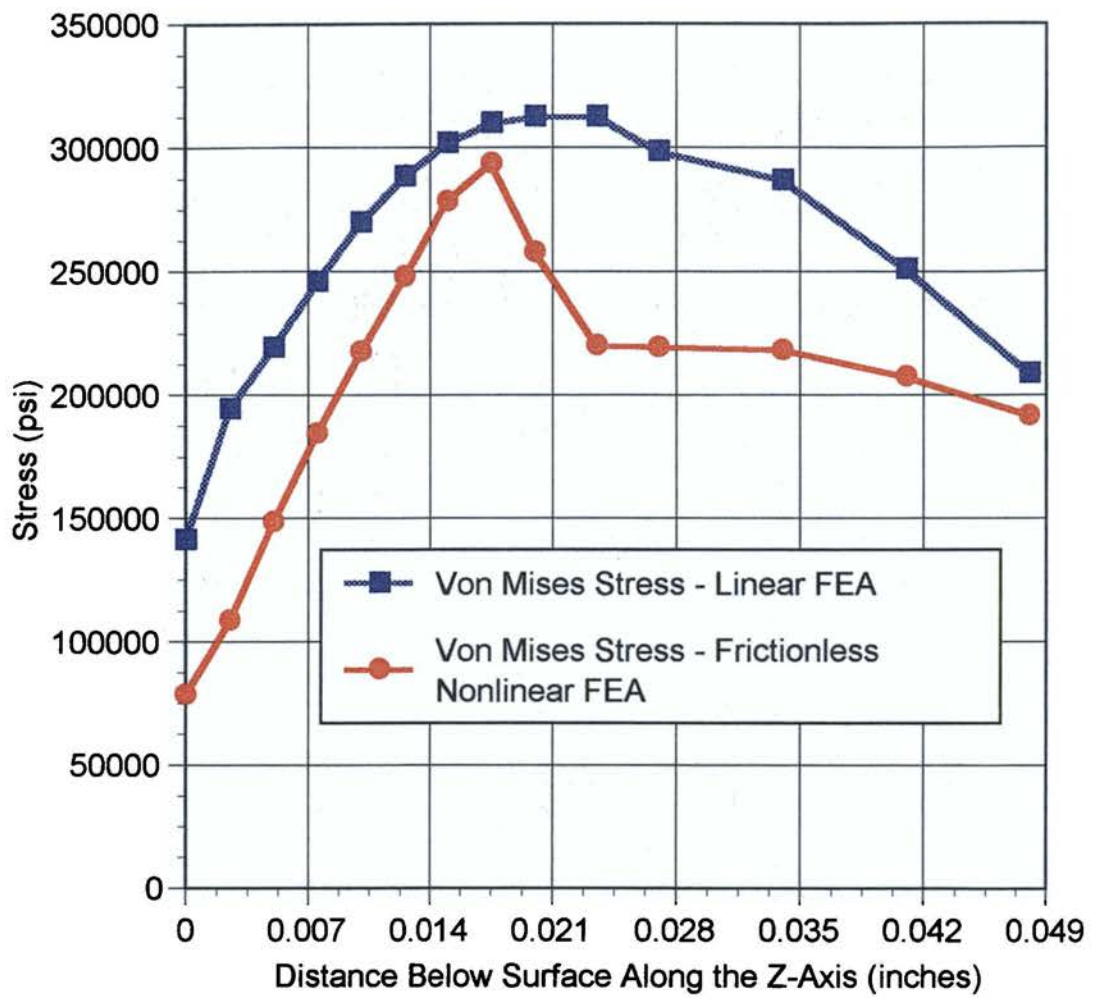


Figure 14: Comparison of von Mises stress along the z-axis for linear and frictionless nonlinear FEA.

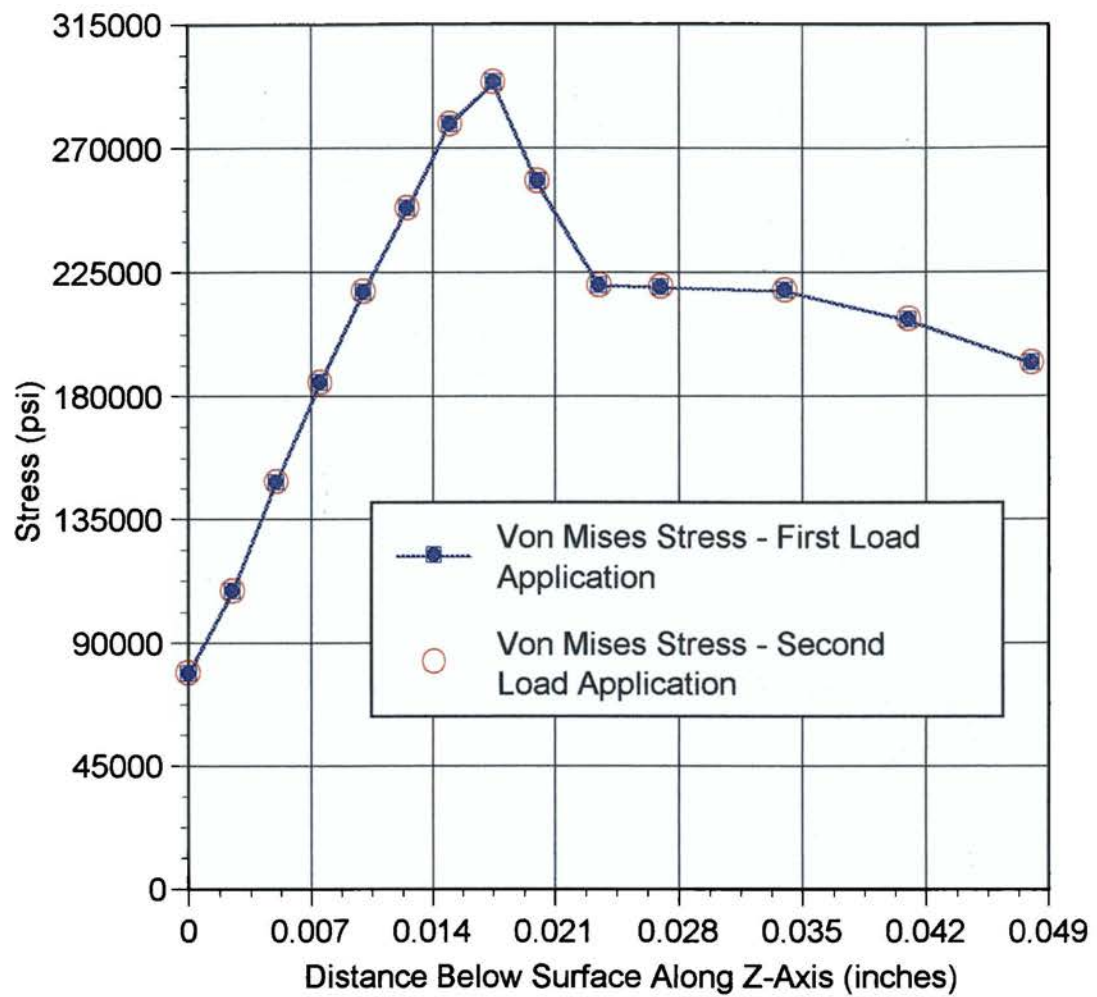


Figure 15: Comparison of von Mises stress along the z-axis after the first and second load applications.

VITA

Julian Raphael was born in New York City on August 1, 1945. He attended public schools in Hewlett, New York, graduating from Hewlett High School in June, 1963. He received the Bachelor of Arts in Mathematics from Adelphi University in January, 1969. Returning to school full time in August, 1980, he received the Bachelor of Science in Mechanical Engineering from Purdue University in August, 1982. He moved to Pennsylvania after graduation, working as a mechanical engineer. He attended Lehigh University part-time and received the Master of Science in Mechanical Engineering in May, 1987. In August, 1996 he returned to The University of Tennessee to pursue the Doctor of Philosophy in Mechanical Engineering. The doctoral degree was received in May, 2000.

He is presently working as Manager of Product Development at Columbus McKinnon Corporation, a manufacturer of both electric and manually operated hoists. He holds several patents in the areas of hoist design and overload protection.