May 20, 1975

To the Graduate Council:

I am submitting herewith a thesis written by Myra L. Schrader entitled "Some Initial-Boundary Value Problems for a Singular Partial Differential Equation." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Philip W. Schaefer

Philip W. Schaefer, Major Professor

We have read this thesis
and recommend its acceptance:

Don B Hinton

John S Bradley

Accepted for the Council:

Hilton A. Smith

Vice Chancellor
Graduate Studies and Research

E

SOME INITIAL-BOUNDARY VALUE PROBLEMS FOR A SINGULAR PARTIAL DIFFERENTIAL EQUATION

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee

Myra L. Schrader

June 1975

ACKNOWLEDGEMENT

The author wishes to express her sincere appreciation to Dr. Philip W. Schaefer for his guidance, encouragement, and assistance throughout the course of this work.

1207510

ABSTRACT

This thesis deals with various mixed problems for the singular partial differential equation

$$u_{tt} + \frac{k}{t} u_t - u_{xx} = 0 \quad ,$$

where k is a positive constant.

Solutions are obtained by using the method of separation of variables and then converted to an integral form. The integral form of the solution is used to verify the existence of a solution. An energy integral argument is used to demonstrate that the solutions are unique.

TABLE OF CONTENTS

CHAPTER		PAGE
I.	INTRODUCTION	1
II.	AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE EULER-POISSON-DARBOUX EQUATION	4
III.	OTHER INITIAL-BOUNDARY VALUE PROBLEMS FOR THE EULER-POISSON-DARBOUX EQUATION	23
IV.	EXTENSION TO PROBLEMS IN TWO-SPACE VARIABLES	40
BIBLIOGRAPHY		60
VITA		62

CHAPTER I

INTRODUCTION

In this paper we seek an explicit solution of various initial-boundary value problems for the singular partial differential equation

$$(*) \qquad u_{tt} + \frac{k}{t} u_t - u_{xx} = 0 \quad ,$$

where k is a constant and the subscript notation denotes partial differentiation.

This equation is called the Euler-Poisson-Darboux (EPD) equation in one-space variable. In recent years this partial differential equation has been widely studied in n-space variables, that is,

$$u_{tt} + \frac{k}{t} u_t - (u_{x_1 x_1} + \ldots + u_{x_n x_n}) = 0 \quad ,$$

and for any k . However, much of this research has concentrated on the Cauchy problem [2, 3, 4, 5, 10, 11] for the EPD equation. Certain mixed problems for the EPD equation in one-space variable have been studied by Weinstein [12], Lieberstein [9], and Fusaro [6].

In the course of this work, we consider the case of $k > 0$. Thus the coefficient of the first derivative term is singular on the initial line $t = 0$. Obviously, when $k = 0$, the equation reduces to the wave equation, in which case there is no singularity. We omit the situation of $k < 0$ from our discussion since the solution is

known to be nonunique in this case (see [13]).

The wave equation

$$u_{tt} - u_{xx} = 0$$

describes the free vibrations of a string. When $k > 0$, (*) may be interpreted as the equation of a vibrating string which is subjected to damping. That is, initially the viscosity of the medium is thought to be infinite.

In Chapter II we shall develop a series solution for the initial-boundary value problem studied by Fusaro [6] by using the method of separation of variables. Applying an integral relation for the Bessel function, we reduce the series to a closed form. In this form the solution is shown to satisfy the requirements of the initial-boundary value problem. Then, utilizing an energy integral argument, we show the solution is unique.

In the next chapter we consider problems with boundary conditions of the second kind and with mixed boundary conditions of the first and second kind. In each of these cases the series solutions can be reduced to an integral form. Consequently, we find the Fusaro procedure holds for these problems.

Finally, in Chapter IV, we examine the initial-boundary value problem with boundary conditions of the first kind for the EPD equation in two-space variables. Again a series solution can be obtained by separation of variables. Here we are unable to reduce the solution

to a closed form for reasons which will become evident. However, we will use Fourier series techniques to show that the series solution converges absolutely and uniformly and satisfies the singular partial differential equation and auxiliary conditions. The uniqueness is again deduced by using an energy integral argument.

It follows that initial-boundary value problems for the singular EPD equation in two-space variables with other boundary conditions can be solved by the Fourier technique. In fact, the extension to n-space dimensions is clear.

Although Fusaro's argument, which used the integral representation of the solution to demonstrate existence, was unsuccessful when applied to a finite region in two-space variables, we remark that in a recent work of Young [13], an explicit solution of a mixed problem for the EPD equation in an infinite region (a quarter-space) in 2-space variables is determined.

CHAPTER II

AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE EULER-POISSON-DARBOUX EQUATION

In his paper, Fusaro [6] obtained a unique solution for the initial-boundary value problem (I-BVP) with boundary conditions of the first kind for the EPD equation. We will present his argument in this chapter.

We consider the (I-BVP)

$$(1)\quad u_{tt} + \frac{k}{t} u_t - u_{xx} = 0 , \qquad 0 < x < \pi, \quad 0 < t ,$$

$$(2)\quad u(x,0) = f(x) , \; u_t(x,0) = 0 , \quad 0 \le x \le \pi ,$$

$$(3)\quad u(0,t) = u(\pi,t) = 0 , \qquad 0 \le t .$$

Here k is a positive constant, f has two continuous derivatives on $0 \le x \le \pi$, that is, $f \in C^2[0,\pi]$, and f satisfies the end conditions $f(0) = f(\pi) = 0$. Because of the $\frac{k}{t}$ term in the partial differential equation, we interpret $u(x,0)$ in the sense that

$$u(x,0) = \lim_{(y,t)\to(x,0^+)} u(y,t) = f(x) .$$

We seek a solution of the form $u(x,t) = X(x)T(t)$ and substitute it in (1) to get

$$XT'' + \frac{k}{t} XT' - X''T = 0 ,$$

where the prime notation indicates differentiation with respect to the argument. Then, dividing by XT, we get

$$\frac{T''}{T} + \frac{k}{t}\frac{T'}{T} - \frac{X''}{X} = 0 ,$$

or

$$\frac{T''}{T} + \frac{k}{t}\frac{T'}{T} = \frac{X''}{X} = -\lambda \quad ,$$

where λ is a constant. Thus, we get the two ordinary differential equations

$$X'' + \lambda X = 0 \tag{4}$$

and

$$T'' + \frac{k}{t} T' + \lambda T = 0 . \tag{5}$$

Because of the boundary data (3), we deduce that $X(x)$ must satisfy the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 , \\ X(0) = 0 = X(\pi) \quad . \end{cases} \tag{6}$$

The values of λ for which (6) has a nontrivial solution (eigenvalues), are $\lambda_n = n^2$, $n = 1, 2, 3, \ldots$. The associated nontrivial solutions (eigenfunctions), are $X_n(x) = \sin nx$, $n = 1, 2, 3, \ldots$.

We now substitute these values of λ into the ordinary differential equation (5) and multiply by t^2 . Thus,

$$t^2T'' + ktT' + \lambda_n t^2 T = 0 \quad . \tag{7}$$

We shall find the solution of equation (7) by letting $T = yx^p$ and $t = \dfrac{x}{\sqrt{\lambda_n}}$, where p is a parameter. Now by the chain rule

$$\frac{dT}{dt} = \sqrt{\lambda_n}\,(pyx^{p-1} + \dot{y}x^p)$$

and

$$\frac{d^2T}{dt^2} = \lambda_n[p(p-1)yx^{p-2} + 2p\dot{y}x^{p-1} + \ddot{y}x^p] \ ,$$

where the dot notation indicates differentiation with respect to x . Hence, substituting in (7) and dividing by x^p , we get

$$x^2[p(p-1)yx^{-2} + 2p\dot{y}x^{-1} + \ddot{y}]$$

$$+ kx[pyx^{-1} + \dot{y}] + x^2y$$

$$= 0 \ ,$$

which, when simplified, becomes

$$x^2\ddot{y} + (2p + k)x\dot{y} + [p(p-1) + kp + x^2]y = 0 \quad .$$

Letting $k = 1 - 2p$, we have, finally,

$$x^2\ddot{y} + x\dot{y} + (x^2 - p^2)y = 0 \quad . \tag{8}$$

We seek a solution of (8) in the form

$$y(x) = \sum_{j=0}^{\infty} C_j x^{r+j} \quad ,$$

where the C_j's are as yet undetermined constants, $C_0 \neq 0$, and r is a parameter. Then

$$\dot{y}(x) = \sum_{j=0}^{\infty} C_j (r+j) x^{r+j-1}$$

and

$$\ddot{y}(x) = \sum_{j=0}^{\infty} C_j (r+j)(r+j-1) x^{r+j-2} \quad .$$

Now substituting in (8), we get

$$\sum_{j=0}^{\infty} C_j(r+j)(r+j-1)x^{r+j} + \sum_{j=0}^{\infty} C_j(r+j)x^{r+j}$$

$$+ \sum_{j=2}^{\infty} C_{j-2}x^{r+j} - \sum_{j=0}^{\infty} p^2 C_j x^{r+j}$$

$$= 0 \quad .$$

We can rewrite this equation as

$$r(r-1)C_0x^r + rC_0x^r - p^2C_0x^r + (r+1)rC_1x^{r+1}$$

$$+ (r+1)C_1x^{r+1} - p^2C_1x^{r+1}$$

$$+ \sum_{j=2}^{\infty} [(r+j)(r+j-1)C_j + (r+j)C_j - p^2C_j + C_{j-2}]x^{r+j}$$

$$= 0 \ ,$$

or

$$(r^2-p^2)C_0x^r + [(r+1)^2-p^2]C_1x^{r+1} + \sum_{j=2}^{\infty} \{[(r+j)^2-p^2]C_j+C_{j-2}\}x^{r+j}$$

$$= 0 \ .$$

Since the coefficients of each power of x must be zero, it follows that $(r^2-p^2)C_0 = 0$. Since $C_0 \neq 0$, we obtain the indicial equation

$$r^2 - p^2 = 0 \quad .$$

The roots of this equation are $r = p$ or $r = -p$. From the coefficient of x^{r+1} we see that $[(r+1)^2 - p^2]C_1 = 0$. We deduce that $C_1 = 0$ unless $p = \frac{1}{2}$ and $r = -p$. However, since $k = 1 - 2p$ and $k > 0$, we have $p < \frac{1}{2}$. Hence $C_1 = 0$. The remaining coefficients are determined by the recurrence relation

$$C_j = \frac{-C_{j-2}}{[(r+j)^2 - p^2]} \quad , \quad j = 2, 3, 4 \ldots \ ,$$

which follows from the vanishing of the coefficients of x^{r+j} , $j = 2, 3, \ldots$, that is, from

$$[(r+j)^2 - p^2]C_j + C_{j-2} = 0 \quad .$$

Now suppose $r = p$. Then $C_1 = 0$ and since

$$C_j = \frac{-C_{j-2}}{j(j+2p)} \quad ,$$

we have that all coefficients with odd indices are zero. In general, the coefficients with even indices are given by

$$C_{2j} = \frac{(-1)^j C_0}{2^{2j} j![(p+1)(p+2) \ldots (p+j-1)(p+j)]}$$

for $j \geq 1$. Thus for $r = p$, we get the series solution

$$y(x) = C_0 \left[x^p + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j+p}}{2^{2j} j!\,(1+p)(2+p)\ \ldots\ (j+p)}\right] .$$

Now using the fact [7] that

$$\Gamma(j+p+1) = (j+p)\ \ldots\ (1+p)\Gamma(1+p) ,$$

we can write

$$y(x) = C_0 2^p \Gamma(1+p) \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{x}{2})^{2j+p}}{\Gamma(j+1)\Gamma(j+p+1)} .$$

Finally, if we substitute $x = t\sqrt{\lambda_n}$, $\sqrt{\lambda_n} = n$, and $y = Tx^{-p}$, we obtain

$$T_n^{(p)}(t) = C_0 (2nt)^p \Gamma(1+p) \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{nt}{2})^{2j+p}}{\Gamma(j+1)\Gamma(j+p+1)} .$$

Now let $r = -p$ so that from the recurrence relation, we have

$$C_j = \frac{-C_{j-2}}{j(j-2p)} .$$

Again the coefficients with odd indices are zero, since $C_1 = 0$, and the coefficients with even indices are given in general by

$$C_{2j} = \frac{(-1)^j C_0}{2^{2j} j![(-p+1)(-p+2)\ \ldots\ (-p+j)]} ,$$

for $j \geq 1$. Hence, for $r = -p$ the series solution in terms of t

is

$$T_n^{(-p)}(t) = C_0\Gamma(1-p)\left(\frac{nt}{2}\right)^p \sum_{j=0}^{\infty} \frac{(-1)^j\left(\frac{nt}{2}\right)^{2j-p}}{\Gamma(j+1)\Gamma(j-p+1)} \quad .$$

Therefore, the general solution of (7) is, where $p = \frac{1-k}{2}$,

$$T_n(t) = A_n(2nt)^p\Gamma(1+p)J_p(nt) + B_n\left(\frac{nt}{2}\right)^p\Gamma(1-p)J_{-p}(nt) \quad ,$$

where A_n and B_n are arbitrary constants and

$$J_p(z) = \sum_{j=0}^{\infty} \frac{(-1)^j\left(\frac{z}{2}\right)^{2j+p}}{\Gamma(j+1)\Gamma(j+p+1)} \tag{9}$$

and

$$J_{-p}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j\left(\frac{z}{2}\right)^{2j-p}}{\Gamma(j+1)\Gamma(j-p+1)} \tag{10}$$

are the Bessel functions of index p and $-p$, respectively.

Consequently, the infinite collection of particular solutions

$$u_n(x,t) = X_n(x)T_n(t) \ ,$$

for $n = 1, 2, 3, \ldots$, solves the partial differential equation (1) and the boundary data (3). Now we take an infinite combination of these functions as our formal solution, that is

$$u(x,t) \sim \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} [A_n(2nt)^p \Gamma(1+p) J_p(nt) + B_n(\tfrac{nt}{2})^p \Gamma(1-p) J_{-p}(nt)] \sin nx \ ,$$

and ask that the arbitrary constants A_n and B_n be chosen so that the initial conditions are satisfied. However, for $p < 0$, $J_p(nt)$ is singular at $t = 0$. Moreover, since $k > 0$, we have $p < \frac{1}{2}$. Thus we take $A_n = 0$. Therefore we ask that

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} B_n \sin nx \ ,$$

where the B_n's are the Fourier sine coefficients of f :

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \ .$$

Finally, we obtain the formal solution

$$(11) \qquad u(x,t) \sim \Gamma(1-p) \sum_{n=1}^{\infty} B_n \sin nx \, (\tfrac{nt}{2})^p J_{-p}(nt) \ .$$

We note that if $k = 0$ then $p = \frac{1}{2}$. In this case

$$(12) \qquad u(x,t) \sim \Gamma(\tfrac{1}{2}) \sum_{n=1}^{\infty} B_n \sin nx \, (\tfrac{nt}{2})^{1/2} J_{-\frac{1}{2}}(nt) \ .$$

However,

$$J_{-\frac{1}{2}}(nt) = \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{nt}{2})^{2j-\frac{1}{2}}}{\Gamma(j+1)\Gamma(j+\frac{1}{2})}$$

$$= \frac{(\frac{nt}{2})^{-1/2}}{\Gamma(\frac{1}{2})} - \frac{(\frac{nt}{2})^{3/2}}{\Gamma(\frac{3}{2})} + \frac{(\frac{nt}{2})^{7/2}}{2\Gamma(\frac{5}{2})} - \frac{(\frac{nt}{2})^{11/2}}{6\Gamma(\frac{7}{2})} + \ldots \quad .$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(x) = \frac{\Gamma(x+1)}{x}$, we have

$$J_{-\frac{1}{2}}(nt) = \sqrt{\frac{2}{\pi nt}} \left[1 - \frac{(nt)^2}{2} + \frac{(nt)^4}{4!} - \frac{(nt)^6}{6!} + \ldots\right]$$

$$= \sqrt{\frac{2}{\pi nt}} \cos nt \ .$$

Therefore, we substitute $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $J_{-\frac{1}{2}}(nt) = \sqrt{\frac{2}{\pi nt}} \cos nt$ into (12) and we get

$$u(x,t) \sim \sum_{n=1}^{\infty} B_n \sin nx \cos nt \ ,$$

which is the known solution for the wave equation with data (2), (3).

We shall now derive an integral relation for the Bessel function (10). With this integral relation we will be able to write (11) in a closed form. To accomplish this, we need to recall the beta function [8]

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \ , \ \text{Re } x > 0 \ , \ \text{Re } y > 0 \ .$$

Furthermore, we know that the beta function can be expressed in terms

of the gamma function, that is,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad .$$

Hence, we have

$$\frac{1}{\Gamma(j-p+1)} = \frac{1}{\Gamma(j+\frac{1}{2})\Gamma(-p+\frac{1}{2})} \int_0^1 t^{j-\frac{1}{2}} (1-t)^{-p-\frac{1}{2}} dt \ ,$$

Re $p < \frac{1}{2}$. We substitute this last relation in (10) and get

$$J_{-p}(z) = \sum_{j=0}^{\infty} \left[\frac{(-1)^j (\frac{z}{2})^{2j-p}}{\Gamma(j+1)\Gamma(j+\frac{1}{2})\Gamma(-p+\frac{1}{2})} \int_0^1 t^{j-\frac{1}{2}} (1-t)^{-p-\frac{1}{2}} dt\right]$$

Reversing the order of summation and integration, we obtain

$$J_{-p}(z) = \frac{(\frac{z}{2})^{-p}}{\Gamma(-p+\frac{1}{2})} \int_0^1 \left[t^{-\frac{1}{2}}(1-t)^{-p-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j (z\sqrt{t})^{2j}}{2^{2j}\Gamma(j+1)\Gamma(j+\frac{1}{2})}\right] dt \quad .$$

But [8],

$$2^{2j}\Gamma(j+1)\Gamma(j+\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\Gamma(2j+1) = \Gamma(\tfrac{1}{2})(2j)! \quad .$$

Hence, it follows that

$$J_{-p}(z) = \frac{(\frac{z}{2})^{-p}}{\Gamma(\frac{1}{2})\Gamma(-p+\frac{1}{2})} \int_0^1 t^{-\frac{1}{2}}(1-t)^{-p-\frac{1}{2}} \cos z\sqrt{t}\ dt \ .$$

Now, letting $t = \sin^2 \theta$, we have

$$\Gamma(\tfrac{1}{2})\Gamma(\tfrac{k}{2})(\tfrac{z}{2})^{p}J_{-p}(z) = 2\int_{0}^{\pi/2}(\cos^{2}\theta)^{-p-\frac{1}{2}}\cos(z\sin\theta)\cos\theta\,d\theta\,,$$

or

$$(13)\qquad (\tfrac{z}{2})^{p}J_{-p}(z) = \frac{2}{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2})}\int_{0}^{\pi/2}\cos^{k-1}\theta\,\cos(z\sin\theta)d\theta\,,$$

which is bounded for $0 < k = 1 - 2p$.

Clearly, since $f \in C^{2}$, $\sum_{n=1}^{\infty} B_{n}\sin nx$ is uniformly convergent [1]. Thus we substitute the integral relation (13) into the formal solution (11) and it follows that (11) is absolutely and uniformly convergent for $0 < t$, $0 \leq x \leq \pi$.

Before we can write (11) in a closed form, we need to use the series representation of f to extend it as an odd function to the interval $-\pi < x < 0$ and then to the entire real line by periodicity of period 2π . That is, we define

$$\tilde{f}(x) = f(x),\ (0,\pi)\ ;\ \tilde{f}(x) = -f(-x),\ (-\pi,0)$$

and

$$F(x) = \tilde{f}(x),\ (-\pi,\pi)\ ;\ F(x) = \tilde{f}(x \pm 2n\pi)$$

elsewhere.

Finally, we can write (11) in closed form. We proceed by substituting the integral relation (13) into (11). We obtain

$$u(x,t) = \Gamma(1-p) \sum_{n=1}^{\infty} B_n \sin(nx) \frac{2}{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2})} \int_0^{\pi/2} \cos(nts) \cos^{k-1} \tau \, d\tau ,$$

where $s = \sin \tau$. Interchanging the order of summation and integration we get

$$(14) \quad u(x,t) = \frac{2}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1} \tau \sum_{n=1}^{\infty} B_n \sin nx \cos nts \, d\tau ,$$

where $\beta(\frac{1}{2},\frac{k}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2})}{\Gamma(1-p)}$. We note that

$$2 \sin nx \cos nts = \sin n(x+ts) + \sin n(x-ts) .$$

When we substitute this sum into (14), we observe that

$$\sum_{n=1}^{\infty} B_n[\sin n(x+st) + \sin n(x-st)]$$

$$= F(x+st) + F(x-st) .$$

Consequently, we have

$$(15) \quad u(x,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(x+st)+F(x-st)]d\tau ,$$

where, as before, $s = \sin \tau$.

Now we will verify that (15) is a solution of (1), (2), (3). In order to show that $u(x,0) = f(x)$, we consider

$$2\int_0^{\pi/2} \sin^{2\sigma-1}\tau \cos^{2\nu-1}\tau \, d\tau \ .$$

Letting $t = \cos^2\tau$, we can write

$$2\int_0^{\pi/2} \sin^{2\sigma-1}\tau \cos^{2\nu-1}\tau \, d\tau = (-)2\int_1^0 \frac{(1-t)^{\sigma-\frac{1}{2}} t^{\nu-\frac{1}{2}}}{2t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} dt$$

$$= \int_0^1 (1-t)^{\sigma-1} t^{\nu-1} dt$$

$$= \beta(\nu,\sigma) = \beta(\sigma,\nu) \ .$$

Thus,

$$2\int_0^{\pi/2} \cos^{k-1}\tau \, d\tau = \beta(\tfrac{1}{2},\tfrac{k}{2}) \ .$$

Hence, we have

$$u(x,0) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(x) + F(x)]d\tau$$

$$= \frac{2f(x)}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, d\tau$$

$$= f(x) \ .$$

We have yet to verify that $u_t(x,0) = 0$. We differentiate (15) with respect to t and obtain

$$u_t(x,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F'(x+st)s + F'(x-st)(-s)]d\tau$$

so that

$$u_t(x,0) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [f'(x) - f'(x)](s)d\tau = 0 .$$

Thus, the initial data are satisfied.

Now we must show that the boundary conditions hold. Since F is an odd function, we have

$$u(0,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(st) + F(-st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(st) - F(st)]d\tau = 0 .$$

Because F is odd and periodic, we can write

$$F(\pi+st) = \tilde{f}(\pi+st - 2\pi) = \tilde{f}(st-\pi) = -f(\pi-st)$$

and

$$F(\pi-st) = \tilde{f}(\pi-st) = f(\pi-st) .$$

Hence, we have

$$u(\pi,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(\pi+st) + F(\pi-st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [-f(\pi-st) + f(\pi-st)]d\tau$$

$$= 0 .$$

Therefore, the boundary data are satisfied.

Now, in order to verify that (15) is a solution of the partial differential equation (1) we consider

$$-t(u_{tt} + \frac{k}{t}u_t - u_{xx}) = t(u_{xx} - u_{tt} - \frac{k}{t}u_t)$$

$$= \frac{t}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \{[F''(x+st) + F''(x-st)]$$

$$- [F''(x+st) + F''(x-st)]s^2$$

$$- \frac{k}{t}[F'(x+st) - F'(x-st)]s\}\cos^{k-1}\tau \, d\tau ,$$

where the prime notation indicates differentiation with respect to the argument. Since $1-s^2 = \cos^2\tau$, the final expression can be written as

$$\frac{t}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \{[F''(x+st) + F''(x-st)]\cos^2\tau$$

$$- \frac{ks}{t}[F'(x+st) - F'(x-st)]\}\cos^{k-1}\tau \, d\tau .$$

This expression, in turn, can be written in the form

$$\frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \{t[F''(x+t\sin\tau) + F''(x-t\sin\tau)]\cos^{k+1}\tau$$

$$-k[F'(x+t\sin\tau) - F'(x-t\sin\tau)]\cos^{k-1}\tau\sin\tau\}d\tau$$

which is clearly

$$\frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \frac{d}{d\tau}\{[F'(x+t\sin\tau) - F'(x-t\sin\tau)]\cos^{k}\tau\}d\tau = 0 \ .$$

Thus, (15) solves the I-BVP (1), (2), (3).

We now state and prove the following theorem concerning the uniqueness of the solution (15).

THEOREM I. The solution of the I-BVP (1), (2), (3), where $|u_t| \leq Kt^{\alpha}$ at $t = 0^+$ for $0 < \alpha$ and K a constant, is uniquely determined.

PROOF. A variant of the energy integral argument will be used. Assume u and v are two solutions of the I-BVP (1), (2), (3). Let $w = u - v$. Then w is a solution of the homogeneous problem

$$w_{tt} + \frac{k}{t}w_t - w_{xx} = 0 \ , \quad 0 < x < \pi, \ 0 < t \ ,$$

$$w(x,0) = 0 = w_t(x,0) \ , \quad 0 \leq x \leq \pi \ ,$$

$$w(0,t) = 0 = w(\pi,t) \ , \quad 0 \leq t \ .$$

We consider

$$0 = 2w_t(w_{tt} + \frac{k}{t}w_t - w_{xx}) \ .$$

After adding and subtracting $2w_x w_{xt}$ to the right side, we have

$$0 = 2w_t w_{tt} + 2w_x w_{xt} - 2w_x w_{tx} - 2w_t w_{xx} + 2\frac{k}{t}w_t^2 \ .$$

We can rewrite this equation as

$$0 = (w_x^2 + w_t^2)_t - 2(w_x w_t)_x + 2\frac{k}{t}w_t^2 \ .$$

This expression is to be integrated over the rectangle $R_\delta = \{(x,t) \mid 0 \leq x \leq \pi \ , \ 0 < \delta \leq t \leq T\}$. We apply Green's theorem to the right side of the equation to get

$$\iint_{R_\delta} [(w_x^2 + w_t^2)_t - 2(w_x w_t)_x + 2\frac{k}{t}w_t^2]dxdt$$

$$= (-1) \oint_{C_\delta} [(w_x^2 + w_t^2)dx + 2w_x w_t dt] + 2k \iint_{R_\delta} \frac{w_t^2}{t} \, dxdt \ ,$$

where C_δ is the boundary of R_δ . The line integral becomes

$$\int_0^\pi [w_x^2(x,T) + w_t^2(x,T)]dx - 2\int_\delta^T w_x(\pi,t)w_t(\pi,t)dt$$

$$- \int_0^\pi [w_x^2(x,\delta) + w_t^2(x,\delta)]dx + 2\int_\delta^T w_x(0,t)w_t(0,t)dt$$

which equals

$$\int_0^{\pi} \{ [w_x^2(x,T) + w_t^2(x,T)] - [w_x^2(x,\delta) + w_t^2(x,\delta)] \} dx ,$$

since $w(0,t) = w(\pi,t) = 0$ implies $w_t(0,t) = w_t(\pi,t) = 0$.

The requirement that $|w_t| \leq Kt^{\alpha}$ is sufficient to induce convergence in $\iint_{R_\delta} \frac{w_t^2}{t} dxdt$. That is

$$\iint_{R_\delta} \frac{w_t^2}{t} dxdt \leq \iint_{R_\delta} L\, t^{2\alpha-1} dxdt ,$$

where L is a constant. The right side of this inequality converges as $\delta \to 0$ since α is positive.

Hence, we take the limit as $\delta \to 0$ and note that $w_t(x,0) = 0$ and that $w(x,0) = 0$ implies $w_x(x,0) = 0$. Therefore we have

$$0 = \int_0^{\pi} (w_x^2 + w_t^2) dx + 2k \int_0^T \int_0^{\pi} \frac{w_t^2}{t} dxdt .$$

Since both integrands are positive, we must have $w_t = w_x = 0$ so that w is equal to a constant. However, by the initial data, $w = 0$. Thus, $u = v$ and the solution is unique.

CHAPTER III

OTHER INITIAL-BOUNDARY VALUE PROBLEMS FOR THE EULER-POISSON-DARBOUX EQUATION

In this chapter we will use Fusaro's procedure to obtain unique solutions of initial-boundary value problems for the EPD equation with various boundary conditions.

First we consider the I-BVP with boundary conditions of the second kind:

$$(1) \qquad u_{tt} + \frac{k}{t}u_t - u_{xx} = 0 , \qquad 0 < x < \pi , \ 0 < t ,$$

$$(2) \qquad u(x,0) = f(x) , \ u_t(x,0) = 0 , \qquad 0 \leq x \leq \pi ,$$

$$(16) \qquad u_x(0,t) = 0 = u_x(\pi,t) , \qquad 0 \leq t .$$

In this problem $k > 0$, $f \in C^2$ on $[0,\pi]$ and we assume $f'(0) = f'(\pi) = 0$.

Again we use the method of separation of variables. Thus, we obtain the two ordinary differential equations (4) and (5). Because of the boundary data (16), we encounter the eigenvalue problem

$$(17) \qquad \begin{aligned} &X'' + \lambda X = 0 \\ &X'(0) = 0 = X'(\pi) \quad . \end{aligned}$$

This problem has eigenvalues $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$, and corresponding eigenfunctions $X_n(x) = \cos nx$, $n = 0, 1, 2, \ldots$. After substituting these values of λ into equation (5), we find that the general solution of (5) is again

$$T_n(t) = A_n(2nt)^p\Gamma(1+p)J_p(nt) + B_n(\frac{nt}{2})^p\Gamma(1-p)J_{-p}(nt) \ .$$

Therefore,

$$u_n(x,t) = X_n(x)T_n(t) \ , \ n = 0, 1, 2, \ldots$$

solves the partial differential equation (1) and the boundary data (16). Thus, we take as our formal solution an infinite combination of these functions, that is,

$$u(x,t) \sim \sum_{n=0}^{\infty} u_n(x,t)$$

$$= \sum_{n=0}^{\infty} [A_n(2nt)^p\Gamma(1+p)J_p(nt) + B_n(\frac{nt}{2})^p\Gamma(1-p)J_{-p}(nt)] \cos nx \ .$$

We ask that the arbitrary constants A_n and B_n be chosen so that the initial conditions are satisfied. Again, we must take $A_n = 0$. Then we ask that

$$u(x,0) = f(x) \sim \sum_{n=0}^{\infty} B_n \cos nx \ ,$$

where the B_n's are the Fourier cosine coefficients of f :

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \ .$$

Finally, our formal solution is

$$(18) \quad u(x,t) \sim \Gamma(1-p) \sum_{n=0}^{\infty} B_n \cos nx \, (\frac{nt}{2})^p \, J_{-p}(nt) \ .$$

Obviously, since $f \in C^2$, $\sum_{n=0}^{\infty} B_n \cos nx$ is uniformly convergent [1]. Therefore, when we use the integral relation (13) in (18), it follows that (18) is absolutely and uniformly convergent for $0 < t$, $0 \leq x \leq \pi$.

In order to write (18) in a closed form, we must use the series representation of f to extend it as an even function to the interval $-\pi < x < 0$, call it $\hat{f}$, and then to the entire real line by periodicity of period 2π . Consequently, $\hat{f}'$ is an odd function. Thus, we have

$$\hat{f}'(x) = f'(x), \ (0,\pi) \ ; \ \hat{f}(x) = -f'(-x), \ (-\pi,0)$$

and

$$F'(x) = \hat{f}'(x), \ (-\pi,\pi) \ ; \ F'(x) = \hat{f}'(x \pm 2n\pi)$$

elsewhere.

Now we substitute the integral relation (13) into (18). We reverse the order of summation and integration to get

$$(19) \qquad u(x,t) = \frac{2}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \sum_{n=0}^{\infty} B_n \cos nx \cos nst \, d\tau \quad ,$$

where $s = \sin \tau$. We recall that

$$2 \cos nx \cos nst = \cos n(x-st) + \cos n(x+st) \quad .$$

When we substitute this sum in (19), we note that

$$\sum_{n=0}^{\infty} B_n[\cos n(x-st) + \cos n(x+st)] = F(x-st) + F(x+st) \quad .$$

Thus,

$$(20) \qquad u(x,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(x+st) + F(x-st)]d\tau \quad .$$

Now we must verify that (20) is a solution of (1), (2), (16). We proceed by demonstrating that the boundary data hold. Since F' is an odd function, we have

$$u_x(0,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F'(-st) + F'(st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [-F'(st) + F'(st)]d\tau = 0 \quad .$$

Because F' is odd and periodic,

$$F'(\pi+st) = \hat{f}'(\pi+st-2\pi) = \hat{f}'(st-\pi) = -f'(\pi-st)$$

and

$$F'(\pi-st) = \hat{f}'(\pi-st) = f'(\pi-st) \ .$$

Therefore, we see that

$$u_x(\pi,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[F'(\pi-st) + F'(\pi+st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[f'(\pi-st) - f'(\pi-st)]d\tau = 0 \ .$$

Hence, the boundary data (16) are satisfied.

The initial data are satisfied by using the same procedure we used to verify them for (15).

In order to show that (20) is indeed a solution of the partial differential equation (1), we proceed in the same manner as we did in showing that (15) is a solution of (1).

We now show that the solution (20) is unique and state this result as Theorem II.

THEOREM II. The solution of the I-BVP (1), (2), (16), where $|u_t| \leq Kt^{\alpha}$ at $t = 0^+$ for $0 < \alpha$ and K a constant, is uniquely determined.

The proof is analogous to the proof of Theorem I except that the line integral terms

$$\int_{\delta}^{T} w_x(\pi,t)w_t(\pi,t)dt \quad \text{and} \quad \int_{\delta}^{T} w_x(0,t)w_t(0,t)dt$$

vanish for different reasons. In Theorem I, $w(0,t) = w(\pi,t) = 0$ implied that $w_t(0,t) = w_t(\pi,t) = 0$ whereas, here the boundary data $w_x(\pi,t) = w_x(0,t) = 0$ causes the vanishing of these terms. Thus, continuing the argument of Theorem I, we arrive at the conclusion of Theorem II.

We thus see that Fusaro's procedure is applicable to the I-BVP for the EPD equation with boundary conditions of the second kind.

We now consider the I-BVP for the EPD equation with mixed boundary conditions:

(1) $u_{tt} + \frac{k}{t} u_t - u_{xx} = 0 , \qquad 0 < x < \pi, \ 0 < t ,$

(2) $u(x,0) = f(x), \ u_t(x,0) = 0 , \qquad 0 \leq x \leq \pi ,$

(21) $u_x(0,t) = 0 = u(\pi,t) , \qquad 0 \leq t .$

Here, $k > 0$, $f \in C^2$ on $[0,\pi]$ and we take $f'(0) = 0 = f(\pi)$.

Again we seek a solution of the form $u(x,t) = X(x)T(t)$ and we obtain equations (4) and (5). As a result of the boundary

data (21), we see that $X(x)$ must satisfy the eigenvalue problem

$$(22) \qquad \begin{aligned} &X' + \lambda X = 0 , \\ &X'(0) = 0 = X(\pi) . \end{aligned}$$

The eigenvalues for (22) are $\lambda_n = (\frac{2n-1}{2})^2$, $n = 1, 2, 3, \ldots$, and the associated eigenfunctions are $X_n(x) = \cos [(\frac{2n-1}{2})x]$, $n = 1, 2, 3, \ldots$. For these new values of λ , the general solution of (5) is

$$T_n(t) = A_n[(2n-1)t]^p \Gamma(1+p) J_p[(\frac{2n-1}{2})t]$$

$$+ B_n[(\frac{2n-1}{4})t]^p \Gamma(1-p) J_{-p}[(\frac{2n-1}{2})t] .$$

Consequently,

$$u_n(x,t) = X_n(x)T_n(t) , \quad n = 1, 2, 3, \ldots ,$$

solves the partial differential equation (1) and the boundary data (21). Hence, we take an infinite combination of these functions as our formal solution, that is,

$$u(x,t) \sim \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \{A_n[(2n-1)t]^p \Gamma(1+p) J_p[(\frac{2n-1}{2})t]$$

$$+ B_n[(\frac{2n-1}{4})t]^p \Gamma(1-p) J_{-p}[(\frac{2n-1}{2})t]\} \cos [(\frac{2n-1}{2})x] .$$

Again, we ask that A_n and B_n be chosen so that the initial conditions are satisfied. Thus, we require that $A_n = 0$. Then

we ask that

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} B_n \cos\left[\left(\frac{2n-1}{2}\right)x\right] ,$$

where the B_n's are the Fourier cosine coefficients of f :

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos\left[\left(\frac{2n-1}{2}\right)x\right]dx \quad .$$

Hence, we arrive at our formal solution

$$(23) \qquad u(x,t) \sim \Gamma(1-p) \sum_{n=1}^{\infty} B_n \cos\left[\left(\frac{2n-1}{2}\right)x\right]\left[\left(\frac{2n-1}{4}\right)t\right]^p J_{-p}\left[\left(\frac{2n-1}{2}\right)t\right] .$$

Clearly, since $f \in C^2$, $\sum_{n=1}^{\infty} B_n \cos\left[\left(\frac{2n-1}{2}\right)x\right]$ is uniformly convergent [1]. Hence, we use the integral relation (13) in (23) and see that (23) is absolutely and uniformly convergent for $0 < t$, $0 \leq x \leq \pi$.

Before we write (23) in a closed form, we need to extend f in the following way:

$$\tilde{f}(x) = f(x), \ [0,\pi) \ ; \ \tilde{f}(x) = f(-x), \ (-\pi,0] \ ,$$

$$\hat{f}(x) = \tilde{f}(x), \ (-\pi,\pi) \ ; \ \hat{f}(x) = -\tilde{f}[-(x-2\pi)] \ , \ (\pi,3\pi) \ ,$$

and

$$F(x) = \hat{f}(x),\ (-\pi,3\pi)\ ;\quad F(x) = \hat{f}(x \pm 4n\pi)$$

elsewhere.

When we substitute the integral relation (13) into (23) and reverse the order of summation and integration, we get

$$(24)\qquad u(x,t) = \frac{2}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \sum_{n=1}^{\infty} B_n \cos[(\tfrac{2n-1}{2})x]\cos[(\tfrac{2n-1}{2})st]d\tau,$$

where $s = \sin\tau$. We note that

$$2\cos(\tfrac{2n-1}{2})x\cos(\tfrac{2n-1}{2})st = \cos[(\tfrac{2n-1}{2})(x-st)] + \cos[(\tfrac{2n-1}{2})(x+st)].$$

When we substitute this sum into (24), we observe that

$$\sum_{n=1}^{\infty} B_n\{\cos[(\tfrac{2n-1}{2})(x-st)] + \cos[(\tfrac{2n-1}{2})(x+st)]\}$$

$$= F(x-st) + F(x+st)\ .$$

Consequently, we can rewrite (24) as

$$(25)\qquad u(x,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[F(x+st) + F(x-st)]d\tau\ .$$

Now we shall verify that (25) is a solution of (1), (2), (21). We begin by showing that the boundary conditions hold. Using the extension of f , we have that

$$F(\pi-st) = f(\pi-st)$$

and

$$F(\pi+st) = -\tilde{f}(\pi-st) = -f(\pi-st) \quad .$$

Thus, we see that

$$u(\pi,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[F(\pi+st) + F(\pi-st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[-f(\pi-st) + f(\pi-st)]d\tau = 0 \ .$$

Since

$$\tilde{f}'(x) = f'(x),\ (0,\pi);\quad \tilde{f}'(x) = -f'(-x),\ (-\pi,0) \ ,$$

$$\hat{f}'(x) = \tilde{f}'(x),\ (-\pi,\pi);\quad \hat{f}'(x) = \tilde{f}'(2\pi-x),\ (\pi,3\pi) \ ,$$

and

$$F'(x) = \hat{f}'(x),\ (-\pi,3\pi);\quad F'(x) = \hat{f}'(x \pm 4n\pi)$$

elsewhere, we have $F'(-st) = -f'(st)$ and $F'(st) = f'(st)$.

Thus, differentiating with respect to x , we get

$$u_x(0,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[F'(-st) + F'(st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau\,[-f'(st) + f'(st)]d\tau = 0 .$$

Therefore, the boundary conditions (21) are satisfied.

The initial data can be verified for (25) in the same way that they were verified for (15).

In order to show that (25) is a solution of (1), we use the same procedure that we used to show that (15) is a solution of (1).

The solution (25) is unique as stated in Theorem III.

THEOREM III. The solution of the I-BVP (1), (2), (21), where $|u_t| \leq Kt^{\alpha}$ at $t = 0^+$ for $0 < \alpha$ and K a constant, is uniquely determined.

The proof is comparable to that of Theorem I except that

$$\int_\delta^T w_x(\pi,t)w_t(\pi,t)dt = 0 = \int_\delta^T w_x(0,t)w_t(0,t)dt$$

because $w(\pi,t) = 0$ implies $w_t(\pi,t) = 0$ and, of course, because $w_x(0,t) = 0$.

In the remainder of this chapter we consider the I-BVP

(1) $$u_{tt} + \frac{k}{t}u_t - u_{xx} = 0 , \qquad 0 < x < \pi , \; 0 < t ,$$

(2) $$u(x,0) = f(x) , \; u_t(x,0) = 0 , \quad 0 \leq x \leq \pi ,$$

(26) $\quad u(0,t) = 0 = u_x(\pi,t) \ , \quad 0 \leq t \ .$

In this problem $k > 0$, $f \in C^2$ on $[0,\pi]$ and we assume $f(0) = 0 = f'(\pi)$.

Again we use the method of separation of variables and we get equations (4) and (5). Because of the boundary data, we obtain the eigenvalue problem

$$(27) \qquad \begin{aligned} X'' + \lambda X &= 0 \\ X(0) = 0 &= X'(\pi) \quad . \end{aligned}$$

We find that its eigenvalues are $\lambda_n = (\frac{2n-1}{2})^2$, $n = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $X_n(x) = \sin[(\frac{2n-1}{2})x]$, $n = 1, 2, 3, \ldots$. For these values of λ , we find that the general solution of (5) is

$$T_n(t) = A_n[(2n-1)t]^p \Gamma(1+p) J_p[(\frac{2n-1}{2})t]$$

$$+ B_n[(\frac{2n-1}{4})t]^p \Gamma(1-p) J_{-p}[(\frac{2n-1}{2})t] \ .$$

Thus,

$$u_n(x,t) = X_n(x)T_n(t) \ , \ n = 1, 2, 3, \ldots \ ,$$

solves the partial differential equation (1) and the boundary data (26). Our formal solution is an infinite combination of these

functions, that is,

$$u(x,t) \sim \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \{A_n[(2n-1)t]^p \Gamma(1+p) J_p[(\frac{2n-1}{2})t] + B_n[(\frac{2n-1}{4})t]^p \Gamma(1-p) J_{-p}[(\frac{2n-1}{2})t]\}$$

$$\cdot \sin (\frac{2n-1}{2})x \quad .$$

We ask that the arbitrary constants A_n and B_n be selected so that the initial conditions are satisfied. Thus, we take $A_n = 0$. Then we ask that

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} B_n \sin [(\frac{2n-1}{2})x] ,$$

where the B_n's are the Fourier sine coefficients of f :

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin [(\frac{2n-1}{2})x] dx \quad .$$

Finally, we obtain the formal solution

$$(28) \quad u(x,t) \sim \Gamma(1-p) \sum_{n=1}^{\infty} B_n \sin [(\frac{2n-1}{2})x][(\frac{2n-1}{4})t]^p J_{-p}[(\frac{2n-1}{2})t] \quad .$$

Since $f \in C^2$, we know $\sum_{n=1}^{\infty} B_n \sin [(\frac{2n-1}{2})x]$ converges uniformly [1]. Thus, we use the integral relation (13) in (28) and see that (28) is absolutely and uniformly convergent for

$0 < t$, $0 \leq x \leq \pi$.

Before we write (28) in a closed form, we must extend f in the following manner:

$$\tilde{f}(x) = f(x),\ (0,\pi);\quad \tilde{f}(x) = -f(-x),\ (-\pi,0)\ ,$$

$$\hat{f}(x) = \tilde{f}(x),\ (-\pi,\pi);\quad \hat{f}(x) = \tilde{f}(2\pi-x),\ (\pi,3\pi)\ ,$$

and

$$F(x) = \hat{f}(x),\ (-\pi,3\pi);\quad F(x) = \hat{f}(x \pm 4n\pi)$$

elsewhere.

Next, we substitute the integral relation (13) in (28). Reversing the order of summation and integration, we get

$$(29)\qquad u(x,t) = \frac{2}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \sum_{n=1}^{\infty} B_n \sin[(\tfrac{2n-1}{2})x]\cos[(\tfrac{2n-1}{2})st]d\tau,$$

where $s = \sin\tau$. We note that

$$2\sin[(\tfrac{2n-1}{2})x]\cos[(\tfrac{2n-1}{2}st] = \sin[(\tfrac{2n-1}{2})(x+st)]+\sin[(\tfrac{2n-1}{2})(x-st)]\ .$$

When we substitute this sum into (29), we observe that

$$\sum_{n=1}^{\infty} B_n \{\sin[(\tfrac{2n-1}{2})(x+st)] + \sin[(\tfrac{2n-1}{2})(x-st]\} = F(x+st) + F(x-st).$$

Consequently, we have

$$(30) \qquad u(x,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(x+st) + F(x-st)]d\tau \ ,$$

where $s = \sin \tau$.

Now we will verify that (30) is a solution of (1), (2), (26). We begin by demonstrating that the boundary data hold. From the extension of f , we see that $F(-st) = -f(st)$ and $F(st) = f(st)$ so that

$$u(0,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F(-st) + F(st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [-f(st) + f(st)]d\tau = 0 \ .$$

Then, since

$$\tilde{f}'(x) = f'(x), \ (0,\pi); \ \tilde{f}'(x) = f'(-x), \ (-\pi,0) \ ,$$

$$\hat{f}'(x) = \tilde{f}'(x), \ (-\pi,\pi); \ \hat{f}'(x) = -\tilde{f}'(2\pi-x), \ (\pi,3\pi) \ ,$$

and

$$F'(x) = \hat{f}'(x), \ (-\pi,3\pi); \ F'(x) = \hat{f}'(x \pm 4n\pi) \quad \text{elsewhere},$$

we have

$$u_x(\pi,t) = \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [F'(\pi-st)+F'(\pi+st)]d\tau$$

$$= \frac{1}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \, [f'(\pi-st)-f'(\pi-st)]d\tau = 0 \, .$$

Thus, we have shown that the boundary data hold.

The initial data are verified for (30) in the same manner as they were for (15).

We verify that (30) is a solution of the partial differential equation (1) by using the same procedure that we used to show that (15) is a solution.

The fact that solution (30) is unique is stated in the following theorem.

THEOREM IV. The solution of the I-BVP (1), (2), (26), where $|u_t| \leq Kt^{\alpha}$ at $t = 0^+$ for $0 < \alpha$ and K a constant, is uniquely determined.

The proof is again analogous to that of Theorem I except that

$$\int_\delta^T w_x(\pi,t)w_t(\pi,t)dt = 0 = \int_\delta^T w_x(0,t)w_t(0,t)dt$$

because $w(0,t) = 0$ implies $w_t(0,t) = 0$ and because $w_x(\pi,t) = 0$.

We have seen that Fusaro's argument holds for these initial-boundary value problems with boundary conditions of the second kind and with mixed boundary conditions.

CHAPTER IV

EXTENSION TO PROBLEMS IN TWO-SPACE VARIABLES

This chapter deals with the I-BVP for the EPD equation in two-space variables. We attempt to use Fusaro's procedure as we did in the previous two chapters but find we are unable to write the formal solution in a closed form. Consequently, we employ other methods to demonstrate that the formal solution actually satisfies the I-BVP.

The problem that we consider is:

$$(31) \qquad u_{tt} + \frac{k}{t} u_t - u_{xx} - u_{yy} = 0 , \qquad 0 < x, y < \pi , 0 < t ,$$

$$(32) \qquad u(x,y,0) = f(x,y), \; u_t(x,y,0) = 0 , \quad 0 \leq x, y \leq \pi ,$$

$$(33) \qquad \begin{cases} u(0,y,t) = 0 = u(\pi,y,t) , & 0 \leq y \leq \pi , 0 \leq t , \\ u(x,0,t) = 0 = u(x,\pi,t) , & 0 \leq x \leq \pi , 0 \leq t . \end{cases}$$

Here k is a positive constant, $f \in C^4$ for $0 \leq x, y \leq \pi$, and f satisfies the boundary conditions $f(0,y) = 0 = f(\pi,y)$ and $f(x,0) = 0 = f(x,\pi)$. The requirement that $f \in C^4$ is sufficient to deduce uniform convergence of the differentiated series solution that we will obtain.

The separation of variables will be done in two stages. First, we assume that $u(x,y,t) = T(t)\phi(x,y)$ and substitute in (31) to get

$$T''\phi + \frac{k}{t} T'\phi - T\phi_{xx} - T\phi_{yy} = 0 ,$$

where, as in the previous problems, the prime notation indicates differentiation with respect to the argument. Then, dividing by $T\phi$, we get

$$\frac{T''}{T} + \frac{k}{t}\frac{T'}{T} - \frac{\phi_{xx}}{\phi} - \frac{\phi_{yy}}{\phi} = 0 ,$$

or

$$\frac{T''}{T} + \frac{k}{t}\frac{T'}{T} = \frac{\phi_{xx}}{\phi} + \frac{\phi_{yy}}{\phi} = -\lambda ,$$

where λ is a constant. Thus, we get our familiar ordinary differential equation (5) and a new partial differential equation

$$\phi_{xx} + \phi_{yy} + \lambda\phi = 0 . \tag{34}$$

Because of the boundary data (33), we conclude that $\phi(x,y)$ must satisfy the two-dimensional eigenvalue problem

$$\begin{aligned} &\Delta\phi + \lambda\phi = 0 , && 0 < x, y < \pi , \\ &\phi(0,y) = 0 = \phi(\pi,y) , && 0 \leq y \leq \pi , \\ &\phi(x,0) = 0 = \phi(x,\pi) , && 0 \leq x \leq \pi . \end{aligned} \tag{35}$$

We undertake to find the eigenvalues and the corresponding

eigenfunctions by further separation of variables. That is, we assume $\phi(x,y) = X(x)Y(y)$ and substitute it in (34) to get

$$X''Y + XY'' + \lambda XY = 0 .$$

Then, dividing by XY, we get

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0$$

or

$$\frac{X''}{X} + \lambda = -\frac{Y''}{Y} = \mu ,$$

where μ is a constant. Thus, X and Y must satisfy

$$\begin{aligned} &X'' + (\lambda-\mu)X = 0 , \\ &X(0) = 0 = X(\pi) , \end{aligned} \tag{36}$$

and

$$\begin{aligned} &Y'' + \mu Y = 0 , \\ &Y(0) = 0 = Y(y) , \end{aligned} \tag{37}$$

respectively. The values of μ for which (37) has a nontrivial solution (eigenvalues) are $\mu_n = n^2$, $n = 1, 2, 3, \ldots$. The

associated eigenfunctions are $Y_n(y) = \sin ny$. Then, with $\mu = \mu_n$ in (36), the values of λ which allow a nontrivial solution of (36) are given by

$$\lambda_{m,n} - \mu_n = m^2 \ , \ m = 1, 2, 3, \ldots \ ,$$

and the corresponding eigenfunctions are $X_m(x) = \sin mx$, $m = 1, 2, 3, \ldots$. Thus, we have found eigenvalues

$$\lambda_{m,n} = m^2 + n^2 \ , \ m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots \ ,$$

and corresponding eigenfunctions

$$\phi_{m,n}(x,y) = X_m(x)Y_n(y) = \sin mx \sin ny$$

for (35).

For these values of λ , the general solution of (5) is

$$T_{m,n}(t) = A_{m,n}(2t\sqrt{m^2+n^2})^p \Gamma(1+p) J_p(t\sqrt{m^2+n^2})$$

$$+ B_{m,n}(t\sqrt{\frac{m^2+n^2}{2}})^p \Gamma(1-p) J_{-p}(t\sqrt{m^2+n^2}) \ .$$

Consequently, the infinite collection of particular solutions

$$u_{m,n}(x,y,t) = T_{m,n}(t)\phi_{m,n}(x,y) \ ,$$

for $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, solves the partial differential equation (31) and the boundary data (33). Now we take an infinite combination of these functions as our formal solution, that is,

$$u(x,y,t) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{m,n}(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{m,n}(2t\sqrt{m^2+n^2})^p \cdot \Gamma(1+p)J_p(t\sqrt{m^2+n^2}) + B_{m,n}(t\frac{\sqrt{m^2+n^2}}{2})^p \Gamma(1-p) \cdot J_{-p}(t\sqrt{m^2+n^2})] \sin mx \sin ny ,$$

and ask that the arbitrary constants $A_{m,n}$ and $B_{m,n}$ be chosen so that the initial conditions are satisfied. Thus, as we saw in Chapter II, we must take $A_{m,n} = 0$. Then we ask that

$$u(x,y,0) = f(x,y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{m,n} \sin mx \sin ny ,$$

where the $B_{m,n}$'s are the double Fourier sine coefficients of f :

$$B_{m,n} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x,y) \sin mx \sin ny \, dxdy .$$

Finally, we obtain the formal solution

$$(38) \quad u(x,y,t) \sim \Gamma(1-p) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{m,n}(t\frac{\sqrt{m^2+n^2}}{2})^p J_{-p}(t\sqrt{m^2+n^2}) \sin mx \sin ny.$$

Now, we will attempt to write (38) in a closed form. We begin by substituting the integral relation (13) in (38). We assume that we can interchange the order of summation and integration. (It will be shown later that (38) is absolutely and uniformly convergent for $0 < t$, $0 \leq x, y \leq \pi$.) Thus we obtain

$$u(x,y,t) \sim \frac{2}{\beta(\frac{1}{2},\frac{k}{2})} \int_0^{\pi/2} \cos^{k-1}\tau \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{m,n} \sin mx \sin ny \cdot \cos\left(ts\sqrt{m^2+n^2}\right)d\tau \ .$$

We know that

$$2 \sin ny \cos\left(ts\sqrt{m^2+n^2}\right) = \sin\left(ny + ts\sqrt{m^2+n^2}\right) + \sin\left(ny-ts\sqrt{m^2+n^2}\right) .$$

Then multiplying by $\sin mx$, we have

$$2 \sin mx \sin ny \cos\left(ts\sqrt{m^2+n^2}\right)$$

$$= \sin mx \sin n\left(y+ts\sqrt{1 + (\tfrac{m}{n})^2}\right) + \sin mx \sin n\left(y - ts\sqrt{1 + (\tfrac{m}{n})^2}\right) .$$

Because of the factor $\sqrt{1 + (\frac{m}{n})^2}$, we are unable to express

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{m,n}\left[\sin mx \sin n\left(y+ts\sqrt{1+(\tfrac{m}{n})^2}\right)+\sin mx \sin n\left(y-ts\sqrt{1+(\tfrac{m}{n})^2}\right)\right]$$

in terms of f . Thus we cannot write (38) in a closed form.

Consequently, we need to show that the series solution (38) converges uniformly. Applying the integral relation (13), we have that $(t\frac{\sqrt{m^2+n^2}}{2})^p J_{-p}(t\sqrt{m^2+n^2})$ is bounded. Now we must show that

$$(39) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n} \sin mx \sin ny$$

actually converges to $f(x,y)$. To show this, we consider

$$\Delta f \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n} \sin mx \sin ny ,$$

where $C_{m,n} = -(m^2 + n^2)B_{m,n}$.

We will verify that the coefficients, $C_{m,n}$, satisfy Bessel's inequality

$$(\frac{\pi}{2})^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}^2 \leq \int_0^{\pi}\int_0^{\pi} (\Delta f)^2 \, dxdy .$$

Since the eigenfunctions are orthogonal on $[0,\pi] \times [0,\pi]$, we can write

$$C_{m,n} = \frac{\int_0^{\pi}\int_0^{\pi} \Delta f \cdot \phi_{m,n}(x,y)dxdy}{\int_0^{\pi}\int_0^{\pi} \phi_{m,n}^2 (x,y)dxdy} ,$$

where $\phi_{m,n}(x,y) = \sin mx \sin ny$. Now we let

$$t_{k,\ell}(x,y) = \sum_{m=1}^{k} \sum_{n=1}^{\ell} \alpha_{m,n}\, \phi_{m,n}(x,y) \ ,$$

where the $\alpha_{m,n}$'s are any real coefficients. Then, we have the identity

$$\int_0^{\pi} \int_0^{\pi} [\Delta f - t_{k,\ell}(x,y)]^2 dxdy = \int_0^{\pi} \int_0^{\pi} (\Delta f)^2 dxdy$$

$$- 2 \sum_{m=1}^{k} \sum_{n=1}^{\ell} \alpha_{m,n} C_{m,n} \int_0^{\pi} \int_0^{\pi} \phi_{m,n}^2(x,y)dxdy$$

$$+ \sum_{m=1}^{k} \sum_{n=1}^{\ell} \alpha_{m,n}^2 \int_0^{\pi} \int_0^{\pi} \phi_{m,n}^2(x,y)dxdy \ .$$

We choose $\alpha_{m,n} = C_{m,n}$ so that $t_{k,\ell}(x,y)$ becomes $s_{k,\ell}(x,y) = \sum_{m=1}^{k} \sum_{n=1}^{\ell} C_{m,n}\phi_{m,n}(x,y)$, the k , ℓth partial sum of the double Fourier series of Δf . Thus, we have

$$\int_0^{\pi} \int_0^{\pi} [\Delta f - s_{k,\ell}(x,y)]^2 dxdy = \int_0^{\pi} \int_0^{\pi} (\Delta f)^2 dxdy$$

$$- \sum_{m=1}^{k} \sum_{n=1}^{\ell} C_{m,n}^2 \int_0^{\pi} \int_0^{\pi} \phi_{m,n}^2(x,y)dxdy \ .$$

Since

$$\int_0^{\pi} \int_0^{\pi} [\Delta f - s_{k,\ell}(x,y)]^2 dxdy \geq 0$$

and

$$\int_0^{\pi}\int_0^{\pi} \sin^2 mx \sin^2 ny \, dxdy = \left(\frac{\pi}{2}\right)^2 ,$$

we deduce that

$$\int_0^{\pi}\int_0^{\pi} (\Delta f)^2 dxdy \geq \left(\frac{\pi}{2}\right)^2 \sum_{m=1}^{k}\sum_{n=1}^{\ell} C_{m,n}^2$$

and then let $k, \ell \to \infty$.

Using the integral test, we find that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^2} < \infty$.

That is, letting $g(x,y) = \frac{1}{(x^2+y^2)^2}$ and using polar coordinates

$$\int_0^{2\pi}\int_1^{\infty} \frac{dr d\theta}{r^3} = \pi .$$

Then, making use of the arithmetic-geometric mean inequality, we get

$$2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |C_{m,n}|\left|\frac{1}{m^2+n^2}\right| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |C_{m,n}|^2 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^2}$$

so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left|\frac{C_{m,n}}{m^2+n^2}\right| < \infty .$$

Consequently,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |B_{m,n}| < \infty .$$

Therefore, the double Fourier sine series (39), by comparison, is absolutely and uniformly convergent. Hence, the series solution (38) converges absolutely and uniformly to $u(x,y,t)$.

Now we wish to show that the solution

$$(40)\quad u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}\left(t\frac{\sqrt{m^2+n^2}}{2}\right)^{p}\Gamma(1-p)\, J_{-p}\left(t\sqrt{m^2+n^2}\right)\sin mx \sin ny$$

satisfies the partial differential equation (31). For convenience, we replace $\sqrt{m^2+n^2}$ by z . Thus, differentiating formally, we have

$$(41)\quad u_x \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}(m)\left(\frac{tz}{2}\right)^{p}\Gamma(1-p)J_{-p}(tz)\cos mx \sin ny ,$$

$$(42)\quad u_{xx} \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}(-m^2)\left(\frac{tz}{2}\right)^{p}\Gamma(1-p)J_{-p}(tz)\sin mx \sin ny ,$$

$$(43)\quad u_y \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}(n)\left(\frac{tz}{2}\right)^{p}\Gamma(1-p)J_{-p}(tz)\sin mx \cos ny ,$$

$$(44)\quad u_{yy} \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}(-n^2)\left(\frac{tz}{2}\right)^{p}\Gamma(1-p)J_{-p}(tz)\sin mx \sin ny ,$$

and

$$(45)\quad u_t \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}(-)z\left(\frac{tz}{2}\right)^{p}\Gamma(1-p)J_{1-p}(tz)\sin mx \sin ny .$$

Series (45) can be rewritten as

$$u_t \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-)zB_{m,n}(\frac{tz}{2})(\frac{tz}{2})^{p-1}\Gamma(1-p)J_{1-p}(tz) \sin mx \sin ny \ .$$

Now using the formula [7]

$$\frac{d}{dx}\left[x^{-n}J_n(\alpha x)\right] = -\alpha x^{-n}J_{n+1}(\alpha x) \ ,$$

we find

$$\frac{d}{dt}\{\frac{tz}{2}[(\frac{z}{2})^{p-1}t^{p-1}J_{1-p}(tz)]\} = (-)z(\frac{tz}{2})^{p}J_{2-p}(tz)+(\frac{tz}{2})^{p}t^{-1}J_{1-p}(tz) \ .$$

Therefore, we determine

$$u_{tt} \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [z^2B_{m,n}(\frac{tz}{2})^{p}\Gamma(1-p)J_{2-p}(tz)$$

$$- zB_{m,n}(\frac{tz}{2})^{p}t^{-1}\Gamma(1-p)J_{1-p}(tz)] \sin mx \sin ny \ ,$$

which can be written

$$(46)\quad u_{tt} \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [z^2B_{m,n}(\frac{tz}{2})^{p}\Gamma(1-p)J_{2-p}(tz)$$

$$- \frac{z^2}{2}B_{m,n}(\frac{tz}{2})^{p-1}\Gamma(1-p)J_{1-p}(tz)] \sin mx \sin ny \ .$$

In order to show that these differentiated series actually converge uniformly, it is sufficient to show that

$$(47) \qquad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2+n^2)B_{m,n} \sin mx \sin ny$$

converges uniformly since the factors

$$(48) \qquad (t\frac{\sqrt{m^2+n^2}}{2})^{\ell} J_q(t\sqrt{m^2+n^2}) ,$$

where $q = -p$, $1 - p$, or $2 - p$ and $\ell = p$ or $p - 1$, are all bounded.

Clearly, when $q = -p$ and $\ell = p$, we know from the integral relation (13), that (48) is bounded. For the other choices of q and ℓ we can see that (48) is bounded by looking at the asymptotic representation of $J_q(x)$, where $x = t\sqrt{m^2+n^2} > 0$. Now for $q = 1 - p$ or $q = 2 - p$ we have $q \geq 0$. Then for large values of x , $J_q(x)$ behaves asymptotically like $\sqrt{\frac{2}{\pi x}} \cos (x - \frac{\pi}{4} - \frac{q\pi}{2})$ and for small values of x , we have the approximation $J_q(x) \sim \frac{x^q}{2^q q!}$ [7].

Now we shall demonstrate that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{m,n} \sin mx \sin ny ,$$

where $C_{m,n} = -(m^2+n^2)B_{m,n}$, converges uniformly to Δf . Our procedure will be analogous to the one we used to show that (39) converges uniformly.

We consider

$$\Delta\Delta f \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{m,n} \sin mx \sin ny \ ,$$

where $D_{m,n} = -(m^2+n^2)\, C_{m,n}$. Since the eigenfunctions are orthogonal on $[0,\pi] \times [0,\pi]$, we can write

$$D_{m,n} = \frac{\int_0^{\pi} \int_0^{\pi} \Delta\Delta f \ \phi_{m,n}(x,y)dxdy}{\int_0^{\pi} \int_0^{\pi} \phi^2_{m,n}(x,y)dxdy} \ ,$$

where $\phi_{m,n}(x,y) = \sin mx \sin ny$. Letting

$t_{k,\ell}(x,y) = \sum_{m=1}^{k} \sum_{n=1}^{\ell} \alpha_{m,n}\, \phi_{m,n}(x,y)$, where the $\alpha_{m,n}$'s are any real coefficients, we obtain the relation

$$\int_0^{\pi} \int_0^{\pi} [\Delta\Delta f - t_{k,\ell}(x,y)]^2 dxdy = \int_0^{\pi} \int_0^{\pi} (\Delta\Delta f)^2 dxdy$$

$$- 2 \sum_{m=1}^{k} \sum_{n=1}^{\ell} \alpha_{m,n} D_{m,n} \int_0^{\pi} \int_0^{\pi} \phi^2_{m,n}(x,y)dxdy$$

$$+ \sum_{m=1}^{k} \sum_{n=1}^{\ell} \alpha^2_{m,n} \int_0^{\pi} \int_0^{\pi} \phi^2_{m,n}(x,y)dxdy \ .$$

We choose $\alpha_{m,n} = D_{m,n}$ so that $t_{k,\ell}(x,y)$ becomes

$$s_{k,\ell}(x,y) = \sum_{m=1}^{k} \sum_{n=1}^{\ell} D_{m,n}\, \phi_{m,n}(x,y) \ .$$

Hence, we find that

$$\int_0^{\pi} \int_0^{\pi} (\Delta\Delta f)^2 dxdy - \sum_{m=1}^{k} \sum_{n=1}^{\ell} D^2_{m,n} \int_0^{\pi} \int_0^{\pi} \phi^2_{m,n}(x,y)dxdy \geq 0$$

since

$$\int_0^{\pi}\int_0^{\pi} [\Delta\Delta f - s_{k,\ell}(x,y)]^2 dxdy \geq 0 \quad .$$

Thus, the coefficients, $D_{m,n}$, satisfy Bessel's inequality

$$(\tfrac{\pi}{2})^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{m,n}^2 \leq \int_0^{\pi}\int_0^{\pi} (\Delta\Delta f)^2 dxdy \quad .$$

Clearly, since we required that $f \in C^4[0,\pi] \times [0,\pi]$, we have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{m,n}^2$ converges. As we have previously shown, $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^2}$ converges. Hence, applying the geometric-arithmetic mean inequality, we get

$$2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |D_{m,n}|\,\left|\frac{1}{m^2+n^2}\right| \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |D_{m,n}|^2 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^2}$$

so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left|\frac{D_{m,n}}{m^2+n^2}\right| < \infty \quad .$$

Consequently,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |(m^2+n^2)B_{m,n}| < \infty \quad .$$

Therefore, we know that series (47), by comparison, is absolutely

and uniformly convergent. Since the series (41), (42), (43), (44), (45), and (46) are majorized by (47), they converge uniformly to u_x , u_{xx} , u_y , u_{yy} , u_t , and u_{tt} , respectively.

Now we can verify that solution (40) satisfies the initial and boundary data. Since the double Fourier sine series converges uniformly to $f(x,y)$, we get $u(x,y,0) = f(x,y)$. From the uniform convergence of (45), we see that

$$u_t(x,y,0) = 0 \quad .$$

Thus, the initial conditions are satisfied. Clearly, the boundary conditions hold since the eigenfunctions, sin mx sin ny , are factors in solution (40).

Finally, we shall verify that (40) satisfies the partial differential equation (31). Again we let $z = \sqrt{m^2+n^2}$. We note that [7]

$$J_{p^*-1}(\alpha x) + J_{p^*+1}(\alpha x) = \frac{2p^*}{\alpha x} J_{p^*}(\alpha x) \quad . \tag{49}$$

Consequently, for $p^* = 1 - p$, we have

$$u_{tt} + \frac{k}{t} u_t - (u_{xx} + u_{yy}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} z^2 B_{m,n}$$

$$\cdot \left(\frac{tz}{2}\right)^p \Gamma(1-p)[J_{-p}(tz) + J_{2-p}(tz)]$$

$$\cdot \sin mx \sin ny - (1+k) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} z\, B_{m,n}$$

$$\cdot \left(\frac{tz}{2}\right)^{p} t^{-1}\Gamma(1-p)J_{1-p}(tz) \sin mx \sin ny$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} z^{2}B_{m,n}\left(\frac{tz}{2}\right)^{p}\Gamma(1-p)\,\frac{2(1-p)}{tz}$$

$$\cdot J_{1-p}(tz) \sin mx \sin ny - (1+k)$$

$$\cdot \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} z\,B_{m,n}\left(\frac{tz}{2}\right)^{p} t^{-1}\Gamma(1-p)J_{1-p}(tz) \sin mx \sin ny$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [2(1-p) - (1+k)]\, z\, B_{m,n}\left(\frac{tz}{2}\right)^{p}$$

$$\cdot t^{-1}\Gamma(1-p)J_{1-p}(tz) \sin mx \sin ny \quad ,$$

where for the second equation we used (49). Now since $2(1-p) - (1+k) = (2-2p) - (1+1-2p) = 0$, we have that $u_{tt} - \frac{k}{t}u_t - (u_{xx} + u_{yy}) = 0$. Therefore, (40) satisfies the partial differential equation (31).

Now we shall show that solution (40) is unique. This result is stated in the following theorem.

THEOREM V. The C^2 solution of the I-BVP (31), (32), (33), where $|u_t| \leq Kt^{\alpha}$ at $t = 0^{+}$ for $0 < \alpha$ and K a constant, is uniquely determined.

PROOF. We will use the energy integral argument. We assume u and v are two solutions of (31), (32), (33), and let $w = u - v$. Then w is a solution of the homogeneous problem

$$w_{tt} + \frac{k}{t} w_t - w_{xx} - w_{yy} = 0 , \quad 0 < x, y < \pi , 0 < t ,$$

$$w(x,y,0) = 0 = w_t(x,y,0) , \quad 0 \leq x , y \leq \pi ,$$

$$w(0,y,t) = 0 = w(\pi,y,t) , \quad 0 \leq y \leq \pi , 0 \leq t ,$$

$$w(x,0,t) = 0 = w(x,\pi,t) , \quad 0 \leq x \leq \pi , 0 \leq t .$$

We consider

$$0 = 2w_t(w_{tt} + \frac{k}{t} w_t - w_{xx} - w_{yy}) .$$

Adding and subtracting $2w_x w_{xt}$ and $2w_y w_{yt}$ from the right side, we get

$$0 = 2w_t w_{tt} - 2w_t w_{xx} - 2w_t w_{yy} + 2w_x w_{xt} - 2w_x w_{tx}$$

$$+ 2w_y w_{yt} - 2w_y w_{ty} + 2 \frac{k}{t} w_t^2 .$$

We can rewrite this equation as

$$0 = (w_x^2 + w_t^2 + w_y)_t^2 - 2(w_x w_t)_x - 2(w_y w_t)_y + 2\frac{k}{t} w_t^2 ,$$

which is to be integrated over the rectangular parallelepiped

$$R_\delta = \{(x,y,t) \mid 0 \le x,\ y \le \pi\ ,\ 0 < \delta \le t \le T\}\ .$$

We can write

$$\iiint_{R_\delta} [(w_x^2 + w_t^2 + w_y^2)_t - 2(w_x w_t)_x - 2(w_y w_t)_y + 2\frac{k}{t} w_t^2]dxdydt$$

$$= \int_0^\pi \{\iint_Q [\tfrac{1}{2}(w_x^2 + w_t^2 + w_y^2)_t - 2(w_x w_t)_x]dxdt\}dy$$

$$+ \int_0^\pi \{\iint_S [\tfrac{1}{2}(w_x^2 + w_t^2 + w_y^2)_t - 2(w_y w_t)_y]dydt\}dx$$

$$+ \iiint_{R_\delta} 2\frac{k}{t} w_t^2\, dxdydt\ ,$$

where $Q = \{(x,y,t) \mid 0 \le x \le \pi\ ,\ 0 < \delta \le t \le T\}$ and

$S = \{(x,y,t) \mid 0 \le y \le \pi\ ,\ 0 < \delta \le t \le T\}$.

We apply Green's theorem to the right side of the equation and obtain

$$+ \int_0^\pi \{\oint_{\tilde{Q}} [\tfrac{1}{2}(w_x^2 + w_y^2 + w_t^2)dx + 2w_x w_t dt]\}dy$$

$$+ \int_0^\pi \{\oint_{\tilde{S}} [\tfrac{1}{2}(w_x^2 + w_y^2 + w_t^2)dy + 2w_y w_t dt]\}dx$$

$$+ \iiint_{R_\delta} 2\frac{k}{t} w_t^2\, dxdydt\ ,$$

where $\tilde{Q}$ and $\tilde{S}$ are the boundaries of Q and S, respectively. On the parallelepiped these integrals become

$$\int_0^{\pi}\int_0^{\pi} [w_x^2(x,y,T) + w_t^2(x,y,T) + w_y^2(x,y,T)]dxdy$$

$$- \int_0^{\pi}\int_0^{\pi} [w_x^2(x,y,\delta) + w_t^2(x,y,\delta) + w_y^2(x,y,\delta)]dxdy$$

$$+ \int_0^{\pi}\int_{\delta}^{T} 2w_x(\pi,y,t)w_t(\pi,y,t)dtdy$$

$$- \int_0^{\pi}\int_{\delta}^{T} 2w_x(0,y,t)w_t(0,y,t)dtdy$$

$$+ \int_0^{\pi}\int_{\delta}^{T} 2w_y(x,\pi,t)w_t(x,\pi,t)dtdx$$

$$- \int_0^{\pi}\int_{\delta}^{T} 2w_y(x,0,t)w_t(x,0,t)dtdx + \iiint_{R_\delta} 2\frac{k}{t} w_t^2 \, dxdydt ,$$

which equals

$$\int_0^{\pi}\int_0^{\pi} \{[w_x^2(x,y,T) + w_t^2(x,y,T) + w_y^2(x,y,T)]$$

$$- [w_x^2(x,y,\delta) + w_t^2(x,y,\delta) + w_y^2(x,y,\delta)]\} \, dxdy$$

$$+ \iiint_{R_\delta} 2\frac{k}{t} w_t^2 \, dxdydt$$

since $w(0,y,t) = w(\pi,y,t) = 0$ implies $w_t(0,y,t) = w_t(\pi,y,t) = 0$ and $w(x,0,t) = w(x,\pi,t) = 0$ implies $w_t(x,0,t) = w_t(x,\pi,t) = 0$. By an argument analogous to the one used in the proof of Theorem I, we deduce the convergence of $\iiint_{R_\delta} \frac{w_t^2}{t} \, dxdydt$.

Then we can take the limit as $\delta \to 0$ and note that $w_t(x,y,0) = 0$ and that $w(x,y,0) = 0$ implies $w_x(x,y,0) = 0 = w_y(x,y,0)$. Hence, we have

$$0 = \int_0^{\pi} \int_0^{\pi} (w_x^2 + w_y^2 + w_t^2)dxdy + \iiint_{R_0} 2\frac{k}{t} w_t^2 \, dxdydt .$$

Since both integrands are positive, we have $w_t = w_x = w_y = 0$ so that w is equal to a constant. Thus, by the initial data, $w = 0$. Hence, $u = v$ and the solution is unique.

The initial boundary value problems consisting of the partial differential equation (31), the initial conditions (32), and boundary conditions of the second kind, or with mixed boundary conditions of the first and second kind, can be solved by a similar procedure.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. P. W. Berg and J. L. McGregor, Elementary Partial Differential Equations, San Francisco: Holden-Day, 1966.

2. E. K. Blum, "The Euler-Poisson-Darboux Equation in the Exceptional Cases," Proc. Amer. Math. Soc. 5 (1954), 511-520.

3. R. Davis, "On a Regular Cauchy Problem for the Euler-Poisson-Darboux Equation," Ann. Mat. Pura Appl. 42 (1956), 205-226.

4. J. B. Diaz and G. S. S. Ludford, "On the Singular Cauchy Problem for a Generalization of the Euler-Poisson-Darboux Equation in Two Space Variables," Ann. Mat. Pura Appl. 38 (1955), 33-50.

5. J. B. Diaz and H. F. Weinberger, "A Solution of the Singular Initial Value Problem for the Euler-Poisson-Darboux Equation," Proc. Amer. Math. Soc. 4 (1953), 703-715.

6. B. A. Fusaro, "A Solution of a Singular Mixed Problem for the Equation of Euler-Poisson-Darboux," Amer. Math. Monthly 73 (1966), 610-613.

7. F. B. Hildebrand, Advanced Calculus for Applications, Englewood Cliffs, New Jersey: Prentice Hall, 1962.

8. N. N. Lebedev, Special Functions and Their Applications, Englewood Cliffs, New Jersey: Prentice Hall, 1965.

9. H. M. Lieberstein, "A Mixed Problem for the Euler-Poisson-Darboux Equation," J. d'Analyse Math. 6 (1958), 357-379.

10. W. Walter, "Uber die Euler-Poisson-Darboux Gleichung," Math. Zeit. 67 (1957), 361-376.

11. A. Weinstein, "Sur le Probleme de Cauchy Pour l'equation de Poisson et l'equation des Ondes," C. R. Acad. Sci. Paris 234 (1952), 2584-2585.

12. A. Weinstein, "On the Wave Equation and the Equation of Euler-Poisson," Proc. Sympos. Appl. Math. 5, (1954), 137-147.

13. E. C. Young, "A Mixed Problem for the Euler-Poisson-Darboux Equation in Two-Space Variables," SIAM J. Math. Anal. 6 (1975), 17-24.

VITA

Myra Ladner Schrader was born in Gulfport, Mississippi on March 22, 1952. She received her elementary education in the Harrison County Schools, and was graduated from Harrison Central High School in 1969. In December, 1972, she received the Bachelor of Arts degree in Mathematics from Mississippi State University, where she graduated as a President's scholar.

She entered the Graduate School of the University of Tennessee in September, 1973, and received the Master of Science degree with a major in Mathematics in June, 1975. She was a teaching assistant in the Mathematics Department during the time of her enrollment. She is a member of Kappa Mu Epsilon and Phi Kappa Phi.

She is married to Edward L. Schrader of Long Beach, Mississippi.