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NONLINEAR UNSTEADY COMBUSTION OF A SOLID PROPELLANT

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Mark William Olles
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To my parents William and Lou Ann Olles

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ABSTRACT

A nonlinear burning model is developed, valid across a wide range of frequencies, to allow for the incorporation of an admittance function into a nonlinear combustion instability (CI) framework. This is accomplished by extending two existing models to incorporate the effects of nonlinearity, the QSHOD (A B) model as described by Culick in his 1968 review paper and the 1973 T'ien model. The classical QSHOD model is marred by major limitations due to the simplifications and assumptions made. The most evident is the exclusion of the unsteady gas phase eliminating the high frequency response. The less recognized T'ien model includes the high frequency unsteady gas phase effects while successfully capturing the low frequency response.

An overlooked and unexpected finding on the sensitivity of a propellant's response on temperature was discovered while recreating the original linear work by T'ien. This temperature sensitivity was first recognized by Brownlee during his T-burner experiments at the Caltech Jet Propulsion Laboratory in 1959, but was not fully understood. Lowering the cold propellant temperature yields an inverse primary admittance peak inferring damping of pressure oscillations. Inversely, by increasing the cold propellant temperature to a value close to that at the burning surface, the low-frequency primary admittance peak is "flattened," also reducing the ability to drive pressure oscillations.

When extended to encompass the second-order nonlinear corrections, the QSHOD model yields a definitive increase in the amplitude of the primary low

frequency response peak. This result is also observed in the extended T'ien model, along with the observed enhancement of wave-amplifying at high frequency due to nonlinear gas phase resonance corrections. The inclusion of the nonlinear high-frequency gas phase interactions explains effects important in understanding the growth of waves from noise to finite-amplitude pressure oscillations observed in Brownlee's data [13], as well as in Shuttle SRB measurements. The occurrence of a mean pressure rise (DC shift) regularly encountered in combustion instability research is found to be partially attributable to the nonlinear steady correction of the burning rate. The appearance of maximum instability occurring in the first tangential mode observed in the Brownlee- Marble data [13] may correspond to the formation of a secondary positive admittance peak due solely to the nonlinear corrections from the extension of T'ien's work. Throughout this work the validity and the limitations of the models are investigated.

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NOMENCLATURE

Symbol	Description
a	burning rate coefficient
A, B	similarity parameters (QSHOD)
A_b	admittance function
B	frequency factor for gas phase reaction
c or C	specific heat
dn	number of moles
E	activation energy
h^0	standard heat of formation
j	stoichiometric oxidizer-fuel ratio
ℓ	characteristic flame length
L	latent heat
m	mass flux
n	burning rate exponent (QSHOD), order of chemical reaction (T'ien)
p	pressure
r	burning rate
$\mathfrak{R}$	gas constant
R_b	response function
q	heat of combustion

$\dot{Q}_d$	average heat release (per unit volume) in the solid
Q_f	average heat release (per unit volume) of reactant
t	time
T	temperature
u, v	velocity
v'	stoichiometric coefficient for reactant
v''	stoichiometric coefficient for product
$v(\infty)$	value of velocity at $x = \infty$, which is the acoustic admittance of the flame
w	dimensionless reaction rate
W_i	molecular weight
x	position in the chamber
$\dot{x}_s$	velocity
Y	species mass fraction
α	thermal diffusivity
β	ratio of solid to gas densities
δ_1	$\lambda^* C_g / \lambda_s^* C_s$
δ_2	λ^* / λ_s^*
ε	perturbation parameter $p' / \bar{p}$
ε_f	fuel mass fraction
$\dot{\varepsilon}$	reaction rate

γ	ratio of specific heats C_p/C_v
λ	thermal conductivity
ρ	density
ω	dimensionless frequency used by T'ien (non-dimensionalized by the reciprocal of gas thermal time $\bar{\alpha}_\infty^*/\bar{u}_\infty^2$)
Ω	dimensionless frequency used by Culick

Subscripts

c	cold propellant
f or ∞	flame edge
g	gas phase
p	solid phase
s	propellant surface
0	moving boundary
$+$	value downstream (gas side)
$-$	value upstream (solid side)

Superscripts

r	real part
i	imaginary part
$(0), (1), \dots$	denotes the perturbation level
$()^*$	dimensional quantity

$\overline{(\quad)}$ mean value

$(\quad)'$ fluctuating value

Acronyms

CI Combustion Instability

QSHOD Quasi-Steady Homogenous One-Dimensional

SRB Solid Rocket Booster

UTSI University of Tennessee Space Institute

CHAPTER I

INTRODUCTION

Combustion Instability (CI)

The presence of undesired oscillations in a combustion chamber is a problem which arises in solid, liquid, and hybrid rockets, as well as in jet engine combustors and afterburners. The observed wave frequencies often match the natural acoustic modes of the chamber and, as a result, the waves are often misinterpreted as sound waves. When the wave structures associated with these oscillations grow to finite amplitudes, they can steepen into shock-like waves. The resulting shock-like waves sometimes result in severe damage to the system. The large amplitude pressure fluctuations grow from acoustic noise that is always present in the flow. The resulting oscillations were first observed during the test firings of solid rocket motors in the late 1930's and were referred to as combustion instabilities.

These pressure oscillations continue to appear in more recent tactical motor experiments by Blomshield [15] (see Figures 1.1 and 1.2 for typical data). Included with the appearance of the pressure oscillations are a rise in the mean pressure, structural vibrations and thrust oscillations. The same holds true for the large shuttle Solid Rocket Boosters SRB, where nonlinear behavior is observed. The waterfall plot shown in Figure 1.3 shows evidence of the aforementioned wave steepening process and displays a nonlinear limit cycle for a large SRB. The nonlinear evolution of pressure wave growth is shown in Figure 1.4. The diagram shows all features of

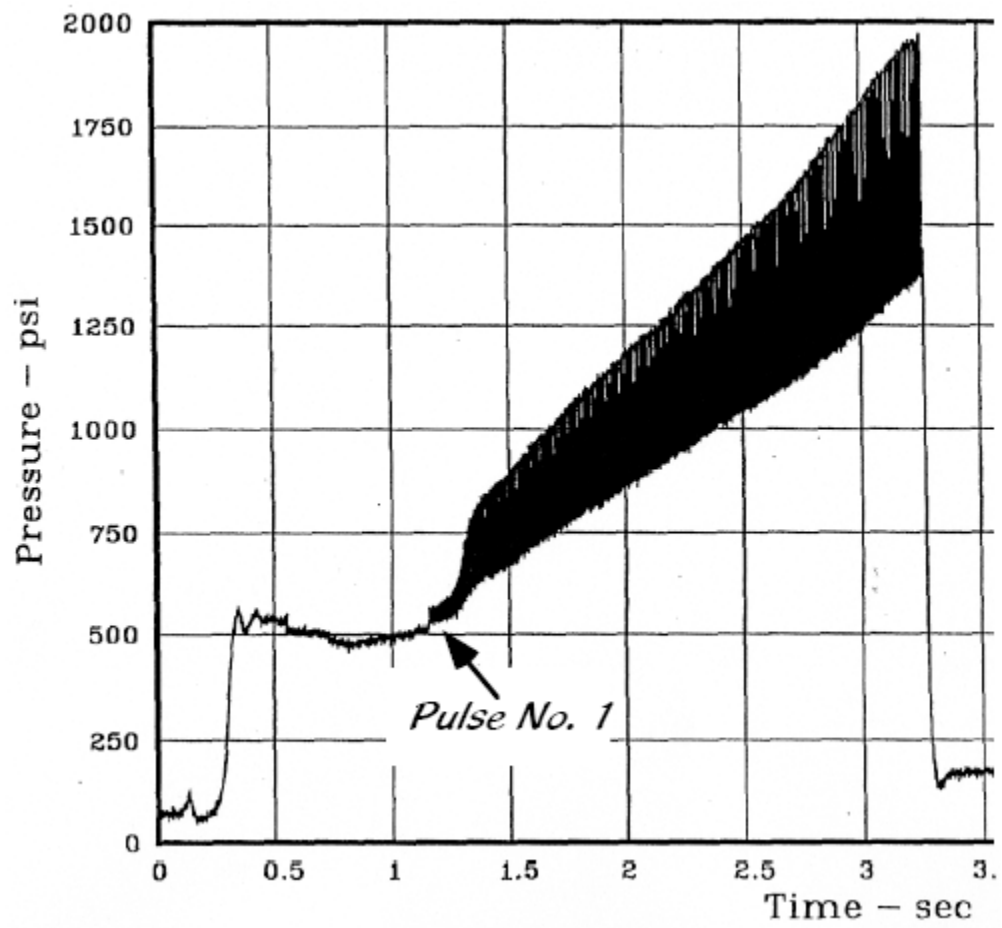


Figure 1.1 Typical Pressure Trace for a Rocket Motor Test (Blomshield 1999) [15]

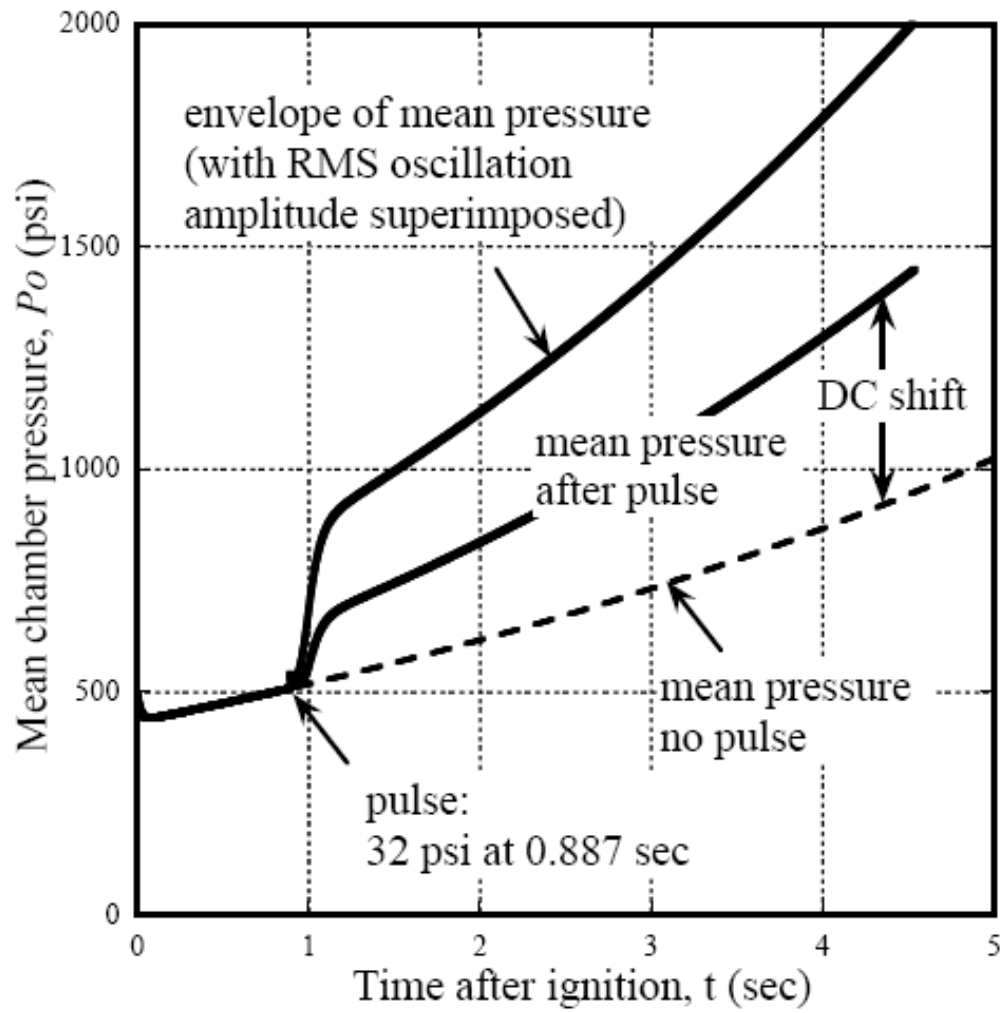


Figure 1.2 Typical Example of a Rocket Motor Test (Blomshield 1999) [15]

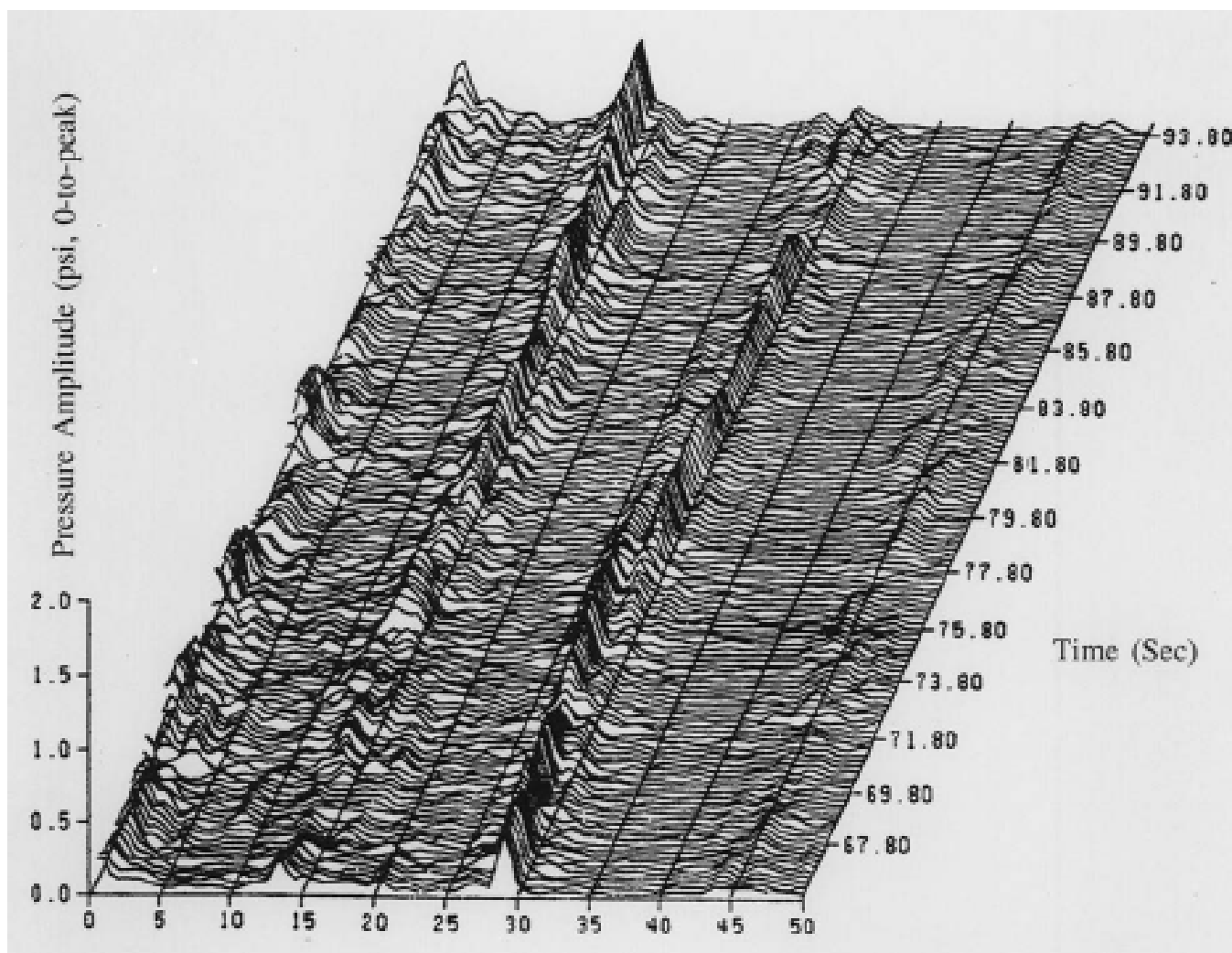


Figure 1.3 SRB Motor Waterfall Plot [14]

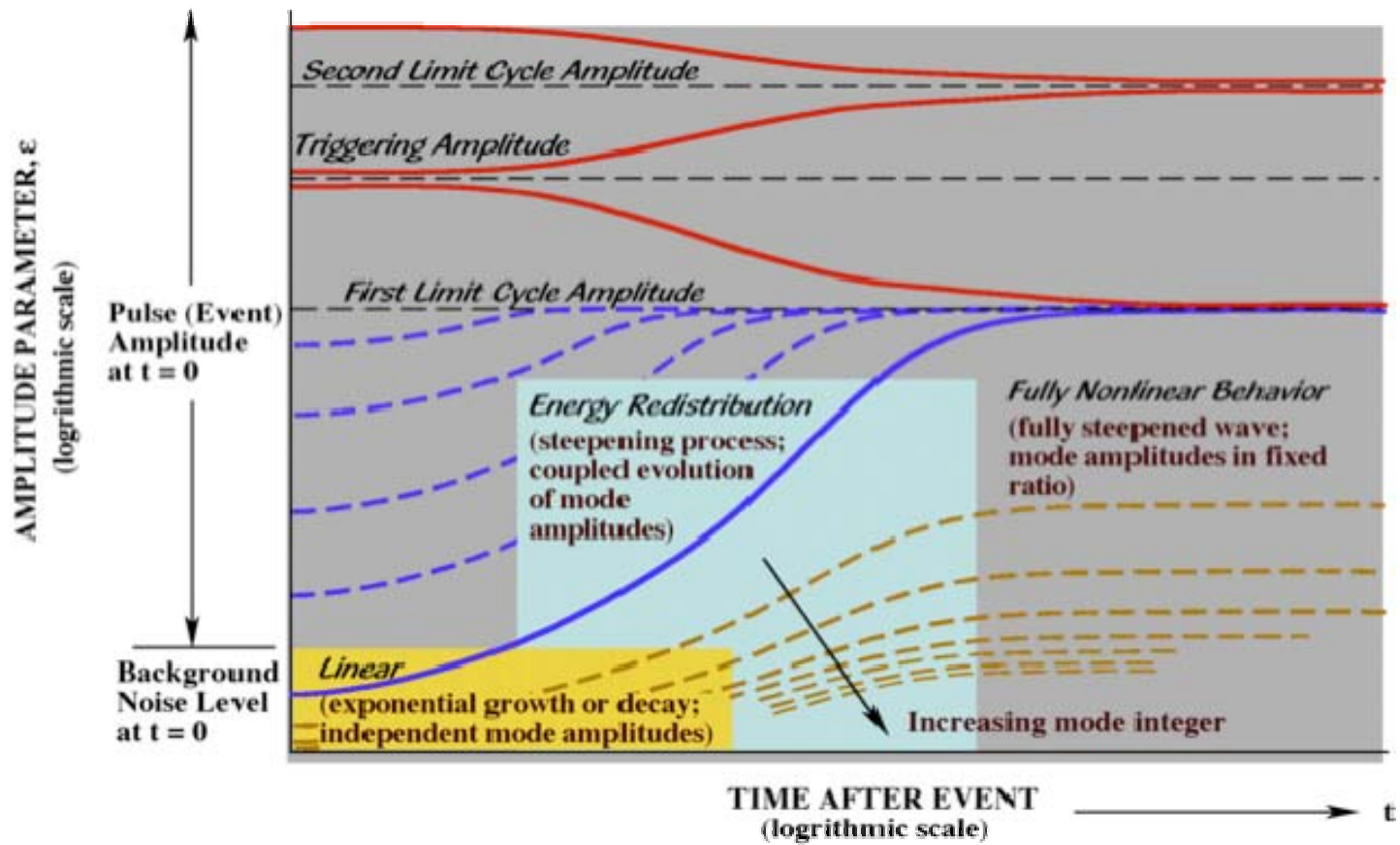


Figure 1.4 Anatomy of Nonlinear Combustion-Driven Pressure Wave Growth [16]

combustion instability that appear experimentally [16]. The figure demonstrates energy transfer from lower to higher modes, the steepening process and the limit cycle. Combustion instabilities also threaten the NASA Ares Program. Even the new five-segment Ares motor exhibits serious vibration and thrust oscillations at 12 Hz, a frequency that can lead to destructive structural resonance.

While there have been considerable advancements in understanding the theory governing combustion instabilities, the ability to predict this behavior is hampered by lack of sufficiently detailed theoretical models that describe the interaction of wave motions in combustion chambers experiencing an unsteady burning process. The continued emergence of these detrimental pressure oscillations indicates that a great deal of work still remains

The term combustion instability can be misleading in gaining an understanding of mechanisms involved with the generation of the associated oscillations. This is due to the fact that combustion instability has very little to do with the combustion process, but more to do with the way that mass is entering the system. The energy released into the system is heavily reliant on the coupling of the fluctuations in the mass entering the system and the unsteady flow processes within the combustion chamber. To clarify how this coupling can occur; assume that the burning rate of a solid rocket motor propellant increases due to a fluctuation in pressure or temperature. If the resulting fluctuation in burning rate is in phase with the initial disturbance, the disturbance grows and the cycle begins again. This example can be modified to describe the behavior of any system subject to combustion instability.

Attempts to understand the underlying mechanisms of the phenomena are based very heavily on classical linear acoustics. The principal restriction of this approach is that the magnitude of the pressure fluctuations must be a small fraction of the ambient chamber pressure. Until recently much of the theoretical work on combustion instability is only focused on the treatment of the linear problems. Note that the solution to the linear problem is a first order approximation, and while it indeed yields a great deal of very useful information, it does not fully describe the complex interactions taking place.

The current work in combustion instabilities is focusing on incorporating nonlinear effects, and remarkable strides have been made. This focus drives the need to find nonlinear components to be incorporated into the CI analysis. The research described here is aimed at just that; extending earlier classical theoretical models to include nonlinear effects. The information provided will prove to be crucial in solving real rocket motor development problems involving combustion instability.

Response and Admittance Functions

To approach the complex coupling effect of mass entering a solid propellant system, and the oscillatory gas effects in the system, a simplified combustion model is necessary. The burning of a solid propellant can be modeled as a one dimensional laminar flame, which is justified by the close proximity of the flame to the burning surface, as seen in Figure 1.5. Heated gases are expelled from the solid surface and releasing energy from solid to gas. This process is referred to as the response

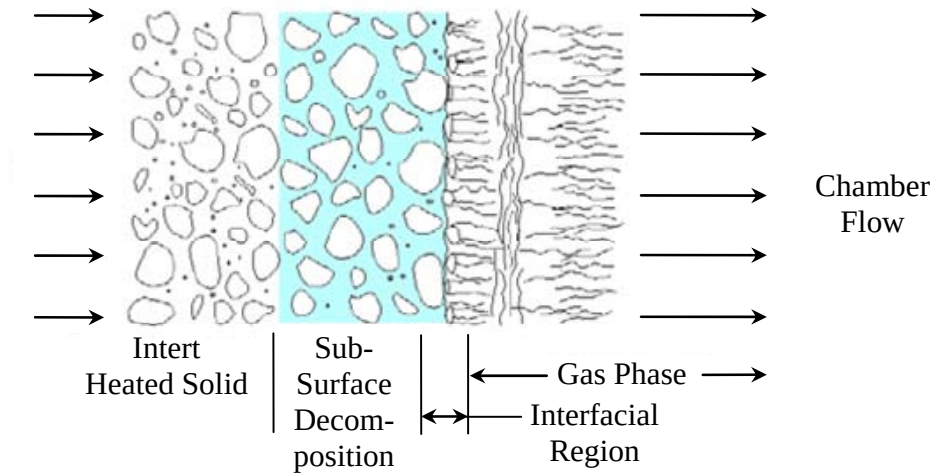


Figure 1.5 Simplified Model of a Burning Solid Propellant [4]

of the propellant. Chemical reactions cause a secondary energy release, which takes place in the flame region. The coupled secondary energy release and surface energy release is referred to as the admittance. The majority of studies on this topic pertain to the situation where the gas oscillations are perpendicular to the combustion zone, which is consistent with the only standardized and cost effective experiment; the T-Burner [9]. The behavior observed is best described as a burning rate oscillation resulting from an imposed pressure oscillation.

The terms “response” and “admittance” functions are commonly used interchangeably in reference to most practical situations involving combustion instability. It should be stressed that the response function does not account for the energy release from the chemical reactions taking place in the flame. Commonly, in practice [5] simply subtract $1/\gamma$ from the response function to acquire the admittance, where γ is the ratio of specific heats. This practice removes the gas phase from the formulation, a great limitation that will be discussed in greater detail in Chapter IV.

The unstable influence of the combustion can more appropriately be characterized by an admittance function characteristic of the propellant [1].

To better understand exactly where the admittance function enters into the combustion instability analysis, a generally accepted growth alpha [9] is defined by:

$$\alpha^{(1)} = \left\{ \begin{array}{l} \frac{1}{2E^2} \iint_{S_b} \bar{M}_b \left(A_b^{(r)} + \frac{1}{\gamma \bar{P}} \right) \left\langle \left(\frac{p'}{\gamma} \right)^2 \right\rangle dS \\ \text{Pressure Coupling} \\ - \frac{1}{2E^2} \iint_{S_b} \bar{M}_N \left(A_N^{(r)} + \frac{1}{\gamma \bar{P}} \right) \left\langle \left(\frac{p'}{\gamma} \right)^2 \right\rangle dS \\ \text{Nozzle Damping} \\ + \frac{1}{2E^2} \iint_{S_{BP}} \bar{M}_b \left\langle \left(\frac{p'}{\gamma} \right)^2 \right\rangle dS \\ \text{Boundary Pumping (Rotational Flow Correction)} \\ - \frac{1}{2E^2} \iint_{S_b} \left(\frac{\delta}{2\gamma \bar{P} \bar{M}_b} \right)^2 \langle \Delta p' \cdot \Delta p' \rangle dS \\ \text{Viscous Damping at Burning Surface} \end{array} \right\} \quad (1.1)$$

where the pressure coupling term contains the admittance function and is the major energy source for oscillations. To devise a fully nonlinear theory it is implicitly necessary to gain knowledge of the nonlinear propellant admittance function A_b .

Defining the Admittance Function

The response function itself is not directly applicable to combustion instability research. Therefore, it is necessary to introduce the admittance function, defined by Culick and Yang [8] as:

$$A_b = \frac{u'/\bar{u}}{p'/\bar{p}} \quad (1.2)$$

Using the identity and assuming isentropic conditions

$$\frac{m'}{\bar{m}} = \frac{u'}{\bar{u}} + \frac{\rho'}{\bar{\rho}} = \frac{u'}{\bar{u}} + \frac{p'}{\gamma \bar{p}} \quad (1.3)$$

rearranging

$$\frac{u'/\bar{u}}{p'/\bar{p}} = \frac{m'/\bar{m}}{p'/\bar{p}} - \frac{1}{\gamma} \quad (1.4)$$

and finally a relation for the admittance as a function of the response function is:

$$A_b^{(r)} = R_b^{(r)} - \frac{1}{\gamma} \quad (1.5)$$

Only the real part of the complex admittance function is retained, this is due to the fact that only the real part of the admittance function is in phase with the oscillating pressure waves.

Experimental Testing

Past studies of unstable combustion in rocket motors were both costly and, for the most part, uninformative. The uninformative nature of the data arose from the lack of an appropriate model to capture the complexity of the phenomenon. If a motor experienced destructive oscillations in full scale testing, the motor designer would draw from a “bag of tricks”, and devise a solution via trial and error. Due to the substantial expense involved in this, a number of relatively inexpensive laboratory experiments were devised to provide less ambiguous conditions and more meaningful

measurements. The most popular and universally accepted experimental setup is the T-Burner [9].

T-Burner

The T-Burner was standardized due to its simplistic geometry that exhibits oscillatory combustion under pre-set conditions of pressure and frequency and its reproducibility. The T-Burner (shown in Figure 1.6) is used in an attempt to measure the response function of propellants, which is particularly important in the field of combustion instabilities. Some of the most complete T-Burner tests were conducted by Perry 1970 [12] and by Brownlee 1959 [13], covering the low frequency solid phase resonance, but failing to cover the necessary frequency range to capture the high frequency gas phase resonance effects. Only one T-burner is still in operation at this time, residing at the Naval Air Warfare Center at China Lake, in California.

Generally, in a T-Burner; a fixed area propellant configuration is used (see Fig. 1.6(b)), but in some cases the so-called “variable area” configuration is utilized (see Figure 1.6(c)). The variable area, or “extended-area”, T-Burner design employs a propellant at the side-wall in addition to the propellant in the conventional end-wall position. The original intent of the variable area was to increase the surface area so that would be easier drive the oscillations. Later work employed a “driver” propellant, with known characteristics, at the head-end with a test propellant as the tubular grain, at the side wall as illustrated in Figure 1.6(c).

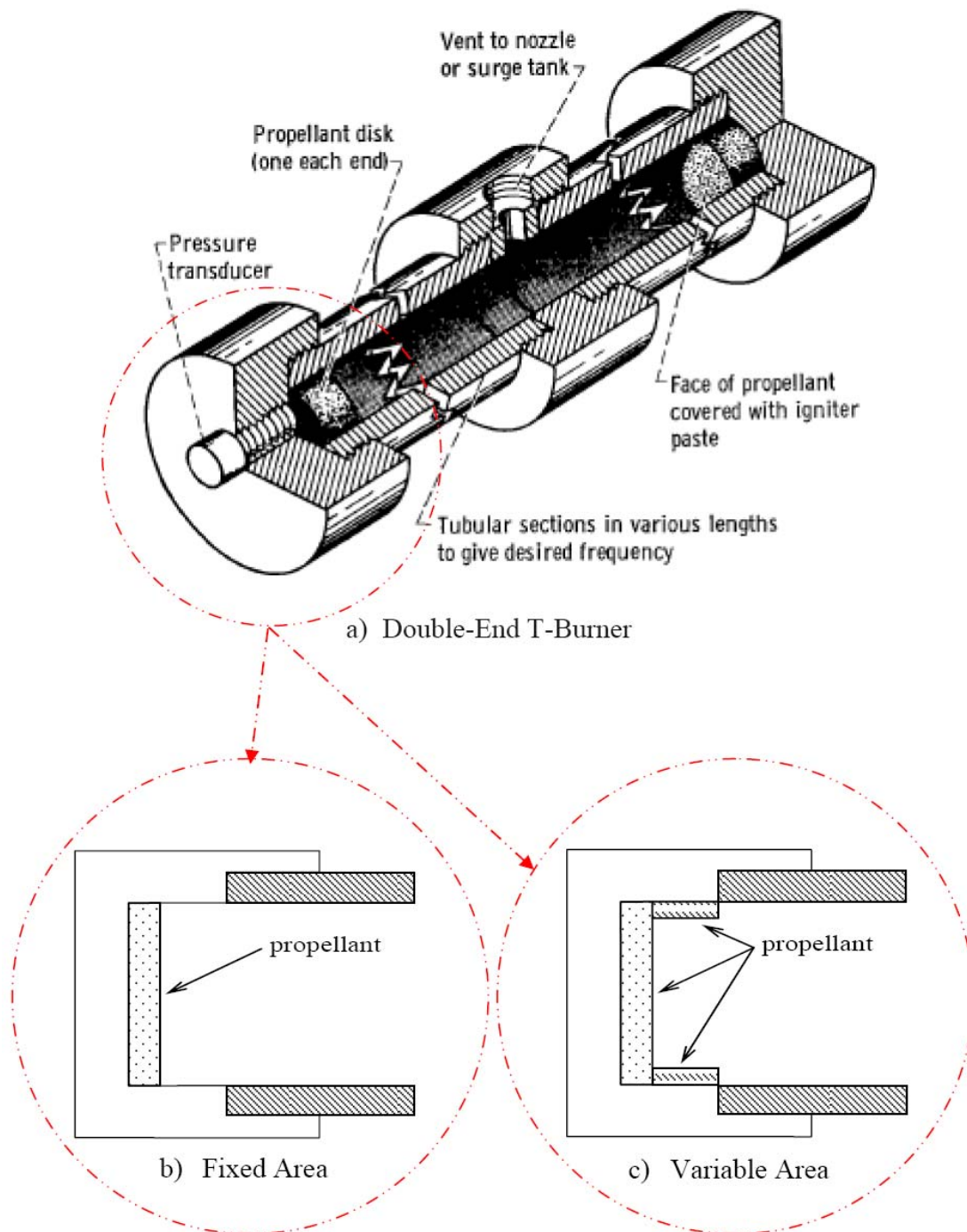


Figure 1.6 Schematic of a T-Burner

Objectives and Thesis Outline

Motivation for this research is to help understand the stability of a solid propellant system, by developing a valid nonlinear burning model to allow for the incorporation of a valid admittance model, across a wide range of frequencies, into nonlinear combustion instability (CI) analysis. The first step is to assess two generally accepted models for the unsteady burning of a solid propellant with the objective of improving them and extending their range of validity into the nonlinear high-frequency range.

An assessment of two key theoretical response models will be made: Culick's QSHOD model (Actually a compilation of over twenty analyses by other investigators), which is based on solid phase resonance, and T'ien's improved analysis including high-frequency gas phase resonance. The outcome will be able to provide predictive capabilities, thus eliminating the need for numerous tests, and act as a tool for selecting propellants due to their stability criteria. The remainder of this section summarizes the main objectives and outlines how and where they are addressed in the chapters that follow.

Chapter II begins by reconstructing a quasi-steady, homogenous, one-dimensional (QSHOD) model as developed by Culick [4]. Throughout the literature, there are discrepancies in the method used to arrive at the final solution, which are made clear in this section. The motivation is to fully understand the model and its limitations, as well as, to gain perspective in attempting to extend the model to encompass the nonlinear corrections.

Chapter III starts by reconstructing and extending the quasi-steady, homogenous, one-dimensional (QSHOD) to incorporate nonlinearity. Again, due to the discrepancies in the literature and difficulty in arriving at the final solution, the method is carried out in full detail recreating the work found in Chapter 2. Our motivation is to gain insight into the methodology to proceed in deriving a nonlinear model that includes the gas phase effects.

Chapter IV is the reconstruction of the linear model constructed by T'ien [1]. As in the previous chapters, the background literature contains discrepancies in the method used to arrive at the final solution, which are made clear in this section. Here, we will fully understand the model and its limitations, as well as to gain perspective in attempting to extend the model to encompass the nonlinear corrections. The final result will be studied parametrically and reviewed.

Chapter V consists of the extension of the linear model constructed by T'ien [1] to include the effects of nonlinearity. The final result will then be studied parametrically and reviewed. Finally, Chapters VI and VII contain concluding remarks and suggestions for future research, respectively.

CHAPTER II

LINEAR QSHOD MODEL

Problem Definition

The following remarks and analysis are based on the model described in the recent book [4] and the 1968 review paper [5] by Culick, which compiles over 20 analyses by other investigators. The steps necessary to derive the model in the reference material are not readily found or necessarily clear. For completeness, the model is described here in full detail.

The model defined is the simplest possible, capturing the dominant solid phase contribution to the combustion dynamics. Only the contribution of the unsteady heat transfer in the condensed phase is assumed to cause dynamical behavior, not attempting to assess the coupling effects associated with the gas phase. The model is defined by the following assumptions:

- (a) **Q**uasi-**S**teady behavior of all processes except unsteady conductive heat transfer in the condensed phase;
- (b) **H**omogeneous and constant material properties, non-reacting condensed phase;
- (c) **Q**ne-**D**imensional variations in space;
- (d) Conversion of condensed material to the gas phase at an infinitesimally thin interface.

The model's acronym QSHOD, is obtained by utilizing the five bold letters in assumptions (a) – (c). The coordinate system used is based on Galilean

transformation such that “observer” moves at the mean burning rate, r . The “moving observer” can be interpreted as cold propellant moving from the left and combustion products exiting the burning zone to the right.

In his review [5] of the works on the unsteady burning of a solid propellant; Culick explained that the ten models referenced from earlier works are for most the part, identical due to the use of the same set of assumptions (a) – (d). Miniscule differences between the models are attributed to the different methods used to represent the gas phase, justifying the use of term QSHOD when referring to the models.

The analysis for this simplified model proceeds through three regions: the temperature field in the solid phase; the temperature field in the gas phase; and finally matching of the two solutions at the interface. Due to the dependence of the model on the temperature field, the final results must be regarded as a thermal theory, lacking description of diffusive processes. A reference system with datum fixed to the average position of the regressing solid surface or interfacial region is chosen. Figure 2.1 depicts this description and Figure 2.2 illustrates the matching conditions to be satisfied at the interface.

Analysis

Solid Phase

The energy equation for the temperature in the solid phase is assumed to have

uniform and constant properties,

$$\lambda_p \frac{\partial^2 T}{\partial x^2} - \bar{m} c_p \frac{\partial T}{\partial x} - \rho_p c_p \frac{\partial T}{\partial t} = \dot{Q}_d \quad (2.1)$$

where $\overline{(\quad)}$ indicates the time-averaged value, the subscript p denotes the solid phase, $\bar{m} = \rho_p \bar{r}$ is the average mass flux in the reference system as defined in Figure 2.1, and $\dot{Q}_d$ is the rate at which energy is released per unit volume due to decomposition of the solid, for simplicity $\dot{Q}_d = 0$. The result of relaxing the $\dot{Q}_d = 0$ assumption has been carried out by Culick [4], but is left out here since it does not add to the underlying theory involved. It is advantageous to make use of the dimensionless variable

$$\xi_p = \frac{\bar{m} c_p}{\lambda_p} x \quad (2.2)$$

substituting this finding into (2.1) yields

$$\frac{\partial^2 T}{\partial \xi_p^2} - \frac{\partial T}{\partial \xi_p} - \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} \frac{\partial T}{\partial t} = 0 \quad (2.3)$$

To seek solutions it is necessary to expand $T = T^{(0)} + T^{(1)} e^{i\omega t}$. Proceeding to collect the steady terms from the expanded (2.3) returns,

$$\frac{d^2 T^{(0)}}{d \xi_p^2} - \frac{dT^{(0)}}{d \xi_p} = 0 \quad (2.4)$$

Integrating (2.4), and applying the boundary conditions $T^{(0)}(0) = T_s^{(0)}$ and $T^{(0)}(-\infty) = T_c^{(0)}$, gives the following formula for the normalized mean temperature

$$T^{(0)}(x) = T_c^{(0)} + (T_s^{(0)} - T_c^{(0)}) \exp\left(\frac{\bar{m}c_p}{\lambda_p} x\right) \quad (2.5)$$

where values at the interface are identified by subscript s and values deep within the cold propellant are defined by subscript c . Collecting the unsteady terms from the expansion of (2.3) presents

$$\frac{d^2 T^{(1)}}{d\xi_p^2} - \frac{dT^{(1)}}{d\xi_p} - i\omega \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} T^{(1)} = 0 \quad (2.6)$$

Equation (2.6) is a Sturm-Liouville boundary value problem that leads to the following equation for $T^{(1)}$,

$$T^{(1)}(x, t) = \tau' \exp\left(\Lambda \frac{\bar{m}c_p}{\lambda_p} x\right) \exp(i\omega t) \quad (2.7)$$

where the eigenvalue Λ must satisfy

$$\Lambda(\Lambda - 1) = i\Omega \quad (2.8)$$

and Ω , is defined as the dimensionless frequency

$$\Omega = \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} \omega^* \quad (2.9)$$

In order to insure that the condition $T^{(1)}(-\infty) \rightarrow 0$, the positive real part of (2.8) must be used. Define $\Lambda = \Lambda_r + i\Lambda_i$ and solve (2.8) for the eigenvalue Λ as a function of Ω

$$\Lambda_r = \frac{\Lambda_i + \Omega}{2\Lambda_i} \quad (2.10)$$

and

$$\Lambda_i = \sqrt{-\frac{1}{8} + \frac{1}{8}\sqrt{1+16\Omega^2}} \quad (2.11)$$

The solution is split into real and imaginary parts because only the real part is in phase with the pressure. Define τ' , from (2.7), as the fluctuation of temperature at the interface ($x = 0$), this is based on the choice of datum.

Attempting to match the conditions at the interface requires knowledge of the values and derivatives of the temperature at the interface itself, referenced to position x_s and velocity $\dot{x}_s$. Values at the interface are calculated by utilizing Taylor series expansions about ($x = 0$), and on the solid side of the interface ($x = x_s$):

$$\begin{aligned} T_s^{(0)}(x_s) &= T_{0-}^{(0)}(0) + x_s \left(\frac{dT^{(0)}}{dx} \right)_{0-} \\ T_s^{(1)}(x_s) &= T_{0-}^{(1)}(0) + x_s \left(\frac{dT^{(0)}}{dx} \right)_{0-} \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \left(\frac{dT^{(0)}}{dx} \right)_{s-} &= \left(\frac{dT^{(0)}}{dx} \right)_{0-} + x_s \left(\frac{d^2T^{(0)}}{dx^2} \right)_{0-} \\ \left(\frac{\partial T^{(1)}}{\partial x} \right)_{s-} &= \left(\frac{\partial T^{(1)}}{\partial x} \right)_{0-} + x_s \left(\frac{d^2T^{(0)}}{dx^2} \right)_{0-} \end{aligned} \quad (2.13)$$

The results for the gas phase side of the interface cannot be completed until the interfacial region is analyzed.

Interfacial Region

The analysis at the interface is governed by three conservation laws: conservation of mass, energy, and the law for conversion of solid to gas

$\left(\rho_p(v_p - \dot{x}_s) = \rho_g(v_g - \dot{x}_s)\right)$. Noting that $\rho_p v_p = \bar{m}$, $\rho_g v_g = m$ and where $m = \bar{m} + m^{(1)} \exp(i\omega t)$, we see that

$$\rho_p \dot{x}_s = \left(1 - \frac{\rho_g}{\rho_p}\right)^{-1} \bar{m} - m \quad (2.14)$$

The mean gas density ρ_g near the surface is much smaller than the density of the condensed phase ρ_p , which yields $\rho_g / \rho_p \ll 1$ allowing the term to be dropped.

$$\rho_p \dot{x}_s \approx -m^{(1)} \exp(i\omega t) \quad (2.15)$$

Isolating the regression rate $\dot{x}_s$ in (2.15) and integrating to find x_s

$$x_s = -\frac{1}{\rho_p} \int m^{(1)} \exp(i\omega t) dt \quad (2.16)$$

Using (2.16) to find x_s the first order approximation is

$$x_s^{(1)} = -\frac{1}{i\omega\rho_p} m^{(1)} \exp(i\omega t) \quad (2.17)$$

The conditions associated with the total mass and energy transfer in the upstream and downstream sides of the interface are found by summing the terms associated with the mass flux and energy transfer from Figure 2.2. More specifically,

$$\begin{aligned} & -\left(\lambda_p \frac{\partial T}{\partial x}\right)_{s-} + \rho_p (v_p - \dot{x}_s) (c_p T_s + L_s + Q_g) \\ & = -\left(\lambda_g \frac{\partial T}{\partial x}\right)_{s+} + \rho_g (v_g - \dot{x}_s) (c_g T_s + Q_g) \end{aligned} \quad (2.18)$$

From mass balance we know that $\rho_p (v_p - \dot{x}_s) = \rho_g (v_g - \dot{x}_s)$ so,

$$\left(\lambda_g \frac{\partial T}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T}{\partial x} \right)_{s-} + \rho_p (v_p - \dot{x}_s) (c_g T_s - c_p T_s + L_s) \quad (2.19)$$

again noting that $\bar{m} = \rho_p v_p$ and using (2.15),

$$\left(\lambda_g \frac{\partial T}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T}{\partial x} \right)_{s-} + m (c_g T_s - c_p T_s + L_s) \quad (2.20)$$

The heat flux $(\lambda_g \partial T / \partial x)_{s+}$ heat flow from the interface to the gas phase and the heat flux $(\lambda_p \partial T / \partial x)_{s-}$ is the heat flow from the interface to the solid phase, note that (2.20) has not yet been split into mean and fluctuating parts. For steady combustion, the energy balance (2.20),

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T^{(0)}}{\partial x} \right)_{s-} + \bar{m} (c_g T_s^{(0)} - c_p T_s^{(0)} + L_s) \quad (2.21)$$

A solution for $(\lambda_p \partial T / \partial x)_{s-}$ can be found by differentiating (2.5) and assuming the gas is ideal,

$$\left(\lambda_p \frac{\partial T^{(0)}}{\partial x} \right)_{s-} = \bar{m} c_p (T_s^{(0)} - T_c^{(0)}) \quad (2.22)$$

Substituting this finding into (2.21) yields

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \bar{m} (c_g T_s^{(0)} - c_p T_c^{(0)} + L_s) \quad (2.23)$$

The linear unsteady part of (2.20), and again assuming the gas is ideal, is

$$\left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T^{(1)}}{\partial x} \right)_{s-} + m^{(1)} L_s + \bar{m} (c_g - c_p) T_s^{(1)} \quad (2.24)$$

Combining (2.5), (2.7) and the proper parts of (2.12) gives the formula for the heat transfer into the condensed phase from the interface:

$$\left(\lambda_p \frac{\partial T^{(1)}}{\partial x} \right)_{s-} = \bar{m} c_p \left[\Lambda T_s^{(1)} - \frac{\bar{m} c_p}{\lambda_p \rho_p i \omega} (T_s^{(0)} - T_c^{(0)}) m^{(1)} \right] \quad (2.25)$$

With the knowledge that $\Lambda(\Lambda - 1) = \lambda_p \rho_p i \omega / \bar{m}^2 c_p$, carrying out a perturbation expansion to leading order,

$$\frac{\bar{m}^2 c_p}{\lambda_p \rho_p i \omega} = \frac{1}{\Lambda(\Lambda - 1)} = -\frac{1}{\Lambda} \quad (2.26)$$

The previous step is cause for concern, as the limitation Λ must be small to allow for this simplification. The concern is based on the inherent dependence of Λ on Ω , by limiting Λ to be small, so must Ω also be small. Proceed by substituting the result in (2.26) into (2.25) yields

$$\left(\lambda_p \frac{\partial T^{(1)}}{\partial x} \right)_{s-} = \bar{m} c_p \left[\Lambda T_s^{(1)} + \frac{1}{\Lambda} (T_s^{(0)} - T_c^{(0)}) \frac{m^{(1)}}{\bar{m}} \right] \quad (2.27)$$

In this result, the approximation in (2.15) has been used. Substitution of (2.27) in (2.24) leads to the condition to be set on the unsteady heat transfer at the downstream side of the interface:

$$\frac{\lambda_g}{\bar{m} c_p} \left(\frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \left(\Lambda + \frac{c_g}{c_p} - 1 \right) T_s^{(1)} + \left[\frac{1}{\Lambda} (T_s^{(0)} - T_c^{(0)}) + \frac{L_s}{c_p} \right] \frac{m^{(1)}}{\bar{m}} \quad (2.28)$$

Gas Phase

To proceed, it is necessary to follow the analysis of a simplified model for steady burning, given by Culick [5]. This is accomplished by assuming constant

thermal properties and uniform combustion throughout the gas phase. Two limits that are contained in this model are an infinitesimally thin flame sheet and uniform combustion beginning at the interface. Figure 2.3 provides a sketch of the model where x_s , x_i and x_f indicate the position of the solid surface, the leading edge of the combustion zone and the downstream edge of the gas phase, respectively.

The equation defining the gas phase is

$$mc_g \frac{dT}{dx} - \frac{d}{dx} \left(\lambda_g \frac{dT}{dx} \right) = \rho_g \dot{\epsilon} Q_f \quad (2.29)$$

where the subscript g denotes values for the gas phase, the constant Q_f is the energy release per unit mass of the reactant mixture and $\dot{\epsilon}$ is the reaction rate. The temperature boundary condition at the downstream edge, $x = x_f$, of the gas phase is

$$T(x_f) = T_f^{(0)} \quad (2.30)$$

where the constant $T_f^{(0)}$ is the adiabatic flame temperature. The derivative condition

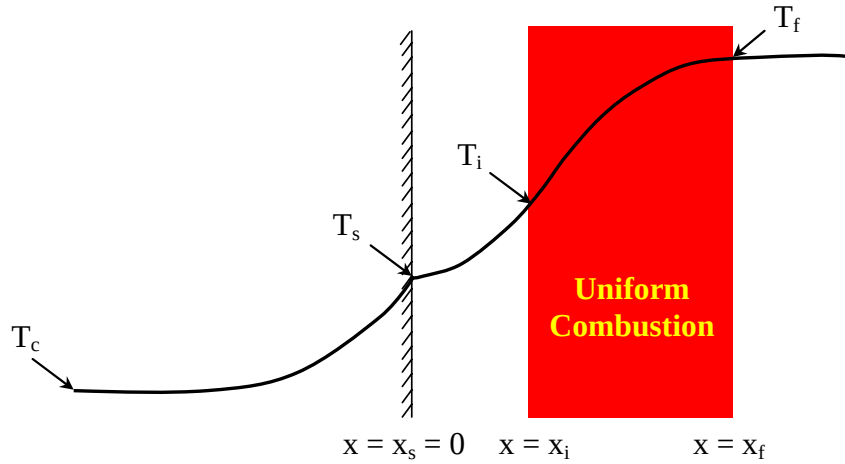


Figure 2.3 Model of a Burning Solid Propellant [4]

at the downstream edge, $x = x_f$, is

$$\frac{dT(x_f)}{dx} = 0 \quad (2.31)$$

and at the interface, ($x = x_s$), the temperature boundary condition is

$$T(x_s) = T_s^{(0)} \quad (2.32)$$

The energy balance at the interface is given by (2.23), assuming steady combustion the energy flow across the gas phase is

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \bar{m} \left[Q_f - c_g (T_f^{(0)} - T_s^{(0)}) \right] \quad (2.33)$$

Alternatively, integrating (2.29) across the combustion zone, and applying boundary conditions (2.30) - (2.32) leads to

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \int_0^\infty (\rho_g Q_f \dot{\epsilon}) dx - \bar{m} c_g (T_f^{(0)} - T_s^{(0)}) \quad (2.34)$$

With the knowledge that Q_f is constant, the reaction rate is found by comparing (2.33) and (2.34), leading to

$$\bar{m} = \int_0^\infty (\rho_g \dot{\epsilon}) dx \quad (2.35)$$

To attempt to solve (2.29) it is convenient to transform x to the dimensionless variable ζ , with the assumption that assuming mc_g/λ_g is constant, namely by

$$\frac{d\zeta}{dx} = \frac{mc_g}{\lambda_g} \zeta \quad (2.36)$$

Substituting the transformation into (2.29) gives

$$-\zeta^2 \frac{d^2 T}{d\zeta^2} = X \quad (2.37)$$

where the eigenvalue X is

$$X = \frac{\lambda_g Q_f w}{m^2 c_g^2} \quad (2.38)$$

and

$$w = \rho_g \dot{\varepsilon} \quad (2.39)$$

The previous steps can only be arrived at if X is assumed to be independent of ζ , which defines the condition of uniform combustion. Integration of (2.37) evaluated over the combustion zone, ζ_i at the leading edge of the combustion zone and ζ_f at the downstream edge of the flame, yields

$$\left(\frac{\partial T}{\partial \zeta} \right)_{s+} = \left(\frac{\zeta_f - \zeta_i}{\zeta_i \zeta_f} \right) X \quad (2.40)$$

Evaluating (2.40), at the leading edge of the combustion phase the dimensionless ζ_i is equal to unity, and $\zeta_f \gg \zeta_i$, which simplifies (2.40) to

$$\left(\frac{\partial T}{\partial \zeta} \right)_{s+} = X \quad (2.41)$$

Substituting the eigenvalue X into (2.41)

$$\left(\lambda_g \frac{\partial T}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f}{c_g} \frac{w}{m} \quad (2.42)$$

The steady form of equation (2.42)

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f}{c_g} \frac{w^{(0)}}{\bar{m}} \quad (2.43)$$

The fluctuation of heat transfer at the surface is given by the unsteady form of (2.42)

$$\left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f}{c_g} \frac{w^{(0)}}{\bar{m}} \left(\frac{w^{(1)}}{w^{(0)}} - \frac{m^{(1)}}{\bar{m}} \right) \quad (2.44)$$

To continue it is necessary to specify an explicit form for w , the reaction rate per unit volume, as a function of the flow variables. To do so, equate (2.23) and (2.43), and the quasi-steady behavior allows the average values of temperature to be replaced by instantaneous values. For this specific special case of steady uniform combustion, the expression for w :

$$w = \frac{c_g m^2}{\lambda_g Q_f} (c_g T_s - c_p T_c + L_s) \quad (2.45)$$

Assuming that the right-hand side is a function of pressure only, it is convenient to introduce the pyrolysis law $m = ap^n$. Where a and n are the burning rate coefficient and the burning rate exponent respectively. Expanding the pyrolysis law appropriately

$$m = \bar{m} + m^{(1)} e^{i\omega t} = 1 + an \frac{p^{(1)}}{p^{(0)}} e^{i\omega t} \quad (2.46)$$

Collecting the terms to the appropriate order, unsteady,

$$\frac{m^{(1)}}{\bar{m}} = an \frac{p^{(1)}}{p^{(0)}} \quad (2.47)$$

Expanding at leading order (2.45) becomes

$$\frac{w}{w^{(0)}} = \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \left(\frac{m}{\bar{m}} \right)^2 \quad (2.48)$$

where

$$H = \frac{L_s}{c_g T_s - c_p T_c} \quad (2.49)$$

and $X^{(0)}$ is found by taking the steady components of (2.38),

$$X^{(0)} = \frac{\lambda_g Q_f w^{(0)}}{\bar{m}^2 c_g^2} \quad (2.50)$$

The fluctuation of the reaction rate is then

$$\frac{w^{(1)}}{w^{(0)}} = 2 \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \frac{m^{(1)}}{\bar{m}} \quad (2.51)$$

Substituting (2.47) into (2.51) the fluctuation of the reaction rate is then

$$\frac{w^{(1)}}{w^{(0)}} = 2an \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \frac{p^{(1)}}{p^{(0)}} \quad (2.52)$$

With all three phases defined, matching of the solid phase to the interface and interface to the gas phase can commence.

Construction of the Linear Response Function

In formulating the response function it is necessary to account for the conversion of a solid to a gas. Proceed by utilizing the commonly used Arrhenius law, which gives the total surface mass flux

$$m = B \exp(-E_s / \Re T_s) \quad (2.53)$$

the perturbed form of (2.53), to first order is

$$\frac{m^{(1)}}{\bar{m}} = E \frac{T_s^{(1)}}{T_s^{(0)}} \quad (2.54)$$

where the dimensionless activation energy is $E = E_s / \mathfrak{R}T_s^{(0)}$. Substitute the Arrhenius law (2.54) into (2.28) defining the interfacial conditions for energy and mass transfer:

$$\frac{\lambda_g}{\bar{m}c_p T_s^{(0)}} \left(\frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \frac{1}{E} \left[\Lambda + \frac{c_g}{c_p} - 1 + \frac{E}{\Lambda} \left(1 - \frac{T_c^{(0)}}{T_s^{(0)}} \right) + \frac{EL_s}{T_s^{(0)}c_p} \right] \frac{m^{(1)}}{\bar{m}} \quad (2.55)$$

letting

$$L = \frac{L_s}{T_s^{(0)}c_p} \quad (2.56)$$

and

$$A = E \left(1 - \frac{T_c^{(0)}}{T_s^{(0)}} \right) \quad (2.57)$$

Substituting (2.56) and (2.57) into (2.55), yields

$$\frac{1}{\bar{m}c_p T_s^{(0)}} \left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \frac{1}{E} \left(\Lambda + \frac{A}{\Lambda} + \frac{c_g}{c_p} - 1 + EL \right) \frac{m^{(1)}}{\bar{m}} \quad (2.58)$$

Substitute the reaction rate (2.52) into the expression (2.44) for the heat loss from the gas phase to the interface:

$$\frac{1}{\bar{m}c_g T_s^{(0)}} \left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = 2an \left(1 - \frac{c_p}{c_g} \frac{T_c^{(0)}}{T_s^{(0)}} \right) (1+H) \frac{p^{(1)}}{p^{(0)}} - \frac{X^{(0)}}{T_s^{(0)}} \frac{m^{(1)}}{\bar{m}} \quad (2.59)$$

Equate the findings from (2.58) and (2.59), and use the Arrhenius law (2.54) to eliminate $T_s^{(1)} / T_s^{(0)}$, leaving the pressure coupled response function:

$$R_b^{(1)} = \frac{\frac{m^{(1)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{2anE \left(1 - \frac{c_p}{c_g} \frac{T_c^{(0)}}{T_s^{(0)}} \right) (1+H)}{\Lambda + \frac{A}{\Lambda} + \frac{c_g}{c_p} - 1 + EL + E \frac{X^{(0)}}{T_s^{(0)}}} \quad (2.60)$$

Finally rewrite (2.60) in the form

$$R_b^{(1)} = \frac{\frac{m^{(1)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{anC_1}{\left(\Lambda + \frac{A}{\Lambda}\right) + C_2} \quad (2.61)$$

Utilizing the unsteady pyrolysis law (2.47), (2.61) can be written as a function of two variables by:

$$an = \frac{anC_1}{\left(\Lambda + \frac{A}{\Lambda}\right) + C_2} \quad (2.62)$$

where Λ can be defined in the limit of zero frequency, $\Omega = 0$, giving the condition

$$\frac{C_1}{1 + A + C_2} = 1 \quad (2.63)$$

define C_1 with

$$C_1 = B \quad (2.64)$$

Substituting this into (2.63), it follows that

$$C_2 = B - (1 + A) \quad (2.65)$$

Now that $C_1 = B$ and $C_2 = B - (1 + A)$ are defined substitute back into (2.61), yielding

$$R_b^{(1)} = \frac{m'}{p'} = \frac{\frac{m^{(1)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{nB}{\left(\Lambda + \frac{A}{\Lambda}\right) + B - (1 + A)} \quad (2.66)$$

Equation (2.66) is the first order pressure coupled response function, as found by Culick [4]. To be incorporated into a combustion instability calculation, the response

function must be put into the form of the admittance function, achieved by using (1.5) and substituting (2.66)

$$A_b^{(r)} = R_b^{(1)} - \frac{1}{\gamma} = \frac{nB}{\left(\Lambda + \frac{A}{\Lambda}\right) + B - (1 + A)} - \frac{1}{\gamma} \quad (2.67)$$

Results and Discussion

With the response and admittance functions defined by equations (2.66) and (2.67), figures can be generated by using values suggested by Culick [8]. Figure 2.4 successfully reproduces the low frequency result for the response obtained by Culick [4], and Figure 2.5 demonstrates the low frequency admittance. The result shown in Figure 2.6 demonstrates a strong correlation of the admittance function to experimental data found by Perry [12], reinforcing that the QSHOD model does capture the low frequency behavior. The Perry data [12] used in Figure 2.6 was recorded using the standardized growth and decay technique [9] for the experiment.

The quasi-steady theory is marred by limitations due to the simplifications and assumptions made. The first is the exclusion of the unsteady gas phase which excludes the high frequency response. The lack in physical meaning of the similarity variables A and B restricts the model to curve fitting tool for low frequency data.

Although the QSHOD model is limited, it does give insight into the solid phase contribution to the response function. The investigation conducted in Chapter III will build on the process used here to gain further understanding into the effects of the second-order wave amplitude.

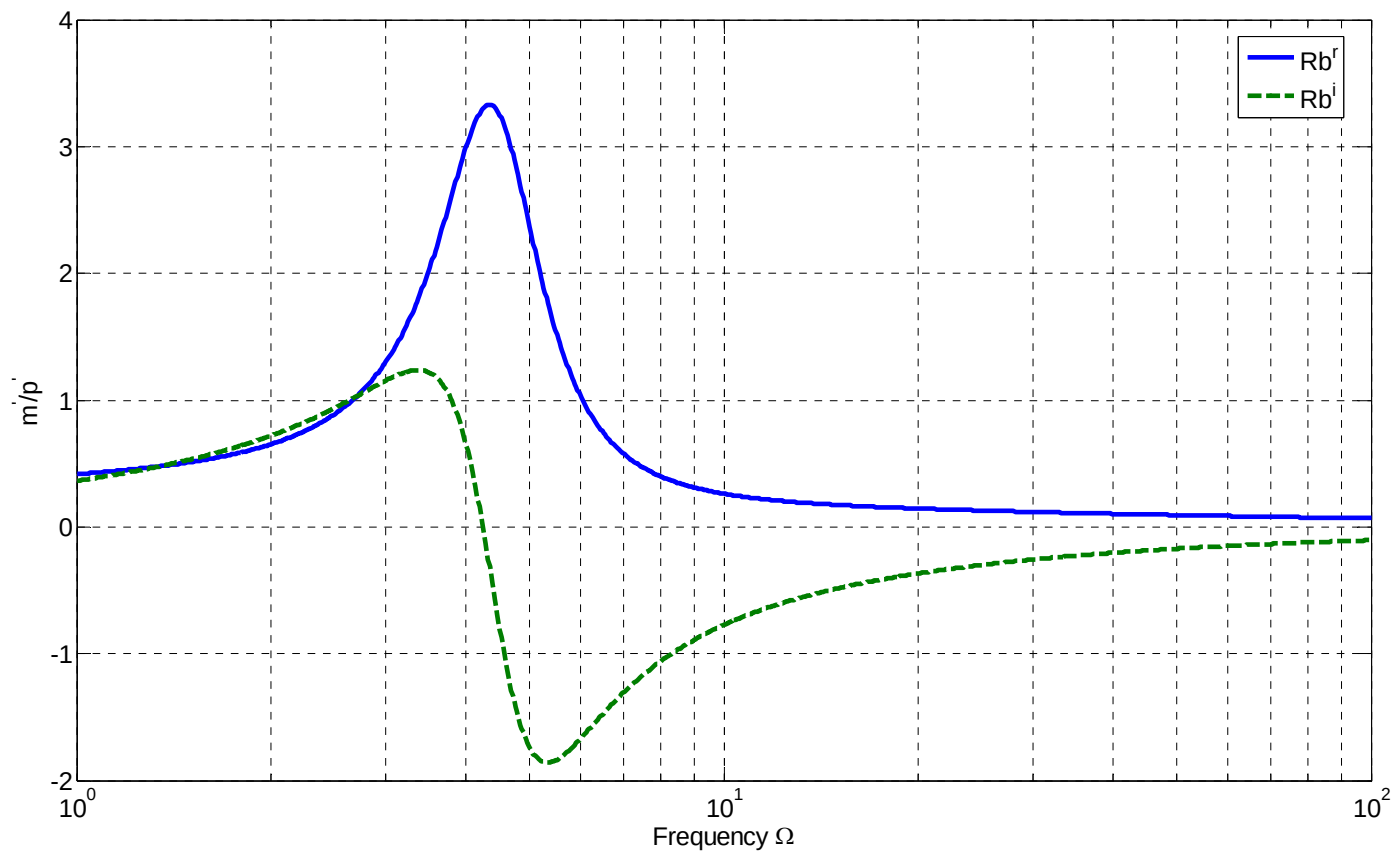


Figure 2.4 Response Function: $n = 0.3$, $A = 6.0$, $B = 3.3$

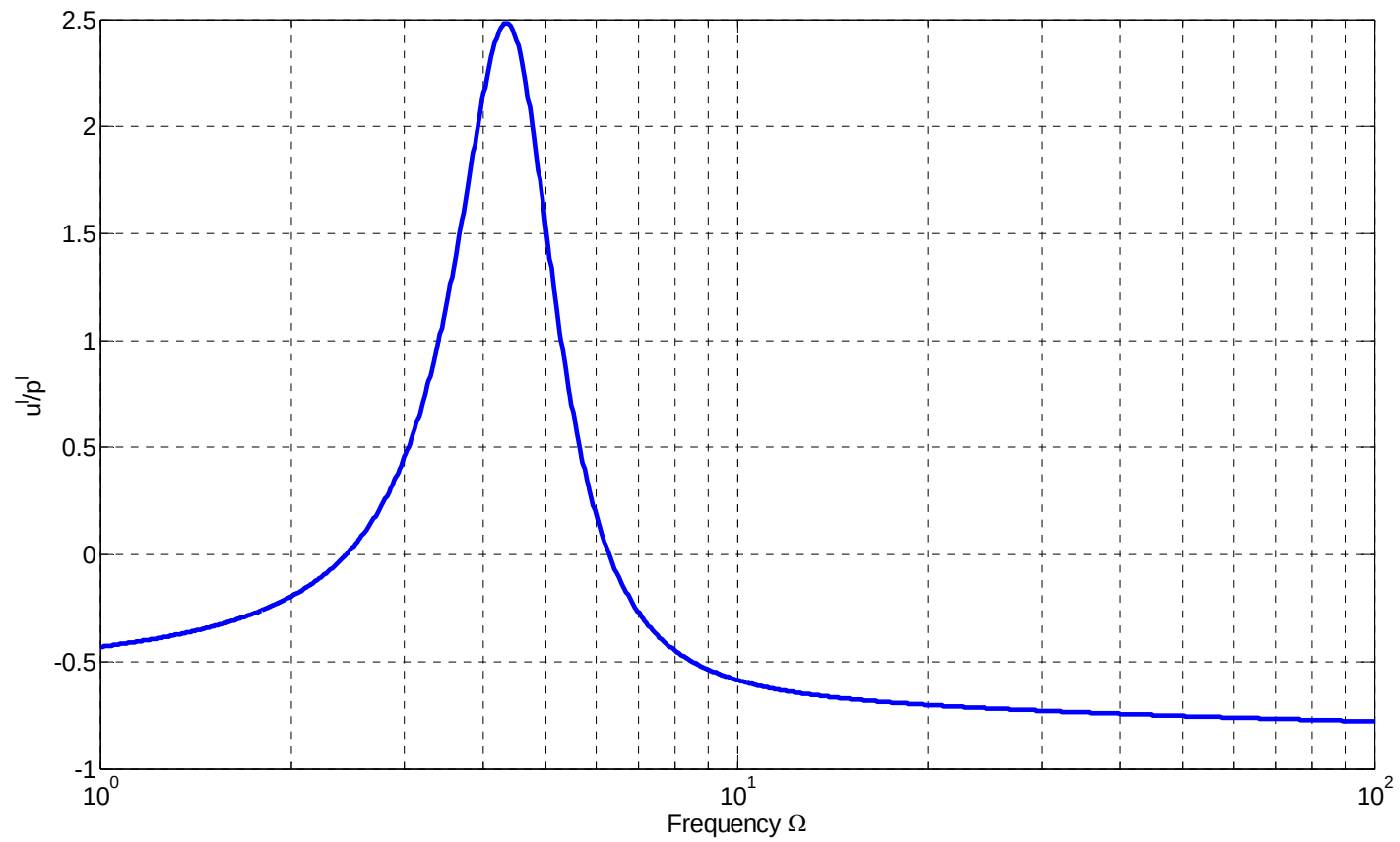


Figure 2.5 Admittance Function: $n = 0.3$, $A = 6.0$, $B = 3.3$, $\gamma = 1.18$

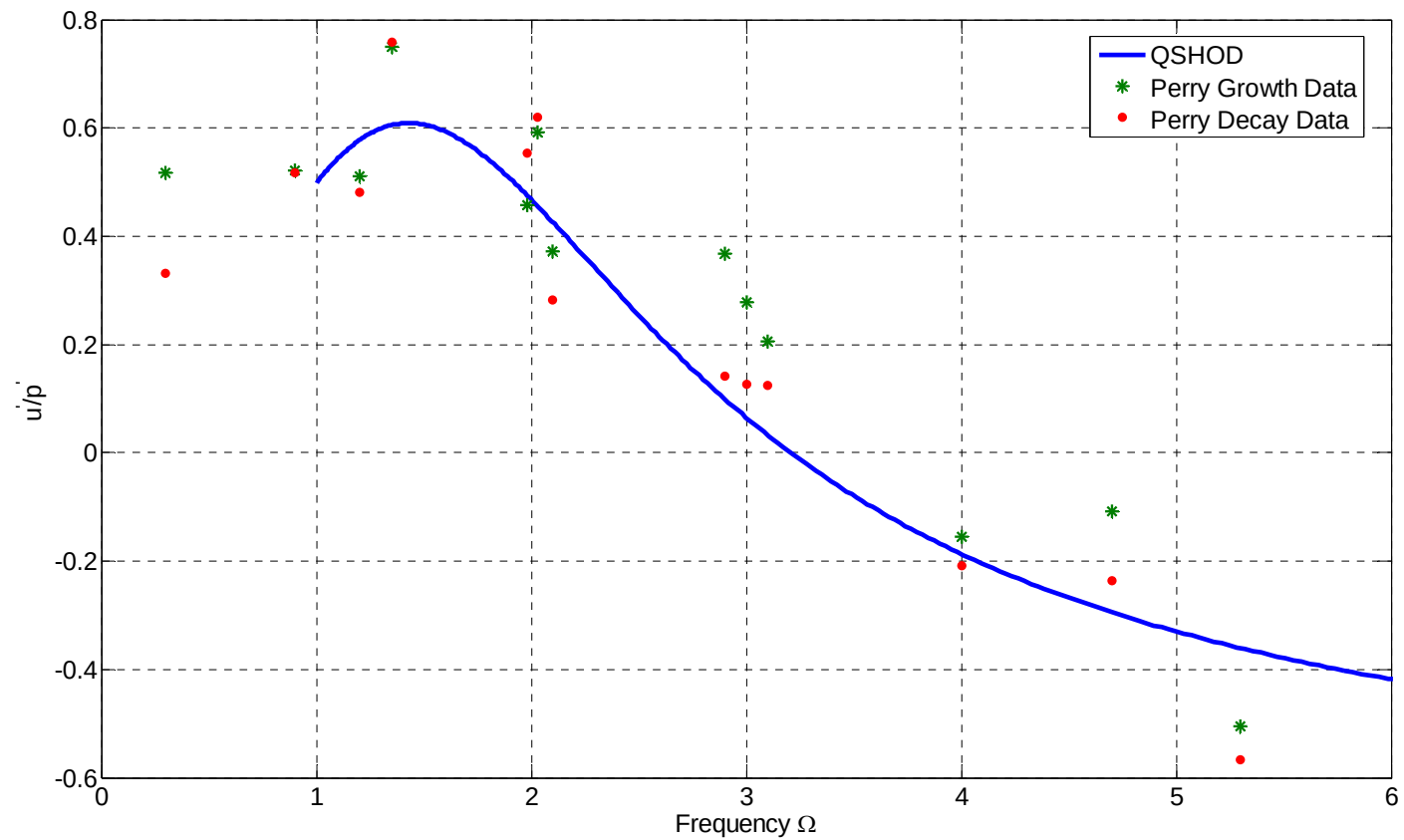


Figure 2.6 Perry's T-17 Data Fit by the QSHOD Model: $n = 0.82$, $A = 2.3$, $B = 1.38$, $\gamma = 1.18$

CHAPTER III

QSHOD MODEL INCORPORATING NONLINEARITY

Problem Definition

The following is an extension of the analysis carried out in Chapter II, to encompass the effects of nonlinearity on the response function by expanding to the perturbation series to second-order in the wave amplitude. The remarks and analysis in this chapter are again based on the model described in the recent book [4] and the 1968 review paper [5] by Culick. Again, for completeness and reproducibility, the model is derived in full detail, including the analysis carried out in Chapter II.

Analysis

Solid Phase

The energy equation for the temperature in the solid phase is assumed to have uniform and constant properties,

$$\lambda_p \frac{\partial^2 T}{\partial x^2} - \bar{m} c_p \frac{\partial T}{\partial x} - \rho_p c_p \frac{\partial T}{\partial t} = \dot{Q}_d \quad (3.1)$$

where $\overline{(\quad)}$ indicates the time-averaged value, the subscript p denotes the solid phase, $\bar{m} = \rho_p \bar{r}$ is the average mass flux in the reference system as defined in Figure 2.1, and $\dot{Q}_d$ is the rate at which energy is released per unit volume due to decomposition of the solid, for simplicity $\dot{Q}_d = 0$. The result of relaxing the $\dot{Q}_d = 0$ assumption has been carried out by Culick [4], but is left out here since it does not add to the

underlying theory involved. It is advantageous to make use of the dimensionless variable

$$\xi_p = \frac{\bar{m}c_p}{\lambda_p} x \quad (3.2)$$

substituting this finding into (3.1) yields

$$\frac{\partial^2 T}{\partial \xi_p^2} - \frac{\partial T}{\partial \xi_p} - \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} \frac{\partial T}{\partial t} = 0 \quad (3.3)$$

To seek solutions it is necessary to expand $T = T^{(0)} + T^{(1)}e^{i\omega t} + T^{(2)}e^{2i\omega t}$. It is important to note at this juncture that the steady component of the nonlinear correction has been excluded due to the assumption that the mean values are constant and the pressure is only being expanded to encompass the measurable physical pressure term $p^{(1)}/p^{(0)}$. The pressure is only carried to order ε as we are seeking the nonlinear, ε^2 effects on the burning process due to the measurable fluctuating pressure (ensures that the model is defined by physical parameters). This becomes clear if the analysis is carried out with inclusion of the steady nonlinear terms, the solution to the steady nonlinear part of the response function will return a zero value. Proceeding to collect the steady terms from the expanded (3.3) returns,

$$\frac{d^2 T^{(0)}}{d \xi_p^2} - \frac{dT^{(0)}}{d \xi_p} = 0 \quad (3.4)$$

Integrating (3.4), and applying the boundary conditions $T^{(0)}(0) = T_s^{(0)}$ and $T^{(0)}(-\infty) = T_c^{(0)}$, gives the following formula for the normalized mean temperature

$$T^{(0)}(x) = T_c^{(0)} + (T_s^{(0)} - T_c^{(0)}) \exp\left(\frac{\bar{m}c_p}{\lambda_p} x\right) \quad (3.5)$$

where values at the interface are identified by subscript s and values deep within the cold propellant are defined by subscript c . Collecting the unsteady terms from the expansion of (3.3) presents

$$\frac{d^2 T^{(1)}}{d\xi_p^2} - \frac{dT^{(1)}}{d\xi_p} - i\omega \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} T^{(1)} = 0 \quad (3.6)$$

easily solved, (3.6) is a Sturm-Liouville boundary value problem that leads to the equation for $T^{(1)}$

$$T^{(1)}(x, t) = \tau' \exp\left(\Lambda \frac{\bar{m}c_p}{\lambda_p} x\right) \exp(i\omega t) \quad (3.7)$$

where the eigenvalue Λ must satisfy

$$\Lambda(\Lambda - 1) = i\Omega \quad (3.8)$$

and Ω is defined as the dimensionless frequency,

$$\Omega = \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} \omega \quad (3.9)$$

In order to insure that $T^{(1)}(-\infty) \rightarrow 0$, the positive real part of (3.8) must be used.

Define $\Lambda = \Lambda_r + i\Lambda_i$ and solve (3.8) for the eigenvalue Λ as a function of Ω

$$\Lambda_r = \frac{\Lambda_i + \Omega}{2\Lambda_i} \quad (3.10)$$

and

$$\Lambda_i = \sqrt{-\frac{1}{8} + \frac{1}{8}\sqrt{1 + 16\Omega^2}} \quad (3.11)$$

The solution is split into real and imaginary parts because only the real part is in phase with the pressure. Define τ' from (3.7); as the fluctuation of temperature at the interface ($x=0$), this is due to the choice of datum. Gathering the nonlinear terms from the expanded (3.3) yields the following differential equation

$$\frac{d^2 T^{(2)}}{d\xi_p^2} - \frac{dT^{(2)}}{d\xi_p} - 2i\omega \frac{\lambda_p \rho_p}{\bar{m}^2 c_p} T^{(2)} = 0 \quad (3.12)$$

solving (3.12) leads to the following equation for $T^{(2)}$

$$T^{(2)}(x, t) = \hat{\tau} \exp\left(\Psi \frac{\bar{m} c_p}{\lambda_p} x\right) \exp(2i\omega t) \quad (3.13)$$

where the eigenvalue Ψ satisfies

$$\Psi(\Psi - 1) = 2i\Omega \quad (3.14)$$

In order to satisfy the condition $T^{(2)}(-\infty) \rightarrow 0$, only the positive real part of (3.14) must be used. Define $\Psi = \Psi_r + i\Psi_i$ and solve (3.14) for Ψ_r and Ψ_i as a function of Ω

$$\Psi_r = \frac{\Psi_i + 2\Omega}{2\Psi_i} \quad (3.15)$$

and

$$\Psi_i = \sqrt{-\frac{1}{8} + \frac{1}{8}\sqrt{1 + 64\Omega^2}} \quad (3.16)$$

Again due to the choice of datum, $\hat{\tau}$ in (3.13) is the nonlinear fluctuation of temperature at the average position of the interface. Attempting to match the

conditions at the interface requires knowledge of the values and derivatives of the temperature at the interface itself, referenced to position x_s and velocity $\dot{x}_s$. Values at the interface are calculated by utilizing Taylor series expansions about $(x = 0)$, and on the solid side of the interface $(x = x_s)$:

$$\begin{aligned} T_s^{(0)}(x_s) &= T_{0-}^{(0)}(0) + x_s \left(\frac{dT^{(0)}}{dx} \right)_{0-} \\ T_s^{(1)}(x_s) &= T_{0-}^{(1)}(0) + x_s \left(\frac{dT^{(0)}}{dx} \right)_{0-} \\ T_s^{(2)}(x_s) &= T_{0-}^{(2)}(0) + x_s \left(\frac{dT^{(0)}}{dx} \right)_{0-} \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \left(\frac{dT^{(0)}}{dx} \right)_{s-} &= \left(\frac{dT^{(0)}}{dx} \right)_{0-} + x_s \left(\frac{d^2T^{(0)}}{dx^2} \right)_{0-} \\ \left(\frac{\partial T^{(1)}}{\partial x} \right)_{s-} &= \left(\frac{\partial T^{(1)}}{\partial x} \right)_{0-} + x_s \left(\frac{d^2T^{(0)}}{dx^2} \right)_{0-} \\ \left(\frac{\partial T^{(2)}}{\partial x} \right)_{s-} &= \left(\frac{\partial T^{(2)}}{\partial x} \right)_{0-} + x_s \left(\frac{d^2T^{(0)}}{dx^2} \right)_{0-} \end{aligned} \quad (3.18)$$

The results for the gas phase side of the interface cannot be completed until the interfacial region is analyzed.

Interfacial Region

The analysis at the interface is governed by three conservation laws: conservation of mass, energy, and the law for conversion of solid to gas

$\left(\rho_p(v_p - \dot{x}_s) = \rho_g(v_g - \dot{x}_s)\right)$. Noting that $\rho_p v_p = \bar{m}$, $\rho_g v_g = m$ and where $m = \bar{m} + m^{(1)} \exp(i\omega t) + m^{(2)} \exp(2i\omega t)$,

$$\rho_p \dot{x}_s = \left(1 - \frac{\rho_g}{\rho_p}\right)^{-1} \bar{m} - m \quad (3.19)$$

The mean gas density ρ_g near the surface is much smaller than the density of the condensed phase ρ_p , which yields $\rho_g / \rho_p \ll 1$ allowing the term to be dropped.

$$\rho_p \dot{x}_s \approx -\left(m^{(1)} \exp(i\omega t) + m^{(2)} \exp(2i\omega t)\right) \quad (3.20)$$

Isolating the regression rate $\dot{x}_s$ in (3.20) and integrating to find x_s

$$x_s = -\frac{1}{\rho_p} \int \left(m^{(1)} \exp(i\omega t) + m^{(2)} \exp(2i\omega t)\right) dt \quad (3.21)$$

Using (3.21) to find x_s to the appropriate orders:

the first order approximation

$$x_s^{(1)} = -\frac{1}{i\omega\rho_p} m^{(1)} \exp(i\omega t) \quad (3.22)$$

and the second order approximation

$$x_s^{(2)} = -\frac{1}{2i\omega\rho_p} m^{(2)} \exp(2i\omega t) \quad (3.23)$$

The conditions associated with the total mass and energy transfer in the upstream and downstream sides of the interface are found by summing the terms associated with the mass flux and energy transfer from Figure 2.1. More specifically,

$$\begin{aligned}
& -\left(\lambda_p \frac{\partial T}{\partial x}\right)_{s-} + \rho_p (v_p - \dot{x}_s)(c_p T_s + L_s + Q_g) \\
& = -\left(\lambda_g \frac{\partial T}{\partial x}\right)_{s+} + \rho_g (v_g - \dot{x}_s)(c_g T_s + Q_g)
\end{aligned} \tag{3.24}$$

From mass balance we know that $\rho_p (v_p - \dot{x}_s) = \rho_g (v_g - \dot{x}_s)$ so,

$$\left(\lambda_g \frac{\partial T}{\partial x}\right)_{s+} = \left(\lambda_p \frac{\partial T}{\partial x}\right)_{s-} + \rho_p (v_p - \dot{x}_s)(c_g T_s - c_p T_s + L_s) \tag{3.25}$$

again noting that $\bar{m} = \rho_p v_p$ and using (3.20),

$$\left(\lambda_g \frac{\partial T}{\partial x}\right)_{s+} = \left(\lambda_p \frac{\partial T}{\partial x}\right)_{s-} + \bar{m}(c_g T_s - c_p T_s + L_s) \tag{3.26}$$

The heat flux $(\lambda_g \partial T / \partial x)_{s+}$ heat flow from the interface to the gas phase and the heat flux $(\lambda_p \partial T / \partial x)_{s-}$ is the heat flow from the interface to the solid phase (note that (3.26) has not yet been split into mean and fluctuating parts). For steady combustion, the energy balance (3.26),

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x}\right)_{s+} = \left(\lambda_p \frac{\partial T^{(0)}}{\partial x}\right)_{s-} + \bar{m}(c_g T_s^{(0)} - c_p T_s^{(0)} + L_s) \tag{3.27}$$

A solution for $(\lambda_p \partial T / \partial x)_{s-}$ can be found by differentiating (3.5) and assuming the gas is ideal,

$$\left(\lambda_p \frac{\partial T^{(0)}}{\partial x}\right)_{s-} = \bar{m} c_p (T_s^{(0)} - T_c^{(0)}) \tag{3.28}$$

Substituting this finding into (3.27) yields

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x}\right)_{s+} = \bar{m}(c_g T_s^{(0)} - c_p T_c^{(0)} + L_s) \tag{3.29}$$

The linear unsteady part of (3.26), and again assuming the gas is ideal, is

$$\left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T^{(1)}}{\partial x} \right)_{s-} + m^{(1)} L_s + \bar{m} (c_g - c_p) T_s^{(1)} \quad (3.30)$$

assuming the gas is ideal, the nonlinear part of (3.26) is

$$\left(\lambda_g \frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T^{(2)}}{\partial x} \right)_{s-} + \left[\begin{aligned} &m^{(2)} L_s + m^{(1)} (c_g - c_p) T_s^{(1)} \\ &+ \bar{m} (c_g - c_p) T_s^{(2)} \end{aligned} \right] \quad (3.31)$$

Combining (3.5), (3.7) and the proper parts of (3.17) gives the formula for the heat transfer into the condensed phase from the interface:

$$\left(\lambda_p \frac{\partial T^{(1)}}{\partial x} \right)_{s-} = \bar{m} c_p \left[\Lambda T_s^{(1)} - \frac{\bar{m} c_p}{\lambda_p \rho_p i \omega} (T_s^{(0)} - T_c^{(0)}) m^{(1)} \right] \quad (3.32)$$

With the knowledge that $\Lambda(\Lambda - 1) = \lambda_p \rho_p i \omega / \bar{m}^2 c_p$, carrying out a perturbation expansion to leading order,

$$\frac{\bar{m}^2 c_p}{\lambda_p \rho_p i \omega} = \frac{1}{\Lambda(\Lambda - 1)} = -\frac{1}{\Lambda} \quad (3.33)$$

The previous step is cause for concern, due to the limitation Λ must be small to allow for this simplification. The concern is due to the inherent dependence of Λ on Ω , by limiting Λ to be small, Ω must also be small. Proceed by substituting the result in (3.33) into (3.32) yields

$$\left(\lambda_p \frac{\partial T^{(1)}}{\partial x} \right)_{s-} = \bar{m} c_p \left[\Lambda T_s^{(1)} + \frac{1}{\Lambda} (T_s^{(0)} - T_c^{(0)}) \frac{m^{(1)}}{\bar{m}} \right] \quad (3.34)$$

In this result, the approximation in (3.20) has been used. Substitution of (3.34) in (3.30) leads to the condition to be set on the unsteady heat transfer at the downstream side of the interface:

$$\frac{\lambda_g}{\bar{m}c_p} \left(\frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \left(\Lambda + \frac{c_g}{c_p} - 1 \right) T_s^{(1)} + \left[\frac{1}{\Lambda} (T_s^{(0)} - T_c^{(0)}) + \frac{L_s}{c_p} \right] \frac{m^{(1)}}{\bar{m}} \quad (3.35)$$

The nonlinear correction from (3.25), and again assuming the gas is ideal, is

$$\left(\lambda_g \frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \left(\lambda_p \frac{\partial T^{(2)}}{\partial x} \right)_{s-} + \left[\frac{m^{(2)}L_s + m^{(1)}(c_g - c_p)T_s^{(1)}}{+\bar{m}(c_g - c_p)T_s^{(2)}} \right] \quad (3.36)$$

Combination of (3.5), (3.13) and the appropriate parts of (3.17) constructs the equation for the heat transfer into the condensed phase from the interface:

$$\left(\lambda_p \frac{\partial T^{(2)}}{\partial x} \right)_{s-} = \bar{m}c_p \left[\Psi T_s^{(2)} + \frac{\bar{m}^2 c_p}{2\lambda_p \rho_p i\omega} (T_s^{(0)} - T_c^{(0)}) \right] \quad (3.37)$$

With the knowledge that $\Psi(\Psi - 1) = \lambda_p \rho_p 2i\omega / \bar{m}^2 c_p$, using a perturbation expansion to leading order

$$\frac{\bar{m}^2 c_p}{2\lambda_p \rho_p i\omega} = \frac{1}{\Psi(\Psi - 1)} = -\frac{1}{\Psi} \quad (3.38)$$

With the same reservations as the result in (3.33), the result in (3.38) again limits the range of valid frequencies Ω . Substituting the result into (3.37),

$$\left(\lambda_p \frac{\partial T^{(2)}}{\partial x} \right)_{s-} = \bar{m}c_p \left[\Psi T_s^{(2)} + \frac{1}{\Psi} (T_s^{(0)} - T_c^{(0)}) \frac{m^{(2)}}{\bar{m}} \right] \quad (3.39)$$

In this result, the approximation in (3.20) has been used. Substitution of (3.39) in (3.31) leads to the boundary condition to be set on the nonlinear heat transfer at the downstream side of the interface:

$$\frac{\lambda_g}{\bar{m}c_p} \left(\frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \left\{ \begin{aligned} & \left(\Psi + \frac{c_g}{c_p} - 1 \right) T_s^{(2)} + T_s^{(1)} \left(\frac{c_g}{c_p} - 1 \right) \frac{m^{(1)}}{\bar{m}} \\ & + \left[\frac{T_s^{(0)} - T_c^{(0)}}{\Psi} + \frac{L_s}{c_p} \right] \frac{m^{(2)}}{\bar{m}} \end{aligned} \right\} \quad (3.40)$$

Gas Phase

To proceed, it is necessary to follow the analysis of a simplified model for steady burning, given by Culick [5]. The model is simplified by the assumption of constant thermal properties and uniform combustion throughout the gas phase. Two limits that are contained in this model are an infinitesimally thin flame sheet and uniform combustion beginning at the interface. Figure 2.3 is a sketch of the model where x_s , x_i and x_f indicate the position of the solid surface, the leading edge of the combustion zone and the downstream edge of the gas phase, respectively.

The equation defining the gas phase is

$$mc_g \frac{dT}{dx} - \frac{d}{dx} \left(\lambda_g \frac{dT}{dx} \right) = \rho_g \dot{\varepsilon} Q_f \quad (3.41)$$

where the subscript g denotes values for the gas phase, the constant Q_f is the energy release per unit mass of the reactant mixture and $\dot{\varepsilon}$ is the reaction rate. The temperature boundary condition at the downstream edge, $x = x_f$, of the gas phase is

$$T(x_f) = T_f^{(0)} \quad (3.42)$$

where the constant $T_f^{(0)}$ is the adiabatic flame temperature. The derivative condition at the downstream edge, $x = x_f$, is

$$\frac{dT(x_f)}{dx} = 0 \quad (3.43)$$

and at the interface, $(x = x_s)$, the temperature boundary condition is

$$T(x_s) = T_s^{(0)} \quad (3.44)$$

The energy balance at the interface is given by (3.29), assuming steady combustion the energy flow across the gas phase is

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \bar{m} \left[Q_f - c_g (T_f^{(0)} - T_s^{(0)}) \right] \quad (3.45)$$

Alternatively, integrating (3.41) across the combustion zone, and applying boundary conditions (3.42) - (3.44) leads to

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \int_0^\infty (\rho_g Q_f \dot{\varepsilon}) dx - \bar{m} c_g (T_f^{(0)} - T_s^{(0)}) \quad (3.46)$$

With the knowledge that Q_f is constant, the reaction rate is found by comparing (3.45) and (3.46), leading to

$$\bar{m} = \int_0^\infty (\rho_g \dot{\varepsilon}) dx \quad (3.47)$$

To attempt to solve (3.41) it is convenient to transform x to the dimensionless variable ζ , with the assumption that assuming mc_g/λ_g is constant, namely by

$$\frac{d\zeta}{dx} = \frac{mc_g}{\lambda_g} \zeta \quad (3.48)$$

Substituting the transformation into (3.41) gives

$$-\zeta^2 \frac{d^2 T}{d\zeta^2} = X \quad (3.49)$$

where the eigenvalue X is

$$X = \frac{\lambda_g Q_f w}{m^2 c_g^2} \quad (3.50)$$

and

$$w = \rho_g \dot{\epsilon} \quad (3.51)$$

The previous steps can only be arrived at if X is assumed to be independent of ζ , which defines the condition of uniform combustion. Integration of (3.49) evaluated over the combustion zone, ζ_i at the leading edge of the combustion zone and ζ_f at the downstream edge of the flame, yields

$$\left(\frac{\partial T}{\partial \zeta} \right)_{s+} = \left(\frac{\zeta_f - \zeta_i}{\zeta_i \zeta_f} \right) X \quad (3.52)$$

Evaluating (3.52), at the leading edge of the combustion phase the dimensionless ζ_i is equal to unity, and $\zeta_f \gg \zeta_i$, which simplifies (3.52) to

$$\left(\frac{\partial T}{\partial \zeta} \right)_{s+} = X \quad (3.53)$$

Substituting the eigenvalue X into (3.53)

$$\left(\lambda_g \frac{\partial T}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f w}{c_g m} \quad (3.54)$$

The steady form of equation (3.54)

$$\left(\lambda_g \frac{\partial T^{(0)}}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f w^{(0)}}{c_g \bar{m}} \quad (3.55)$$

The fluctuation of heat transfer at the surface is given by the unsteady form of (3.54)

$$\left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f}{c_g} \frac{w^{(0)}}{\bar{m}} \left(\frac{w^{(1)}}{w^{(0)}} - \frac{m^{(1)}}{\bar{m}} \right) \quad (3.56)$$

The nonlinear portion of equation (3.54)

$$\left(\lambda_g \frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \frac{\lambda_g Q_f}{c_g} \frac{w^{(0)}}{\bar{m}} \left[\frac{w^{(2)}}{w^{(0)}} - \frac{m^{(2)}}{\bar{m}} - \frac{w^{(1)}}{w^{(0)}} \frac{m^{(1)}}{\bar{m}} + \left(\frac{m^{(1)}}{\bar{m}} \right)^2 \right] \quad (3.57)$$

To continue it is necessary to specify an explicit form for w , the reaction rate per unit volume, as a function of the flow variables. To do so, equate (3.29) and (3.55), and the quasi-steady behavior allows the average values of temperature to be replaced by instantaneous values. For this specific special case of steady uniform combustion, the expression for w :

$$w = \frac{c_g m^2}{\lambda_g Q_f} (c_g T_s - c_p T_c + L_s) \quad (3.58)$$

Assuming that the right-hand side is a function of pressure only, it is convenient to introduce the pyrolysis law $m = ap^n$. Where a and n are the burning rate coefficient and the burning rate exponent respectively. Expanding the pyrolysis law appropriately

$$m = \bar{m} + m^{(1)} e^{i\omega t} + m^{(2)} e^{2i\omega t} = 1 + an \frac{p^{(1)}}{p^{(0)}} e^{i\omega t} + a \frac{n(n-1)}{2} \left(\frac{p^{(1)}}{p^{(0)}} \right)^2 e^{2i\omega t} \quad (3.59)$$

Collecting the terms to the appropriate order, unsteady,

$$\frac{m^{(1)}}{\bar{m}} = an \frac{p^{(1)}}{p^{(0)}} \quad (3.60)$$

and nonlinear

$$\frac{m^{(2)}}{\bar{m}} = a \frac{n(n-1)}{2} \left(\frac{p^{(1)}}{p^{(0)}} \right)^2 \quad (3.61)$$

Expanding at leading order (3.58) becomes

$$\frac{w}{w^{(0)}} = \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \left(\frac{m}{\bar{m}} \right)^2 \quad (3.62)$$

where

$$H = \frac{L_s}{c_g T_s - c_p T_c} \quad (3.63)$$

and $X^{(0)}$ is found by taking the steady components of (3.50),

$$X^{(0)} = \frac{\lambda_g Q_f w^{(0)}}{\bar{m}^2 c_g^2} \quad (3.64)$$

The fluctuation of the reaction rate is then

$$\frac{w^{(1)}}{w^{(0)}} = 2 \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \frac{m^{(1)}}{\bar{m}} \quad (3.65)$$

and the fluctuation of the reaction rate is then

$$\frac{w^{(2)}}{w^{(0)}} = \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \left[2 \frac{m^{(2)}}{\bar{m}} + \left(\frac{m^{(1)}}{\bar{m}} \right)^2 \right] \quad (3.66)$$

Substituting (3.60) into (3.65) the fluctuation of the reaction rate is then

$$\frac{w^{(1)}}{w^{(0)}} = 2an \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \frac{p^{(1)}}{p^{(0)}} \quad (3.67)$$

Similarly, substituting (3.61) into (3.66) the nonlinear fluctuation of the reaction rate is then

$$\frac{w^{(2)}}{w^{(0)}} = \frac{c_g T_s - c_p T_c}{c_g X^{(0)}} (1 + H) \left[an(n-1) \left(\frac{p^{(1)}}{p^{(0)}} \right)^2 + \left(\frac{m^{(1)}}{\bar{m}} \right)^2 \right] \quad (3.68)$$

With all three phases defined, matching of the solid phase to the interface and interface to the gas phase can commence.

Construction of the Linear Response Function

In formulating the response function, it is necessary to account for the conversion of a solid to a gas. Proceed by utilizing the commonly used Arrhenius law, which gives the total surface mass flux

$$m = B \exp(-E_s / \Re T_s) \quad (3.69)$$

the perturbed form of (3.69), to first order is

$$\frac{m^{(1)}}{\bar{m}} = E \frac{T_s^{(1)}}{T_s^{(0)}} \quad (3.70)$$

where the dimensionless activation energy is $E = E_s / \Re T_s^{(0)}$. Perturbing (3.69) to second order

$$\frac{m^{(2)}}{\bar{m}} = E \left[\frac{T_s^{(2)}}{T_s^{(0)}} + \left(\frac{E}{2} - 1 \right) \left(\frac{T_s^{(1)}}{T_s^{(0)}} \right)^2 \right] \quad (3.71)$$

Substitute the Arrhenius law (3.70) into (3.35) defining the interfacial conditions for energy and mass transfer:

$$\frac{\lambda_g}{\bar{m} c_p T_s^{(0)}} \left(\frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \frac{1}{E} \left[\Lambda + \frac{c_g}{c_p} - 1 + \frac{E}{\Lambda} \left(1 - \frac{T_c^{(0)}}{T_s^{(0)}} \right) + \frac{E L_s}{T_s^{(0)} c_p} \right] \frac{m^{(1)}}{\bar{m}} \quad (3.72)$$

letting

$$L = \frac{L_s}{T_s^{(0)} c_p} \quad (3.73)$$

and

$$A = E \left(1 - \frac{T_c^{(0)}}{T_s^{(0)}} \right) \quad (3.74)$$

Substituting (3.73) and (3.74) into (3.72), yields

$$\frac{1}{\bar{m} c_p T_s^{(0)}} \left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = \frac{1}{E} \left(\Lambda + \frac{A}{\Lambda} + \frac{c_g}{c_p} - 1 + EL \right) \frac{m^{(1)}}{\bar{m}} \quad (3.75)$$

Substitute the reaction rate (3.67) into the expression (3.56) for the heat loss from the gas phase to the interface:

$$\frac{1}{\bar{m} c_g T_s^{(0)}} \left(\lambda_g \frac{\partial T^{(1)}}{\partial x} \right)_{s+} = 2an \left(1 - \frac{c_p}{c_g} \frac{T_c^{(0)}}{T_s^{(0)}} \right) (1+H) \frac{p^{(1)}}{p^{(0)}} - \frac{X^{(0)}}{T_s^{(0)}} \frac{m^{(1)}}{\bar{m}} \quad (3.76)$$

Equate the findings from (3.75) and (3.76), and use the Arrhenius law (3.70) to eliminate $T_s^{(1)}/T_s^{(0)}$, leaving the equation for the pressure coupled response function:

$$R_b^{(1)} = \frac{\frac{m^{(1)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{2anE \left(1 - \frac{c_p}{c_g} \frac{T_c^{(0)}}{T_s^{(0)}} \right) (1+H)}{\Lambda + \frac{A}{\Lambda} + \frac{c_g}{c_p} - 1 + EL + E \frac{X^{(0)}}{T_s^{(0)}}} \quad (3.77)$$

Finally rewrite (3.77) in the form

$$R_b^{(1)} = \frac{\frac{m^{(1)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{anC_1}{\left(\Lambda + \frac{A}{\Lambda} \right) + C_2} \quad (3.78)$$

Utilizing the unsteady pyrolysis law (3.60), (3.78) can be written as a function of two variables by:

$$an = \frac{anC_1}{\left(\Lambda + \frac{A}{\Lambda}\right) + C_2} \quad (3.79)$$

where Λ can be defined in the limit of zero frequency, $\Omega = 0$, giving the condition

$$\frac{C_1}{1 + A + C_2} = 1 \quad (3.80)$$

define C_1 with

$$C_1 = B \quad (3.81)$$

Substituting this into (3.80), it follows that

$$C_2 = B - (1 + A) \quad (3.82)$$

Now that $C_1 = B$ and $C_2 = B - (1 + A)$ are defined substitute back into (3.78), yielding

$$R_b^{(1)} = \frac{m'}{p'} = \frac{\frac{m^{(1)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{nB}{\left(\Lambda + \frac{A}{\Lambda}\right) + B - (1 + A)} \quad (3.83)$$

Equation (3.83) is the first order pressure coupled response function, as found by Culick [4].

Construction of the Nonlinear Response Function

Similar to the formulation of the linear response function, the nonlinear response function is found by substituting the nonlinear portion of the Arrhenius law(3.71) into (3.40), and also recalling (3.70), (3.73), (3.74),

$$\frac{\lambda_g}{\bar{m}c_p T_s^{(0)}} \left(\frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \frac{1}{E} \left\{ \left[\left(\frac{1}{E} - \frac{1}{2} \right) \left(\Psi + \frac{c_g}{c_p} - 1 \right) + \frac{c_g}{c_p} - 1 \right] \left(\frac{m^{(1)}}{\bar{m}} \right)^2 + \left[\Psi + \frac{A}{\Psi} + \left(\frac{c_g}{c_p} - 1 \right) + EL \right] \frac{m^{(2)}}{\bar{m}} \right\} \quad (3.84)$$

and using the first order (unsteady) result, namely

$$\frac{m^{(1)}}{\bar{m}} = \frac{p^{(1)}}{p^{(0)}} R_b^{(1)} \quad (3.85)$$

Equation (3.84) becomes

$$\frac{\lambda_g}{\bar{m}c_p T_s^{(0)}} \left(\frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \frac{1}{E} \left\{ \left[\left(\frac{1}{E} - \frac{1}{2} \right) \left(\Psi + \frac{c_g}{c_p} - 1 \right) + \frac{c_g}{c_p} - 1 \right] \left(\frac{p^{(1)}}{p^{(0)}} R_b^{(1)} \right)^2 + \left[\Psi + \frac{A}{\Psi} + \left(\frac{c_g}{c_p} - 1 \right) + EL \right] \frac{m^{(2)}}{\bar{m}} \right\} \quad (3.86)$$

Substitution the reaction rate (3.68) into the expression (3.57), and again using (3.85), yields the nonlinear heat loss from the gas phase to the interface,

$$\frac{\lambda_g}{\bar{m}c_g} \left(\frac{\partial T^{(2)}}{\partial x} \right)_{s+} = \left\{ \frac{an(n-1) \left(1 - \frac{c_p T_c^{(0)}}{c_g T_s^{(0)}} \right) (1+H) \left(\frac{p^{(1)}}{p^{(0)}} \right)^2 - \frac{X^{(0)}}{T_s^{(0)}} \frac{m^{(2)}}{\bar{m}}}{\left[\frac{X^{(0)}}{T_s^{(0)}} - \left(1 - \frac{c_p T_c^{(0)}}{c_g T_s^{(0)}} \right) (1+H) \right] \left(\frac{p^{(1)}}{p^{(0)}} R_b^{(1)} \right)^2} \right\} \quad (3.87)$$

Equate (3.86) and (3.87), leaving an equation which can be rearranged to give the nonlinear part of the pressure coupled response function,

$$R_b^{(2)} = \frac{\frac{m^{(2)}}{p^{(1)}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{p^{(1)}}{p^{(0)}} \frac{\left[\left[\left(\frac{1}{2} - \frac{1}{E} \right) \Psi \frac{EX^{(0)}}{T_s^{(0)}} - \left(\frac{1}{E} + \frac{1}{2} \right) \left(\frac{c_g}{c_p} - 1 \right) \right] \left(R_b^{(1)} \right)^2 + \left[n(n-1) - \left(R_b^{(1)} \right)^2 \right] E(1+H) \left(1 - \frac{c_p T_c^{(0)}}{c_g T_s^{(0)}} \right) \right]}{\Psi + \frac{A}{\Psi} + \frac{c_g}{c_p} - 1 + EL + \frac{EX^{(0)}}{T_s^{(0)}}} \quad (3.88)$$

Rewrite (3.88) in the form

$$R_b^{(2)} = \frac{\frac{m^{(2)}}{p^{(1)}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{p^{(1)}}{p^{(0)}} \frac{\frac{n(n-1)}{2} C_1 + \left[\left(\frac{1}{2} - \frac{1}{E} \right) \Psi + C_3 \right] \left(R_b^{(1)} \right)^2}{\Psi + \frac{A}{\Psi} + C_2} \quad (3.89)$$

Constants C_1 and C_2 are known from previous order calculations, substituting these back into (3.89), yields

$$R_b^{(2)} = \frac{\frac{m^{(2)}}{p^{(1)}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{p^{(1)}}{p^{(0)}} \frac{\frac{n(n-1)}{2} B + \left[\left(\frac{1}{2} - \frac{1}{E} \right) \Psi + C_3 \right] \left(R_b^{(1)} \right)^2}{\Psi + \frac{A}{\Psi} + B - (1+A)} \quad (3.90)$$

Utilizing the nonlinear pyrolysis law (3.61), (3.90) can be simplified and written as:

$$\frac{\frac{n(n-1)}{2} B + \left[\left(\frac{1}{2} - \frac{1}{E} \right) \Psi + C_3 \right] \left(R_b^{(1)} \right)^2}{\Psi + \frac{A}{\Psi} + B - (1+A)} = \frac{n(n-1)}{2} \quad (3.91)$$

where Ψ can be defined in the limit of zero frequency, $\Omega = 0$, giving the condition

$$\frac{\frac{n(n-1)}{2} B + \left[\left(\frac{1}{2} - \frac{1}{E} \right) + C_3 \right] \left(R_b^{(1)} \right)^2}{1 + \frac{A}{1} + B - 1 - A} = \frac{n(n-1)}{2} \quad (3.92)$$

defining $C_3 = \frac{1}{E} - \frac{1}{2}$ as,

$$C_3 = \frac{1}{E} - \frac{1}{2} \quad (3.93)$$

substitute this finding back into (3.90), yielding the nonlinear correction to the pressure coupled response function

$$R_b^{(2)} = \frac{\frac{m^{(2)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} = \frac{p^{(1)}}{p^{(0)}} \frac{\frac{n(n-1)}{2} B + \left(\frac{1}{2} - \frac{1}{E}\right) (\Psi - 1) (R_b^{(1)})^2}{\Psi + \frac{A}{\Psi} + B - (1 + A)} \quad (3.94)$$

Combining results (3.83) and (3.94) yields the total solution for the pressure coupled response function,

$$R_b = \frac{m'}{p'} = \frac{\frac{m^{(1)} + m^{(2)}}{\bar{m}}}{\frac{p^{(1)}}{p^{(0)}}} \quad (3.95)$$

more specifically,

$$R_b = \frac{nB}{\Lambda + \frac{A}{\Lambda} + B - 1 - A} + \varepsilon \frac{\frac{n(n-1)}{2} B + \left(\frac{1}{2} - \frac{1}{E}\right) (\Psi - 1) (R_b^{(1)})^2}{\Psi + \frac{A}{\Psi} + B - 1 - A} \quad (3.96)$$

To be incorporated into a combustion instability calculation the response function must be put into the form of the admittance function, this is achieved by using (1.5)

$$A_b^{(r)} = \frac{nB}{\Lambda + \frac{A}{\Lambda} + B - 1 - A} + \varepsilon \frac{\frac{n(n-1)}{2} B + \left(\frac{1}{2} - \frac{1}{E}\right) (\Psi - 1) (R_b^{(1)})^2}{\Psi + \frac{A}{\Psi} + B - 1 - A} - \frac{1}{\gamma} \quad (3.97)$$

Results and Discussion

With the response and admittance function, defined figures can be generated by using values suggested by Culick [8]. With ε set to zero, Figure 3.1 successfully reproduces the low frequency result for the response obtained by Culick [4], which should be expected. The nonlinear corrections lead to significant increase in amplitude and a frequency shift of the primary low frequency response peak, as observed in Figure 3.1. It is important to note that this result shows that the waves receive enhanced energy flux as amplitude increases. Figure 3.2 demonstrates the low frequency admittance, also showing the effects of nonlinearity.

The result shown in Figure 3.3 demonstrates a strong correlation of the admittance function to experimental data found by Perry [12]. The inclusion of the nonlinear correction further reinforces that the QSHOD model does capture the low frequency behavior. The Perry data [12] used in Figure 3.3 was recorded using the standardized growth and decay technique [9] for the experiment.

The nonlinear quasi-steady theory is marred by the same limitations as the linear theory, as it is an extension of the linear model. Although the nonlinear QSHOD model is limited, the model does give insight into the solid phase contribution to the nonlinear response function and the treatment of the boundary conditions, thus making it important to understand the analysis. The limitations discussed here drive the need for a more complete model, one that includes unsteady gas phase behavior and unsteady conduction heat transfer in the solid.

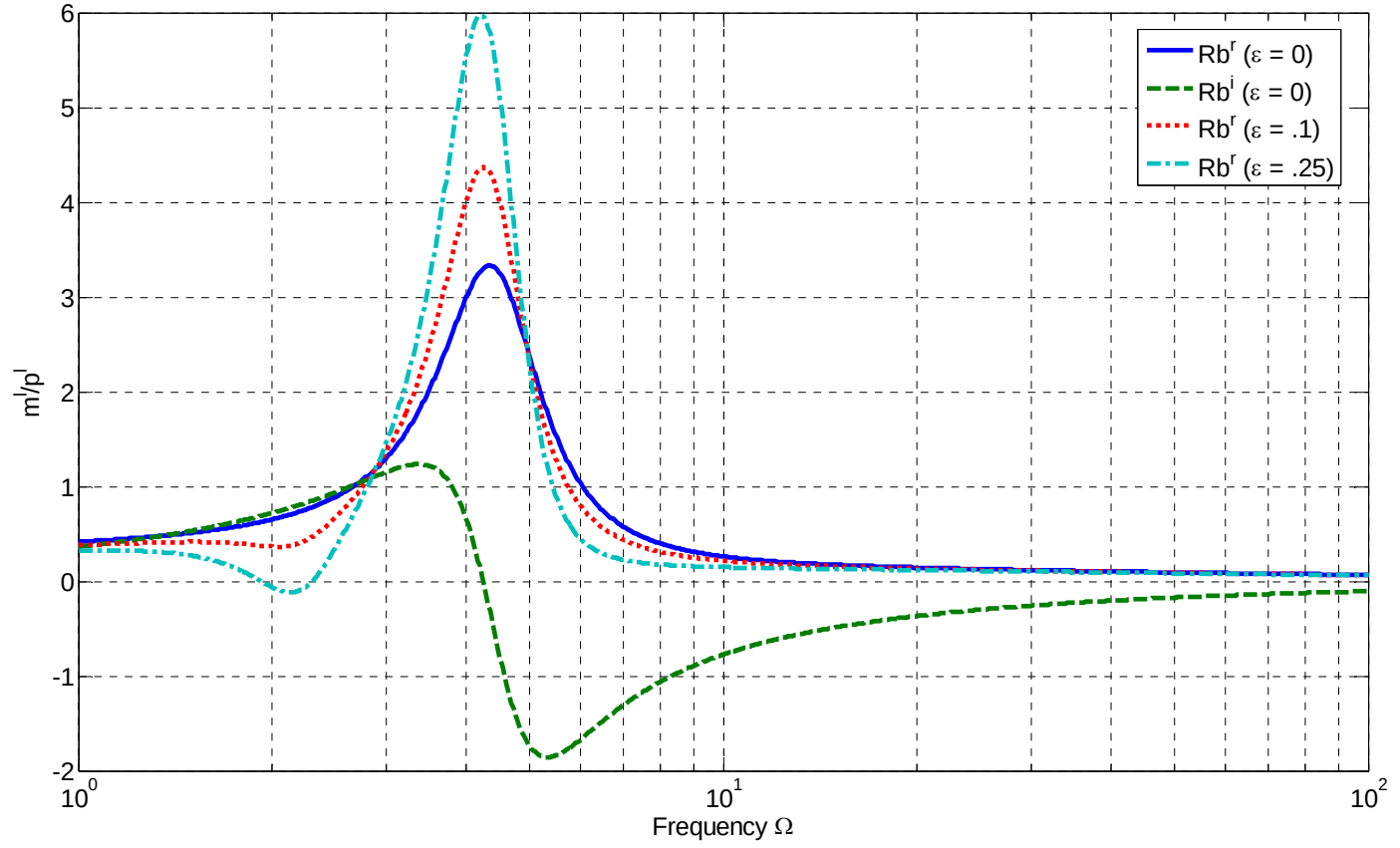


Figure 3.1 Response Function: $n = 0.3$, $A = 6.0$, $B = 3.3$, $E_s = 9$

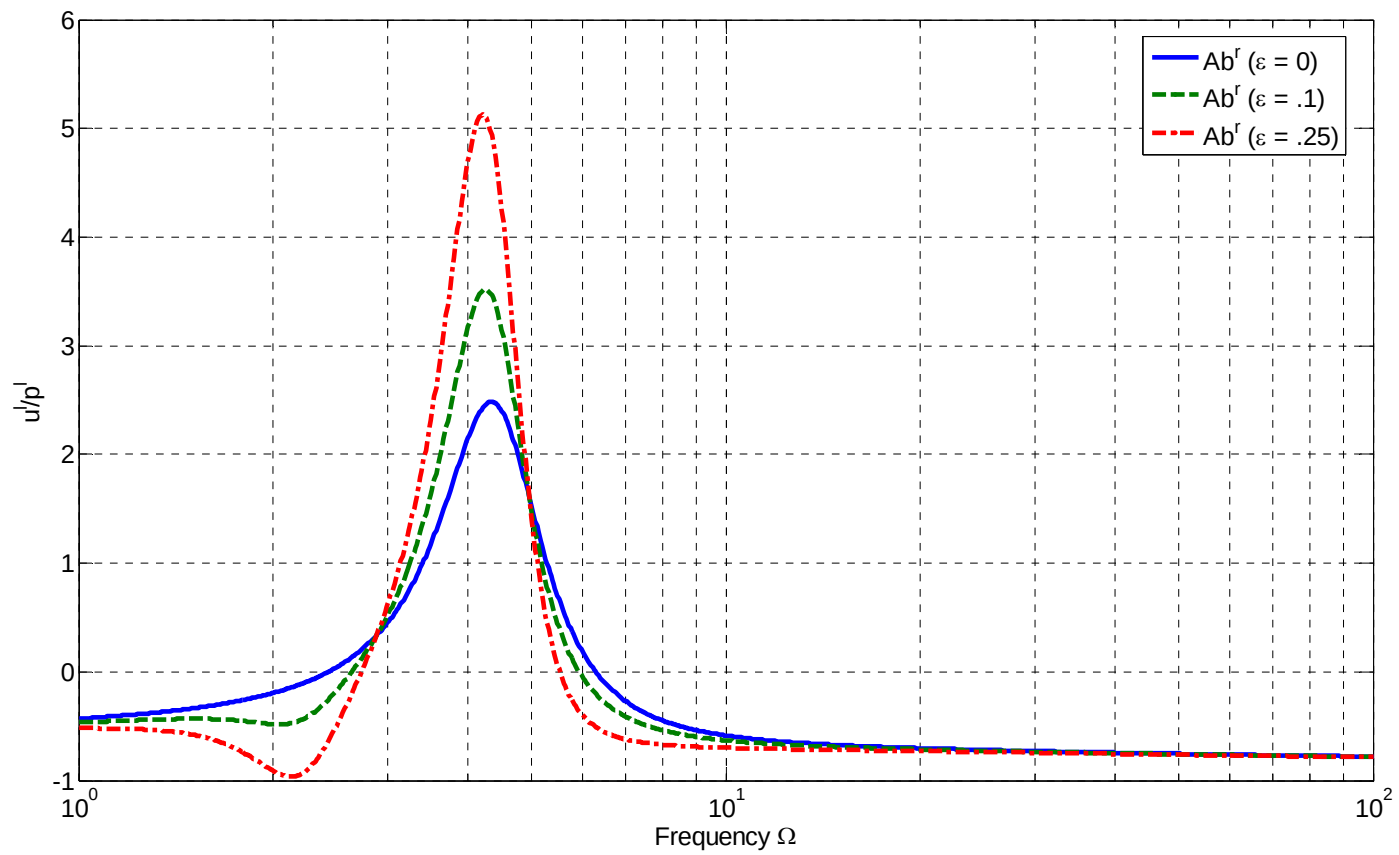


Figure 3.2 Admittance Function: $n = 0.3$, $A = 6.0$, $B = 3.3$, $\gamma = 1.18$, $E_s = 9$

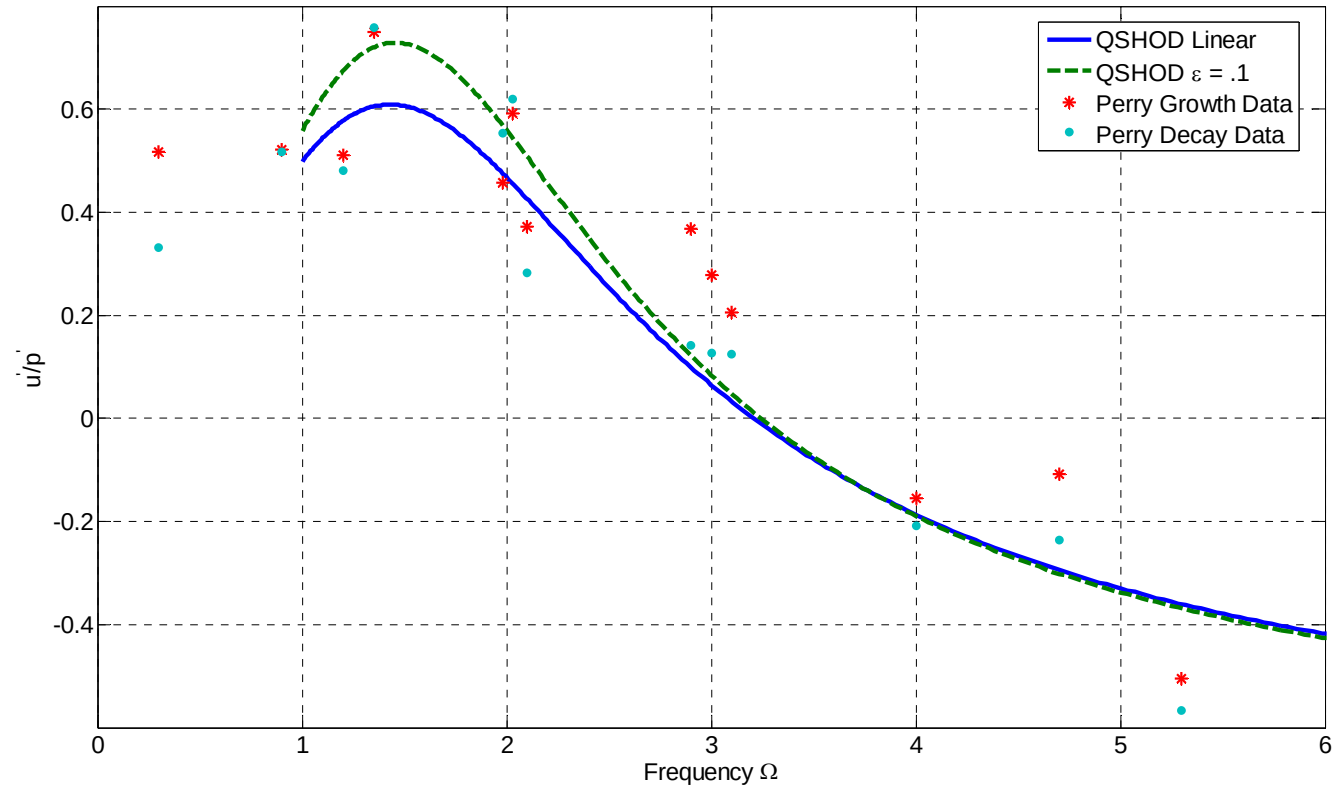


Figure 3.3 Perry's T-17 Data Fit by the QSHOD Model : $n = 0.82$, $A = 2.3$, $B = 1.38$, $E_s = 9$, $\gamma = 1.18$

CHAPTER IV

LINEAR T' IEN MODEL

Problem Definition

Making use of the less well known approach as laid out by T'ien will remove several limitations of the QSHOD model. Doing so will fill in some of the limiting assumptions by including the unsteady gas phase behavior and unsteady conduction heat transfer in the solid while retaining the one dimensional, homogeneous constant material properties assumption. The Galilean coordinate system used in the QSHOD models is retained.

The following analysis and remarks are based on a paper by T'ien [1]. Throughout the reference material the steps necessary to derive the model are not readily found or necessarily clear, for clarity, the model is described here in full detail including intermediate steps.

It can be noted from the review by Culick [5]; that most of the previous investigations of unsteady burning of a solid propellant have been based on the quasi-steady approach, yielding a result with several limiting outcomes. The exceptions are the paper by T'ien [1] and the paper by Calvin and Lazmi [11], whose models the gas phase to respond to the pressure oscillations, introducing a coupling effect. The importance of including this effect becomes apparent at higher frequencies.

To study the higher frequency range not captured by the quasi-steady model, both the solid and gas phases are treated in an unsteady manner. The governing

equations are linearized for small amplitude oscillations, and then the resulting perturbed systems of equations are solved by numerical integration.

Analysis

Gas Phase

The flame in the gas phase is assumed to be premixed, laminar, one-dimensional, and to proceed with a one-step forward chemical reaction. The specific heats, molecular weights, and coefficient of heat conduction are taken to be constant, and the Lewis number equal to unity [1]. Applying these assumptions, the dimensional continuity, energy, specie equations, and perfect gas law yields

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial \rho^* u^*}{\partial x^*} = 0 \quad (4.1)$$

$$\rho^* C_g^* \frac{\partial T^*}{\partial t^*} + \rho^* u^* C_g^* \frac{\partial T^*}{\partial x^*} - \lambda^* \frac{\partial^2 T^*}{\partial x^{*2}} - \frac{\partial p^*}{\partial t^*} = - \sum_{i=1}^N h_i^0 w_i^* \quad (4.2)$$

$$\rho^* \frac{\partial Y_i}{\partial t^*} + \rho^* u^* \frac{\partial Y_i}{\partial x^*} - \frac{\lambda^*}{C_p^*} \frac{\partial^2 Y_i}{\partial x^{*2}} = w_i^* \quad i = 1, 2, \dots, N \quad (4.3)$$

$$p^* = \rho^* \Re T^* \quad (4.4)$$

where dimensional quantities are denoted by the superscript *. For equations (4.1) - (4.4) the range of x^* is from 0, fixed at the propellant surface, to ∞ .

For a one-step forward chemical reaction, the conversion of fuel and oxidizer into products is defined by,



where the subscripts f , o and p , represent fuel oxidizer and products respectively.

The law of mass action for an arbitrary chemical reaction:

$$\sum_{i=1}^N \nu_i' W_i \rightarrow \sum_{i=1}^N \nu_i'' W_i \quad (4.6)$$

where, W_i is the molecular weight. The change in number of moles dn_i , of each species is given by:

$$w_i^* = W_i \frac{dn_i}{dt}, \quad \frac{dn_i}{(\nu_i'' - \nu_i')} = \frac{dn_j}{(\nu_j'' - \nu_j')} \quad (4.7)$$

The reaction rate w_i^* for the species i

$$w_i^* = W_i \frac{\nu_i'' - \nu_i'}{\nu_j'' - \nu_j'} \frac{dn_j}{dt} \quad (4.8)$$

for $j = f$:

$$w^* = W_f \frac{dn_f}{dt}, \quad \nu_f'' = 0 \quad (4.9)$$

Substitution of (4.9) into (4.8) with $j = f$, yields

$$w_i^* = -\frac{W_i}{W_f} \frac{\nu_i'' - \nu_i'}{\nu_j'} w^* \quad (4.10)$$

The reaction rate of the fuel can be expressed by

$$w^* = B^* T^{*\delta} \left(\frac{p^*}{\Re T^*} \right)^n \prod_{j=1}^N \left(\frac{Y_j}{W_j} \right)^{n_j} e^{-\left(\frac{E^*}{\Re T^*} \right)} \quad (4.11)$$

To define the reaction rate for oxidizer and products

$$w_i^* = \frac{\nu_i'' - \nu_i'}{\nu_f'} w^*, \quad i = 0, p \quad (4.12)$$

The right hand side of equation (4.2) becomes

$$-\sum_{i=1}^N h_i^0 w_i^* = w^* q^* \quad (4.13)$$

where, the heat of combustion per unit mass of fuel consumed is

$$q^* = \sum \frac{W_i (v_i' - v_i'') h_i^0}{W_f v_f'} \quad (4.14)$$

It is useful at this juncture to introduce the following dimensionless variables:

$$\begin{aligned} \rho &= \frac{\rho^*}{\bar{\rho}_\infty^*} & T &= \frac{T^*}{\bar{T}_\infty^*} & x &= \frac{x^* \bar{\rho}_\infty^* C_g \bar{u}_\infty^*}{\bar{\lambda}_\infty^*} \\ u &= \frac{u^*}{\bar{u}_\infty^*} & p &= \frac{p^*}{\bar{p}_\infty^*} & t &= \frac{x^* \bar{\rho}_\infty^* C_g \bar{u}_\infty^{*2}}{\bar{\lambda}_\infty^*} \\ q &= \frac{q^*}{C_g \bar{T}_\infty^*} & w &= \frac{w^* \bar{\lambda}_\infty^*}{\bar{\rho}_\infty^* C_g \bar{u}_\infty^{*2}} \end{aligned} \quad (4.15)$$

Substituting (4.11), (4.13) and (4.14) into (4.1) - (4.4), and using the dimensionless variables yields,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \quad (4.16)$$

$$\rho \frac{\partial T}{\partial t} + \rho u \frac{\partial T}{\partial x} - \frac{\partial^2 T}{\partial x^2} - \frac{\gamma - 1}{\gamma} \frac{\partial p}{\partial t} = w q \quad (4.17)$$

$$\rho \frac{\partial Y_i}{\partial t} + \rho u \frac{\partial Y_i}{\partial x} - \frac{\partial^2 Y_i}{\partial x^2} = v_i w \quad i = f, o, p \quad (4.18)$$

$$p = \rho T \quad (4.19)$$

where v_i is defined by:

$$v_i = \frac{W_i (v_i' - v_i'')}{W_f v_f'} \quad (4.20)$$

The mass fraction equations (4.18); for the fuel, the oxidizer, and the products, can be made similar, with the following relation

$$\frac{Y_i}{v_i} - \frac{Y_f}{v_f} = \text{const} \quad (4.21)$$

With the knowledge that both the fuel and the oxidizer are completely expended at the flame edge, the following mathematical statement can be made:

$$Y_o = \frac{v_o}{v_f} Y_f = -v_o Y_f \quad (4.22)$$

This important step allows for the treatment of the fuel equation alone, allowing for two important steps to take place, (4.18) becomes

$$\rho \frac{\partial Y_f}{\partial t} + \rho u \frac{\partial Y_f}{\partial x} - \frac{\partial^2 Y_f}{\partial x^2} = -w \quad (4.23)$$

and the non-dimensional reaction rate (4.11), can be written as

$$w = BjT^\delta \left(\frac{p}{T} \right)^n Y_f^n e^{-E/T} \quad (4.24)$$

where

$$j = \frac{W_o v'_o}{W_f v'_f} = \text{oxidizer - fuel ratio} \quad (4.25)$$

$$B = \frac{B^* \lambda_g^* \bar{T}_\infty^{*\beta} \bar{p}_\infty^{*n}}{\bar{m}^{*2} C_g \prod_{j=1}^N (W_j^{n_j})} \quad (4.26)$$

$$E = \frac{E^*}{\Re \bar{T}_\infty^*} \quad (4.27)$$

and n is the order of chemical reaction.

Under the assumption of small amplitude oscillations, a perturbation analysis is appropriate. The pressure wavelength in the combustor is always much larger than the flame thickness, so that the pressure can be assumed to be a function of time only. Expanding the pressure, temperature, velocity, species and burning rate in terms of the small parameter ε , yields,

$$p(t) = 1 + \varepsilon e^{i\omega t} \quad (4.28)$$

$$T(x, t) = T^{(0)} + \varepsilon T^{(1)} e^{i\omega t} + \mathcal{O}(\varepsilon^2) \quad (4.29)$$

$$u(x, t) = u^{(0)} + \varepsilon u^{(1)} e^{i\omega t} + \mathcal{O}(\varepsilon^2) \quad (4.30)$$

$$Y_f(x, t) = Y^{(0)} + \varepsilon Y^{(1)} e^{i\omega t} + \mathcal{O}(\varepsilon^2) \quad (4.31)$$

$$r(x, t) = r^{(0)} + \varepsilon r^{(1)} e^{i\omega t} + \mathcal{O}(\varepsilon^2) \quad (4.32)$$

The frequency ω here can be referenced to the classical dimensionless frequency as shown in the QSHOD analysis by:

$$\Omega = \delta_1 \beta \omega \quad (4.33)$$

and the frequency can be made dimensional by the following as seen in (2.9)

$$\omega^* = \frac{\bar{m}^2 c_p}{\lambda_p \rho_p} \Omega \quad (4.34)$$

To eliminate the density we now expand $\rho(x, t)$, and make use of the perfect gas law,

$$\rho(x, t) = \frac{p(t)}{T(x, t)} = \frac{1}{T^{(0)}} \left[1 + \varepsilon \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) e^{i\omega t} - \varepsilon^2 \frac{\bar{T}^{(2)}}{T^{(0)}} \right] \quad (4.35)$$

In addition, it is useful at this time to expand $m(x, t) = \rho(x, t)u(x, t)$

$$m(x, t) = \frac{1}{T^{(0)}} \left[u^{(0)} + \varepsilon \left(u^{(1)} + u^{(0)} - \frac{u^{(0)} T^{(1)}}{T^{(0)}} \right) e^{i\omega t} \right] \quad (4.36)$$

Expanding the reaction rate (4.24) is algebraically intensive and is more manageable when broken into its components,

$$p^n = 1 + \varepsilon n e^{i\omega t} \quad (4.37)$$

$$Y_f^n = \left(Y^{(0)} \right)^n \left[1 + \varepsilon n \frac{Y^{(1)}}{Y^{(0)}} e^{i\omega t} \right] \quad (4.38)$$

$$T^{\delta-n} = \left(T^{(0)} \right)^{\delta-n} \left[1 + \varepsilon (\delta - n) \frac{T^{(1)}}{T^{(0)}} e^{i\omega t} \right] \quad (4.39)$$

$$T^{-1} = \frac{1}{T^{(0)}} \left(1 - \varepsilon \frac{T^{(1)}}{T^{(0)}} e^{i\omega t} \right) \quad (4.40)$$

$$e^{-E/T} = e^{-\frac{E}{T^{(0)}}} \left(1 + \varepsilon \frac{E}{T^{(0)}} \frac{T^{(1)}}{T^{(0)}} e^{i\omega t} \right) \quad (4.41)$$

Recombining and collecting by increasing order, the steady reaction rate,

$$w^{(0)} = B j \left(T^{(0)} \right)^{\delta-n} \left(Y^{(0)} \right)^n e^{-\frac{E}{T^{(0)}}} \quad (4.42)$$

unsteady reaction rate

$$w^{(1)} = w^{(0)} \left[\left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} + n \left(1 + \frac{Y^{(1)}}{Y^{(0)}} \right) \right] e^{i\omega t} \quad (4.43)$$

Expanding the governing equations (4.16), (4.17) and (4.23), collecting terms of leading order (steady-state)

$$\frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial \rho^{(0)} u^{(0)}}{\partial x} = 0 \quad (4.44)$$

$$\rho^{(0)} \frac{\partial T^{(0)}}{\partial t} + \rho^{(0)} u^{(0)} \frac{\partial T^{(0)}}{\partial x} - \frac{\partial^2 T^{(0)}}{\partial x^2} = w^{(0)} q \quad (4.45)$$

$$\rho^{(0)} \frac{\partial Y^{(0)}}{\partial t} + \rho^{(0)} u^{(0)} \frac{\partial Y^{(0)}}{\partial x} - \frac{\partial^2 Y^{(0)}}{\partial x^2} = -w^{(0)} \quad (4.46)$$

Making use of (4.35) and (4.36) to eliminate the density terms

$$m^{(0)} = \rho^{(0)} u^{(0)} = \frac{u^{(0)}}{T^{(0)}} = 1 \quad (4.47)$$

$$\frac{\partial T^{(0)}}{\partial x} - \frac{\partial^2 T^{(0)}}{\partial x^2} = w^{(0)} q \quad (4.48)$$

$$\frac{\partial Y^{(0)}}{\partial x} - \frac{\partial^2 Y^{(0)}}{\partial x^2} = -w^{(0)} \quad (4.49)$$

It is important at this juncture to note that from (4.47), $u^{(0)} = T^{(0)}$, and that equations (4.48) and (4.49) can be made similar by

$$Y^{(0)} = \frac{1}{q} (1 - T^{(0)}) \quad (4.50)$$

Substituting (4.50) into (4.42) returns an equation for the steady reaction rate:

$$w^{(0)} = \frac{Bj}{q^n} \left(T^{(0)} \right)^{\delta-n} \left(1 - T^{(0)} \right)^n e^{-E/T^{(0)}} \quad (4.51)$$

In doing so the steady-state problem, is reduced to the solution of a single equation, equation (4.48), where $w^{(0)}$ is defined by (4.51).

The focus is now turned to the unsteady component of the problem. Eliminating density by using the proper components of (4.35), and also recalling that $u^{(0)} = T^{(0)}$, the unsteady perturbed governing equations can be written as:

$$\frac{\partial u^{(1)}}{\partial x} = \frac{\partial T^{(1)}}{\partial x} - \left(\frac{T^{(1)}}{T^{(0)}} - \frac{u^{(1)}}{T^{(0)}} \right) \frac{\partial T^{(0)}}{\partial x} - i\omega \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) \quad (4.52)$$

$$\frac{\partial^2 T^{(1)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial T^{(1)}}{\partial x} + \left(1 + \frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial T^{(0)}}{\partial x} + i\omega \left(\frac{T^{(1)}}{T^{(0)}} - \frac{\gamma-1}{\gamma} \right) \\ & -w^{(0)}q \left[n \left(1 - \frac{Y^{(1)}}{Y^{(0)}} \right) + \left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} \right] \end{aligned} \right\} \quad (4.53)$$

$$\frac{\partial^2 Y^{(1)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial Y^{(1)}}{\partial x} + \left(1 + \frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial Y^{(0)}}{\partial x} + \frac{i\omega}{T^{(0)}} Y^{(1)} \\ & +w^{(0)} \left[n + n \frac{Y^{(1)}}{Y^{(0)}} + \left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} \right] \end{aligned} \right\} \quad (4.54)$$

Equations (4.52) - (4.54) constitute a fifth order system, that will be solved by numerical integration.

Boundary Conditions

To solve the systems of equations, it is necessary to specify the appropriate number of boundary conditions. There are two boundary conditions at the flame edge, $x \rightarrow \infty$, the first is that the fuel must vanish,

$$Y^{(0)}(\infty) = Y^{(1)}(\infty) = 0 \quad (4.55)$$

and the second is a condition for the temperature and is worthy of a detailed discussion.

The flame temperature, in general, oscillates, and a temperature wave is convected away from the flame. When the period of oscillation is close to the gas thermal time, the temperature wave length is of the same order as the flame length. Deriving a suitable boundary condition depends on the flow condition in the

combustion zone. For this one-dimensional problem, the product gas is convected away in the direction perpendicular to the flame front, it is reasonable to assume that the flow just outside the flame edge is isentropic. Therefore, the entropy is conserved, $DS/Dt = 0$, for each fluid particle convected away from the flame. Writing this expression in terms of temperature and pressure yields [1]

$$T^*(X^*, t^*) - T^*\left(0, t^* - \frac{X^*}{u_\infty^*}\right) = \frac{\gamma-1}{\gamma} \left[p^*(t^*) - p^*\left(t^* - \frac{X^*}{u_\infty^*}\right) \right] \quad (4.56)$$

where X^* is the distance measured from the flame edge into the combustion zone ($X^* = 0$ coincides with $x^* = \infty$). Differentiation yields:

$$\frac{\partial T^*(X^*, t^*)}{\partial X^*} = -\frac{1}{u_\infty^*} \frac{\partial T^*(0, \eta)}{\partial \eta} + \frac{\gamma-1}{\gamma} \frac{1}{u_\infty^*} \frac{\partial p^*(\eta)}{\partial \eta} \quad (4.57)$$

where

$$\eta = t^* - \frac{X^*}{u_\infty^*} \quad (4.58)$$

As X^* approaches zero and η approaches t^* , equation (4.57) yields

$$\frac{\partial T^*(X^* = 0, t^*)}{\partial X^*} = -\frac{1}{u_\infty^*} \frac{\partial T^*(X^* = 0, t^*)}{\partial t^*} + \frac{\gamma-1}{\gamma} \frac{1}{u_\infty^*} \frac{\partial p^*(t^*)}{\partial t^*} \quad (4.59)$$

Remembering that $X^* = 0$ is equivalent to $x^* = \infty$, equation (4.59) is made dimensionless with the use of the variables in (4.15),

$$\frac{\partial T(\infty, t)}{\partial x} = -\frac{\partial T(\infty, t)}{\partial t} + \frac{\gamma-1}{\gamma} \frac{\partial p(t)}{\partial t} \quad (4.60)$$

Perturbing (4.60) and collecting the appropriate orders, the steady-state solution yields two conditions that are equivalent for use in (4.48), namely

$$\frac{dT^{(0)}(\infty)}{dx} = 0 \quad (4.61)$$

or

$$T^{(0)}(\infty) = 1 \quad (4.62)$$

The oscillatory unsteady boundary condition from (4.60) is,

$$\frac{\partial T^{(1)}(\infty)}{\partial x} = -i\omega T^{(1)}(\infty) + \frac{\gamma-1}{\gamma} i\omega \quad (4.63)$$

Equation (4.63) is the oscillatory temperature boundary condition at the flame edge, and is valid for high frequency oscillations.

Three boundary conditions remain, and all are located at the solid surface $x = 0$. The first condition comes from the fact that the mass flux fraction of the fuel and oxidizer are fixed by the composition of the propellants. The fuel mass flux fraction is expressed by

$$\varepsilon_f = Y_f - \frac{1}{m} \frac{\partial Y_f}{\partial x} \quad (4.64)$$

Evaluating (4.64) at $x = 0$, surrenders

$$\varepsilon_f = \varepsilon_{fs}^{(0)} \quad (4.65)$$

and

$$\varepsilon_f^{(1)} = 0 \quad (4.66)$$

To seek out the boundary conditions, perturb (4.64), and utilize (4.65) and (4.66).

Which yields, for the steady part

$$\varepsilon_{fs}^{(0)} = Y^{(0)} - \frac{\partial Y^{(0)}}{\partial x} \quad (4.67)$$

the unsteady part

$$\varepsilon_f^{(1)} = 0 = Y^{(1)} - \frac{\partial Y^{(1)}}{\partial x} + \left(1 + \frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial Y^{(0)}}{\partial x} \quad (4.68)$$

To proceed with finding the other two boundary conditions, it is necessary to consider the heat transfer process in the solid phase and the law governing the mass conversion of a solid to a gas at $x = 0$.

Solid Phase

The solid propellant is assumed to be homogeneous throughout and to not have an associated condensed-phase chemical reaction. The dimensionless heat transfer equation in the solid, with $x = 0$ located at the solid surface, is

$$\beta \frac{\partial T}{\partial t} + r \frac{\partial T}{\partial x} - \delta_1 \frac{\partial^2 T}{\partial x^2} = 0, \quad -\infty < x < 0 \quad (4.69)$$

where

$$\delta_1 = \frac{\lambda_s^* c_g}{\lambda^* c_p} \quad (4.70)$$

$$r = \frac{r^*}{\bar{r}^*} \quad (4.71)$$

and the ratio of solid density to steady-state gas flame density is given by

$$\beta = \frac{\rho_s^*}{\rho_\infty^*} \quad (4.72)$$

The boundary condition at the gas-surface interface $x = 0$ is

$$\left(\frac{\partial T}{\partial x}\right)_{x=0^-} = \delta_2 \left(\frac{\partial T}{\partial x}\right)_{x=0^+} - \delta_2 r L \quad (4.73)$$

where

$$\delta_2 = \frac{\lambda^*}{\lambda_s^*} \quad (4.74)$$

$$L = \frac{\ell}{c_g \bar{T}_\infty^*} \quad (4.75)$$

and at $x = -\infty$,

$$T(-\infty) = T_c \quad (4.76)$$

To account for the conversion of a solid to a gas, the solid propellant is assumed to decompose according to Arrhenius Law. In dimensionless form,

$$r = \exp\left(-\frac{E_s}{T}\right) \quad (4.77)$$

where

$$E_s = \frac{E_s^*}{\Re \bar{T}_\infty^*} \quad (4.78)$$

Perturbing (4.77) and collecting the result at increasing orders, for the steady case

$$r^{(0)} = e^{-\frac{E_s}{T_s^{(0)}}} \quad (4.79)$$

unsteady

$$r^{(1)} = \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} \quad (4.80)$$

Seeking the steady state temperature distribution, return to the heat transfer equation and perturb (4.69), which yields,

$$\delta_1 \frac{\partial^2 T^{(0)}}{\partial x^2} - \frac{\partial T^{(0)}}{\partial x} = 0 \quad (4.81)$$

Solving (4.81) and using the boundary condition $T^{(0)}(0) = T_s^{(0)}$,

$$T^{(0)} = T_c^{(0)} + (T_s^{(0)} - T_c^{(0)}) e^{\left(\frac{x}{\delta_1}\right)} \quad (4.82)$$

This result is used to calculate the temperature gradient in the gas phase, by substituting (4.82) into (4.73):

$$\left(\frac{dT^{(0)}}{dx} \right)_{x=0^+} = \frac{1}{\delta_1 \delta_2} (T_s^{(0)} - T_c^{(0)}) + L \quad (4.83)$$

Seeking the unsteady temperature distribution, collect the appropriate terms from the perturbed (4.69),

$$\delta_1 \frac{\partial^2 T^{(1)}}{\partial x^2} - \frac{\partial T^{(1)}}{\partial x} - \beta i \omega T^{(1)} - r^{(1)} \chi e^{\left(\frac{x}{\delta_1}\right)} = 0 \quad (4.84)$$

where

$$\chi = \frac{1}{\delta_1} (T_s^{(0)} - T_c^{(0)}) \quad (4.85)$$

First, finding the particular integral

$$T_{\text{particular}}^{(1)} = A e^{\left(\frac{x}{\delta_1}\right)} \quad (4.86)$$

and substituting into (4.84), the result yields a value for A

$$A = -\frac{r^{(1)} \chi}{i \omega \beta} = -\frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{i \omega \beta} \quad (4.87)$$

Next, a complementary solution must be found, by using the roots of the characteristic equation

$$m^2 - m - i\delta_1\omega\beta = 0 \rightarrow m = \frac{1}{2}\left(1 + \sqrt{1 + 4i\delta_1\omega\beta}\right) \quad (4.88)$$

Only the positive root must be taken to insure that the fluctuations will decay for any ω at $x \rightarrow -\infty$. Using this result a complete solution is defined

$$T^{(1)} = C_1 e^{\left(\frac{m}{\delta_1}x\right)} - \frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{i\omega\beta} e^{\left(\frac{x}{\delta_1}\right)} \quad (4.89)$$

Employing the condition $T^{(1)}(0) = T_s^{(1)}$

$$C_1 = \left(1 + \frac{1}{T_s^{(0)}} \frac{E_s}{T_s^{(0)}} \frac{\chi}{i\omega\beta}\right) T_s^{(1)} \quad (4.90)$$

Substituting this finding into (4.89)

$$T^{(1)} = T_s^{(1)} e^{\left(\frac{m}{\delta_1}x\right)} + \frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{i\omega\beta} \left[e^{\left(\frac{m}{\delta_1}x\right)} - e^{\left(\frac{x}{\delta_1}\right)} \right] \quad (4.91)$$

Taking the derivative of (4.91)

$$\frac{\partial T^{(1)}}{\partial x} = \frac{m}{\delta_1} T_s^{(1)} e^{\left(\frac{m}{\delta_1}x\right)} + i \frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{\omega\beta} \left[\frac{m}{\delta_1} e^{\left(\frac{m}{\delta_1}x\right)} - \frac{1}{\delta_1} e^{\left(\frac{x}{\delta_1}\right)} \right] \quad (4.92)$$

Using the unsteady form of the boundary conditions from the perturbed (4.73) and (4.76)

$$\left(\frac{\partial T^{(1)}}{\partial x} \right)_{x=0^-} = \delta_2 \left(\frac{\partial T^{(1)}}{\partial x} \right)_{x=0^+} - \delta_2 r^{(1)} L \quad (4.93)$$

where $r^{(1)}$ is given by (4.80) and

$$T^{(1)}(-\infty) = 0 \quad (4.94)$$

Equation (4.93) can be solved to yield a relationship between the surface temperature and temperature gradient,

$$T^{(1)}(0) = T_s^{(1)} = \frac{i\omega\beta\delta_2 \left(\frac{\partial T^{(1)}}{\partial x} \right)_{x=0^+}}{\frac{m}{\delta_1} \left(i\omega\beta + \frac{E_s}{T_s^{(0)}} \frac{\chi}{T_s^{(0)}} \right) - \frac{E_s}{T_s^{(0)}} \frac{\chi}{\delta_1 T_s^{(0)}} + \frac{E_s}{T_s^{(0)}} \frac{i\omega\beta\delta_2}{T_s^{(0)}} L} \quad (4.95)$$

The remaining boundary conditions are derived by using the Arrhenius Law (4.77) and satisfying continuity across the propellant surface, more specifically;

$$\rho_s r^* = \rho u^* \quad (4.96)$$

Making (4.96) dimensionless through the use of the variables defined in (4.15) and perturbing the result, yields

$$u_s^{(1)} = \left(\frac{E_s}{T_s^{(0)}} + 1 \right) T_s^{(1)} - T_s^{(0)} \quad (4.97)$$

With the systems of equations and their associated boundary conditions fully defined, a numerical integration can take place.

Numerical Method

In order to proceed with the analysis, it is necessary at this stage to solve the resulting systems of equations through numerical integration. The first step is to numerically integrate (4.48) using fourth order Runge-Kutta. This is accomplished by guessing the eigenvalue B and specifying the boundary condition $T^{(0)}(0) = T_s^{(0)}$. The boundary condition at $x = \infty$ can only be satisfied when the correct eigenvalue is chosen, so an iterative process is employed. The iterative process proceeds until the

boundary condition (4.61) is satisfied to a specified error of $0.4E-4$. Typical profiles of the temperature, reaction rate and species are shown in Figure 4.1.

The remaining unsteady equations constitute a fifth order system. They are solved using Runge-Kutta, by writing each system as a system of first order equations, and incorporating a double shooting technique to satisfy the coupled boundary conditions. Typical profiles of the velocity, temperature, and species for the unsteady system is shown in Figure 4.2. Please note that Figures 4.1 and 4.2 are for a single frequency, $\omega = 1$.

Calculation for the response and admittance functions is a time consuming process since the numerical integration of the unsteady system must be preformed for each frequency over a large range of frequencies. The response of the propellant is obtained by plotting the real and imaginary parts of (4.80). This is accomplished through the use of temperature values at the surface, $x = 0$, found from the numerical integration at each value of ω . The admittance is acquired by plotting the real part of the unsteady velocity component at the flame edge, $x = \infty$, for each value of ω .

Results and Discussion

For the reference or base case, the following values were used: second order chemical reaction $n = 2$, oxidizer-fuel ratio of $j = 1$, temperature deep within the cold propellant of $T_c = 0.15$, surface temperature of $T_s = 0.35$, $\gamma = 1.17$, $\delta_1 = \delta_2 = 1$, $\delta = 0$, $q = 0.65$, $L = 0.15$, $E = 10$, $E_s = 4$, and $\beta = 1000$. A parametric study on the variables is performed changing one parameter at a time from the base case.

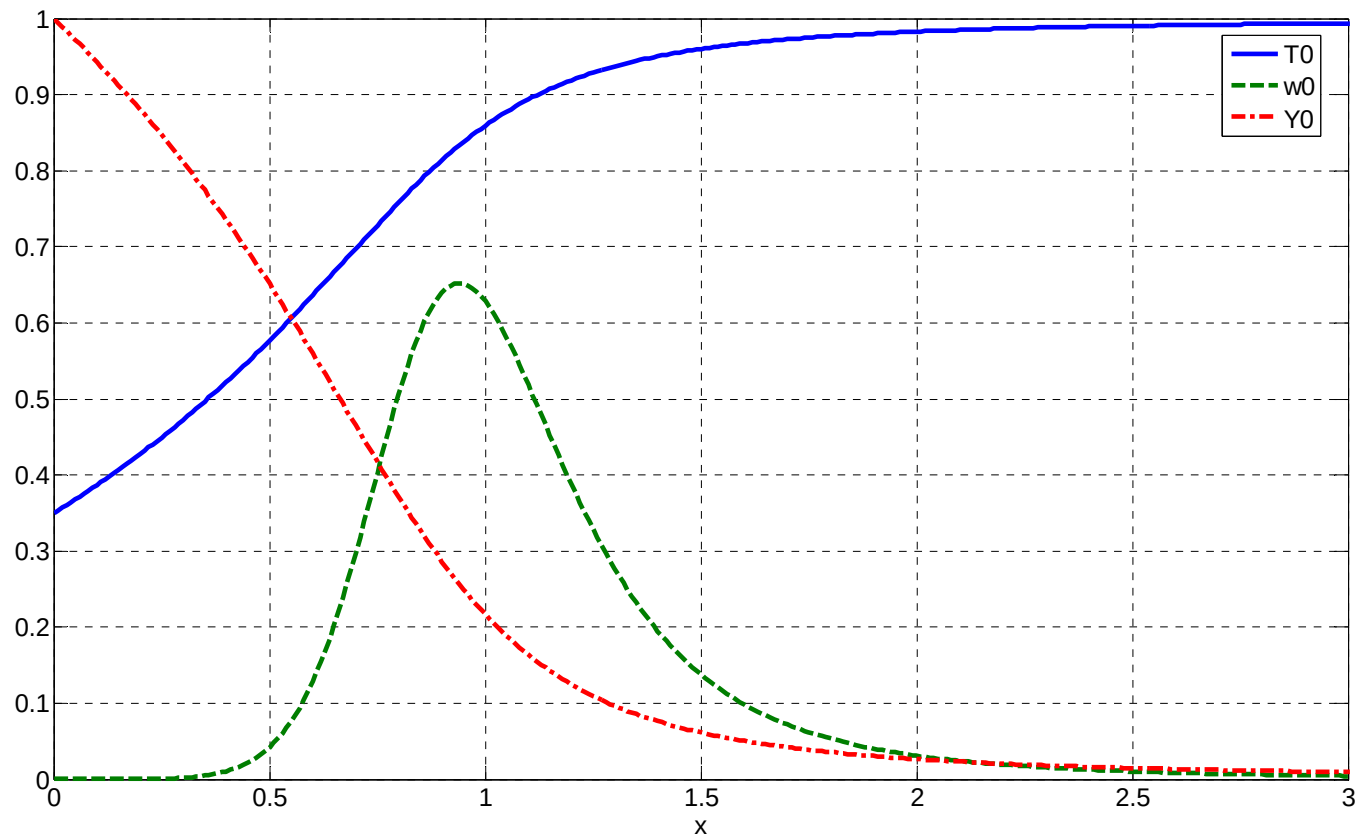


Figure 4.1 Steady-State Distributions of Dimensionless Temperature, Reaction Rate and Species

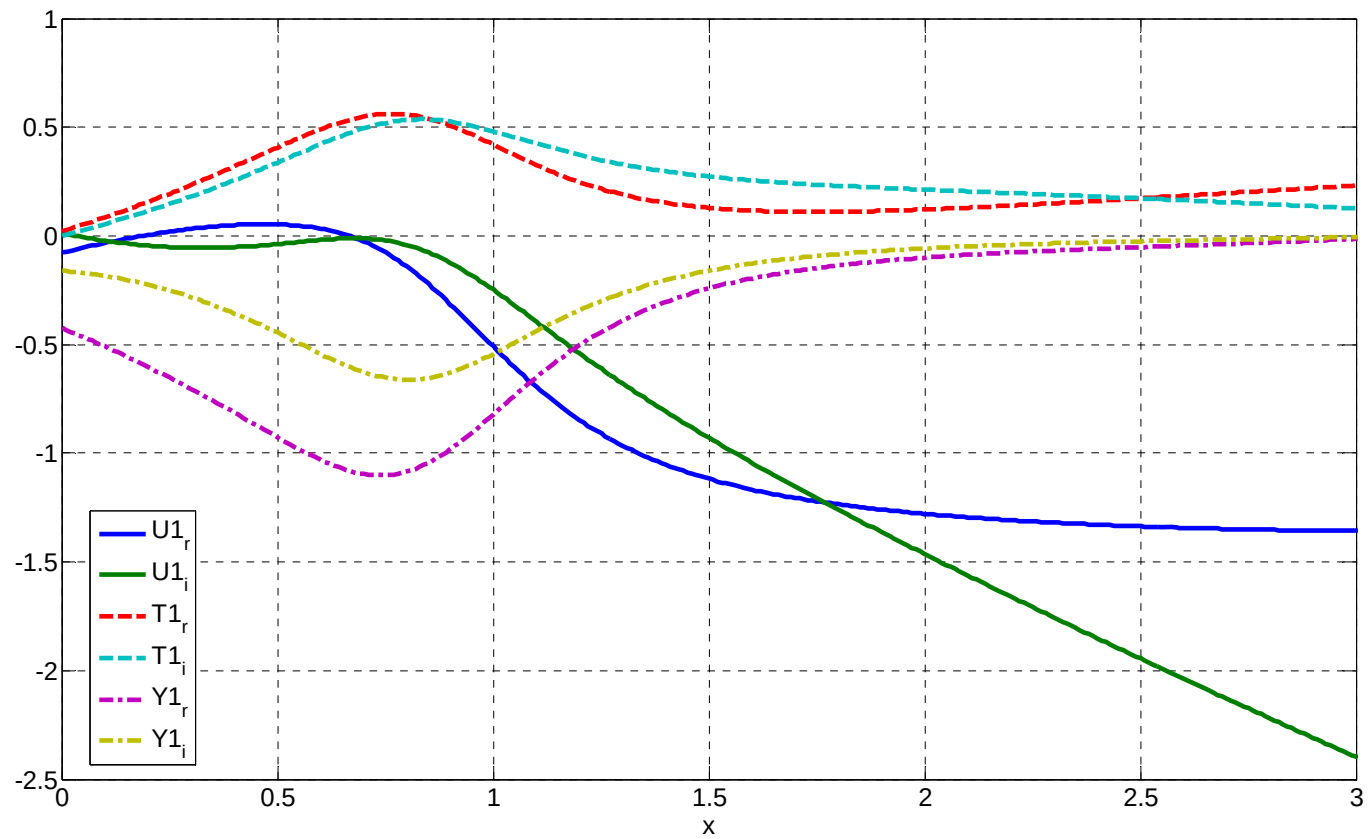


Figure 4.2 Unsteady Distributions of Dimensionless Velocity, Temperature and Species

Figures 4.1 and 4.2. are dimensionless distributions created based on values for the base case at a frequency of $\omega = 1$, or $\Omega = 1000$.

In Figure 4.3, the response and the real part of acoustic admittance for the base case is plotted for the frequency ω , exactly reproducing the figure generated by T'ien [1]. Also from this figure, the importance of including the high frequency effects due to the gas phase becomes strikingly apparent. Figure 4.4 is a successful attempt to fit the T-17 data from Perry's dissertation [12] by using the linear T'ien model, verifying its ability to model the low frequency behavior of a propellant, and was accomplished by only changing one parameter, namely E_s . The T'ien model demonstrates the same ability as the QSHOD model. When used to fit the data, the main difference between the two models is that the T'ien model allows for the actual parameters to change whereas, the non-physical similarity parameters change in the QSHOD model.

Figure 4.5 displays the impact of the ratio of solid to gas densities, β , on the admittance of the propellant. It can be noted from both Figures 4.3 and 4.5 that the low frequency regions of the curves appear to be in agreement with the QSHOD findings, giving another form of validity to the quasi-steady analysis. The importance of inclusion of the gas phase effects can be seen in Figure 4.6, where at higher frequencies there is a dramatic wave damping effect followed by a high frequency wave augmentation. One major assumption of QSHOD model is the use of the restricting relation in (3.97), the restrictiveness becomes apparent when examining Figure 4.6, where after a dimensionless frequency of about ten the adjusted surface response no longer captures the physics of the problem. Simply assuming that the

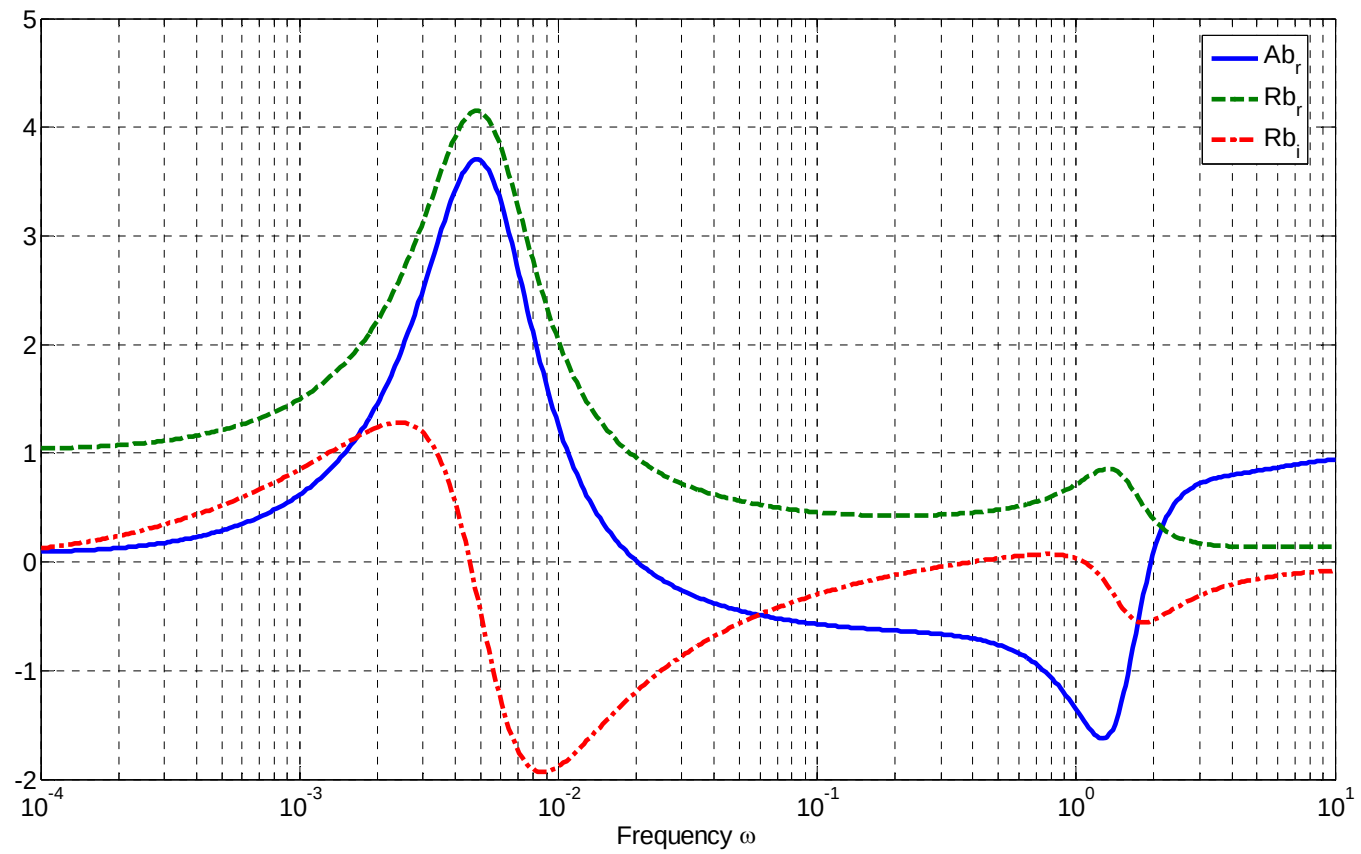


Figure 4.3 Acoustic Admittance and Response Function (Base Case)

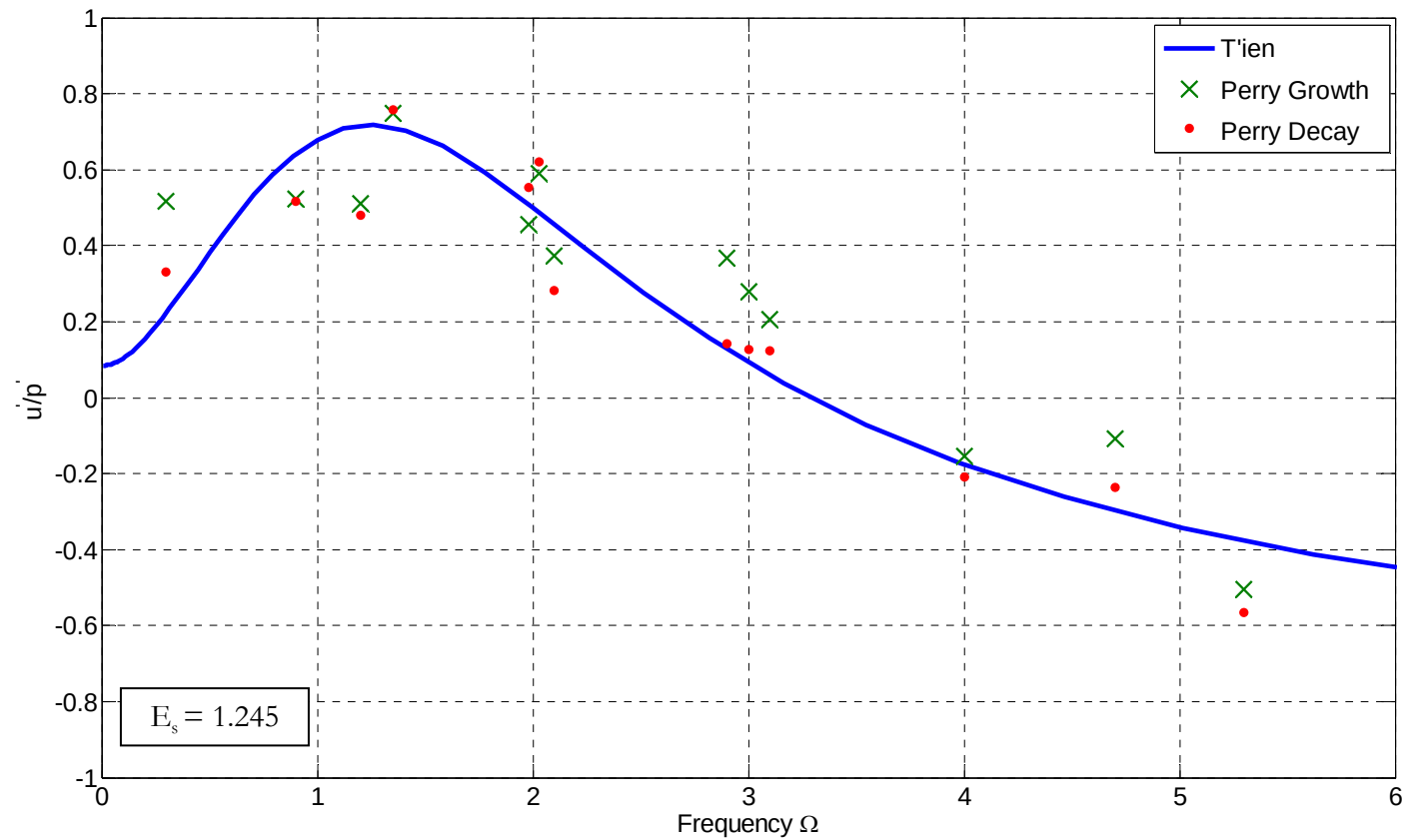


Figure 4.4 Perry's T-17 Data Fit by the T'ien Model by Varying one Parameter E_s

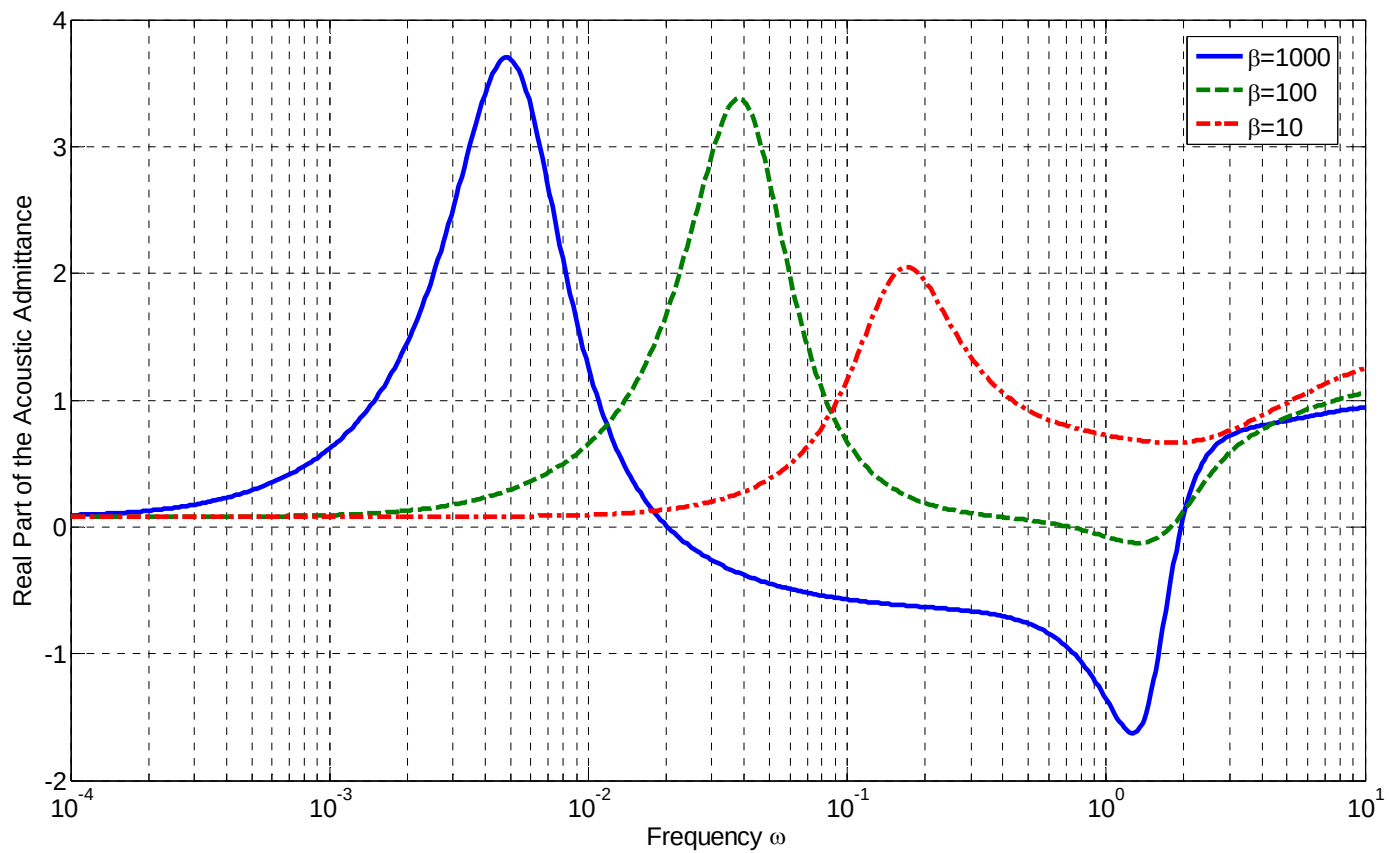


Figure 4.5 Dependence of Acoustic Admittance on the Ratio of Solid to Gas Densities

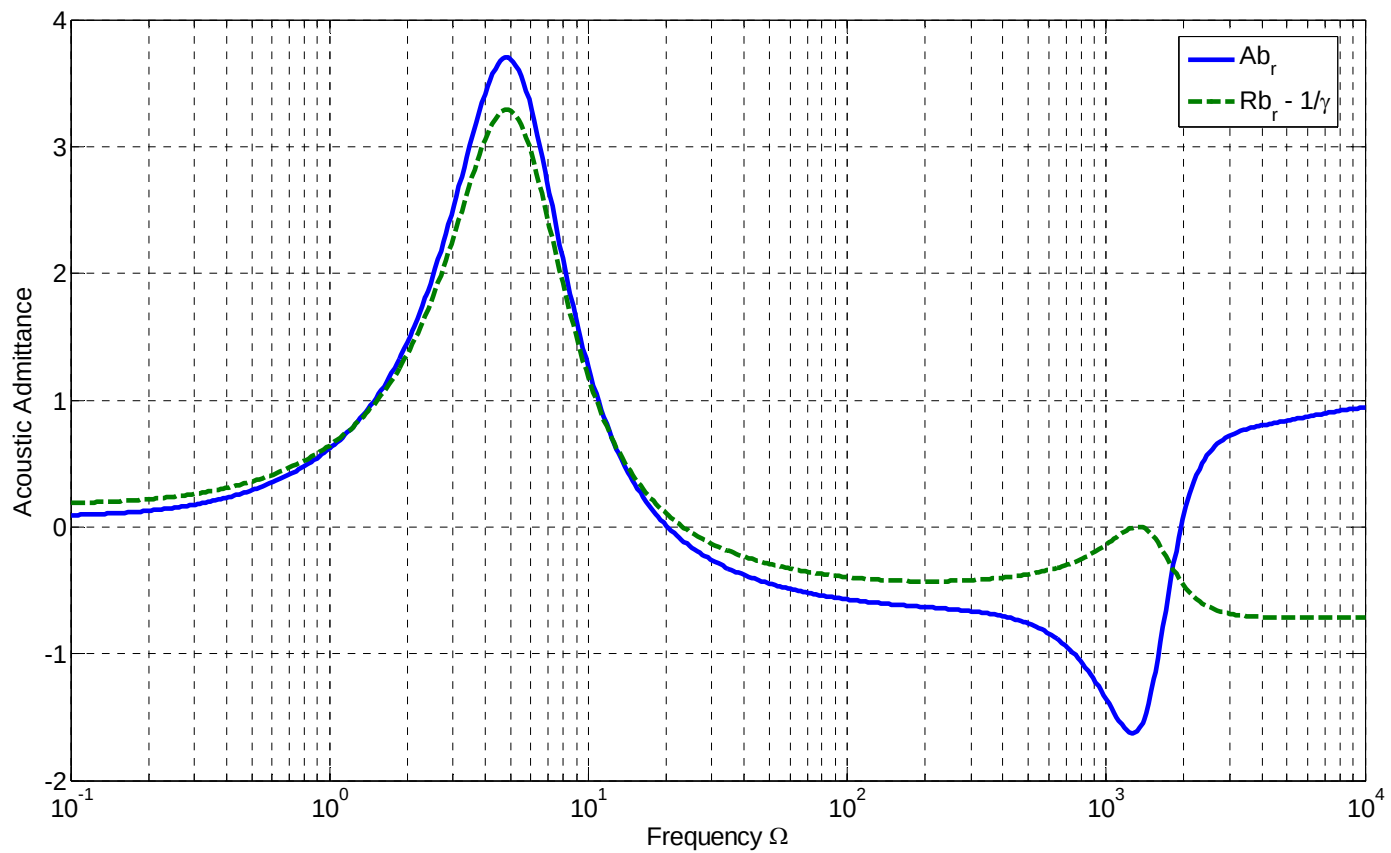


Figure 4.6 Acoustic Admittance and Adjusted Response Function (Base Case)

admittance can be found by subtracting the inverse of γ is not valid or recommended. The importance of the high frequency is undeniable and can not be left out of a complete analysis.

Continuing with the parametric study of the constants, the activation energies are studied. Figure 4.7 shows that increasing the solid phase surface activation energy, E_s , results in a decrease in the magnitude of both the primary peak and the high frequency inverse peak. Also shown in Figure 4.7, is the effect of decreasing E_s , which results in a decrease and an increase in the magnitudes of the primary and the inverse high frequency peaks, respectively. Figure 4.8 shows that increasing the gas phase activation energy, E , increases the magnitudes of both the primary peak and the inverse high frequency peak. Also shown in Figure 4.8, is the effect of decreasing E , which results in a decrease in the magnitudes of both the primary peak and the inverse high frequency peak.

Figure 4.9 demonstrates the consequences of varying the parameter L , the latent heat, yields minimal changes to the low frequency peak. The effects are seen at higher frequency, in the form a proportional moderate amplitude change and an intriguing frequency shift. Variation of the parameter q , yields no change to the linear admittance.

An exciting finding that appears to be overlooked in the work conducted by T'ien is the parametric study of the cold temperature deep within the propellant, but is seen in Figure 4.10 to be of great significance. This was first discussed by Brownlee during

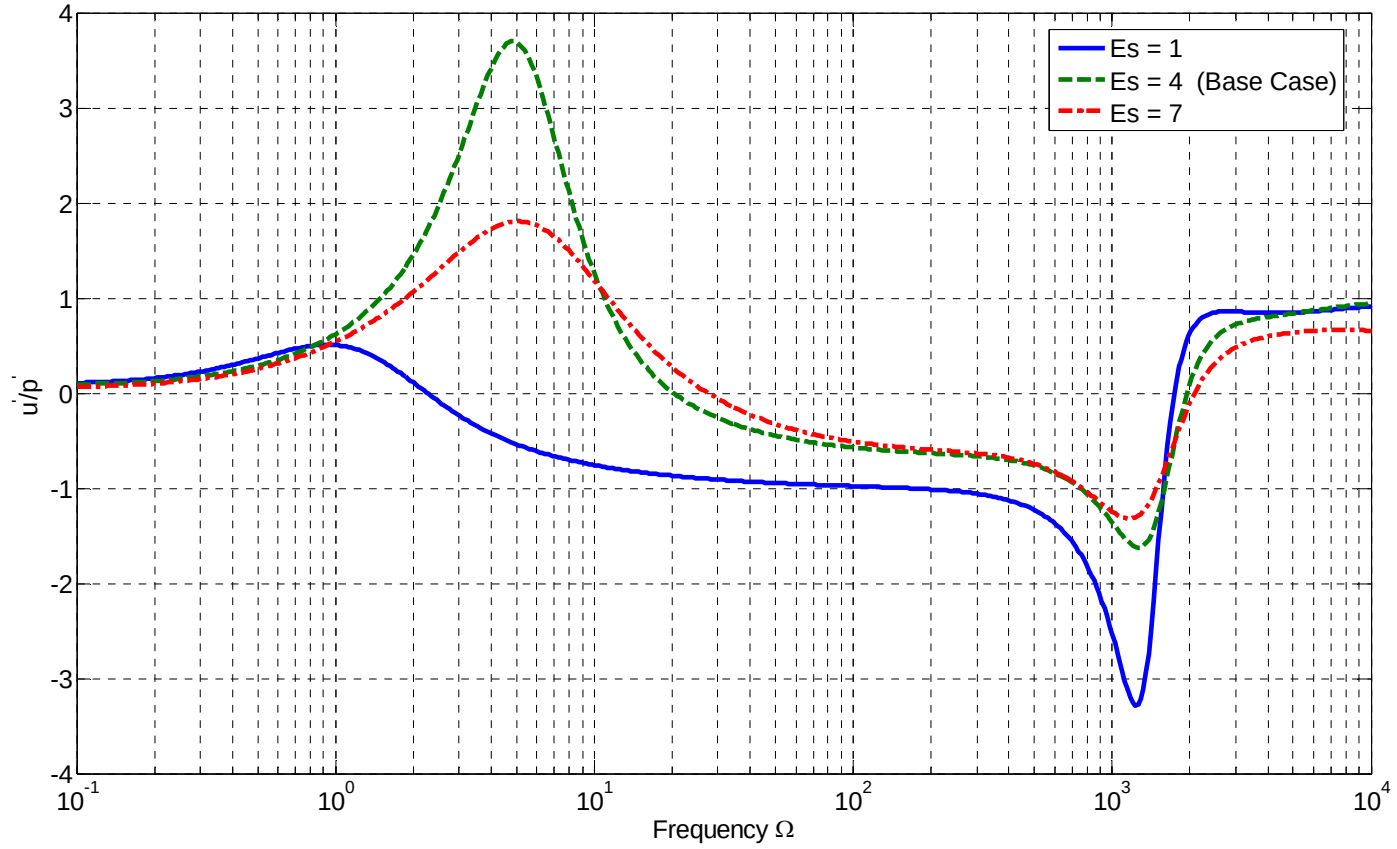


Figure 4.7 Effect of Varying the Dimensionless Solid Phase Activation Energy on the Linear Admittance

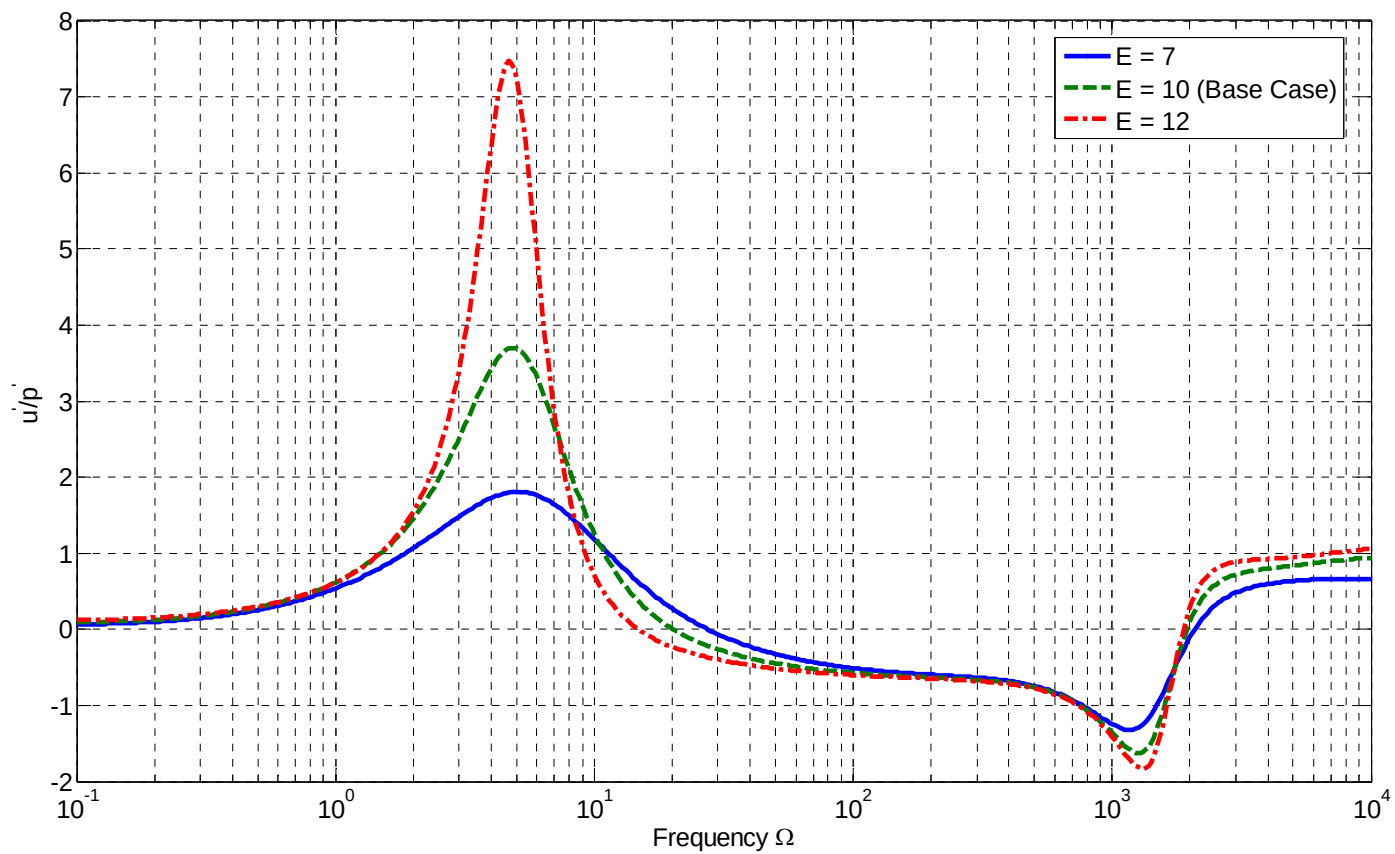


Figure 4.8 Effect of Varying the Dimensionless Gas Phase Activation Energy on the Linear Admittance

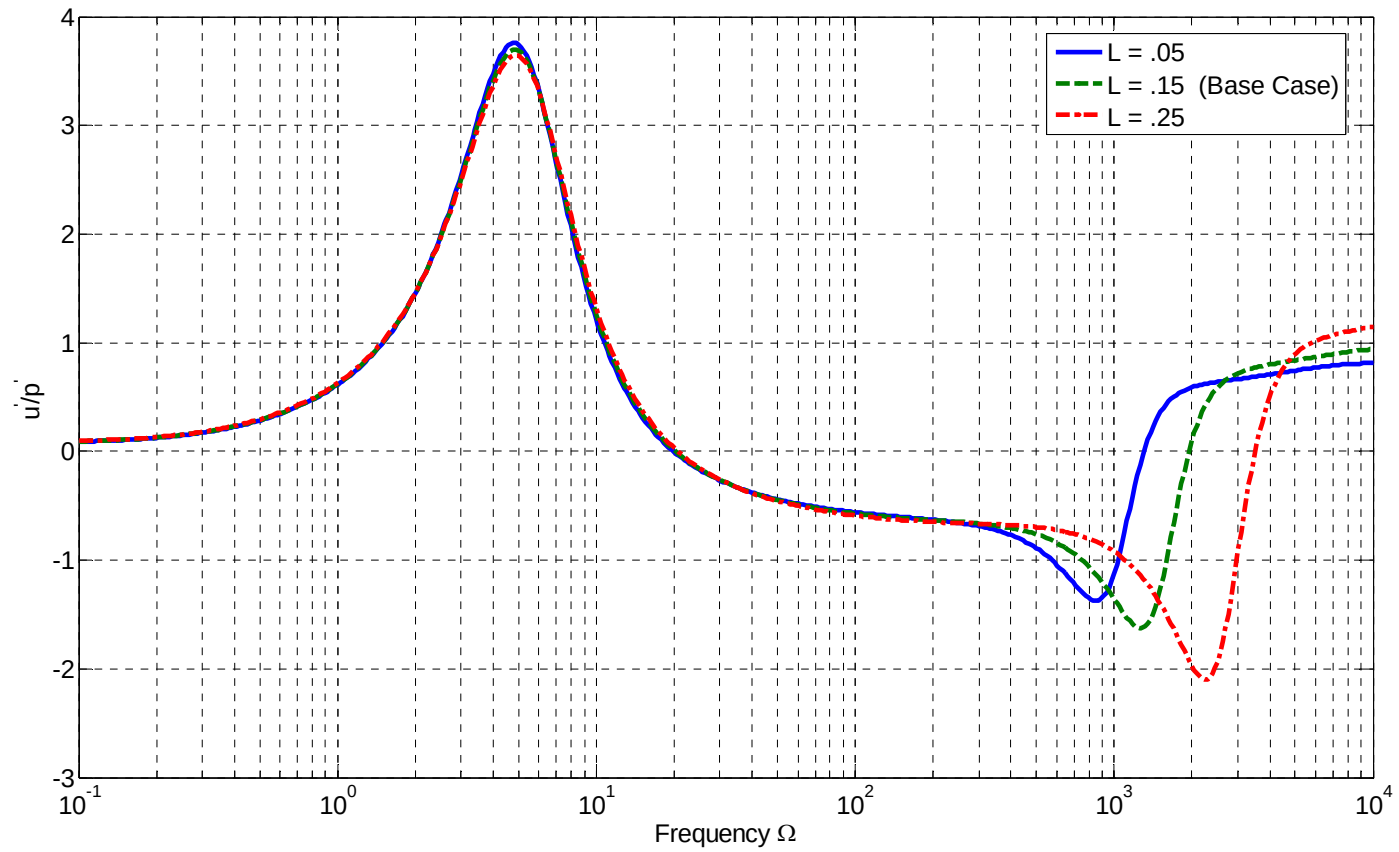


Figure 4.9 Effect of Varying the Dimensionless Latent Heat on the Linear Admittance

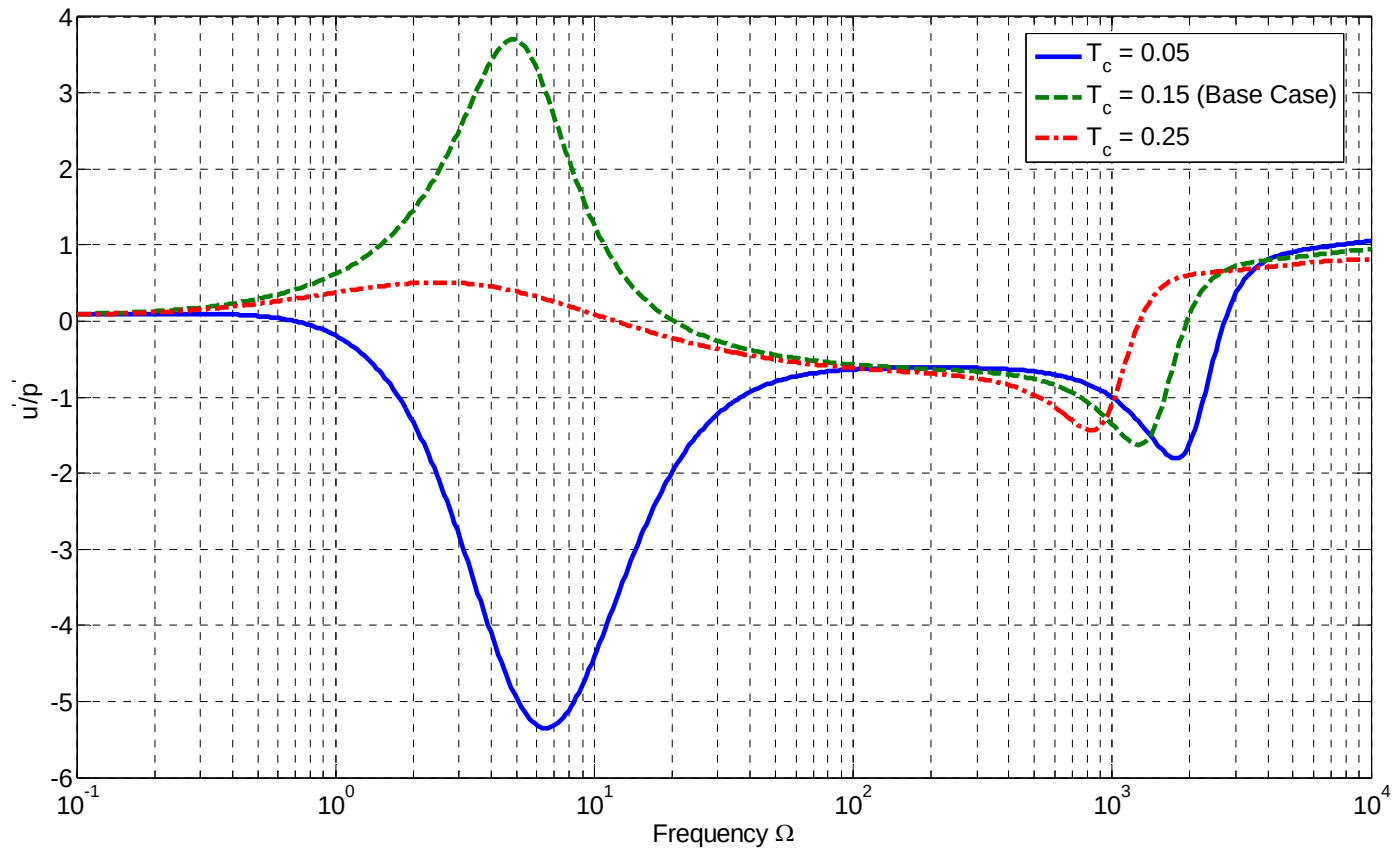


Figure 4.10 Effect of Varying the Dimensionless Cold Propellant Temperature on the Linear Admittance

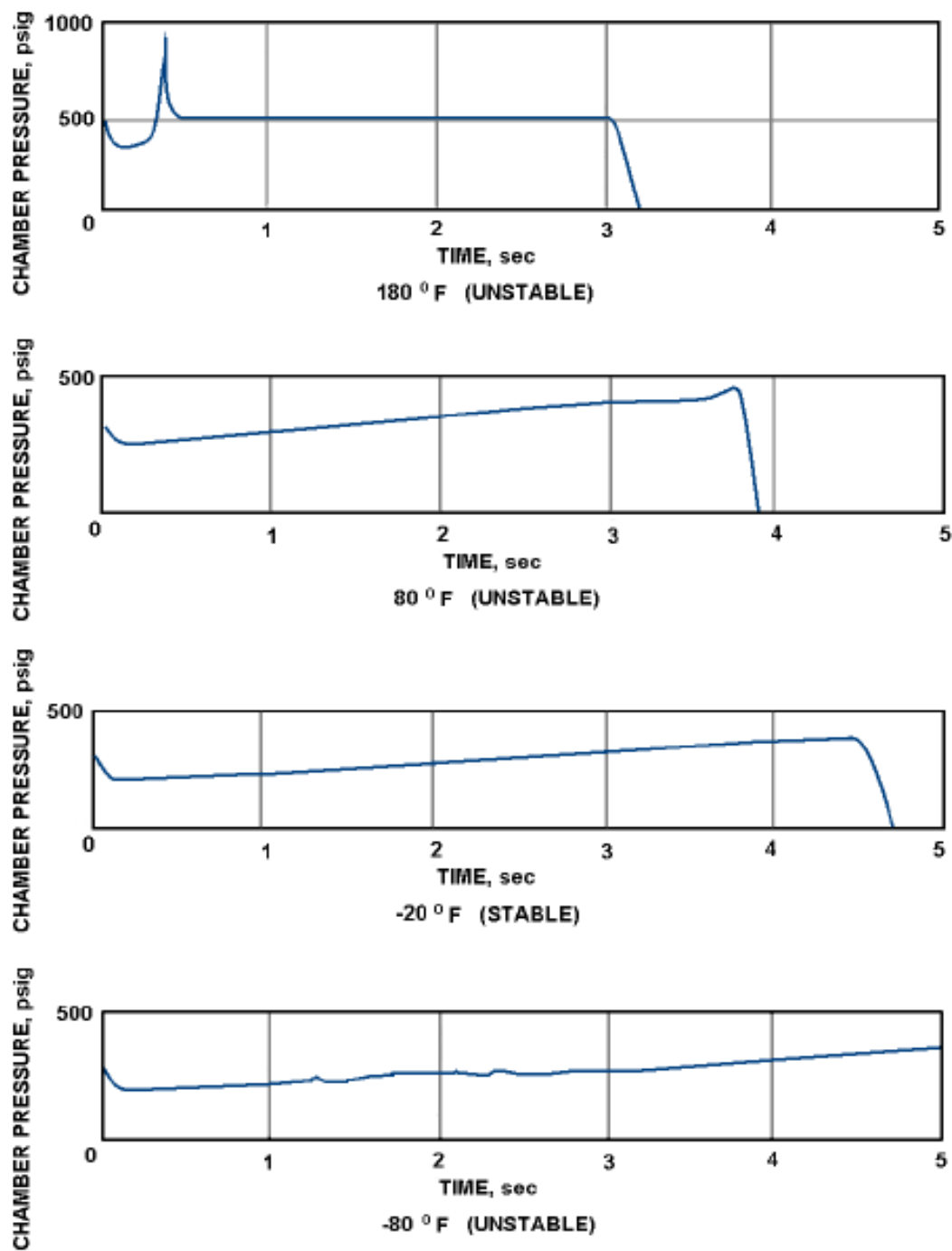


Figure 4.11 Brownlee Experimental Data and Findings [13]

his experiments using a T-burner in 1959 [13] (see Figure 4.11). Brownlee studied this temperature sensitivity while conducting T-Burner experiments, and went on to note that, this sensitivity had a direct impact on the stability boundary associated with the propellant. The Brownlee observation was accomplished by conditioning the cold propellant temperature in a temperature range of -80 to 180 degrees Fahrenheit. The finding here that needs to be stressed is the sensitivity of the linear admittance to the cold propellant temperature. By decreasing the propellant temperature, a negative (damping) primary peak is captured. Also by increasing the temperature, the admittance loses its ability to drive the oscillations at low frequency. This finding shows promise in attempting to control combustion instability by just temperature conditioning the propellant before a firing.

Variation of the parameters studied only leads to very limited conclusions, but do give a sense for the effect of each parameter and its importance. The motivation for this study; the need for a nonlinear model, aligns perfectly with the previous work as it is a logical extension to the perturbation analysis, which will be carried out in the next chapter.

CHAPTER V

T'IEN MODEL INCORPORATING NONLINEARITY

Problem Definition

The following analysis is based on the work in the paper by T'ien [1], and will reference Chapter II as the analysis will be extended by carrying the perturbation series out to second-order to encompass the effects of nonlinearity on the unsteady burning of a solid propellant. The pressure series expansion is only carried to order ε as we are seeking the nonlinear, ε^2 , effects on the burning process due to a well-defined and measurable fluctuating pressure disturbance. For the purpose of clarity, the model is again described here in full detail including intermediate steps.

Analysis

Gas Phase

The flame in the gas phase is assumed to be premixed, laminar, one-dimensional, and to proceed with a one-step forward chemical reaction. The specific heats, molecular weights, and coefficient of heat conduction are taken to be constant, and the Lewis number equal to unity [1]. Applying these assumptions, the dimensional continuity, energy, specie equations, and perfect gas law can be written as

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial \rho^* u^*}{\partial x^*} = 0 \quad (5.1)$$

$$\rho^* C_g^* \frac{\partial T^*}{\partial t^*} + \rho^* u^* C_g^* \frac{\partial T^*}{\partial x^*} - \lambda^* \frac{\partial^2 T^*}{\partial x^{*2}} - \frac{\partial p^*}{\partial t^*} = - \sum_{i=1}^N h_i^0 w_i^* \quad (5.2)$$

$$\rho^* \frac{\partial Y_i}{\partial t^*} + \rho^* u^* \frac{\partial Y_i}{\partial x^*} - \frac{\lambda^*}{C_p^*} \frac{\partial^2 Y_i}{\partial x^{*2}} = w_i^* \quad i = 1, 2, \dots, N \quad (5.3)$$

$$p^* = \rho^* \Re T^* \quad (5.4)$$

where dimensional quantities are denoted by the superscript ()^{*}. For equations (5.1)

- (5.4) the range of x^* is from 0, fixed at the propellant surface, to ∞ .

For a one-step forward chemical reaction, the conversion of fuel and oxidizer into products is defined by,



where the subscripts f , o and p , represent fuel oxidizer and products respectively.

The law of mass action for an arbitrary chemical reaction:

$$\sum_{i=1}^N v_i' W_i \rightarrow \sum_{i=1}^N v_i'' W_i \quad (5.6)$$

where, W_i is the molecular weight. The change in number of moles dn_i , of each species is given by:

$$w_i^* = W_i \frac{dn_i}{dt}, \quad \frac{dn_i}{(v_i'' - v_i')} = \frac{dn_j}{(v_j'' - v_j')} \quad (5.7)$$

The reaction rate w_i^* for the species i

$$w_i^* = W_i \frac{v_i'' - v_i'}{v_j'' - v_j'} \frac{dn_j}{dt} \quad (5.8)$$

for $j = f$:

$$w^* = W_f \frac{dn_f}{dt}, \quad v_f^* = 0 \quad (5.9)$$

Substitution of (5.9) into (5.8) with $j = f$, yields

$$w_i^* = -\frac{W_i}{W_f} \frac{v_i'' - v_i'}{v_j'} w^* \quad (5.10)$$

The reaction rate of the fuel can be expressed by

$$w^* = B^* T^{*\delta} \left(\frac{p^*}{\Re T^*} \right)^n \prod_{j=1}^N \left(\frac{Y_j}{W_j} \right)^{nj} e^{-\left(\frac{E^*}{\Re T^*} \right)} \quad (5.11)$$

To define the reaction rate for oxidizer and products

$$w_i^* = \frac{v_i'' - v_i'}{v_f'} w^*, \quad i = 0, p \quad (5.12)$$

The right hand side of equation (5.2) becomes

$$-\sum_{i=1}^N h_i^0 w_i^* = w^* q^* \quad (5.13)$$

where, the heat of combustion per unit mass of fuel consumed is

$$q^* = \sum \frac{W_i (v_i' - v_i'') h_i^0}{W_f v_f'} \quad (5.14)$$

It is useful at this juncture to introduce the following dimensionless variables:

$$\begin{aligned} \rho &= \frac{\rho^*}{\bar{\rho}_\infty^*} & T &= \frac{T^*}{\bar{T}_\infty^*} & x &= \frac{x^* \bar{\rho}_\infty^* C_g \bar{u}_\infty^*}{\bar{\lambda}_\infty^*} \\ u &= \frac{u^*}{\bar{u}_\infty^*} & p &= \frac{p^*}{\bar{p}_\infty^*} & t &= \frac{x^* \bar{\rho}_\infty^* C_g \bar{u}_\infty^{*2}}{\bar{\lambda}_\infty^*} \\ q &= \frac{q^*}{C_g \bar{T}_\infty^*} & & & w &= \frac{w^* \bar{\lambda}_\infty^*}{\bar{\rho}_\infty^* C_g \bar{u}_\infty^{*2}} \end{aligned} \quad (5.15)$$

Substituting (5.11), (5.13) and (5.14) into (5.1) - (5.4), and using the dimensionless variables yields,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \quad (5.16)$$

$$\rho \frac{\partial T}{\partial t} + \rho u \frac{\partial T}{\partial x} - \frac{\partial^2 T}{\partial x^2} - \frac{\gamma - 1}{\gamma} \frac{\partial p}{\partial t} = wq \quad (5.17)$$

$$\rho \frac{\partial Y_i}{\partial t} + \rho u \frac{\partial Y_i}{\partial x} - \frac{\partial^2 Y_i}{\partial x^2} = v_i w \quad i = f, o, p \quad (5.18)$$

$$p = \rho T \quad (5.19)$$

where v_i is defined by:

$$v_i = \frac{W_i (v'_i - v''_i)}{W_f v'_f} \quad (5.20)$$

The mass fraction equations (5.18); for the fuel, the oxidizer, and the products, can be made similar, with the following relation

$$\frac{Y_i}{v_i} - \frac{Y_f}{v_f} = \text{const} \quad (5.21)$$

With the knowledge that both the fuel and the oxidizer are completely expended at the flame edge, the following mathematical statement can be made:

$$Y_o = \frac{v_o}{v_f} Y_f = -v_o Y_f \quad (5.22)$$

This important step allows for the treatment of the fuel equation alone, allowing for two important steps to take place, (5.18) becomes

$$\rho \frac{\partial Y_f}{\partial t} + \rho u \frac{\partial Y_f}{\partial x} - \frac{\partial^2 Y_f}{\partial x^2} = -w \quad (5.23)$$

and the non-dimensional reaction rate (5.11), can be written as

$$w = BjT^\delta \left(\frac{p}{T} \right)^n Y_f^n e^{-E/T} \quad (5.24)$$

where

$$j = \frac{W_o v_o'}{W_f v_f'} = \text{oxidizer - fuel ratio} \quad (5.25)$$

$$B = \frac{B^* \lambda_g^* \bar{T}_\infty^{*\beta} \bar{p}_\infty^{*n}}{\bar{m}^{*2} C_g \prod_{j=1}^N (W_j^{nj})} \quad (5.26)$$

$$E = \frac{E^*}{\Re \bar{T}_\infty^*} \quad (5.27)$$

and n is the order of chemical reaction.

When assuming that the oscillations are of small amplitude, a perturbation analysis is appropriate. The pressure wavelength in the combustor is always much larger than the flame thickness, so that the pressure can be assumed to be a function of time only. It is important to notice that the pressure term $p(t)$ is only expanded to order ε . This is due to the need to find the ε^2 effects on the burning process to a well-defined and measurable fluctuating pressure disturbance. That is the linear pressure fluctuation is the input quantity with the nonlinear velocity, temperature and species mass fraction fluctuations as the desired output. In essence, the result sought is the effect of the measurable fluctuating pressure on the nonlinear reaction of the system. Expanding the pressure, temperature, velocity, species and burning rate in terms of the small parameter ε , yields,

$$p(t) = 1 + \varepsilon e^{i\omega t} \quad (5.28)$$

$$T(x, t) = T^{(0)} + \varepsilon T^{(1)} e^{i\omega t} + \varepsilon^2 \left(\bar{T}^{(2)} + T^{(2)} e^{2i\omega t} \right) + \mathcal{O}(\varepsilon^3) \quad (5.29)$$

$$u(x, t) = u^{(0)} + \varepsilon u^{(1)} e^{i\omega t} + \varepsilon^2 \left(\bar{u}^{(2)} + u^{(2)} e^{2i\omega t} \right) + \mathcal{O}(\varepsilon^3) \quad (5.30)$$

$$Y_f(x, t) = Y^{(0)} + \varepsilon Y^{(1)} e^{i\omega t} + \varepsilon^2 \left(\bar{Y}^{(2)} + Y^{(2)} e^{2i\omega t} \right) + \mathcal{O}(\varepsilon^3) \quad (5.31)$$

$$r(x, t) = r^{(0)} + \varepsilon r^{(1)} e^{i\omega t} + \varepsilon^2 \left(\bar{r}^{(2)} + r^{(2)} e^{2i\omega t} \right) + \mathcal{O}(\varepsilon^3) \quad (5.32)$$

The frequency ω here can be referenced to the classical dimensionless frequency as shown in the QSHOD analysis by:

$$\Omega = \delta_1 \beta \omega \quad (5.33)$$

To eliminate the density we now expand $\rho(x, t)$, and make use of the perfect gas law,

$$\rho(x, t) = \frac{p(t)}{T(x, t)} = \frac{1}{T^{(0)}} \left\{ \begin{aligned} &1 + \varepsilon \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) e^{i\omega t} - \varepsilon^2 \frac{\bar{T}^{(2)}}{T^{(0)}} \\ &+ \varepsilon^2 \left[\left(\frac{T^{(1)}}{T^{(0)}} \right)^2 - \frac{T^{(1)}}{T^{(0)}} - \frac{T^{(2)}}{T^{(0)}} \right] e^{2i\omega t} \end{aligned} \right\} \quad (5.34)$$

In addition, it is useful at this time to expand $m(x, t) = \rho(x, t)u(x, t)$

$$m(x, t) = \frac{1}{T^{(0)}} \left\{ \begin{aligned} &u^{(0)} + \varepsilon \left(u^{(1)} + u^{(0)} - \frac{u^{(0)}T^{(1)}}{T^{(0)}} \right) e^{i\omega t} + \varepsilon^2 \left(\bar{u}^{(2)} - \frac{u^{(0)}\bar{T}^{(2)}}{T^{(0)}} \right) \\ &+ \varepsilon^2 \left[\begin{aligned} &u^{(2)} - \frac{u^{(0)}T^{(2)}}{T^{(0)}} + \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) u^{(1)} \\ &+ \left(\frac{T^{(1)}}{T^{(0)}} - 1 \right) \frac{u^{(0)}T^{(1)}}{T^{(0)}} \end{aligned} \right] e^{2i\omega t} \end{aligned} \right\} \quad (5.35)$$

Expanding the reaction rate (5.24) is algebraically intensive and is more manageable when broken into its components,

$$p^n = 1 + \varepsilon n e^{i\omega t} + \varepsilon^2 \frac{n(n-1)}{2} e^{2i\omega t} \quad (5.36)$$

$$Y_f^n = \left(Y^{(0)}\right)^n \left\{ \begin{aligned} &1 + \varepsilon n \frac{Y^{(1)}}{Y^{(0)}} e^{i\omega t} + \varepsilon^2 n \frac{\bar{Y}^{(2)}}{Y^{(0)}} \\ &+ \varepsilon^2 \left[n \frac{Y^{(2)}}{Y^{(0)}} + \frac{n(n-1)}{2} \left(\frac{Y^{(1)}}{Y^{(0)}} \right)^2 \right] e^{2i\omega t} \end{aligned} \right\} \quad (5.37)$$

$$T^{\delta-n} = \left(T^{(0)}\right)^{\delta-n} \left\{ \begin{aligned} &1 + \varepsilon (\delta-n) \frac{T^{(1)}}{T^{(0)}} e^{i\omega t} + \varepsilon^2 (\delta-n) \frac{\bar{T}^{(2)}}{T^{(0)}} \\ &+ \varepsilon^2 (\delta-n) \left[\frac{(\delta-n-1)}{2} \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 + \frac{T^{(2)}}{T^{(0)}} \right] e^{2i\omega t} \end{aligned} \right\} \quad (5.38)$$

$$T^{-1} = \frac{1}{T^{(0)}} \left\{ 1 - \varepsilon \frac{T^{(1)}}{T^{(0)}} e^{i\omega t} - \varepsilon^2 \frac{\bar{T}^{(2)}}{T^{(0)}} + \varepsilon^2 \left[\left(\frac{T^{(1)}}{T^{(0)}} \right)^2 - \frac{T^{(2)}}{T^{(0)}} \right] e^{2i\omega t} \right\} \quad (5.39)$$

$$e^{-E/T} = e^{-\frac{E}{T^{(0)}}} \left\{ \begin{aligned} &1 + \varepsilon \frac{E}{T^{(0)}} \frac{T^{(1)}}{T^{(0)}} e^{i\omega t} + \varepsilon^2 \frac{E}{T^{(0)}} \frac{\bar{T}^{(2)}}{T^{(0)}} \\ &+ \varepsilon^2 \frac{E}{T^{(0)}} \left[\frac{T^{(2)}}{T^{(0)}} + \left(\frac{1}{2} \frac{E}{T^{(0)}} - 1 \right) \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 \right] e^{2i\omega t} \end{aligned} \right\} \quad (5.40)$$

Recombining and collecting by increasing order, the steady reaction rate,

$$w^{(0)} = Bj \left(T^{(0)}\right)^{\delta-n} \left(Y^{(0)}\right)^n e^{-\frac{E}{T^{(0)}}} \quad (5.41)$$

unsteady reaction rate

$$w^{(1)} = w^{(0)} \left[\left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} + n \left(1 + \frac{Y^{(1)}}{Y^{(0)}} \right) \right] e^{i\omega t} \quad (5.42)$$

steady component of the nonlinear reaction rate is

$$\bar{w}^{(2)} = w^{(0)} \left[\left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{\bar{T}^{(2)}}{T^{(0)}} + n \frac{\bar{Y}^{(2)}}{Y^{(0)}} \right] \quad (5.43)$$

and the oscillatory component of the nonlinear reaction rate

$$w^{(2)} = w^{(0)} \left\{ \begin{aligned} & \left[\frac{n(n-1)}{2} \left[\left(\frac{Y^{(1)}}{Y^{(0)}} \right)^2 + 1 \right] + (\delta - n) \left(\frac{\delta - n - 1}{2} \right) \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 \right] \\ & + \frac{E}{T^{(0)}} \left(\frac{1}{2} \frac{E}{T^{(0)}} - 1 + \delta - n \right) \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 + n \frac{Y^{(2)}}{Y^{(0)}} + n^2 \frac{Y^{(1)}}{Y^{(0)}} \\ & + \left(\delta - n + \frac{E}{T^{(0)}} \right) \left[n \left(\frac{Y^{(1)}}{Y^{(0)}} + 1 \right) \frac{T^{(1)}}{T^{(0)}} + \frac{T^{(2)}}{T^{(0)}} \right] \end{aligned} \right\} e^{2i\omega t} \quad (5.44)$$

Expanding the governing equations (5.16), (5.17) and (5.23), collecting terms of leading order (steady-state)

$$\frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial \rho^{(0)} u^{(0)}}{\partial x} = 0 \quad (5.45)$$

$$\rho^{(0)} \frac{\partial T^{(0)}}{\partial t} + \rho^{(0)} u^{(0)} \frac{\partial T^{(0)}}{\partial x} - \frac{\partial^2 T^{(0)}}{\partial x^2} = w^{(0)} q \quad (5.46)$$

$$\rho^{(0)} \frac{\partial Y^{(0)}}{\partial t} + \rho^{(0)} u^{(0)} \frac{\partial Y^{(0)}}{\partial x} - \frac{\partial^2 Y^{(0)}}{\partial x^2} = -w^{(0)} \quad (5.47)$$

Making use of (5.34) and (5.35) to eliminate the density terms

$$m^{(0)} = \rho^{(0)} u^{(0)} = \frac{u^{(0)}}{T^{(0)}} = 1 \quad (5.48)$$

$$\frac{\partial T^{(0)}}{\partial x} - \frac{\partial^2 T^{(0)}}{\partial x^2} = w^{(0)} q \quad (5.49)$$

$$\frac{\partial Y^{(0)}}{\partial x} - \frac{\partial^2 Y^{(0)}}{\partial x^2} = -w^{(0)} \quad (5.50)$$

It is important at this juncture to note that from (5.48), $u^{(0)} = T^{(0)}$, and that equations (5.49) and (5.50) can be made similar by

$$Y^{(0)} = \frac{1}{q} (1 - T^{(0)}) \quad (5.51)$$

Substituting (5.51) into (5.41) returns an equation for the steady reaction rate:

$$w^{(0)} = \frac{Bj}{q^n} (T^{(0)})^{\delta-n} (1 - T^{(0)})^n e^{-E/T^{(0)}} \quad (5.52)$$

In doing so the steady-state problem, is reduced to the solution of a single equation, equation (5.49), where $w^{(0)}$ is defined by (5.52).

The focus is now turned to the unsteady component of the problem. Eliminating density by using the proper components of (5.34), and also recalling that $u^{(0)} = T^{(0)}$, the unsteady perturbed governing equations can be written as:

$$\frac{\partial u^{(1)}}{\partial x} = \frac{\partial T^{(1)}}{\partial x} - \left(\frac{T^{(1)}}{T^{(0)}} - \frac{u^{(1)}}{T^{(0)}} \right) \frac{\partial T^{(0)}}{\partial x} - i\omega \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) \quad (5.53)$$

$$\frac{\partial^2 T^{(1)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial T^{(1)}}{\partial x} + \left(1 + \frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial T^{(0)}}{\partial x} + i\omega \left(\frac{T^{(1)}}{T^{(0)}} - \frac{\gamma-1}{\gamma} \right) \\ & - w^{(0)} q \left[n \left(1 - \frac{Y^{(1)}}{Y^{(0)}} \right) + \left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} \right] \end{aligned} \right\} \quad (5.54)$$

$$\frac{\partial^2 Y^{(1)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial Y^{(1)}}{\partial x} + \left(1 + \frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial Y^{(0)}}{\partial x} + \frac{i\omega}{T^{(0)}} Y^{(1)} \\ & + w^{(0)} \left[n + n \frac{Y^{(1)}}{Y^{(0)}} + \left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} \right] \end{aligned} \right\} \quad (5.55)$$

At this juncture it is convenient to expand the governing equations to the second order to encompass the steady nonlinear correction,

$$\frac{\partial \bar{u}^{(2)}}{\partial x} = \frac{\partial \bar{T}^{(2)}}{\partial x} + \frac{\bar{u}^{(2)} - \bar{T}^{(2)}}{T^{(0)}} \frac{\partial T^{(0)}}{\partial x} \quad (5.56)$$

$$\frac{\partial^2 \bar{T}^{(2)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial \bar{T}^{(2)}}{\partial x} + \left(\frac{\bar{u}^{(2)}}{T^{(0)}} - \frac{\bar{T}^{(2)}}{T^{(0)}} \right) \frac{\partial T^{(0)}}{\partial x} \\ & - w^{(0)} q \left[n \frac{\bar{Y}^{(2)}}{Y^{(0)}} + \left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{\bar{T}^{(2)}}{T^{(0)}} \right] \end{aligned} \right\} \quad (5.57)$$

$$\frac{\partial^2 \bar{Y}^{(2)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial \bar{Y}^{(2)}}{\partial x} + \left(\frac{\bar{u}^{(2)}}{T^{(0)}} - \frac{\bar{T}^{(2)}}{T^{(0)}} \right) \frac{\partial Y^{(0)}}{\partial x} \\ & + w^{(0)} \left[n \frac{\bar{Y}^{(2)}}{Y^{(0)}} + \left(\delta - n + \frac{E}{T^{(0)}} \right) \frac{\bar{T}^{(2)}}{T^{(0)}} \right] \end{aligned} \right\} \quad (5.58)$$

again recalling the result from (5.48), $u^{(0)} = T^{(0)}$. As in the steady state equations (5.57) and (5.58) can be made similar by:

$$\bar{Y}^{(2)} = \frac{1}{q} (1 - \bar{T}^{(2)}) \quad (5.59)$$

Finally, carrying out the perturbation expansion to the governing equations to include the unsteady nonlinear correction,

$$\frac{\partial u^{(2)}}{\partial x} = \left\{ \begin{aligned} & \frac{\partial^2 T^{(2)}}{\partial x^2} - \frac{T^{(2)} - u^{(2)}}{T^{(0)}} \frac{\partial T^{(0)}}{\partial x} - \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial u^{(1)}}{\partial x} \\ & - 2i\omega \left[\left(\frac{T^{(1)}}{T^{(0)}} \right)^2 - \frac{T^{(1)}}{T^{(0)}} - \frac{T^{(2)}}{T^{(0)}} \right] - \left(2 \frac{T^{(1)}}{T^{(0)}} - \frac{u^{(1)}}{T^{(0)}} - 1 \right) \frac{\partial T^{(1)}}{\partial x} \\ & - \left(2 \frac{T^{(1)}}{T^{(0)}} - 1 \right) \left(\frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial T^{(0)}}{\partial x} \end{aligned} \right\} \quad (5.60)$$

$$\frac{\partial^2 T^{(2)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial T^{(2)}}{\partial x} + 2i\omega \frac{T^{(2)}}{T^{(0)}} + \frac{u^{(2)}}{T^{(0)}} \frac{\partial Y^{(0)}}{\partial x} + i\omega \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{T^{(1)}}{T^{(0)}} - w^{(0)} q n \frac{Y^{(2)}}{Y^{(0)}} \\ & + \left[\frac{u^{(1)}}{T^{(0)}} - \frac{T^{(2)}}{T^{(0)}} + \left(\frac{T^{(1)}}{T^{(0)}} - \frac{u^{(1)}}{T^{(0)}} - 1 \right) \frac{T^{(1)}}{T^{(0)}} \right] \frac{\partial T^{(0)}}{\partial x} \\ & + \left(\frac{u^{(1)}}{T^{(0)}} + 1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial T^{(1)}}{\partial x} - w^{(0)} q \frac{n(n-1)}{2} \left[\left(\frac{Y^{(1)}}{Y^{(0)}} \right)^2 + 1 \right] \\ & - w^{(0)} q \left(\delta - n + \frac{E}{T^{(0)}} \right) \left[n \left(\frac{Y^{(1)}}{Y^{(0)}} + 1 \right) \frac{T^{(1)}}{T^{(0)}} + \frac{T^{(2)}}{T^{(0)}} \right] - w^{(0)} q n^2 \frac{Y^{(1)}}{Y^{(0)}} \\ & - w^{(0)} q \left[(\delta - n) \left(\frac{\delta - n - 1}{2} + \frac{E}{T^{(0)}} \right) + \frac{E}{T^{(0)}} \left(\frac{1}{2} \frac{E}{T^{(0)}} - 1 \right) \right] \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 \end{aligned} \right\} \quad (5.61)$$

$$\frac{\partial^2 Y^{(2)}}{\partial x^2} = \left\{ \begin{aligned} & \frac{\partial Y^{(2)}}{\partial x} + 2i\omega \frac{Y^{(2)}}{T^{(0)}} + i\omega \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{Y^{(1)}}{T^{(0)}} + w^{(0)} n \frac{Y^{(2)}}{Y^{(0)}} \\ & + \left[\frac{u^{(2)}}{T^{(0)}} - \frac{T^{(2)}}{T^{(0)}} + \frac{u^{(1)}}{T^{(0)}} + \left(\frac{T^{(1)}}{T^{(0)}} - \frac{u^{(1)}}{T^{(0)}} - 1 \right) \frac{T^{(1)}}{T^{(0)}} \right] \frac{\partial Y^{(0)}}{\partial x} \\ & + \left(\frac{u^{(1)}}{T^{(0)}} + 1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial Y^{(1)}}{\partial x} + w^{(0)} \frac{n(n-1)}{2} \left[\left(\frac{Y^{(1)}}{Y^{(0)}} \right)^2 + 1 \right] \\ & + w^{(0)} \left(\delta - n + \frac{E}{T^{(0)}} \right) \left[n \left(\frac{Y^{(1)}}{Y^{(0)}} + 1 \right) \frac{T^{(1)}}{T^{(0)}} + \frac{T^{(2)}}{T^{(0)}} \right] + w^{(0)} n^2 \frac{Y^{(1)}}{Y^{(0)}} \\ & + w^{(0)} \left[(\delta - n) \left(\frac{\delta - n - 1}{2} + \frac{E}{T^{(0)}} \right) + \frac{E}{T^{(0)}} \left(\frac{1}{2} \frac{E}{T^{(0)}} - 1 \right) \right] \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 \end{aligned} \right\} \quad (5.62)$$

Equations (5.53) - (5.55) and (5.60) - (5.62) constitute two separate fifth order systems, and (5.56) - (5.58) constitute a third order system that will each be solved by numerical integration, to be discussed in further detail.

Boundary Conditions

To solve the systems of equations, it is necessary to specify the appropriate number of boundary conditions. There are two boundary conditions at the flame edge, $x \rightarrow \infty$, the first is that the fuel must vanish,

$$Y^{(0)}(\infty) = Y^{(1)}(\infty) = \bar{Y}^{(2)}(\infty) = Y^{(2)}(\infty) = 0 \quad (5.63)$$

and the second is a condition for the temperature and is worthy of a detailed discussion.

The flame temperature, in general, oscillates, and a temperature wave is convected away from the flame. When the period of oscillation is close to the gas thermal time, the temperature wave length is of the same order as the flame length. Deriving a suitable boundary condition depends on the flow condition in the combustion zone. For this one-dimensional problem, the product gas is convected away in the direction perpendicular to the flame front, it is reasonable to assume that the flow just outside the flame edge is isentropic. Therefore, the entropy is conserved, $DS/Dt = 0$, for each fluid particle convected away from the flame. Writing this expression in terms of temperature and pressure yields [1]

$$T^*(X^*, t^*) - T^*\left(0, t^* - \frac{X^*}{u_\infty^*}\right) = \frac{\gamma-1}{\gamma} \left[p^*(t^*) - p^*\left(t^* - \frac{X^*}{u_\infty^*}\right) \right] \quad (5.64)$$

where X^* is the distance measured from the flame edge into the combustion zone ($X^* = 0$ coincides with $x^* = \infty$). Differentiation yields:

$$\frac{\partial T^*(X^*, t^*)}{\partial X^*} = -\frac{1}{u_\infty^*} \frac{\partial T^*(0, \eta)}{\partial \eta} + \frac{\gamma-1}{\gamma} \frac{1}{u_\infty^*} \frac{\partial p^*(\eta)}{\partial \eta} \quad (5.65)$$

where

$$\eta = t^* - \frac{X^*}{u_\infty^*} \quad (5.66)$$

As X^* approaches zero and η approaches t^* , equation (5.65) yields

$$\frac{\partial T^*(X^*=0, t^*)}{\partial X^*} = -\frac{1}{u_\infty^*} \frac{\partial T^*(X^*=0, t^*)}{\partial t^*} + \frac{\gamma-1}{\gamma} \frac{1}{u_\infty^*} \frac{\partial p^*(t^*)}{\partial t^*} \quad (5.67)$$

Remembering that $X^*=0$ is equivalent to $x^*=\infty$, equation (5.67) is made dimensionless with the use of the variables in (5.15),

$$\frac{\partial T(\infty, t)}{\partial x} = -\frac{\partial T(\infty, t)}{\partial t} + \frac{\gamma-1}{\gamma} \frac{\partial p(t)}{\partial t} \quad (5.68)$$

Perturbing (5.68) and collecting the appropriate orders, the steady-state solution yields two conditions that are equivalent for use in (5.49), namely

$$\frac{dT^{(0)}(\infty)}{dx} = 0 \quad (5.69)$$

or

$$T^{(0)}(\infty) = 1 \quad (5.70)$$

The oscillatory unsteady boundary condition from (5.68) is,

$$\frac{\partial T^{(1)}(\infty)}{\partial x} = -i\omega T^{(1)}(\infty) + \frac{\gamma-1}{\gamma} i\omega \quad (5.71)$$

the nonlinear steady boundary condition is

$$\frac{\partial \bar{T}^{(2)}(\infty)}{\partial x} = 0 \quad (5.72)$$

and finally, the nonlinear unsteady boundary condition is

$$\frac{\partial T^{(2)}(\infty)}{\partial x} = -2i\omega T^{(2)}(\infty) \quad (5.73)$$

Equations (5.71) and (5.73) are the oscillatory temperature boundary conditions at the flame edge, and are valid for high frequency oscillations.

Three boundary conditions remain, and all are located at the solid surface $x = 0$. The first condition comes from the fact that the mass flux fraction of the fuel and oxidizer are fixed by the composition of the propellants. The fuel mass flux fraction is expressed by

$$\varepsilon_f = Y_f - \frac{1}{m} \frac{\partial Y_f}{\partial x} \quad (5.74)$$

Evaluating (5.74) at $x = 0$, surrenders

$$\varepsilon_f = \varepsilon_{fs}^{(0)} + \varepsilon^2 \bar{\varepsilon}_{fs}^{(2)} \quad (5.75)$$

and

$$\varepsilon_f^{(1)} = \varepsilon_f^{(2)} = 0 \quad (5.76)$$

To seek out the boundary conditions, perturb (5.74), and utilize (5.75) and (5.76).

Which yields, for the steady part

$$\varepsilon_{fs}^{(0)} = Y^{(0)} - \frac{\partial Y^{(0)}}{\partial x} \quad (5.77)$$

the unsteady part

$$\varepsilon_f^{(1)} = 0 = Y^{(1)} - \frac{\partial Y^{(1)}}{\partial x} + \left(1 + \frac{u^{(1)}}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial Y^{(0)}}{\partial x} \quad (5.78)$$

the nonlinear steady component

$$\bar{\varepsilon}_{fs}^{(2)} = \bar{Y}^{(2)} - \frac{\partial \bar{Y}^{(2)}}{\partial x} + \left(\frac{\bar{u}^{(2)}}{T^{(0)}} - \frac{\bar{T}^{(2)}}{T^{(0)}} \right) \frac{\partial Y^{(0)}}{\partial x} \quad (5.79)$$

and the nonlinear unsteady part

$$\mathcal{E}_f^{(2)} = \left\{ \begin{aligned} & \bar{Y}^{(2)} - \frac{\partial Y^{(2)}}{\partial x} + \left(\frac{u^{(1)}}{T^{(0)}} + 1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{\partial Y^{(1)}}{\partial x} \\ & + \left[\frac{u^{(2)}}{T^{(0)}} - \frac{T^{(2)}}{T^{(0)}} + \left(1 - \frac{T^{(1)}}{T^{(0)}} \right) \frac{u^{(1)}}{T^{(0)}} \right. \\ & \left. + \left(\frac{T^{(1)}}{T^{(0)}} - 1 \right) \frac{T^{(1)}}{T^{(0)}} - \left(\frac{u^{(1)}}{T^{(0)}} + 1 - \frac{T^{(1)}}{T^{(0)}} \right)^2 \right] \frac{\partial Y^{(0)}}{\partial x} \end{aligned} \right\} = 0 \quad (5.80)$$

To proceed with finding the other two boundary conditions, it is necessary to consider the heat transfer process in the solid phase and the law governing the mass conversion of a solid to a gas at $x = 0$.

Solid Phase

The solid propellant is assumed to be homogeneous throughout and to not have an associated condensed-phase chemical reaction. The dimensionless heat transfer equation in the solid, with $x = 0$ located at the solid surface, is

$$\beta \frac{\partial T}{\partial t} + r \frac{\partial T}{\partial x} - \delta_1 \frac{\partial^2 T}{\partial x^2} = 0, \quad -\infty < x < 0 \quad (5.81)$$

where

$$\delta_1 = \frac{\lambda_s^* c_g}{\lambda^* c_p} \quad (5.82)$$

$$r = \frac{r^*}{\bar{r}^*} \quad (5.83)$$

and the ratio of solid density to steady-state gas flame density is given by

$$\beta = \frac{\rho_s^*}{\rho_\infty^*} \quad (5.84)$$

The boundary condition at the gas-surface interface $x = 0$ is

$$\left(\frac{\partial T}{\partial x}\right)_{x=0^-} = \delta_2 \left(\frac{\partial T}{\partial x}\right)_{x=0^+} - \delta_2 r L \quad (5.85)$$

where

$$\delta_2 = \frac{\lambda^*}{\lambda_s^*} \quad (5.86)$$

$$L = \frac{\ell}{c_g \bar{T}_\infty^*} \quad (5.87)$$

and at $x = -\infty$,

$$T(-\infty) = T_c \quad (5.88)$$

To account for the conversion of a solid to a gas, the solid propellant is assumed to decompose according to Arrhenius Law. In dimensionless form,

$$r = \exp\left(-\frac{E_s}{T}\right) \quad (5.89)$$

where

$$E_s = \frac{E_s^*}{\Re \bar{T}_\infty^*} \quad (5.90)$$

Perturbing (5.89) and collecting the result at increasing orders, for the steady case

$$r^{(0)} = e^{-\frac{E_s}{T_s^{(0)}}} \quad (5.91)$$

unsteady

$$r^{(1)} = \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} \quad (5.92)$$

nonlinear steady

$$\bar{r}^{(2)} = \frac{E_s}{T_s^{(0)}} \frac{\bar{T}^{(2)}}{T_s^{(0)}} \quad (5.93)$$

and nonlinear unsteady

$$r^{(2)} = \frac{E_s}{T_s^{(0)}} \left[\frac{T^{(2)}}{T_s^{(0)}} + \left(\frac{1}{2} \frac{E}{T_s^{(0)}} - 1 \right) \left(\frac{T^{(1)}}{T_s^{(0)}} \right)^2 \right] \quad (5.94)$$

Seeking the steady state temperature distribution, return to the heat transfer equation and perturb (5.81), which yields,

$$\delta_1 \frac{\partial^2 T^{(0)}}{\partial x^2} - \frac{\partial T^{(0)}}{\partial x} = 0 \quad (5.95)$$

Solving (5.95) and using the boundary condition $T^{(0)}(0) = T_s^{(0)}$,

$$T^{(0)} = T_c^{(0)} + (T_s^{(0)} - T_c^{(0)}) e^{\left(\frac{x}{\delta_1} \right)} \quad (5.96)$$

This result is used to calculate the temperature gradient in the gas phase, by substituting (5.96) into (5.85):

$$\left(\frac{dT^{(0)}}{dx} \right)_{x=0^+} = \frac{1}{\delta_1 \delta_2} (T_s^{(0)} - T_c^{(0)}) + L \quad (5.97)$$

Seeking the unsteady temperature distribution, collect the appropriate terms from the perturbed (5.81),

$$\delta_1 \frac{\partial^2 T^{(1)}}{\partial x^2} - \frac{\partial T^{(1)}}{\partial x} - \beta i \omega T^{(1)} - r^{(1)} \chi e^{\left(\frac{x}{\delta_1} \right)} = 0 \quad (5.98)$$

where

$$\chi = \frac{1}{\delta_1} (T_s^{(0)} - T_c^{(0)}) \quad (5.99)$$

First, finding the particular integral

$$T_{\text{particular}}^{(1)} = A e^{\left(\frac{x}{\delta_1}\right)} \quad (5.100)$$

and substituting into (5.98), the result yields a value for A

$$A = -\frac{r^{(1)} \chi}{i \omega \beta} = -\frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{i \omega \beta} \quad (5.101)$$

Next, a complementary solution must be found, by using the roots of the characteristic equation

$$m^2 - m - i \delta_1 \omega \beta = 0 \rightarrow m = \frac{1}{2} \left(1 + \sqrt{1 + 4i \delta_1 \omega \beta} \right) \quad (5.102)$$

Only the positive root must be taken to insure that the fluctuations will decay for any ω at $x \rightarrow -\infty$. Using this result a complete solution is defined

$$T^{(1)} = C_1 e^{\left(\frac{m x}{\delta_1}\right)} - \frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{i \omega \beta} e^{\left(\frac{x}{\delta_1}\right)} \quad (5.103)$$

Employing the condition $T^{(1)}(0) = T_s^{(1)}$

$$C_1 = \left(1 + \frac{1}{T_s^{(0)}} \frac{E_s}{T_s^{(0)}} \frac{\chi}{i \omega \beta} \right) T_s^{(1)} \quad (5.104)$$

Substituting this finding into (5.103)

$$T^{(1)} = T_s^{(1)} e^{\left(\frac{m x}{\delta_1}\right)} + \frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{i \omega \beta} \left[e^{\left(\frac{m x}{\delta_1}\right)} - e^{\left(\frac{x}{\delta_1}\right)} \right] \quad (5.105)$$

Taking the derivative of (5.105)

$$\frac{\partial T^{(1)}}{\partial x} = \frac{m}{\delta_1} T_s^{(1)} e^{\left(\frac{m x}{\delta_1}\right)} + i \frac{E_s}{T_s^{(0)}} \frac{T_s^{(1)}}{T_s^{(0)}} \frac{\chi}{\omega \beta} \left[\frac{m}{\delta_1} e^{\left(\frac{m x}{\delta_1}\right)} - \frac{1}{\delta_1} e^{\left(\frac{x}{\delta_1}\right)} \right] \quad (5.106)$$

Using the unsteady form of the boundary conditions from the perturbed (5.85) and (5.88)

$$\left(\frac{\partial T^{(1)}}{\partial x} \right)_{x=0^-} = \delta_2 \left(\frac{\partial T^{(1)}}{\partial x} \right)_{x=0^+} - \delta_2 r^{(1)} L \quad (5.107)$$

where $r^{(1)}$ is given by (5.92) and

$$T^{(1)}(-\infty) = 0 \quad (5.108)$$

Equation (5.107) can be solved to yield a relationship between the surface temperature and temperature gradient,

$$T^{(1)}(0) = T_s^{(1)} = \frac{i\omega\beta\delta_2 \left(\frac{\partial T^{(1)}}{\partial x} \right)_{x=0^+}}{\frac{m}{\delta_1} \left(i\omega\beta + \frac{E_s}{T_s^{(0)}} \frac{\chi}{T_s^{(0)}} \right) - \frac{E_s}{T_s^{(0)}} \frac{\chi}{\delta_1 T_s^{(0)}} + \frac{E_s}{T_s^{(0)}} \frac{i\omega\beta\delta_2}{T_s^{(0)}} L} \quad (5.109)$$

Collecting the steady nonlinear terms of (5.81) yields

$$\delta_1 \frac{\partial^2 \bar{T}^{(2)}}{\partial x^2} - \frac{\partial \bar{T}^{(2)}}{\partial x} - \bar{r}^{(2)} \chi e^{\left(\frac{x}{\delta_1} \right)} = 0 \quad (5.110)$$

finding the particular integral

$$\bar{T}_{\text{particular}}^{(2)} = A \chi e^{\left(\frac{x}{\delta_1} \right)} + B e^{\left(\frac{x}{\delta_1} \right)} \quad (5.111)$$

The result of substituting (5.111) into (5.110) returns an expression for A

$$A = \bar{r}^{(2)} \chi = \frac{E_s}{T_s^{(0)}} \frac{\bar{T}_s^{(2)}}{T_s^{(0)}} \chi \quad (5.112)$$

using the roots of the characteristic equation to find a complementary solution

$$m^2 - \frac{m}{\delta_1} = 0 \rightarrow m = 0, \quad m = \frac{1}{\delta_1} \quad (5.113)$$

a complete solution is defined by

$$\bar{T}^{(2)} = C_1 e^{\left(\frac{x}{\delta_1}\right)} + C_2 + \chi \frac{E_s}{T_s^{(0)}} \frac{\bar{T}_s^{(2)}}{T_s^{(0)}} x e^{\left(\frac{x}{\delta_1}\right)} + B e^{\left(\frac{x}{\delta_1}\right)} \quad (5.114)$$

Making use of the condition $\bar{T}^{(2)}(-\infty) = 0$, one finds

$$C_2 = 0 \quad (5.115)$$

Employing the condition $\bar{T}^{(2)}(0) = \bar{T}_s^{(2)}$

$$C_1 = \bar{T}_s^{(2)} - B \quad (5.116)$$

Substituting this finding into (5.114)

$$\bar{T}^{(2)} = \bar{T}_s^{(2)} \left[1 + \frac{E_s}{T_s^{(0)}} \frac{\chi}{T_s^{(0)}} x \right] e^{\left(\frac{x}{\delta_1}\right)} \quad (5.117)$$

Taking the derivative of (5.117)

$$\frac{\partial \bar{T}^{(2)}}{\partial x} = \bar{T}_s^{(2)} \left[\frac{1}{\delta_1} + \left(1 + \frac{x}{\delta_1} \right) \frac{E_s}{T_s^{(0)}} \frac{\chi}{T_s^{(0)}} \right] e^{\left(\frac{x}{\delta_1}\right)} \quad (5.118)$$

and using the unsteady form of the boundary conditions from (5.85) and (5.88)

$$\left(\frac{\partial \bar{T}^{(2)}}{\partial x} \right)_{x=0^-} = \delta_2 \left(\frac{\partial \bar{T}^{(2)}}{\partial x} \right)_{x=0^+} - \delta_2 \bar{r}^{(2)} L \quad (5.119)$$

where $\bar{r}^{(2)}$ is given by (5.93). Equation (5.119) can be solved to yield a relationship between the surface temperature and temperature gradient,

$$\bar{T}_s^{(2)} = \frac{\delta_2 \left(\frac{\partial \bar{T}^{(2)}}{\partial x} \right)_{x=0^+}}{\frac{1}{\delta_1} + \frac{E_s}{T_s^{(0)}} \frac{\chi + \delta_2 L}{T_s^{(0)}}} \quad (5.120)$$

The nonlinear unsteady component of (5.81) is

$$\delta_1 \frac{\partial^2 T^{(2)}}{\partial x^2} - \frac{\partial T^{(2)}}{\partial x} - 2i\omega\beta T^{(2)} - r^{(1)} \frac{\partial T^{(1)}}{\partial x} - r^{(2)} \chi e^{\left(\frac{x}{\delta_1}\right)} = 0 \quad (5.121)$$

finding the particular integral

$$T_{\text{particular}}^{(2)} = A \exp\left(\frac{x}{\delta_1}\right) + B \quad (5.122)$$

Substituting this finding into (5.121), yields

$$A = \frac{i\chi}{2\omega\beta} r^{(2)} = \frac{i\chi}{2\omega\beta} \frac{E_s}{T_s^{(0)}} \left[\frac{T^{(2)}}{T_s^{(0)}} + \left(\frac{1}{2} \frac{E}{T_s^{(0)}} - 1 \right) \left(\frac{T^{(1)}}{T_s^{(0)}} \right)^2 \right] \quad (5.123)$$

and

$$B = \frac{i\chi}{2\omega\beta} \frac{\partial T^{(1)}}{\partial x} r^{(1)} = \frac{i\chi}{2\omega\beta} \frac{\partial T^{(1)}}{\partial x} \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} \quad (5.124)$$

using the roots of the characteristic equation to find a complementary solution

$$m^2 - \frac{m}{\delta_1} - \frac{2i\omega\beta}{\delta_1} = 0 \rightarrow m = \frac{1 + \sqrt{1 + 8i\omega\beta\delta_1}}{2\delta_1} \quad (5.125)$$

Only the positive root must be taken to insure that the fluctuations will decay for any ω at $x \rightarrow -\infty$. Using this result a complete solution is defined

$$T^{(2)} = \left\{ C_1 e^{\left(\frac{m x}{\delta_1}\right)} + \frac{i\chi}{2\omega\beta} \frac{\partial T^{(1)}}{\partial x} \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} + \frac{i\chi}{2\omega\beta} \frac{E_s}{T_s^{(0)}} \left[\left(\frac{1}{2} \frac{E}{T_s^{(0)}} - 1 \right) \left(\frac{T^{(1)}}{T_s^{(0)}} \right)^2 + \frac{T^{(2)}}{T_s^{(0)}} \right] e^{\left(\frac{x}{\delta_1}\right)} \right\} \quad (5.126)$$

Remembering that $\partial T^{(1)}/\partial x$ is defined by (5.106), and employing the condition

$$T^{(2)}(0) = T_s^{(2)}$$

$$C_1 = \left\{ \begin{aligned} &T_s^{(2)} - \frac{i\chi}{2\omega\beta} \frac{\partial T^{(1)}}{\partial x} \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} \\ &-\frac{i\chi}{2\omega\beta} \frac{E_s}{T_s^{(0)}} \left[\left(\frac{1}{2} \frac{E}{T_s^{(0)}} - 1 \right) \left(\frac{T^{(1)}}{T_s^{(0)}} \right)^2 + \frac{T^{(2)}}{T_s^{(0)}} \right] \end{aligned} \right\} \quad (5.127)$$

Substituting this finding into (5.126)

$$T^{(2)} = \left\{ \begin{aligned} &T_s^{(2)} e^{\left(\frac{m}{\delta_1} x\right)} + \left[1 - e^{\left(\frac{m}{\delta_1} x\right)} \right] \frac{i\chi}{2\omega\beta} \frac{\partial T^{(1)}}{\partial x} \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} \\ &+ \frac{i\chi}{2\omega\beta} \frac{E_s}{T_s^{(0)}} \left[\frac{T^{(2)}}{T_s^{(0)}} + \left(\frac{1}{2} \frac{E_s}{T_s^{(0)}} - 1 \right) \left(\frac{T^{(1)}}{T_s^{(0)}} \right)^2 \right] \left[e^{\left(\frac{x}{\delta_1}\right)} - e^{\left(\frac{m}{\delta_1} x\right)} \right] \end{aligned} \right\} \quad (5.128)$$

Taking the derivative of (5.128) and using the unsteady form of the boundary conditions from (5.85) and (5.88)

$$\left(\frac{\partial T^{(2)}}{\partial x} \right)_{x=0^-} = \delta_2 \left(\frac{\partial T^{(2)}}{\partial x} \right)_{x=0^+} - \delta_2 r^{(2)} L \quad (5.129)$$

where $r^{(2)}$ is given by (5.94) and

$$T^{(2)}(-\infty) = 0 \quad (5.130)$$

Equation (5.119) can be solved to yield a relationship between the surface temperature and temperature gradient,

$$T_s^{(2)} = \frac{T_s^{(0)} \left[C_2 m + \delta_1 \delta_2 \left(\frac{\partial T^{(2)}}{\partial x} \right)_{x=0^+} + C_3 C_4 (m - 1 - \delta_1 \delta_2 L) \right]}{C_3 - C_3 m + m T_s^{(0)} + C_3 \delta_1 \delta_2 L} \quad (5.131)$$

where

$$C_2 = \frac{i\chi}{2\omega\beta} \frac{\partial T^{(1)}}{\partial x} \frac{E_s}{T_s^{(0)}} \frac{T^{(1)}}{T_s^{(0)}} \quad (5.132)$$

$$C_3 = \frac{i\chi}{2\omega\beta} \frac{E_s}{T_s^{(0)}} \quad (5.133)$$

$$C_4 = \left(\frac{1}{2} \frac{E_s}{T_s^{(0)}} - 1 \right) \left(\frac{T_s^{(1)}}{T_s^{(0)}} \right)^2 \quad (5.134)$$

Equations (5.109), (5.120) and (5.131) provide the relations at each order being sought. The remaining boundary conditions are derived by using the Arrhenius Law (5.89) and satisfying continuity across the propellant surface, more specifically;

$$\rho_s r^* = \rho u^* \quad (5.135)$$

Making (5.135) dimensionless through the use of the variables defined in (5.15) and perturbing the result, yields

$$u_s^{(1)} = \left(\frac{E_s}{T_s^{(0)}} + 1 \right) T_s^{(1)} - T_s^{(0)} \quad (5.136)$$

$$\bar{u}_s^{(2)} = \left(\frac{E_s}{T_s^{(0)}} + 1 \right) \bar{T}_s^{(2)} \quad (5.137)$$

$$u_s^{(2)} = \left[\begin{aligned} & \left(\frac{E_s}{T_s^{(0)}} + 1 \right) T_s^{(2)} + \left(1 - \frac{T_s^{(1)}}{T_s^{(0)}} \right) \left(T_s^{(1)} - u_s^{(1)} \right) \\ & + E_s \left(\frac{1}{2} \frac{E_s}{T_s^{(0)}} - 1 \right) \left(\frac{T_s^{(1)}}{T_s^{(0)}} \right)^2 \end{aligned} \right] \quad (5.138)$$

Equations (5.136) - (5.138) are the velocity boundary conditions; unsteady, steady nonlinear and unsteady nonlinear respectively. With the systems of equations and their associated boundary conditions fully defined, a numerical integration can take place.

Numerical Method

The iterative method discussed in the linear analysis from the previous chapter is again employed to find the steady or leading order results. The remaining unsteady, nonlinear steady and nonlinear unsteady equations constitute a fifth order system, a third order and another fifth order system, respectively. Again as in Chapter II, they are solved by using Runge-Kutta and incorporating a double shooting technique to satisfy the coupled boundary conditions. The typical dimensionless distributions of the velocity, temperature, and species for the final three systems are shown in Figures 5.1 – 5.4. Please note that Figures 5.1 – 5.4 are for a single frequency $\omega = 1$.

The calculation of the response and admittance functions is a previously stated in Chapter II, is a time consuming process, where the numerical integration of the unsteady system must be performed for each frequency over a large range of frequencies. The response of the propellant is obtained by plotting the real and imaginary parts of (5.92) - (5.94) at each value of ω . Finally, the admittance is obtained by plotting the real part of the fluctuating velocity components at the flame edge, $x = \infty$, for each value of ω .

Results and Discussion

Reiterating the values for the base case from Chapter II: second order chemical reaction $n = 2$, oxidizer-fuel ratio of $j = 1$, temperature deep within the cold propellant of $T_c = 0.15$, surface temperature of $T_s = 0.35$, $\gamma = 1.17$, $\delta_1 = \delta_2 = 1$, $\delta = 0$, $q = 0.65$, $L = 0.15$, $E = 10$, $E_s = 4$, and $\beta = 1000$. A parametric study on

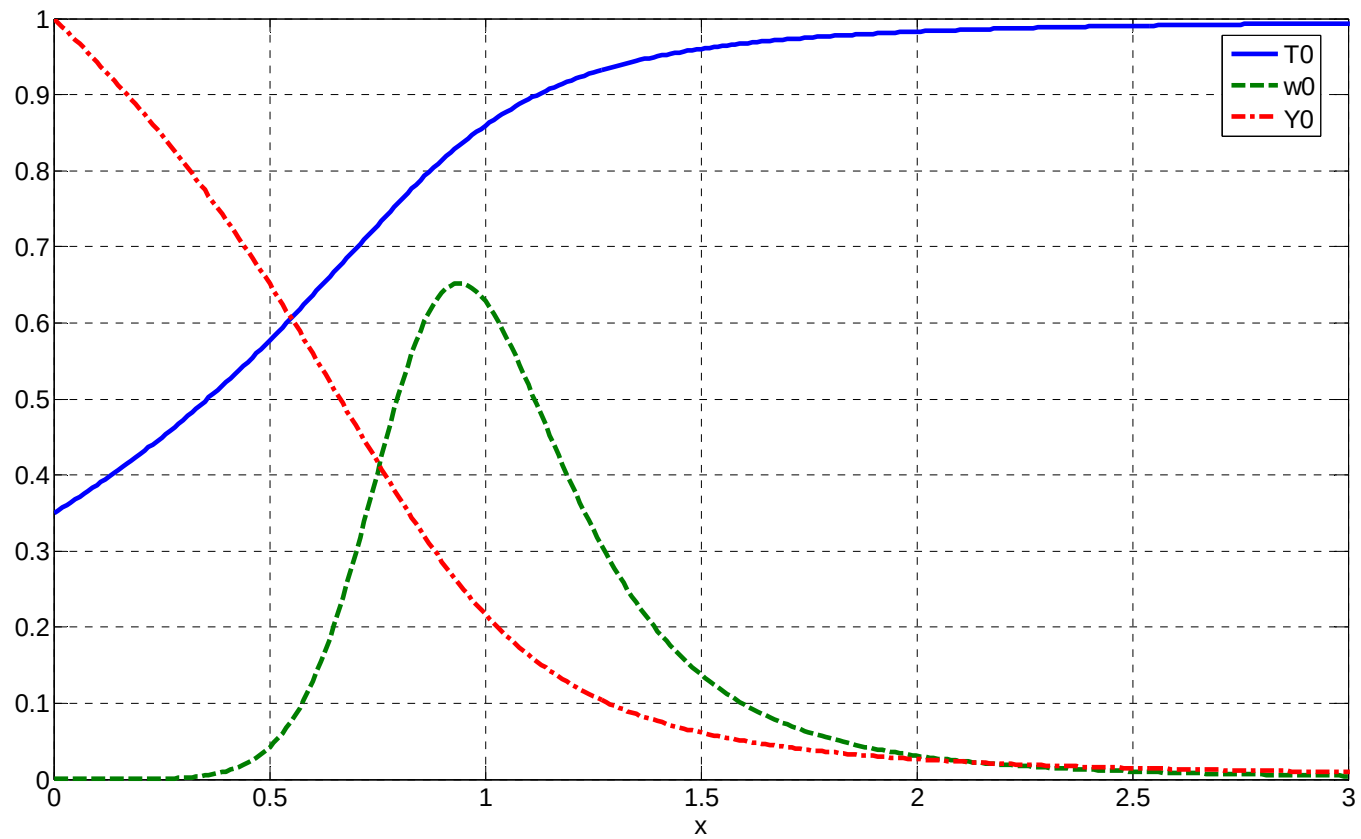


Figure 5.1 Steady-State Distributions of Dimensionless Temperature, Reaction Rate and Species

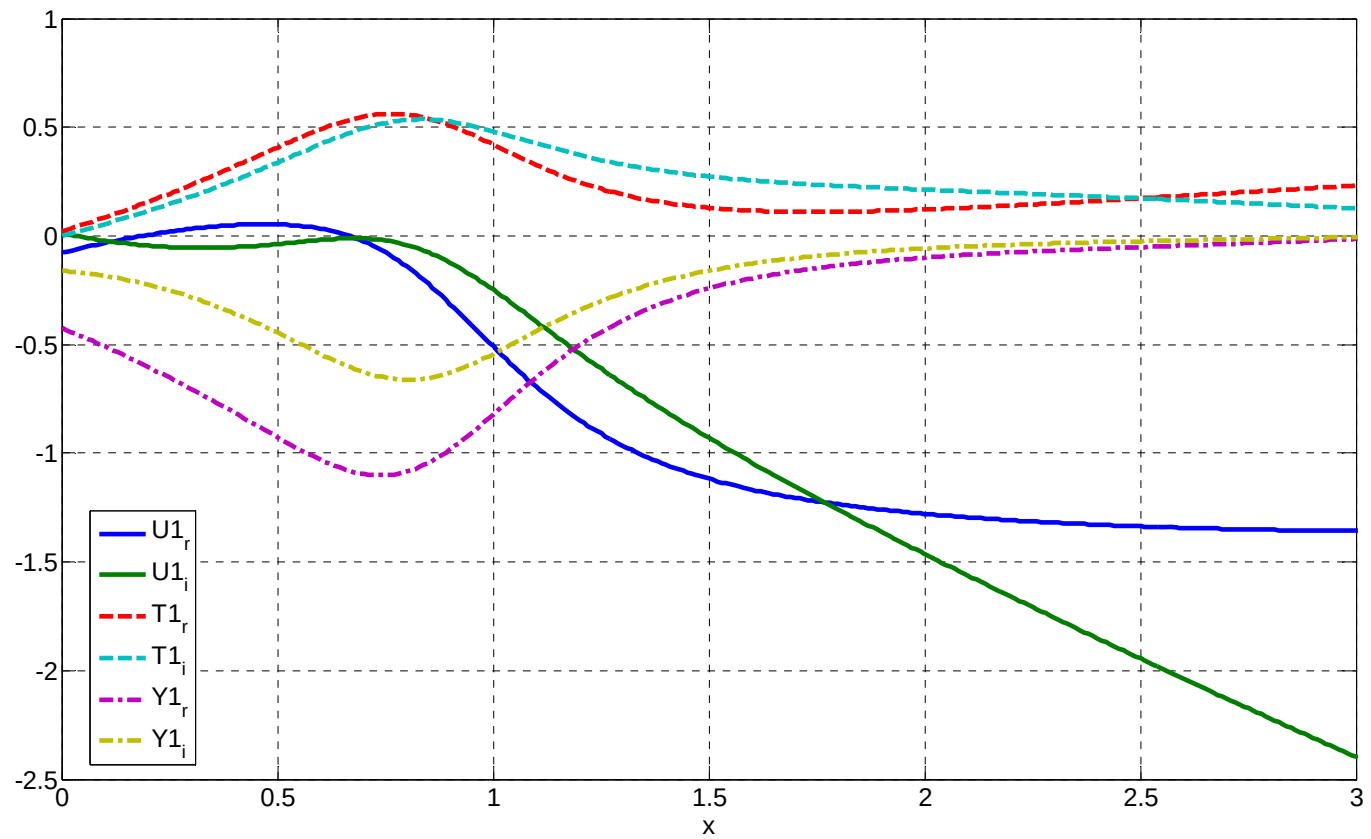


Figure 5.2 Unsteady Distributions of Dimensionless Velocity, Temperature and Species

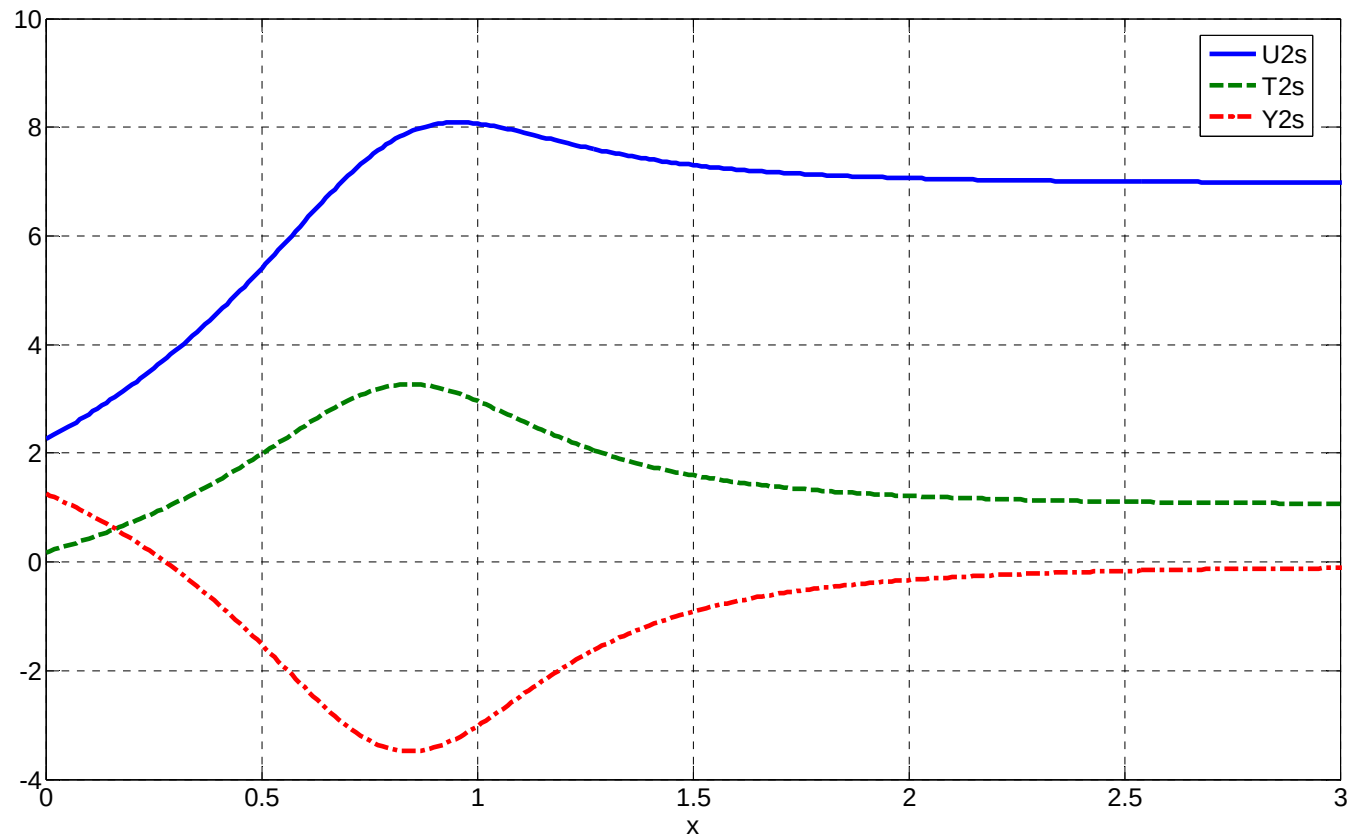


Figure 5.3 Nonlinear Steady Distributions of Dimensionless Velocity, Temperature and Species

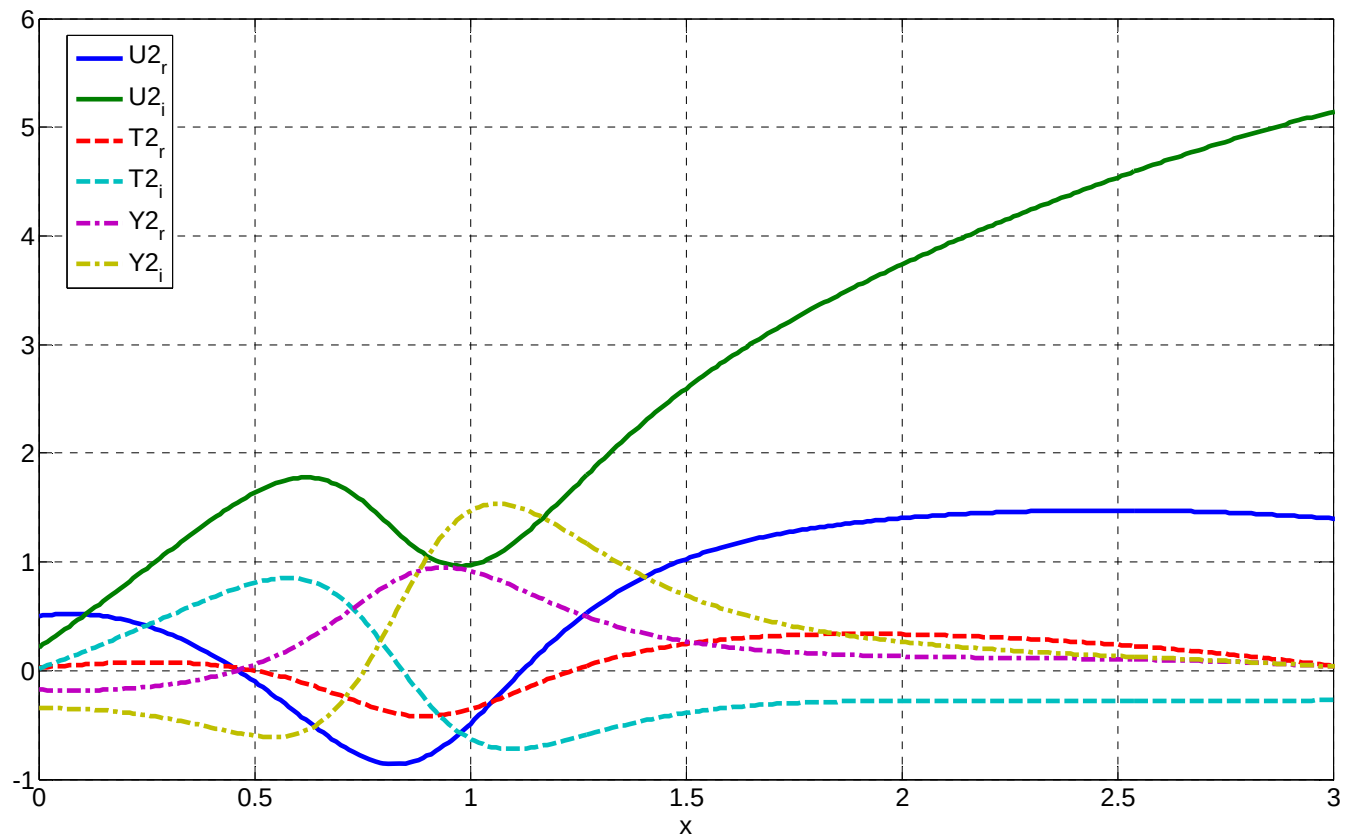


Figure 5.4 Nonlinear Unsteady Distributions of Dimensionless Velocity, Temperature and Species

the variables is performed changing one parameter at a time from the base case. Figures 5.1 – 5.4 are created based on values for the base case at a frequency of $\omega = 1$, or $\Omega = 1000$.

The effects of nonlinearity seen in the QSHOD model (i.e. increased primary admittance peak) are again reproduced in this nonlinear model with the inclusion of the gas phase resonance, Figure 5.5 depicts the base case varying ε from zero to 0.3. There is a small nonlinear contribution to the amplification at the low frequency peak and a secondary resonant peak is noticed around a dimensionless frequency of 300. This secondary peak is due solely to the nonlinear correction and important because it may correspond to the first tangential mode instability experienced during test firings in the Brownlee data [13] (see Figure 5.6), the reason for which had gone unnoticed until now. The high frequency amplification, associated with the higher mode oscillations, is also shown to be dramatically enhanced by the nonlinear effects. Inclusion of the nonlinear high-frequency gas phase interactions yields an observed increase in the amplitude of the high frequency peak, this finding will help clarify nonlinear experimental data (as observed in Brownlee's data [13] and Shuttle SRB measurements). The effect of nonlinearity on the burning rate can be seen in Figure 5.7, the increase in real part of the burning rate is due solely to the nonlinear steady correction. This rise in the burning rate is a contributing factor to the mean pressure rise (DC shift) frequently experienced in combustion instability research.

Increasing the parameter q , the heat of combustion, from 0.65 to 0.85 yields an increase to the secondary amplifying peak, seen in Figure 5.8, there are decreases in the low and high frequency peaks accompanied by an increase in the damping peak

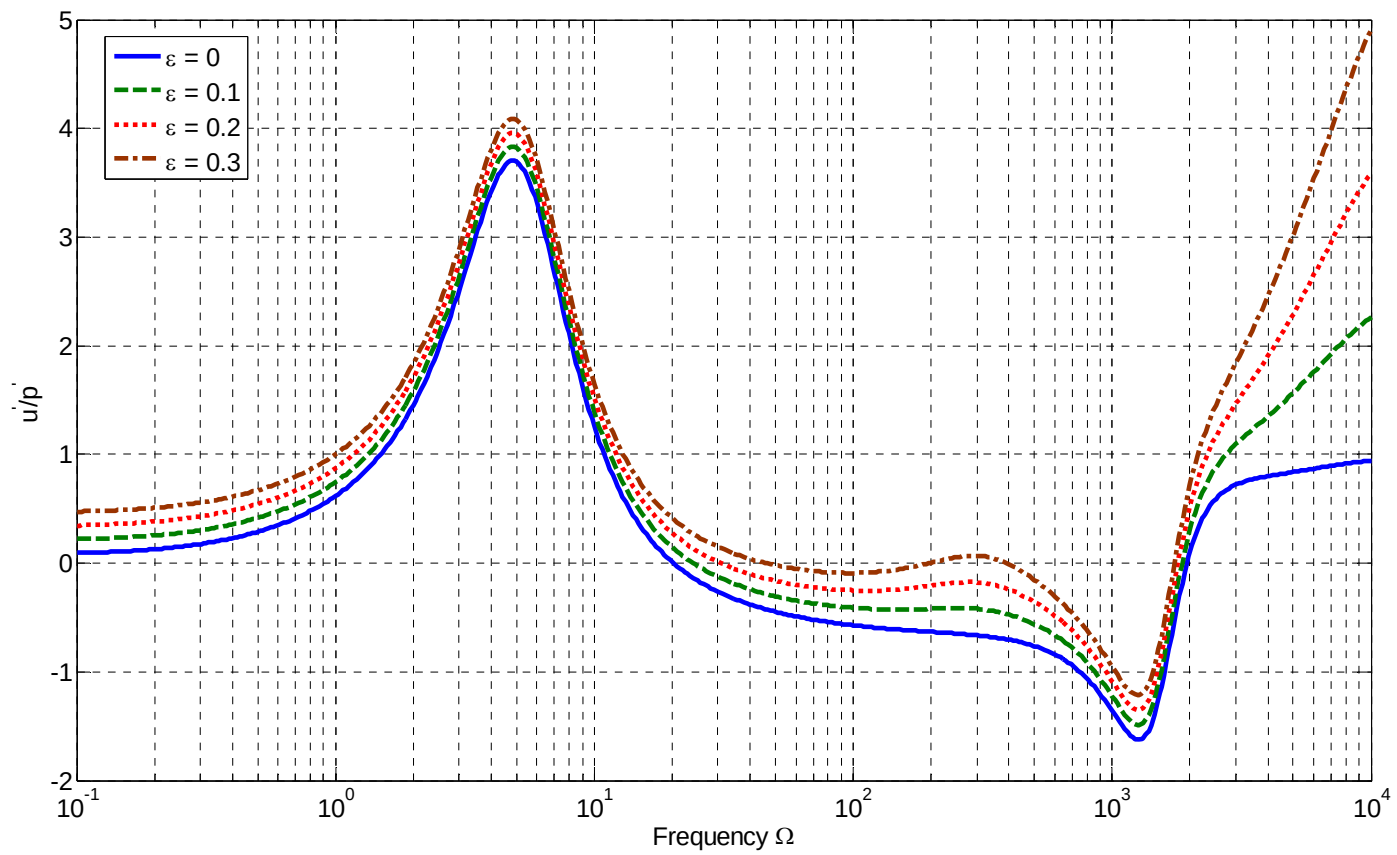


Figure 5.5 Effect of Nonlinearity on the Acoustic Admittance (Base Case)

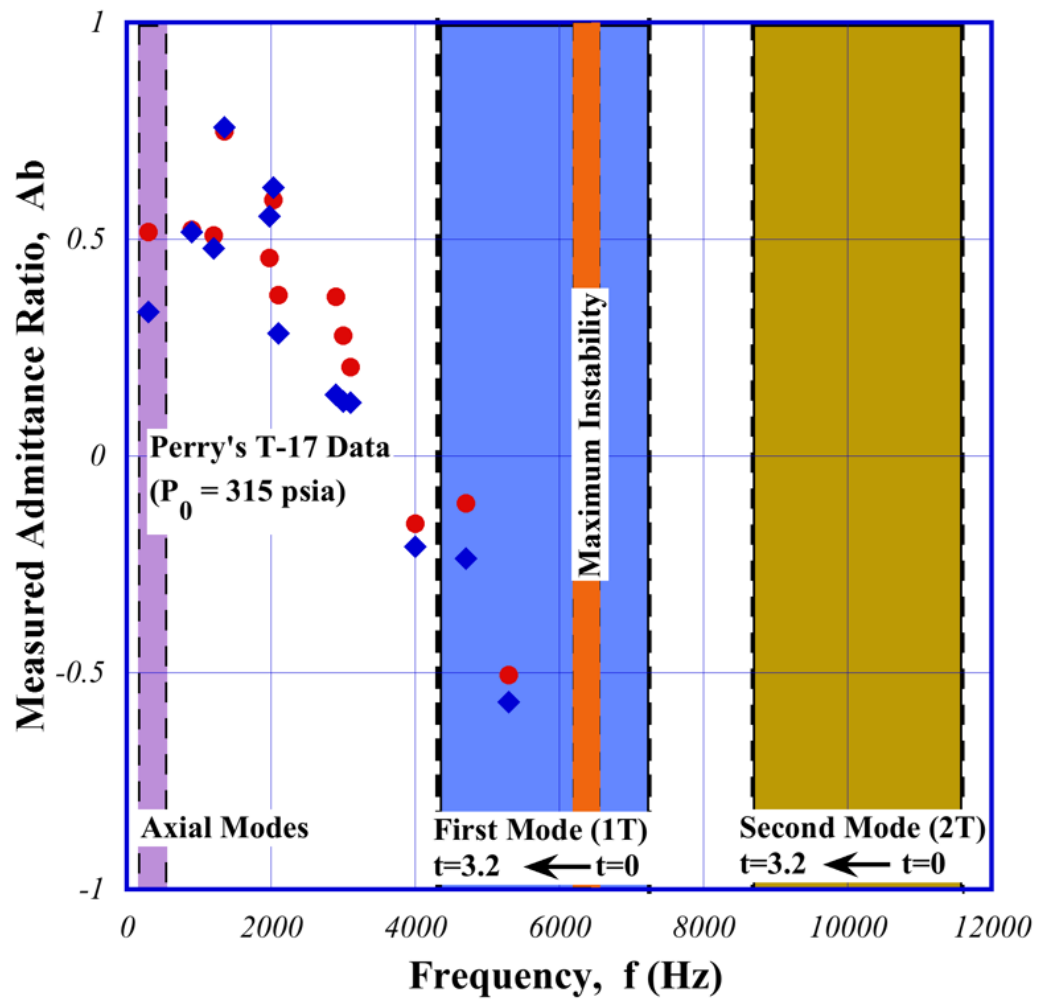


Figure 5.6 Perry's T-17 Data Shown with Typical Instability Regions [14]

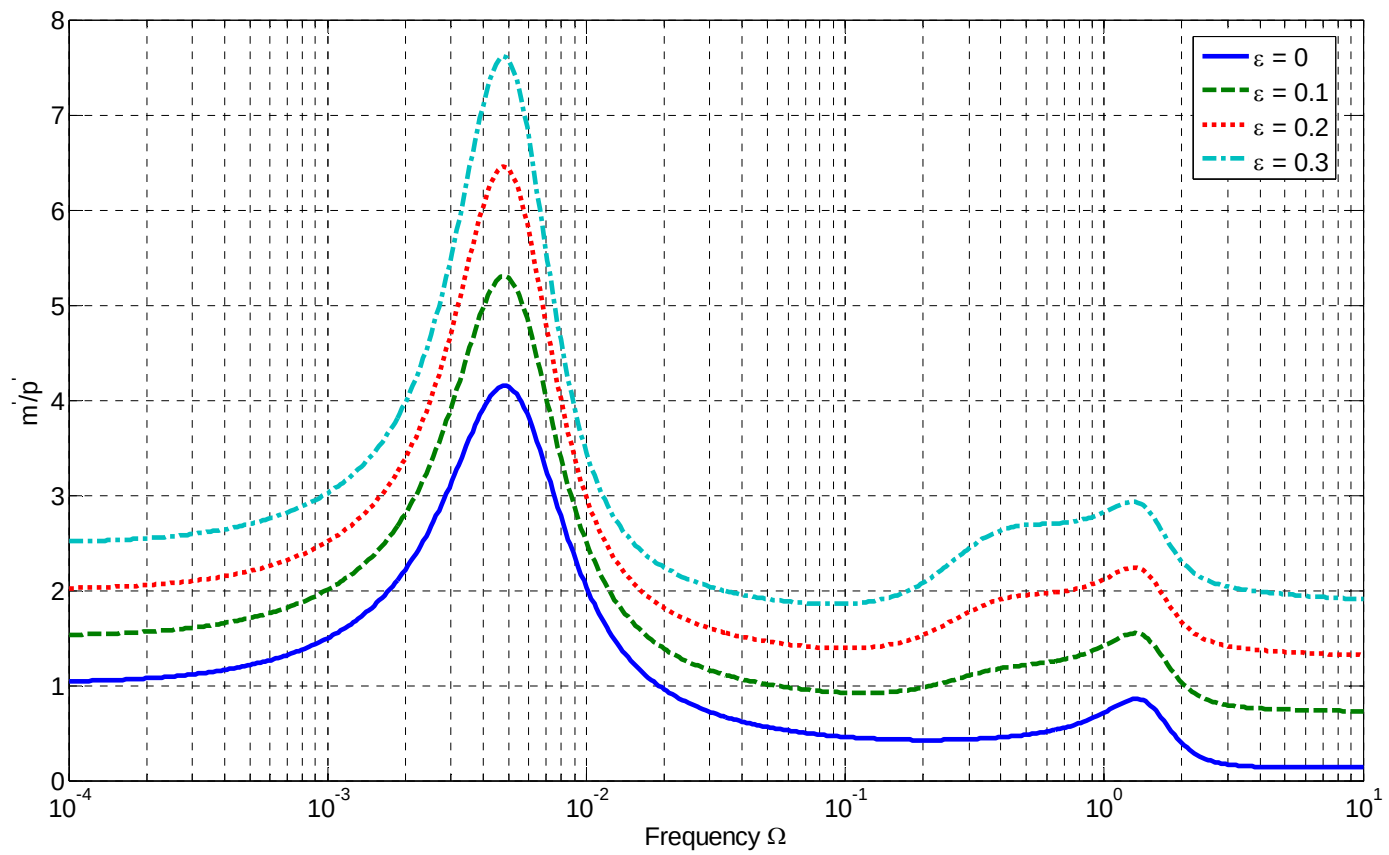


Figure 5.7 Effect of Nonlinearity on the Real Part of the Burning Rate (Base Case)

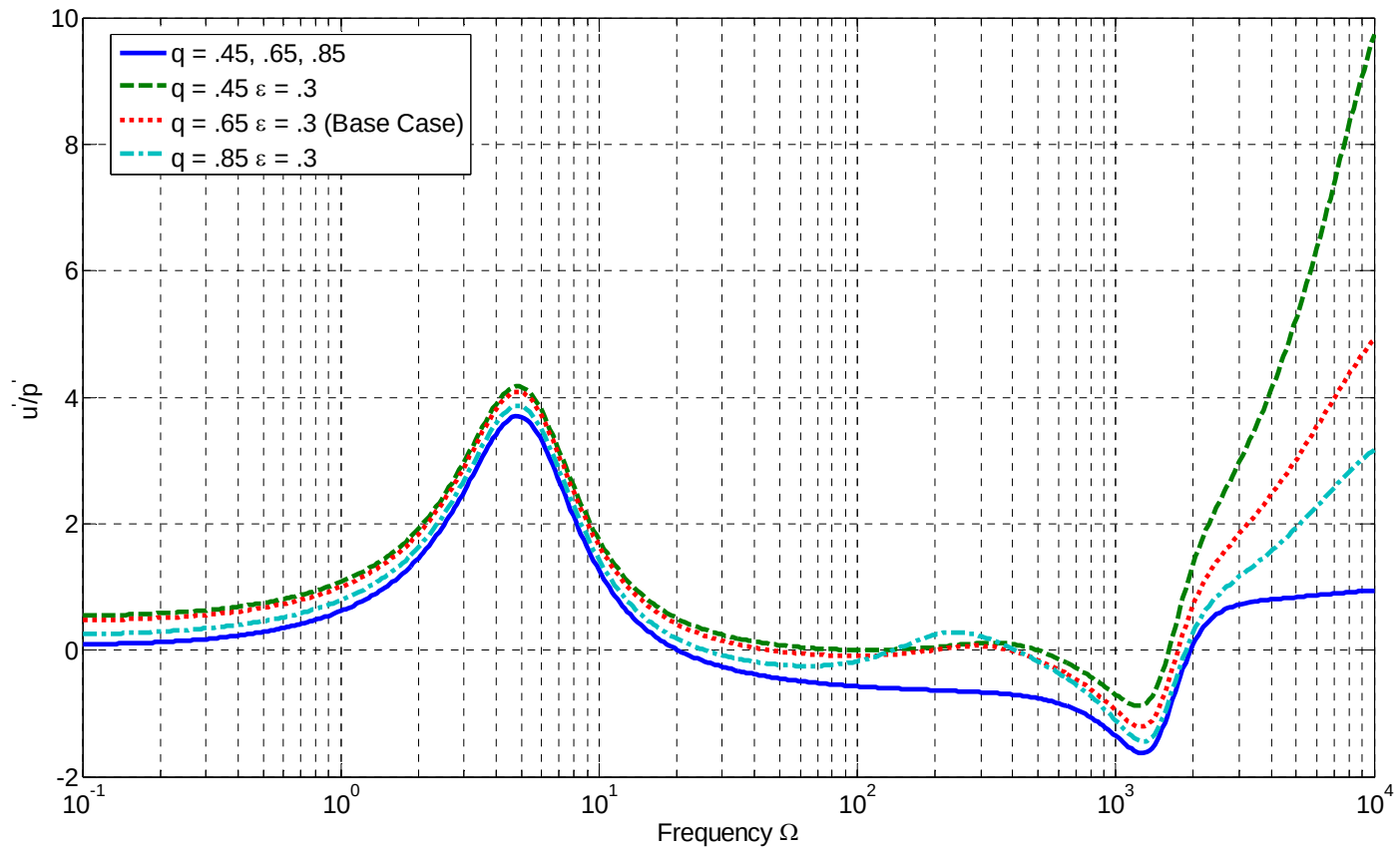


Figure 5.8 Effects of Nonlinearity and Varying the Dimensionless Heat of Combustion on the Admittance

and a significant increase in the secondary peak associated with the first tangential mode maximum instability region.

The exciting finding in Chapter IV of the observed this temperature sensitivity is again seen here. The incorporation of nonlinearity (seen in Figure 5.9) yields an overall increase in the magnitude of the admittance across all frequencies, with the most dramatic results again appearing in the high frequency range.

A similar trend to that in Figure 5.9 begins to form and can be seen in Figures 5.10 and 5.12. The effects of nonlinearity on the solid phase surface activation energy, the gas phase activation energy, and the latent heat all invariably yield the same overall increase in the magnitude of the admittance across all frequencies as seen in the cold propellant temperature figure.

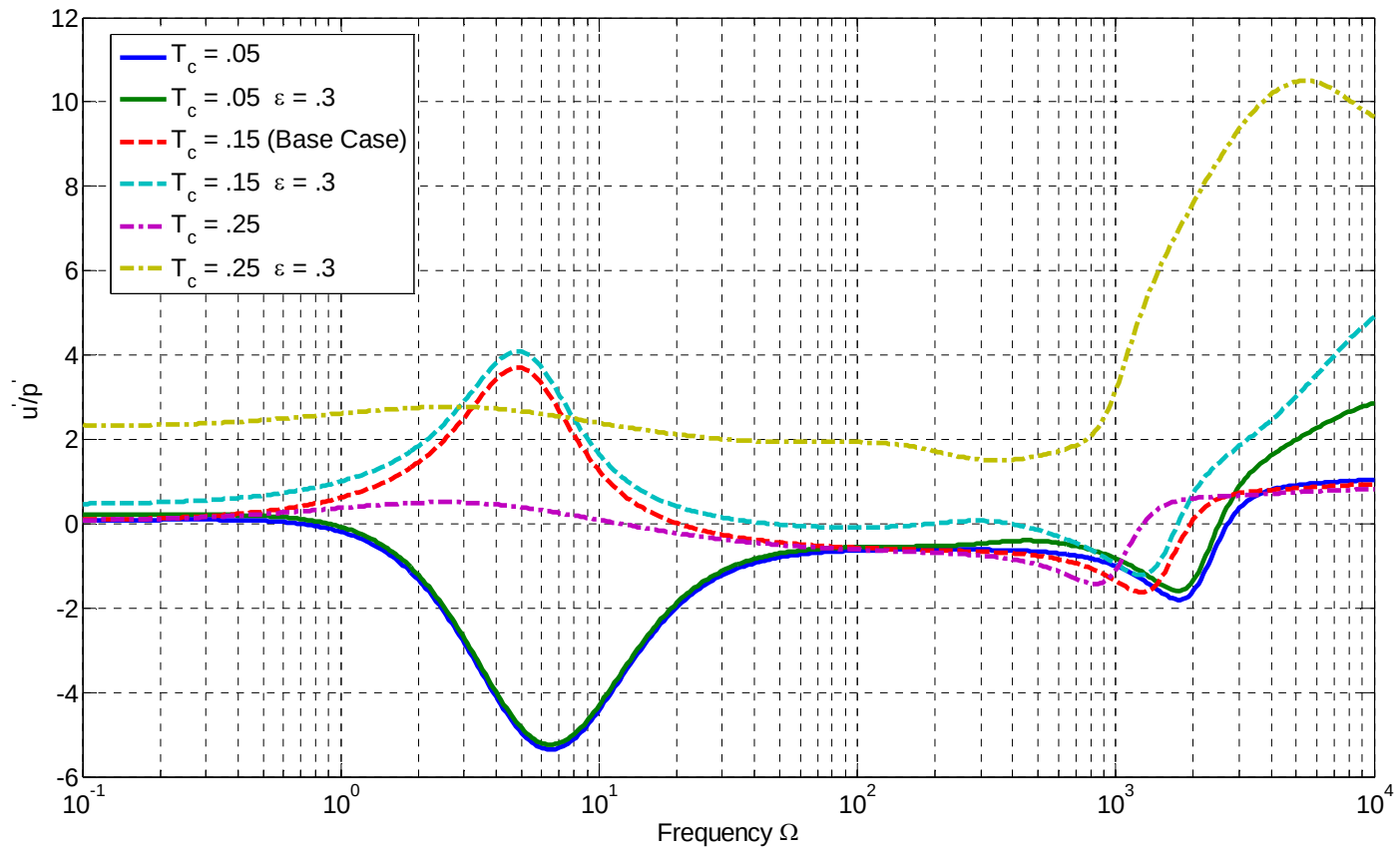


Figure 5.9 Effects of Nonlinearity and Varying the Dimensionless Cold Propellant Temperature on the Admittance

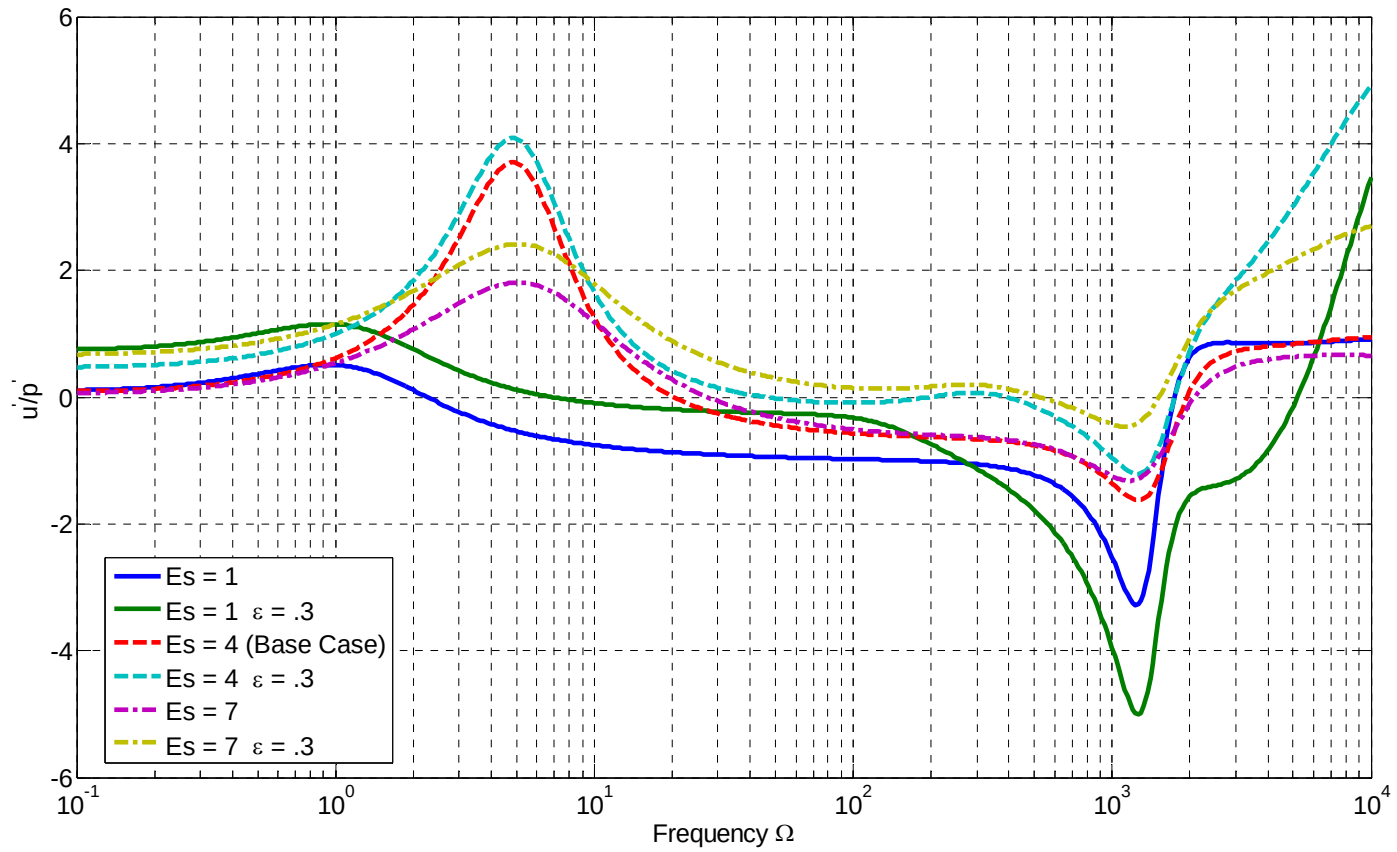


Figure 5.10 Effects of Nonlinearity and Varying the Dimensionless Solid Phase Activation Energy on the Admittance

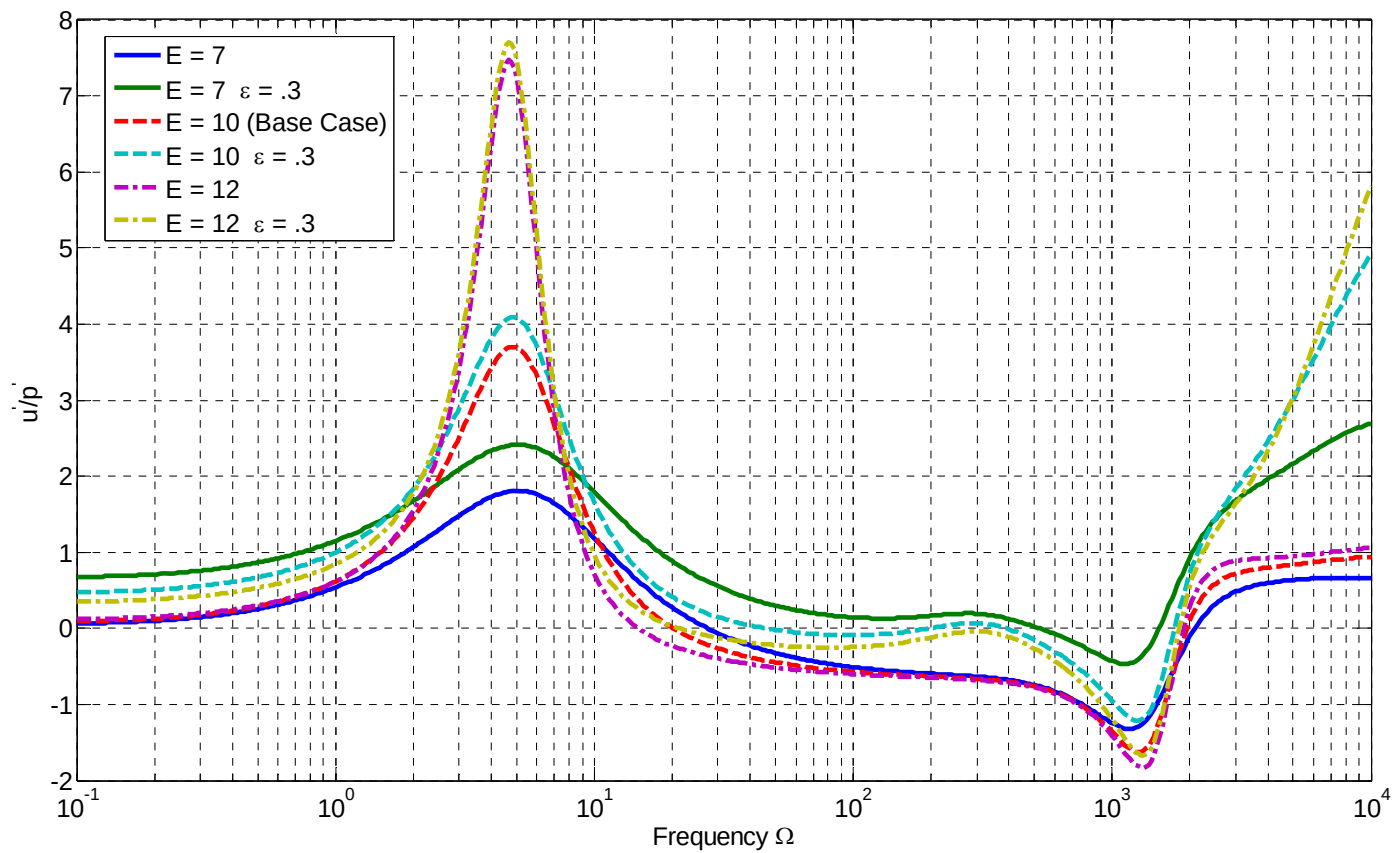


Figure 5.11 Effects of Nonlinearity and Varying the Dimensionless Gas Phase Activation Energy on the Admittance

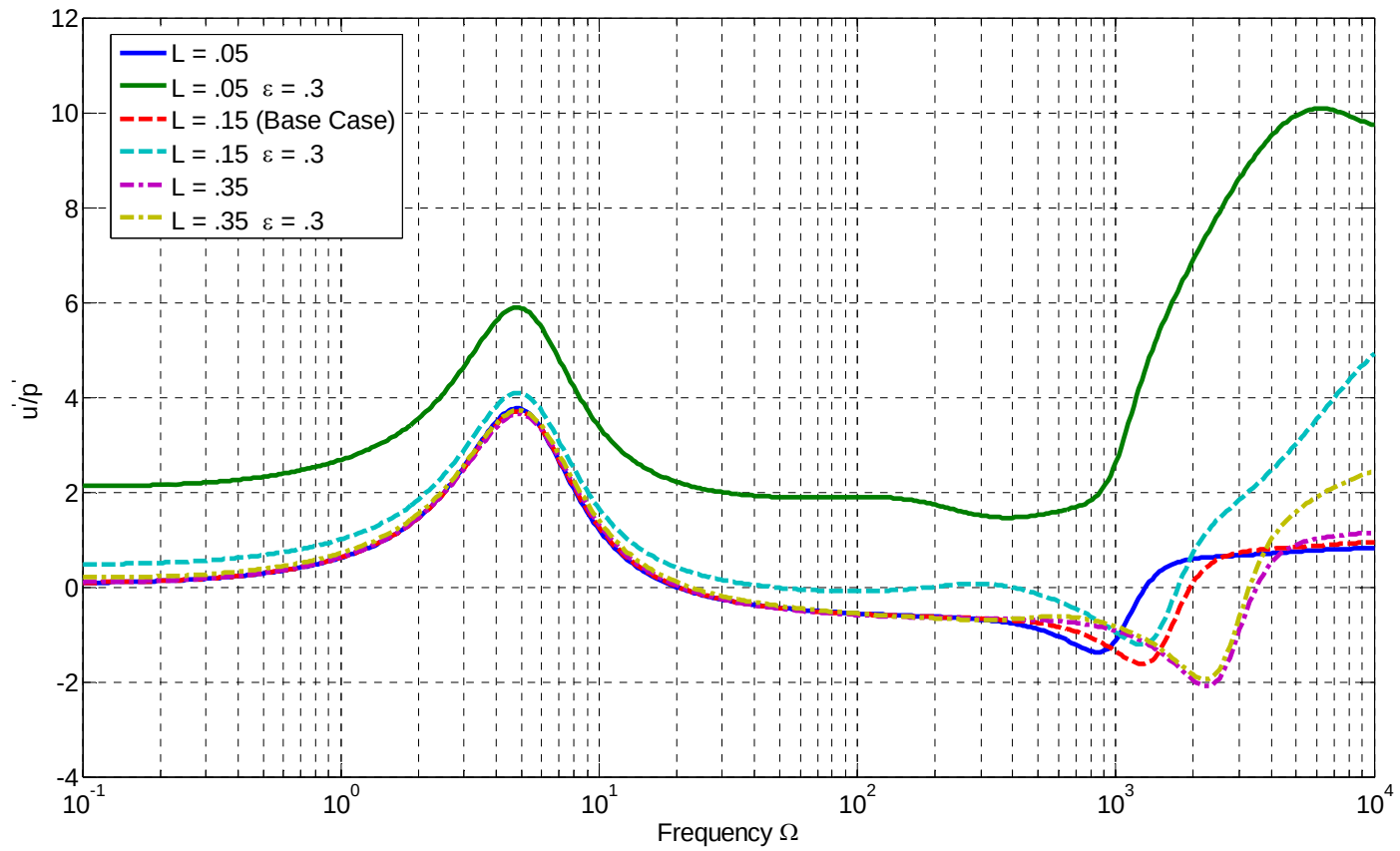


Figure 5.12 Effects of Nonlinearity and Varying the Dimensionless Latent Heat on the Admittance

CHAPTER VI

CONCLUSION

QSHOD Linear and Extension to Nonlinear

The quasi-steady theory is limited by assumptions. The most evident is the exclusion of the unsteady gas phase eliminating the high frequency response. When examining the work required arriving at a proper QSHOD curve fit for experimental data, it becomes apparent that the similarity variables A and B fail to explain the physical meaning of the variables they contain. Another shortcoming of the QSHOD model is the lack of dependence of the mean burning rate on oscillatory pressure, an effect that is observed in experimental firings. The lack in physical meaning makes the model a method to curve fit low frequency data, and has not been successfully implemented into a combustion instability analysis. Although the classical linear QSHOD model is limited, the model does give a framework to extend the response function to incorporate nonlinearity.

The new second-order nonlinear corrections to the QSHOD model show a definitive increase in the amplitude of the primary low frequency response peak is observed and is entirely due to the incorporation of nonlinear effects. This result yields a major improvement in understanding of growth of waves from noise to finite-amplitude pressure oscillations and will play a major role in new Combustion Instability Framework under development by Flandro and his research group at UTSL.

The limitations applied to the classical QSHOD linear theory, naturally appear in the nonlinear quasi-steady theory, as the nonlinear is an extension of the linear

model. As it stands the nonlinear QSHOD model does contain limitations, the model yields valuable information about the solid phase contribution to the nonlinear response function, making it a valuable tool. The limitations discussed here drive the need for a more complete model, one that includes unsteady gas phase behavior and unsteady conduction heat transfer in the solid.

T'ien Linear and Extension to Nonlinear

The limitations in the QSHOD model drive the need for a more complete model, one that includes unsteady gas phase behavior and unsteady conduction heat transfer in the solid. The inclusion of the unsteady gas phase effects is found to be of importance even when only looking to the low frequency response of the propellant. Simply assuming that the admittance can be found by subtracting the inverse of γ as suggested in the QSHOD model is not valid or recommended. The importance of the high frequency is undeniable and can not be left out of a complete analysis.

An exciting finding, first discussed by Brownlee during his experiments using a T-burner in 1959 [13], is the temperature sensitivity of a propellant's stability boundary. From the linear study it is evident that by lowering the propellant temperature a inverse primary admittance peak inferring damping of pressure oscillations. Alternatively by raising the cold propellant temperature to a value approaching that of the burning propellant surface, the surface the primary admittance peak is flattened reducing the ability of the amplification of pressure oscillations. This finding demonstrates promise in attempting to control combustion instability by just temperature conditioning the propellant before a firing.

The increased primary admittance peak due to effects of nonlinearity seen in the QSHOD model is again reproduced in the new nonlinear extension of the T'ien model with the inclusion of the gas phase resonance corrections. Inclusion of the high-frequency gas phase interactions explains effects important in nonlinear calculations involving many high-frequency harmonics (as observed in Brownlee's data [13] and Shuttle SRB measurements).

The mean pressure rise (DC shift), commonly experienced in combustion instability can partially be attributed to the nonlinear steady correction of the burning rate. A secondary peak due solely to the nonlinear correction is noted; this give insight into the high-amplitude first tangential mode instability region observed in the Brownlee- Marble data [13]. An observed increase in the amplitude of the high frequency peak, due to nonlinear contributions, will prove to be essential in interpreting nonlinear experimental data.

CHAPTER VII

RECOMMENDATIONS FOR FURTHER STUDY

The QSHOD model should only be used as a stepping stone for understanding the complex problem of unsteady burning of a solid propellant, as the final result has no predictive capability. Limitations of the T'ien model, such as a one-step forward chemical reaction taking place in the gas phase, can be side stepped by using test propellants whose behavior closely resembles the model. To further understand the problem data from inherently nonlinear experiments needs to be gathered, during this process values for the parameters set forth in the T'ien model need to be defined and verified. Allowing for comparison and verification of the model, at this time no such data exists. So comparisons to data alone would revert to only being a curve fitting exercise, yielding no benefits.

Testing at the high frequencies will be difficult and expensive but, once the model can be verified, predictive capabilities would follow, eliminating the need for numerous tests. The admittance data from the T'ien model can, and should, be incorporated into nonlinear theoretical framework for combustion instability analysis, where no proper model for the admittance exists at this time.

What can be accomplished with the findings presented? Presently the model should be used to understand the data that has been overlooked in the past, due to the inability to understand the complexities observed. Propose new T-burner experiments based on propellants for which properties of the propellant can be accurately established over a wide range of frequencies and mean pressures. The

findings described here must be correlated with actual motor instability data from current programs exhibiting combustion instability (i.e. the five-segment ARES SRB). The model could also be used to assist in the design stable motors, or at least using it as a criterion for selection of propellants and operating conditions. On a larger scale, it should be used in the development of new propellants that are less susceptible to instability.

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