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I am submitting herewith a thesis written by Michael J. Scheidler entitled "A Finite Element Method for the Two-Dimensional Helmholtz Equation in Unbounded Regions." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Kenneth R. Kimble

Kenneth R. Kimble, Major Professor

We have read this thesis and recommend its acceptance:

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A FINITE ELEMENT METHOD FOR THE TWO-DIMENSIONAL HELMHOLTZ EQUATION IN UNBOUNDED REGIONS

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Michael J. Scheidler

August 1979

DEDICATION

The author dedicates this work to his fiancee and to his mother for their sacrifices and understanding.

ACKNOWLEDGMENTS

The author extends his sincerest thanks to Dr. Kenneth R. Kimble for his guidance during the past two years. Special thanks are also extended to Dr. K. C. Reddy and Dr. Trevor Moulden for their constructive criticisms and suggestions.

The work reported herein was supported under a grant from NASA Langley Research Center. The grant, NASA NSG 1224, was for A Study of Unsteady Transonic Flow Analysis and was monitored by Dr. Sam Bland.

The author also wishes to thank Mrs. June Jarrell for her excellent typing of this thesis, Mr. Sorris Sims for his fine printing and Mr. Joseph Chu for managing the printing in the author's absence.

ABSTRACT

A finite element method for finding approximate solutions of boundary value problems for the Helmholtz equation in two-dimensional unbounded regions is studied. The method is based on a technique developed by R. W. Thatcher for solving the Laplace equation in unbounded regions. The method uses a grid consisting of an infinite number of triangular elements constructed in a systematic way. A system of linear equations must be solved for the solution at nodes near the boundary. Values at the remaining nodes in the infinite region are calculated from a simple recurrence relation. The method appears very promising but has not yet been tested. Possible extensions are pointed out and discussions of the outward radiation condition and of other methods for solving the Helmholtz equation are included.

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to study a finite element method for finding approximate solutions of boundary value problems for the Helmholtz equation

$$\nabla^2 u + \omega^2 u = 0$$

in two-dimensional unbounded regions. In particular, the type of region being considered is one which is exterior to a finite closed curve. Such problems usually arise in relation to the time-dependent wave equation. Examples are the radiation of acoustic waves from an oscillating body and the scattering of an incident acoustic wave by a body.

Problems in unbounded regions present certain theoretical and computational difficulties. To obtain a unique solution, the behavior of the solution must be specified not only on the boundary but also at infinity. Since most physical applications involve solving for a radiated or scattered wave traveling outward from the body, the condition imposed on the solution at infinity is one of outward radiation. Because an understanding of the radiation condition and of the asymptotic behavior of the solutions is essential in developing numerical methods, these topics are discussed in the next chapter.

The numerical methods for problems in bounded regions usually involve solving a system of linear equations for the approximate values of the solution at a discrete set of points distributed throughout the region. These methods cannot be directly applied to problems in unbounded regions, and a variety of techniques have been developed to overcome the difficulties associated with unbounded regions. Several of these are discussed in Appendix C.

The particular method being studied here is based on a finite element method developed by R. W. Thatcher [1] for solving the Laplace equation in unbounded regions. His method involves covering the exterior region with a grid consisting of an infinite number of triangular elements and solving for the values at the nodes of these elements. What results is an infinite system of linear equations for these unknowns. Once the behavior of the solution at infinity is specified, the infinite system of equations can be reduced to a finite system of equations involving only unknowns at nodes near the boundary. Values at the remaining nodes in the infinite region can then be found from a simple recurrence relation.

The major purpose of this study has been to attempt to extend Thatcher's method to the Helmholtz equation. Although in principle his method is not restricted to the Laplace equation, several difficulties were encountered in applying the method to the Helmholtz equation. Most of these difficulties have been overcome and the method appears very promising, although it has not yet been applied to a test problem.

In most of the following discussions, a Dirichlet boundary condition is assumed. This is for simplicity only; it should be emphasized that Thatcher's method is not restricted to a particular type of boundary condition.

CHAPTER II

THE HELMHOLTZ EQUATION

Problem Statement

Let C be a simple closed curve in two-dimensional Euclidean space and let C^- and C^+ denote the regions interior and exterior to C respectively. The closure of C^+, i.e., $C \cup C^+$, is denoted by $\overline{C}^+$. Let (x,y) and (r,θ) denote the rectangular and polar coordinates of a point in the plane with respect to some coordinate system of arbitrary origin. Let $i = \sqrt{-1}$ and let ∇^2 denote the Laplacian operator. The following Dirichlet problem for the Helmholtz equation will be considered. Given a continuous complex valued function u_0 on C and a real number $\omega > 0$, find the complex valued function u which satisfies:

(a) u is continuous in $\overline{C}^+$;

(b) $u(\overline{x}_0) = u_0(\overline{x}_0)$, $\overline{x}_0 \in C$; (2-1)

(c) u is a solution of the Helmholtz equation

$$(\nabla^2 + \omega^2)u(\overline{x}) = 0 \ , \ \overline{x} \in C^+ \ ; \tag{2-2}$$

(d) u satisfies Sommerfeld's outward radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial}{\partial r} - i\omega\right) u(r,\theta) = 0 \tag{2-3}$$

uniformly with respect to θ .

In some of the literature additional constraints are placed on the problem such as:

(e) The second derivatives of u are continuous in C^+ ;

(f) $\lim_{r \to \infty} \sqrt{r}\, u(r,\theta)$ is bounded uniformly with respect to θ .

The condition (e) is discussed by Levine [2] for the three-dimensional problem. He defines a class of "regular closed surfaces" and proves that if the boundary belongs to this class then conditions (a) through (d) (appropriately modified for three dimensions) are sufficient to guarantee a unique solution and that this solution satisfies (e). Levine's definition of a regular closed surface is somewhat involved and will not be given here; however, it includes a fairly general class of piecewise smooth surfaces. It will be assumed for this problem that the boundary curve C is of sufficient smoothness that the problem, as specified by (a) through (d), is well posed. Note that if C is composed of several smooth curves, then at the edge points joining these curves the derivatives of u will generally be unbounded. Levine also points out that condition (f) is superfluous, but it was not clear whether this follows from (c) alone or from both (c) and (d).

It should be pointed out that there are several other outward radiation conditions which could be used in place of Sommerfeld's condition (2-3). Wilcox [3] discusses equivalent radiation conditions for the three-dimensional problem.

Relation to the Wave Equation

The Helmholtz equation (2-2) is often referred to as the reduced wave equation. Boundary value problems for the Helmholtz equation involving Sommerfeld's radiation condition (2-3) arise in the solution of the wave equation with harmonic time dependence. In this connection it is helpful to discuss a two-dimensional exterior problem that occurs in the propagation of acoustic waves in a homogeneous medium.

Consider an infinite cylindrical surface of uniform cross section which intersects any plane parallel to its axis in a curve C, and let C^+ denote the region in that plane exterior to C. Assume the three-dimensional region outside the cylinder is occupied by a homogeneous fluid initially at rest. Starting at time $t = 0$ the fluid is disturbed by small amplitude oscillations of the cylindrical surface. These oscillations are assumed to be harmonic in time with frequency ω and to be uniform along the axis of the cylinder. Acoustic approximations allow the fluid velocity to be expressed as the gradient of a real valued velocity potential U [4, pp. 159-163]. It can be shown [5, pp. 6-8, pp. 45-48] that in any plane parallel to the cylinder axis, U satisfies the two-dimensional wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(\overline{x},t) = 0 \ , \ \overline{x} \in C^+ \ , \ t > 0 \ , \tag{2-4}$$

where c is the speed of sound in the fluid. The condition that the original state of the fluid is one of rest is given by

$$U(\overline{x},0) = \frac{\partial}{\partial t} U(\overline{x},0) = 0 \ , \ \overline{x} \in C^+ \ . \tag{2-5}$$

Also, assume the oscillations are such that the potential at the cylindrical surface is given by

$$U(\overline{x}_0,t) = U_0(\overline{x}_0) \sin(\omega t) \ , \ \overline{x}_0 \in C \ , \ t \geq 0 \ , \tag{2-6}$$

where the function U_0 is known. Then the wave equation (2-4), the initial conditions (2-5) and the boundary condition (2-6) uniquely determine the potential U in C^+ for all time $t \geq 0$.

The potential is obviously not harmonic in time. However we are often only interested in the state of the fluid after the oscillations have acted for a very long time. Assuming the oscillations have acted for an infinite length of time, then any transient waves produced when the oscillations began will have vanished. This permits the assumption that the potential in C^+ is time harmonic with frequency ω and thus has the general form

$$U(\bar{x},t) = u_1(\bar{x})\,\sin(\omega t) + u_2(\bar{x})\,\cos(\omega t)\ , \qquad \bar{x} \in C^+ \ . \tag{2-7}$$

Note that in spite of the fact $U = U_0 \sin(\omega t)$ on C, it would be incorrect to assume $U = u_1 \sin(\omega t)$ in C^+ because there is a shift in phase as the cylindrical waves spread outward from the surface.

The assumption (2-7) is a great simplification in that the original initial-boundary value problem for U can now be replaced by a pure boundary value problem in which U is required to satisfy (2-4), (2-6) and (2-7) for all $-\infty < t < \infty$. Substituting (2-7) into (2-4) reduces this problem to 2 real boundary value problems for the Helmholtz equation,

$$\left.\begin{aligned} &\left(\nabla^2 + \frac{\omega^2}{c^2}\right) u_1(\bar{x}) = 0 \ , \ \bar{x} \in C^+ \\ &u_1(\bar{x}_0) = U_0(\bar{x}_0) \ , \ \bar{x}_0 \in C \end{aligned}\right\} \tag{2-8}$$

and

$$\left.\begin{aligned} &\left(\nabla^2 + \frac{\omega^2}{c^2}\right) u_2(\bar{x}) = 0 \ , \ \bar{x} \in C^+ \\ &u_2(\bar{x}_0) = 0 \ , \ \bar{x}_0 \in C \end{aligned}\right\} \ . \tag{2-9}$$

This apparent simplification is not without cost, however. Neither boundary value problem (2-8) nor (2-9) has a unique solution. This loss of uniqueness can be explained on physical grounds. The original initial-boundary value problem clearly involves a process of outward radiation of acoustic waves from a finite source into an unbounded region. There are no mechanisms, such as sources at infinity or reflection from a distant boundary, for producing incoming or standing waves. In the process of simplifying the original initial-boundary value problem, the initial conditions, and thus the time bias, have been lost so that outward propagating waves are no longer guaranteed. The resulting equations (2-7), (2-8) and (2-9) apply equally well to problems arising from incoming or standing waves. To eliminate these possibilities it is necessary to impose an additional condition of outward radiation.

For the problem of finding a real function U which satisfies (2-7), (2-8) and (2-9), it can be shown that Summerfeld's outward radiation condition takes the form

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_1}{\partial r} - \frac{\omega}{c} u_2 \right) = 0 , \tag{2-10}$$

or equivalently

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_2}{\partial r} + \frac{\omega}{c} u_1 \right) = 0 . \tag{2-11}$$

Note that the radiation condition imposes a relationship between the solutions u_1 of (2-8) and the solutions u_2 of (2-9).

To avoid the coupling, through a radiation condition, of two real boundary value problems, the exterior problems for the Helmholtz equation are usually formulated in terms of complex functions. The problem is then to find the complex valued potential U which represents an outward propagating wave and which satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(\overline{x},t) = 0 \ , \ \overline{x} \in C^+ \ , \ -\infty < t < \infty \ , \tag{2-12}$$

$$U(\overline{x}_0,t) = U_0(\overline{x}_0)\ e^{-i\omega t} \ , \ \overline{x}_0 \in C \ , \ -\infty < t < \infty \ , \tag{2-13}$$

and

$$U(\overline{x},t) = u(\overline{x})\ e^{-i\omega t} \ , \ \overline{x} \in C^+ \ , \ -\infty < t < \infty \ . \tag{2-14}$$

The functions U_0 and u in (2-13) and (2-14) are now complex. Substituting (2-14) into (2-12) gives a single boundary value problem for the Helmholtz equation,

$$\left.\begin{aligned} &\left(\nabla^2 + \frac{\omega^2}{c^2}\right) u(\overline{x}) = 0 \ , \ \overline{x} \in C^+ \\ &u(\overline{x}_0) = U_0(\overline{x}_0) \ , \ \overline{x}_0 \in C \end{aligned}\right\} . \tag{2-15}$$

Problem (2-15) is made well posed by adding Sommerfeld's outward radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - i\,\frac{\omega}{c}\,u\right) = 0 \ . \tag{2-16}$$

When the solution $u(\overline{x})$ of (2-15) and (2-16) is substituted into (2-14), the resulting function $U(\overline{x},t)$ behaves like an outgoing wave. It is customary to refer to u itself as an outgoing wave, even though u is a function of position only. What is meant here is that u , when multiplied by the assumed time factor $e^{-i\omega t}$, represents an outgoing wave. Note that instead of $e^{-i\omega t}$, a harmonic time dependence of the form $e^{+i\omega t}$ could have been assumed in (2-13) and (2-14). In this case Sommerfeld's outward radiation condition becomes

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} + i \frac{\omega}{c} u \right) = 0 \ . \tag{2-17}$$

In most of the literature the time factor $e^{-i\omega t}$ is assumed. This convention will be followed here.

It can be shown [2, 3] that an equivalent form of Sommerfeld's outward radiation condition is

$$\lim_{R \to \infty} \int_{C_R} \left| \frac{\partial u}{\partial r} - i \frac{\omega}{c} u \right|^2 = 0 \ , \tag{2-18}$$

where C_R is a circle of radius R . This form, originally due to Magnus, has a simple physical interpretation [6, pp. 313-317]. Let C_R^- be the annular region between the boundary C and C_R . Then in terms of the velocity potential U , the acoustic energy E_R in the region C_R^- at any time t is (to within a proportionality constant)

$$E_R(t) = \frac{1}{2} \int_{C_R^-} \left[\frac{1}{c^2} \left| \frac{\partial U}{\partial t} \right|^2 + \left| \frac{\partial U}{\partial x} \right|^2 + \left| \frac{\partial U}{\partial y} \right|^2 \right] . \tag{2-19}$$

From (2-12), (2-14), (2-15) and Green's formula it can be shown that the time rate of change of the acoustic energy is

$$\frac{dE_R}{dt} = -\omega \int_C \operatorname{Im}\left(\bar{u} \frac{\partial u}{\partial n}\right) - \omega \int_{C_R} \operatorname{Im}\left(\bar{u} \frac{\partial u}{\partial n}\right) = 0 , \tag{2-20}$$

where Im denotes the imaginary component. Therefore the energy E_R is constant. The first line integral in (2-20) can be interpreted as the energy flux through the inner boundary C . The second line integral,

$$F = -\omega \int_{C_R} \operatorname{Im}\left(\bar{u} \frac{\partial u}{\partial n}\right) , \tag{2-21}$$

can be interpreted as the energy flux through the outer boundary C_R . From (2-20) it follows that the flux F is independent of R . Now a phenomenon of outward radiation requires a positive energy flux into C_R^- through the boundary C . From (2-20), the energy balance can be maintained only if there is an outward flux of energy through the outer boundary C_R , that is, if $F < 0$. Imposing the radiation condition (2-18) and noting that this can be written as

$$\frac{2}{c} F + \lim_{R \to \infty} \int_{C_R} \left[\left| \frac{\partial u}{\partial r} \right|^2 + \frac{\omega^2}{c^2} \left| u \right|^2 \right] = 0 \ , \qquad (2\text{-}22)$$

it follows that $F \leq 0$. Thus (2-18) indeed requires that there be an outward flux of energy through any circle C_R enclosing the boundary C .

Finally, since the complex potential U is a solution of the wave equation, is harmonic in time with frequency ω and behaves like an outgoing wave, both its real and imaginary parts also satisfy these conditions. The complex problem can then be interpreted as solving two real problems simultaneously.

There is a very interesting paper by Wilcox [3] which treats the initial-boundary value problem for the wave equation and various outward radiation conditions. Some of the points in the above discussion have been taken from his paper.

Solutions Outside a Circular Boundary

Exact solutions to the exterior Dirichlet problem for the Helmholtz equation are available in the special case when the boundary C is a circle of radius r_0 . The behavior of these solutions is a valuable guide for developing numerical methods for the Helmholtz equation. The asymptotic behavior of solutions outside circular boundaries can also be used to study the asymptotic behavior of solutions outside noncircular boundaries. A derivation of the results stated below is given in Appendix B.

Using separation of variables in polar coordinates, the general solution of the Helmholtz equation (2-2) outside a circular boundary is

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} \alpha_m \, e^{im\theta} \left[H_m^{(1)}(\omega r) + \beta_m \, H_m^{(2)}(\omega r)\right] , \tag{2-23}$$

where $H_m^{(1)}$ and $H_m^{(2)}$ are Hankel functions of the first and second kind of order m , and α_m and β_m are arbitrary complex constants. The asymptotic expressions for the Hankel functions, when multiplied by the time factor $e^{-i\omega t}$, are

$$H_m^{(1)}(\omega r) \; e^{-i\omega t} \underset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}} \; e^{i\omega(r-t) - i\frac{\pi}{2}(m+\frac{1}{2})} \tag{2-24}$$

and

$$H_m^{(2)}(\omega r) \; e^{-i\omega t} \underset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}} \; e^{-i\omega(r+t) + i\frac{\pi}{2}(m+\frac{1}{2})} \tag{2-25}$$

It follows that $H_m^{(1)}$ behaves like an outgoing wave and $H_m^{(2)}$ like an incoming wave, so that $H_m^{(2)}$ should be disregarded as a possible solution of an outward radiation problem. Setting the β_m to zero, the α_m are then determined from the boundary condition (2-1), which requires that $u(r_0,\theta) = u_0(\theta)$. Note that in the simple case of a circular boundary the incoming waves have been filtered out by inspection, rather than by using an outward radiation condition such as Sommerfeld's or Magnus'. This technique will also be used in the numerical method developed later.

In general the solution $u(r,\theta)$ is a very complicated function of the radial and angular coordinates. However, for large r we have the simpler asymptotic formula

$$u(r,\theta) \underset{r \to \infty}{\sim} \hat{\Theta}(\theta) \frac{e^{i\omega r}}{\sqrt{\omega r}} \tag{2-26}$$

where $\hat{\Theta}$ is still a complicated function of θ. Equation (2-26) leads to the conclusion that the solutions of the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ are, along any radial line, approximately periodic with a frequency ω and an amplitude which decays at the rate of $1/\sqrt{\omega r}$. Note that this result also applies to solutions outside noncircular boundaries since any boundary curve C can be enclosed in a circle of radius r_0. In this case the values of u on $r = r_0$ will not be known, but the separation of variables method can still be applied to the region outside $r = r_0$, although the coefficients α_m in (2-23) can no longer be determined.

As pointed out above, the function $\hat{\Theta}$ in (2-26) is in general a very complicated function of θ. However, for large r it is possible to obtain a simple formula for the angular dependence in the high frequency limit.

This is

$$u(r,\theta) \underset{\substack{r \to \infty \\ \omega \to \infty}}{\sim} \sqrt{\frac{r_0}{r}}\ e^{i\omega(r - r_0)}\ u_0(\theta) \ . \tag{2-27}$$

Equation (2-27) leads to the following conclusion about the solutions u of the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ for very high frequencies ω . If the boundary C is enclosed in a circle of radius r_0 , then for $r >> r_0$ the amplitude of u at the point (r,θ) is equal to the amplitude of u at the point (r_0,θ) radially below (r,θ) , diminished by the factor $\sqrt{r_0/r}$. Referring to the example of acoustic wave propagation, this is just what would be expected for high frequency radiation from a circular cylindrical surface of radius r_0 . The high frequency acoustic waves tend to radiate in the direction of the radial velocity of the oscillating surface rather than spread out in all directions. In other words, diffraction is not noticeable.

On the other hand, diffraction is significant for low frequency waves. Low frequency waves tend to spread out uniformly in all directions. It follows that at large distances from the boundary, the solutions of the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ for low frequencies ω should not reproduce the details of the wave pattern near the boundary. Rather, the amplitude should be independent of θ and represent an average of all sources along the boundary.

This is verified in Morse and Feshbach [7, pp. 1371-1376], from which much of the above discussion has been taken.

There is an exception to this last result for low frequencies in the case of a circular boundary. If the boundary condition is such that

$$u(r_0,\theta) = u_0(\theta) = \alpha e^{iM\theta} + \beta e^{iM\theta} \tag{2-28}$$

for some integer M and complex constants α and β, then the exact solution of the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ outside the circle is

$$u(r,\theta) = u_0(\theta)\, \frac{H_M^{(1)}(\omega r)}{H_M^{(1)}(\omega r_0)} \, . \tag{2-29}$$

Equation (2-29), which is valid for any frequency ω, shows that for periodic boundary conditions of the form (2-28), the value u at the point (r,θ) is equal to the value of u at the boundary point (r_0,θ) radially below (r,θ), diminished by a factor depending only on r.

<u>Galerkin Weak Form</u>

The exterior Dirichlet problem for the Helmholtz equation, as stated in (2-1), (2-2) and (2-3), can be reformulated in several ways, not all of which are equivalent to the original problem. Some of these formulations, such as the Galerkin weak form, are more general in that they may be applied in cases where the original problem is not even defined. Other formulations, such as those arising from boundary integral equations, may fail to have solutions or may not have unique solutions in certain cases where a unique solution exists to the original problem. Some formulations are particularly useful as starting points for numerical methods. In the finite element method to be developed later, the approximating equations are derived from the Galerkin weak form. For this reason the Galerkin weak form is discussed below. Other formulations, and some of the numerical methods based on these formulations, are discussed in Appendix C.

A function is said to have finite or compact support if it is zero outside some circle of finite radius. Let ψ be a complex valued function with finite support which is zero on the boundary C,

$$\psi(\overline{x}_0) = 0 \ , \ \overline{x}_0 \in C \ . \tag{2-30}$$

Multiplying the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ by ψ and integrating gives

$$\int_{C^+} \psi \nabla^2 u + \int_{C^+} \omega^2 \psi u = 0 \ . \tag{2-31}$$

Both of these integrals exist since ψ has finite support. The term on the left can be integrated by parts using Green's formula,

$$\int_{C^+} \psi \nabla^2 u = \lim_{R \to \infty} \int_{C_R^-} \psi \nabla^2 u$$

$$= - \lim_{R \to \infty} \int_{C_R^-} \nabla u \cdot \nabla \psi^* + \int_{C} \psi \frac{\partial u}{\partial n} + \lim_{R \to \infty} \int_{C_R} \psi \frac{\partial u}{\partial n} \quad . \tag{2-32}$$

Recall that C_R^- is the annular region between the boundary C and the circle C_R of radius R . The * denotes complex conjugate and the • denotes the standard scalar product for the vector space of ordered pairs of complex numbers, so that

$$\nabla u \cdot \nabla \psi^* = \frac{\partial u}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \psi}{\partial y} \quad . \tag{2-33}$$

Since $\psi = 0$ on C , the line integral over C vanishes. Since ψ has finite support, the integral over C_R^- exists in the limit while the line integral over C_R vanishes. Thus from (2-31) and (2-32) it follows that

$$\int_{C^+} (\nabla u \cdot \nabla \psi^* - \omega^2 u \psi) = 0 \quad . \tag{2-34}$$

Reversing the above procedure, it can be shown that if u is required to satisfy (2-34) for all test functions ψ which are zero on the boundary and have finite support, then u satisfies the Helmholtz equation (2-2). If, in addition, u is required to satisfy the boundary condition (2-1) and the radiation (2-3), we have an equivalent formulation of the original problem. Actually, these results hold only if u and ψ are of sufficient smoothness for Green's formula (2-32) to be applicable. However, the integral in (2-34) exists under less stringent conditions on u and ψ . This leads to the formulation of the following problem.

Let $H^1_{B,\infty}$ be the set of functions which satisfy the boundary condition (2-1), the radiation condition (2-3) and which are square integrable and have square integrable first derivatives in any bounded subset of C^+ . Define a set of test functions which are zero on the boundary, have finite support and which are square integrable and have square integrable first derivatives in C^+ . Then the Galerkin weak form of the original problem is to find the function $u \in H^1_{B,\infty}$ which satisfies equation (2-34) for all test functions ψ . If the solution is sufficiently smooth it will agree with the solution of the problem as originally posed. But in cases where a solution with second derivatives does not exist, the weak form may still have a solution even though the original problem is not well defined in such cases. The solution is then referred to as a generalized solution.

CHAPTER III

THATCHER'S METHOD

R. W. Thatcher [1] has developed a finite element method for the solution of the two-dimensional Laplace equation in unbounded regions. In principle, his method is not restricted to the Laplace equation. The major purpose of this study has been to attempt to extend Thatcher's method to the Helmholtz equation. The method has several unique features as well as some features in common with other methods discussed in Appendix C. Some of the advantages of Thatcher's method, and also its primary difficulty, will be pointed out below following a brief outline of the method.

Outline of the Method

This section is intended to give only a brief introduction to Thatcher's method for the Laplace equation, followed by some of the modifications that were required to extend his method to the Helmholtz equation. Each of the points touched on below will be discussed in detail in the sections following.

Perhaps the most striking feature of Thatcher's method is the use of an infinite grid. The plane region C^+ outside the boundary curve C is covered with a grid consisting of an infinite number of triangular elements as shown in Figure 1, Appendix A. An approximate solution $\tilde{u}$ of an exterior boundary value problem for the Laplace equation is sought in the form of a continuous function which is piecewise linear over this infinite grid. The problem is restated in variational form, and $\tilde{u}$ is required to minimize a functional I over a set of continuous piecewise linear functions satisfying an approximation to the boundary condition on C and a boundedness condition at infinity. Imposing at first only the boundary condition on C leads to an infinite system of linear equations in which the unknowns are the values of $\tilde{u}$ at the nodes of the grid. This infinite system of equations does not have a unique solution. A unique solution is obtained by requiring that the piecewise linear function $\tilde{u}$ corresponding to the solution of these equations be bounded at infinity. A way must be found to impose the boundedness condition and reduce the infinite system of equations to a finite system which can then be solved on the computer. The key lies in a systematic triangulation of the infinite region.

A grid of the type used by Thatcher is shown in Figure 1. The irregular boundary C is enclosed by a sequence of concentric circles C_0 , C_1 , C_2 , . . . , whose radii approach infinity. On each circle N equally spaced nodes are placed as shown in the figure.
A consistent nodal numbering scheme results in an infinite system of linear equations in which the coefficient matrix

is of block tridiagonal form [see, for example, equation (3-23)]. Furthermore, the spacing of the grid circles can be chosen so that the matrices along a given diagonal are identical, except for the first matrix which is associated with the irregular grid between C and C_0 . The regular pattern of the coefficient matrix allows the unknowns at the nodes on the circles C_1 , C_2 , C_3 , . . . , to be expressed as the solution of a three-term constant coefficient matrix recurrence relation. The general solution of this recurrence relation can be obtained by an eigensystem technique. This solution involves 2N arbitrary parameters which can be associated with the 2N unknowns on C_1 and C_2 . Imposing the boundedness condition at infinity on this general solution necessarily imposes a relationship between the parameters, reducing the number of arbitrary parameters to at most N . This imposes a relationship between the unknowns on C_1 and C_2 . This relationship can then be used to obtain a finite system of equations involving only the unknowns on and inside C_2 . The unknowns on C_3 , C_4 , C_5 , . . . , are then found from the solution of the recurrence relation once the values on C_2 have been determined.

The resulting solution gives the best approximation (with respect to a certain norm) to the exact solution, in the sense of minimizing the functional I over the set of all continuous piecewise linear functions defined on the infinite grid and satisfying the boundary conditions. Thatcher's method has the advantage of requiring the solution of a system of equations involving only unknowns near the boundary. The value of the solution at other nodes of the infinite grid can then be easily obtained. Thatcher has proved convergence of the solutions as the grid becomes more refined, provided the elements do not become extreme in shape. He has also given a theoretical error bound.

In attempting to extend Thatcher's method to the Helmholtz equation, complex solutions were sought and a similar infinite grid was used, again resulting in an infinite system of equations with a block tridiagonal coefficient matrix. However, it was found that there was no spacing of the grid circles which would give rise to identical matrices along the diagonals. A constant spacing was chosen in order to correctly model the oscillatory behavior of solutions of the Helmholtz equation (Figures 2 and 3, Appendix A). The general solution of a variable coefficient matrix recurrence relation was now required. The eigensystem technique used by Thatcher to solve the constant coefficient matrix recurrence relation required some modification. It was noticed that all of the matrices involved were circulant and thus had the same linearly independent set of eigenvectors. A similarity transformation was made which reduced all matrices in the recurrence relation to diagonal matrices. The matrix recurrence relation could then be reduced to N uncoupled difference equations in the new variables.

This last result was a considerable simplification in the problem. The general solution of a linear, second order, variable coefficient difference equation was now required. A method was devised for generating two approximate, linearly independent solutions--one of which had the characteristics of an outgoing wave, the other the characteristics of an incoming wave. The approximate general solution of the original matrix recurrence relation could then be constructed from the general solutions of the N difference equations. Imposing the outward radiation condition required that N of the 2N arbitrary parameters in this general solution be set to zero, thus imposing a relationship between the unknowns on C_1 and C_2 . The method could then proceed as in the case of the Laplace equation.

It is seen that the primary difficulty in applying Thatcher's method to the Helmholtz equation lies in finding the general solution of the matrix recurrence relation. This is used to derive a relationship between the values of $\tilde{u}$ at nodes on C_1 and C_2 which characterizes outward radiation. The problem can then be reduced to solving for the unknowns on the annular subregion between C and C_2 . It seems reasonable to ask whether the necessary relationship between the values on C_1 and C_2 could be obtained by one of the methods discussed in Appendix C. The matrix recurrence relation in its original form could then be used to calculate the values on C_3 , C_4 , C_5 , . . . , so there would be no need to find the general solution of the matrix recurrence relation. As will be discussed later, it appears that this approach would work for the Helmholtz equation but would be unstable in the case of the Laplace equation.

Grid Construction

The details of Thatcher's method for the Helmholtz equation can be illustrated most simply by considering solutions of the Helmholtz equation outside circular boundaries. The extension to irregular boundaries is straightforward and will be discussed later. It will be assumed then that the boundary C is a circle of radius r_0 with center at the point O . This circle will hereafter be denoted by C_0 , with the region exterior to C_0 denoted by C_0^+ . Then from (2-1) and (2-2) the complex function u is sought which satisfies

$$u(\overline{x}_0) = u_0(\overline{x}_0) \ , \ \overline{x}_0 \ \varepsilon \ C_0 \ , \tag{3-1}$$

$$(\nabla^2 + \omega^2) \ u(\overline{x}) = 0 \ , \ \overline{x} \ \varepsilon \ C_0^+ \ , \tag{3-2}$$

and which satisfies Sommerfeld's outward radiation condition (2-3).

The details of the grid construction are as follows. Let C_1 , C_2 , C_3 , . . . , be a sequence of concentric circles with center at O and radii r_1 , r_2 , r_3 , . . . , which satisfy

$$r_{k+1} > r_k \qquad k = 0, 1, 2, \ldots , \tag{3-3}$$

and

$$\lim_{k \to \infty} r_k = \infty . \tag{3-4}$$

At this time no other restrictions are placed on the r_k , thus allowing quite arbitrary radial spacings between the grid circles C_k . From the center O draw N equally spaced (i.e., equi-angular) radial lines extending to infinity. These radial lines are numbered in a clockwise (CW) sense or a counterclockwise (CCW) sense depending on a condition to be stated below. The point of intersection of circle C_k with radial line j will be denoted by $P_k^{\,j}$ (see Figure 2, Appendix A). These points are the nodes of the grid. Let P_k be the regular N-sided polygon formed by connecting adjacent nodes on C_k . Adjacent polygons and adjacent radial lines form a trapezoidal element which can be triangulated by drawing one of the two diagons (see Figure 3, Appendix A). These diagonals must be drawn in the same sense for all trapezoids, permitting only two types of triangulation. A CW triangulation is one in which moving outward along a diagonal results in a net CW movement; a CCW triangulation is one in which moving outward along a diagonal results in a net

CCW movement. It will be assumed that the radial lines are numbered in the sense of the triangulation.

The circular boundary $C = C_0$ has now been approximated by the polygon P_0 . Let P_0^+ denote the region exterior to P_0 and $\overline{P}_0^+ = P_0 \cup P_0^+$ denote the closure of P_0^+ .

Finite Element Formulation

A continuous piecewise linear function f over the infinite grid in $\overline{P}_0^+$ is a function which is continuous in $\overline{P}_0^+$ and which is linear over each triangular element of the grid (i.e., $f(x,y) = a_E x + b_E y + c_E$ over element E). When discussing piecewise linear functions in $\overline{P}_0^+$, reference to the grid will usually not be made since it is assumed the function is piecewise linear with respect to the grid under consideration. The set of all continuous piecewise linear complex functions in P_0^+ is a linear space $\tilde{H}^1$ with a countably infinite basis. The simplest basis for this space is the set of real functions ψ_k^j , where ψ_k^j is the continuous piecewise linear function in P_0^+ which has the value one at node P_k^j and zero at all other nodes. Thus $\psi_k^j(\overline{x})$ is nonzero only at those points $\overline{x}$ lying in elements which have P_k^j as a node (see Figure 4, Appendix A). The basis function ψ_k^j is said to have its support at the node P_k^j where its value is one. In terms of the basis ψ_k^j , any function $\tilde{u} \in \tilde{H}^1$ can be written as

$$\tilde{u}(\bar{x}) = \sum_{k=0}^{\infty} \sum_{j=1}^{N} u_k^j \psi_k^j(\bar{x}) \ , \ \bar{x} \in \bar{P}_0^+ \ , \tag{3-5}$$

where the u_k^j are the components or generalized coordinates of $\tilde{u}$ with respect to the basis ψ_k^j . From the properties of the ψ_k^j it follows that u_k^j is the value of $\tilde{u}$ at node P_k^j ,

$$\tilde{u}(P_k^j) = u_k^j \ . \tag{3-6}$$

Referring to the boundary condition (3-1), let $\tilde{H}_B^1$ be the subset of $\tilde{H}^1$ consisting of those functions which agree with u_0 at the nodes of the polygon P_0 . That is,

$$\tilde{H}_B^1 = \left\{ \tilde{u} \in \tilde{H}^1 \ : \ u_0^j = \tilde{u}(P_0^j) = u_0(P_0^j) \ ; \ j = 1,2, \ldots, N \right\} . \tag{3-7}$$

To find an approximate solution of the exterior Dirichlet problem by the finite element method, a function from $\tilde{H}_B^1$ is sought which, in some approximate sense, satisfies the Helmholtz equation (3-2) and the outward radiation condition (2-3). Since the functions in $\tilde{H}_B^1$ are continuous and piecewise linear, their first derivatives are piecewise constant and discontinuous at the element edges, so their second derivatives are delta-functions. Thus these functions cannot be substituted directly into the differential equation.

Recall, however, that the Galerkin weak form of the problem involves only first derivatives and requires only that they be square integrable. This is the formulation that will be applied to functions in $\tilde{H}_B^{\,1}$ to find approximate solutions of the Helmholtz equation. From (2-34), functions $\tilde{u} \in \tilde{H}_B^{\,1}$ are sought which satisfy

$$\int_{P_0^+} \left(\nabla\tilde{u} \cdot \nabla\psi^* - \omega^2 \tilde{u}\,\psi \right) = 0 \tag{3-8}$$

for each ψ belonging to a suitable space of test functions. It is to be expected that more than one function $\tilde{u} \in \tilde{H}_B^{\,1}$ exists which satisfies (3-8), since only the boundary condition and not the radiation condition is being imposed at this point.

There is considerable flexibility in the choice of the space of test functions for (3-8); however, since a countably infinite set of components $u_k^{\,j}$ is to be determined, the set of test functions must be countably infinite. The most advantageous choice is to take the space of test functions to be the set of basis functions

$$\psi_m^{\,n} \; ; \; m = 1,2,3, \ldots \; ; \; n = 1,2, \ldots, N \, .$$

These functions have square integrable first derivatives, have finite support and are zero on P_0 as required. This choice leads to a banded system of equations and to simple integrals for evaluating the coefficients. From (3-8), substituting $\psi_k{}^j$ for ψ and (3-5) for $\tilde{u}$ gives the system of equations

$$\left.\begin{array}{l} \sum_{k=0}^{\infty} \sum_{j=1}^{N} u \int_{P_0^+} \left(\nabla\psi_k{}^j \cdot \nabla\psi_m{}^n - \omega^2 \psi_k{}^j \psi_m{}^n\right) = 0 \\ m = 1,2,3, \ldots \qquad n = 1,2, \ldots , N \end{array}\right\} . \tag{3-9}$$

This can be written more compactly as

$$\sum_{k=1}^{\infty} \sum_{j=1}^{N} A_{mk}^{nj} u_k^j = h_m^n \quad ; \quad m = 1,2,3, \ldots \quad ; \quad n = 1,2, \ldots N \tag{3-10}$$

where the real coefficients A_{mk}^{nj} are defined by

$$A_{mk}^{nj} = A_{km}^{jn} = \int_{P_0^+} \left(\nabla\psi_m^n \cdot \nabla\psi_k^j - \omega^2 \psi_m^n \psi_k^j\right) \tag{3-11}$$

and the complex right-hand side terms h_m^n are defined by

$$h_m^n = - \sum_{j=1}^{N} A_{m0}^{nj} u_0^j \quad . \tag{3-12}$$

Recall from (3-7) that the u_0^j are known from the boundary condition (3-1).

The system of equations (3-10) is an infinite system of linear equations for the unknown values $u_k{}^j$ of $\tilde{u}$ at the nodes $P_k{}^j$ outside P_0 . Again, this system does not have a unique solution until some condition is imposed on the behavior of the $u_k{}^j$ as $k \to \infty$. As a first step in this direction, the structure of the coefficient matrix of the system of equations (3-10) must be examined.

Infinite System of Equations

The infinite system of equations (3-10) is written in matrix form as

$$A\bar{u} = \bar{h} \ . \tag{3-13}$$

The vector of unknowns $\bar{u}$ and the right-hand side vector $\bar{h}$ are partitioned into column vectors of length N as follows:

$$\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \vdots \end{bmatrix} \ , \qquad \bar{h} = \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \\ \bar{h}_3 \\ \vdots \end{bmatrix} \ , \tag{3-14}$$

where

$$\bar{u}_k = \begin{bmatrix} u_k^{\ 1} \\ u_k^{\ 2} \\ \vdots \\ u_k^{\ N} \end{bmatrix} \ ; \qquad k = 1,2,3, \ . \ . \ . \tag{3-15}$$

and

$$\bar{h}_m \begin{bmatrix} h_m^1 \\ h_m^2 \\ \vdots \\ h_m^N \end{bmatrix} \quad ; \quad m = 1,2,3, \ldots . \tag{3-16}$$

Since from (3-6) $u_k{}^j$ is the value of $\tilde{u}$ at node $P_k{}^j$ and (see Figure 3, Appendix A) $P_k{}^1$, $P_k{}^2$, . . . , $P_k{}^N$ are the N nodes on polygon P_k , $\bar{u}_k$ in (3-15) is the vector whose components are the unknowns at nodes on P_k .
The coefficient matrix A in (3-13) is defined by

$$A = \left[A_{mk}^{nj} \right] \quad , \tag{3-17}$$

where the ordering of the coefficients A_{mk}^{nj} is by definition that which is compatible with (3-10) and (3-13) through (3-16). The matrix A can be partitioned into $N \times N$ matrices A_{IJ} as follows:

$$A = \left[A_{IJ} \right] = \begin{bmatrix} A_{11} & \cdots & A_{1J} & \cdots \\ \vdots & & \vdots & \\ A_{I1} & \cdots & A_{IJ} & \cdots \\ \vdots & & \vdots & \end{bmatrix} \tag{3-18}$$

where

$$A_{IJ} = \left[A_{IJ}^{nj}\right] = \begin{bmatrix} A_{IJ}^{11} & \cdots & A_{IJ}^{1j} & \cdots & A_{IJ}^{1N} \\ \vdots & & \vdots & & \vdots \\ A_{IJ}^{n1} & \cdots & A_{IJ}^{nj} & \cdots & A_{IJ}^{nN} \\ \vdots & & \vdots & & \vdots \\ A_{IJ}^{N1} & \cdots & A_{IJ}^{Nj} & \cdots & A_{IJ}^{NN} \end{bmatrix} . \qquad (3\text{-}19)$$

From (3-11) it follows that the coefficient matrix A is symmetric and that

$$A_{IJ} = A_{JI}^{T} \quad , \qquad (3\text{-}20)$$

where the T denotes transpose.

Referring again to (3-11) shows that the coefficient A_{mk}^{nj} can be nonzero only when $\psi_m{}^n$ and $\psi_k{}^j$ are both nonzero over at least one common element. But since $\psi_k{}^j$ is nonzero only over those elements which have $P_k{}^j$ as a node, it follows that A_{mk}^{nj} is nonzero only when nodes $P_m{}^n$ and $P_k{}^j$ both lie on the same element. Therefore (see Figure 3, Appendix A) the coefficients in the infinite system (3-10) satisfy

$$A_{mk}^{nj} = 0 \quad \text{if} \quad |m-k| > 1 \quad \text{or} \quad 1 < |n-j| < N-1 \qquad (3\text{-}21)$$

which, from (3-19), gives

$$A_{IJ} = 0 \quad , \quad |I-J| > 1 \; . \qquad (3\text{-}22)$$

Then from (3-12) through (3-22), the infinite system of equations (3-10) can be written in the block tridiagonal form

$$\begin{bmatrix} A_{11} & A_{12} & & & & \\ A_{12}^T & A_{22} & A_{23} & & & \\ & A_{23}^T & A_{33} & A_{34} & & \\ & & A_{34}^T & A_{44} & A_{45} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} h_1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} . \tag{3-23}$$

The infinite system of equations (3-23) can also be written as

$$A_{11}\ \bar{u}_1 + A_{12}\ \bar{u}_2 = \bar{h}_1 \tag{3-24}$$

$$\left.\begin{array}{l} A_{12}^T\ \bar{u}_1 + A_{22}\ \bar{u}_2 + A_{23}\ \bar{u}_3 = 0 \\ A_{23}\ \bar{u}_2 + A_{33}\ \bar{u}_3 + A_{34}\ \bar{u}_4 = 0 \\ \qquad\qquad\vdots \end{array}\right\} . \tag{3-25}$$

From (3-25) it follows that once $\bar{u}_1$ and $\bar{u}_2$ are known, then $\bar{u}_3$ can be determined from $\bar{u}_1$ and $\bar{u}_2$, $\bar{u}_4$ can be determined from $\bar{u}_2$ and $\bar{u}_3$, etc. However, (3-24) cannot determine $\bar{u}_1$ and $\bar{u}_2$ uniquely. What we would like to do here is introduce an artificial boundary on polygon P_2 , find a condition of the form

$$B_1\ \bar{u}_1 + B_2\ \bar{u}_2 = \bar{b} \tag{3-26}$$

which characterizes outward radiation, and then solve (3-24) and (3-26) for the unknowns $\bar{u}_1$ and $\bar{u}_2$. The unknowns $\bar{u}_3$, $\bar{u}_4$, $\bar{u}_5$, . . . , could then be found from the recurrence relation (3-25). To find a radiation condition of the form (3-26) by Thatcher's method, we first attempt to find the general solution $\bar{u}_k$ of the recurrence relation (3-25). From the behavior of the general solution $\bar{u}_k$ as $k \to \infty$, the relationship between $\bar{u}_1$ and $\bar{u}_2$ which gives rise only to outgoing waves can then be determined. The unknowns $\bar{u}_3$, $\bar{u}_4$, $\bar{u}_5$, . . . , are then easily calculated from the general solution $\bar{u}_k$, rather than from the recurrence relation in its original form (3-25).

Matrices from the Infinite System

Before attempting to solve the recurrence relation (3-25), the matrices A_{kk}, $A_{k,k+1}$: $k = 1,2,3, \ldots$, occurring in (3.25) must be studied. From (3-11) and (3-19), A_{kk} is the matrix of coefficients A_{kk}^{nj} involving the integral of products of the basis functions $\psi_k{}^n$ and $\psi_k{}^j$. These functions have support at the nodes $P_k{}^n$ and $P_k{}^j$ on polygon P_k (see Figure 3, Appendix A), so that A_{kk}^{nj} is zero if $P_k{}^n$ and $P_k{}^j$ are not identical or adjacent nodes on P_k. This can be stated concisely as

$$A_{kk}^{nj} = 0 \quad \text{if} \quad 1 < |n-j| < N-1 . \qquad (3\text{-}27)$$

It follows that A_{kk} has the form

$$A_{kk} = \begin{bmatrix} A_{kk}^{11} & A_{kk}^{12} & & & A_{kk}^{1N} \\ A_{kk}^{21} & A_{kk}^{22} & A_{kk}^{23} & & \\ & A_{kk}^{32} & A_{kk}^{33} & A_{kk}^{34} & \\ & & \ddots & \ddots & \ddots \\ & & A_{kk}^{N-1,N-2} & A_{kk}^{N-1,N-1} & A_{kk}^{N-1,N} \\ A_{kk}^{N1} & & & A_{kk}^{N,N-1} & A_{kk}^{NN} \end{bmatrix} . \tag{3-28}$$

This can be simplified even further by taking into account the grid symmetry resulting from equally spaced radial lines and regular polygons. For example, the diagonal coefficients A_{kk}^{jj} are all equal. They each involve the integral of products of $\psi_k{}^j$ with itself, and the triangular elements over which $\psi_k{}^j$ is nonzero are of the same shape for each j . Similarly, the off-diagonal coefficients in (3-28) are all equal, though not to the ones on the diagonal. It follows that A_{kk} must have the form

$$A_{kk} = \begin{bmatrix} a_k & b_k & & & b_k \\ b_k & a_k & b_k & & \\ & b_k & a_k & b_k & \\ & & \ddots b_k & \ddots a_k & \ddots b_k \\ b_k & & & b_k & a_k \end{bmatrix}, \qquad (3\text{-}29)$$

where the a_k and b_k are still to be determined.

Similarly, $A_{k,k+1}$ is the matrix of coefficients $A^{n\ j}_{k,k+1}$ involving the integral of products of the basis functions $\psi_k{}^n$ and ψ^j_{k+1} which have their support at nodes on the polygons P_k and P_{k+1} respectively. Arguing as above, it can be shown that $A_{k,k+1}$ has the form

$$A_{k,k+1} = \begin{bmatrix} \alpha_k & \beta_k & & & \\ & \alpha_k & \beta_k & & \\ & & \ddots \alpha_k & \ddots \beta_k & \\ & & & \alpha_k & \beta_k \\ \beta_k & & & & \alpha_k \end{bmatrix}. \qquad (3\text{-}30)$$

Unlike A_{kk} , the matrix $A_{k,k+1}$ is not symmetric. This is due to the lack of symmetry in the grid (see Figure 3, Appendix A), resulting from the fact that the trapezoidal regions have been triangulated by drawing only one of the diagonals. Thus, for example, $\psi_1{}^1 \psi_2{}^2 \neq 0$ but $\psi_1{}^2 \psi_2{}^1 = 0$.

Referring to the section on grid construction, it should be noted that equations (3-29) and (3-30) and the results to follow are valid for both CW and CCW triangulations. This simply follows from the convention that radial lines be numbered in the sense of the triangulation. Also note that, aside from conditions (3-3) and (3-4), no restrictions have been placed as yet on the spacing of the grid circles. Thatcher [1] used a variable spacing of a certain form for the Laplace equation (see Figure 1, Appendix A), while for the Helmholtz equation a constant spacing is appropriate (see Figures 2 and 3, Appendix A). This subject will be discussed in more detail in the next section. It will prove worthwhile to derive 3 different formulas for each of the coefficients a_k , b_k , α_k , and β_k in (3-29) and (3-30) in terms of different but equivalent sets of grid parameters.

Aside from translations and rotations, the infinite grid is completely specified by the following parameters:

(i) N--the number of radial lines;

(ii) r_0--the distance from the center O to the circle C_0 ;

(iii) $\{r_1 , r_2 , r_3 , \ldots\}$--the distances from O to the circles C_1 , C_2 , C_3 , ...

The radial spacing as defined by the sequence of radii $\{r_1, r_2, r_3, \ldots\}$ can also be specified by the sequence $\{\Delta r_1, \Delta r_2, \Delta r_3, \ldots\}$ or the sequence $\{R_1, R_2, R_3, \ldots\}$ where

$$\Delta r_k = r_k - r_{k-1} \qquad k = 1,2,3, \ldots \tag{3-31}$$

is the radial distance between circles C_k and C_{k-1} and

$$R_k = \frac{r_k}{r_{k-1}} \qquad k = 1,2,3, \ldots . \tag{3-32}$$

It is convenient in the formulas that follow to introduce the parameters ε_1 and ε_2 defined by

$$\varepsilon_1 = 2 \tan\left(\frac{\pi}{N}\right) \quad , \qquad \varepsilon_2 = \sin\left(\frac{2\pi}{N}\right) \quad , \tag{3-33}$$

where $2\pi/N$ is the angle between adjacent radial lines.

The derivation of the formulas for the a_k, b_k, α_k, and β_k in (3-29) and (3-30) is tedious but straightforward. The details will not be given here; only the essential steps will be outlined. Recall that a_k, b_k, α_k, and β_k are coefficients in the infinite system of Equations (3-10). For example, from (3-11), (3-19), (3-28) and (3-29),

$$a_k = A_{kk}^{11} = \int_{P_0^+} \left(\nabla\psi_k^{\,1} \cdot \nabla\psi_k^{\,1} - \omega^2\, \psi_k^{\,1}\, \psi_k^{\,1}\right)$$

$$= \sum_E \int_E \left(\nabla\psi_k^{\,1} \cdot \nabla\psi_k^{\,1} - \omega^2\, \psi_k^{\,1}\, \psi_k^{\,1}\right) \tag{3-34}$$

and

$$b_k = A^{12}_{kk} = \sum_E \int_E \left(\nabla\psi_k{}^1 \cdot \nabla\psi_k{}^2 - \omega^2 \psi_k{}^1 \psi_k{}^2\right) , \tag{3-35}$$

where E denotes a typical triangular element and $\sum_E$ denotes summation over all elements. However, since the integrand in (3-34) is nonzero only over those elements for which $P_k{}^1$ is a node, it is seen from Figure 3 (Appendix A) that the integral in (3-34) need be evaluated only over 6 triangular elements. Similarly, the integral in (3-35) need be evaluated only over the 2 triangular elements for which both $P_k{}^1$ and $P_k{}^2$ are nodes. The integrals are evaluated over one element at a time and then summed. The integration formulas for a single triangular element in terms of the element geometry are given in Appendix D. Formulas for the a_k, b_k, α_k and β_k were first derived in terms of the grid parameters ε_1, ε_2 and r_k and of course the frequency ω. For later use, some of the expressions involving the r_k were rewritten in terms of Δr_k and R_k using (3-31) and (3-32). The results are listed below.

In terms of ε_1, ε_2, r_k and ω the formulas are

$$a_k = \varepsilon_1\left(\frac{r_{k+1}}{r_{k+1} - r_k} + \frac{r_{k-1}}{r_k - r_{k-1}}\right) + \frac{1}{\varepsilon_2}\left(\frac{r_{k+1} - r_{k-1}}{r_k}\right)$$

$$- \frac{\omega^2\varepsilon_2}{12}\left(r_{k+1} - r_{k-1}\right)\left(r_{k+1} + r_k + r_{k-1}\right) , \tag{3-36}$$

$$b_k = -\frac{1}{2\varepsilon_2}\left(\frac{r_{k+1} - r_{k-1}}{r_k}\right) - \frac{\omega^2\varepsilon_2}{24} r_k\left(r_{k+1} - r_{k-1}\right) \quad , \tag{3-37}$$

$$\alpha_k = -\frac{\varepsilon_1}{2}\left(\frac{r_{k+1} + r_k}{r_{k+1} - r_k}\right) + \beta_k \quad , \tag{3-38}$$

and

$$\beta_k = -\frac{\omega^2\varepsilon_2}{24}\left(r_{k+1}^2 - r_k^2\right) \quad . \tag{3-39}$$

In terms of ε_1 , ε_2 , r_k , Δr_k and ω the formulas are

$$a_k = \varepsilon_1\left(\frac{1}{\Delta r_{k+1}} + \frac{1}{\Delta r_k}\right) r_k + \frac{1}{\varepsilon_2}\left(\Delta r_{k+1} + \Delta r_k\right)\frac{1}{r_k}$$

$$- \frac{\omega^2\varepsilon_2}{12}\left(\Delta r_{k+1} + \Delta r_k\right)\left(3r_k + \Delta r_{k+1} - \Delta r_k\right) \quad , \tag{3-40}$$

$$b_k = -\frac{1}{2\varepsilon_2}\left(\Delta r_{k+1} + \Delta r_k\right)\frac{1}{r_k} - \frac{\omega^2\varepsilon_2}{24}\left(\Delta r_{k+1} + \Delta r_k\right) r_k \tag{3-41}$$

$$\alpha_k = -\varepsilon_1\left(\frac{r_k}{\Delta r_{k+1}} + \frac{1}{2}\right) + \beta_k \quad , \tag{3-42}$$

and

$$\beta_k = -\frac{\omega^2\varepsilon_2}{24}\Delta r_{k+1}\left(2r_k + \Delta r_{k+1}\right) \quad . \tag{3-43}$$

In terms of ε_1 , ε_2 , r_k , R_k and ω the formulas are

$$a_k = \varepsilon_1\left(\frac{R_{k+1}}{R_{k+1} - 1} + \frac{1}{R_k - 1}\right) + \frac{1}{\varepsilon_2}\left(R_{k+1} - \frac{1}{R_k}\right)$$

$$- \frac{\omega^2\varepsilon_2}{12}\left(R_{k+1} - R_k\right)\left(R_{k+1} + 1 + \frac{1}{R_k}\right) r_k^2 \quad , \tag{3-44}$$

$$b_k = -\frac{1}{2\varepsilon_2}\left(R_{k+1} - \frac{1}{R_k}\right) - \frac{\omega^2\varepsilon_2}{24}\left(R_{k+1} - \frac{1}{R_k}\right)r_k^2 \quad , \qquad (3\text{-}45)$$

$$\alpha_k = -\frac{\varepsilon_1}{2}\left(\frac{R_{k+1}+1}{R_{k+1}-1}\right) + \beta_k \quad , \qquad (3\text{-}46)$$

and

$$\beta_k = -\frac{\omega^2\varepsilon_2}{24}\left(R_{k+1}^2 - 1\right)r_k^2 \quad . \qquad (3\text{-}47)$$

Radial Spacing

Although the N radial lines in the infinite grid have been assumed to be equally spaced, the radial spacing between the concentric grid circles or polygons has been left arbitrary, subject only to conditions (3-3) and (3-4). Before determining the radial spacing best suited for the Helmholtz equation, it is worthwhile to discuss the spacing used by Thatcher [1] for the Laplace equation.

Referring to equation (3-32), Thatcher chose R_k to be a constant R greater than 1 . That is,

$$R_k = R > 1 \quad ; \quad k = 1,2,3,\ldots \quad . \qquad (3\text{-}48)$$

The radii r_k then satisfy

$$r_k = R\, r_{k-1} \qquad (3\text{-}49)$$

which gives

$$r_k = R^k\, r_0 \quad , \quad R > 1 \quad . \qquad (3\text{-}50)$$

With the r_k as defined above, it can be shown that as k increases, the radial spacing between polygons P_{k-1} and P_k increases in such a way that all of the trapezoids in the infinite grid are similar (i.e., equal angles and proportional sides). Triangulating the trapezoids then gives two sets of similar triangles. For an example of such a grid see Figure 1, Appendix A. Since the Laplace equation $\nabla^2 u = 0$ is a special case of the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$, the equation (3-11) for the coefficients A_{mk}^{nj} can be applied to the Laplace equation by setting $\omega = 0$. This gives

$$A_{mk}^{nj} = \sum_E \int_E \nabla\psi_m{}^n \cdot \nabla\psi_k{}^j \quad . \tag{3-51}$$

As shown in Appendix D, these integrals depend only on the shape of the triangular element E and not on its size. Since the triangles in the grid are similar, it follows that the coefficients A_{kk}^{nj} and $A_{k,k+1}^{nj}$ are actually independent of k and thus the matrices A_{kk} and $A_{k,k+1}$ in the infinite system of equations (3-23) are independent of k .

The above result can be verified directly by referring to equations (3-44) through (3-47). Setting $\omega = 0$ and $R_k = R$ in these equations gives

$$a = \varepsilon_1 \left(\frac{R+1}{R-1}\right) - 2b \quad , \tag{3-52}$$

$$b = -\frac{1}{2\varepsilon_2}\left(R - \frac{1}{R}\right) \quad , \tag{3-53}$$

$$\alpha = -\frac{\varepsilon_1}{2}\left(\frac{R+1}{R-1}\right) \quad , \tag{3-54}$$

and

$$\beta = 0 \quad . \tag{3-55}$$

The k subscripts have been suppressed since each of these coefficients is independent of k .

Thatcher found the general solution of the matrix recurrence relation (3-25) using an eigensystem technique which relied on the fact that the matrices A_{kk} and $A_{k,k+1}$ are constant and the fact that $A_{k,k+1}$ is symmetric. This last property is trivially satisfied since, from (3-30) and (3-55), the matrix $A_{k,k+1}$ is diagonal. Most of the details of the eigensystem technique used by Thatcher are not included in his paper [1]. The technique is discussed in more detail in two earlier papers dealing with singularities.

To determine the proper radial spacing for the Helmholtz equation, recall from (2-26) that along any radial line $\theta = \theta_0$, the solutions of $\nabla^2 u + \omega^2 u = 0$ which represent outgoing waves are asymptotic to $\hat{\Theta}(\theta)\, e^{i\omega r}/\sqrt{\omega r}$. To model the oscillatory behavior of $e^{i\omega r}$ it is necessary that the radial spacing Δr_k be no greater than half the wavelength of the wave $e^{i\omega r}$. Thus for the Helmholtz equation the Δr_k should always satisfy

$$\Delta r_k \leq \frac{\pi}{\omega} \quad . \tag{3-56}$$

The simplest choice is to take Δr_k to be a constant Δr , so that

$$\Delta r_k = \Delta r \qquad k = 1,2,3, \ldots \quad . \tag{3-57}$$

From (3-56) it follows that Δr should decrease as the frequency ω increases, and should always be such that

$$\omega \, \Delta r \leq \pi \quad . \tag{3-58}$$

From (3-31) it follows that

$$r_k = r_{k-1} + \Delta r = r_0 + k \, \Delta r \quad . \tag{3-59}$$

With the grid as defined by (3-57) or (3-59), equations (3-40) through (3-43) give

$$a_k = \frac{2\varepsilon_1}{\Delta r} r_k + \frac{2\Delta r}{\varepsilon_2} \frac{1}{r_k} - \frac{\omega^2 \varepsilon_2 \Delta r}{2} r_k \quad , \tag{3-60}$$

$$b_k = - \frac{\Delta r}{\varepsilon_2} \frac{1}{r_k} - \frac{\omega^2 \varepsilon_2 \Delta r}{12} r_k \quad , \tag{3-61}$$

$$\alpha_k = - \varepsilon_1 \left(\frac{r_k}{\Delta r} + \frac{1}{2} \right) + \beta_k \quad , \tag{3-62}$$

and

$$\beta_k = - \frac{\omega^2 \varepsilon_2 \Delta r}{24} (2r_k + \Delta r) \quad . \tag{3-63}$$

Unfortunately, all of these coefficients depend on k so that the method used by Thatcher for solving the constant coefficient matrix recurrence relation in the case of the Laplace equation cannot be used for the Helmholtz equation. A technique for solving the matrix recurrence relation (3-25) is developed in the next chapter.

The question naturally arises as to whether there is any grid spacing for the Helmholtz equation which gives coefficients a_k , b_k , α_k and β_k which are independent of k . The answer is no, as can be seen by examining the sets of equations (3-36) through (3-47) for different choices of the r_k , Δr_k and r_k . Also, with reference to the equation $(\nabla^2 + \omega^2)u = 0$, the terms in (3-60) through (3-63) with an ω^2 coefficient obviously come from the $\omega^2 u$ term in the differential equation while the terms without an ω^2 coefficient come from the $\nabla^2 u$ term in the differential equation. It is interesting to note that for both a_k and α_k , which are the coefficients on the main diagonal of the matrices A_{kk} and $A_{k,k+1}$ respectively, the ∇^2 terms and the ω^2 terms both are of the same order r_k as $k \to \infty$. This is only true for a constant radial spacing or a spacing which approaches a constant value in the limit. For other spacings either the ω^2 terms dominate the ∇^2 terms as $k \to \infty$, or visa-versa. However, even for a constant spacing, the ω^2 terms dominate in the off-diagonal coefficients b_k

and β_k in (3-61) and (3-63). The significance of this is not understood.

Finally, a question that has not been mentioned yet is whether the angular behavior of the solutions of the Helmholtz equation can be correctly modeled by the type of grid being considered. Each polygon P_k has N equally spaced nodes. As $k \to \infty$ the distance between adjacent nodes on P_k becomes infinite, although the angle between the radial lines passing through these nodes remains a constant $2\pi/N$. It follows that such a grid can allow accurate approximations at large radial distances only if the solution u is such that $|\partial u/\partial\theta|$ is bounded as $r \to \infty$. As discussed in Chapter II, the most pronounced angular behavior in solutions of the Helmholtz equation occurs either at very high frequencies ω or for a particular type of periodic boundary condition. However, as seen from (2-27), (2-28) and (2-29), even in these extreme cases $|\partial u/\partial\theta|$ is a decreasing function of r. It follows that if the number of radial lines N is sufficient to model the angular behavior of the solution near the boundary, then the angular behavior far from the boundary should also be modeled correctly.

CHAPTER IV

SOLUTION OF THE RECURRENCE RELATION

The problem at this point has been reduced to solving the matrix recurrence relation (3-25). The general solution $\overline{u}_k$ of this recurrence relation is sought. By examining the behavior of this general solution as $k \to \infty$, we hope to obtain the necessary relationship between $\overline{u}_1$ and $\overline{u}_2$ which gives rise only to outgoing waves. The matrix recurrence relation (3-25) cannot be solved directly in its present form. The reduction of this relation to a simpler form by means of a similarity transformation is based on the circulant properties of the matrices A_{kk} and $A_{k,k+1}$. The next section gives a summary of the properties of circulant matrices which are needed to transform (3-25) to a simpler form.

Circulant Matrices

An Nth order circulant matrix D is an $N \times N$ matrix of the form

$$D = \begin{bmatrix} d_0 & d_1 & d_2 & d_3 \cdots & d_{N-2} & d_{N-1} \\ d_{N-1} & d_0 & d_1 & d_2 \cdots & d_{N-3} & d_{N-2} \\ d_{N-2} & d_{N-1} & d_0 & d_1 \cdots & d_{N-4} & d_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_2 & d_3 & d_4 & d_5 \cdots & d_0 & d_1 \\ d_1 & d_2 & d_3 & d_4 \cdots & d_{N-1} & d_0 \end{bmatrix} , \qquad (4-1)$$

where the d_n can be any real or complex numbers. Note that each row after the first can be obtained from the preceding row by shifting it one column to the right, the entry in the last column being moved to the first column. Since any circulant matrix is completely determined by its first row, it is convenient to adopt the notation

$$D = \operatorname{circ}\left[d_0 , d_1 , d_2 , \ . \ . \ . \ , d_{N-1}\right] = \operatorname{circ}\left[d_n\right] = \operatorname{circ}\left[\bar{d}^T\right] \tag{4-2}$$

for the circulant matrix D in (4-1). Here $\bar{d}$ is the column vector whose transpose is the first row of D .

Circulant matrices have the following remarkable property. Every Nth order circulant matrix has the same linearly independent set of eigenvectors, regardless of the entries in the first row, although the eigenvalues do depend on these entries. Both the eigenvectors and the eigenvalues are given by very simple formulas involving the N roots of unity. In particular, if z is a root of $z^N = 1$, then the vector $\bar{z}$ defined by

$$\bar{z} = \begin{bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \vdots \\ z^{N-1} \end{bmatrix} \tag{4-3}$$

is an eigenvector of any Nth order circulant matrix. With D as defined in (4-1) or (4-2), the eigenvalue λ corresponding to the eigenvector $\overline{z}$ in (4-3) is

$$\lambda = \sum_{n=0}^{N-1} d_n z^n = \overline{d}^T z \ . \tag{4-4}$$

A simple proof of these results is given in Bellman [9, pp. 234-235]. Also note that the transpose of a circulant matrix is again a circulant matrix.

The complete eigensystem of an Nth order circulant matrix D can be constructed as follows. Since

$$1 \ , \ e^{\frac{2\pi i}{N}} \ , \ e^{\frac{2\pi i}{N}\cdot 2} \ , \ e^{\frac{2\pi i}{N}\cdot 3} \ , \ . \ . \ . \ , \ e^{\frac{2\pi i}{N}(N-1)}$$

are the N roots of unity, define the N vectors

$$\overline{z}_j = \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{N}(j-1)} \\ e^{\frac{2\pi i}{N}(j-1)\cdot 2} \\ e^{\frac{2\pi i}{N}(j-1)\cdot 3} \\ \vdots \\ e^{\frac{2\pi i}{N}(j-1)(N-1)} \end{bmatrix} \qquad j = 1, 2, \ldots, N \tag{4-5}$$

and the N numbers

$$\lambda_j = \sum_{n=0}^{N-1} d_n \, e^{\frac{2\pi i}{N}(j-1)n} = \bar{d}^T \bar{z}_j$$

$$j = 1, 2, \ldots, N \quad . \tag{4-6}$$

Then with D as defined in (4-2), it follows that the $\bar{z}_j$ are the N linearly independent eigenvectors of D, with corresponding eigenvalues λ_j .

For later use, let Z be the N×N matrix whose jth column is the eigenvector $\bar{z}_j$,

$$Z = \left[\bar{z}_1 \; , \; \bar{z}_2 \; , \; \bar{z}_3 \; , \; \ldots \; , \; \bar{z}_N\right] \quad . \tag{4-7}$$

Since the $\bar{z}_j$ are linearly independent, the matrix Z is nonsingular. Also, from (4-5) and (4-7) it follows that

$$Z = \left[Z_{jn}\right] = e^{\frac{2\pi i}{N}(j-1)(n-1)} \quad , \tag{4-8}$$

so that Z is a symmetric matrix.

Transformation of the Recurrence Relation

Referring to equations (3-29) and (3-30) and to the definition (4-1) of a circulant matrix, it is obvious that both A_{kk} and $A_{k,k+1}$ are Nth order circulant matrices. It follows that for all k these matrices have the same linearly independent set of eigenvectors $\{\bar{z}_j : j = 1,2, \ldots, N\}$ as defined in (4-5). This property can be used to reduce the matrix recurrence relation (3-25) to a much simpler form.

The recurrence relation (3-25) can be written as

$$A^T_{k-1,k}\,\bar{u}_{k-1} + A_{kk}\,\bar{u}_k + A_{k,k+1}\,\bar{u}_{k+1} = 0 \qquad k = 2,3,4,\ldots \,. \tag{4-9}$$

Premultiplying (4-9) by Z^{-1} gives

$$\left(Z^{-1}A^T_{k-1,k}\right)\bar{u}_{k-1} + \left(Z^{-1}A_{kk}\right)\bar{u}_k + \left(Z^{-1}A_{k,k+1}\right)\bar{u}_{k+1} = 0 \qquad k = 2,3,4,\ldots \,. \tag{4-10}$$

Recall from (4-7) that the columns of Z are the eigenvectors of the circulant matrices $A^T_{k-1,k}$, A_{kk} and $A_{k,k+1}$. Thus, for example, $Z^{-1}A_{kk}Z$ would define a similarity transformation for the matrix A_{kk} [10, p. 344]. It is advantageous, then, to introduce the transformation

$$\bar{u}_k = Z\,\bar{v}_k \qquad k = 1,2,3,\ldots \,. \tag{4-11}$$

Substituting (4-11) into (4-10) gives a matrix recurrence relation for $\bar{v}_k$,

$$\left(Z^{-1}A^T_{k-1,k}Z\right)\bar{v}_{k-1} + \left(Z^{-1}A_{kk}Z\right)\bar{v}_k + \left(Z^{-1}A_{k,k+1}Z\right)\bar{v}_{k+1} = 0 \qquad k = 2,3,4,\ldots \,. \tag{4-12}$$

Let Λ_k be the $N \times N$ diagonal matrix whose jth diagonal element is the eigenvalue $\lambda_k^{\,j}$ of A_{kk} corresponding to the eigenvector $\bar{z}_j$,

$$\Lambda_k = \mathrm{diag}\left[\lambda_k^{1} \ , \ \lambda_k^{2} \ , \ . \ . \ . \ , \ \lambda_k^{N}\right]$$

$$k = 1,2,3, \ldots \quad . \qquad (4\text{-}13)$$

From the definition of the matrix Z in (4-7) and the properties of similarity transformations [10, pp. 344-347], it follows that

$$Z^{-1} A_{kk} Z = \Lambda_k \quad . \qquad (4\text{-}14)$$

Let Γ_k be the diagonal matrix whose jth diagonal element is the eigenvalue γ_k^{j} of $A_{k,k+1}$ corresponding to the eigenvector $\overline{z}_j$. Then

$$\Gamma_k = \mathrm{diag}\left[\gamma_k^{1} \ , \gamma_k^{2} \ , \ . \ . \ . \ , \gamma_k^{N}\right]$$

$$k = 1,2,3, \ldots \qquad (4\text{-}15)$$

and

$$Z^{-1} A_{k,k+1} Z = \Gamma_k \quad . \qquad (4\text{-}16)$$

It will be shown below that the eigenvalues of $A_{k,k+1}^{T}$ are the complex conjugates of the corresponding eigenvalues of $A_{k,k+1}$. Denoting by $\overset{*}{\gamma}_k{}^{j}$ the complex conjugate of γ_k^{j} and by Γ_k^{*} the matrix whose elements are the complex conjugates of the elements in Γ_k , then

$$\Gamma_k^{*} = \mathrm{diag}\left[\overset{*}{\gamma}_k{}^{1} \ , \overset{*}{\gamma}_k{}^{2} \ , \ . \ . \ . \ , \overset{*}{\gamma}_k{}^{N}\right]$$

$$k = 1,2,3, \ldots \qquad (4\text{-}17)$$

and

$$Z^{-1} A^{T}_{k-1,k} Z = \Gamma^{*}_{k-1} \qquad k = 2,3,4,\ldots \quad . \tag{4-18}$$

From (4-14), (4-16) and (4-18), the matrix recurrence relation (4-12) reduces to

$$\Gamma_k \bar{v}_{k+1} + \Lambda_k \bar{v}_k + \Gamma^{*}_{k-1} \bar{v}_{k-1} = 0$$
$$k = 2,3,4,\ldots \quad . \tag{4-19}$$

The transformation $\bar{u}_k = Z \bar{v}_k$ has reduced the matrix recurrence relation (4-9) for $\bar{u}_k$ to the matrix recurrence relation (4-19) for $\bar{v}_k$. This is a considerable simplification since the matrices in (4-19) are diagonal. Denote the N components of $\bar{v}_k$ by $v_k{}^j$ so that

$$\bar{v}_k = \begin{bmatrix} v_k{}^1 \\ v_k{}^2 \\ \vdots \\ v_k{}^N \end{bmatrix} \qquad k = 1,2,3,\ldots \quad . \tag{4-20}$$

Then from (4-13), (4-15), (4-17) and (4-20), it follows that the matrix recurrence relation (4-19) reduces to N uncoupled difference equations for the components of $\bar{v}_k$,

$$\left.\begin{array}{l}
\gamma_k^{\ 1}\ v_{k+1}^{\ 1} + \lambda_k^{\ 1}\ v_k^{\ 1} + \overset{*}{\gamma}{}_{k-1}^{\ 1}\ v_{k-1}^{\ 1} = 0 \\
\qquad\qquad\qquad\qquad k = 2,3,4,\ldots \\
\gamma_k^{\ 2}\ v_{k+1}^{\ 2} + \lambda_k^{\ 2}\ v_k^{\ 2} + \overset{*}{\gamma}{}_{k-1}^{\ 2}\ v_{k-1}^{\ 2} = 0 \\
\qquad\qquad\qquad\qquad k = 2,3,4,\ldots \\
\vdots \\
\gamma_k^{\ N}\ v_{k+1}^{\ N} + \lambda_k^{\ N}\ v_k^{\ N} + \overset{*}{\gamma}{}_{k-1}^{\ N}\ v_{k-1}^{\ N} = 0 \\
\qquad\qquad\qquad\qquad k = 2,3,4,\ldots
\end{array}\right\} \quad . \tag{4-21}$$

These are linear, second order, homogeneous difference equations with variable coefficients. It is to be emphasized that these N difference equations are uncoupled, so that each may be solved separately.

Recall that the coefficients $\lambda_k^{\ j}$ in (4-21) are the eigenvalues of the matrix A_{kk} corresponding to the eigenvectors $\overline{z}_j$. From (3-29), (4-2) and (4-6), these eigenvalues are given by

$$\lambda_k^{\ j} = a_k + b_k\ e^{\frac{2\pi i}{N}(j-1)} + b_k\ e^{\frac{2\pi i}{N}(j-1)(N-1)} \quad , \tag{4-22}$$

which reduces to the real numbers

$$\lambda_k^{\ j} = a_k + 2b_k \cos(2\delta_j) \qquad \begin{array}{l} j = 1,2,\ldots,N \\ k = 1,2,3,\ldots, \end{array} \tag{4-23}$$

where

$$\delta_j = (j-1)\ \frac{\pi}{N} \quad . \tag{4-24}$$

The coefficients $\gamma_k^{\,j}$ in (4-21) are the eigenvalues of the matrix $A_{k,k+1}$ so that from (3-30), (4-2), (4-6) and (4-24),

$$\gamma_k^{\,j} = \alpha_k + \beta_k \, e^{2i\delta_j} \quad . \tag{4-25}$$

Similarly, from (3-30) it follows that the eigenvalues of $A_{k,k+1}^{T}$ are

$$\alpha_k^{\,j} + \beta_k \, e^{2i(N-1)\delta_j} \quad ,$$

which reduces to $\overset{*}{\gamma}_k^{\,j}$ as stated earlier.

Solution for the Laplace Equation

The next step in this method is to find the general solution of each of the difference equations in (4-21). From these general solutions, the general solution $\bar{u}_k$ of the original matrix recurrence relation (3-25) can be constructed. By imposing the boundary condition at infinity on this solution, we hope to obtain a relationship between the unknowns $\bar{u}_1$ and $\bar{u}_2$ on the circles C_1 and C_2 which leads to the proper behavior at infinity. This process is most easily illustrated for the case of the Laplace equation, so this case will be considered first.

The Laplace equation is simpler in two respects. First, the appropriate boundary condition at infinity is that the solution be bounded [11, pp. 82-83]. Since from

(4-11) $\bar{u}_k = Z\,\bar{v}_k$, it follows that the values $\bar{u}_k$ on the grid circles C_k will be bounded as $k \to \infty$ iff $\bar{v}_k$ is bounded as $k \to \infty$. From (4-20) this implies that each of the components $v_k{}^j : j = 1,2, \ldots , N$ must be bounded as $k \to \infty$. Also, the difference equations to be solved have constant coefficients. This follows from the fact that the matrices A_{kk} and $A_{k,k+1}$ in the original matrix recurrence relation (4-9) are constant (i.e., independent of k).

From (3-52) through (3-55) and (4-23) and (4-25), the eigenvalues of A_{kk} and $A_{k,k+1}$ are

$$\lambda_k{}^j = \lambda_j = -\ 2\gamma_j + \frac{1}{\varepsilon_2}\left(R - \frac{1}{R}\right)\left[1 - \cos\left(2\delta_j\right)\right] \qquad j = 1,2, \ldots , N , \qquad (4\text{-}26)$$

$$\gamma_k{}^j = \gamma_j = -\ \frac{\varepsilon_1}{2}\left(\frac{R+1}{R-1}\right) \qquad j = 1,2, \ldots , N . \qquad (4\text{-}27)$$

The k subscripts have been suppressed since the eigenvalues are independent of k . Also note that these eigenvalues are real. It follows from (4-21), (4-26) and (4-27) that the N difference equations to be solved in the case of the Laplace equation are

$$v_{k+1}^j - 2\,\alpha_j\,v_k{}^j + v_{k-1}^j = 0 \qquad k = 2,3,4, \ldots \qquad (4\text{-}28)$$

for each $j = 1,2, \ldots , N$, where

$$a_j = -\frac{\lambda_j}{2\gamma_j} = 1 + \frac{2(R-1)^2 \sin^2(\delta_j)}{\varepsilon_1 \; \varepsilon_2 \; R} \geq 1 \quad . \tag{4-29}$$

The equality holds only for j = 1 . Note in (4-28) that the subscript k is the difference variable whereas the j indices merely indicate one of the N difference equations. The procedure for solving linear, homogeneous, constant coefficient difference equations is well established (see, for example [12, pp. 152-154]). It can be shown that the general solutions of the N difference equations (4-28) are

$$v_k^{\,j} = \sigma_j \, s_k^{\,j} + \tau_j \, t_k^{\,j}$$

$$k = 1,2,3, \ldots$$

$$j = 1,2, \ldots , N \quad , \tag{4-30}$$

where the coefficients σ_j and τ_j are arbitrary constants independent of k . The linearly independent solutions $s_k^{\,j}$ and $t_k^{\,j}$ are defined as follows:

$$s_k^{\,j} = (u_j)^k \qquad k = 1,2,3,\ldots \qquad j = 1,2, \ldots , N \;; \tag{4-31}$$

$$0 < u_j = a_j - \sqrt{(a_j)^2 - 1} \leq 1 \qquad j = 1,2, \ldots , N \;; \tag{4-32}$$

$$t_k^{\,1} = k \qquad k = 1,2,3, \ldots \;; \tag{4-33}$$

$$t_k^{\,j} = (\nu_j)^k \qquad k = 1,2,3, \ldots \qquad j = 2,3, \ldots , N \;; \tag{4-34}$$

$$\nu_j = a_j + \sqrt{(a_j)^2 - 1} > 1 \qquad j = 2,3, \ldots , N \;. \tag{4-35}$$

The equality in (4-32) holds only for j = 1 .

Having obtained the general solutions (4-30) of the difference equations (4-28), the boundedness condition at infinity is now imposed. Recall that this requires the $v_k{}^j$ to be bounded as $k \to \infty$. From (4-30) through (4-35) it follows that the $v_k{}^j$ are bounded as $k \to \infty$ iff

$$\tau_j = 0 \qquad\qquad j = 1,2,\ldots,N \, . \qquad (4\text{-}36)$$

This gives

$$v_k{}^j = \sigma_j \, s_k{}^j \qquad k = 1,2,3,\ldots \qquad j = 1,2,\ldots,N \, , \qquad (4\text{-}37)$$

from which it follows that $\sigma_j = v_1{}^j / s_1{}^j$ and thus

$$v_k{}^j = v_1{}^j \, \frac{s_k{}^j}{s_1{}^j} \qquad k = 1,2,3,\ldots \qquad j = 1,2,\ldots,N \, . \qquad (4\text{-}38)$$

Then from (4-31) and (4-38), the only solutions of the difference equations (4-28) which are bounded at infinity are

$$v_k{}^j = v_1{}^j \, (\mu_j)^{k-1} \qquad k = 1,2,3,\ldots \qquad j = 1,2,\ldots,N \, , \qquad (4\text{-}39)$$

where the first terms $v_1{}^j$ are arbitrary. From (4-20) and (4-39), the solution of the matrix recurrence relation (4-19) which is bounded at infinity is

$$\bar{v}_k = \Omega_k \, \bar{v}_1 \quad , \tag{4-40}$$

where $\bar{v}_1$ is arbitrary and Ω_k is the diagonal matrix

$$\Omega_k = \text{diag}\left[(\mu_1)^{k-1} \, , \, (\mu_2)^{k-1} \, , \, . \, . \, . \, , \, (\mu_N)^{k-1} \right] \quad . \tag{4-41}$$

Finally, from (4-41) and $\bar{u}_k = Z \, \bar{v}_k$, it follows that the solution of the original matrix recurrence relation (4-9) which is bounded at infinity is

$$\bar{u}_k = Z \, \Omega_k \, \bar{v}_1 = \sum_{j=1}^{N} v_1{}^j \, (\mu_j)^{k-1} \, \bar{z}_j \quad , \quad k = 1,2,3, \ldots \tag{4-42}$$

which can also be written as

$$\bar{u}_k = Z \, \Omega_k \, Z^{-1} \, \bar{u}_1 \qquad k = 1,2,3, \ldots \quad . \tag{4-43}$$

The relationship between the unknowns $\bar{u}_1$ and $\bar{u}_2$ on the grid circles C_1 and C_2 which leads to bounded solutions at infinity is given implicitly from (4-42) by

$$\left. \begin{aligned} \bar{u}_1 &= Z \, \bar{v}_1 = \sum_{j=1}^{N} v_1{}^j \, \bar{z}_j \\ \bar{u}_2 &= Z \, \Omega_2 \, \bar{v}_1 = \sum_{j=1}^{N} v_1{}^j \, \mu_j \, \bar{z}_j \end{aligned} \right\} \tag{4-44}$$

and explicitly from (4-43) by

$$\overline{u}_2 = Z\ \Omega_2\ Z^{-1}\ \overline{u}_1 \quad . \tag{4-45}$$

The first equation (3.24) of the infinite system of equations (3-23), when added to (4-45), gives the following system of equations for $\overline{u}_1$ and $\overline{u}_2$ for the case of a circular boundary:

$$\left.\begin{aligned} A_{11}\ \overline{u}_1 + A_{12}\ \overline{u}_2 &= \overline{h}_1 \\ \overline{u}_2 &= Z\ \Omega_2\ Z^{-1}\ \overline{u}_1 \end{aligned}\right\} \quad . \tag{4-46}$$

The unknowns $\overline{u}_3$, $\overline{u}_4$, $\overline{u}_5$, . . . , on the grid circles C_3 , C_4 , C_5 , . . . , can then be determined from (4-43). Note that (4-43) and (4-46) require the inverse of the Z matrix. Although this has not been given, an explicit formula should be obtainable due to the simple structure (4-8) of Z . Actually it is better to work with the form (4-42) since fewer operations are required. Substituting $\overline{u}_1$ and $\overline{u}_2$ from (4-44) into (3-24) gives, for the case of a circular boundary,

$$\sum_{j=1}^{N} \left[v_1{}^j \left(A_{11}\ \overline{z}_j \right) + v_1{}^j\ \mu_j \left(A_{12}\ \overline{z}_j \right) \right] = \overline{h}_1 \quad . \tag{4-47}$$

This reduces to the system of equations

$$\sum_{j=1}^{N} v_1{}^j \left(\lambda_j + \mu_j\ \gamma_j \right) \overline{z}_j = \overline{h}_1 \quad , \tag{4-48}$$

since the $\overline{z}_j$ are eigenvectors of A_{11} and A_{12} . The system of equations (4-48) can also be written as

$$\bar{v}_1 = \Pi_2 \; Z^{-1} \; \bar{h}_1 \quad , \tag{4-49}$$

where

$$\Pi_2 = (\Lambda_1 + \Gamma_1 \; \Omega_2)^{-1}$$

$$= \mathrm{diag}\left[\frac{1}{\lambda_1 + \mu_1 \; \gamma_1} , \frac{1}{\lambda_2 + \mu_2 \; \gamma_2} , \; . \; . \; . \; , \frac{1}{\lambda_N + \mu_N \; \gamma_N}\right] . \tag{4-50}$$

Once $\bar{v}_1$ has been solved for, the $\bar{u}_k$ can be determined from the simple formula (4-42).

Solution for the Helmholtz Equation

The general solutions of the N difference equations (4-21) are now sought for the case of the Helmholtz equation. The coefficients are more complicated than in the case of the Laplace equation, but the overall procedure is the same. By imposing the outward radiation condition on the solutions $\bar{v}_k$, we hope to eventually obtain a relationship between $\bar{u}_1$ and $\bar{u}_2$ which leads to solutions satisfying the outward radiation condition.

From (3.60) through (3-63) and (4.23) through (4.25), it can be shown that for the case of the Helmholtz equation with constant radial spacing Δr , the eigenvalues of A_{kk} and $A_{k,k+1}$ are

$$\lambda_k{}^j = \frac{2\,\varepsilon_1}{\Delta r}\, r_k + \frac{4\,\Delta r}{\varepsilon_2} \sin^2(\delta_j)\, \frac{1}{r_k} - \frac{\omega^2\, \varepsilon_2\, \Delta r}{3} \left(1 + \cos^2(\delta_j)\right) r_k \tag{4-51}$$

and

$$\gamma_k^{\ j} = - \varepsilon_1 \left(\frac{r_k}{\Delta r} + \frac{1}{2} \right) - \frac{\omega^2 \ \varepsilon_2 \ \Delta r}{24} \left(1 + e^{2i\delta_j}\right) \left(2 \ r_k + \Delta r\right) \qquad (4\text{-}52)$$

for $k = 1,2,3, \ldots$, and $j = 1,2, \ldots , N$. From (3-59), $r_k = r_0 + k \ \Delta r$. Define the grid parameter ρ to be the ratio of the radius of the circle C_0 to the radial spacing Δr , so that

$$\rho = \frac{r_0}{\Delta r} \qquad (4\text{-}53)$$

and

$$r_k = \frac{1}{\Delta r} (k + \rho) \quad . \qquad (4\text{-}54)$$

Then from (4-51), (4-52) and (4-53), the eigenvalues can be written as

$$\lambda_k^{\ j} = - 2 \ a_j (k + \rho) + \frac{b_j}{k + \rho} \qquad k = 1,2,3, \ldots \qquad (4\text{-}55)$$

$$\gamma_k^{\ j} = d_j (k + \rho + \tfrac{1}{2}) \qquad k = 1,2,3, \ldots \qquad (4\text{-}56)$$

where

$$a_j = - \varepsilon_1 + \frac{\varepsilon_2}{6} (\omega \Delta r)^2 \left[1 + \cos^2(\delta_j)\right] \quad , \qquad (4\text{-}57)$$

$$b_j = \frac{4}{\varepsilon_2} \sin^2(\delta_j) \quad , \qquad (4\text{-}58)$$

and

$$d_j = - \varepsilon_1 - \frac{\varepsilon_2}{12} (\omega \Delta r)^2 \left[1 + e^{2i\delta_j}\right] \quad , \qquad (4\text{-}59)$$

for $j = 1,2, \ldots, N$. From (4-55) and (4-56) the N difference equations (4-21) can be written as

$$d_j(k+\rho+\tfrac{1}{2})v_{k+1}^j - \left[2\,a_j(k+\rho) - b_j(k+\rho)^{-1}\right]v_k^j$$
$$+ d_j^*\,(k+\rho-\tfrac{1}{2})v_{k-1}^j = 0 \qquad k = 2,3,4,\ldots \tag{4-60}$$

for $j = 1,2, \ldots, N$. Again note that the subscript k is the difference variable whereas the j subscript merely indicates one of the N difference equations.

It will be assumed that for each $j = 1,2,\ldots, N$, there exists 2 real numbers θ_j and $\hat{\theta}_j$ and 2 unique solutions, s_k^j and t_k^j, of (4-60) satisfying:

$$|s_1^j| = |t_1^j| = 1 \quad ; \quad 0 \leq \arg(s_1^j)\ ,\ \arg(t_1^j) < \pi \quad ; \tag{4-61}$$

$$s_k^j \neq 0 \quad , \quad t_k^j \neq 0 \qquad k = 2,3,4,\ldots\ ; \tag{4-62}$$

$$\lim_{k\to\infty} |s_k^j| = 0 \quad , \quad \lim_{k\to\infty} |t_k^j| = 0 \quad ; \tag{4-63}$$

$$\arg(s_k^j) \underset{k\to\infty}{\sim} \theta_j\, k \qquad 0 < \theta_j < \pi \quad ; \tag{4-64}$$

$$\arg(t_k^j) \underset{k\to\infty}{\sim} -\hat{\theta}_j\, k \quad , \quad 0 < \hat{\theta}_j < \pi \quad . \tag{4-65}$$

Here $|s_k^j|$ denotes the modulus or amplitude of s_k^j and $\arg\,(s_k^j)$ denotes the argument or phase of s_k^j. Conditions (4-62) through (4-65) do not determine s_k^j and t_k^j uniquely. For this reason, and for later convenience, condition (4-61) has been included. The validity of assuming solutions of this form will be discussed in the next section. It follows from the asymptotic properties (4-64) and (4-65) that these solutions are linearly independent. Therefore any solution $v_k^{\,j}$ of (4-60) can be written as a linear combination

$$v_k^{\,j} = \sigma_j \, s_k^{\,j} + \tau_j \, t_k^{\,j} \qquad k = 1,2,3, \ldots \qquad j = 1,2, \ldots , N \quad . \tag{4-66}$$

From (4-63), (4-64) and (4-65),

$$s_k^{\,j} \underset{k \to \infty}{\sim} |s_k^{\,j}| \, e^{i\theta_j k} \quad , \quad 0 < \theta_j < \pi \qquad j = 1,2, \ldots , N \tag{4-67}$$

and

$$t_k^{\,j} \underset{k \to \infty}{\sim} |t_k^{\,j}| \, e^{-i\hat{\theta}_j k} \quad , \quad 0 < \hat{\theta}_j < \pi \qquad j = 1,2, \ldots , N \quad , \tag{4-68}$$

where $|s_k^{\,j}|$ and $|t_k^{\,j}|$ decay to zero as $k \to \infty$. Multiplying by the harmonic time factor $e^{-i\omega t}$ gives

$$s_k^{\,j} \, e^{-i\omega t} \underset{k \to \infty}{\sim} |s_k^{\,j}| \, e^{i(\theta_j k - \omega t)} \qquad j = 1,2, \ldots , N \tag{4-69}$$

and

$$\tau_k^{\,j} \, e^{-i\omega t} \underset{k \to \infty}{\sim} |\tau_k^{\,j}| \, e^{-i(\hat{\theta}_j k + \omega t)} \qquad j = 1,2, \ldots , N \quad . \tag{4-70}$$

Thus $s_k^{\,j}$ behaves like an outgoing wave and $t_k^{\,j}$ behaves like an incoming wave. It follows from (4-66) that $v_k^{\,j}$ is an outgoing wave iff $\tau_j = 0$. Furthermore, from (4-11) and (4-20), $\bar{u}_k$ is an outgoing wave iff each of the $v_k^{\,j}$: $j = 1,2, \ldots, N$ are outgoing waves. Therefore the solution $\bar{u}_k$ of the original matrix recurrence relation (4-9) behaves like an outgoing wave iff in (4-66)

$$\tau_j = 0 \qquad\qquad j = 1,2, \ldots, N \quad . \tag{4-71}$$

As in the case of the Laplace equation, (4-11), (4-20), (4-61), (4-66) and (4-71) require that, for arbitrary $\bar{v}_1$,

$$\bar{u}_k = Z\ \Omega_k\ \bar{v}_1 = \sum_{j=1}^{N} v_1^{\,j}\ \frac{s_k^{\,j}}{s_1^{\,j}}\ \bar{z}_j \quad , \tag{4-72}$$

which can also be written, for arbitrary $\bar{u}_1$, as

$$\bar{u}_k = Z\ \Omega_k\ Z^{-1}\ \bar{u}_1 \qquad k = 1,2,3, \ldots \quad . \tag{4-73}$$

Here Ω_k is the diagonal matrix

$$\Omega_k = \mathrm{diag}\left[\frac{s_k^{\,1}}{s_1^{\,1}}\ ,\ \frac{s_k^{\,2}}{s_1^{\,2}}\ ,\ \ldots\ ,\ \frac{s_k^{\,N}}{s_1^{\,N}}\right]$$

$$k = 1,2,3, \ldots \quad . \tag{4-74}$$

Equation (4-72) defines an implicit relationship between $\bar{u}_1$ and $\bar{u}_2$ whereas (4-73) gives $\bar{u}_2$ directly in terms of $\bar{u}_1$. As before, for the case of circular boundary the first equation (3-24) of the infinite system of equations (3-23), when added to (4-73) for $k = 2$, gives the system of equations

$$\left.\begin{aligned} A_{11}\,\bar{u}_1 + A_{12}\,\bar{u}_2 &= \bar{h}_1 \\ \bar{u}_2 &= Z\,\Omega_2\,Z^{-1}\,\bar{u}_1 \end{aligned}\right\} \quad . \tag{4-75}$$

Once the unknowns $\bar{u}_1$ and $\bar{u}_2$ on the grid circles C_1 and C_2 have been solved for, the remaining unknowns $\bar{u}_3$, $\bar{u}_4$, $\bar{u}_5$, . . . , can be determined from (4-73). However, as previously noted, the form (4-72) actually gives a simpler expression for the $\bar{u}_k$, so that it would be worthwhile to solve instead for $\bar{v}_1$. For the case of a circular boundary, equation (3-24), coupled with (4-72) for $k = 1$ and $k = 2$, gives the following system of equations for the $v_1{}^j$,

$$\sum_{j=1}^{N} v_1{}^j \left(\lambda_1{}^j + \frac{s_j{}^2}{s_1{}^j}\,\gamma_1{}^j\right)\bar{z}_j = \bar{h}_1 \quad . \tag{4-76}$$

This can be expressed in matrix form as

$$\bar{v}_1 = \Pi_2\,Z^{-1}\,\bar{h}_1 \quad , \tag{4-77}$$

where

$$\begin{aligned} \Pi_2 &= (\Lambda_1 + \Gamma_1\,\Omega_2)^{-1} \\ &= \operatorname{diag}\left[\frac{1}{\lambda_1{}^1 + \frac{s_2{}^1}{s_1{}^1}\gamma_1{}^1}\,,\ \frac{1}{\lambda_1{}^2 + \frac{s_2{}^2}{s_1{}^2}\gamma_1{}^2}\,,\ \ldots\,,\ \frac{1}{\lambda_1{}^N + \frac{s_2{}^N}{s_1{}^N}\gamma_1{}^N}\right] . \end{aligned} \tag{4-78}$$

Asymptotic Form of the Difference Equations

The results of the previous section are based on the assumption that for each $j = 1,2, \ldots, N$ the difference equation (4-60) has 2 solutions s_k^j and t_k^j which satisfy conditions (4-61) through (4-65). Furthermore, in order to apply the method, it is necessary to actually construct a solution s_k^j of (4-60) which satisfies (4-61) through (4.64). An analytical proof that solutions of the required form exist has not yet been found. However, based on the assumption that they did exist, a method for obtaining approximations to such solutions was developed. The approximations to s_k^j and t_k^j are in very good agreement with conditions (4-60) through (4-65), thus giving some validity to the above assumptions. The method for generating the approximations to s_k^j and t_k^j is discussed in the next section.

One reason for assuming the difference equations (4-60) have solutions s_k^j and t_k^j satisfying (4-61) through (4-65) is that this allows the outward radiation condition to be easily applied to the finite element approximations. In fact, as shown in the last section, solutions behaving like outgoing waves or incoming waves can then be determined by inspection. The technique is similar to that used for obtaining exact solutions of the Helmholtz equation outside circular boundaries (see Chapter II). In that case it was determined by inspection that Hankel functions of the first kind represent outgoing waves and Hankel functions of the second kind represent incoming waves.

The primary motivation behind the assumption that solutions of (4-60) satisfying (4-61) through (4-65) do exist has come from an examination of the asymptotic form of the difference equations (4-60). Dividing by $k + \rho + 1/2$ and taking the limit as $k \to \infty$ gives the constant coefficient difference equations

$$d_j \, v_{k+1}^j - 2 \, a_j \, v_k^j + d_j^* \, v_{k-1}^j = 0$$

$$j = 1,2, \ldots , N \, . \qquad (4\text{-}79)$$

It should not be expected that, as $k \to \infty$, the solutions of the original difference equations (4-60) will approach the solutions of the asymptotic difference equations (4-79). However, for large k the behavior of the solutions of (4-60) should in some ways be similar to the solutions of (4-79).

It can be shown that if $|a_j| \neq |d_j|$, then each of the N difference equations (4-79) has 2 linearly independent solutions s_k^j and t_k^j of the form

$$s_k^j = (\mu_j)^k \quad , \quad t_k^j = (\nu_j)^k$$

$$j = 1,2, \ldots , N \, , \qquad (4\text{-}80)$$

where

$$\mu_j = \frac{a_j - \sqrt{|a_j|^2 - |d_j|^2}}{d_j}$$

$$j = 1,2, \ldots, N \qquad (4\text{-}81)$$

and

$$\nu_j = \frac{a_j + \sqrt{|a_j|^2 - |d_j|^2}}{d_j}$$

$$j = 1,2, \ldots, N \quad . \qquad (4\text{-}82)$$

If

$$|a_j| < |d_j| \qquad j = 1,2, \ldots, N \qquad (4\text{-}83)$$

then

$$\mu_j = \frac{a_j - i\sqrt{|d_j|^2 - |a_j|^2}}{d_j}$$

$$j = 1,2, \ldots, N \qquad (4\text{-}84)$$

and

$$\nu_j = \frac{a_j + i\sqrt{|d_j|^2 - |a_j|^2}}{d_j}$$

$$j = 1,2, \ldots, N \quad , \qquad (4\text{-}85)$$

so that

$$|\mu_j| = |\nu_j| = 1 \qquad j = 1,2, \ldots, N \quad . \qquad (4\text{-}86)$$

It follows from (4-80), (4-86) and De Moivre's formula that each of the N difference equations (4-79) has 2 linearly independent solutions

$$s_k^{\,j} = e^{i\theta_j k} \quad , \quad t_k^{\,j} = e^{-i\hat{\theta}_j k}$$

$$j = 1,2, \ . \ . \ . \ , N \ , \qquad (4\text{-}87)$$

where

$$\theta_j = \arg(\mu_j) \quad , \quad \hat{\theta}_j = -\arg(\nu_j)$$

$$j = 1,2, \ . \ . \ . \ , N \ . \qquad (4\text{-}88)$$

Equations (4-86), (4-87) and (4-88) depend on the assumption (4-83). From (3-33), (4-24), (4-57) and (4-59) it can be shown that

$$\frac{|a_j|^2 - |d_j|^2}{\left(\frac{\varepsilon_2}{6}\right)^2 (\omega\Delta r)^4} = 1 + \cos^2(\delta_j) + \cos^4(\delta_j)$$

$$- \left[1 + 2\cos^2(\delta_j)\right] \frac{12}{(\omega\Delta r)^2 \cos^2\left(\frac{\pi}{N}\right)}$$

$$\leq \left[1 + 2\cos^2(\delta_j)\right] \left[1 - \frac{12}{(\omega\Delta r)^2 \cos^2\left(\frac{\pi}{N}\right)}\right]$$

$$(4\text{-}89)$$

for each $j = 1,2, \ . \ . \ . \ , N$, where the equality holds for $j = 1$. It follows that

$$\frac{12}{(\omega\Delta r)^2 \cos^2\left(\frac{\pi}{N}\right)} > 1 \quad \Rightarrow \quad |a_j| < |d_j| \quad , \qquad (4.90)$$

and thus

$$(\omega\Delta r) \cos\left(\frac{\pi}{N}\right) < 2\sqrt{3} \quad \Rightarrow \quad |a_j| < |d_j|$$

$$j = 1,2, \ . \ . \ . \ , N \ . \qquad (4\text{-}91)$$

Since $\cos\left(\frac{\pi}{N}\right) \to 1$ as $N \to \infty$, condition (4-91) is satisfied for any number of radial lines N if

$$\omega\Delta r < 2\sqrt{3} = 3.4641 \ . \ . \ . \qquad . \qquad (4\text{-}92)$$

However, recall from (3-58) that in order to correctly model the oscillatory behavior of the solutions, Δr should be chosen small enough that $\omega\Delta r \leq \pi$, which obviously satisfies (4-92). Therefore equation (4-83) is valid for any number of radial lines N and any reasonable radial spacing Δr , in particular for Δr satisfying $\omega\Delta r \leq \pi$.

Referring to (4-88), it can also be shown that under the same conditions as in (4-91),

$$0 < \theta_j < \pi \quad , \quad 0 < \hat{\theta}_j < \pi$$

$$j = 1,2, \ . \ . \ . \ , N \ . \qquad (4\text{-}93)$$

Then from (4-87) and (4-93) it follows on multiplication by $e^{-i\omega t}$ that $s_k{}^j$ behaves like an outgoing wave and $t_k{}^j$ behaves like an incoming wave, except for the fact that they do not decay to zero. Referring to conditions (4-61) through (4-65) and to (4-67) and (4-68), the following assumptions are being made. For each $j = 1,2, \ldots, N$, there exists 2 solutions s_k^j and t_k^j of the original difference equation (4.60) which decay to zero as $k \to \infty$. The oscillatory behavior of these solutions approaches the oscillatory behavior of the corresponding solutions (4-87) of the asymptotic difference equation (4-79). In particular, the θ_j and $\hat{\theta}_j$ in (4-64) and (4-65) are as defined in (4-84), (4-85) and (4-88).

Approximate Solutions of the Difference Equations

The purpose of this section is to describe a method for obtaining solutions s_k^j and t_k^j of (4-60) which satisfy conditions (4-61) through (4-65). Recall that s_k^j behaves like an outgoing wave and t_k^j like an incoming wave. For any $j = 1,2, \ldots, N$, multiplication of (4-60) by $k + \rho$ gives a difference equation whose coefficients are polynomials in k. Methods for finding exact solutions of such equations do exist. However, these solutions are usually expressed in terms of an infinite series. The series solutions obtained by such methods do not necessarily behave like incoming or outgoing waves. Given two linearly independent series solutions, a study of the properties of the infinite series is needed to determine the linear combination that gives solutions of the desired type.

Finally, in actual computations only a finite number of terms in the infinite series can be retained. It was felt, then, that it would not be practical to seek exact solutions of the difference equations (4-60) having the desired characteristics. Rather, a method was needed which directly generated accurate approximations to the solutions s_k^j and t_k^j representing outgoing and incoming waves, as specified by (4-61) through (4-65).

Recall from (4-72) that only the outgoing waves s_k^j are actually used in constructing solutions of the Helmholtz equation. Therefore we will concentrate on obtaining approximations to the solutions s_k^j of (4-60) satisfying (4-61) through (4-64). Referring to (4-21), (4-55), (4-56) and (4-60), the difference equations for the s_k^j can be written concisely as

$$\gamma_k^j \, s_{k+1}^j + \lambda_k^j \, s_k^j + \overset{*}{\gamma}{}_k^j \, s_{k-1}^j = 0 \qquad \begin{array}{l} k = 2,3, \ldots \\ j = 1,2, \ldots , N \; . \end{array} \tag{4-94}$$

For convenience, make the substitutions

$$s_k^j = S_k^j \, e^{i\theta_j k} \qquad j = 1,2, \ldots , N \; , \tag{4-95}$$

where θ_j is as defined in (4-64), (4-84) and (4-88). Then the S_k^j must satisfy the difference equations

$$\left. \begin{array}{l} \left(\gamma_k^{\ j}\, e^{i\theta_j}\right) S_{k+1}^{\ j} + \lambda_k^{\ j}\, S_k^{\ j} + \left(\gamma_k^{\ j}\, e^{i\theta_j}\right)^* S_{k-1}^{\ j} = 0 \\ \qquad k = 2,3,\ldots \qquad\qquad j = 1,2,\ldots,N \end{array} \right\} \quad . \tag{4-96}$$

One reason for the substitution (4-95) is that, from (4-64), the $S_k^{\ j}$ satisfy

$$\arg(S_k^{\ j}) \underset{k\to\infty}{\sim} 0 \qquad j = 1,2,\ldots \;, \tag{4-97}$$

which is equivalent to the simple condition

$$\operatorname{Im}(S_k^{\ j}) \underset{k\to\infty}{\sim} 0 \qquad j = 1,2,\ldots,N \;. \tag{4-98}$$

Also from (4-61) and (4-95) it follows that the $S_k^{\ j}$ must satisfy

$$S_1^{\ j} = e^{-i\theta_j} \qquad j = 1,2,\ldots,N \;. \tag{4-99}$$

Thus, for each $j = 1,2,\ldots,N$, a unique solution of (4-96) is sought which satisfies (4-98) and (4-99). A method for obtaining approximate solutions will now be described. For simplicity, the j superscript has been suppressed in the discussion below.

Two linearly independent solutions T_k and W_k of (4-96) can be generated by specifying the first two terms T_1, T_2 and W_1, W_2 and requiring that

$$T_1 W_2 - T_2 W_1 \neq 0 \;. \tag{4-100}$$

The remaining terms T_3, T_4, ..., and W_3, W_4, ..., can then be determined recursively from (4-96). Any solution S_k of (4-96) is necessarily a linear combination of T_k and W_k. Let

$$S_k = \alpha\, T_k + \beta\, W_k \qquad k = 1,2,3,\ \ldots\ , \tag{4-101}$$

where the complex constants α and β are to be determined from (4-98) and (4-99). For some $k = \bar{k}$, impose the following approximation to (4-98):

$$\mathrm{Im}(S_{\bar{k}-1}) = \mathrm{Im}(S_{\bar{k}}) = \mathrm{Im}(S_{\bar{k}+1}) = 0 \quad . \tag{4-102}$$

These conditions, together with (4-99), determine α and β uniquely. Using R and I subscripts to denote real and imaginary parts, condition (4-102) can be used to determine α_I, β_R, β_I in terms of α_R. From (4-101) and (4-102) the α_I, β_R, β_I satisfy

$$\begin{bmatrix} T_{\bar{k}-1,R} & W_{\bar{k}-1,I} & W_{\bar{k}-1,R} \\ T_{\bar{k},R} & W_{\bar{k},I} & W_{\bar{k},R} \\ T_{\bar{k}+1,R} & W_{\bar{k}+1,I} & W_{\bar{k}+1,R} \end{bmatrix} \begin{bmatrix} \alpha_I \\ \beta_R \\ \beta_I \end{bmatrix} = -\,\alpha_R \begin{bmatrix} T_{\bar{k}-1,I} \\ T_{\bar{k},I} \\ T_{\bar{k}+1,I} \end{bmatrix} \ . \tag{4-103}$$

For arbitrary α_R, the solutions of this system of equations have the form

$$\alpha_I = \alpha_I'\,\alpha_R \ , \quad \beta_R = \beta_R'\,\alpha_R \ , \quad \beta_I = \beta_I'\,\alpha_R \quad . \tag{4-104}$$

From (4-101) and (4-104), α_R can then be determined so that (4-99) is satisfied. Different values of $\bar{k}$

in (4-102) give different coefficients in the equations (4-103) and thus different values for α_I' , β_R' , β_I' in the solutions (4-104). However, as $\bar{k}$ increases, condition (4-102) gives a better approximation to (4-98). It is reasonable to expect that the coefficients α_I' , β_R' , β_I' in (4-104) will converge to some limiting values as $\bar{k} \to \infty$. This appears to be the case, subject to conditions to be stated later.

An example with $\omega = 1$, $r_0 = 1$, $\Delta r = 0.1$ and $N = 12$ is illustrated in Figures 5 and 6 (Appendix A), which plot α_I' , β_R' , β_I' as functions of $\bar{k}$ for $j = 2$. The values of α_I' , β_R' , β_I' for $\bar{k} = 900$ were used, giving the solution S_j^2 shown in Figure 7 (Appendix A). This is in excellent agreement with condition (4-98). The procedure outlined above is carried out for each of the N difference equations. The resulting solutions S_k^j are substituted into (4-95), giving solutions s_k^j of the difference equations (4-60) which satisfy (4-61), (4-62) and (4-63) and give good approximations to (4-64). Note that condition (4-63), which requires $\lim_{k \to \infty} | s_k^j | = 0$, has never been imposed. As discussed below, it appears that all solutions of (4-60) decay to zero.

Under certain conditions, the method outlined above for generating solutions S_k^j of (4-96) satisfying (4-98) may be numerically unstable. In particular, if there are any solutions W_k of (4-96) which do not decay to zero, then due to rounding errors it is impossible to recursively generate a solution T_k of (4-96) which decays to zero by specifying the first 2 terms, even if such a solution exists [13]. Stable techniques are available in these cases, but they involve some additional work [13, 14]. Fortunately, it appears that for all values of the grid parameters in the range of interest, the coefficients in (4-96) are such that all solutions decay to zero. Recall that periodic solutions of the asymptotic difference equations (4-79) exist only when the coefficients a_j and d_j satisfy (4-83), that is, when $|a_j| < |d_j|$. Numerical computations with equations (4-96) indicate that all solutions decay to zero iff $|a_j| < |d_j|$. Recall from (4-92) that this condition is satisfied for any number of radial lines when $\omega\Delta r \leq \pi$.

It is interesting to note that the method used here for generating approximate solutions of the difference equations would not have worked for the case of the Laplace equation since unbounded solutions exist, as shown in (4-28) through (4-35). In this case, if the exact solutions of the difference equations were not known, some of the special techniques for avoiding numerical instability [13, 14] would have to be employed.

CHAPTER V

EXTENSIONS, RECOMMENDATIONS AND CONCLUSIONS

Extension of the Method

As pointed out in Chapters I and III, Thatcher's method is not restricted to Dirichlet boundary conditions or circular boundaries. The purpose of this section is to outline the extension of the method to irregular boundaries and various types of boundary conditions. Some other generalizations are also discussed.

When the boundary C is noncircular, it is enclosed in a circle C_0 of radius r_0 . The infinite grid outside C_0 is the same as before (see Chapter III). In the annular region between C_0 and C , an irregular grid is constructed to fit the boundary C , which is approximated by a polygonal curve. This grid must mate with the infinite grid on the circle C_0 (see Figure 1, Appendix A). The vector of unknowns at the N nodes on C_0 is denoted by $\overline{u}_0$, and the vector of unknowns at the N^- nodes inside C_0 is denoted by $\overline{u}_{-1}$. The infinite system of equations (3-23) is expanded to

$$\left[\begin{array}{cc|ccc} A_{-1-1} & A_{-10} & & & \\ A_{-10}^T & A_{00} & A_{01} & & \\ \hline & A_{01}^T & A_{11} & A_{12} & \\ & & A_{12}^T & A_{22} & A_{23} \\ & & & & \end{array}\right] \begin{bmatrix} \bar{u}_{-1} \\ \bar{u}_0 \\ \bar{u}_1 \\ \bar{u}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \bar{h}_{-1} \\ \bar{h}_0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{5-1}$$

where the matrices inside the dashed lines are those associated with the original system (3-23). In many cases $\bar{h}_0 = 0$. Like (3-23), the infinite system (5-1) is symmetric and block tridiagonal. The matrix A_{01} has the form given in (3-30). The matrix A_{00} has the form given in (3-28); due to the irregular grid its diagonals are no longer constant. The matrices A_{-1-1} and A_{-10} have an irregular form, but if a consistent nodal numbering scheme is used they will be banded. The matrix recurrence relation for the $\bar{u}_k : k = 1,2,3, \ldots$, is now replaced by a matrix recurrence relation for the $u_k : k = 0,1,2, \ldots$. Thus the 1 subscripts in (4-61), (4-72) and related equations are replaced by zero subscripts. Substituting $\bar{u}_0$ and $\bar{u}_1$ from (4-72) into (5-1), and noting that Ω_0 is the identity matrix, gives the finite system of equations

$$\begin{bmatrix} A_{-1-1} & A_{-10}\, Z \\ A_{-10}^{T} & (A_{00}\, Z + A_{01}\, Z\, \Omega_1) \end{bmatrix} \begin{bmatrix} \overline{u}_{-1} \\ \overline{v}_0 \end{bmatrix} = \begin{bmatrix} \overline{h}_{-1} \\ \overline{h}_0 \end{bmatrix} . \qquad (5\text{-}2)$$

The unknowns $\overline{u}_1$, $\overline{u}_2$, $\overline{u}_3$, . . . , are then determined from

$$\overline{u}_k = Z\, \Omega_k\, \overline{v}_0 = \sum_{j=1}^{N} v_0^{\,j}\, s_k^{\,j}\, \overline{z}_j \quad . \qquad (5\text{-}3)$$

The system of equations (5-2) involves $N^- + N$ unknowns. Since, from (4-8), Z is a full matrix, some of the banded structure in the coefficient matrix has been lost.

Thatcher's method can easily be extended to problems in which the boundary is a set of several closed curves. The first grid circle C_0 must then be chosen to enclose all of these boundary curves. The method is not directly applicable to regions outside infinite boundaries.

Extension of the method to different types of boundary conditions on C requires only minor changes. For example, if Neumann or mixed boundary conditions are applied then additional unknowns are introduced at the nodes on C . These unknowns are inserted in the vector $\overline{u}_{-1}$ in (5-1) and (5-2). The right-hand-side vectors $\overline{h}_{-1}$ and $\overline{h}_0$ then involve integrals along C .

Consider the nonhomogeneous Helmholtz equation

$$(\nabla^2 + \omega^2)\, u(\overline{x}) = f(\overline{x}) \quad , \quad \overline{x} \in C^+ \quad . \tag{5-4}$$

Particular solutions can be constructed using the Green's function (C-2) [6, p. 186]. Equation (5-4) can then be reduced to a homogeneous equation. However, if f has finite support then r_0 can be chosen large enough so that $f = 0$ outside the first grid circle C_0 . Thatcher's method can then be applied directly since the nonhomogeneous term f contributes only to the right-hand-side vectors $\overline{h}_{-1}$ and $\overline{h}_0$. If f does not have finite support then there are sources at infinity and a different radiation condition would be required. Assuming f has finite support, taking r_0 large enough so that $f = 0$ outside C_0 may introduce too large an annular region over which the problem has to be solved. Bringing C_0 closer to C results in nonhomogeneous difference equations for the first few equations. With slight modifications, the method outlined in Chapter IV could still be applied in this case.

Variable coefficient equations of the Helmholtz type arise in the propagation of acoustic waves in an inhomogeneous medium [21, pp. 11-12]. Thatcher's method can be directly applied to such equations provided the coefficients reach limiting values inside C_0 , so that

$\nabla^2 u + \omega^2 u = 0$ still applies outside C_0 . In general, if the coefficients in the differential equation vary outside C_0 , the matrices in the recurrence relation will no longer be circulant, since the diagonals will not be constant. However, variable coefficients with radial symmetry will give rise to circulant matrices so that Thatcher's method is applicable in this case.

It has been assumed throughout that the frequency ω in the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ is a real number different from zero. The special case where $\omega = 0$ reduces to the Laplace equation. This had to be considered separately since the solutions have a completely different behavior. If complex ω satisfying

$$\mathrm{Re}(\omega) \geq 0 \quad , \quad \mathrm{Im}(\omega) \geq 0 \quad , \quad \omega \neq 0 \tag{5-5}$$

are considered, then Sommerfeld's outward radiation condition (2-3) is applicable [2]. If $\omega = i\tilde{\omega}$ where $\tilde{\omega}$ is real, then (2-2) becomes $\nabla^2 u - \tilde{\omega}^2 u = 0$. The solutions of this equation behave like the solutions of the Laplace equation. In this case the condition at infinity is merely that the solution decay to zero [11, p. 112]. Assuming complex ω satisfying (5-5), the solutions (4-87) of the asymptotic difference equations (4-79) would no longer be valid. These solutions depend on the fact that the coefficients α_j in (4-79) are real, as shown in (4-57). For complex ω

this is no longer true. Thus some modifications are required in this case.

Finally, the extension of Thatcher's method to three-dimensional problems might also be considered. Thatcher [1] does not discuss this point, and it is not apparent whether the method can be applied at all in this case. In two dimensions, each grid circle C_k circumscribes a regular N-sided polygon P_k, where the number of radial lines N can be chosen arbitrarily. If the circles are replaced by spherical surfaces and the triangular elements by tetrahedral elements, this approach does not appear to generalize to three dimensions.

Miscellaneous Approaches

In extending Thatcher's method to the Helmholtz equation, several approaches were tried which have not been mentioned in Chapters III and IV. In this section some of these approaches are discussed briefly.

Referring to Figure 3, Appendix A, there is a lack of symmetry in the infinite grid due to the fact that the trapezoidal elements are triangulated by drawing only one of the diagonals. This lack of symmetry can be removed if, instead of triangular elements, the original trapezoidal elements are used. In this case the basis functions are no longer linear over an element, but have a distorted bilinear shape. The integrals for determining the coefficients in the infinite system of equations are now much more complicated, and the coefficients in the difference equations involve the logarithm of the difference variable k . Aside from these complications, the trapezoidal elements have the advantage of leading to symmetric circulant matrices. This gives rise to difference equations with real coefficients which can be put in a self-adjoint form. Symmetry can also be obtained by triangulating the trapezoidal elements along both diagonals. Although this introduces a node inside the trapezoid, the unknowns at inside nodes can be eliminated by static condensation.

In an attempt to simplify the difference equations (4-60), a variable spacing Δr_k which approaches a constant value Δr in the limit was investigated. This approach was not successful.

Referring to the matrix recurrence relation (3-25) as restated in (5-9) shows that

$$\bar{u}_{k+1} = - A_{k,k+1}^{-1} \left(A_{kk} \bar{u}_k + A_{k-1,k}^{T} \bar{u}_{k-1}\right)$$

$$k = 2, 3, \ldots \qquad (5\text{-}6)$$

Circulant matrices have the property that the inverse of a circulant matrix is also a circulant matrix. An explicit formula for the inverse of the circulant matrix $A_{k,k+1}$ has been determined but will not be given here. It follows that if a relationship between $\overline{u}_1$ and $\overline{u}_2$ of the form (3-26) can be determined which characterizes outward radiation, then the unknowns on and inside C_2 can be solved for with the remaining unknowns determined from (5-6). In Thatcher's method, the required relationship between $\overline{u}_1$ and $\overline{u}_2$ is determined by finding the general solution of the matrix recurrence relation (5-9). Instead of using (5-6), the solutions are then determined from the simpler formula (4-72). However, if one of the methods discussed in Appendix C were used to determine the relationship (3-26), then the problem of finding the general solution of the matrix recurrence relation (5-9) could have been bypassed. Based on the results in Chapter IV, it appears that (5-6) would be a stable method for calculating $\overline{u}_3$, $\overline{u}_4$, . . . , once $\overline{u}_1$ and $\overline{u}_2$ are determined. Such an approach would probably not be as accurate as the method described in Chapter IV, but further research is needed to verify this. Note that this alternate approach would be numerically unstable in the case of the Laplace equation, since unbounded solutions exist.

Conclusions and Recommendations

In Chapters III and IV techniques were developed to extend Thatcher's method from the Laplace to the Helmholtz equation. All of the essential steps have been outlined, and the method appears very promising. Final judgment as to the success of this method cannot be made until it is applied to several test problems for which the exact solutions are known. Due to time limitations, the program for running such tests has not yet been written. It is highly recommended that further work be directed towards this area.

Further study of the difference equations (4-72) is also recommended. Results concerning the behavior of the solutions of these equations have thus far been based primarily on numerical computations.

Another area which could be explored is the relationship of this method to the discrete Fourier transform. Recall that the method developed here is based on the transformation $\bar{u}_k = Z\,\bar{v}_k$. Referring to the form of the Z matrix as given in (4-8) shows that this transformation has the characteristics of a discrete Fourier transform. Further study in this direction might lead to a better understanding of Thatcher's method.

Other areas for further research have been discussed in the first two sections of this chapter. In particular, the relationship between Thatcher's method and some of the methods described in Appendix C should be investigated.

The primary conclusion of this study is that Thatcher's method can be extended to the Helmholtz equation in two-dimensional unbounded regions. The particular method developed appears to give a consistent way of imposing the outward radiation condition and a very efficient means for determining the solution throughout the infinite region, although the accuracy of the method has not yet been tested.

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APPENDIXES

APPENDIX A

FIGURES

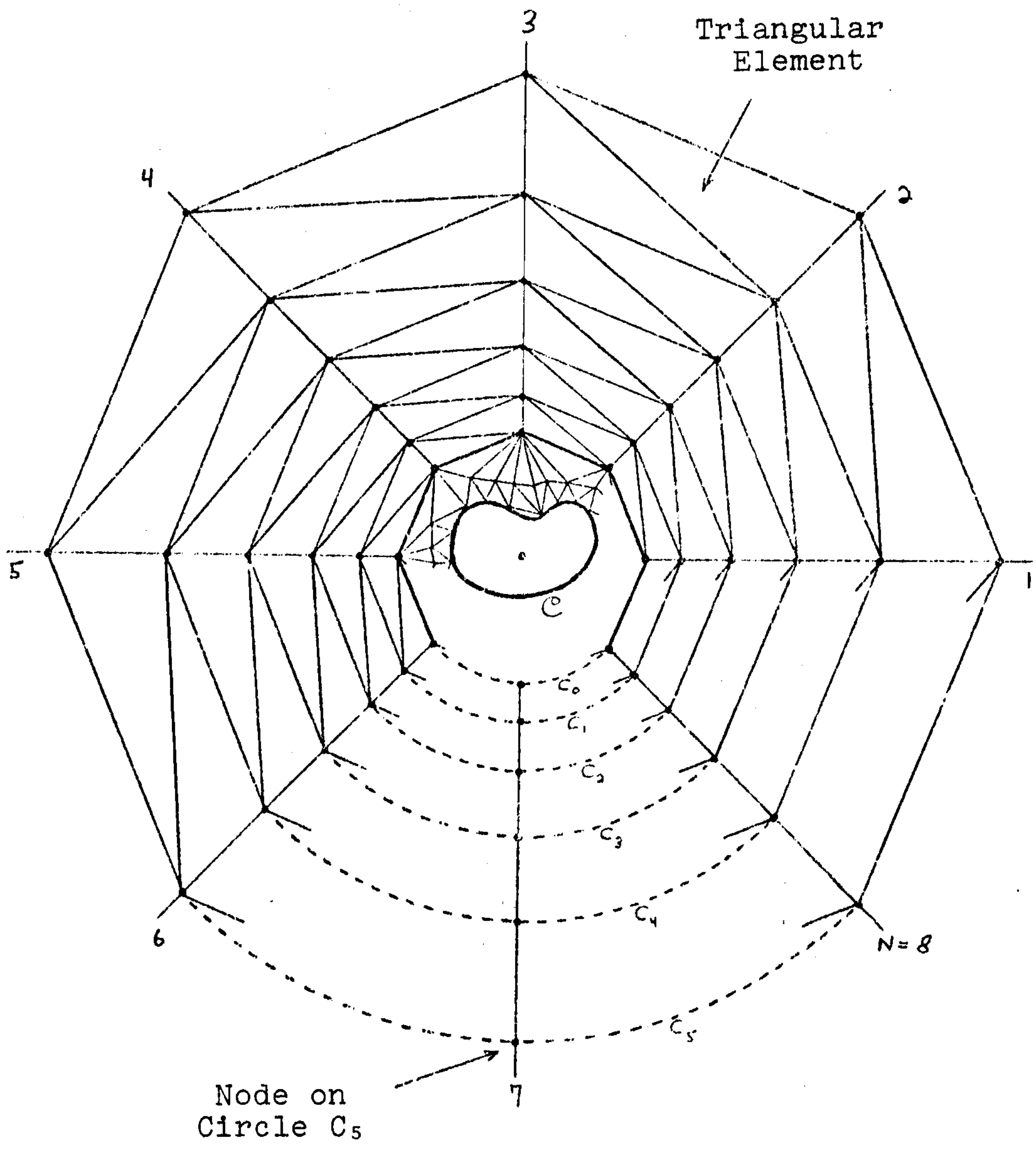

Figure 1. Thatcher's Infinite Grid for Laplace Equation.

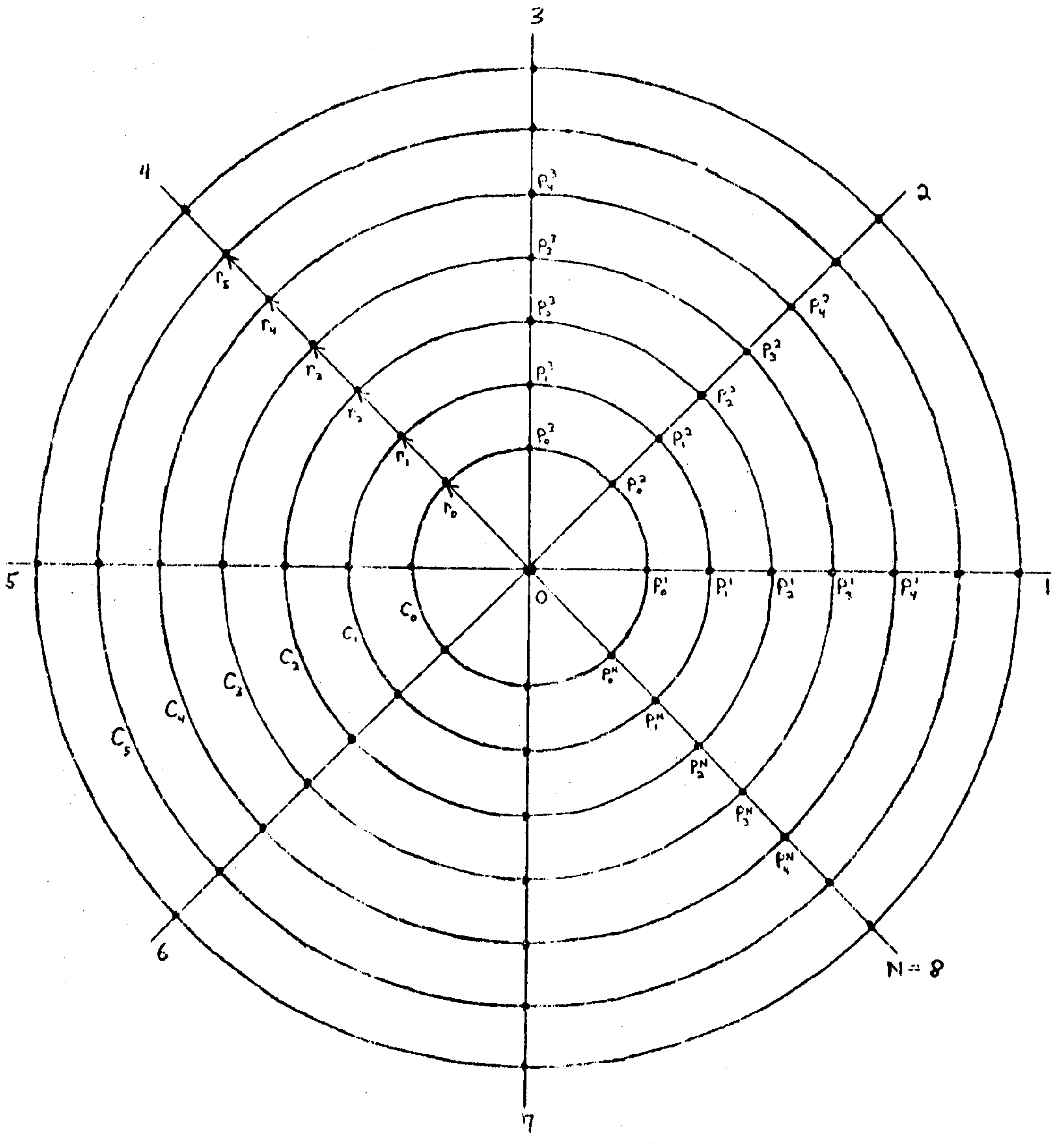

Figure 2. First Stage of Grid Construction.

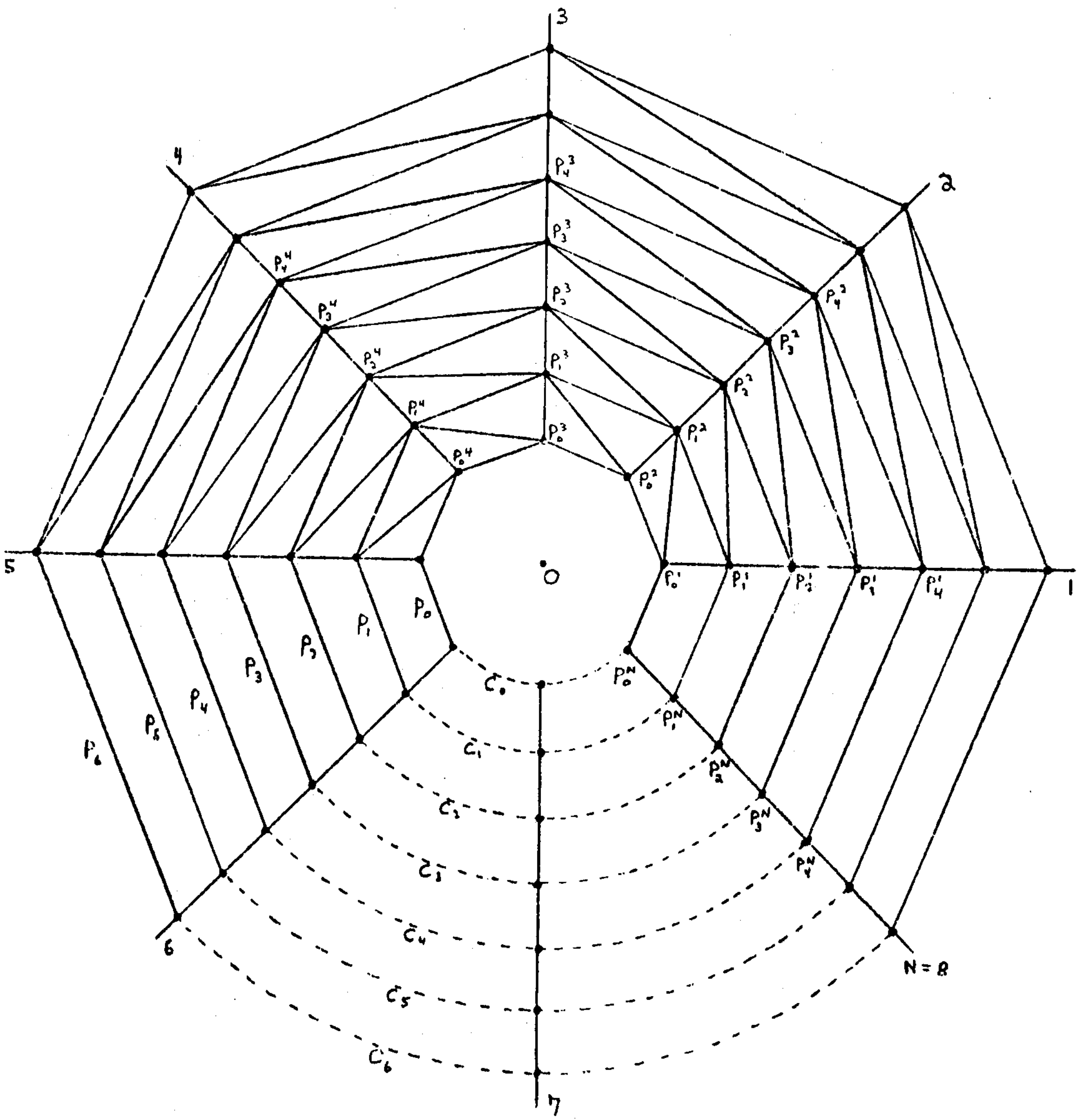

Figure 3. Final Stage of Grid Construction.

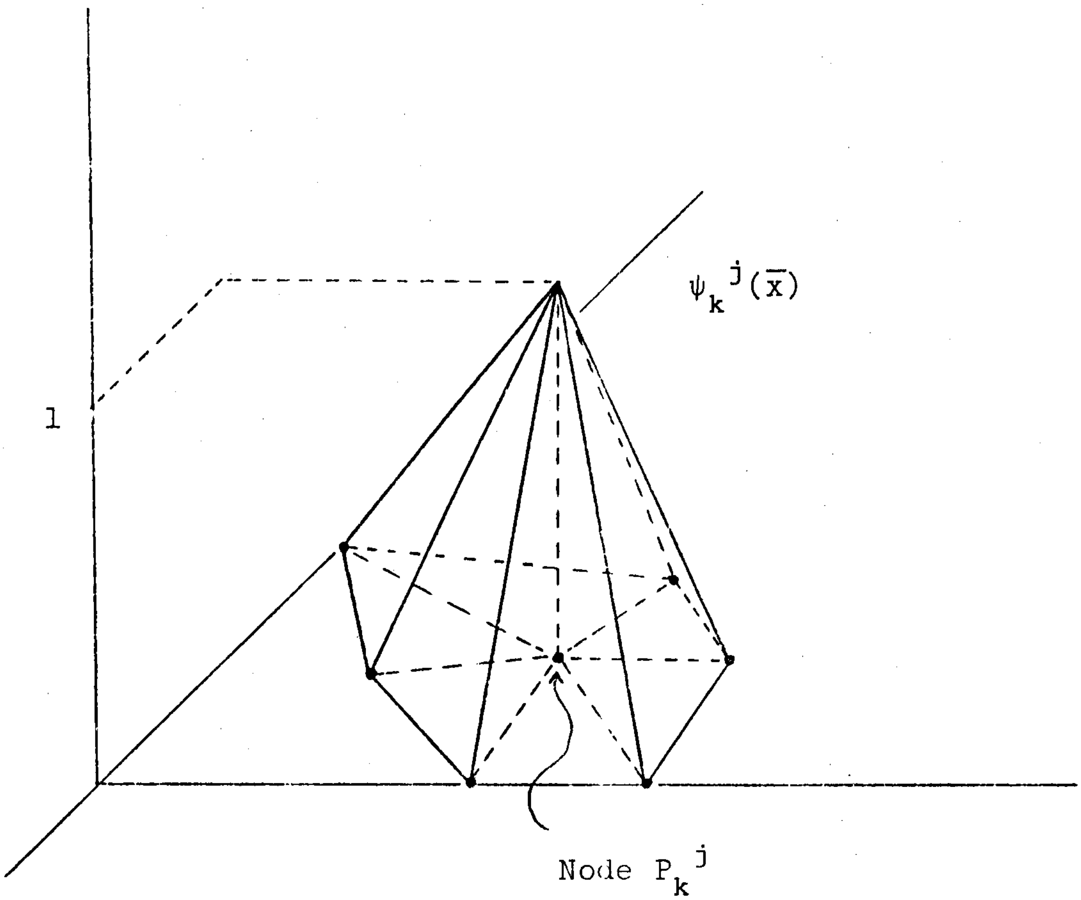

Figure 4. Typical Basis Function ψ_k^j with Support at Node P_k^j.

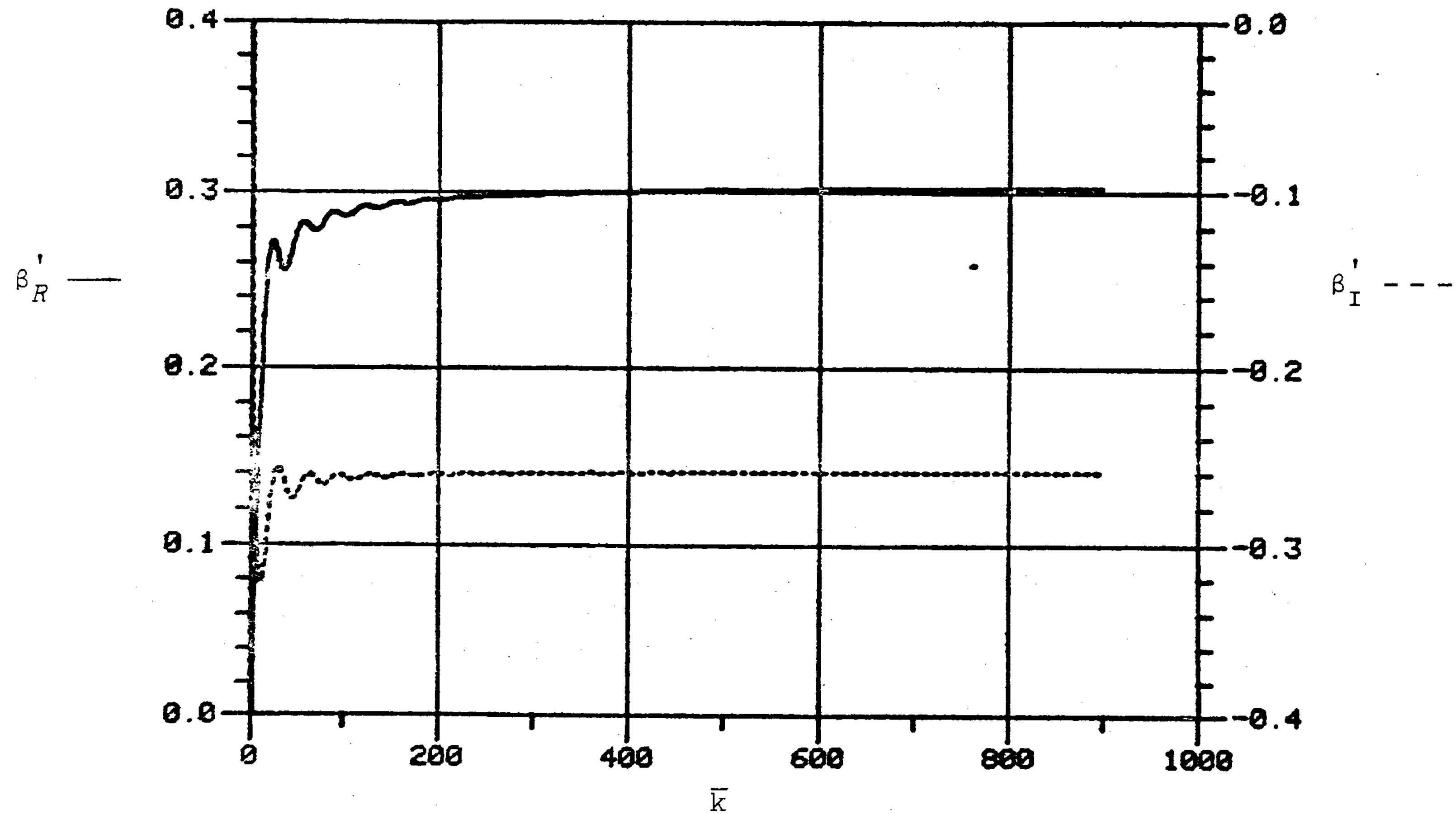

Figure 5. Coefficients β'_R and β'_I in Equation (4-104).

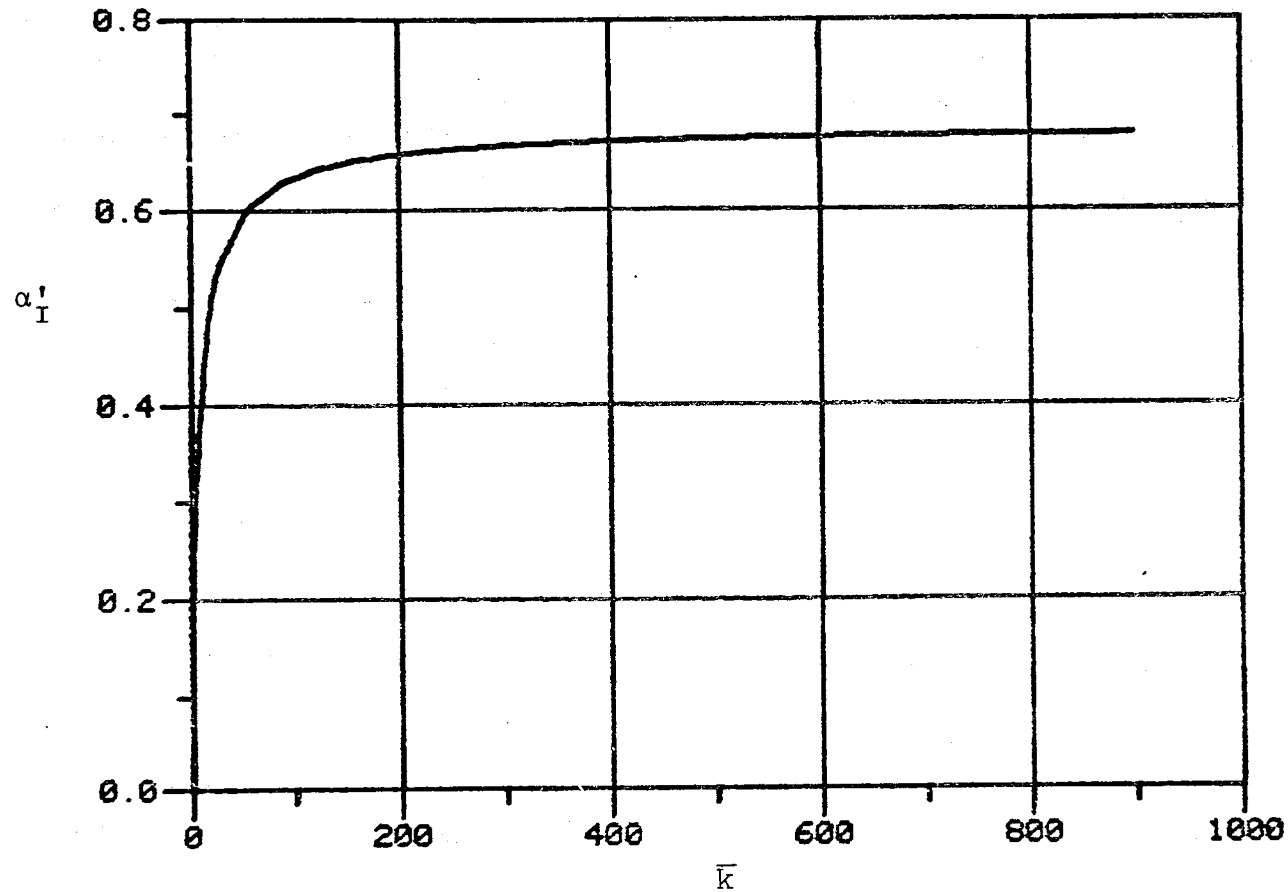

Figure 6. Coefficient α_I' in Equation (4-104).

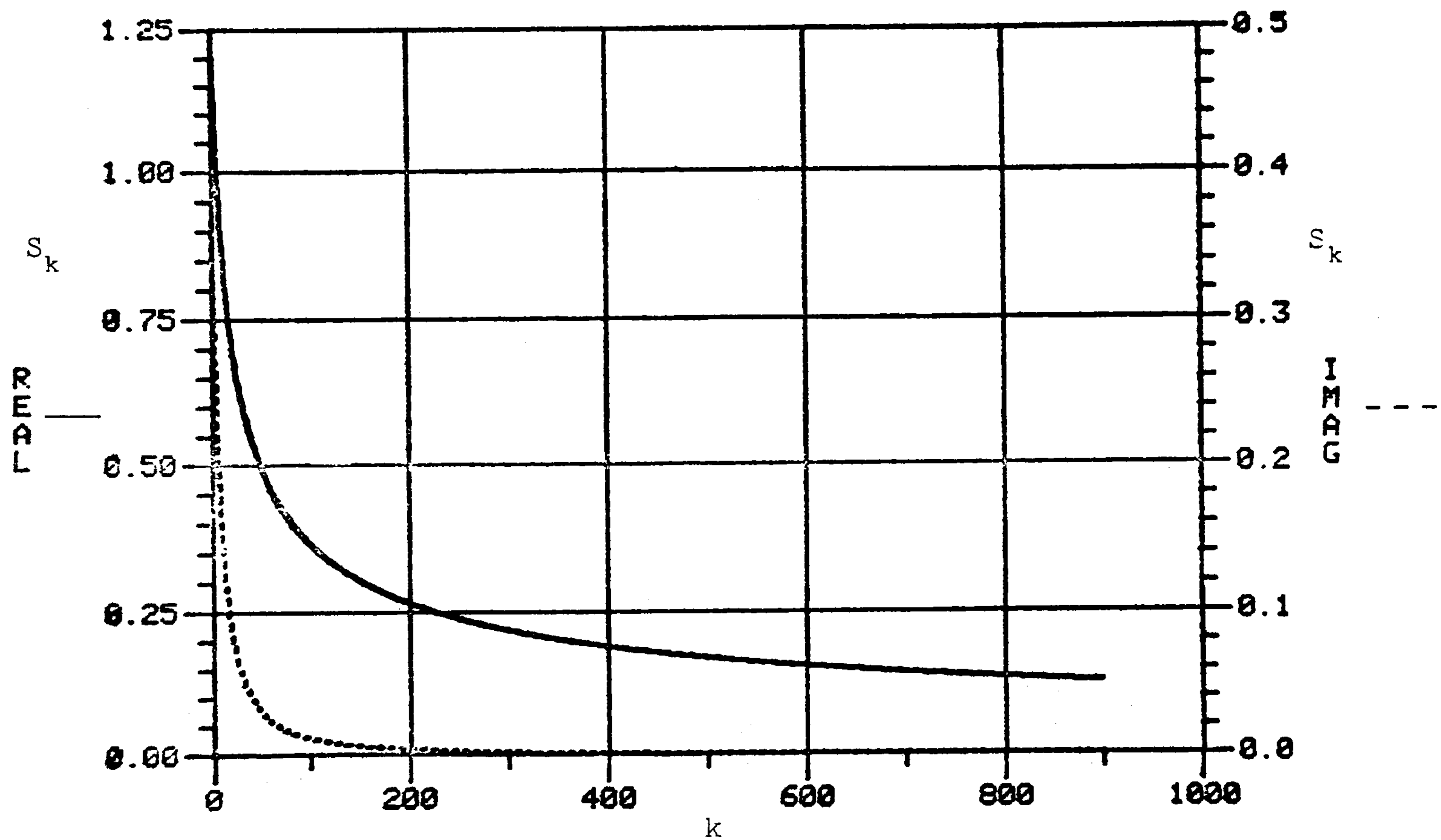

Figure 7. Solution S_k of (4-96) and (4-98) for $j = 2$.

APPENDIX B

DERIVATION OF THE SOLUTIONS OUTSIDE A CIRCULAR BOUNDARY

Using separation of variables in polar coordinates, solutions of the form

$$u(r,\theta) = R(r)\ \Theta(\theta) \tag{B-1}$$

are sought. Substitution into $\nabla^2 u + \omega^2 u = 0$ gives

$$\Theta'' + m^2\ \Theta = 0 \tag{B-2}$$

and

$$r^2 R'' + r\ R' + (\omega^2 r^2 - m^2)R = 0\ , \tag{B-3}$$

where m must be an integer. Equation (B-2) has the general solution

$$\Theta_m(\theta) = \alpha_m e^{i m\theta} + \alpha_m'\ e^{-i m\theta}\ , \tag{B-4}$$

where α_m and α_m' are arbitrary complex constants. Upon the substitution $\tilde{r} = \omega r$, Equation (B-3) reduces to Bessel's equation of order m in $\tilde{r}$. Thus the general complex solution of (B-3) is [7, pp. 619-624; 8, p. 358]

$$R_m(r) = \beta_m'\ H_m^{(1)}(\omega r) + \beta_m\ H_m^{(2)}(\omega r)\ , \tag{B-5}$$

where $H_m^{(1)}$ and $H_m^{(2)}$ are the Hankel functions of the first and second kind of order m , and β_m' and β_m are arbitrary complex constants. From (B-1), (B-4) and

(B-5) the general solution outside a circular boundary is

$$u(r,\theta) = \sum_{m=0}^{\infty} \left(\alpha_m e^{im\theta} + \alpha'_m e^{-im\theta}\right)\left[\beta'_m H_m^{(1)}(\omega r) + \beta_m H_m^{(2)}(\omega r)\right] . \tag{B-6}$$

Using the property [8, p. 358] that for integer m ,

$$H_{-m}^{(1)}(r) = (-1)^m H_m^{(1)}(r) \tag{B-7}$$

and

$$H_{-m}^{(2)}(r) = H_m^{(2)}(r) , \tag{B-8}$$

the infinite series (B-6) can be written more compactly as

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} \alpha_m e^{im\theta} \left[H_m^{(1)}(\omega r) + \beta_m H_m^{(2)}(\omega r)\right] . \tag{B-9}$$

The asymptotic expressions for the Hankel functions are

$$H_m^{(1)}(r) \underset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi r}}\, e^{i(r - m\frac{\pi}{2} - \frac{\pi}{4})} \tag{B-10}$$

and

$$H_m^{(2)}(r) \underset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi r}}\, e^{-i(r - m\frac{\pi}{2} - \frac{\pi}{4})} . \tag{B-11}$$

As shown in Chapter II, $H_m^{(1)}$ is an outgoing wave and $H_m^{(2)}$ is an incoming wave, so that the coefficients β_m in

(B-9) should be set to zero. This gives

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} \alpha_m \, e^{im\theta} \, H_m^{(1)}(\omega r) \; . \tag{B-12}$$

The boundary condition (2-1) requires that

$$u(r_0,\theta) = u_0(\theta) \; , \tag{B-13}$$

so that the coefficients α_m in (B-12) are given by

$$\alpha_m = \frac{1}{2\pi \, H_m^{(1)}(\omega r_0)} \int_0^{2\pi} u_0(\phi) \, e^{-im\phi} \, d\phi \; . \tag{B-14}$$

From (B-10), (B-12) and (B-14), u has the asymptotic form

$$u(r,\theta) \underset{r\to\infty}{\sim} \hat{\Theta}(\theta) \frac{e^{i\omega r}}{\sqrt{\omega r}} \; , \tag{B-15}$$

where

$$\hat{\Theta}(\theta) = \sqrt{\frac{2}{\pi}} \, e^{-i\frac{\pi}{4}} \sum_{m=-\infty}^{\infty} \alpha_m \, e^{im(\theta - \frac{\pi}{2})} . \tag{B-16}$$

An asymptotic formula for high frequencies can also be obtained. From (B-10) and (B-14),

$$\alpha_m \underset{\omega\to\infty}{\sim} \frac{1}{2\pi}\sqrt{\frac{\pi\omega r_0}{2}} \, e^{-i(\omega r_0 - m\frac{\pi}{2} - \frac{\pi}{4})} \int_0^{2\pi} u_0(\phi) \, e^{-im\phi} \, d\phi \; . \tag{B-17}$$

Then from (B-15), B-16) and (B-17),

$$u(r,\theta) \underset{\substack{r\to\infty \\ \omega\to\infty}}{\sim} \sqrt{\frac{r_0}{r}}\; e^{i\omega(r - r_0)}\; \overline{\Theta}(\theta) \;, \qquad \text{(B-18)}$$

where

$$\overline{\Theta}(\theta) = \sum_{m=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_0^{2\pi} u_0(\phi)\; e^{-im\phi}\; d\phi\right] e^{im\theta} \;. \qquad \text{(B-19)}$$

But the expression in brackets in (B-19) is the coefficient a_m in the Fourier series expansion

$$\sum_{m=-\infty}^{\infty} a_m\; e^{im\theta}$$

of $u_0(\theta)$. Therefore $\overline{\Theta}(\theta) = u_0(\theta)$ and (B-18) reduces to

$$u(r,\theta) \underset{\substack{r\to\infty \\ \omega\to\infty}}{\sim} \sqrt{\frac{r_0}{r}}\; e^{i\omega(r - r_0)}\; u_0(\theta) \;. \qquad \text{(B-20)}$$

If, for some integer M, the boundary condition has the special periodic form

$$u(r_0,\vartheta) = u_0(\theta) = \alpha\; e^{iM\theta} + \beta\; e^{-iM\theta} \;, \qquad \text{(B-21)}$$

then from (B-14) it follows that $\alpha_m = 0$ if $|m| \neq |M|$. The exact solution in this case is

$$u(r,\theta) = u_0(\theta)\; \frac{H_M^{(1)}(\omega r)}{H_M^{(1)}(\omega r_0)} \;. \qquad \text{(B-22)}$$

Finally, it is interesting to derive Sommerfeld's outward radiation condition (2-3) from the properties of solutions outside a circular boundary. It has already been determined by inspection of the asymptotic expressions (B-10) and (B-11) that the Hankel functions of the first kind represent outgoing waves and the Hankel functions of the second kind represent incoming waves. This simple approach for distinguishing outward radiation does not work for solutions outside noncircular boundaries since separation of variables in polar coordinates cannot be applied in these cases. What is needed for more general problems is a radiation condition which does not involve the Hankel functions directly, but which is based on their asymptotic properties. Such a condition can be derived as follows.

From (B-4) and (B-6), the general solution of the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$ outside a circular boundary can be written as

$$u(r,\theta) = \sum_{m=0}^{\infty} u_m(r,\theta) \tag{B-23}$$

where

$$u_m(r,\theta) = \Theta_m(\theta)\left[\beta_m' H_m^{(1)}(\omega r) + \beta_m H^{(2)}(\omega r)\right] \quad . \tag{B-24}$$

From the asymptotic expressions (B-10) and (B-11) it follows that as $r \to \infty$,

$$H_m^{(1)}(\omega r) \sim \sqrt{\frac{2}{\pi\omega r}}\; e^{i(\omega r - m\frac{\pi}{2} - \frac{\pi}{4})} , \tag{B-25}$$

$$\frac{d}{dr} H_m^{(1)}(\omega r) \sim i\omega\, H_m^{(1)}(\omega r) - \frac{1}{2r} H_m^{(1)}(\omega r) , \tag{B-26}$$

$$\left(\frac{d}{dr} - i\omega\right) H_m^{(1)}(\omega r) \sim -\frac{1}{2r} H_m^{(1)}(\omega r) , \tag{B-27}$$

$$H_m^{(2)}(\omega r) \sim \sqrt{\frac{2}{\pi\omega r}}\; e^{-i(\omega r - m\frac{\pi}{2} - \frac{\pi}{4})} , \tag{B-28}$$

$$\frac{d}{dr} H_m^{(2)}(\omega r) \sim -i\omega\, H_m^{(2)}(\omega r) - \frac{1}{2r} H_m^{(2)}(\omega r) , \tag{B-29}$$

and

$$\left(\frac{d}{dr} - i\omega\right) H_m^{(2)}(\omega r) \sim -2i\omega\, H_m^{(2)}(\omega r) . \tag{B-30}$$

Then from (B-25) and (B-27),

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{d}{dr} - i\omega\right) H_m^{(1)}(\omega r) = 0 , \tag{B-31}$$

while from (B-28) and (B-30),

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{d}{dr} - i\omega\right) H_m^{(2)}(\omega r) \neq 0 . \tag{B-32}$$

It follows from (B-24), (B-31) and (B-32) that

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial}{\partial r} - i\omega\right) u_m(r,\theta) = 0 \iff \beta_m = 0 . \tag{B-33}$$

Thus (B-33) can be used to filter out the inward radiation term $H_m^{(2)}(\omega r)$ from u_m in (B-24), since it requires the coefficient β_m of $H_m^{(2)}(\omega r)$ to be zero. Since β_m should be zero for all m , it seems reasonable from (B-23), (B-24) and (B-33) to assume that

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial}{\partial r} - i\omega\right) u(r,\theta) = 0 \tag{B-34}$$

is a sufficient condition to characterize outward radiation. This is Sommerfeld's outward radiation condition (2-3) and it should hold for solutions outside noncircular boundaries since they have a similar asymptotic behavior. Proofs of this result for the three-dimensional problem can be found in Courant and Hilbert [6, pp. 312-318], Dettman [15, pp. 210-213], Levine [2] and Wilcox [3] to mention just a few. Proofs for the two-dimensional problem are given in Copson [16, pp. 200-203] and Smirnov [17, pp. 679-683].

APPENDIX C

ALTERNATE FORMULATIONS AND SOME NUMERICAL METHODS

Variational Principle

Exterior problems for the Helmholtz equation can be stated as a variational principle [18, 19, 20]. Define the functional

$$I(u) = \int_{C^+} (\nabla u \cdot \nabla u^* - \omega^2 u^2) \qquad \text{(C-1)}$$

where the domain of I is the set of sufficiently smooth functions satisfying the boundary condition (2-1) and the radiation condition (2-3). Then I is stationary at the function u which satisfies the Helmholtz equation (2-2). The variational form and the Galerkin weak form have many similarities. Both formulations are more general than the problem as originally posed. Both impose the boundary condition (2-1) and the radiation condition (2-3) as essential boundary conditions. And both formulations can be used to arrive at the same set of approximating equations in the finite element method.

Boundary Integral Equation Method

Another approach to exterior problems is the boundary integral equation (BIE) method. One of the advantages of this method is that it allows the problem to be reduced from one involving an infinite region to one involving only its boundary, so that the dimension of the problem is reduced by one. A second advantage is that the outward radiation condition is incorporated as a natural boundary condition in the integral formulation and need not be imposed separately. A typical disadvantage of this method is that the process of reduction from the exterior domain to the boundary may give rise to difficulties of nonexistence or nonuniqueness not inherent in the original problem. Three examples of a BIE formulation are given below.

For the two-dimensional Helmholtz equation $\nabla^2 u + \omega^2 u = 0$, the fundamental solution, or free space Green's function, representing an outgoing wave can be taken as [21]

$$G(\bar{x},\bar{y}_0) = -\frac{i}{4} H_0^{(1)}(\omega R) \quad , \tag{C-2}$$

where

$$R = |\bar{x} - \bar{y}_0| \tag{C-3}$$

is the distance from the point $\bar{x}$ to the parameter point $\bar{y}_0$. One BIE method represents the solution u of the exterior problem as a simple source distribution of density μ along the boundary. That is,

$$u(\bar{x}) = \int_C \mu(\bar{y}_0)\ G(\bar{x},\bar{y}_0)\ ds_{y_0}\ ,\ \bar{x} \varepsilon C^+\ , \qquad (C\text{-}4)$$

where the function μ is to be determined. It can be shown that for any μ , u as defined in (C-4) is continuous at each point $\bar{x}$ in the plane, satisfies the Helmholtz equation (2-2) and satisfies the outward radiation condition (2-3). Letting $\bar{x}$ approach the boundary point $\bar{x}_0$, it follows from the boundary condition $u = u_0$ on C that the density μ must satisfy

$$\int_C \mu(\bar{y}_0)\ G(\bar{x}_0,\bar{y}_0)\ ds_{y_0} = u_0(\bar{x}_0)\ ,\ \bar{x}_0\ \varepsilon\ C\ . \qquad (C\text{-}5)$$

Equation (C-5) is a Fredholm integral equation of the first kind for μ with a singularity at $\bar{y}_0 = \bar{x}_0$. The solution μ of (C-5), when substituted into (C-4), can be used to determine u at any point $\bar{x}$ in the infinite region C^+ . Note that the problem has been reduced from solving a two-dimensional boundary value problem in C^+ to solving a one-dimensional integral equation on C . This BIE method is discussed by Hsiao and Maccamy [22].

A more common approach, since it leads to a Fredholm integral equation of the second kind, is to represent u as a double source distribution of density ν . That is,

$$u(\bar{x}) = \int_C \nu(\bar{y}_0) \frac{\partial}{\partial n_{y_0}} G(\bar{x},\bar{y}_0)\, ds_{y_0} \quad , \ \bar{x} \in C^+ \qquad \text{(C-6)}$$

where $\partial/\partial n_{y_0}$ denotes differentiation in the direction of the outward pointing normal to C at $\bar{y}_0$. Letting $\bar{x}$ approach the boundary point $\bar{x}_0$, it follows from the boundary condition $u = u_0$ on C and from the jump relation for a double source [21, pp. 2-4] that the density ν must satisfy

$$-\frac{1}{2}\nu(\bar{x}_0) + \int_C \nu(\bar{y}_0) \frac{\partial}{\partial n_{y_0}} G(\bar{x}_0,\bar{y}_0)\, ds_{y_0} = u_0(\bar{x}_0) \quad , \ \bar{x}_0 \in C \ . \qquad \text{(C-7)}$$

A third BIE method uses the Helmholtz representation theorem [16, pp. 200-202], which states that if u satisfies the Helmholtz equation (2-2) and the radiation condition (2-3), then

$$u(\bar{x}) = \int_C \frac{\partial u}{\partial n}(\bar{y}_0)\, G(\bar{x},\bar{y}_0)\, ds_{y_0} - \int_C u(\bar{y}_0) \frac{\partial}{\partial n_{y_0}} G(\bar{x},\bar{y}_0)\, ds_{y_0} \quad , \ \bar{x} \in C^+ \ . \qquad \text{(C-8)}$$

To make use of (C-8), the normal derivative of u on C must first be determined. This can be done in two ways. Letting $\bar{x}$ approach the boundary and using the continuity of a single source and the jump relation for a double source gives a Fredholm integral equation of the first kind for $\partial u/\partial n$,

$$\int_C \frac{\partial u}{\partial n}(\bar{y}_0)\, G(\bar{x}_0,\bar{y}_0)\, ds_{y_0} = \int_C u_0(\bar{y}_0) \frac{\partial}{\partial n_{y_0}} G(\bar{x}_0,\bar{y}_0)\, ds_{y_0}$$

$$+ \frac{1}{2} u_0(\bar{x}_0) \;,\; \bar{x}_0 \in C \;. \qquad \text{(C-9)}$$

Also, differentiating (C-8) in the direction normal to $\bar{x}_0 \in C$ and using the continuity of the normal derivative of a double source [6, p. 259] and the jump relation for the normal derivative of a single source [21, pp. 15-16] gives a Fredholm integral equation of the second kind for $\partial u/\partial n$,

$$-\frac{1}{2}\frac{\partial u}{\partial n}(\bar{x}_0) + \int_C \frac{\partial u}{\partial n}(\bar{y}_0) \frac{\partial}{\partial n_{x_0}} G(\bar{x}_0,\bar{y}_0)\, ds_{y_0}$$

$$= \int_C u_0(\bar{y}_0) \frac{\partial}{\partial n_{x_0}} \frac{\partial}{\partial n_{y_0}} G(\bar{x}_0,\bar{y}_0)\, ds_{y_0} \;,\; \bar{x}_0 \in C \;. \qquad \text{(C-10)}$$

After solving either (C-9) or (C-10) for $\partial u/\partial n$, the solution u in C^+ can be determined from (C-8) .

It turns out that all of the above formulations break down for a discrete set of values of the frequency ω . These values are the eigenvalues of the Dirichlet (Neumann) problem for the interior region C^- in the case of a Fredholm integral equation of the first (second) kind [22, 23]. For these frequencies a simple (double) source formulation is usually not possible. The Helmholtz representation (C-8), on the other hand, is valid at any frequency ω .

The problem at interior eigenvalues of the Dirichlet (Neumann) problem is that the integral Equation (C-9) [(C-10)] does not have a unique solution for $\partial u/\partial n$. However, as shown by Kleinman and Roach [23], the 2 integral equations (C-9) and (C-10) taken together will always have a unique solution, apparently because each fails at a different set of frequencies. Kleinman and Roach give a thorough discussion of the connection between interior and exterior problems for both Dirichlet and Neumann boundary conditions, although the only source formulations considered are those leading to integral equations of the second kind. They also give references to other methods that have been developed to overcome difficulties at interior eigenvalues. Baker and Copson [5] discuss the physical interpretation of the Helmholtz representation and of simple and double sources in terms of acoustic waves. Along these lines, Copley [24] discusses the failure of the source method as a resonance phenomenon. It should be pointed out that in the papers by Hsiao and Maccamy [22] and Kleinman and Roach [23] mentioned above, a smooth boundary was assumed. It was not clear what modifications are necessary for extension to piecewise smooth boundaries.

Artificial Boundaries

Problems in infinite regions can also be solved by introducing an artificial boundary C_∞ which encloses the original boundary C , and then solving a boundary value problem on the finite annular subregion C_∞^- between these two boundaries. See Figure 8 below.

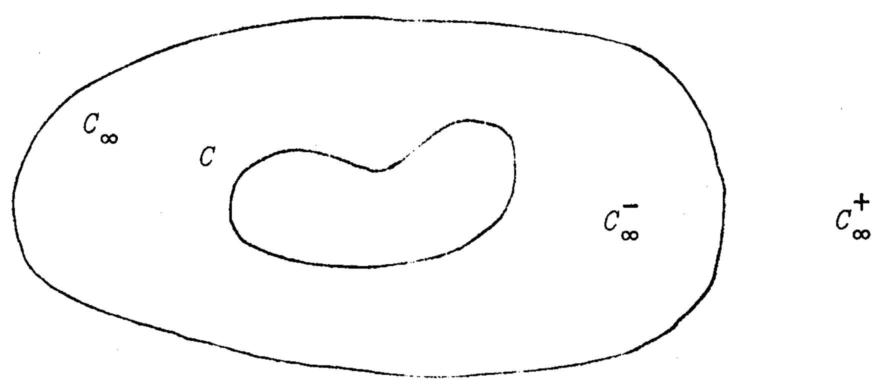

Figure 8. Artificial Boundary.

The catch here is to find a boundary condition on C_∞ which guarantees that the solution of the problem on the annular region agrees, on the annular region, with the solution of the original problem. This approach has long been used to obtain numerical solutions to problems in exterior regions, although the boundary condition applied on C_∞ is usually only approximate, even before the discretization. Recently, attention has been given to the derivation of theoretical or exact boundary conditions for the artificial boundary.

When compared with a straightforward application of the BIE method, reformulating the exterior problem as a problem over an annulus has certain advantages. First of all, the boundary condition imposed on C_∞ should depend only on the fact that the solution must satisfy the outward radiation condition (2-3) and the Helmholtz equation (2-2) in the region C_∞^+ exterior to C_∞. Such a condition is independent of the type of boundary condition imposed on C, and should work equally well for Dirichlet, Neumann or mixed boundary conditions on C. Also the same condition on the artificial boundary can be used for variable coefficient equations, provided these coefficients reach limiting values inside the annulus so that $\nabla^2 u + \omega^2 u = 0$ still applies in C_∞^+. In contrast, the BIE methods require the solution of a different type of integral equation for each type of boundary condition on C. And for variable coefficient equations a different free space Green's function must be found.

Marin [21] develops an exact condition for the artificial boundary C_∞ starting from a simple source formulation for u in C_∞^+,

$$u(\bar{x}) = \int_{C_\infty} \mu(\bar{y}_0)\, G(\bar{x},\bar{y}_0)\, ds_{y_0} \quad , \quad \bar{x} \in C_\infty^+ . \tag{C-11}$$

This condition cannot be imposed directly on C_∞ since the unknown source density μ depends on the unknown values of u on C_∞. However, using (C-11) and the jump relation for the normal derivative of a simple source gives 2 integral equations

$$u(\bar{x}_0) = \int_{C_\infty} \mu(\bar{y}_0)\ G(\bar{x}_0,\bar{y}_0)\ ds_{y_0}\ , \quad \bar{x}_0 \in C_\infty \tag{C-12}$$

$$\frac{\partial u}{\partial n}(\bar{x}_0) = \frac{1}{2}\mu(\bar{x}_0) + \int_{C_\infty} \mu(\bar{y}_0)\ \frac{\partial}{\partial n_{x_0}} G(\bar{x}_0,\bar{y}_0)\ ds_{y_0}\ , \quad \bar{x}_0 \in C_\infty \tag{C-13}$$

which establish an implicit relationship between the unknown values of u and $\partial u/\partial n$ on C_∞. Eliminating the unknown density μ from these equations yields a relationship F of the form

$$\frac{\partial u}{\partial n}(\bar{x}_0) = F(u|C_\infty,\bar{x}_0)\ , \quad \bar{x}_0 \in C_\infty \tag{C-14}$$

which is nonlocal in the sense that the value of $\partial u/\partial n$ at a single point $\bar{x}_0 \in C_\infty$ is related to the values of u over the entire boundary C_∞.

Marin applies (C-14) on the artificial boundary C_∞ to reduce the exterior problem to a problem over the annular region C_∞^- (see Figure 8). He restates the problem in the annular region in a Galerkin weak form which incorporates (C-14) as a natural boundary condition. Although the source formulation (C-11) is invalid if the frequency ω is an eigenvalue for the region interior to

C_∞ , Marin states that, for a given ω , this difficulty can be avoided in practice by adjusting C_∞ accordingly.

Engquist and Majda [25, 26] have developed a theoretical boundary condition for artificial boundaries for the wave equation based on Fourier transform techniques. Their condition is nonlocal in both space and time.

Numerical Methods

The purpose of this section is to outline some of the numerical methods currently being used to solve exterior boundary value problems for the Helmholtz equation. The extension of Thatcher's method [1] to the Helmholtz equation will not be discussed here as this topic is covered in Chapters III and IV. Although there is quite a variety of methods being used, most of them fall into one of two categories: (a) direct application of the BIE method; (b) solution on an annular subregion by the introduction of an artificial boundary.

A direct application of the BIE method requires the solution of the appropriate integral equation on the boundary C . This can be done by choosing N points on the boundary and applying the integral equation at each of these points. Approximations are made which result in a system of linear equations for the values of the unknown source function at the N selected boundary points. One of the advantages of the BIE method is that a relatively small number of unknowns appears since the approximation involved is only along the boundary. The approximate scurce function is then substituted into the line integral expression for the solution u in C^+. This line integral must be evaluated for each point in C^+ where the solution is desired. Since the integrals involved in each step of this process are singular, special care must be taken or the process can become numerically unstable [27]. Recalling the difficulties of the basic BIE methods at interior eigenvalues, Copley [24] comments "Except at quite low frequencies, these eigenvalues are so close together that it would not appear practicable to use the methods in their basic form." Thus it seems that one of the modified formulations [23] would have to be used in most cases. A good discussion of the general procedures involved in applying the BIE method can be found in the paper by Zeinkiewicz, Kelly and Bettess [28].

If an artificial boundary C_∞ is introduced, the problem in the annular region can be solved by standard finite element or finite difference methods once an appropriate boundary condition is applied on C_∞. Such boundary conditions are often derived using a BIE method. A good discussion of the general procedures involved in the coupling of the finite element method and the BIE method can again be found in Zeinkiewicz, Kelly and Bettess [28]. This paper contains 96 references.

Boundary conditions for the artificial boundary can also be derived from the asymptotic behavior of solutions of the Helmholtz equation. Although solutions to outward radiation problems decay to zero as $r \to \infty$, the boundary condition

$$u(\overline{x}_0) = 0 \ , \ \overline{x}_0 \ \varepsilon \ C_\infty \tag{C-15}$$

will obviously not work since it is insufficient to characterize outward radiation. From Sommerfeld's outward radiation condition (2-3), it follows that if C_∞ is a circular boundary of radius r_0 , then the approximation

$$\left(\frac{\partial}{\partial r} - i\omega\right) u(r_0,\theta) = 0 \ , \ (r_0,\theta)\varepsilon \ C_\infty \tag{C-16}$$

should be sufficient to characterize outward radiation provided the artificial boundary C_∞ is far enough removed from the inner boundary C . Condition (C-16) has the advantage of being local, i.e., the value of $\partial u/\partial n$ at $(r_0,\theta)\varepsilon \ C_\infty$ is related only to the value of u at (r_0,θ) . But this condition has the disadvantage that it becomes a poorer approximation as C_∞ is taken closer to C . This is in contrast to some of the boundary conditions derived by the BIE method. For example, in the method of Marin [21] and the method of McDonald and Wexler [29] to be discussed below, the condition imposed on the artificial boundary C_∞ is nonlocal but its effectiveness is for the most part independent of the location of C_∞ with respect to C . This allows for smaller annular regions and thus fewer unknowns to be solved for.

It is possible to improve the accuracy of condition (C-16) by adding on a term which depends on the radius r_0 . Recall that the derivation of Sommerfeld's radiation condition given in Appendix B was based on the asymptotic expressions (B-10) and (B-11) for the Hankel functions. These expressions are only the first terms in the general asymptotic expansions given by Smirnov [30, p. 432]. For the Hankel function of the first kind, keeping the first two terms gives

$$H_m^{(1)}(\omega r) \underset{r \to \infty}{\sim} \left(\frac{2}{\pi \omega r}\right)^{1/2} \left[1 + \left(m^2 - \frac{1}{4}\right)\left(\frac{i}{2\omega r}\right)\right] e^{i(\omega r - m\frac{\pi}{2} - \frac{\pi}{4})} \; . \tag{C-17}$$

Following the same procedure as in Appendix B, it can be shown that

$$\lim_{r \to \infty} r^{3/2} \left(\frac{d}{dr} - i\omega + \frac{1}{2r}\right) H_m^{(1)}(\omega r) = 0 \; . \tag{C-18}$$

This leads to the outward radiation condition

$$\lim_{r \to \infty} r^{3/2} \left(\frac{\partial}{\partial r} - i\omega + \frac{1}{2r}\right) u(r,\theta) = 0 \; . \tag{C-19}$$

Equation (C-19) is equivalent to Sommerfeld's radiation condition (2-3) and offers no advantage from a theoretical standpoint. However when the artificial boundary C_∞ is a circle of radius r_0 , (C-19) yields the approximate radiation condition

$$\left(\frac{\partial}{\partial r} - i\omega + \frac{1}{2r_0}\right)u(r_0,\theta) = 0 \ , \ (r_0,\theta)\varepsilon\ C_\infty \ . \qquad \text{(C-20)}$$

It follows from (2-3) and (C-19) that (C-20) is a more accurate approximation than (C-16). Marin [21, pp.22-23, 90-92, 100] carried out computations on a simple problem in which results using (C-16) and (C-20) were compared. As expected, condition (C-20) gave better results.

It is reasonable to ask if more accurate conditions of the type (C-20) can be obtained by retaining more terms in the asymptotic expansion for the Hankel functions. This doesn't appear to work since when more terms are retained, the resulting outward radiation condition depends on the order m of the Hankel function, unlike conditions (B-31) and (C-18). Marin [21, pp. 22-23] refers to the Keller asymptotic expansion of the solution with respect to the frequency ω , and indicates that this can be used to obtain a sequence of local radiation conditions in which (C-16) and (C-20) are special cases. Engquist and Majda [25, 26] use their exact nonlocal condition for artificial boundaries for the wave equation to derive a hierarchy of local boundary conditions which approximate the exact condition. They show that these conditions guarantee stable finite difference approximations. Their results can be applied to the Helmholtz equation by assuming harmonic time dependence of the form $e^{-i\omega t}$. The first approximating condition (1-27) in their paper [25] then reduces to equation (C-20). Finally, it appears that equations (2-26) and (2-27) could also be used to obtain an approximate local boundary condition for the artificial boundary.

An example of a finite element method for the annular region coupled with a BIE method on the artificial boundary is given by McDonald and Wexler [29]; see Figure 9. They introduce a curve C_A lying inside the annular region

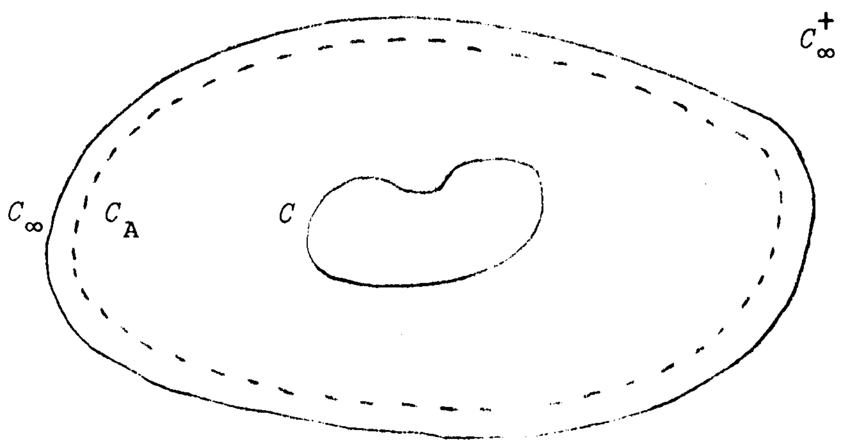

Figure 9. Method of McDonald and Wexler.

but close to C_∞ . They use the Helmholtz representation (C-8) to apply a constraint on the outer boundary of the form

$$u(\overline{x}) = \int_{C_A} \frac{\partial u}{\partial n}(\overline{y}_0)\, G(\overline{x},\overline{y}_0)\, ds_{y_0}$$

$$- \int_{C_A} u(\overline{y}_0) \frac{\partial}{\partial n_{y_0}} G(\overline{x},\overline{y}_0)\, ds_{y_0} \;, \; \overline{x} \in C_\infty \; . \tag{C-21}$$

This condition relates the unknown values of u on the artificial boundary C_∞ to the unknown values of u and $\partial u/\partial n$ on C_A . It is evaluated at each nodal point $\overline{x}$ on C_∞ . After the solution in the annular region is obtained, the formulation (C-21) is used to determine the values of u at points $\overline{x}$ in the region C_∞^+ outside the annulus. Singularity problems with the integrals are avoided since each point $\overline{x}$ on or outside C_∞ is at a finite distance from the curve C_A .

Chen and Mei [19] and Yue, Chen and Mei [31] use a circular artificial boundary C_∞ . The exact solution outside this boundary can be represented as an infinite series of complex exponential functions and Hankel functions of the first kind, as shown in Appendix B. They use only the first M terms in this series so that the solution outside C_∞ is approximated by the function

$$\hat{u}(r,\theta) = \sum_{m=0}^{M} (\alpha_m e^{im\theta} + \alpha'_m e^{-im\theta}) H_m^{(1)}(\omega r) \; , \tag{C-22}$$

where the coefficients α_m and α'_m are to be determined. This function $\hat{u}$ necessarily satisfies the Helmholtz equation in C_∞^+ and also Sommerfeld's outward radiation condition. If $\tilde{u}$ denotes the (unknown) finite element solution in the annulus, then the inner and outer solutions $\tilde{u}$ and $\hat{u}$ can be matched on C_∞ by requiring that

$$\tilde{u} = \hat{u} \ , \ \frac{\partial \tilde{u}}{\partial r} = \frac{\partial \hat{u}}{\partial r} \tag{C-23}$$

on C_∞ . This matching condition is then incorporated as a natural boundary condition in a variational principle for the solution $\tilde{u}$ in the annular region. The coefficients α_m and α_m' in (C-22) appear with the other unknowns in the resulting linear system of equations. Yue, Chen and Mei [31] refer to their method as a hybrid element method.

Zienkiewicz, Kelly and Bettess [28] discuss a generalization of the type of variational principle used by Yue, Chen and Mei [19, 31]. They apply this variational principle to the coupling of finite element solutions in the annular region with BIE formulations outside the artificial boundary. Herrera [32] has developed a method for generating 3 types of variational principles for the coupling of solutions on adjoining regions. His method applies to time dependent problems.

In deriving their variational principle for the annular region, Chen and Mei [19] started from a variational principle involving the entire infinite region C^+ . In the context of the finite element method, the infinite region C_∞^+ exterior to the artificial boundary C_∞ can be interpreted as an infinite element which is coupled, at the boundary C_∞ , to the finite elements in the annular region. Each of the terms in the truncated series $\hat{u}$ used to approximate the solution in C_∞^+ can be viewed as a basis function with support on C_∞^+ . Since these functions satisfy the Helmholtz equation and the radiation condition, integration by parts reduced the area integrals over C_∞^+ to line integrals over C_∞ .

Zienkiewicz and Bettess [18, 20] have developed a similar method which divides the infinite region C_∞^+ into several infinite elements with a finite number of nodes per element (see Figure 10). The basis functions defined on these elements satisfy the radiation condition (2-3) but not the Helmholtz equation. As a result, the variational principle that they formulate for the infinite region C^+ involves integrations over these infinite elements. The authors point out that despite this complication, their method has an advantage over some of those discussed above. Since the basis functions on the infinite elements do not have to satisfy the differential equation, each basis function can be chosen to be zero over most elements. This allows for a narrower coupling at the interface C_∞ (see Figure 10). In contrast, the BIE methods and the

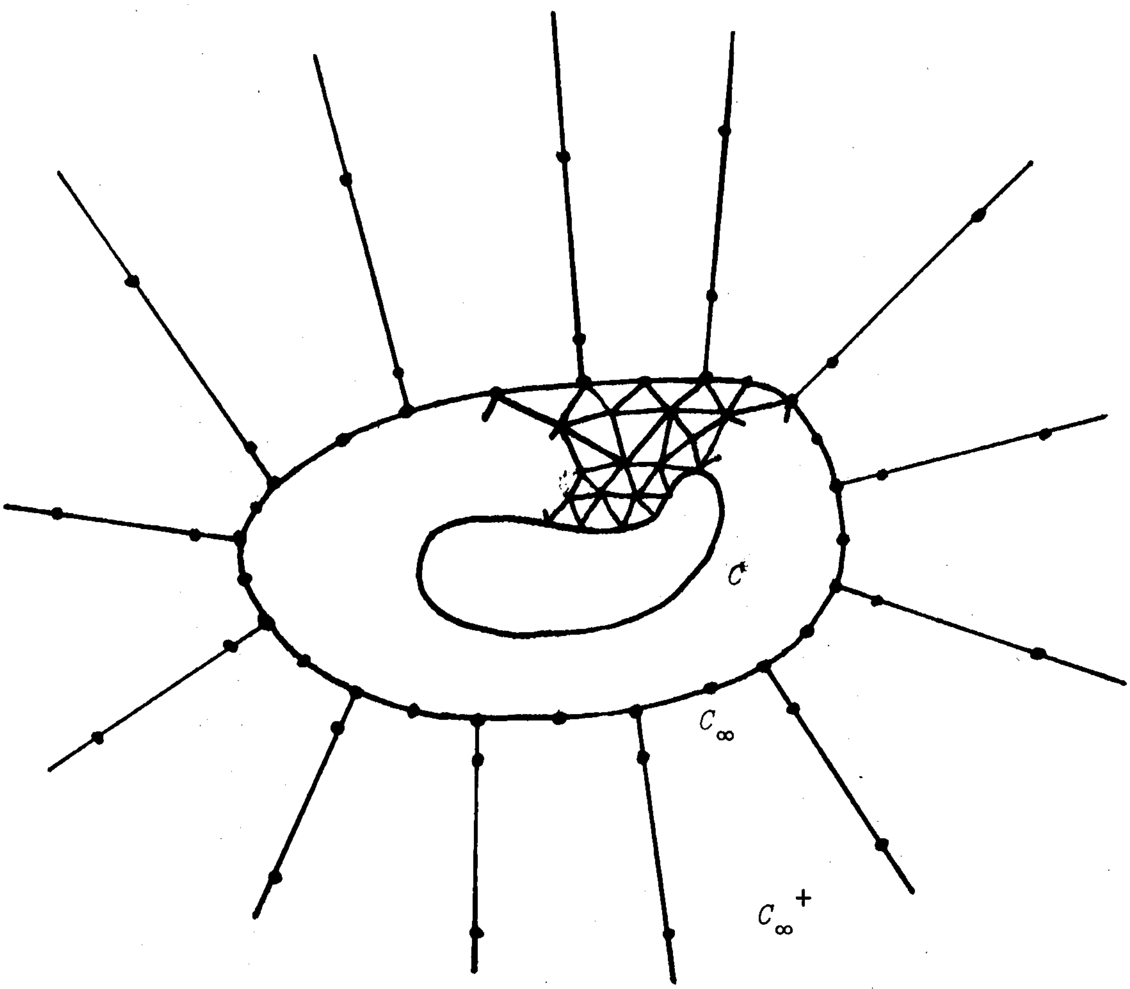

Figure 10. Infinite Elements of Zienkiewicz and Bettess.

series solution method of Yue, Chen and Mei [19, 31] result in a coupling of all elements along C_∞ giving rise to larger bandwidth equations.

APPENDIX D

INTEGRATION FORMULAS

Consider an arbitrary triangular element E labeled as shown below.

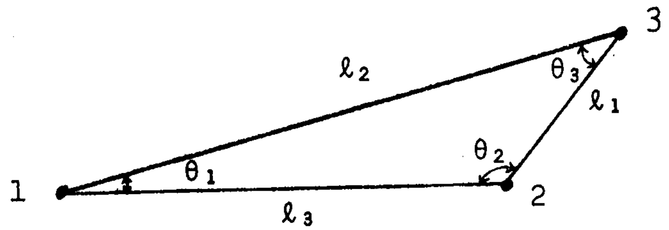

Figure 11. Typical Triangular Element.

The area A of the triangle E is given by

$$A = \frac{1}{2}\, \ell_i\, \ell_j\, \sin(\theta_k) \;, \; ijk \text{ permutation of } 1\,2\,3 \;.$$

Let ψ_i be the piecewise linear basis function with support at node i , so that $\psi_i = 1$ at node i and $\psi_i = 0$ along the side opposite node i . Then

$$\int_E \psi_i\, \psi_j = \begin{cases} i = j : \frac{A}{6} \\ i \neq j : \frac{A}{12} \end{cases} \;.$$

Note that the above integral depends only on the area of the triangle and not on its shape. Also,

$$\int_E \nabla \psi_i \cdot \nabla \psi_j = \begin{cases} i = j : \dfrac{\ell_i^2}{4A} = \dfrac{1}{2\cos(\theta_k)} \dfrac{\ell_i}{\ell_j} \,, \text{ ijk perm } 1\,2\,3 \\[2ex] i \neq j : -\dfrac{\ell_i\,\ell_j\,\cos(\theta_k)}{4A} = -\dfrac{1}{2}\cot(\theta_k) \,, \\ \qquad\qquad\qquad\qquad \text{ijk perm } 1\,2\,3\,. \end{cases}$$

Note that the above integral depends only on the shape of the triangle and not on its size.

VITA

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