

Towards the production of a self-consistent phase space distribution

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Abstract

A self-consistent phase space distribution is a charged particle beam in which the electric field has a linear dependence on the particle coordinates, and in which the linearity of the electric field is conserved as the beam is transported through arbitrary linear focusing fields. These features could increase the possible beam intensity in a circular accelerator by minimizing/eliminating the space charge tune shift/spread. Additionally, the uniform density of known self-consistent distributions would be ideal for fixed-target applications. Finally, certain self-consistent distributions can be flattened by exploiting the relationships between their phases space coordinates and would therefore be useful in a collider.

Although self-consistent distributions are often used in theoretical studies, they are not assumed to be realistic. Yet simulations predict that at least one — the Danilov distribution — could be approximately produced in a real machine using a method called elliptical painting. This dissertation contributes to efforts to test this prediction in the Spallation Neutron Source (SNS). First, the beam envelope model was used to calculate the matched solutions of the Danilov distribution in periodic focusing channels, placing constraints on the elliptical painting method. Second, several methods to indirectly measure the four-dimensional (4D) phase space distribution of an accumulated beam in the SNS were identified, implemented using existing diagnostics, and optimized, allowing the comparison of real beams to the Danilov model in minimal time. Finally, three initial experiments to produce a Danilov distribution in the SNS were carried out. Although the experiments were performed under suboptimal conditions due to current hardware constraints, the measured reduction in 4D emittance was not insignificant in the final experiment, indicating that the beam was closer to the desired self-consistent case than a typical beam in the SNS. Simulations were included to benchmark the measurements, resulting in qualitative agreement and recommendations for future experiments. Small modifications to the SNS ring lattice are expected to bring the beam closer to a self-consistent state.

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Chapter 1

Introduction

High-intensity particle accelerators are important to many areas of scientific research; for example, they serve as drivers for bright secondary beam production (neutrons, neutrinos, muons, etc.) [1] and could be used for nuclear waste processing [2]. In particular, spallation neutron sources have become vital to material science [3].

The maximum beam intensity in these machines is often limited by nonlinear space charge forces — forces between the charged particles in the beam [4]. Such forces lead to beam loss that is difficult to predict and minimize. If the uncontrolled beam loss is significant, it can lead to radio-activation of the accelerator components, making hands-on maintenance unsafe [5]; thus, reducing uncontrolled losses is of primary concern. The present standard design criterion is to keep losses below one watt per meter; for beam pulses in current spallation neutron sources, which contain more than 10^{14} particles per pulse at a beam power above 1 megawatt, this corresponds to a fractional loss of $\approx 10^{-6}$ [6]. The prediction and mitigation of space charge effects will become even more important in the design of future ten-megawatt accelerators [7].

One method to produce a high-intensity hadron beam is to repeatedly inject particles from a linear accelerator (linac) into a circular accelerator (ring), accumulating charge in the ring over many turns. Rings are designed to produce stable motion, but the combination of the periodic electromagnetic fields of the accelerator *and beam* may produce unstable motion. Additionally, if the beam's electric field has a strong nonlinear dependence on the particle coordinates, the number of stable machine configurations decreases. This places an upper limit on the beam intensity. Thus, it is desirable to produce a beam in which the electric field has a linear dependence on the particle coordinates; for example, a uniform density ellipsoid.

Uniform density beams producing linear space charge forces are frequently used in analytical and computational studies of high-intensity beam dynamics [8]. These studies are possible because of the existence of special distributions of particles in position-momentum space (phase space) that maintain linear space charge forces as they are transported through arbitrary linear focusing fields. We refer to these distributions as *self-consistent*.

It is, however, difficult to produce such a distribution: real beam transport involves nonlinear external fields, for which no known self-consistent distributions exist, and control of the phase space distribution at early stages of acceleration is limited. Nonetheless,

several recently derived self-consistent distributions are possible to produce in a circular accelerator in the linear approximation [9]. Furthermore, simulations predict that one of these distributions — the Danilov distribution — could be approximately produced in a real machine [10]. This dissertation contributes to efforts to test this prediction in the Spallation Neutron Source (SNS).

The structure of this introductory chapter is as follows: the relevant theory of beam dynamics is reviewed in Section 1.1; the definition and properties of self-consistent distributions are discussed in Section 1.2; a method to generate an approximate Danilov distribution in a ring, as well as the implementation of the method in the SNS, is presented in Section 1.3; the structure and goals of this dissertation are laid out in Section 1.4.

1.1 Beam dynamics

1.1.1 Single-particle motion

We begin by describing the motion of a single particle in a ring. We assume the existence of a closed orbit and use coordinates in which s is the location along the orbit and x and y are the horizontal and vertical transverse displacements. We then study oscillations in the transverse plane with the assumption of constant longitudinal velocity $\beta_s c$, where c is the speed of light.

Magnetic fields are preferred for transverse steering and focusing when the kinetic energy is significant. The magnetic field $\mathbf{B} = (B_x, B_y)$ in a vacuum may be written as an infinite sum:

$$B_x - iB_y = \sum_{n=1}^{\infty} (b_n - ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1}, \quad (1.1)$$

where r_0 is a constant, $i = \sqrt{-1}$, $\{b_n\}$ are the multipole coefficients, and $\{a_n\}$ are the skew multipole coefficients. The b_n term in the expansion is produced by $2n$ symmetrically arranged magnetic poles; the skew terms are obtained by a $\pi/2n$ rotation. Assuming the transverse velocities are much smaller than $\beta_s c$, the equations of motion for x and y are

$$\begin{aligned} x'' &= -\frac{q}{mc\beta_s\gamma_s} B_y, \\ y'' &= +\frac{q}{mc\beta_s\gamma_s} B_x, \end{aligned} \quad (1.2)$$

where q is the particle charge, m is the particle mass, $\gamma_s = (1 - \beta_s^2)^{-1/2}$, and the prime represents differentiation with respect to s .¹ A curved coordinate system modifies the horizontal equation of motion, but we have assumed a straight coordinate system here for simplicity.

¹Since the small-angle approximation is used, x' is usually reported in radians.

Linear dynamics

Accelerators employ dipole fields (b_1) for bending and quadrupole fields (b_2) for focusing. Keeping only these terms, Eq. (1.2) becomes

$$x'' + k(s)x = 0. \quad (1.3)$$

Eq. (1.3) is of general interest [11, 12]. It describes a one-dimensional (1D) parametric oscillator — an oscillator whose physical properties change with time. Its solution in the Courant-Snyder theory [13] is

$$x(s) = \sqrt{2J\beta(s)} \cos(\mu(s) + \delta), \quad (1.4)$$

with J constant, δ constant, and the phase advance $\mu(s)$ given by

$$\mu(s) = \int_0^s \frac{ds'}{\beta(s')}. \quad (1.5)$$

$\beta(s)$ is referred to as the “beta function”. In a ring of length C , $k(s) = k(s + C)$ and there is a unique periodic solution with $\beta(s) = \beta(s + C)$; otherwise, $\beta(s)$ depends on the initial conditions.

It is helpful to view the motion in phase space (x - x') at a fixed location in the ring on a turn-by-turn basis as in Fig. 1.1a. The particle jumps around an ellipse. The so-called Twiss parameters β , $\alpha = -\beta'/2$, and $\gamma = (1 + \alpha^2)/\beta$ determine the ellipse dimensions. J , which is proportional to the area of the ellipse, is called the Courant-Snyder invariant:

$$J = \frac{x^2 + (\alpha x + \beta x')^2}{\beta}. \quad (1.6)$$

We define the tune ν as the number of phase space oscillations per turn; i.e.,

$$\nu = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}, \quad (1.7)$$

where the integral is around the entire ring.

Thus, motion between two locations in the ring is equivalent to an area-preserving linear transformation of a phase space ellipse, plus rotation of the particle around the ellipse. This is more clear in the transfer matrix formulation of the dynamics, writing $\mathbf{x}(s) = \mathbf{M}(s)\mathbf{x}(0)$ where

$$\begin{aligned} \mathbf{M}(s) &= \begin{bmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\frac{1}{\beta(s)}} \end{bmatrix} \begin{bmatrix} \cos \mu(s) & \sin \mu(s) \\ -\sin \mu(s) & \cos \mu(s) \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{\beta(0)}} & 0 \\ \frac{\alpha(0)}{\sqrt{\beta(0)}} & \sqrt{\beta(0)} \end{bmatrix} \\ &= \mathbf{V}(s) \mathbf{P}(s) \mathbf{V}(0)^{-1} \end{aligned} \quad (1.8)$$

and $\mathbf{x} = (x, x')^T$. As illustrated in Fig. 1.1b, $\mathbf{V}(0)^{-1}$ transforms the phase space ellipse into a circle while preserving its area, $\mathbf{P}(s)$ rotates the coordinates around the circle according to the phase advance, and $\mathbf{V}(s)$ transforms the circle back into an ellipse [14]. Eq. (1.8) motivates the definition of normalized phase space coordinates $\mathbf{x}_n(s) = \mathbf{V}(s)^{-1}\mathbf{x}(s)$ in

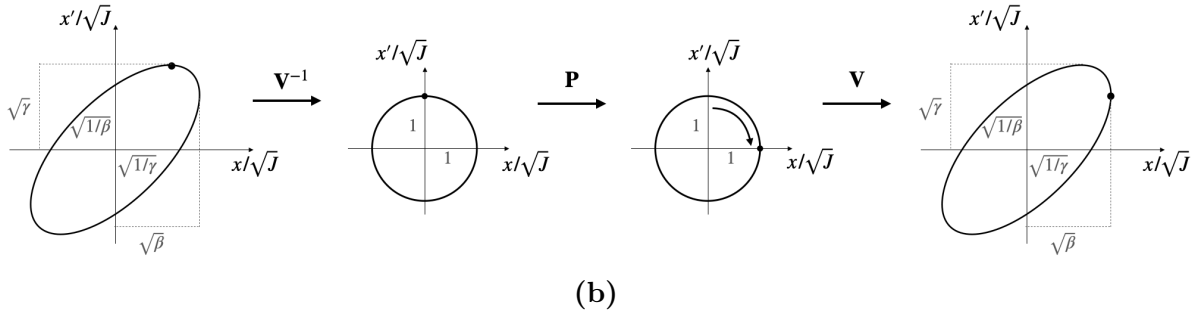
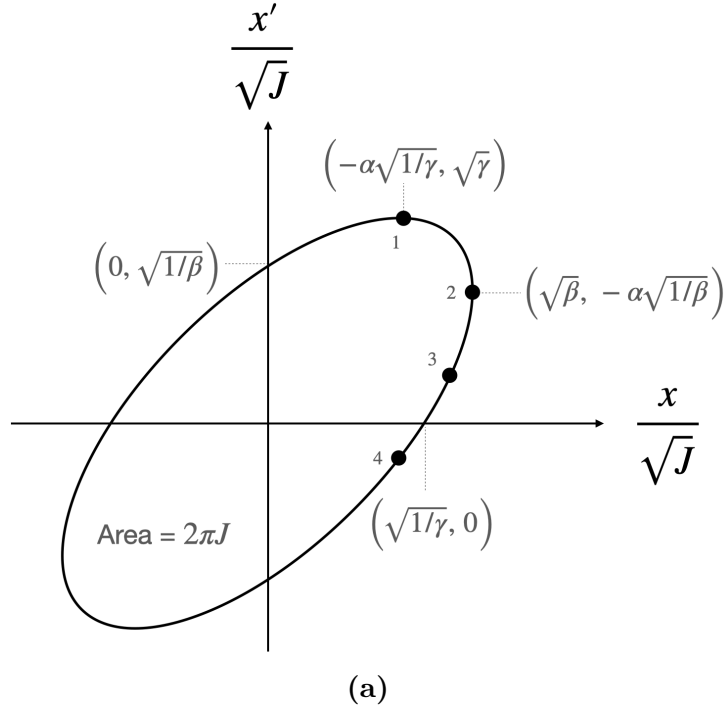


Figure 1.1: (a) Turn-by-turn motion of a single particle along the Courant-Snyder ellipse in horizontal phase space. (b) Alternative view of the motion using the factored transfer matrix.

which the particle performs simple harmonic oscillations, i.e., rotates in a circle of area J at frequency $2\pi\nu$.

Linear (coupled) dynamics

In the presence of linear coupling, the equations of motion take the following form:

$$\mathbf{x}'' + \mathbf{K}_0(s)\mathbf{x} + \mathbf{K}_1(s)\mathbf{x}' = 0, \quad (1.9)$$

where $\mathbf{x} = (x, y)^T$ and $\mathbf{K}_{0,1}$ are 2×2 matrices.² The particle now moves on an ellipsoid in 4D phase space $(x-x'-y-y')$, the volume of which is conserved. For example, Fig. 1.2 shows the turn-by-turn trajectory of a single particle in the presence of linear coupling from a rotated (skew) quadrupole. The motion is most simply described using transfer matrices. Consider the eigenvectors and eigenvalues of the 4×4 symplectic one-turn transfer matrix $\mathbf{M}$. There are four eigenvectors — $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1^*, \mathbf{v}_2^*$ — and four eigenvalues — $\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*$ — with $\lambda_i \lambda_j^* = 1$ (* denotes the complex conjugate). The eigenvalue equation is written

$$\mathbf{M}\mathbf{v}_l = e^{-i\mu_l} \mathbf{v}_l, \quad (1.10)$$

with $l = 1, 2$. The phase space coordinate vector $\mathbf{x} = (x, x', y, y')^T$ at one position in the ring is a linear combination of the eigenvectors:

$$\mathbf{x} = \text{Re} \left\{ \sqrt{2J_1} \mathbf{v}_1 e^{-i\psi_1} + \sqrt{2J_2} \mathbf{v}_2 e^{-i\psi_2} \right\}, \quad (1.11)$$

where $J_{1,2}$ are constant amplitudes, $\psi_{1,2}$ are initial phases, and $\text{Re}\{z\}$ selects the real component of z . Application of the transfer matrix advances the phases:

$$\mathbf{M}\mathbf{x} = \text{Re} \left\{ \sqrt{2J_1} \mathbf{v}_1 e^{-i(\psi_1+\mu_1)} + \sqrt{2J_2} \mathbf{v}_2 e^{-i(\psi_2+\mu_2)} \right\}. \quad (1.12)$$

The old invariants $J_{x,y}$ are replaced by $J_{1,2}$ and the phase advances $\mu_{x,y}$ are replaced by $\mu_{1,2}$. A new normalized phase space is defined by rewriting Eq. (1.12) as $\mathbf{x}_n = \mathbf{V}^{-1}\mathbf{x}$ with

$$\mathbf{V} = [\text{Re}\{\mathbf{v}_1\}, -\text{Im}\{\mathbf{v}_1\}, \text{Re}\{\mathbf{v}_2\}, -\text{Im}\{\mathbf{v}_2\}]. \quad (1.13)$$

Particles perform simple harmonic oscillations in normalized phase space, moving in circles of area J_1 in the $x_n-x'_n$ plane and J_2 in the $y_n-y'_n$ plane.

We would like to parameterize the eigenvectors as in the uncoupled case. There are currently several parameterizations in existence [15–19]; we will use the parameterization of Lebedev and Bogacz [18]:

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\frac{\alpha_{1x}+i(1-u)}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}e^{i\nu_1} \\ -\frac{\alpha_{1y}+iu}{\sqrt{\beta_{1y}}}e^{i\nu_1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}e^{i\nu_2} \\ -\frac{\alpha_{2x}+iu}{\sqrt{\beta_{2x}}}e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\frac{\alpha_{2y}+i(1-u)}{\sqrt{\beta_{2y}}} \end{bmatrix}. \quad (1.14)$$

²I will occasionally switch between $\mathbf{x} = (x, x', y, y')^T$ and $\mathbf{x} = (x, y)^T$. The correct definition should be clear from context.

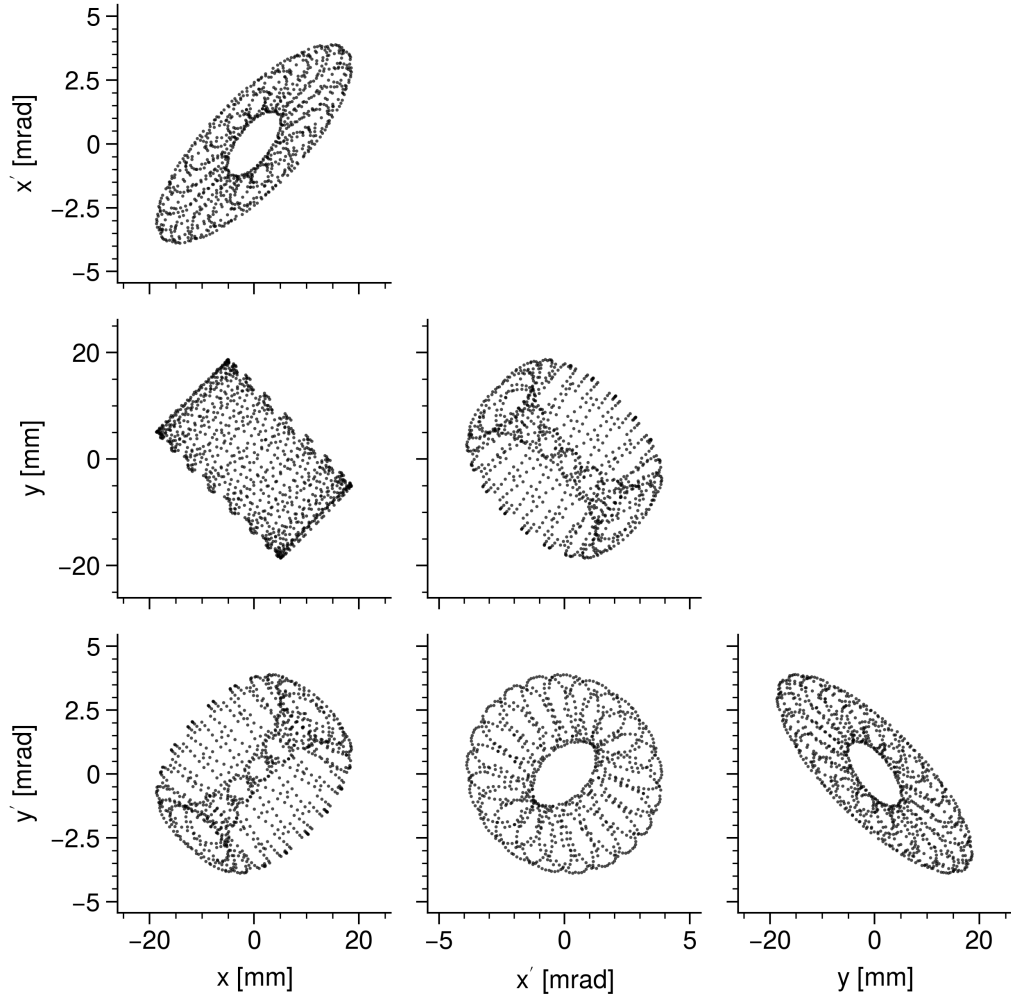


Figure 1.2: Turn-by-turn trajectory of a particle in a linear focusing system with the addition of a skew quadrupole.

The meaning of the new parameters is illustrated in Fig. 1.3. The motion is the sum of two eigenvectors, each of which traces an ellipse when projected onto any 2D subspace. The horizontal and vertical amplitudes J_x and J_y are exchanged because the eigenvectors rotate at different frequencies. The parameterization assigns a β and α parameter to each ellipse. The parameters ν_1 and ν_2 are the phase differences between the horizontal ($x-x'$) and vertical ($y-y'$) parts of the eigenvectors, which determines the tilt angle of the ellipses traced in the cross-plane projections ($x-y$, $x-y'$, $y-x'$, $x'-y'$). Finally, u determines the area of the ellipse traced by the eigenvectors in horizontal phase space relative to the ellipse in vertical phase space.

Nonlinear resonances

Nonlinear terms in Eq. (1.1) are generally small but nonzero in reality. Furthermore, they are periodic since they occur once per turn. As detailed in Appendix A, perturbation analysis shows that these terms may drive a resonance when

$$M_x \nu_x + M_y \nu_y = N, \quad (1.15)$$

where $\nu_{x,y}$ are the single-particle tunes, M_x , M_y , and N are integers, and $|M_x| + |M_y|$ is the order of the resonance. The single-particle tunes must be precisely controlled to avoid these resonance lines; otherwise, particles may be driven to large amplitudes and eventually fall outside the machine aperture. The strength of the resonance varies inversely with the order: fourth-order and below are the primary concern in most machines, but higher-order effects may be important when the number of stored turns is large.

1.1.2 Collective beam description

A beam is a distribution of particles in phase space. In the limit of many particles, we define a distribution function $f(\mathbf{x})$ such that $f(\mathbf{x})d\mathbf{x}$ is the number of particles in an infinitesimal volume of phase space $d\mathbf{x}$. It is often sufficient to characterize a distribution by its covariance matrix $\Sigma = \langle \mathbf{x}\mathbf{x}^T \rangle$, where $\langle \dots \rangle$ represents the expected value. In the transverse plane:

$$\Sigma = \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle xy \rangle & \langle x'y \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle xy' \rangle & \langle x'y' \rangle & \langle yy' \rangle & \langle y'y' \rangle \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy}^T & \sigma_{yy} \end{bmatrix}. \quad (1.16)$$

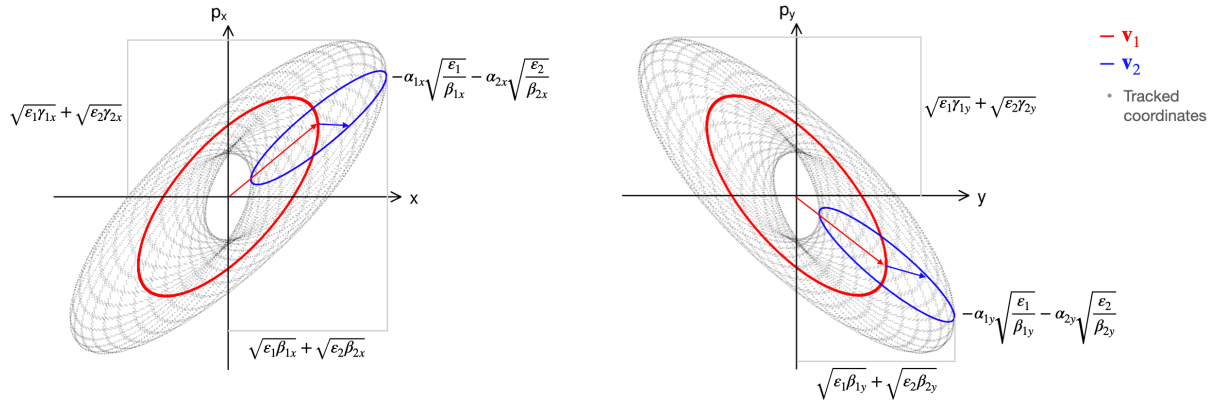
If a symplectic linear transformation $\mathbf{x} \rightarrow \mathbf{M}\mathbf{x}$ is applied to the coordinates, the covariance matrix transforms as

$$\Sigma \rightarrow \mathbf{M} \Sigma \mathbf{M}^T. \quad (1.17)$$

There are several invariant quantities under such a transformation. The first is the 4D emittance ε_{4D} , which is proportional to the volume of the ellipsoid defined by the covariance matrix:

$$\varepsilon_{4D} = |\Sigma|^{1/2} = \varepsilon_1 \varepsilon_2 \leq \varepsilon_x \varepsilon_y, \quad (1.18)$$

where $|\dots|$ is the determinant. The *intrinsic* emittances $\varepsilon_{1,2}$ are individually conserved; they are found by a symplectic diagonalization of Σ ; i.e., they are the imaginary compo-



- α_{lw} and β_{lw} describe ellipses traced by $\mathbf{v}_l$ in the w - p_w plane ($w = x, y$)
- Area of $\mathbf{v}_1$ ellipse is $|1 - u| \epsilon_1$ in horizontal phase space and $|u| \epsilon_2$ in vertical phase space
- Area of $\mathbf{v}_2$ ellipse is $|u| \epsilon_1$ in horizontal phase space and $|1 - u| \epsilon_2$ in vertical phase space

$$\begin{aligned} \gamma_{1x} \beta_{1x} &= (1 - u)^2 + \alpha_{1x}^2 \\ \gamma_{2x} \beta_{2x} &= u^2 + \alpha_{2x}^2 \\ \gamma_{1y} \beta_{1y} &= u^2 + \alpha_{1y}^2 \\ \gamma_{2y} \beta_{2y} &= (1 - u)^2 + \alpha_{2y}^2 \end{aligned}$$

Figure 1.3: Lebedev-Bogacz parameterization of coupled motion. The grey markers are the turn-by-turn trajectory of a single particle. The red and blue lines are the ellipses traced by the transfer matrix eigenvectors.

nents of the eigenvalues of $\Sigma\mathbf{U}$, where $\mathbf{U}$ is the unit symplectic matrix:

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (1.19)$$

Equivalently [20],

$$\varepsilon_{1,2} = \frac{1}{2} \sqrt{-\text{tr}[(\Sigma\mathbf{U})^2] \pm \sqrt{\text{tr}^2[(\Sigma\mathbf{U})^2] - 16|\Sigma|}}. \quad (1.20)$$

In the absence of cross-plane correlations ($\sigma_{xy} = 0$), the intrinsic emittances are equal to the *apparent* emittances $\varepsilon_x = |\sigma_{xx}|^{1/2}$ and $\varepsilon_y = |\sigma_{yy}|^{1/2}$ [21]. The apparent emittances correspond to the areas of the projected ellipses in the x - x' and y - y' planes and are only conserved if $\mathbf{M}$ is block-diagonal, i.e., if there is no coupling between the horizontal and vertical motion.

It is challenging to generate realistic initial distributions for simulations due to the current lack of high-resolution 6D phase space measurements [22, 23]; however, a simple and often-justified strategy is to assume elliptical symmetry and construct a distribution from the single-particle invariants $J_{x,y}$.³ We define the ellipsoid parameter $T = J_x/\varepsilon_x + J_y/\varepsilon_y$ and stack ellipsoids to create the distribution, writing $f = f(T)$. One option is a Gaussian distribution: $f \propto \exp(-T/2)$; another is the Waterbag distribution, which is a uniformly filled ellipsoid: $f \propto \Theta(1 - T)$, where Θ is the Heaviside step function; another is the KV distribution, which is a uniformly populated ellipsoidal shell: $f \propto \delta(1 - T)$. The 1D and 2D projections of these 4D distributions are shown in Fig. 1.4a, Fig. 1.4b, and Fig. 1.4c. The black ellipses show the projections of the covariance matrix ellipsoid (multiplied by a factor of four).

1.1.3 Space charge

Particle motion is also influenced by space charge — the charge density of the beam in free space. The beam's electric field $\mathbf{E} = (E_x, E_y)^T$ modifies the single-particle equation of motion:

$$\mathbf{x}'' + \mathbf{K}_0(s)\mathbf{x} + \mathbf{K}_1(s)\mathbf{x}' = \frac{q}{m\gamma_s^3\beta_s^2c^2}\mathbf{E}. \quad (1.21)$$

Due to the attractive magnetic force between co-moving charges in the lab frame, the space charge force approaches zero as $\beta_s \rightarrow 1$. We will make the coasting beam approximation — infinite length, uniform density, and constant momentum in the longitudinal plane — to reduce the problem to two dimensions.⁴

Following Hofmann [4], we divide space charge effects into two categories: incoherent effects involving the motion of single particles, and coherent effects involving the self-consistent motion of the entire beam. Although the two effects may be difficult to isolate during the beam evolution [24], the distinction is clear in some cases.

³Alternatively, the generalized invariants $J_{1,2}$ can be used.

⁴This is generally invalid for linacs but locally valid for a transverse slice of a long distribution in a ring. It is equivalent to replacing particles with infinitely long uniform density charged rods.

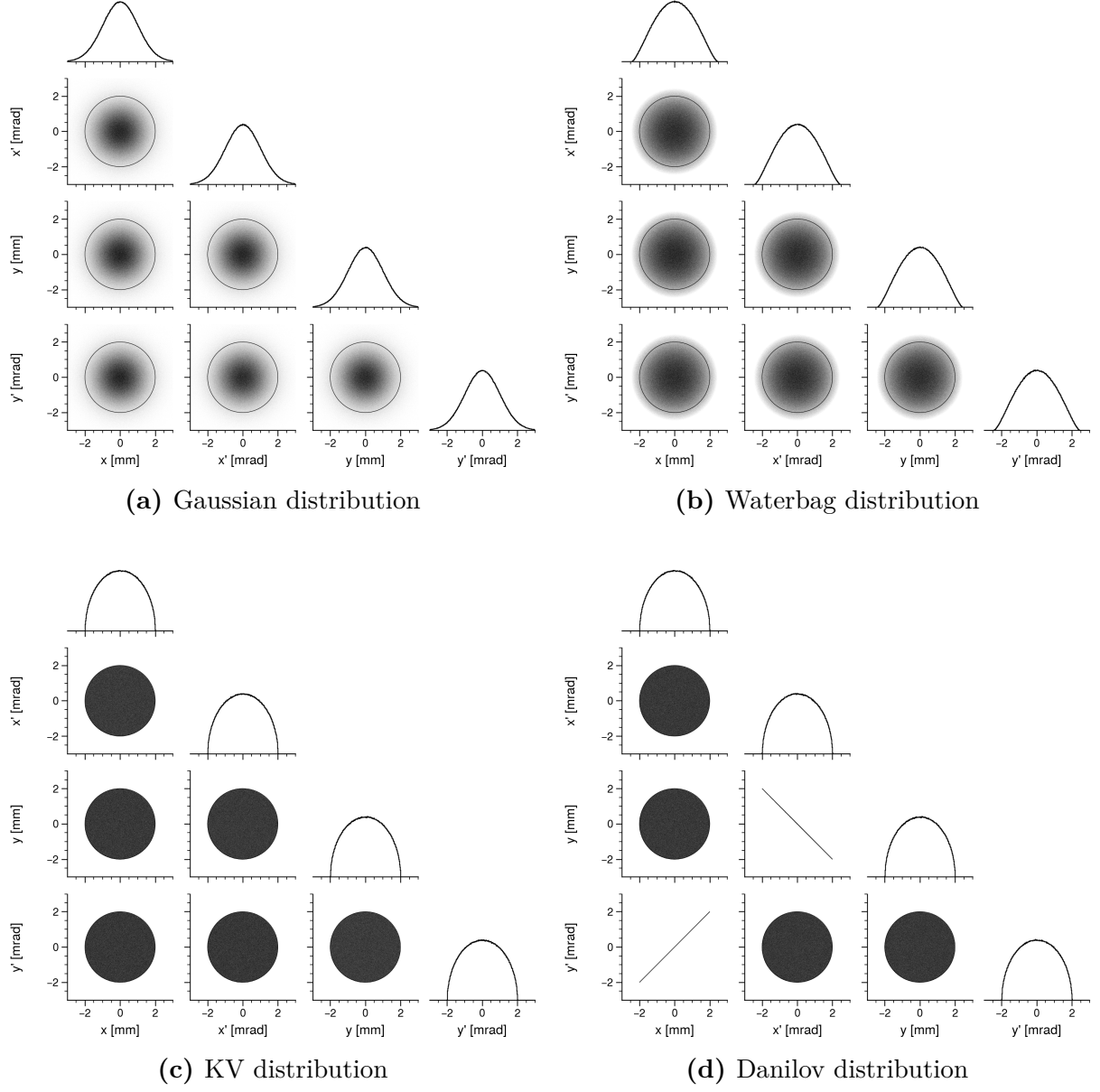


Figure 1.4: 1D and 2D projections of various 4D phase space distributions. The black ellipses are defined by four times the distribution covariance matrix.

Incoherent effects

We first assume that the beam is matched — i.e., oscillates with the same periodicity as the external focusing — and track a particle in the field of the matched beam. This may be justified if space charge is weak or if the particle is very far from the beam core. If the beam’s electric field has a linear dependence on the transverse coordinates, it will simply modify the external linear focusing, reducing the single-particle tunes. A primary concern in rings is that the depressed tunes cross one of the low-order resonance lines in Fig. A.1. Approximate analytical formulas for the tune shift can be obtained [25] but are not presented here.

If the electric field is nonlinear, the tune shift will depend on the particle amplitude, leading to a spread of tunes. An intuitive explanation is that large-amplitude particles experience a weaker average electric field throughout one turn in the ring [26]. A recent study of the space charge tune spread in rings is found in [27]; Fig. 1.5, taken from the paper, shows the simulated tune spread in the Japan Proton Accelerator Research Center (J-PARC).

Thus, the beam intensity in a circular accelerator is limited by the incoherent space charge tune shift/spread. A rough guideline is that the maximum tune shift should be kept below 0.25 to avoid fourth-order resonance lines [28], but specific requirements depend on the application.⁵

Coherent effects

Coherent space charge effects involve self-consistent oscillations of the entire beam [28, 29, 32]. We may model the beam as a smooth distribution in phase space $f(\mathbf{x}, \mathbf{x}', s)$; neglecting collisions between particles, the evolution of f is given by the Vlasov equation [33]:

$$\frac{d}{ds}f(\mathbf{x}, \mathbf{x}', s) = \frac{\partial f}{\partial s} + \mathbf{x}' \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{x}'' \cdot \frac{\partial f}{\partial \mathbf{x}'} = 0, \quad (1.22)$$

Hidden in Eq. (1.22) is the single-particle equation of motion, which we rewrite as:

$$\mathbf{x}'' + \mathbf{K}_0\mathbf{x} + \mathbf{K}_1\mathbf{x}' = -\frac{q}{m\gamma_s^3\beta_s^2c^2} \frac{\partial \Phi}{\partial \mathbf{x}}, \quad (1.23)$$

with the space charge potential Φ determined from the Poisson equation [34]:

$$\frac{\partial^2 \Phi}{\partial \mathbf{x}^2} = -\frac{q}{\epsilon_0} \int_{-\infty}^{\infty} f d\mathbf{x}'. \quad (1.24)$$

Analysis of the Vlasov equation is difficult in the general case of time-dependent external forces; thus, computer simulation must be used to understand the beam evolution. Solutions exist, but are rare (see Section 1.2). Perturbations of the Vlasov equation around these solutions can be used to derive stability conditions for the coherent oscillations of the beam, albeit this is only feasible in simple cases. This idea is explored in Appendix

⁵This condition is likely too conservative, and research in the area of intensity limits in circular accelerators is ongoing [4, 29–31]. As described in Appendix B, it is also possible for the beam’s electric field to drive single-particle resonances in the frozen-beam model, or for the entire beam to resonantly oscillate when space charge is strong.

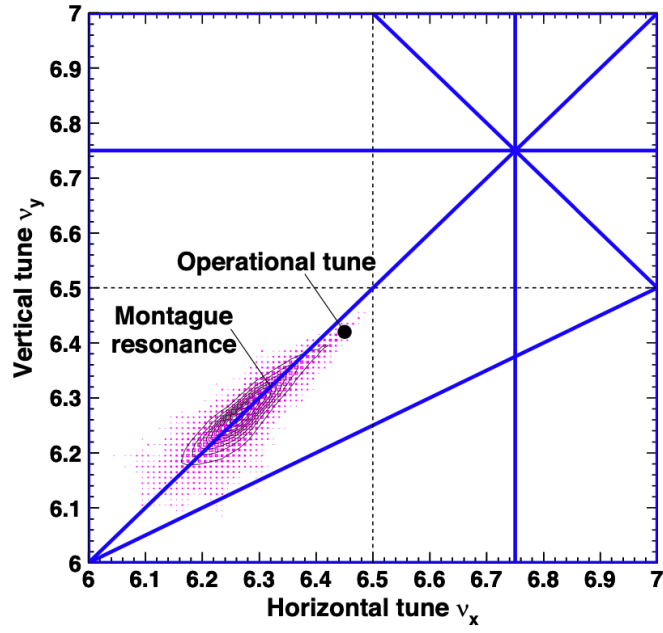


Figure 1.5: Simulated tune footprint in the JPARC accelerator. (From [27].)

B.

1.2 Self-consistent phase space distributions

1.2.1 Definition and properties

Any function constructed from single-particle invariants $\{C_i\}$ is a solution of the Vlasov equation:

$$\frac{d}{ds}f(\{C_i\}) = \sum_i \frac{df}{dC_i} \frac{dC_i}{ds} = 0. \quad (1.25)$$

One example of a single-particle invariant when the focusing is linear and time-dependent is the Courant-Snyder invariant of Eq. (1.6). The inclusion of space charge complicates the identification of invariants. Known solutions are found by the following procedure: (1) identify single-particle invariants $\{C_i\}$ in the presence of time-dependent linear forces; (2) construct a distribution from $\{C_i\}$ such that the distribution produces linear space charge forces under any linear transformation of the phase space coordinates. In this way, $\{C_i\}$ remain invariant even with space charge, and the linearity of the space charge force is conserved as the beam is transported through arbitrary linear focusing fields. We label such distributions as *self-consistent* [9].

Self-consistent distributions possess several notable properties. First, the integro-differential system of equations in Eq. (1.22) is reduced to a system of ordinary differential equations. Second, nonlinear space charge effects are minimized: the emittance is conserved, the maximum space charge tune shift is minimized, and the space charge tune spread is eliminated. Third, known self-consistent distributions have a uniform charge density. Fourth, higher-order coherent instabilities may be present in self-consistent distributions due to their small tune spread.

1.2.2 Known solutions

Danilov et al. enumerated a class of self-consistent distributions in $2n$ -dimensional phase space [9]:

$$f(\mathbf{x}, \mathbf{x}') = g(H - H_b) \prod_{i=1}^m \delta(\mathbf{e}_i \cdot \mathbf{x} + \mathbf{e}'_i \cdot \mathbf{x}'), \quad (1.26)$$

where $\mathbf{x}, \mathbf{x}'$ are the n dimensional vector coordinates and momenta, g is a function of H — a quadratic positively defined function of the phase space coordinates — and H_b — an upper bound on H — δ is the Dirac delta function, and $\mathbf{e}_i$ and $\mathbf{e}'_i$ are vectors of constants. This is labeled as the $\{n, m\}$ case. It was proven that the electric field within any uniformly filled ellipsoid is linear, and that any distribution of the form of (1.26) that produces a linear electric field will do so under any linear transformation of the phase space coordinates.

In other words, one class of self-consistent distributions consists of those that generate linear space charge forces and are constructed from quantities that are invariant in the

presence of linear focusing. We now focus on the $\{2, 0\}$ and $\{2, 2\}$ cases.⁶

The KV distribution

The $\{2, 0\}$ case corresponds to the KV distribution derived by Kapchinskij and Vladimirskij in 1959 [36]. The distribution is a function of the Courant-Snyder invariants J_x and J_y :

$$f(\mathbf{x}, \mathbf{x}') = \frac{\lambda}{\pi^2 \varepsilon_x \varepsilon_y} \delta \left(\frac{J_x}{\varepsilon_x} + \frac{J_y}{\varepsilon_y} - 1 \right), \quad (1.27)$$

where λ is the longitudinal particle density. Particles in the KV distribution are evenly distributed on an ellipsoidal shell in 4D phase space. As shown in Fig. 1.4c, any 2D projection of the distribution is a uniform density ellipse. Of particular importance is the x - y projection, which remains upright and uniform density under any uncoupled transformation. The electric field within such an ellipse is

$$\mathbf{E}(x, y) = \frac{\lambda}{\pi \epsilon_0} \left[\frac{x}{c_x (c_x + c_y)} \hat{x} + \frac{y}{c_y (c_x + c_y)} \hat{y} \right], \quad (1.28)$$

where c_x and c_y are the horizontal and vertical semi-axes and ϵ_0 is the permittivity of free space. Since the space charge force is linear and uncoupled, $J_{x,y}$ remains invariant for every particle, and the emittances $\varepsilon_{x,y}$ are conserved. The KV distribution does not exist in three spatial dimensions [37].

As mentioned in the previous section, the preservation of the linearity of the space charge force leads to a self-consistent set of differential equations for the evolution of the beam envelope. The KV envelope equations read:

$$\begin{aligned} \tilde{x}'' + k_x(s)\tilde{x} - \frac{\varepsilon_x^2}{\tilde{x}^3} - \frac{Q}{2(\tilde{x} + \tilde{y})} &= 0, \\ \tilde{y}'' + k_y(s)\tilde{y} - \frac{\varepsilon_y^2}{\tilde{y}^3} - \frac{Q}{2(\tilde{x} + \tilde{y})} &= 0. \end{aligned} \quad (1.29)$$

The RMS widths of the beam $\tilde{x} = \sqrt{\langle x^2 \rangle}$ and $\tilde{y} = \sqrt{\langle y^2 \rangle}$ are used instead of the true widths c_x and c_y . They are related by a factor of two for a uniform density ellipse. The perveance Q is a dimensionless measure of space charge strength:

$$Q = \frac{2\lambda r_0}{\beta^2 \gamma^3}, \quad (1.30)$$

where $r_0 = e^2/4\pi\epsilon_0 mc^2$ is the classical proton radius. Eqs. (1.29) provide many insights into dynamical beam behavior and serve as a benchmark for computer simulations. A remarkable fact is that Eqs. (1.29) are exact for distributions with elliptical symmetry even if the space charge force is nonlinear [37], although they are not closed since the root-mean-square emittances become time-dependent; thus, the KV envelope equations provide a good approximation to the evolution of more realistic distributions in the limit of elliptical symmetry and small emittance growth [8].

⁶Qin and Davidson derived a self-consistent distribution in two spatial dimensions in [35] using their recent parameterization of coupled motion but did not refer to [9]. The connection between Danilov's work and Qin and Davidson's work should be explored in the future.

The Danilov distribution

The focus of this dissertation is on the $\{2, 2\}$ case of Eq. (1.26) which will be referred to as the Danilov distribution:

$$f(\mathbf{x}, \mathbf{x}') \propto \Theta(1 - \mathbf{x}^T \boldsymbol{\sigma}^{-1} \mathbf{x}) \delta(\mathbf{x} - \mathbf{D}\mathbf{x}') \quad (1.31)$$

with

$$\boldsymbol{\sigma} = 4 \begin{bmatrix} \langle xx \rangle & \langle xy \rangle \\ \langle xy \rangle & \langle yy \rangle \end{bmatrix} \quad (1.32)$$

and $\mathbf{D}$ a 2×2 matrix. Similar to the KV distribution, any 2D projection of the Danilov distribution is a uniform density ellipse; however, the ellipses in the cross-plane projections (x - y , x - y' , y - x' , x' - y') are not necessarily upright and may collapse to lines. For example, the projections are shown in Fig. 1.4d for $\mathbf{D}_{11} = \mathbf{D}_{22} = 0$ and $\mathbf{D}_{12} = -\mathbf{D}_{21} = 1$, which corresponds to a rigidly rotating disk. The electric field in a uniform density ellipse with semi-axes $c_{x,y}$ tilted at an angle ϕ in the x - y plane is:

$$\begin{aligned} E_x &\propto \left(\frac{\cos^2 \phi}{c_x} + \frac{\sin^2 \phi}{c_y} \right) \frac{x}{c_x + c_y} + \sin \phi \cos \phi \left(\frac{1}{c_y} - \frac{1}{c_x} \right) \frac{y}{c_x + c_y}, \\ E_y &\propto \left(\frac{\cos^2 \phi}{c_y} + \frac{\sin^2 \phi}{c_x} \right) \frac{y}{c_x + c_y} + \sin \phi \cos \phi \left(\frac{1}{c_y} - \frac{1}{c_x} \right) \frac{x}{c_x + c_y}. \end{aligned} \quad (1.33)$$

The field is linear in x and y , as required. And it is worth repeating: the linearity of the electric field is maintained under any linear transformation of the phase space coordinates. The key difference from the KV distribution is that space charge linearly couples the horizontal and vertical motion of individual particles. The Courant-Snyder invariants $J_{x,y}$ are therefore replaced by the more general invariants $J_{1,2}$, which are conserved even with the inclusion of space charge.

The delta functions in the Danilov distribution function cause the four-dimensional emittance, and therefore one of the intrinsic emittances, to be zero. The following relationship has been found to hold:

$$\varepsilon_1 = \varepsilon_x + \varepsilon_y, \quad \varepsilon_2 = 0 \quad (1.34)$$

or vice versa.

Discussion of the Danilov envelope equations is reserved for Chapter 2.

1.3 Producing a self-consistent distribution

1.3.1 Motivation and previous work

The following points motivate the physical realization of a Danilov distribution.

1. The properties listed in section 1.2 — conservation of emittance, reduced space charge tune shift, and reduced space charge tune spread — have the potential to increase the maximum beam intensity in a ring.
2. It is an interesting challenge to bring a real distribution closer to a self-consistent analytical model which is generally taken to be unrealistic.

3. Beams with a uniform charge density are ideal for fixed-target applications such as spallation neutron production. SNS targets are complex and expensive [38] and considerable research and development go into reducing the peak density on the target. This issue will become even more important if future machines are built on the intensity horizon with similar targets.
4. There has been recent interest in an angular-momentum-dominated (AMD) beam — a beam with a small 4D emittance. Such a beam can be transformed to a round state ($\varepsilon_x \approx \varepsilon_y$) or a flat state ($\varepsilon_x \ll \varepsilon_y$) using coupled linear optics that preserve the intrinsic emittances [39]. They may find use in colliders, relativistic electron cooling, low-energy hadron cooling, muon ionization cooling, and/or radiation generation [40]. The Danilov distribution is an AMD beam with the special property of self-consistency.

The application that is most relevant to this work is the use of an AMD beam in a collider, as suggested by Burov in [41] for the Large Hadron Collider (LHC). The primary benefit would be the increase in luminosity when the beam is converted to a flat state.⁷ Flat beams could also suppress beam-beam resonances and allow for luminosity leveling [41]. A promising approach to generating such a distribution is to maintain a round state at lower energies, such as in the LHC booster rings, then transform to a flat state at higher energies. The challenge is to accumulate the initial round AMD beam, which is complicated by nonlinear focusing fields and space charge forces. One way to address this challenge is to maintain an approximate Danilov distribution to linearize the space charge force during accumulation.

Luiten et al. proposed a method to create a self-consistent $\{3, 3\}$ distribution (see Eq. (1.26)) of electrons [43]. They observed that since a uniform density oblate spheroid⁸ will collapse under its own gravity into a flat disk [44], a flat transverse disk of electrons will longitudinally expand into a uniform density ellipsoid. This “pancake” distribution can be created using ultrashort pulsed-laser photoemission with an appropriate radial intensity profile. Musumeci et al. experimentally demonstrated this method in [45].

Unfortunately, these methods do not apply to high-intensity proton rings where distributions are much longer, often resembling 2D coasting beams, and are built up over many turns. A method to produce an approximate Danilov distribution in a high-intensity proton ring was identified in [9]. Before describing this method, we must introduce the concept of phase space painting.

1.3.2 Phase space painting

Accelerators are often broken into stages. A common pattern is the injection of a beam from a linac into a ring followed by eventual extraction to a different section of the machine. Injection and extraction are accomplished using kicker magnets — dipole magnets with fast rise times.

⁷Consider the head-on collision of two Gaussian beams, each with $\varepsilon_x = \varepsilon_y$, with equal β functions at the interaction point. The luminosity in this case is inversely proportional to $\sqrt{\varepsilon_x \varepsilon_y}$ [42]. Now suppose the cross-plane correlations in the beam are removed before the collision: the new luminosity is inversely proportional to $\sqrt{\varepsilon_1 \varepsilon_2}$. Thus, the fractional increase in luminosity is $\sqrt{\varepsilon_x \varepsilon_y} / \varepsilon_1 \varepsilon_2$, which diverges as $\varepsilon_{4D} = \varepsilon_1 \varepsilon_2 \rightarrow 0$.

⁸ $(x/A)^2 + (y/B)^2 + (z/C)^2 = 1$ with $A = B > C$

In direct multi-turn injection, multiple beam pulses are injected into the same stable region of longitudinal phase space in the ring. This process is limited by Liouville’s theorem in the sense that the phase space density in the ring cannot be increased. In charge-exchange injection, an ion beam from the linac is stripped of its electrons upon entering the ring, leaving only protons.⁹ This produces a higher phase space density because charge-exchange is a non-Liouvillian process. While normal multi-turn injection is usually performed over tens of turns, charge-exchange injection is performed over hundreds of turns [47].¹⁰

Phase space painting, or simply “painting”, is the time-dependent variation of the transverse distance and angle between the injected beam and the circulating beam; as such, it allows time-dependent control over the phase space distribution in the ring. Painting is a vital tool for space charge mitigation. The free parameters are the painting path — the path of the injection point in phase space — and the speed at which this path is traversed.

Correlated painting

Correlated painting proceeds as

$$\begin{aligned}x(t) &= x_{max}\sqrt{t}, \\y(t) &= y_{max}\sqrt{t}, \\x'(t) &= y'(t) = 0,\end{aligned}\tag{1.35}$$

where x , x' , y , and y' are the coordinates of the injected beam in the phase space of the circulating beam and t is a time variable normalized to the range $[0, 1]$. In the linear approximation and without space charge, correlated painting generates uniform density ellipses in the x - x' and y - y' planes and a rectangular distribution in the x - y plane. This is illustrated in Fig. 1.6, in which non-interacting particles are tracked in the linear approximation and plotted during accumulation in normalized phase space. New particles are injected at the intersection of the faint horizontal and vertical lines. Notice that this method produces a non-uniform density in the x - y plane. At the SNS, where the size and peak density of the x - y distribution on the spallation target is a concern [49], a modified correlated painting scheme is used:

$$\begin{aligned}x(t) &= x_0 + x_{max}\sqrt{t}, \\y(t) &= y_0 + y_{max}\sqrt{t}, \\x'(t) &= y'(t) = 0.\end{aligned}\tag{1.36}$$

The initial distribution is a donut in the x - x' and y - y' planes. Space charge and other nonlinear forces eventually cause the distribution to fill in its hollow center, reducing the peak density.

⁹Electrons are currently stripped using foils. Laser stripping is a potential alternative [46].

¹⁰There is, however, renewed interest in direct multi-turn injection for future accelerators to avoid the issues associated with charge exchange injection, such as foil scattering and unstripped particles, and it seems possible to extend the number of injected turns into the hundreds using novel techniques [48].

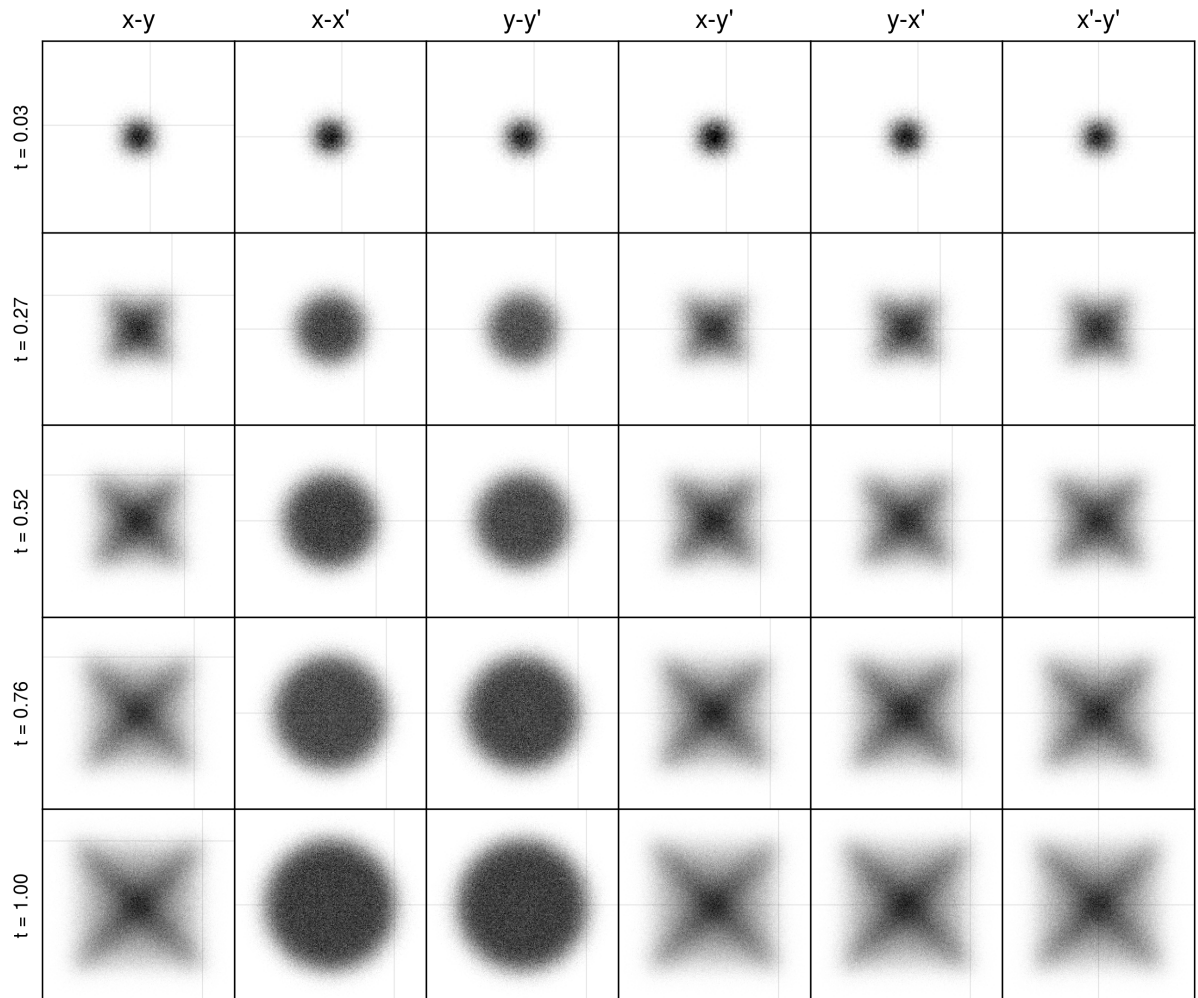


Figure 1.6: Correlated painting (linear approximation, non-interacting particles).

Anti-correlated painting

Anti-correlated painting is equivalent to correlated painting reversed in one of the planes:

$$\begin{aligned}x(t) &= x_{max}\sqrt{t}, \\y(t) &= y_{max}\sqrt{1-t}.\end{aligned}\tag{1.37}$$

The initial distribution is a point in the x - x' plane and a donut in the y - y' plane. The painting path in this method follows the line $J_x + J_y = \text{constant}$, which is the condition of particles in the KV distribution; thus, in the linear approximation without space charge, the final distribution is a KV distribution. This is illustrated in Fig. 1.7. However, the space charge forces are nonlinear throughout injection and the KV structure will not be maintained [50]. How to overcome this limitation is an open question. Nonetheless, anti-correlated painting can have benefits over correlated painting [27].

Elliptical painting

We now present a method called elliptical painting that produces a Danilov distribution in the linear approximation, even with space charge. Refer back to Eq. (1.12) in which the single-particle motion is written as the sum of two modes. Elliptical painting proceeds by scaling the injection point $\mathbf{x} = (x, x', y, y')$ along one of the eigenvectors:

$$\mathbf{x}(t) = \text{Re} \left\{ \sqrt{2J_l} \mathbf{v}_l e^{-i\psi_l} \right\} \sqrt{t},\tag{1.38}$$

with $l = 1, 2$. The first injected pulse does not move since it is injected onto the closed orbit. The second pulse traces a small ellipse in every 2D projection of the phase space on a turn-by-turn basis. The third pulse traces a slightly larger ellipse enclosing the second, and so on. The square root time-dependence ensures that the beam is always a uniform density ellipse in every 2D projection of the 4D phase space. Thus, in the linear approximation, a Danilov distribution is maintained at all times, even with space charge.¹¹ This is illustrated in Fig. 1.8.

The final distribution depends strongly on the eigenvectors (modes) of the ring transfer matrix. If the ring is uncoupled, the two modes correspond to horizontal and vertical motion — we refer to these modes as *planar* [40]. Injection into a planar mode results in a flat beam. If the ring is coupled, each mode corresponds to elliptical motion in the transverse plane — we refer to these modes as *elliptical*. Finally, there is the special case of circular motion in the transverse plane — we refer to these modes as *circular*. Injection into an elliptical/circular mode results in an elliptical/circular beam.

One way to generate elliptical modes in a ring is to use coupled focusing elements such as skew quadrupoles and solenoids. To create circular modes, locally rotational-invariant optics can be used [40]. The second way to generate elliptical modes in a ring is to equate the horizontal and vertical tunes, keeping all focusing elements uncoupled. In this case, the transfer matrix has degenerate eigenvalues and any linear combination of eigenvectors is itself an eigenvector; i.e., the turn-by-turn coordinates of a single particle will trace an ellipse no matter the initial coordinates.

¹¹The method is limited even in the linear approximation due to the finite emittance of the beam from the linac.

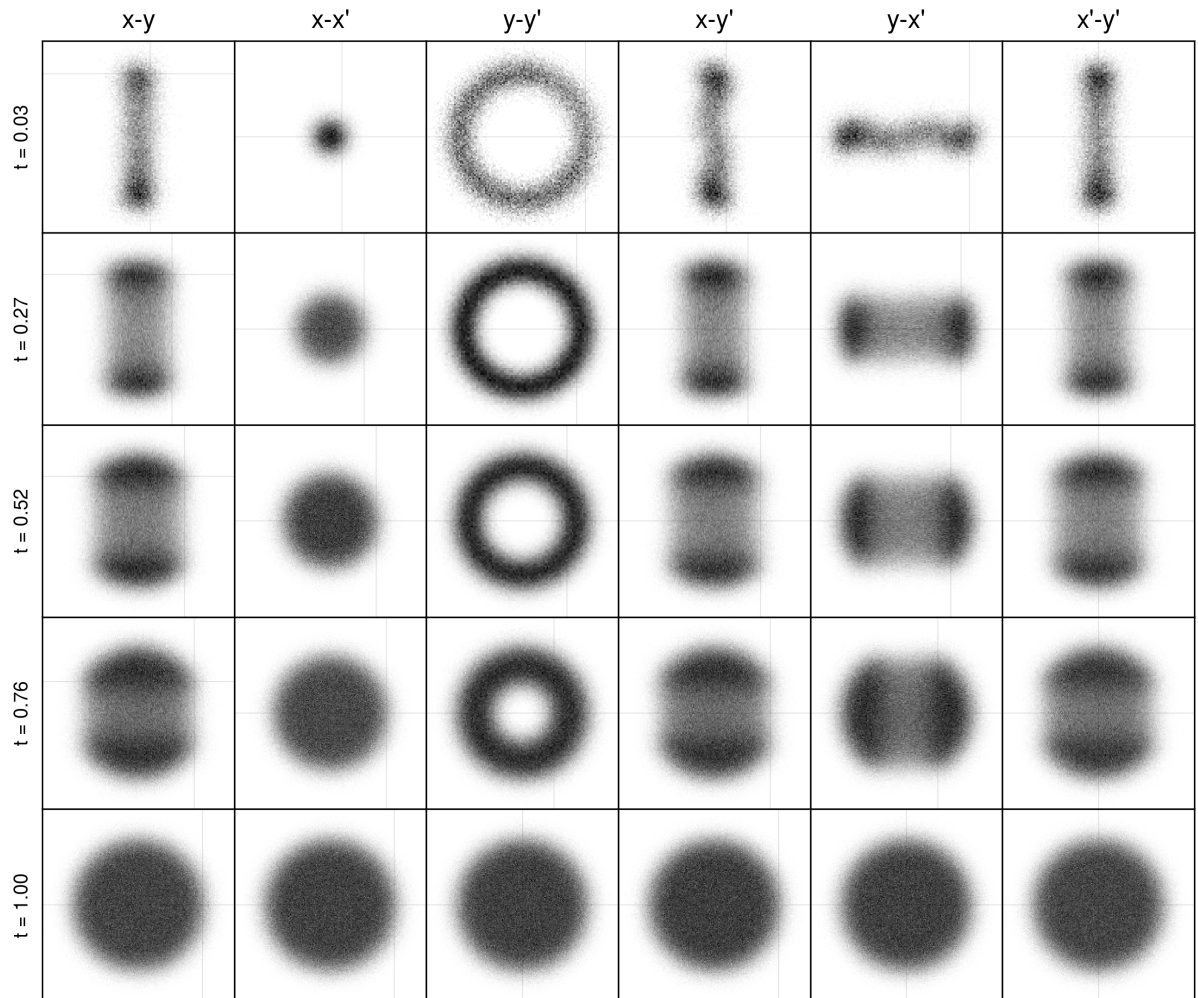


Figure 1.7: Anti-correlated painting (linear approximation, non-interacting particles).

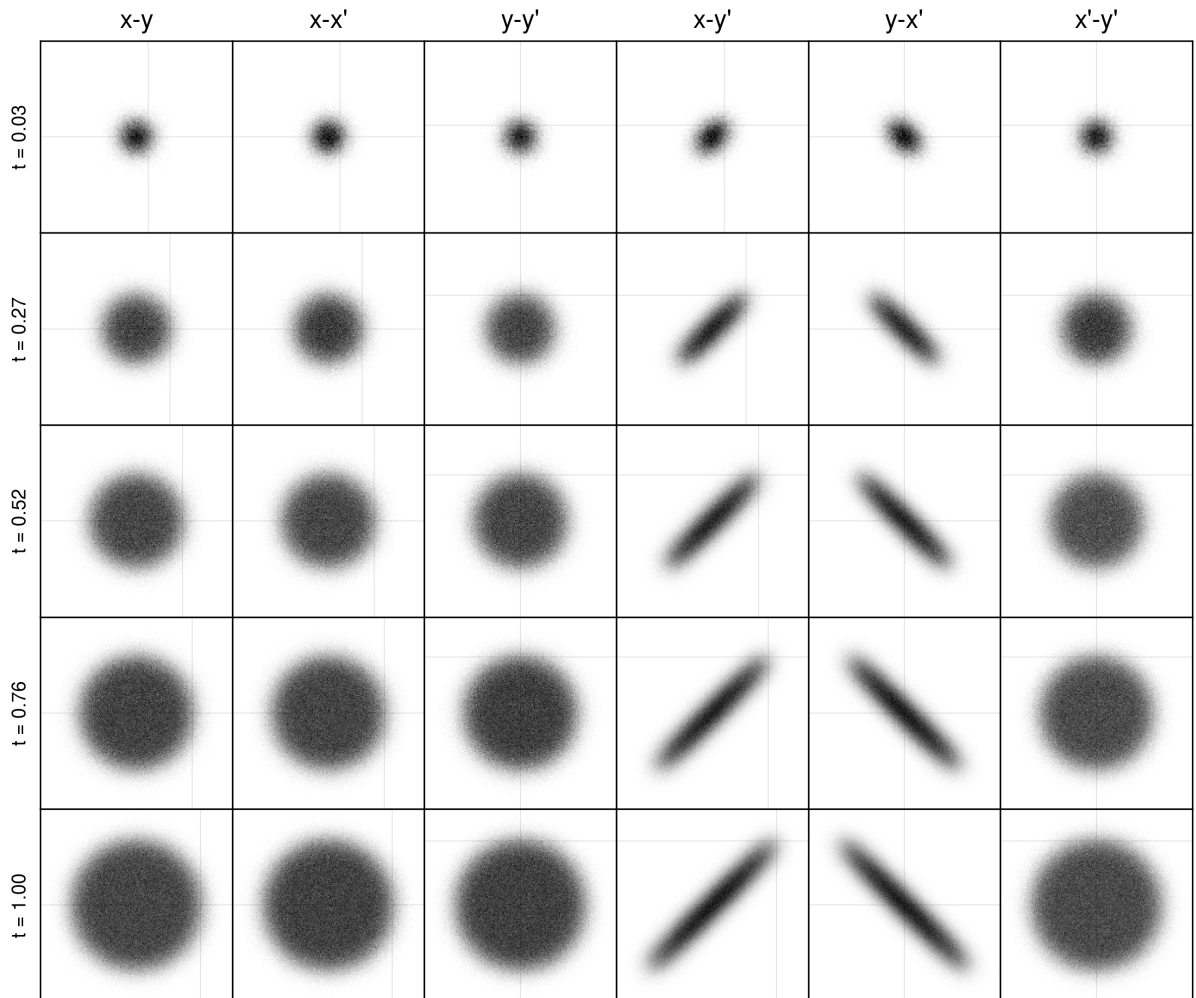


Figure 1.8: Elliptical painting (linear approximation, non-interacting particles).

1.3.3 Elliptical painting in the Spallation Neutron Source

Elliptical painting is possible in the SNS. The requirements are (i) the generation of elliptical modes in the ring and (ii) time-dependent control of the position and angle of the injected beam relative to the circulating beam. Before discussing how to implement (i) and (ii), a brief description of the SNS is warranted.¹²

Description of the Spallation Neutron Source

The SNS is a neutron scattering facility. Sixty times per second, a microsecond-long proton beam collides with a liquid mercury target at 1 GeV kinetic energy, producing neutrons by the process of spallation [53]. The original beam is a continuous wave of H^- ions which is chopped and then bunched in a 402.5 MHz radio-frequency quadrupole (RFQ), forming microsecond-long minipulses. Each minipulse is accelerated to 1 GeV through a normal-conducting, then superconducting linac, then transported to the injection region through the high-energy beam transport (HEBT). The electrons are stripped using a carbon foil, and the remaining protons continue their journey in the ring. A second minipulse is injected after the first minipulse completes one turn, and so on until one thousand minipulses, or 1.5×10^{14} protons, are accumulated over 10^{-3} seconds. Finally, the beam is extracted and guided through the ring-target beam transport line (RTBT) to the target. A comprehensive description of the SNS is given in [6]. Fig. 1.9 shows an overview of the machine.

Fig. 1.10 zooms in on the injection region. Four Chicane dipole magnets provide a time-independent bump to align the horizontal orbit with the H^- trajectory at the foil. Painting is performed by eight time-dependent kicker magnets — four per plane — which modify the transverse position and angle of the circulating beam. The injected beam trajectory is held fixed so that remaining H^0 or H^- particles can be reliably guided to a dump.

Generation of elliptical modes in the ring

Although the transfer matrix of the SNS ring is uncoupled, elliptical modes can be created by setting equal horizontal and vertical tunes. Tune measurement and control is standard practice in most accelerators [54] and is not described here. For reasons described in later chapters, it is advantageous to add solenoid magnetic fields to the ring. Solenoids are planned to be installed in the SNS ring in late 2022.

Ring Injection Control

Once elliptical modes are created in the ring, particles must be injected into one of the modes. Recall the free parameters: the painting path and the time-dependence with which this path is traversed. Eq. 1.38 requires fixed ratios between the injected coordinates (x, x', y, y') — the coordinates of the injected beam in the frame of the circulating beam at the injection point — and square root scaling of every coordinate. In the SNS, these coordinates are determined by the eight injection kicker magnet voltages. Once the correct initial and final voltages are determined, each kicker is given a waveform that scales its voltage during injection. The square root scaling of each coordinate is

¹²Experiments at the SNS will be relevant to the production of a flat beam in a collider; LHC booster rings now use H^- charge-exchange injection and phase space painting [51, 52].

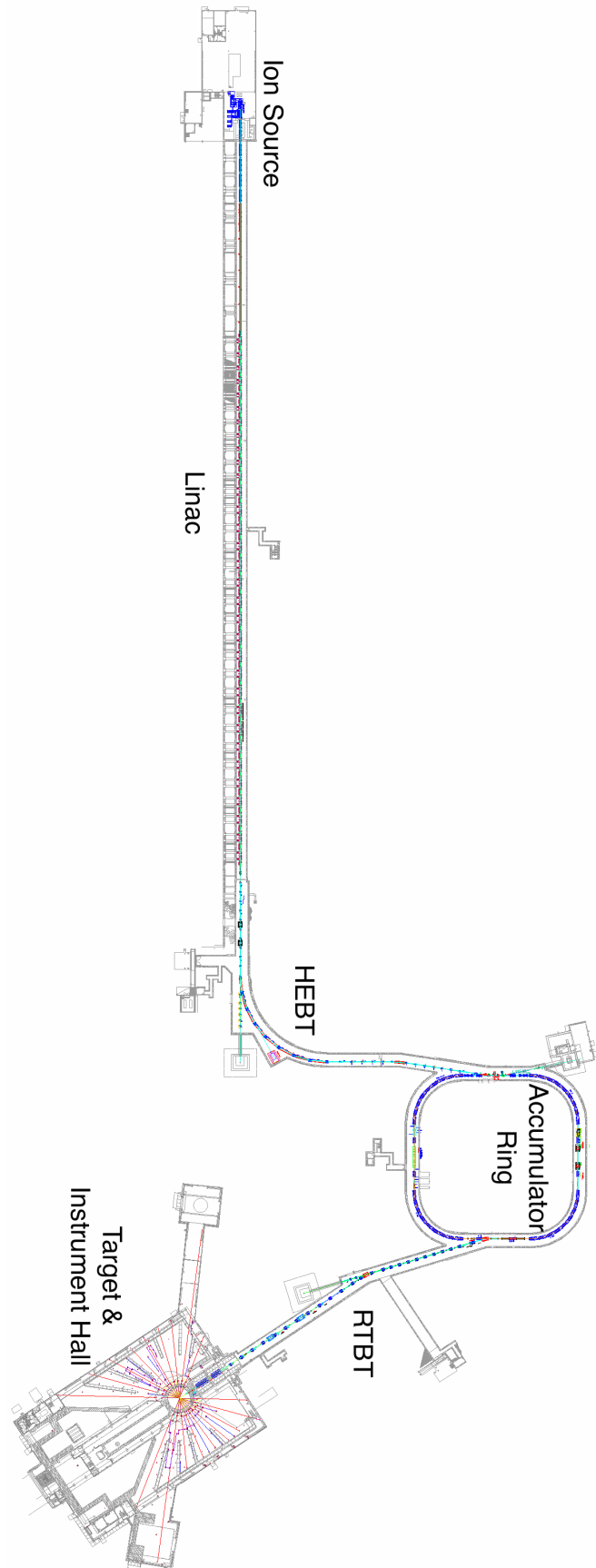


Figure 1.9: The Spallation Neutron Source (SNS).

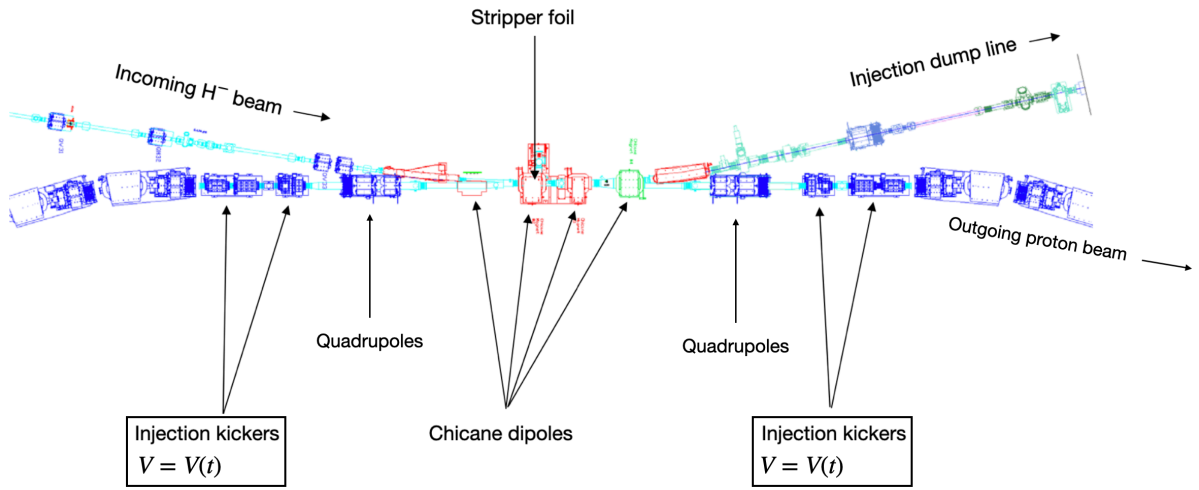


Figure 1.10: The SNS injection region.

ensured by using a square root waveform. Thus, what remains is to determine the initial and final kicker voltages.

It is first necessary to measure the phase space coordinates at the injection point. This can be done indirectly as follows. A single minipulse is injected and stored in the ring, and its turn-by-turn mean transverse position is measured using a beam-position-monitor (BPM). This is repeated for several minipulses and the average is taken. In the linear approximation, the mean position performs the pseudo-harmonic oscillations of Eq. (1.4); however, energy spread in the minipulse eventually sends the mean position to zero. A damped sine wave is an accurate model of this process for Gaussian energy spread [55]:

$$x(t) = A_0 + Ae^{kt^2} \cos(\mu + \mu_0), \quad (1.39)$$

where t is the turn number. The parameter A gives the betatron amplitude, $\mu/2\pi$ gives the fractional tune, and μ_0 gives the particle phase at the BPM. The phase space coordinates are recovered by combining these parameters with the linear ring model:

$$\begin{aligned} x_{bpm} &= A \cos \mu_0 \\ x'_{bpm} &= -A \left(\sin \mu_0 + \frac{\alpha}{\beta} \cos \mu_0 \right). \end{aligned} \quad (1.40)$$

The coordinates are then transported to the injection point using the model transfer matrix. Repeating this for each BPM gives an estimated mean and standard deviation of the phase space coordinates at the injection point. Examples of measured individual and averaged BPM signals in the SNS ring are shown in Fig. 1.11a along with the damped-sinusoid fit. Additionally, a simulated minipulse in the (linearized) SNS ring is shown in Fig. 1.11b.

The next issue is how to control the phase space coordinates at the injection point. Each kicker magnet is calibrated by applying a voltage difference to the magnet, measuring the orbit response using the ring BPMs, and varying the angular kick associated with the magnet until the model orbit agrees with the measured orbit. It was found that slight quadrupole corrections are necessary for this to occur. The standard deviation of the measured phase space coordinates is usually small after this calibration. One can then ask the model for a change in coordinates, update the kickers accordingly, and measure the new coordinates, iterating if necessary. This is currently done manually, and kicker power supplies need to be visually checked to make sure they are not beyond physical limits. Once this setup is complete, the kicker voltages are saved to a file and fed to a different script which scales sets the kicker waveforms. These steps were implemented as part of the Ring Injection Control (RIC) application in the OpenXAL framework [56] prior to this work.

The SNS injection kicker magnets have limited strengths and are unipolar, which limits the maximum injection angle. This is potentially disadvantageous because it could restrict the final apparent beam emittances, which should be maximized to reduce the effect of space charge and better approximate a pencil-beam from the linac, i.e., a small ratio of the minipulse emittances to the final pulse emittances. For example, one painting scheme is to vary the relative horizontal distance and vertical angle between the beams while fixing the vertical distance and horizontal angle; in this case, the final vertical emittance depends only on the maximum vertical angle. Previous simulations indicate

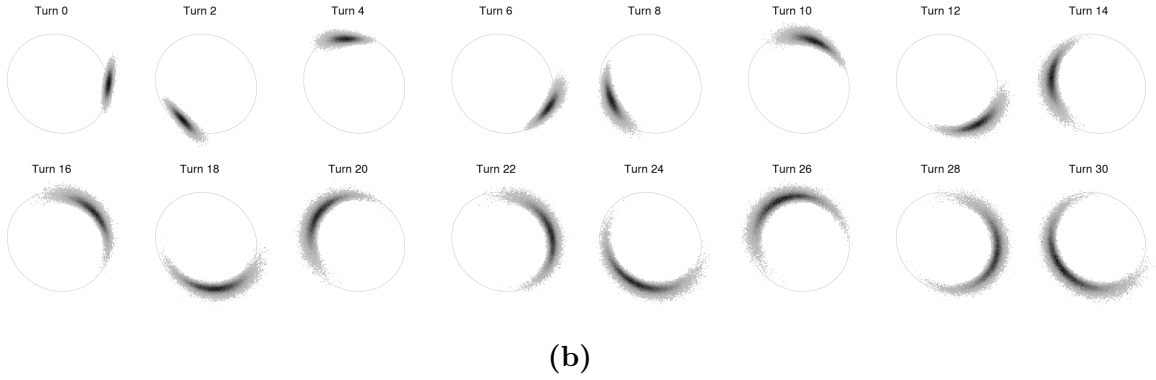
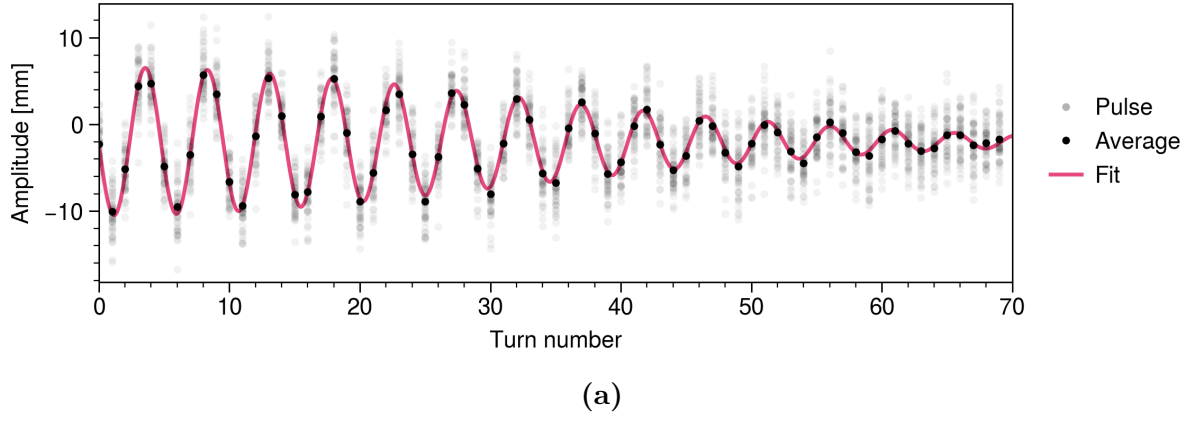


Figure 1.11: (a) Measured turn-by-turn BPM signal in the SNS ring — averaged over 50 pulses and fit with Eq. (1.39). (b) Simulated minipulse in the (linearized) SNS ring. The x - x' distribution is plotted at the injection point along with the Courant-Snyder ellipse.

that this angle will be quite small at the nominal beam energy of 1.0 GeV. Several tricks to increase the effective strength of the injection kicker magnets are discussed in Chapter 5. An example trajectory using this painting scheme is shown in Fig. 1.12.

1.4 Structure and goals of this dissertation

The primary goal of this dissertation is to contribute to efforts to produce an approximate Danilov distribution in the SNS ring. The secondary goal is to improve the current understanding of the dynamics of the Danilov distribution with space charge.

In Chapter 2, the envelope equations describing the linear transport of the Danilov distribution are used to calculate the matched beam envelope with the inclusion of space charge forces. Such a calculation is critical to producing a Danilov distribution in a ring using the elliptical painting method. In Chapter 3, the computational model used for realistic simulations of beam dynamics in the SNS ring is described. Previously published simulation results are re-examined and the significance of updated experimental constraints is discussed. In Chapter 4, several methods to measure the similarity between a painted distribution and a Danilov distribution are identified, implemented using existing diagnostics in the SNS, and optimized. In Chapter 5, the results of three initial experimental studies of elliptical painting in the SNS are presented. Simulations are provided for comparison. In Chapter 6, implications and extensions of this work are discussed.

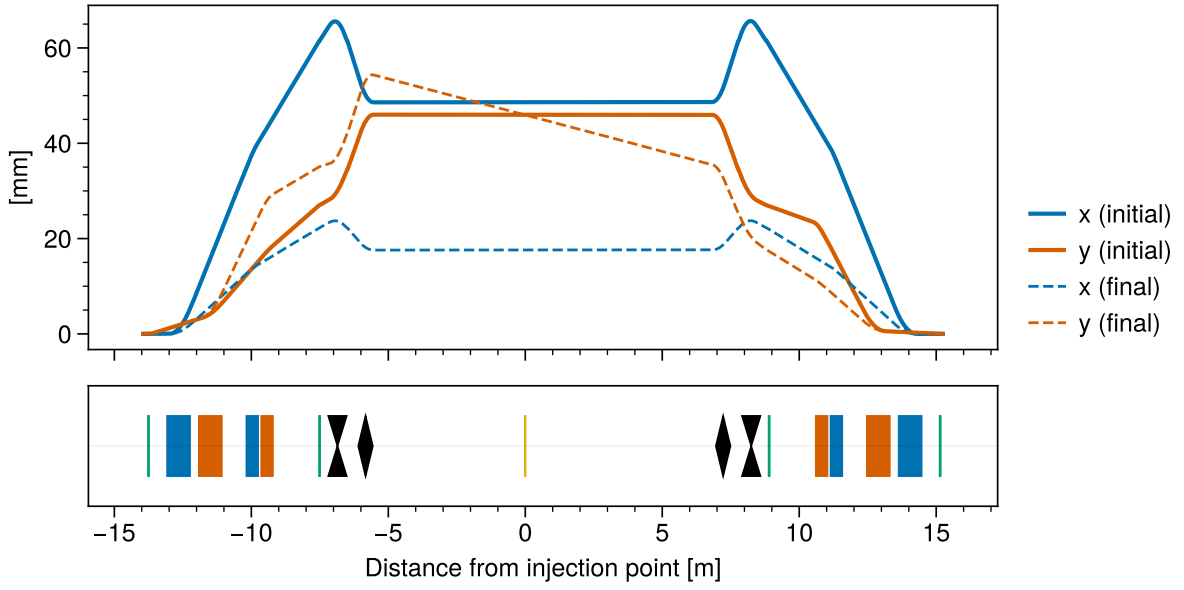


Figure 1.12: Example initial/final trajectories of the closed orbit in the SNS injection region for the elliptical painting method. The initial injected coordinates are $(x, x', y, y') = (0 \text{ mm}, 0 \text{ mrad}, 0 \text{ mm}, 0 \text{ mrad})$ and the final injected coordinates are $(x, x', y, y') = (31 \text{ mm}, 0 \text{ mrad}, 0 \text{ mm}, 1.5 \text{ mrad})$. The bottom plot shows the elements along the beamline (excluding the Chicane dipoles): stripper foil (gold), quadrupoles (black), horizontal kickers (blue), vertical kickers (orange), and vertical orbit corrector dipoles (green). The vertical orbit corrector dipoles provide a closed bump to assist the kickers.

Chapter 2

Beam envelope equations

In this chapter, the dynamics of the Danilov distribution are investigated using the envelope model. First, the envelope equations for the Danilov distribution are presented. Second, a method to find the matched beam envelope with space charge is developed. The method is demonstrated in several simple focusing systems and the properties of the solutions are examined. Third, the matched beam envelope in a linearized version of the SNS ring is calculated and the practical importance of the calculation is discussed.¹

2.1 The Danilov envelope equations

We seek a self-consistent set of differential equations for the elliptical boundary containing the beam particles, similar to Eq. (1.29). Although Chernin's equations satisfy this requirement [58], we adopt the equations derived in [9]. There, the coordinates of the ellipse in real space are parameterized as $\mathbf{x} = \mathbf{W}\mathbf{c}$ where $\mathbf{x} = (x, y)^T$, $\mathbf{W}$ is the 2×2 envelope matrix, $\mathbf{c} = (\cos \psi, \sin \psi)^T$, and ψ is a free parameter running from 0 to 2π . The envelope matrix evolves according to

$$\mathbf{W}'' + (\mathbf{K}_0 - \mathbf{K}_{sc})\mathbf{W} + \mathbf{K}_1\mathbf{W}' = 0, \quad (2.1)$$

where $\mathbf{K}_0$, $\mathbf{K}_1$, and $\mathbf{K}_{sc}$ are time-dependent 2×2 matrices. Linear external focusing is encompassed by $\mathbf{K}_{0,1}$, and linear space charge defocusing is encompassed by $\mathbf{K}_{sc}$. If the beam ellipse in real space has semi-axes c_x and c_y and is tilted at an angle ϕ below the x axis, then

$$\mathbf{K}_{sc} = \frac{2Q}{c_x + c_y} \mathbf{R}(\phi) \begin{bmatrix} 1/c_x & 0 \\ 0 & 1/c_y \end{bmatrix} \mathbf{R}(\phi)^T. \quad (2.2)$$

$\mathbf{R}$ is the rotation matrix and Q is the beam perveance (Eq. (1.30)). When Eq. (2.1) is expanded, the evolution of each envelope matrix element resembles that of a single particle in a linear system with a nonlinear driving term proportional to Q . Thus, the equations can be easily integrated using an existing tracking code such as PyORBIT [59]. The four elements of $\mathbf{W}$ are represented using two bunch particles, and nodes are added to the accelerator model to perform the nonlinear kicks.

The integration of Eq. (2.1) reveals that the Danilov distribution tilts in real space even

¹The majority of this chapter has been published in [57].

without coupled forces; this is illustrated in Fig. 2.1. The tilt angle is a function of the difference between the horizontal and vertical phase advances, which are calculated from

$$\mu_x(s) = \int_0^s \frac{\varepsilon_x(s')}{\tilde{x}(s')^2} ds', \quad (2.3)$$

where $\tilde{x}^2 = \langle x^2 \rangle$ and x and y can be interchanged.

2.2 Matched envelope computation

The distribution function f of a matched beam in a lattice of period length L satisfies $f(s) = f(s + L)$ for all s . In practice, matching usually refers only to the second-order moments contained in the beam covariance matrix: $\Sigma(s) = \Sigma(s + L)$. The two notions are equivalent for the Danilov distribution, for which all higher-order moments vanish.

The problem of computing the matched beam is as follows:

$$\begin{aligned} & \underset{\Sigma}{\text{minimize}} && C(\Sigma) = \|\mathbf{M}\Sigma\mathbf{M}^T - \Sigma\|^2 \\ & \text{subject to} && |\Sigma| = 0, \end{aligned} \quad (2.4)$$

where $\mathbf{M} = \mathbf{M}(\Sigma)$ is a linear transfer matrix connecting the initial and final covariance matrix, and $\|\dots\|$ is the matrix norm. The constraint that the covariance matrix is singular comes from the definition of the Danilov distribution. Additionally, we would like to hold the nonzero intrinsic emittance fixed so that a unique solution is found for a given lattice and beam perveance. An iterative approach is needed since space charge causes $\mathbf{M}$ to depend on Σ in a potentially complicated way which is unknown before tracking the beam.

2.2.1 Motivation

The calculation of the matched beam envelope is of practical importance for space-charge-dominated beams. It is a common first step in lattice design [60]. The beam current able to be transported through a periodic focusing channel with a given aperture is maximum when the beam is matched [28]. The free energy available in a mismatched beam may result in emittance growth and halo formation [28]. Calculation of the matched envelope is generally the first step in a stability analysis of the KV envelope equations [60]. The task has been performed for the KV distribution in both uncoupled and coupled focusing systems [58, 60–64], and it would be beneficial to extend this analysis to the Danilov distribution. Furthermore, the calculation of the matched envelope of the Danilov distribution is of critical importance to the experimental realization of such a distribution: the result of the elliptical painting method only approaches a Danilov distribution if the circulating beam is matched.

The purpose of this section is to calculate and describe the properties of the matched envelope of the Danilov distribution in simple focusing systems as space charge is increased. In addition to the ability to calculate the matched envelope in more complicated focusing systems such as the SNS accumulator ring, a better understanding of the beam dynamics under the influence of coupled internal and external forces should fall out of this analysis.

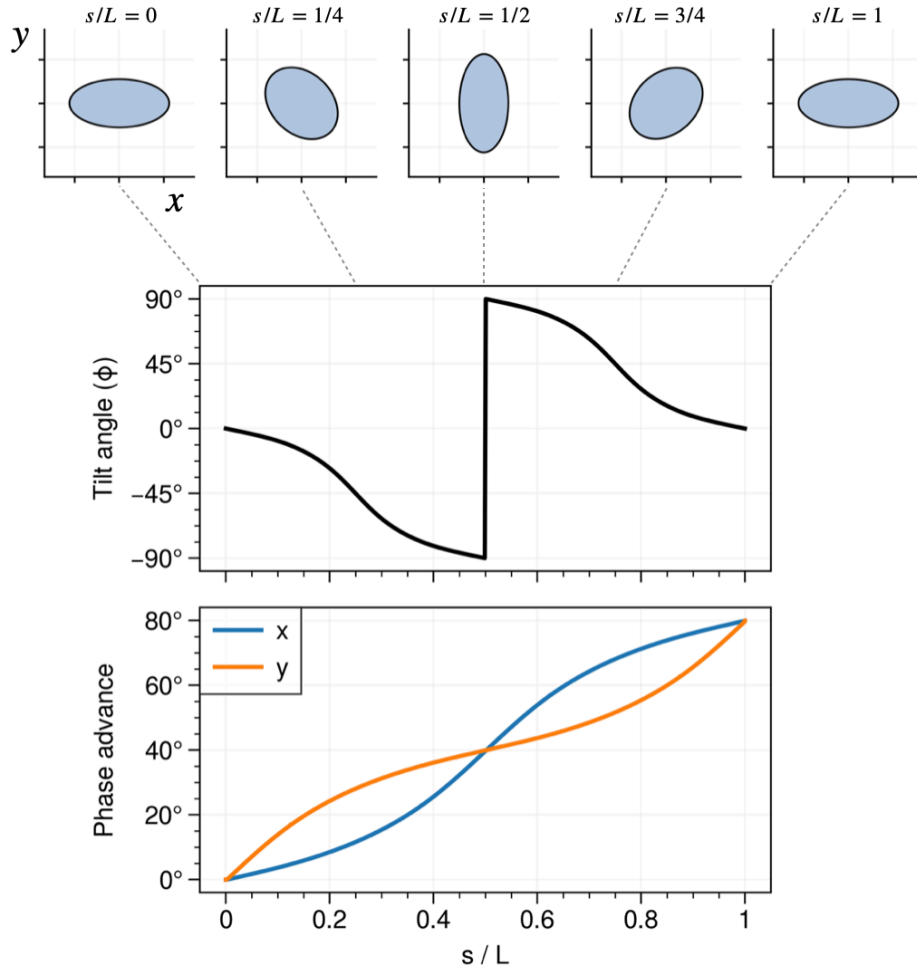


Figure 2.1: Relationship between the tilt angle in real space and the x and y phase advances for a Danilov distribution in a FODO cell of length L . The beam is tracked without space charge by integrating the envelope equations.

Before presenting our solution, we highlight the space-charge-driven mismatch oscillations that are to be corrected. Fig. 2.2 shows the turn-by-turn evolution of a beam that is matched to a lattice without space charge and then tracked with space charge. There are two frequencies in the mismatch oscillations: a larger frequency near twice the zero-current tune corresponding to the breathing oscillation of the beam sizes, and a smaller frequency corresponding to the emittance exchange from space-charge-driven linear coupling.

2.2.2 Solution

The problem of computing the matched envelope can be approached in the following way. The effect of the linear beam space charge is to modify the linear focusing strength at every position; we call this modified linear focusing system the *effective lattice*. Generating a beam that is matched to a lattice with space charge is equivalent to generating a beam that is matched to an effective lattice without space charge. The latter task is straightforward using an existing parameterization of coupled motion. The correct effective lattice is unknown a priori, so a search must be performed over the parameters. Fig. 2.3 illustrates the concept of the effective lattice.

Zero space charge

We begin by demonstrating how to compute the matched envelope without space charge. We write Eq. (1.12) again for convenience:

$$\mathbf{M}\mathbf{x} = \text{Re} \left\{ \sqrt{2J_1} \mathbf{v}_1 e^{-i(\psi_1 + \mu_1)} + \sqrt{2J_2} \mathbf{v}_2 e^{-i(\psi_2 + \mu_2)} \right\}. \quad (2.5)$$

The turn-by-turn trajectory of a particle with a given $J_{1,2}$ forms a closed surface in phase space, and a group of particles distributed uniformly over this surface will appear to be invariant. A matched distribution is a collection of these surfaces with different amplitudes.

We now switch to the collective description of the beam using its covariance matrix. The symplectic normalization matrix $\mathbf{V}$ (from Eq. (1.13)) can be used to express the matched beam covariance matrix as

$$\Sigma = \mathbf{V} \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix} \mathbf{V}^T. \quad (2.6)$$

In the uncoupled case, $\mathbf{V}^{-1}$ transforms a tilted ellipse in the x - x' plane into a circle. In the coupled case, $\mathbf{V}^{-1}$ transforms a “tilted” 4D ellipsoid into an “upright” 4D ellipsoid. A parameterization of $\mathbf{V}$ was introduced in Fig. 1.3. The number of parameters can be reduced to six by observing that the Danilov distribution is a function of only one eigenvector. We now set one of the intrinsic emittances to zero in Eq. (2.6) and display the connection between the parameters and the covariance matrix. The beta functions give the ratios between the beam size and intrinsic emittance:

$$\beta_{lx} = \frac{\langle x^2 \rangle}{\varepsilon_l}, \quad \beta_{ly} = \frac{\langle y^2 \rangle}{\varepsilon_l} \quad (2.7)$$

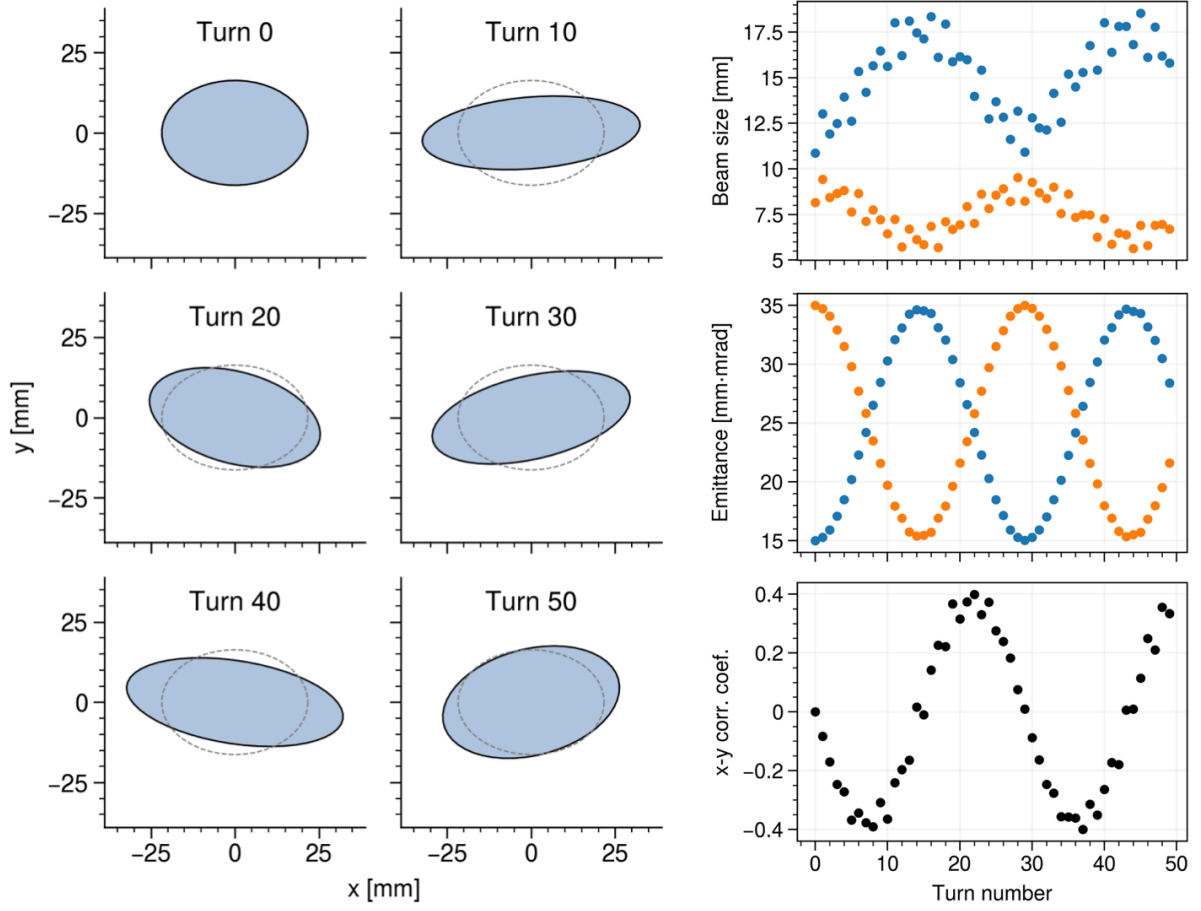


Figure 2.2: Turn-by-turn mismatch oscillations of the Danilov distribution at the entrance of an uncoupled lattice. The beam would be matched to the lattice without space charge. In the right column, blue (orange) corresponds to x (y).

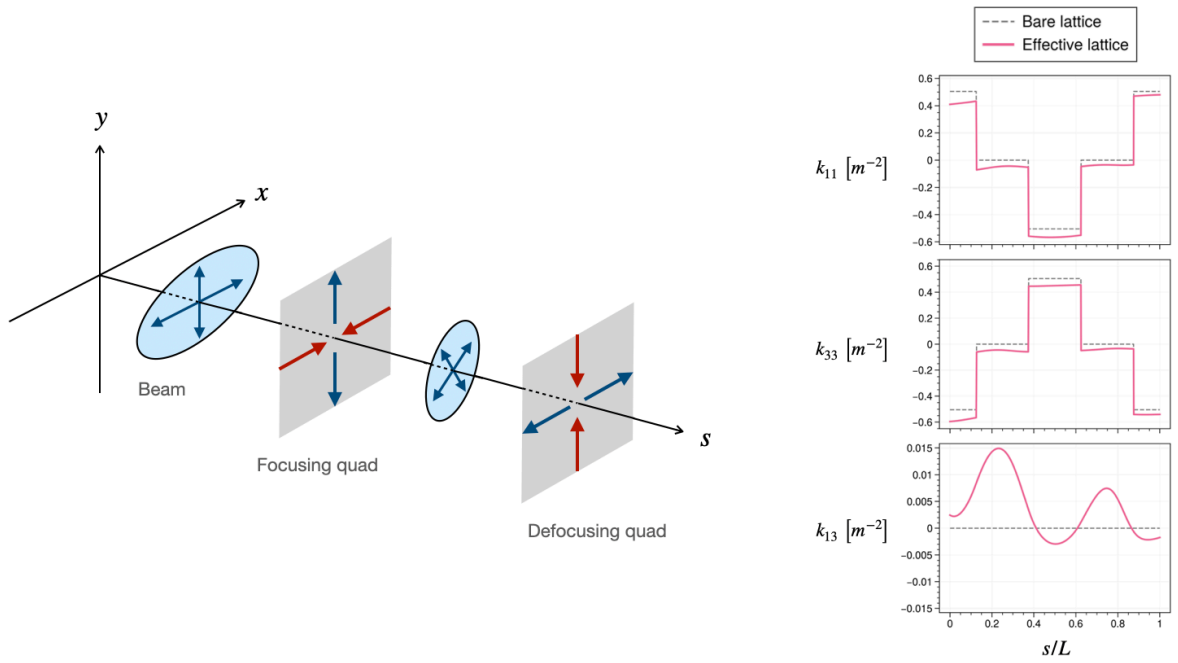


Figure 2.3: Illustration of the effective lattice — the net linear focusing after space charge is included. The coefficients k_{ij} are defined by $x'' + k_{11}x + k_{13}y = 0$; $y'' + k_{33}y + k_{31}x = 0$.

where $l = 1$ or 2 . Similarly, the alpha functions give the ratios between the beam divergence and the intrinsic emittance:

$$\alpha_{lx} = -\frac{\langle xx' \rangle}{\varepsilon_l}, \quad \alpha_{ly} = -\frac{\langle yy' \rangle}{\varepsilon_l} \quad (2.8)$$

Next, u gives the ratio between the apparent emittance and the intrinsic emittance. When $l = 1$, $u = \varepsilon_y/\varepsilon_l$, or when $l = 2$, $u = \varepsilon_x/\varepsilon_2$. Finally, ν_l is related to the x - y correlation coefficient:

$$\cos \nu_l = \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}}. \quad (2.9)$$

The subscript will be dropped from now on since it has no effect. As mentioned in Fig. 2.1, ν will vary even without the presence of coupled forces. For example, Fig. 2.4 shows the turn-by-turn x - y projection of a beam whose x - x' and y - y' ellipses are matched to an uncoupled lattice with a tune separation of 0.01, along with the value of ν at each frame. In summary, the matched beam is described by a vector of parameters $\mathbf{p}$, where

$$\mathbf{p} = (\alpha_{lx}, \alpha_{ly}, \beta_{lx}, \beta_{ly}, u, \nu) \quad (2.10)$$

with $l = 1$ or 2 depending on which intrinsic emittance is zero.

Nonzero space charge

We denote the choice $\varepsilon_2 = 0$ as solution 1 and $\varepsilon_1 = 0$ as solution 2. Once this emittance is chosen, $\mathbf{p}$ is initialized using the bare lattice parameters. We then perform the following procedure: (1) generate a beam envelope from $\mathbf{p}$, (2) track the beam through one lattice period and compute the cost function, (3) update $\mathbf{p}$, (4) stop if the relative change in C or $|\mathbf{p}|$ is below a given tolerance, otherwise repeat from step 1. A trust-region minimization algorithm [65] is used to determine the update strategy for $\mathbf{p}$. If necessary, the process can be repeated at multiple steps so that the seed envelope remains close to the matched solution. In one case during our studies, this optimizer failed to converge and it became necessary to use a custom update method; in this method, the beam is tracked for several turns and $\mathbf{p}$ is updated to its average over those turns. The method does not need to worry about bounds on the parameters since every update is based on an existing beam. An example of the progress of this method is shown in Fig. 2.5.

2.2.3 Method demonstration

The matching routine was applied to an equally spaced, periodic quadrupole (FODO) lattice. The horizontal focusing strength in this lattice is shown in Fig. 2.6a as a function of s . Several variants of the FODO lattice were also considered to include external coupling: in Fig. 2.6b, the focusing and defocusing quadrupoles are rotated by 3° in opposite directions in the transverse plane, and in Fig. 2.6c solenoid magnets are inserted in the drift spaces between the quadrupoles. This section examines the matched solutions in each lattice as space charge is increased. Previous studies indicate that the KV envelope equations have a unique matched solution for each choice of lattice, beam perveance, and apparent emittances. Although there is no known proof of this conjecture, it seems to be true based on numerical evidence [60]. Thus, for the Danilov distribution, it was expected that each choice of lattice and beam perveance would lead to two matched

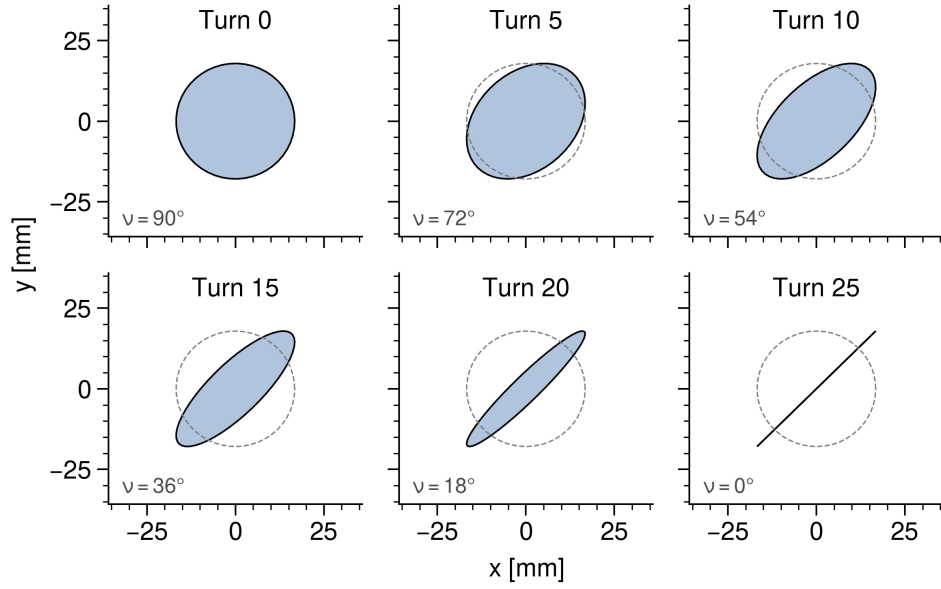


Figure 2.4: Turn-by-turn x - y projections of a Danilov distribution in a lattice with a tune split of 0.01. The x - x' and y - y' projections are matched to the lattice.

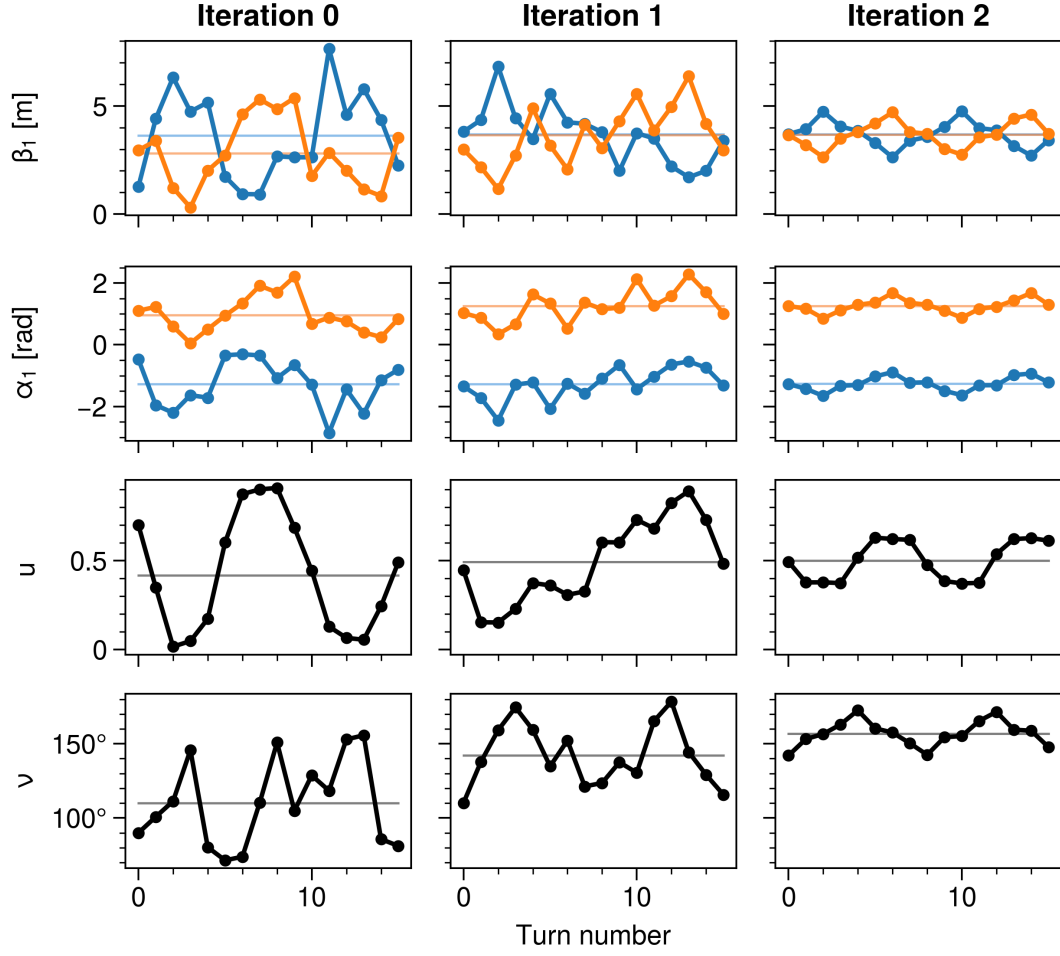


Figure 2.5: Turn-by-turn oscillations of the beam parameters after the first few iterations of the matching routine. The custom update method is used. Faint horizontal lines give the average of the oscillations. Blue (orange) corresponds to x (y).

solutions depending on which intrinsic emittance is set to zero. No evidence to the contrary was found.

Uncoupled lattice

We now apply the matching routine to an uncoupled FODO lattice. First, a note about the matched solution without space charge. The eigenvectors of the transfer matrix are uncoupled, meaning that $\mathbf{v}_1$ has no component in the y - y' plane and $\mathbf{v}_2$ has no component in the x - x' plane. A matched beam is formed by generating particles uniformly in phase along either of these eigenvectors; therefore, the matched beam is flat ($\varepsilon_x = 0$ or $\varepsilon_y = 0$). An exception occurs when the transfer matrix has degenerate eigenvalues, i.e., when the horizontal and vertical tunes are equal. In this case, any linear combination of eigenvectors forms another eigenvector. Thus, without space charge, there are an infinite number of matched solutions in a lattice with equal tunes. The free parameters from Eq. (2.10) are u and ν .

We now include the self-force of the beam in the matching routine. Fig. 2.7 shows the matched beam sizes, apparent emittances, and ν parameter within the lattice for a range of linearly increasing space charge strengths.² It also shows the phase space projections at the lattice entrance. The following properties of the matched solutions are worth noting. First, except for the oscillatory apparent emittances, the beam evolution within the lattice when space charge is nonzero is very similar to the case of zero space charge. Second, two solutions are found which differ in the sign of their angular momentum. This is seen in the opposite signs of the slopes defining the linear relationships between the position in one plane and the momentum in the other; it is a consequence of the opposite directions of rotation of the two eigenvectors. The third property to note is how the solutions scale with increased space charge: the average width and height of the matched beam within the lattice grow approximately linearly, and the variation in the difference between the horizontal and vertical phase advances is suppressed — hence the decreased oscillation of the ν parameter.

The same analysis can be performed when the horizontal and vertical tunes are split. We chose to increase the horizontal phase advance and decrease the vertical phase advance, both by 5° . The results are displayed in Fig. 2.8.³ Due to the unequal tunes in the lattice, the zero space charge solution is flat, with maximal separation of the apparent emittances. Space charge generates a non-flat matched beam by decreasing the horizontal and vertical tunes by unequal amounts such that they are equal in the matched beam. Since the original horizontal tune is larger than the vertical tune, the average ε_x is always larger than the average ε_y to create stronger defocusing in the horizontal plane. Additionally, since the tune depression scales nonlinearly with the space charge strength, the average difference between ε_x and ε_y must also decrease nonlinearly with the space charge strength. The beam evolution within the lattice is similar to the previous case.

Coupled lattice

The same information as in Fig. 2.7 and Fig. 2.8 is plotted in Fig. 2.9 for the skew quadrupole lattice from Fig. 2.6b. The locations of the skew quadrupoles are evident

²For the zero space charge solution, we chose $u = 0.5$ and $\nu = \pi/2$.

³ ν is undefined when the beam is flat, but we chose to draw a flat line at $\nu = \pi/2$.

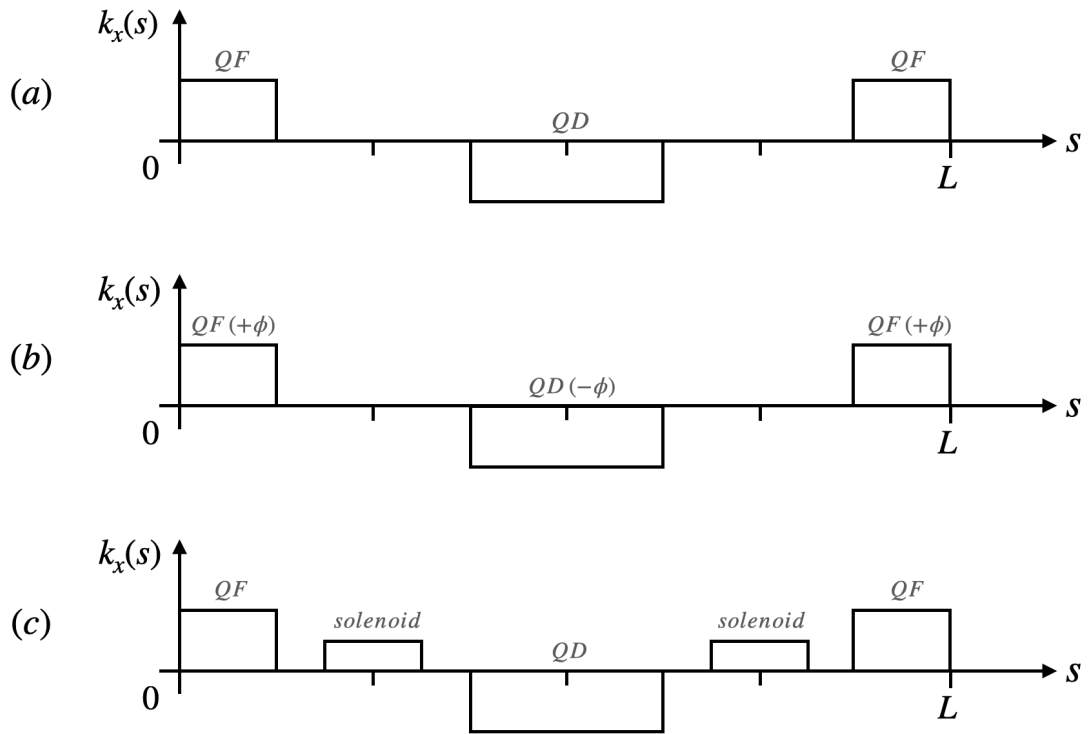
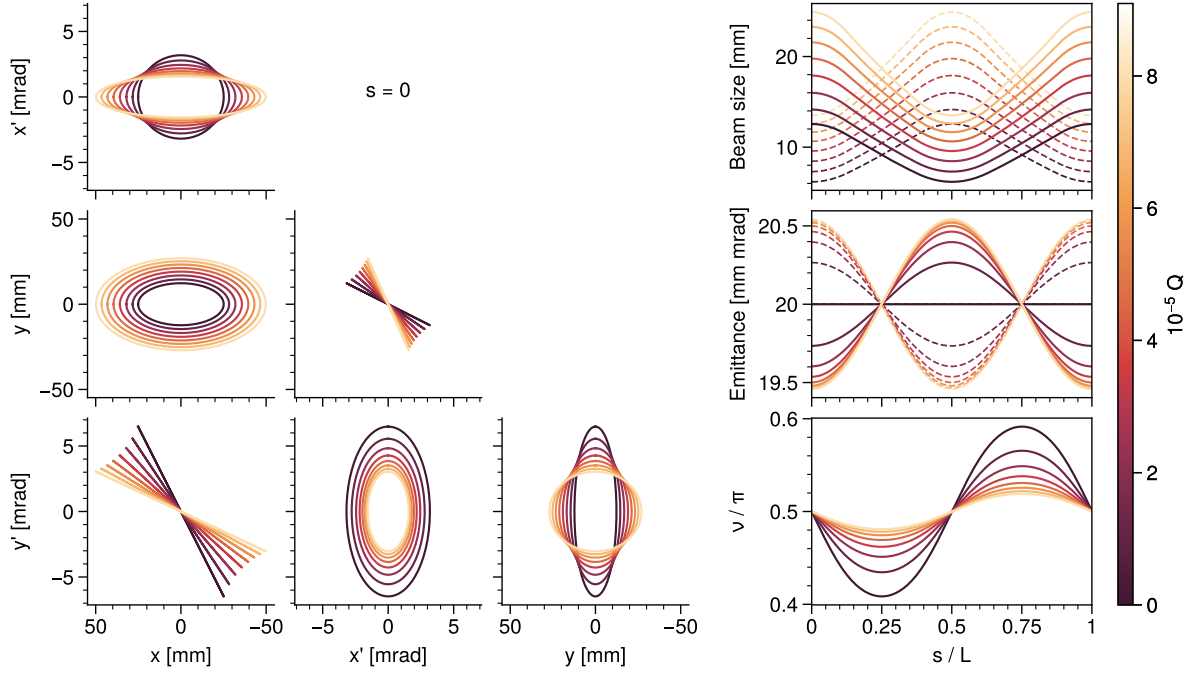
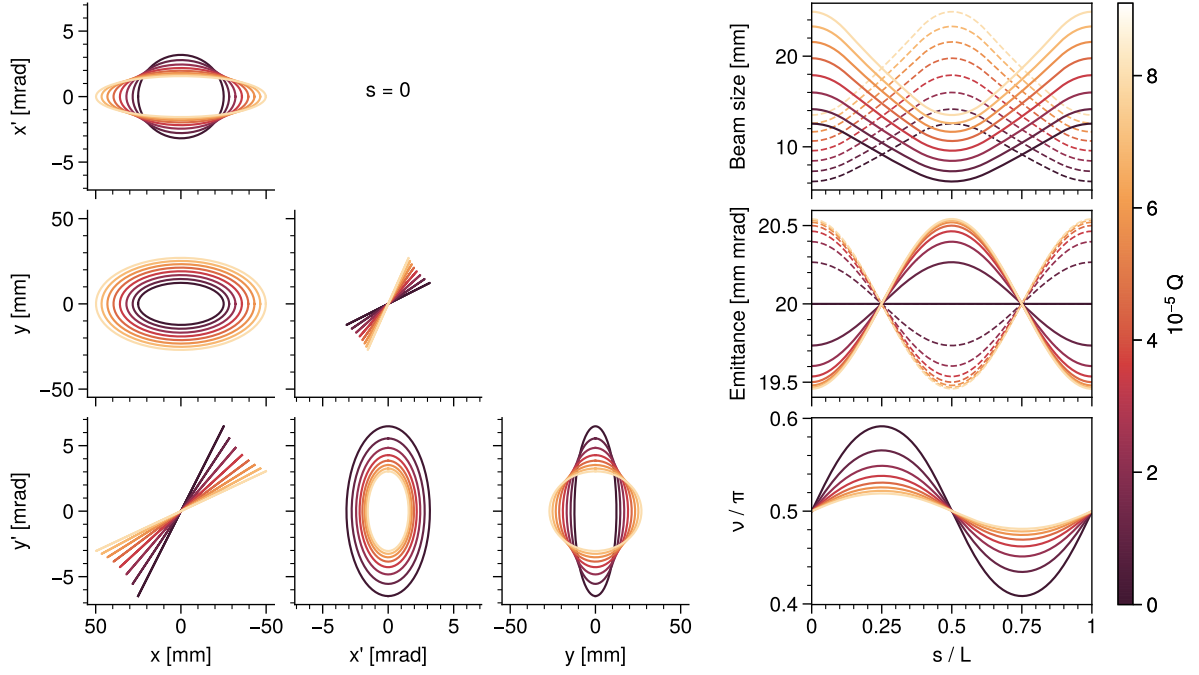


Figure 2.6: Horizontal focusing strength as a function of s in a FODO lattice with period length L . Quadrupoles have length $L/4$ and are equally spaced. (a) Upright quadrupoles with 80° phase advance in both planes. (b) The quadrupoles are rotated by $\phi = 3^\circ$ in the transverse plane (QF and QD are rotated in opposite directions). (c) Solenoid magnets are inserted between the quadrupoles in (a).

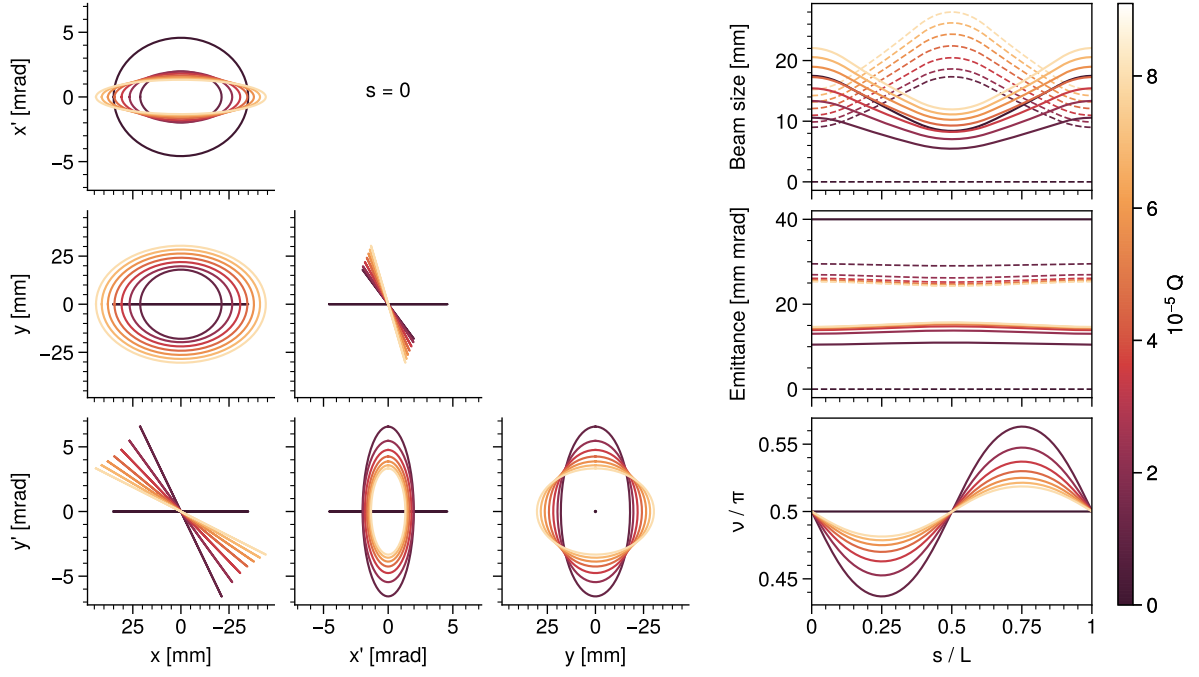


(a) Solution 1

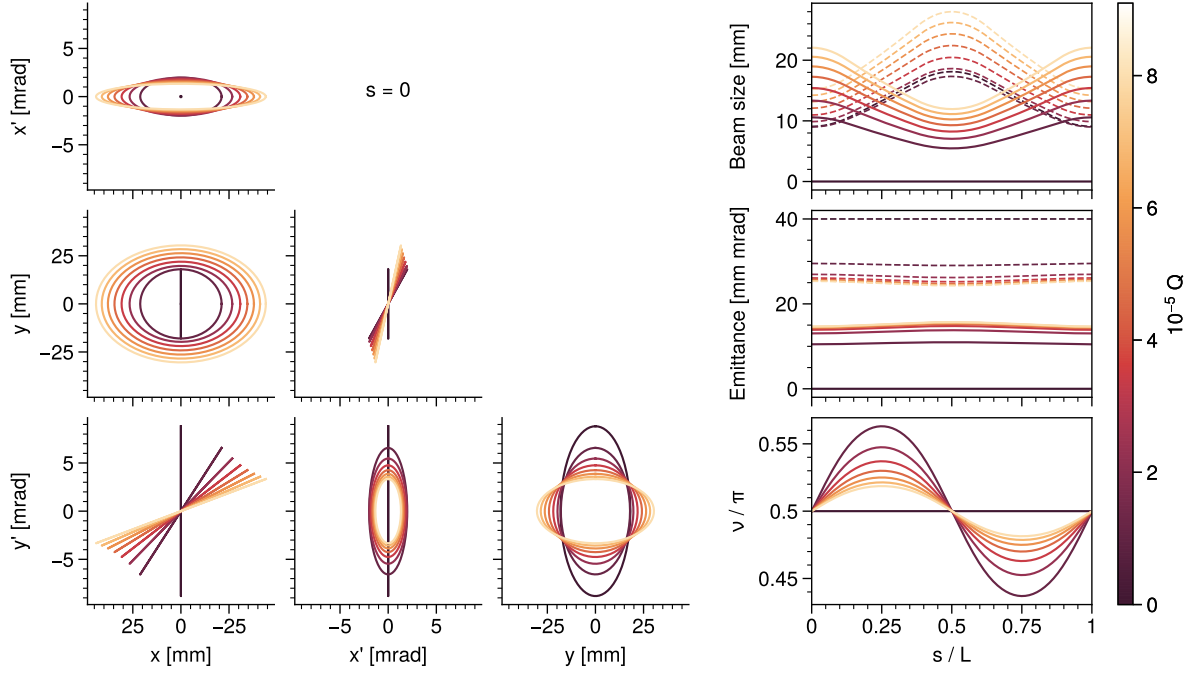


(b) Solution 2

Figure 2.7: Matched envelope of the Danilov distribution in an uncoupled FODO lattice as space charge is increased. Left: phase space projections at the lattice entrance. Right: beam parameters within the lattice. Solid lines are for x and dashed lines are for y in the top two plots.

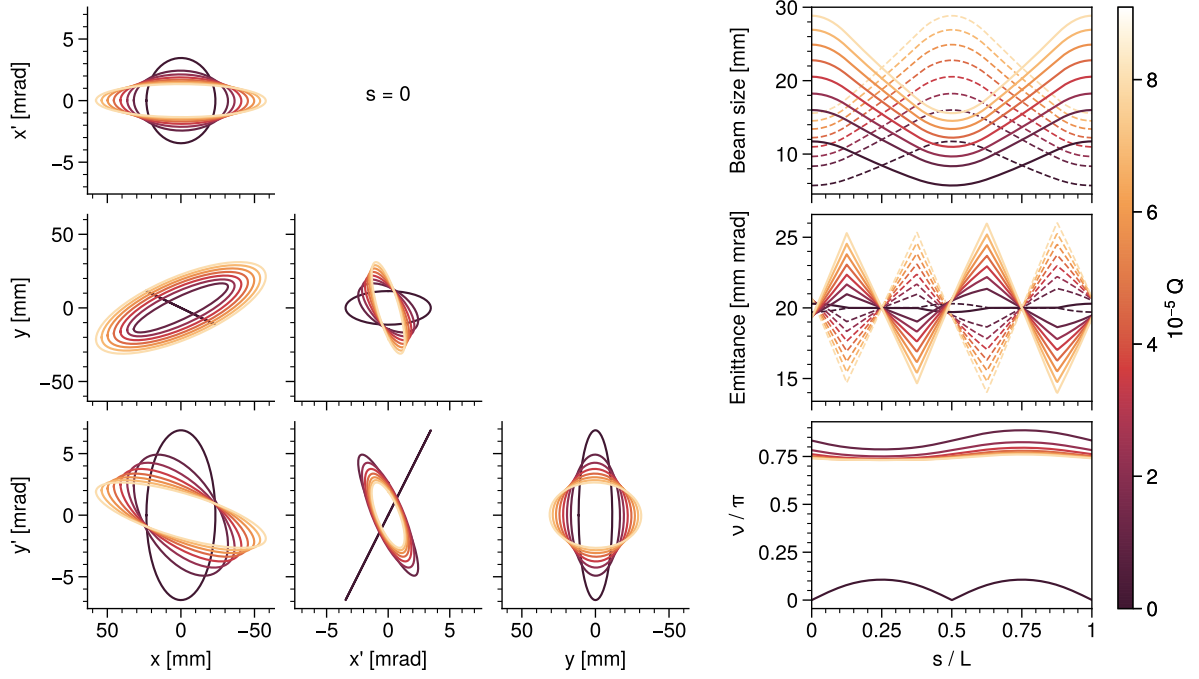


(a) Solution 1

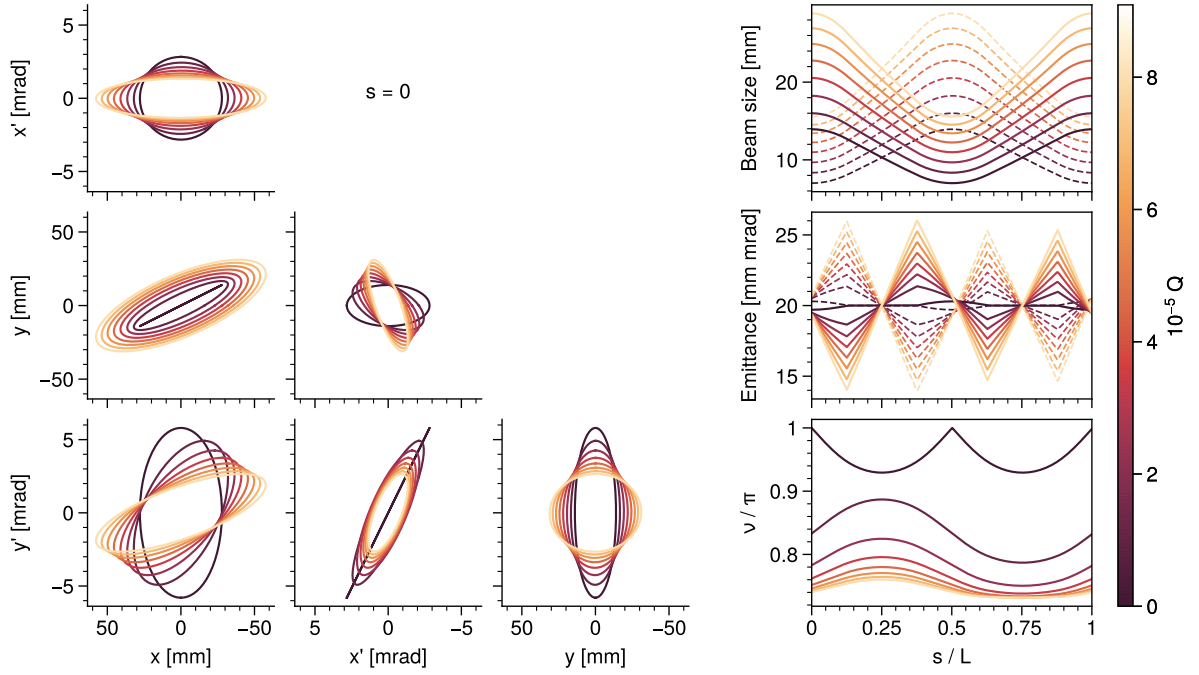


(b) Solution 2

Figure 2.8: Matched envelope of the Danilov distribution in an uncoupled FODO lattice with unequal tunes as space charge is increased. Left: phase space projections at the lattice entrance. Right: beam parameters within the lattice. Solid lines are for x and dashed lines are for y in the top two plots.



(a) Solution 1



(b) Solution 2

Figure 2.9: Matched envelope of the Danilov distribution in a coupled FODO lattice as space charge is increased. The lattice is coupled due to skew quadrupoles. Left: phase space projections at the lattice entrance. Right: beam parameters within the lattice. Solid lines are for x and dashed lines are for y in the top two plots.

from the small arcs in the emittance curves. Without space charge, the matched beam at the center of the quadrupoles projects to a diagonal line in real space ($\nu = 0^\circ$ or 180°) with zero angular momentum, and the two solutions differ in the sign of the tilt angle of this line. The inclusion of space charge pulls ν away from these extreme values, resulting in a nonzero beam area. The cross-plane correlations between the positions and slopes also become nonzero, again revealing the opposite signs of the angular momentum between the two solutions.

The presence of space charge leads to two solutions with the same tilt angle in the x - y plane, as opposed to opposite tilt angles without space charge. This abrupt change in the matched beam orientation in solution 1 caused the optimizer to struggle for low space charge, often terminating due to a lack of progress. Fig. 2.10a shows the final cost as a function of the beam perveance, and Fig. 2.10b shows the turn-by-turn oscillations of the ν parameter for a subset of these cases. The matching routine is not run when $Q = 0$ since the beam is already matched to the bare lattice; this corresponds to the bottom line in both panels of Fig. 2.10b at $\nu = 0$. The final cost is therefore the same for the two algorithms at this point. For small but nonzero perveance, the optimizer converges to a beam with $\nu \approx 0$ which exhibits very small mismatch oscillations (yellow region). The oscillations become more severe as the perveance is increased (red region), which corresponds to lines starting at $\nu \approx 25^\circ$ in Fig. 2.10b. An exact match is eventually found (green region) when the perveance is sufficiently large with $\nu \approx 150^\circ$. The averaging method, on the other hand, finds the exact match in nearly all cases. (Note that the circles and crosses on the far right of Fig. 2.10b represent the same beam; the algorithms have just terminated at different final costs.) This discussion simply illustrates that some care must be taken for certain values of the beam perveance when skew quadrupoles are present.

As a final demonstration of the method, coupling was included by the insertion of solenoid magnets in Fig. 2.6c. The results are shown in Fig. 2.11. The matched beam resembles that of the uncoupled FODO lattice in Fig. 2.7; most notably, it is round at the symmetry points in the lattice. The differences are found in the apparent emittances: their oscillation amplitude is larger within the drift spaces and quadrupoles, and there is a significant additional emittance exchange within the solenoids.

Effective transfer matrices

The effective linear transfer matrix generated by the lattice and matched beam can be calculated by tracking test particles subject to the internal fields of the matched beam.⁴ The eigenvalues of each effective transfer matrix are plotted in the complex plane as space charge is increased in Fig. 2.12. We observe that the difference between the phase advances $\Delta = |\mu_2 - \mu_1|$, which is a measure of the coupling strength in the effective lattice, is never zero when space charge is nonzero. (In the top two rows, Δ is very small and the plotted eigenvalues lie nearly on top of one another.) We also observe that the matched beam space charge cancels out some of the bare lattice coupling; for example,

⁴The matched beam is a function of one of the transfer matrix eigenvectors; the second eigenvector does not necessarily correspond to a matched solution in the same lattice. For example, the two solutions in the uncoupled FODO lattice share the same effective transfer matrix, but the two solutions in the skew quadrupole lattice do not share the same effective transfer matrix. In the latter case, the unused eigenvector is a matched solution in a lattice in which the sign of each skew term is reversed (a mirror reflection in one plane).

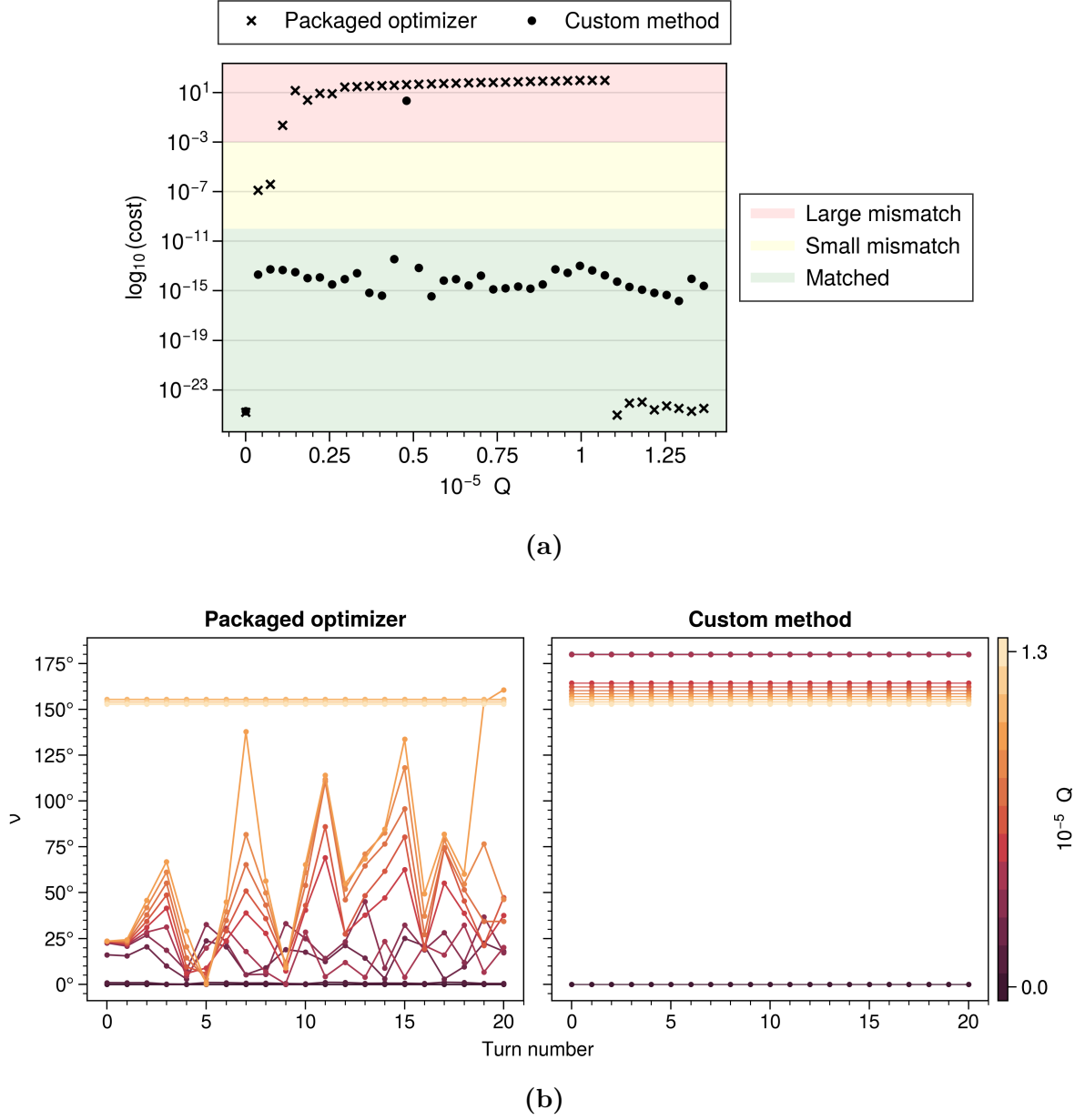
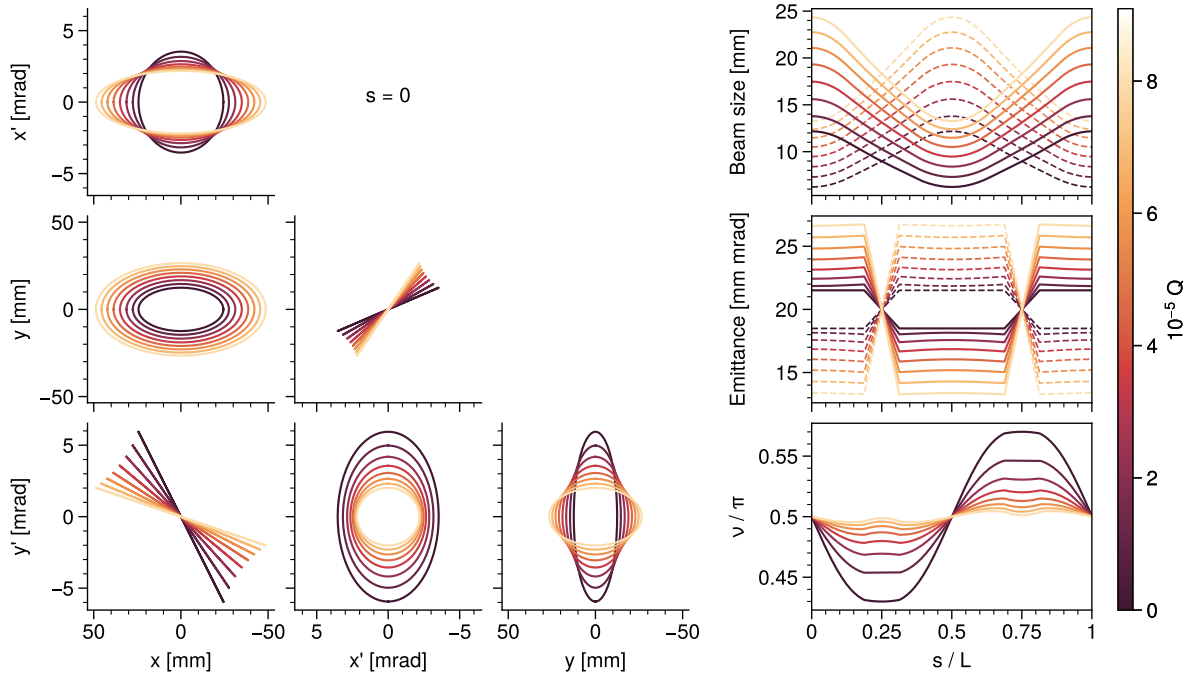
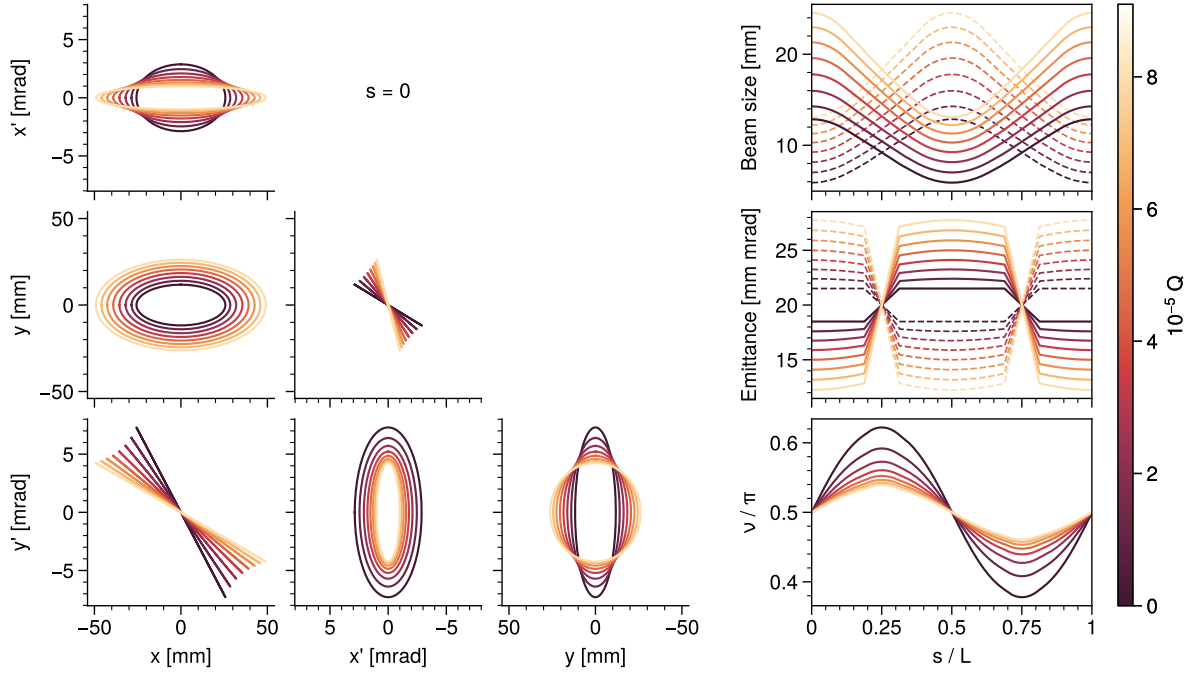


Figure 2.10: Performance of the matching algorithm in a skew quadrupole lattice, corresponding to solution 1 in Fig. 2.9a. (a) Final value of the cost function as a function of the beam perveance. (b) Turn-by-turn oscillations of the ν parameter after running each algorithm.



(a) Solution 1



(b) Solution 2

Figure 2.11: Matched envelope of the Danilov distribution in a coupled FODO lattice as space charge is increased. The lattice is coupled due to solenoid magnets inserted between the quadrupoles. Left: phase space projections at the lattice entrance. Right: beam parameters within the lattice. Solid lines are for x and dashed lines are for y in the top two plots.

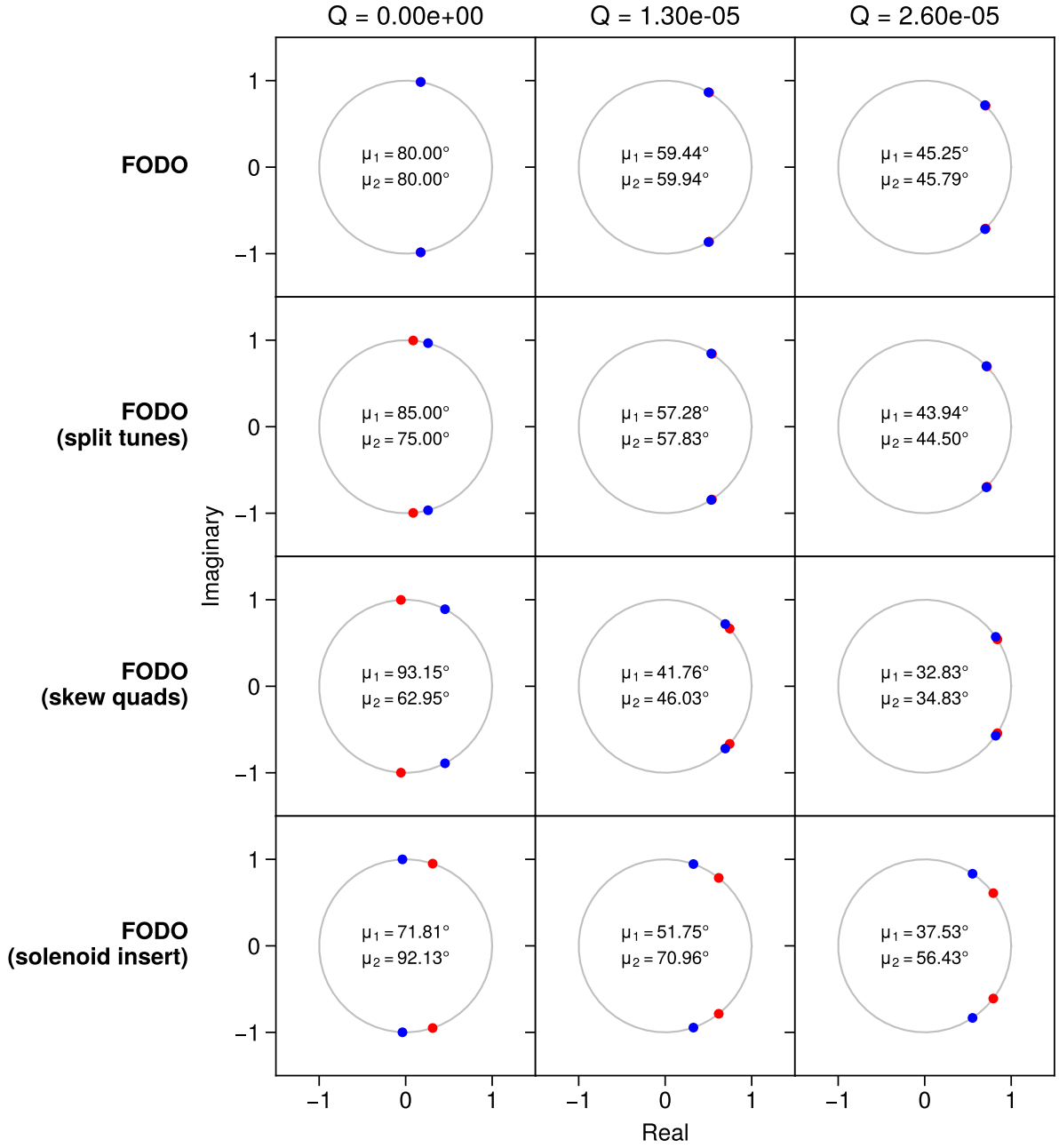


Figure 2.12: Eigenvalues of the transfer matrix of the effective lattice generated by the matched beam, plotted in the complex plane.

in the skew quadrupole lattice, Δ is large in the left column but nearly zero in the right column. This is not true when coupling is included using solenoid magnets; Δ instead remains relatively constant.

2.3 Relevance to experiment

These studies pave the way for future research on the stability of the Danilov distribution using perturbations around the matched envelope [64], as well as halo formation using the particle-core model [66–68]. Our findings are also relevant to future experiments which will aim to produce an approximate Danilov distribution in the SNS ring using the elliptical painting method. Recall the definition of elliptical painting: the injection point is scaled along an eigenvector of the transfer matrix. It is critical to account for the beam’s electric field when computing these eigenvectors; the painting must proceed along an eigenvector of the *effective* transfer matrix generated by the matched beam.

The matching procedure was therefore applied to the SNS ring lattice. Using the simple focusing channels in the previous section, we found that the matched beam at a symmetry point ($\alpha_x = \alpha_y = 0$) in an uncoupled lattice tends to be upright ($\nu \approx \pi/2$); since the injection point in the SNS ring is close to a symmetry point, we expect this tendency to hold. The top row of Fig. 2.13 shows the matched envelope at the SNS injection point with realistic parameters: $\varepsilon_1 = 0$ mm mrad, $\varepsilon_2 = 20$ mm mrad, energy = 0.8 GeV, bunch length $\approx 3/4$ ring length, intensity = 0.75×10^{14} , and equal tunes of 6.18. The matched cross-plane parameters are $u \equiv \varepsilon_x/\varepsilon_2 \approx 0.5$ and $\nu \approx \pi/2$. The FFT of the turn-by-turn oscillations of a particle on the edge of the beam over 200 turns is included on the right side of the figure, which shows that the tunes are equal. A mismatched initial beam is shown in the bottom row of Fig. 2.13. The initial beam envelope, represented by the pink ellipses, has $\nu = \pi/4$ and $u = 0.7$, with all other beam parameters unchanged. The beam ellipses over the next ten turns in the ring are overlayed on the same plot. The second peak in the Fourier spectrum represents a fast space-charge-driven emittance exchange, manifesting in the turn-by-turn rotation of the beam in the x - y plane.

If we attempted to paint the beam represented by the pink ellipses, the final distribution would resemble the superposition of the beams represented by the grey ellipses. As revealed by the blurred x - y' and y - x' correlations, the superposition of these beams is not a self-consistent distribution. Additionally, notice that the superposition has approximately equal apparent emittances; we therefore expect that even if the chosen painting parameters *should* produce a larger beam size in one plane, the final beam will be approximately round ($\varepsilon_x \approx \varepsilon_y$). Finally, a uniform density ellipse would not be maintained in this scenario because particles would not always be injected onto the beam edge. We conclude that the painting path should be a line in the x - y' (or y - x') plane and that x_{max} and y'_{max} (or y_{max} and x'_{max}) should be chosen such that the apparent emittances are equal.

These calculations complement the work of Holmes et al. [10], who used realistic injection simulations to determine the feasibility of elliptical painting in the SNS.⁵ One of their findings was that the ideal painting path in the SNS is a line in the x - y' plane. This conclusion was reached empirically: simulations were repeated as the difference between

⁵These simulations are described in more detail in Chapter 3.

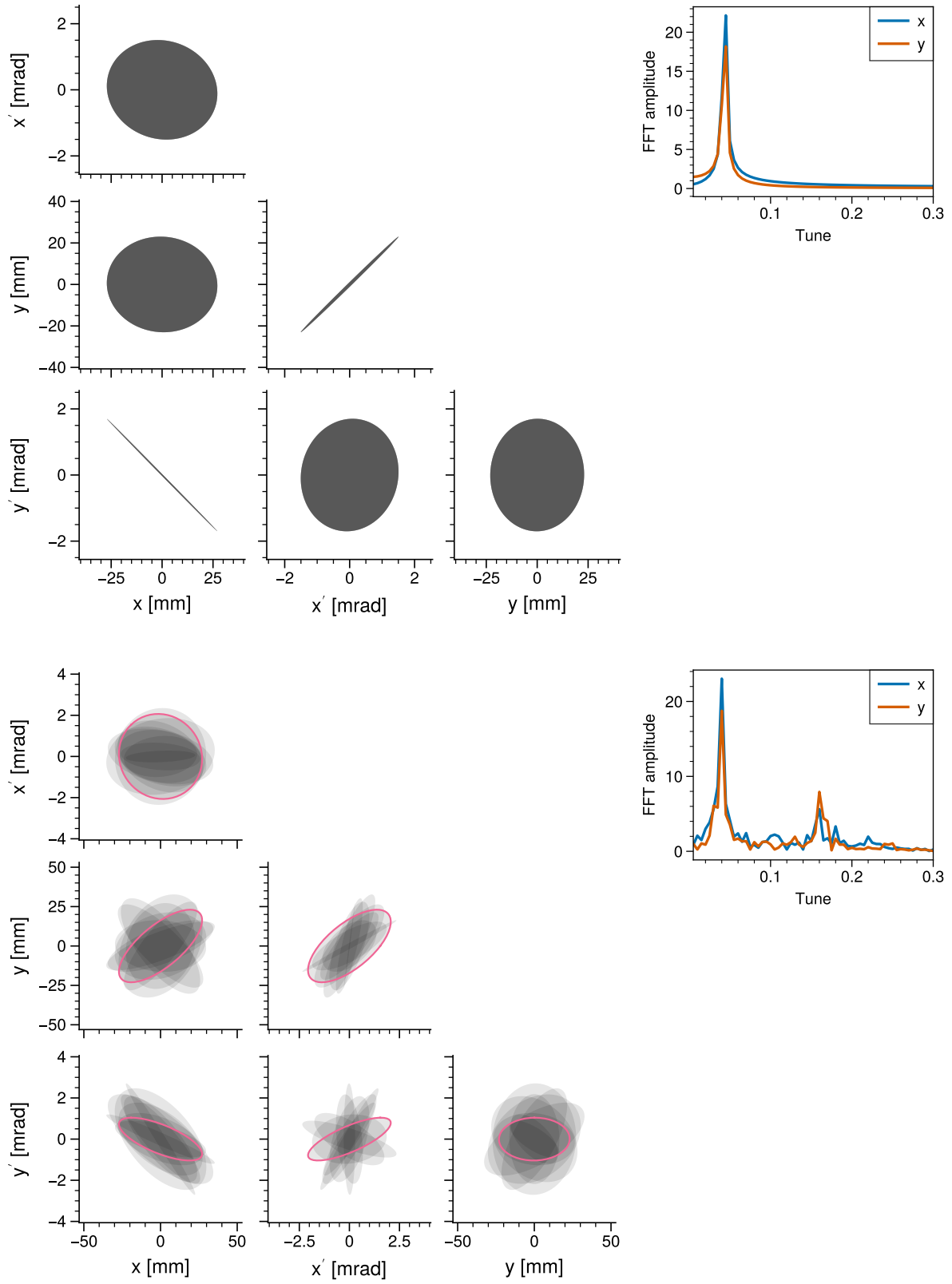


Figure 2.13: Matched (top) and mismatched (bottom) envelopes at the SNS ring injection point.

the horizontal/vertical phases of the injected particles was varied, effectively changing the intended ν parameter of the painted beam, and the fraction of lost particles was recorded in each case. The major driver of losses is the geometry of the injection region, which restricts x'_{max} .

Another finding from [10] was that when a solenoid is added to an uncoupled lattice with tune split $\nu_x - \nu_y$, the final beam quality is, to a degree, insensitive to the tune split. It was suggested that the beam adjusts its shape such that the depressed horizontal and vertical tunes are equal. Using the envelope model and a simple lattice, we have confirmed that such a process can occur by varying the ratio of apparent emittances. This may be a contributing factor to the simulated beam behavior. Another important factor to consider is that the transfer matrix of the ring with the solenoid produces two eigenvectors, each of which rotates in a circle in the x - y plane at the injection point. This does not depend on the tune split in the original lattice and remains true when linear space charge forces are included in the transfer matrix.

The above analysis demonstrates that envelope tracking is a valuable tool for the purposes of this work. Its main function is to provide fast insights into beam behavior and place rough constraints on the experimental parameters. We note that there remain unexplored modifications to the ring that may improve the painting method, and that some of these modifications could be tested using envelope tracking; for example, perhaps skew quadrupole correctors could be used to change the shape of the matched beam such that the required angular kicks, which introduce technical challenges as well as opportunities for beam loss, are minimized.

2.4 Summary

The evolution of the Danilov distribution is given by its envelope equations. An iterative procedure to calculate the matched envelope was developed by observing that the matched beam is a function of a single eigenvector of an unknown coupled transfer matrix. The method was demonstrated in a simple FODO lattice, which was then modified to study the effects of unequal tunes and linear coupling. Two matched solutions were obtained for each lattice and space charge strength. The primary difference between these solutions was the sign of their angular momentum. A common finding among nearly all the cases was that the shape of the matched beam in phase space remained approximately the same as space charge was increased; the main effect of space charge was to increase the average beam area within the lattice, as well as to introduce an exchange of the apparent emittances.

The matching routine was then applied to the SNS ring. We found that the variation of the cross-plane beam parameters can generate large cross-plane mismatch oscillations, making the elliptical painting method impossible. It is therefore critical to account for the electric field of the matched beam when determining the painting path. We found that the matched beam at the SNS injection point is round ($\varepsilon_x \approx \varepsilon_y$) with a $\pi/2$ difference between the horizontal and vertical particle phases, constraining the painting path to a line in the x - y' plane. These findings complemented the work in [10], where the x - y' painting path was recommended for independent reasons.

Chapter 3

Simulations

One purpose of this chapter is to briefly describe the computational model used in the PyORBIT code before displaying any simulation results. The second purpose of this chapter is to review the work of Holmes et al., who performed particle-in-cell (PIC) simulations in the original ORBIT code to determine the feasibility of elliptical painting in the SNS [10].

3.1 Computational model

3.1.1 Single-particle tracking

PyORBIT tracks a Bunch object containing a set of 6D phase space coordinates. The accelerator is modeled as a series of nodes, each of which modifies the coordinates in some way. The core of PyORBIT is comprised of single-particle tracking routines. For dipoles, quadrupoles, and solenoids, full linear transport is used. The approach to handling nonlinear elements is inspired by TEAPOT (Thin-Element Accelerator Program for Optics and Tracking), where particles pass through a series of drifts and "kicks" from thin elements — elements with infinitesimal length [69]. Each drift changes the particle's position without changing its momentum, while each kick changes the particle's momentum without changing its position. Thick elements are then approximated as a series of thin elements connected with drifts. The momentum kicks are derived from the Hamiltonian of a charged particle in an electromagnetic field. It can be shown that for a Hamiltonian H that does not explicitly depend on s (the time-like variable), then over a distance L , a function g of the canonical coordinates will transform as

$$g \rightarrow e^{-L:H:} g = \sum_{n=0}^{\infty} \frac{L^n}{n!} (-:H:)^n g, \quad (3.1)$$

where the operator $:H:$ is defined by

$$\{H, g\} \equiv :H:g = \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{q}} \quad (3.2)$$

[70, 71]. Symplectic maps connecting the initial and final phase space coordinates are then derived for various elements.

Fringe fields need to be taken into account for elements of finite length, that is, the fields outside the start and end of the element. For example, the magnetic field in a solenoid has a transverse component near the edges that vanishes in the limit of infinite length. The strength of fringe fields generally increases with the transverse distance from the center of the magnet; they are especially important in the injection region, where the closed orbit is significantly displaced from the quadrupole centers. PyORBIT handles fringe fields using a hard-edge model in which a nonlinear mapping is applied at the start and end of the magnet [71].

Longitudinal dynamics have not yet been discussed. Longitudinal focusing in the SNS ring is provided by four RF cavities. The harmonic number h , which is defined as the RF frequency divided by the revolution frequency of the beam, takes the value $h = 1$ in three cavities and $h = 2$ in one cavity, resulting in two peaks in the distribution. The cavities were designed to operate at 40 kV total for the first harmonic and 20 kV for the second harmonic, but after years of operational experience, they were lowered to their current values of approximately 5 kV. The energy gain for a particle passing through the cavity is approximated as $qV \sin(h\phi + \phi_0)$ where the particle phase ϕ is zero for the synchronous particle.¹ Since particles see different accelerating voltages depending on their arrival time, they oscillate in a stable region of longitudinal phase space. The longitudinal tune is four orders of magnitude smaller than the transverse tune in the SNS ring.

3.1.2 Collective effects

The calculation of collective effects is a major component of high-intensity beam physics simulations. We focus first on space charge. PyORBIT uses the particle-in-cell (PIC) approach in which an N particle bunch is represented by M macroparticles, where $M \ll N$. The macroparticles are tracked according to the single-particle equation of motion (Eq. (1.21)). The electric field is obtained by solving the Poisson equation (Eq. (1.24)) on a grid. The key step is transforming between the discrete and continuous representations of the bunch.

The PIC loop is shown in Fig. 3.1. First, the charge density $\rho_{i,j}$ is obtained on a grid. A common method is to treat each macroparticle as a rectangular, uniform density cloud of charge with dimensions equal to the grid spacing, assigning a fractional charge to each bin according to the fraction of the cloud overlapping with that bin [72]. Second, the Poisson equation is solved on the grid. The method used in PyORBIT follows [73], where the potential is written as the convolution ($*$) of a Green’s function² $G(\mathbf{x})$ with the charge density $\rho(\mathbf{x})$:

$$\Phi(\mathbf{x}) = G(\mathbf{x}) * \rho(\mathbf{x}). \quad (3.3)$$

We then exploit the convolution theorem [75] to write the Fourier transform ($\mathcal{F}$) of the potential as the direct product of the Fourier transforms of the Green’s function and charge density:

$$\mathcal{F}[\Phi(\mathbf{x})] = \mathcal{F}[G(\mathbf{x})] \cdot \mathcal{F}[\rho(\mathbf{x})]. \quad (3.4)$$

For a grid with N total points, the time-complexity of the convolution is $O(N^2)$. The Fourier transform reduces this to $O(N \log N)$. To create periodic boundary conditions, the grid is doubled in each dimension; in the new regions, the Green’s function is mirror-

¹Since there is no net acceleration in the SNS ring, $\phi_0 = 0$.

²Here we take a stand on the “Green’s function” vs. “Green function” debate [74].

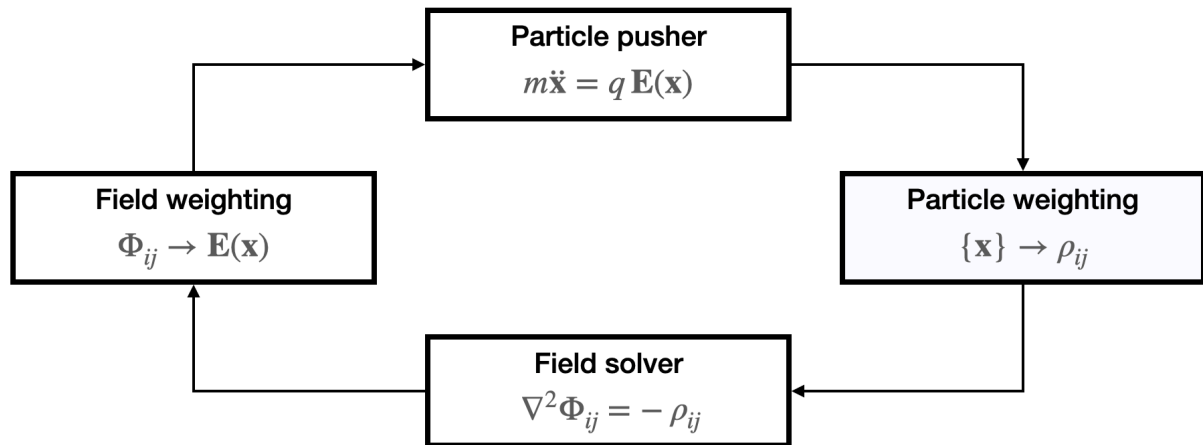


Figure 3.1: The particle-in-cell loop.

reflected and the charge density is set to zero. The potential is then solved for on the doubled grid, after which only the original region is kept. An example is shown in Fig. 3.2. Third, the gradient of the potential is calculated, then interpolated at the particle positions using the same weighting method as in the first step. Finally, the particle momenta are updated using an appropriate integration scheme.

In rings where the coasting beam approximation is valid, the longitudinal and transverse dimensions can be treated separately. In PyORBIT, a longitudinal space charge node, which solves the Poisson equation for the 1D projection of the distribution onto the z axis, acts on the bunch once per turn. Transverse space charge nodes are distributed throughout the ring. The model used in this work is known as the sliced model: the bunch is sliced into longitudinal segments, and the Poisson equation is individually solved for each segment's projection onto the x - y plane.

In the discussion of space charge thus far, the beam was assumed to be in free space; in reality, the beam is in a vacuum chamber such as a conducting pipe. This has two effects. First, the boundary conditions must be applied when solving the Poisson equation. Second, each charged particle interacts with the vacuum chamber, leaving a wake field that affects trailing particles or the same particle on subsequent turns, possibly leading to instability. The treatment of wake fields is introduced in [76]. No details are described here; we only mention that PyORBIT takes these effects into account and has been benchmarked against experiments [77].

3.1.3 Simulation procedure

PyORBIT employs a two-language scheme: the core algorithms are written in C++ but are accessed at the Python level. All PyORBIT scripts are written in Python. The scripts used in this work are modified versions of previously benchmarked injection scripts.

For space charge calculations, a $128 \times 128 \times 64$ grid is used with the sliced model. No more than 1 meter is allowed between solver nodes. The final number of particles is 5×10^5 .³ With these settings, a 1000-turn injection simulation takes several days to run on a 2014 MacBook Pro with a 2.5 GHz Intel Core i7 processor, 16 GB of memory, and the macOS Mojave operating system.

In all simulations, the turn-by-turn covariance matrix of the distribution at the injection point is saved to a file. If space permits, the turn-by-turn 6D phase space coordinates at the injection point are saved to a single file as an array of size $nT(T+1)/2 \times 6$, where n is the number of macroparticles injected per turn and T is the total number of turns. (The array is reshaped during analysis to obtain a T -element list of coordinate arrays.) Additionally, the distribution is periodically saved at the ring extraction point so that it can be transported to the target in a separate simulation.

³These settings are not hard-wired into PyORBIT.

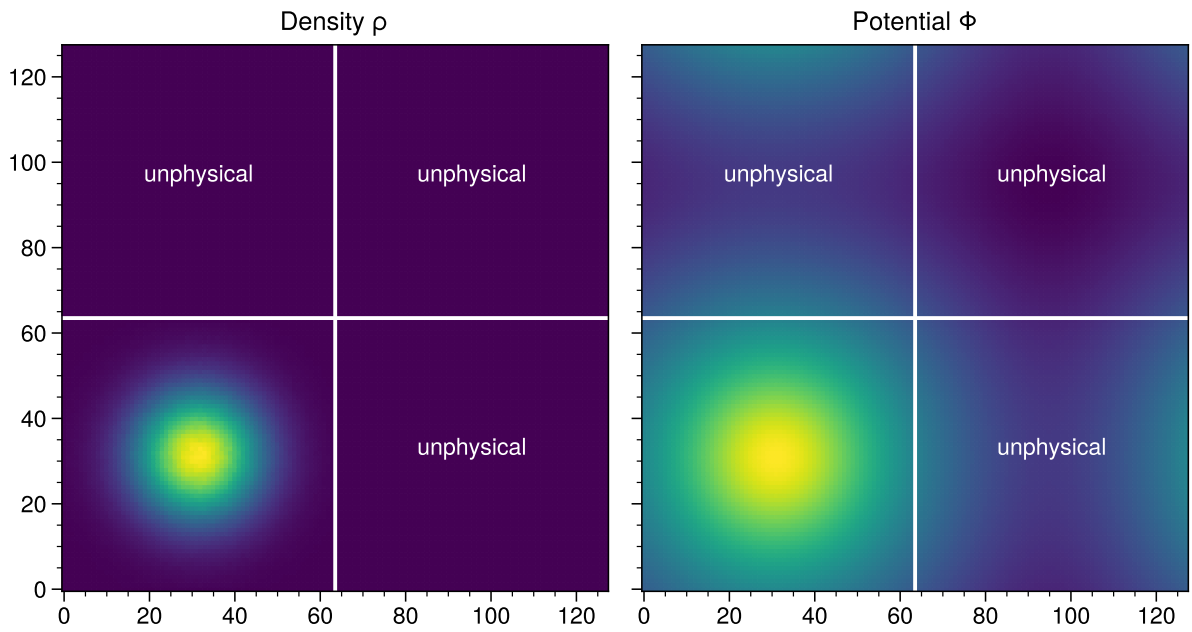


Figure 3.2: Solution of the Poisson equation on a doubled grid.

3.2 Painting simulations

3.2.1 Injection

Injection is simulated by adding particles to the bunch at the foil location on each turn. Since each pulse is the sum of many minipulses from the linac, it is only necessary to use an approximate representation of the minipulse. The distribution used in this work resembles a truncated Gaussian in the transverse plane. The design RMS emittance is approximately 0.3 mm mrad, but recent measurements indicate that the emittance may be closer to 0.5 mm mrad. Additionally, although the Twiss parameters are usually assumed to be matched to the ring ($\beta_x \approx \beta_y \approx 10$ m, $\alpha_x \approx \alpha_y \approx 0$), the measurements indicated that the β functions may be closer to 4 m/rad and the alpha functions may be closer to -0.5. We use values close to these measurements. In the longitudinal plane, the spatial distribution is uniform and the energy distribution is a Gaussian with a mean of 1 GeV and a standard deviation of less than 1 MeV.

An important effect during charge-exchange injection is the scattering and energy loss that occurs during passage through the stripper foil, which increases the effective emittance of the injected bunch. The relevant processes are multiple Coloumb scattering, ionization energy loss, and single scattering interactions such as Rutherford elastic scattering, nuclear elastic scattering, and nuclear inelastic absorption [29, 78]. These are implemented as Monte Carlo routines in a foil node that acts on the bunch once per turn.

Phase space painting can be simulated by adding an artificial time-dependent bump to the coordinates on each turn; however, it is more realistic to vary the strengths of the eight injection kickers in the model with the inclusion of element offsets from the closed orbit. The following procedure is used to set the coordinates at the injection point: Let (x_o, x'_o, y_o, y'_o) be the desired coordinates. First, a particle is launched from the injection point with $x = x_o$, $x' = x'_o$, and $y = y' = 0$. The particle is tracked to the end of the injection region. An optimizer varies two kicker strengths to flatten the downstream orbit, i.e., obtain $x = x' = 0$ after tracking. Then, the particle is launched with $x = x_o$, $x' = -x'_o$ and tracked backwards to the start of the injection region. Two kickers are varied to flatten the upstream orbit, i.e., obtain $x = x' = 0$ at the start of the injection region. The process is repeated with the vertical orbit.

3.2.2 Best-case scenario

Holmes et al. performed realistic simulations of elliptical painting in the SNS using the original ORBIT code [10]. Their final simulation produced an approximate Danilov distribution. Only the final 1D projections of the distribution were displayed in the publication, and the intrinsic emittances were not calculated, so it is worth revisiting these results.

The beam emittances and the projections of the 4D phase space distribution during accumulation are shown in Fig. 3.3. The faint pink ellipses represent the projections of an ideal Danilov distribution with the same Twiss parameters and apparent emittances as the final simulated distribution. The x - y projection has an elliptical shape and an approximately uniform density with small halo. The most striking features are the x - y' and y - x' projections, which, although broadened by nonlinear effects, exhibit strong correlations. As a consequence, the smaller intrinsic emittance ε_2 remains close to its

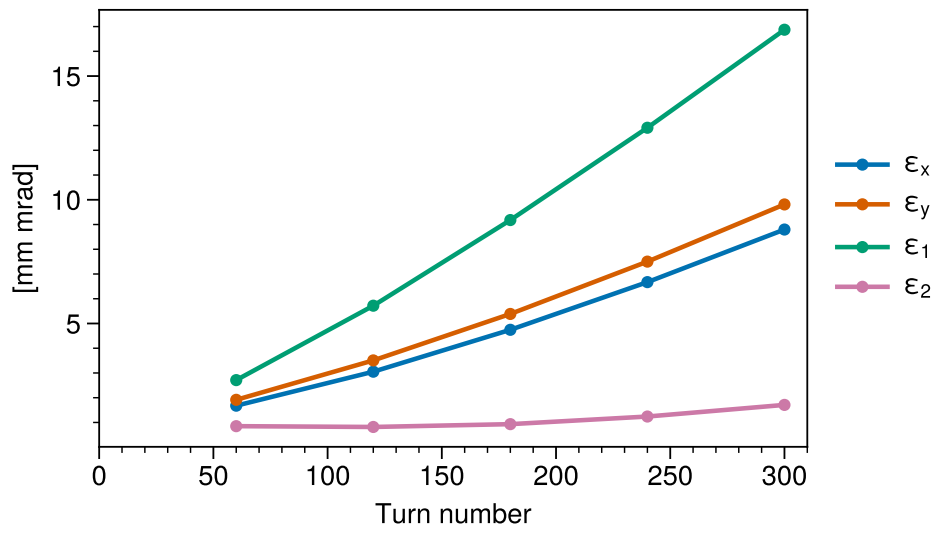
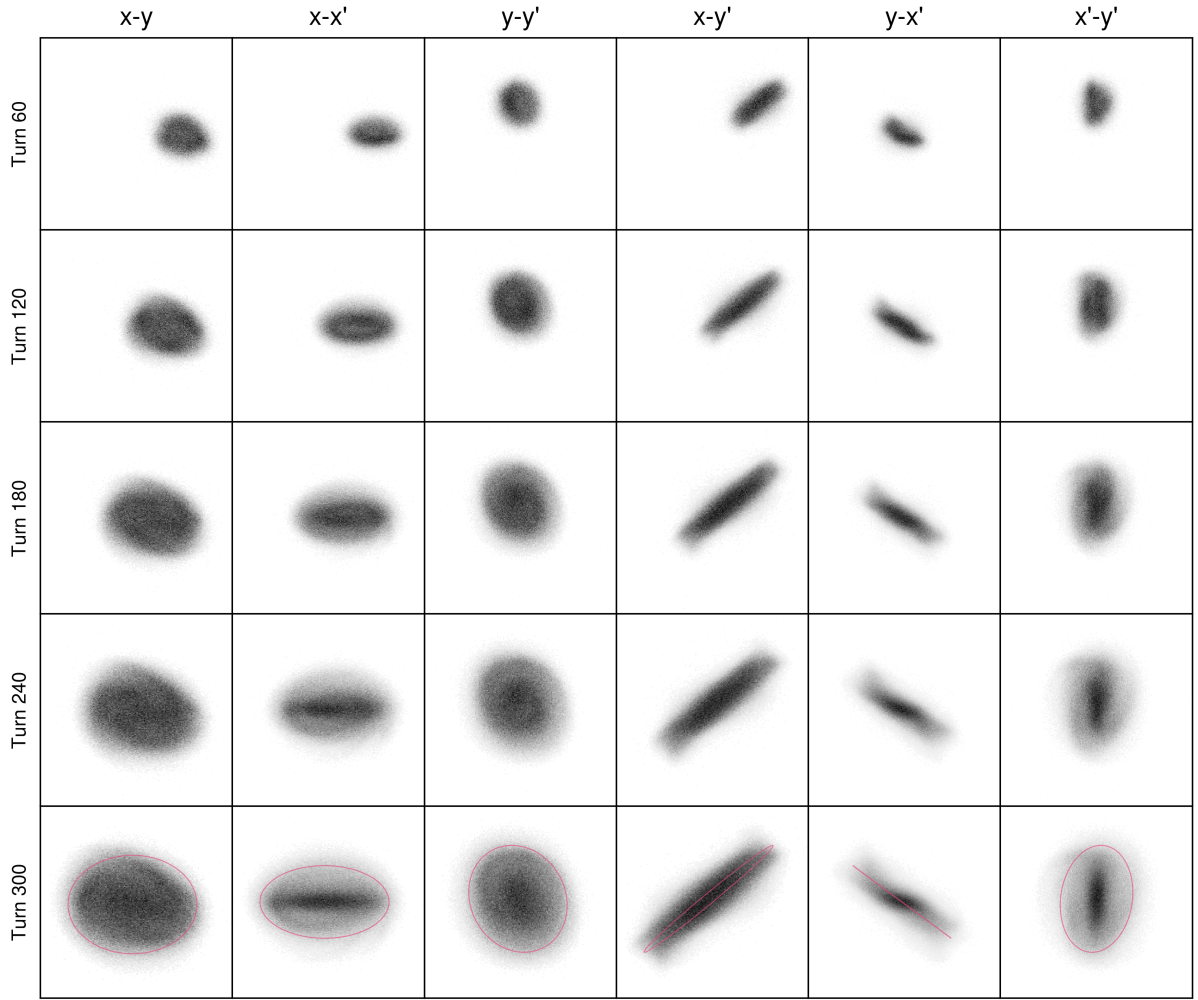


Figure 3.3: Simulation of elliptical painting in the SNS from [10].

lower limit throughout injection; particles have been injected mostly into one mode, as intended.

Several steps were necessary to achieve this simulated result. First, the painting path was chosen to follow a line in the x - y' plane and the apparent emittances were kept somewhat equal; both these steps were discussed in Chapter 2. Third, the ring RF cavity voltages were decreased to better approximate a coasting beam. Fourth, the beam energy was lowered to 0.6 GeV to increase the effective kicker strength; additionally, orbit corrector dipoles in the injection region provided a closed vertical bump. These modifications allowed a vertical injection angle of 1.58 mrad at the injection point. Fifth, the number of injected turns was reduced from 1000 to 300 to compensate for the increased space charge strength at the lower energy. Finally, a solenoid magnet was added to the ring to mitigate the effect of fringe fields. (The stabilizing effect of solenoids on the motion is examined in Appendix C.)

It is encouraging that even with the inclusion of space charge, wake fields, nonlinear external fields, beam bunching, foil scattering, chromaticity, limited kicker strengths, etc., simulations predict that an approximate Danilov distribution can be produced. This is likely the best-case scenario in the SNS. But as will be discussed in Chapter 5, new experimental constraints have come to light. The SNS cannot easily reach 0.6 GeV, and the use of orbit correctors in the injection region is difficult. The net result is that the maximum vertical injection angle may be limited (at least in initial experiments). Additionally, the minipulse may be mismatched at the injection point, but we expect this to have only a small effect if a sizeable beam is painted. Finally, solenoid magnets will not be installed in the SNS ring until a later date, so initial experiments will create elliptical modes in the ring by setting equal horizontal and vertical tunes. The distribution is expected to be less stable with such a machine configuration.⁴ Simulations with these constraints will be shown after several of the experiments in Chapter 5.

⁴Holmes found that the beam should be somewhat insensitive to the tune split in the ring, but as mentioned in Chapter 3, his studies were performed with a solenoid to the ring — the tune split was calculated without the solenoid.

Chapter 4

Diagnostics

Determining the similarity between a painted distribution in the SNS and a Danilov distribution requires measurement of the 4D transverse phase space distribution. A direct measurement using a slit-scan [79] is not possible at high energy, so the distribution must be reconstructed from lower-dimensional projections. In this chapter, we first describe the available hardware to measure such projections in the SNS. We then describe several methods to perform the reconstruction using 1D and/or 2D projections, as well as the implementation of these methods in the SNS.

4.1 Available hardware and constraints

The phase space measurement must be performed in the ring-target beam transfer (RTBT) section of the SNS after the beam has been accumulated in the ring. The RTBT is effectively an extension of the ring that is traversed once. It is straightforward to vary the number of accumulated turns to measure the beam at any time during injection.

The RTBT optics are shown in Fig. 4.1 along with the locations of four wire-scanners — WS20, WS21, WS23, and WS24 — near the target. Each wire-scanner consists of three thin tungsten wires mounted on a fork — one wire is vertical, another is horizontal, and another is tilted at a forty-five-degree angle. The 1D projections of the distribution onto axes perpendicular to the wires are generated by moving the fork across the beam and measuring the charge induced by secondary emission from the wires [6]. The four wire-scanners can be run in parallel and take approximately five minutes to move across the beam and return to their original positions. Their step size is 3 mm and their dynamic range is approximately 100. They are run at a beam pulse frequency of 1 Hz.¹ The measured profiles can be used to estimate $\langle xx \rangle$, $\langle yy \rangle$, and $\langle uu \rangle$, where the u axis is tilted at angle $\phi = \pi/4$ above the x axis, as well as $\langle xy \rangle$ from

$$\langle xy \rangle = \frac{\langle uu \rangle - \langle xx \rangle \cos^2 \phi - \langle yy \rangle \sin^2 \phi}{2 \sin \phi \cos \phi}. \quad (4.1)$$

The SNS employs a target imaging system (TIS) to measure the 2D projection of the

¹Each data point corresponds to a separate beam pulse, so the measurement relies on pulse-to-pulse stability.

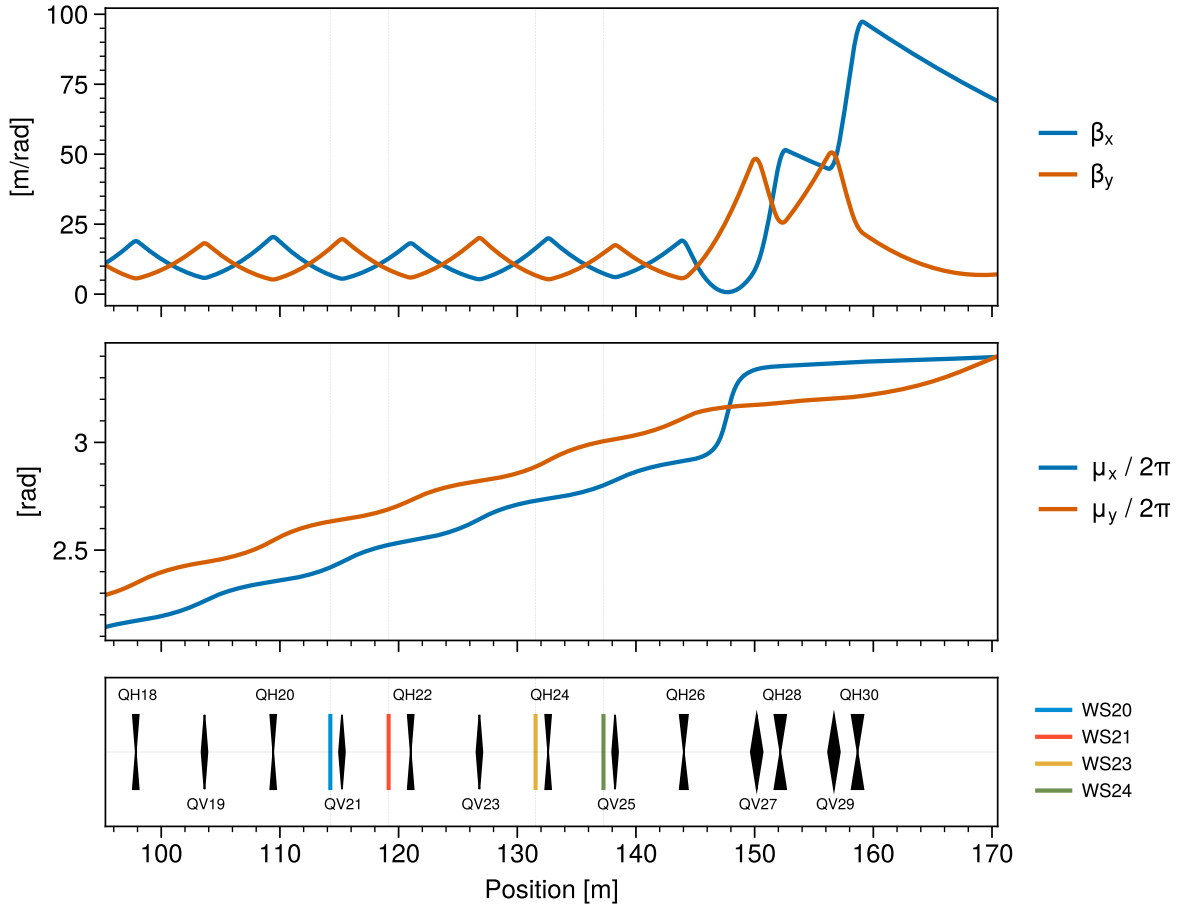


Figure 4.1: β functions, phase advances, and quadrupole/wire-scanner locations in the second half of the RTBT. The plot ends at the spallation target.

distribution on the target [80]. The SNS target is a stainless steel vessel containing liquid mercury. Its nose is prepared with a Cr:Al₂O₃ coating that releases light when impacted by the proton beam. Due to the high-radiation environment, the light is collected by a mirror, deflected, and focused onto an optical fiber bundle which guides the light to a camera some distance away. The TIS configuration is shown in Fig. 4.2.

The optics in the RTBT can be modified, but there are constraints. The β functions should be kept below ≈ 30 m/rad in the wire-scanner region and below ≈ 100 m/rad closer to the target to avoid excess beam loss. At the target, it is best to keep the β functions near their default values of $\beta_x \approx 60$ m/rad and $\beta_y \approx 6$ m/rad to satisfy peak density and beam size requirements on the target. In addition to these constraints, quadrupoles in the wire-scanner region share power supplies: there is a horizontal group {QH18, QH20, QH22, QH24} and a vertical group {QV19, QV21, QV23, QV25}. The last five magnets — QH26, QV27, QH28, QV29, and QH30 — are individually controlled.

4.2 4D phase space reconstruction from 1D projections

4.2.1 Method description

The covariance matrix Σ can be reconstructed from 1D projections [54, 81, 82]. We seek to reconstruct Σ at position a by measuring $\langle xx \rangle$, $\langle yy \rangle$ and $\langle xy \rangle$ at position b , downstream of a . Assuming linear transport, the two covariance matrices are related by

$$\Sigma_b = \mathbf{M}\Sigma_a\mathbf{M}^T, \quad (4.2)$$

where $\mathbf{M}$ is the linear transfer matrix from a to b . We repeat the measurement at least four times with different transfer matrices — either by changing the measurement location or by changing the machine optics — and write

$$\begin{bmatrix} \langle xx \rangle^{(1)} \\ \langle xy \rangle^{(1)} \\ \langle yy \rangle^{(1)} \\ \langle xx \rangle^{(2)} \\ \langle xy \rangle^{(2)} \\ \langle yy \rangle^{(2)} \\ \langle xx \rangle^{(3)} \\ \langle xy \rangle^{(3)} \\ \langle yy \rangle^{(3)} \\ \vdots \end{bmatrix}_b = \mathbf{A} \begin{bmatrix} \langle xx \rangle \\ \langle xx' \rangle \\ \langle xy \rangle \\ \langle xy' \rangle \\ \langle x'x' \rangle \\ \langle x'y \rangle \\ \langle x'y' \rangle \\ \langle yy \rangle \\ \langle yy' \rangle \\ \langle y'y' \rangle \end{bmatrix}_a. \quad (4.3)$$

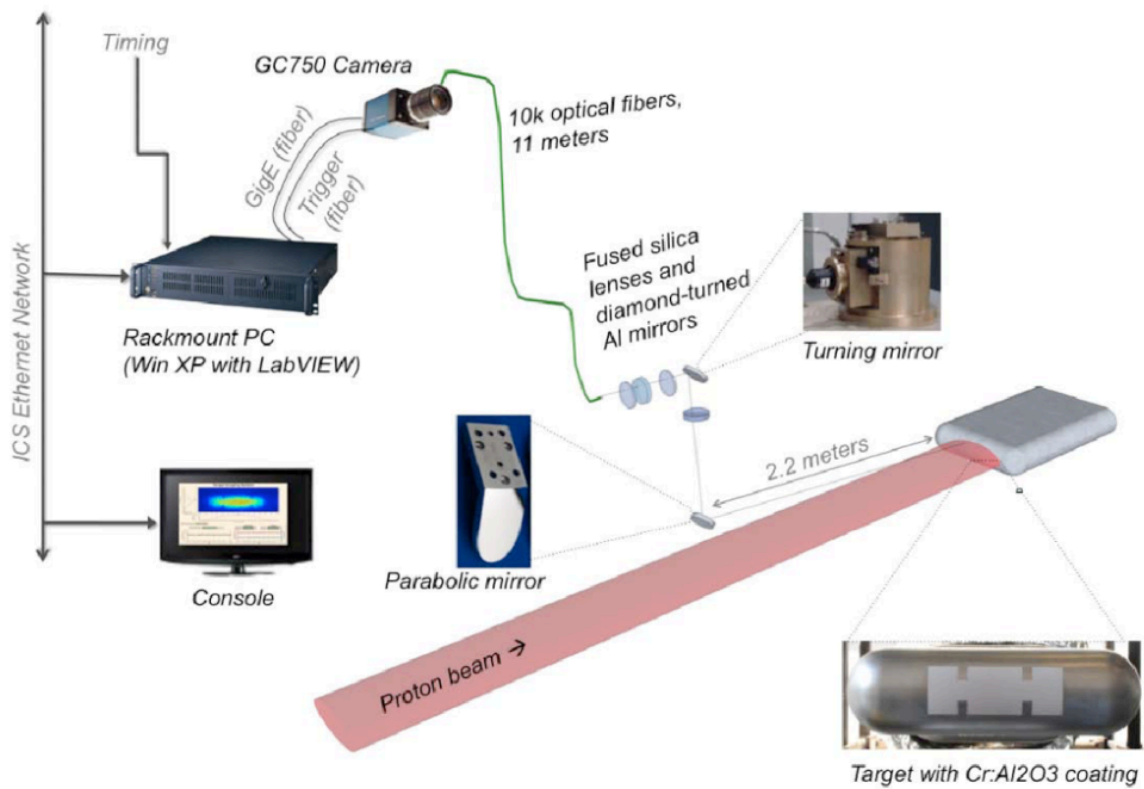


Figure 4.2: Configuration of the SNS target imaging system. (From [80].)

The superscripts represent the measurement index. The transpose of the coefficient matrix $\mathbf{A}$ for a single measurement is

$$\mathbf{A}^T = \begin{bmatrix} M_{11}M_{11} & M_{11}M_{31} & M_{31}M_{31} \\ 2M_{11}M_{12} & M_{12}M_{31} + M_{11}M_{32} & 2M_{31}M_{32} \\ 2M_{11}M_{13} & M_{13}M_{31} + M_{11}M_{33} & 2M_{31}M_{33} \\ 2M_{11}M_{14} & M_{14}M_{31} + M_{11}M_{34} & 2M_{31}M_{34} \\ M_{12}M_{12} & M_{12}M_{32} & M_{32}M_{32} \\ 2M_{12}M_{13} & M_{13}M_{32} + M_{12}M_{33} & 2M_{32}M_{33} \\ 2M_{12}M_{14} & M_{14}M_{32} + M_{12}M_{34} & 2M_{32}M_{34} \\ M_{13}M_{13} & M_{13}M_{33} & M_{33}M_{33} \\ 2M_{13}M_{14} & M_{14}M_{33} + M_{13}M_{34} & 2M_{33}M_{34} \\ M_{14}M_{14} & M_{14}M_{34} & M_{34}M_{34} \end{bmatrix} \quad (4.4)$$

where M_{ij} is the i,j element of the transfer matrix for that measurement. The system is solved using linear least squares (LLSQ).

The measurement has a geometric interpretation, illustrated in Fig. 4.3 for the 2D case. Each measured beam size defines two lines in the $x-x'$ plane at b ; when the lines are transported back to a , their intersection bounds the phase space ellipse. In the general case, each measurement at b defines a 2D surface in 4D phase space and the intersection of these surfaces at a bounds the phase space ellipsoid.

4.2.2 Implementation in the SNS

To perform the wire scans, the beam is set to a pulse frequency of 1 Hz, and the beam loss monitors in the wire-scanner region are masked due to the higher-than-normal losses when the wires cross the beam core. Wire-scanner data acquisition is performed by the Profile Tools and Analysis (PTA) application. After the four wire-scanners complete their scan, a time-stamped file is produced containing the measured profiles.

The four wire-scanners produce four equations, exactly determining the cross-plane moments, so no optics changes are needed in principle; however, additional measurements should reduce the error. In the 2D case, it is typically recommended to space n measurements by π/n in phase advance [54]. This may be due to the geometric interpretation of Fig. 4.3: in normalized phase space, the rotation angle of the measurement lines is equivalent to the phase advance and the lines are evenly spaced around the phase space ellipse. The four wire-scanners in the RTBT are already somewhat evenly spaced in phase advance, and it was determined that a 30° window around each wire-scanner would provide sufficient coverage.

Due to the shared power supplies of the quadrupoles in the wire-scanner region, there is limited control of the phase advances between the wire-scanners. We instead vary the phase advances from QH18 (the first varied quadrupole) to WS24 (the last wire-scanner), which changes the phase advances at WS20, WS21, and WS23 by similar amounts. To set the phase advances at WS24 while constraining the beam size in the wire-scanner region, two power supplies (eight quadrupoles) upstream of WS24 were varied using an optimizer that minimizes the following cost function:

$$C(\mathbf{g}) = \|\tilde{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2 + \epsilon \left\| \Theta \left(\tilde{\boldsymbol{\beta}}_{max} - \boldsymbol{\beta}_{max} \right) \right\|^2. \quad (4.5)$$

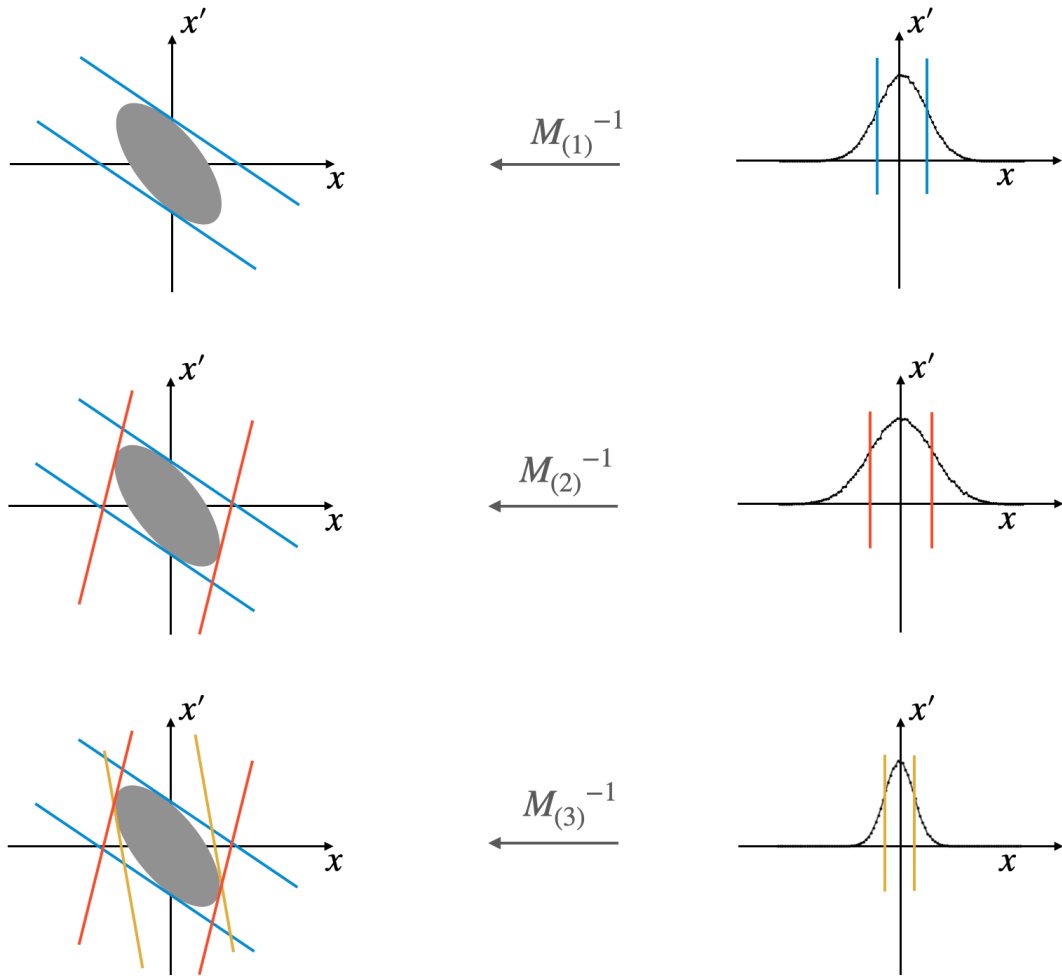


Figure 4.3: Illustration of 2D emittance measurement.

The quadrupole field strengths are contained in the vector $\mathbf{g}$. The calculated and desired phase advances at WS24 are $\boldsymbol{\mu} = (\mu_x, \mu_y)$, and $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_x, \tilde{\mu}_y)$, respectively. The maximum calculated and allowed β functions in the wire-scanner region are $\boldsymbol{\beta}_{max} = (\beta_{x_{max}}, \beta_{y_{max}})$ and $\tilde{\boldsymbol{\beta}}_{max} = (\tilde{\beta}_{x_{max}}, \tilde{\beta}_{y_{max}})$, respectively. Θ is the Heaviside step function. Finally, ϵ is a constant.²

After the model optics are computed, the live quadrupole settings must be changed. The SNS employs a machine protection system (MPS) that will cause the machine to trip if the RTBT quadrupole strengths wander outside a certain window, so this window is extended beforehand. Additionally, the MPS will activate if the fractional change in field strength is too large; to solve this problem, the field strength is changed in small steps.

A GUI application to perform the above tasks was developed in the OpenXAL framework for use in the SNS control room. In the first pane of the application, the user can set the phase advances at WS24 and view the model optics and phase advances throughout the RTBT. In the second pane, the user can load wire-scanner output files and choose the reconstruction location. These files contain the wire-scanner profiles, RMS parameters, and Gaussian fit parameters. They also contain an integer that defines the machine state at the time of the measurement. The application reads this number, synchronizes the model with the machine state, and computes the transfer matrices from the wire-scanners to the reconstruction location. The RMS moments and transfer matrices are then used to reconstruct the covariance matrix. The resulting beam parameters are printed and compared to the model lattice parameters. The 2D projections of the covariance ellipsoid are plotted along with the measurement lines, with the option to view in normalized coordinates.

4.2.3 Measurement of a production beam

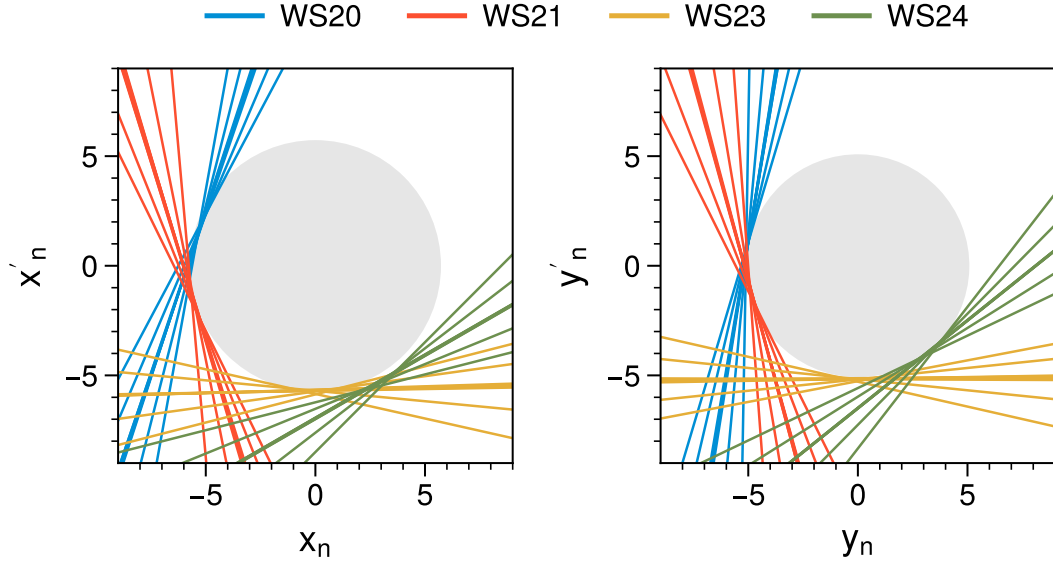
The multi-optics method was tested on a fully accumulated production beam. The phase advances at WS24 were varied in a 30° range over ten steps: the first half of the scan held the vertical phase advance fixed while varying the horizontal phase advance, and the second half of the scan held the horizontal phase advance fixed while varying the vertical phase advance. The result of the reconstruction is shown in Fig. 4.4 for a location just before QH18. The best-fit ellipses in the x - x' and y - y' planes are normalized by the reconstructed Twiss parameters. The uncertainties in the beam parameters were calculated by propagating the standard deviations of the ten reconstructed moments obtained from the LLSQ estimator.³ The reconstructed Twiss parameters are close to the model parameters computed from the linear transfer matrices of the ring and RTBT, showing that the beam is matched. The intrinsic emittances are almost equal to the apparent emittances, showing that there is very little cross-plane correlation in the beam. This is expected for a production beam.

4.2.4 Sensitivity to errors

A comprehensive study of errors in the multi-optics 4D emittance measurement was completed at the SwissFEL Injector Test Facility (SITF) by Prat and Aiba in [82]. They

²We are assuming that the beam is approximately matched to the lattice optics so that the calculated phase advances are close to the true phase advances.

³See Appendix A of [83].



Parameter	Measurement	Model
β_x [m/rad]	22.06 ± 0.29	22.00
β_y [m/rad]	4.01 ± 0.02	3.81
α_x	2.33 ± 0.04	2.37
α_y	-0.49 ± 0.01	-0.60
ε_1 [mm mrad]	33.02 ± 0.05	-
ε_2 [mm mrad]	25.67 ± 1.03	-
ε_x [mm mrad]	32.85 ± 0.05	-
ε_y [mm mrad]	25.87 ± 0.12	-

Figure 4.4: Reconstructed beam parameters and graphical output from a multi-optics emittance measurement of a production beam.

considered errors in the measured moments, quadrupole field and alignment errors, beam energy errors, beam mismatch at the reconstruction point, and dispersion/chromaticity [84], concluding that the multi-optics measurement remained accurate, reporting $< 5\%$ uncertainty in the intrinsic emittances. We initially performed similar studies in Py-ORBIT using envelope tracking to estimate the reconstruction errors in the RTBT, also concluding that the method should remain accurate [85]. Space charge forces, which can render the method invalid for high-perveance beams [86], can be neglected: the space charge tune shift in the ring is around 3%, and the distance between the reconstruction and measurement locations is much smaller than the length of the ring.

In summary, the multi-optics emittance measurement is feasible in the SNS. The only downside is the long measurement time, for the following reasons. First, we are not only interested in the beam emittances at a single time but are also interested in the growth and evolution of the emittances throughout accumulation. For example, it could be possible for ε_2 to remain small (as desired) in the first half of accumulation before growing to a much larger value in the second half of accumulation. Measurement of this emittance evolution would convey valuable information about the beam dynamics and allow for qualitative comparison with computer simulation. Second, it would be beneficial to quickly evaluate various machine states, the best of which will be unknown during initial experiments. Finally, a practical point: the time reserved for accelerator physics (AP) experiments at the SNS is limited, and the setup for initial experiments will be much longer than typical AP experiments for reasons discussed in Chapter 5. Thus, the fixed-optics method — in which only four profiles are used in the reconstruction — is preferred.⁴ A modest reduction in accuracy for the increase in speed is warranted since weak cross-plane correlations ($\varepsilon_1\varepsilon_2 \approx \varepsilon_x\varepsilon_y$) are uninteresting for our purposes and do not need to be resolved.

Using only one set of optics from the previous scan resulted in a covariance matrix that was not positive-definite, producing imaginary intrinsic emittances. We label this a failed fit. A nonlinear solver [87] or Cholesky decomposition [88] can be used to ensure a valid covariance matrix, but we found that the answer depended strongly on the initial guess provided to the solver and on which measurement in the scan was used in the reconstruction.

To investigate the failure of the fixed-optics method, a covariance matrix was generated with cross-plane moments set to zero and within-plane moments matched to the lattice optics, then tracked to the wire-scanners using the known transfer matrices. The reconstruction was then performed 1000 times with 3% random noise added to the moments.⁵ Failed fits were discarded. Fig. 4.5 shows that the reconstructed intrinsic emittances in the successful trials are very sensitive to changes to the “measured” moments. Unlike the apparent emittances, the intrinsic emittances are strongly correlated and are not centered on the correct values. We refer to the difference between the mean emittances and the

⁴An additional benefit of the fixed-optics method is that there is no potential for steering errors in the RTBT, which could enhance beam loss.

⁵To select the proper noise level, the wire-scanners were run seven times without changing the machine optics, producing seven estimated moments for each of the three wires on each of the four wire-scanners. The profiles are highly reproducible: for each wire, the maximum difference between any two moments was always less than 3% of the mean moment. Therefore, the noisy moments were sampled within $\pm 3\%$ of the true moments.

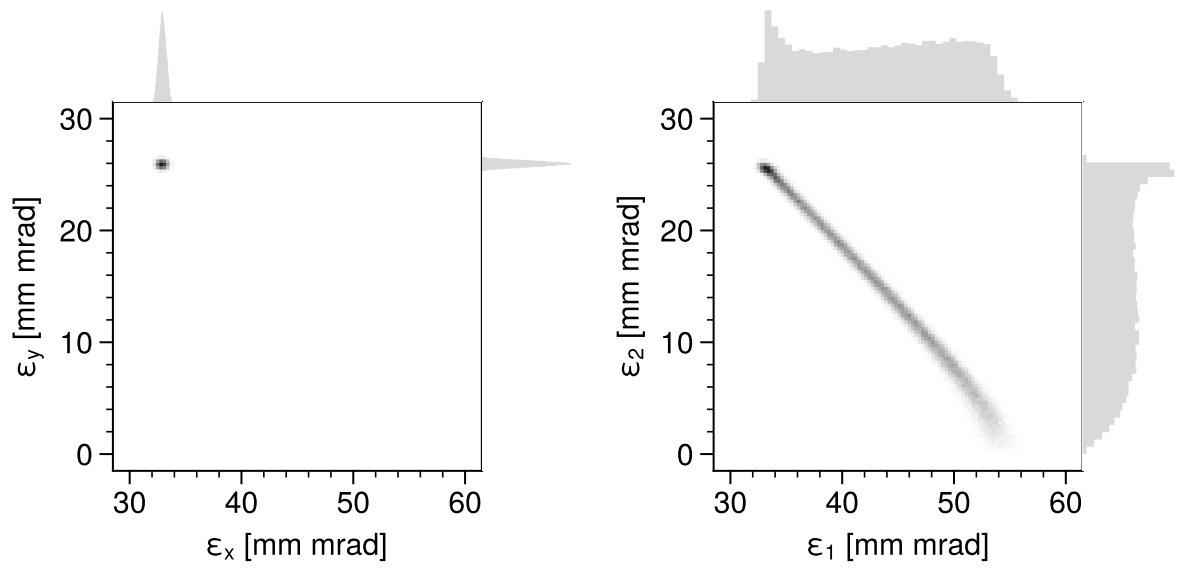


Figure 4.5: Monte Carlo simulation of a fixed-optics emittance measurement in the RTBT. Trials were repeated until several thousand successful fits were obtained. 3% noise was assumed for the measured moments. Transfer matrix errors were ignored. The correct values are $\varepsilon_1 = \varepsilon_x = 32$ mm mrad, $\varepsilon_2 = \varepsilon_y = 20$ mm mrad.

true emittances as the *bias*.⁶

Sensitivity of fixed-optics 4D emittance measurements was observed by Woodley and Emma [81] and studied more recently by Agapov, Blair, and Woodley [88] as well as Faus-Golfe et al. [83], all in the context of design studies for a future International Linear Collider (ILC). The motivation for these studies is that a flat beam ($\varepsilon_y \ll \varepsilon_x$) would be ideal for the ILC, and that ε_y can be minimized by measuring and removing any cross-plane correlation in the beam. Woodley and Emma proposed to abandon the fixed-optics method due to the bias in the reconstructed intrinsic emittances introduced by large errors in the measured moments, suggesting to instead measure the 2D emittance and iteratively minimize ε_y .

Agapov, Blair, and Woodley revisited this problem and showed that the linear system used to reconstruct the cross-plane moments can easily become ill-conditioned. The sensitivity of a linear system $\mathbf{Ax} = \mathbf{b}$ to errors in $\mathbf{b}$ is determined by the condition number $C = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$ (or the pseudo-inverse $\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ if $\mathbf{A}$ is not square) where $\|\dots\|$ is a matrix norm [89]. As an example, consider four wire-scanners that are evenly spaced in phase advance and connected by rotation matrices. Since the transfer matrices are uncoupled, there are three independent subsystems to solve: x - x' , y - y' , and the cross-plane moments. Let the coefficient matrices for these subsystems be $\mathbf{A}_{xx}$, $\mathbf{A}_{yy}$, and $\mathbf{A}_{xy}$, respectively, and the condition numbers be C_{xx} , C_{yy} , and C_{xy} . Recall that the within-plane moments are overdetermined while the cross-plane moments are exactly determined. Fig. 4.6 plots the inverse of these condition numbers as a function of the wire-scanner spacing. C_{xx} and C_{yy} approach ∞ when the spacing is $\pi/2$ in their respective planes, in which case two pairs of measurements provide degenerate information, while C_{xy} depends on the difference between the phase advances. The pattern will be more complicated for different optics and/or additional wire-scanners. The error and uncertainty in the emittance reconstruction, as well as the number of failed fits, mirrors these condition numbers. Using this framework, Faus-Golfe et al. developed analytical formulas to determine whether a given system can accurately measure the intrinsic emittances. They also suggested that the planned ILC emittance measurement station, which contained four wire-scanners, could be modified to reduce the sensitivity.

We performed a similar modification to the RTBT wire-scanner region. To find a new set of optics, the phase advances at WS24 (μ_x, μ_y) were varied in a 90° window centered on their nominal values (μ_{x0}, μ_{y0}); at each setting, the condition numbers were calculated and the reconstruction was simulated with true emittances $\varepsilon_x = \varepsilon_y = \varepsilon_1 = \varepsilon_2 = 20$ mm mrad. Failed trials were discarded. The resulting biases and standard deviations of the reconstructed emittances are plotted in Fig. 4.7. Settings that produced no successful trials appear as white cells. The apparent emittances are not displayed because they remained within 1% of their true values at every optics setting. Modifying the optics so that $\mu_x = \mu_{x0} + 45^\circ$, $\mu_y = \mu_{y0} - 45^\circ$ reduces the bias to $\approx 7\%$ and the standard deviation to $\approx 5\%$. The fraction of failed fits, which is very large along the diagonal in the figure, is reduced to zero.

It is also important to examine the effect of mismatched beam parameters on the accuracy of the reconstruction. Recall that the phase advance is the integral of the inverse of the β function. In a periodic system, there is a unique periodic solution for the β function, but

⁶The bias and strong correlation between the intrinsic emittances stems from the fact that $\varepsilon_1 \varepsilon_2 \leq \varepsilon_x \varepsilon_y$. In the case at hand, the reconstructed ε_x and ε_y are essentially constant in each trial.

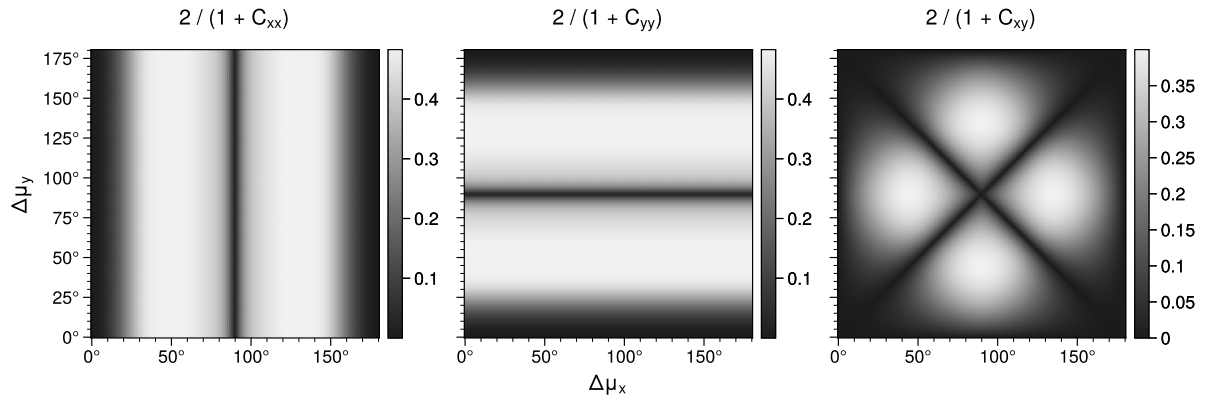


Figure 4.6: Condition numbers of the coefficient matrices produced by four wire-scanners that are evenly spaced in phase advance and connected by rotation matrices.

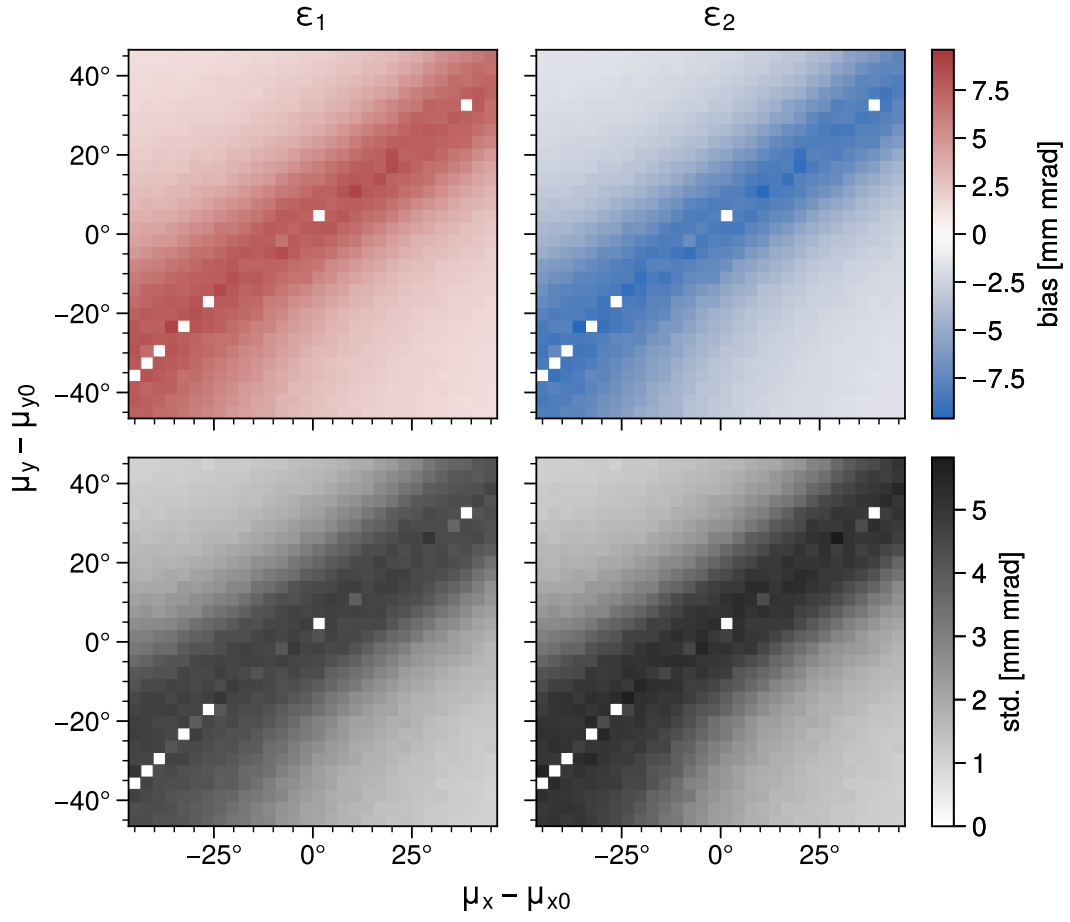


Figure 4.7: Simulated 4D emittance reconstruction errors as a function of the phase advances at WS24.

this is not true in a transfer line such as the RTBT; thus, the phase advance in the RTBT depends on the Twiss parameters at the ring extraction point — the RTBT entrance.

All previous phase advance calculations have assumed that the beam Twiss parameters are the same as the ring Twiss parameters at extraction. This is generally a safe assumption since turn-by-turn mismatch oscillations are washed out during painting. It is possible, however, for space charge to effectively modify the ring Twiss parameters, resulting in mismatch when entering the RTBT. This modification is small during production painting, as shown in Fig. 4.4, but it is expected (from simulations) that more significant mismatch could occur if the space charge density is increased and/or if the beam energy is decreased.

To examine the effect of mismatch, we first moved the operating point to $\mu_x = \mu_{x0} + 45^\circ$, $\mu_y = \mu_{y0} - 45^\circ$, then varied the initial Twiss parameters at BPM17 in the RTBT and repeated the Monte Carlo trials. There are four parameters: α_x , α_y , β_x , and β_y . We based the range of each parameter on a measurement in which the reconstructed Twiss parameters were different than the nominal Twiss parameters, shown in Table 4.1. The beam mismatch is unlikely to exceed these values in future experiments.⁷ Therefore, to examine the effect of mismatch, we first moved the operating point to $\mu_x = \mu_{x0} + 45^\circ$, $\mu_y = \mu_{y0} - 45^\circ$, then varied β_x and β_y within a $\pm 20\%$ window around their model values, α_x within a $\pm 15\%$ window, and α_y within a $-40\%, +10\%$ window to extend beyond the measured discrepancies, and repeated the Monte Carlo trials for each initial beam, thus producing a collection of means and standard deviations for the reconstructed intrinsic emittances. The left plot in Fig. 4.8 displays the standard deviations and biases for ε_1 (pink) and ε_2 (blue). Although most of the points are clustered near the original bias and standard deviation of 7% and 5%, respectively, the bias increases to nearly 15% in some cases, which may make it difficult to resolve weak cross-plane correlation; however, the measurement should still resolve strong cross-plane correlation. This is demonstrated in the rest of the plots in Fig. 4.8, in which the entire process is repeated with $\varepsilon_1/\varepsilon_2 > 1$. The bias in the reconstruction quickly decreases — the emittances are clustered around their true values.

We conclude that with small modifications to the RTBT optics, the fixed-optics method should be sufficient for fast 4D emittance measurements in the SNS. As detailed in Chapter 5, such measurements will be needed to evaluate various machine settings within a single study period, especially in initial experiments, as well as to measure the emittance growth during accumulation for qualitative comparison with simulation. The multi-optics method should be used once a promising machine state is found (or if time allows) to reduce the uncertainty.

Other uses of 1D projections

To close this section, we mention that there is information to be gained from 1D projections in addition to the root-mean-square reconstruction just described. First, the measured projections can be compared to the ideal “half-circle” projections of a uniform

⁷Details about the measurement are left for Chapter 5.

Table 4.1: Reconstructed and model Twiss parameters at BPM 17 in the RTBT (see Experiment 2 in Chapter 5.)

Parameter	Measured	Model
β_x [m/rad]	6.26	5.49
β_y [m/rad]	20.82	19.25
α_x	-0.89	-0.78
α_y	1.17	1.91

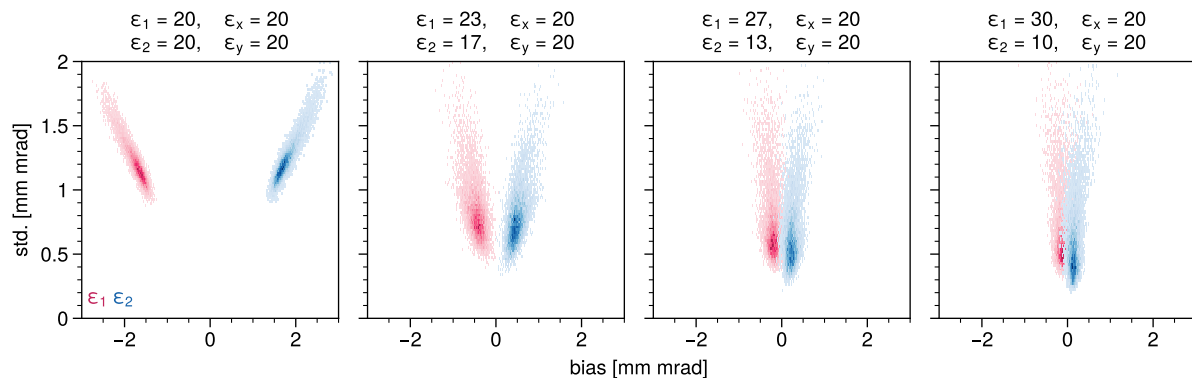


Figure 4.8: Bias and standard deviation of ε_1 (pink) and ε_2 (blue) from simulated reconstructions in the RTBT. In each plot, the collection of points is generated by varying the initial beam Twiss parameters. The true values of the emittances are printed on the top of the figures.

density ellipse.⁸ Second, the projections can be used to reconstruct the x - x' or y - y' distribution using the tomographic methods described in the next section; it may be possible to include cross-plane information in the reconstruction using diagonal projections.

4.3 4D phase space reconstruction from 2D projections

Tomographic methods are well-established for the reconstruction of 2D phase space distributions from 1D projections in transverse phase space [90] and longitudinal phase space [91]. The concept has recently been extended to the reconstruction of the 4D transverse phase space, both in theory and in practice [92–94]. This section begins with a brief discussion of tomography in two dimensions as applied to beam diagnostics, then moves on to describe the accuracy and limitations of several 4D reconstruction algorithms. Finally, the use of tomography to reconstruct the 4D phase space distribution from beam images on the SNS target is discussed.

4.3.1 Tomography for beam diagnostics

Several algorithms exist to reconstruct 2D images from 1D projections, such as filtered back-projection (FBP), algebraic reconstruction (ART) [95], and maximum entropy (MENT) [96]. Projections of an object are normally obtained by illuminating the object at different angles. Although the measured projections of a 2D phase space distribution (x - x') are always along the x axis, we can take advantage of the known transfer matrix $\mathbf{M}$ between the measurement location b and the reconstruction location a to obtain the projections at different angles in x - x' at the reconstruction location. The measured projection of the x - x' distribution at b is

$$p_b(x_b) = \int_{-\infty}^{\infty} f(x_b, x'_b) dx'_b. \quad (4.6)$$

When the distribution is transported back to a , the projection will be along axis $\tilde{x}_a$, which is rotated at angle θ above the x_a axis. The projection angle is computed from the transfer matrix [97]:

$$\tan \theta = \frac{M_{12}}{M_{11}}. \quad (4.7)$$

The distance along the projection axis will be scaled:

$$r = \frac{x_b}{\tilde{x}_a} = \sqrt{M_{11}^2 + M_{12}^2}. \quad (4.8)$$

The projection must then be scaled to conserve its area. The projections at a and b are related by

$$p_a(\tilde{x}_a) = r p_b(x_b). \quad (4.9)$$

⁸The best expected case is a uniform density core with small nonlinear tails, the 1D projection of which is distinguishable from a Gaussian curve, but it may be difficult to distinguish intermediate cases with larger tails. The method we employ in the next chapter is to calculate the standard deviation of the measured profile, plot the projections of an ideal Gaussian and uniform density elliptical distribution with the same standard deviation, and visually compare the three curves. More quantitative methods may be used in the future.

Standard tomography algorithms can be applied to the scaled projections.

Reconstructing the distribution in normalized phase space can reduce errors [98]. Recall the normalization matrix $\mathbf{V}$ from Eq. (1.8). Note that

$$\mathbf{x}_b = \mathbf{M}\mathbf{x}_a = \mathbf{M}\mathbf{V}(\mathbf{V}^{-1}\mathbf{x}_a), \quad (4.10)$$

where $\mathbf{V}$ depends on the Twiss parameters at a . Eq. (4.7), Eq. (4.8), and Eq. (4.9) can be applied to the matrix $\mathbf{M}\mathbf{V}$ to obtain the projections in the normalized phase space at a . After the image is reconstructed, the true distribution can be obtained by transforming the grid coordinates using $\mathbf{V}$ and interpolating at the transformed coordinates. Any Twiss parameters can be used to form $\mathbf{V}$; if the Twiss parameters are matched to the distribution, the rotation angle of the projection will be the phase advance from a to b , and the reconstructed distribution will be circular in the normalized phase space.

4.3.2 4D reconstruction as a series of 2D reconstructions

Recent work by Hock et al. reduces 4D reconstruction to a series of 2D reconstructions when the x - y projections are available [97]. The method, which we refer to as Hock's method, is as follows. Assume that the rotation angles in x - x' and y - y' can be independently controlled. Let the angles in x - x' be $\{\theta_{x_1}, \dots, \theta_{x_k}, \dots, \theta_{x_K}\}$ and the angles in y - y' be $\{\theta_{y_1}, \dots, \theta_{y_l}, \dots, \theta_{y_L}\}$. The projections are stored in an array $\mathbf{S}$, where $\mathbf{S}_{i,j,k,l}$ is the intensity at point (x_i, y_j) on the screen for angles $\theta_{x_k}, \theta_{y_l}$. Consider a single row of an image, fixing y_j , which gives a 1D projection of a slice of the distribution onto the x -axis at the screen. If we fix θ_{y_j} and vary θ_{x_k} , we produce a set of 1D projections that can be used to reconstruct the x - x' phase space distribution for this slice using any 1D $\rightarrow$ 2D reconstruction method. This is repeated for each y_j and θ_{y_l} . Now, for each x and x' in the reconstruction grid, we have set of projections of the y - y' distribution onto the y -axis at the screen for different θ_{y_j} ; thus, for each x and x' in the reconstruction grid, we can reconstruct the y - y' distribution. This completes the reconstruction.

To test the method, the 600,000-particle distribution from Fig. 3.3 was used. In [97], filtered back-projection (FBP) was used for the 2D reconstructions. FBP requires many projections — something that is not always possible in the context of beam diagnostics. Simultaneous algebraic reconstruction (SART) is a possible alternative when the number of projections is small. The accuracy of SART will depend on the number of projections and the range of projection angles. Fig. 4.9 demonstrates this by reconstructing the y - y' distribution from 1D projections as these numbers are varied. It appears that if the projection angles are distributed over a significant range, the accuracy does not improve much beyond 10-15 projections. As discussed later, 15 projections are likely near the maximum possible in the SNS for each 2D reconstruction if using Hock's method. In the following simulated 4D reconstruction, the phase advances in both planes were scanned over 180° in 12 steps; at each step, the distribution was transported to and then binned on a virtual screen. The reconstruction was performed in normalized phase space, and it was assumed that the distribution was matched to the lattice parameters. Fig. 4.10 shows the simulated images in normalized space with a screen resolution of 75×75 . Three SART iterations were used for each 2D reconstruction. The 2D projections of the reconstructed distribution are compared to those of the original distribution in Fig. 4.11 in normalized phase space, which shows good agreement. Notice that the projections of

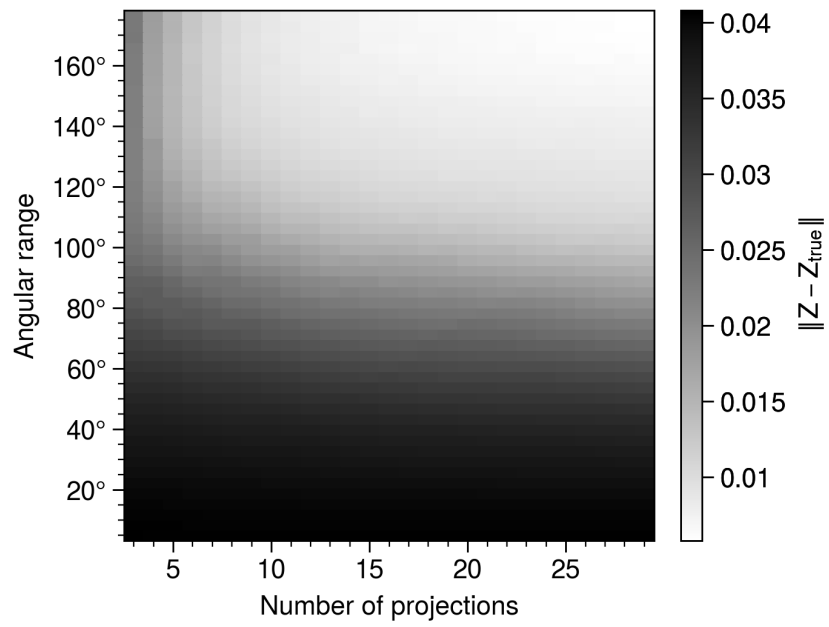


Figure 4.9: SART accuracy as a function of number of projections and range of projection angles.

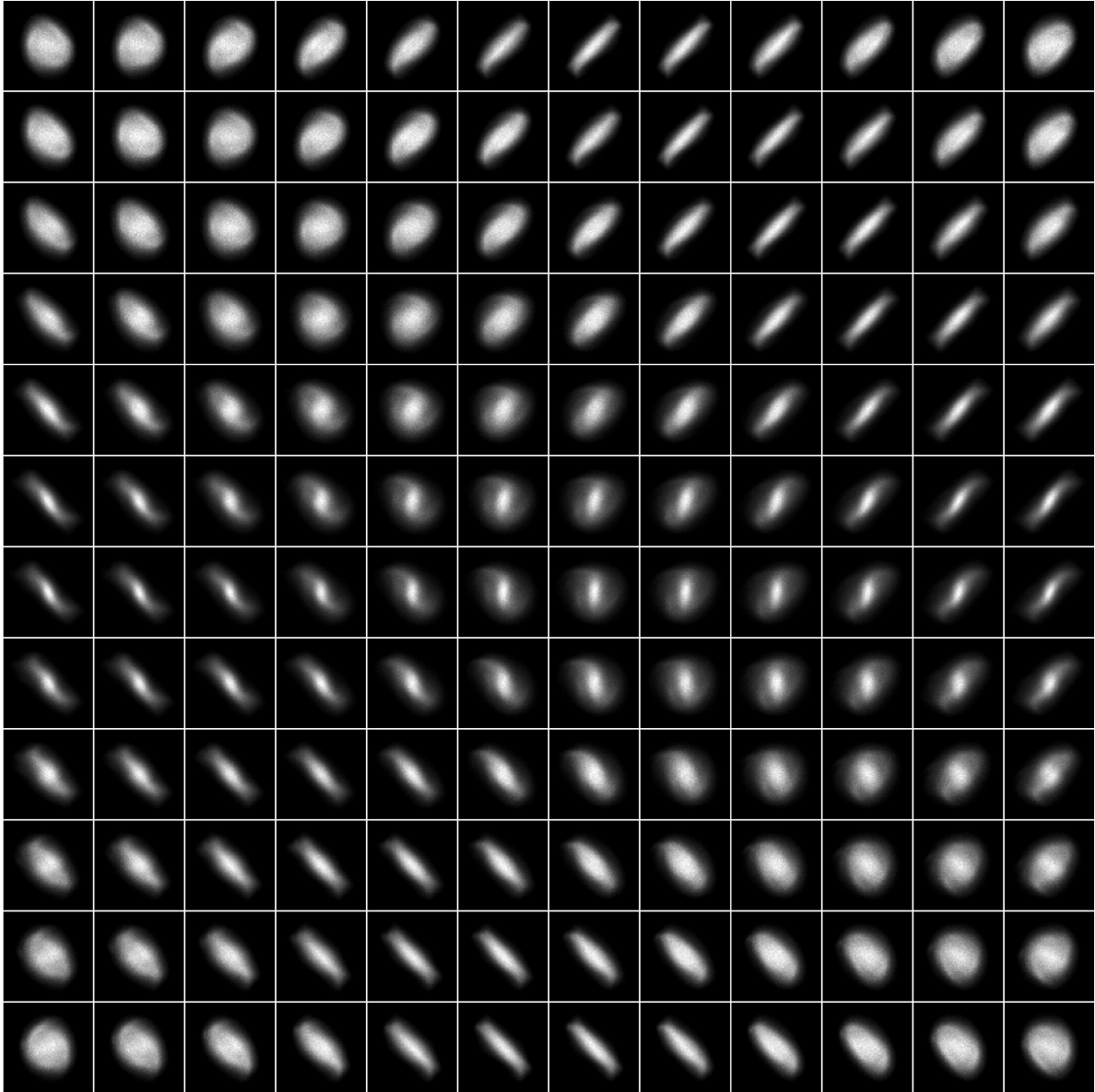


Figure 4.10: Simulated x - y projections as the horizontal (rows) and vertical (columns) phase advances are varied.

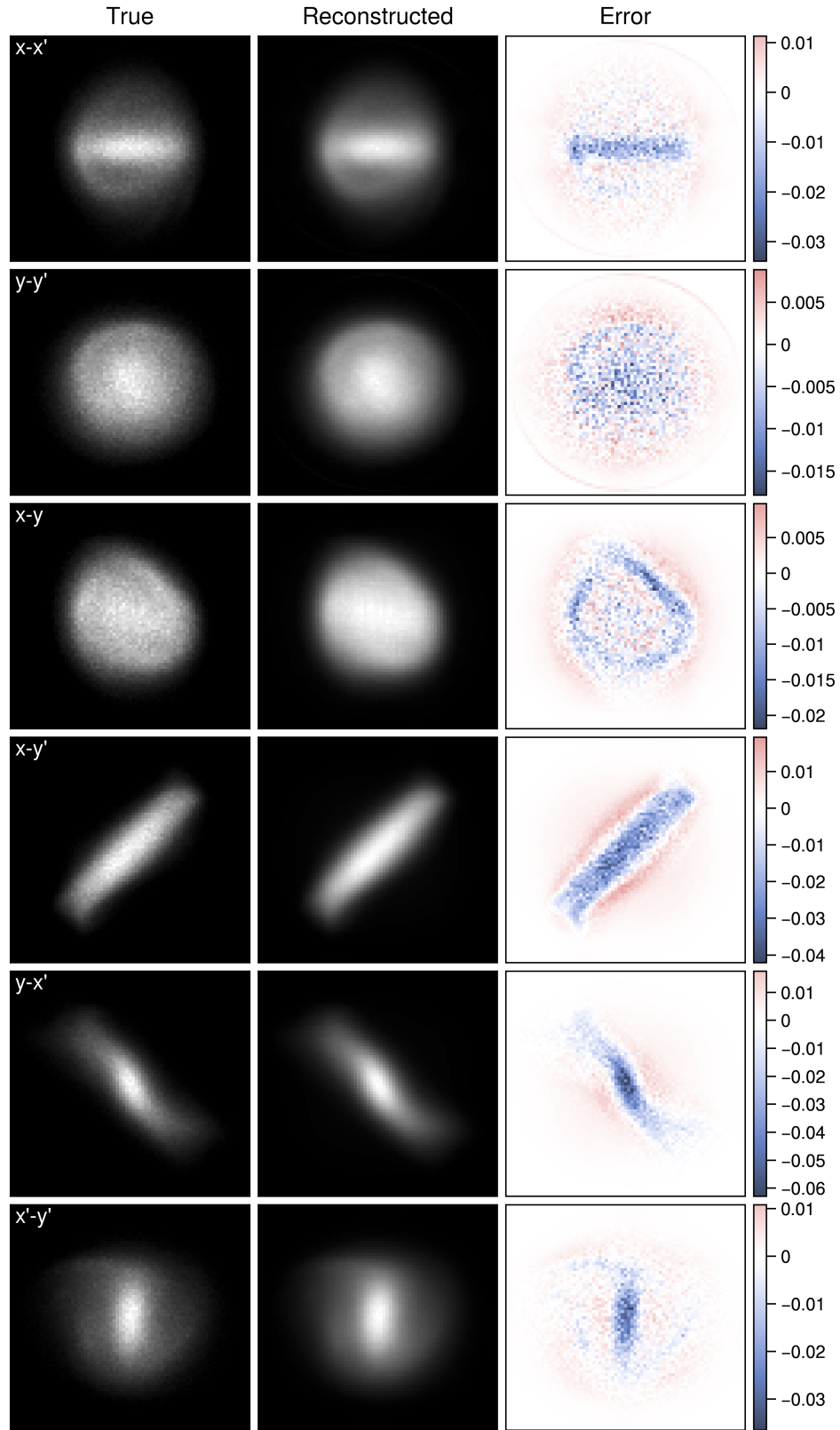


Figure 4.11: Simulated reconstruction using Hock's method (normalized phase space).

the reconstructed distribution, such as $x-y'$, are present in the simulated $x-y$ images in Fig. 4.10. The reason is straightforward: if the phase advance in the vertical plane is $\pi/2$, then $y \rightarrow y'$ and $f(x, y) \rightarrow f(x, y')$ [97].

This method is preferred because it leverages 2D reconstruction algorithms. Open-source implementations of these algorithms are widely available and the conditions needed for accurate reconstructions are well-understood, primarily due to the use of tomography in medical imaging.

4.3.3 Direct 4D reconstruction

If the phase advances cannot be independently controlled or if only a very small number of projections can be collected, 2D reconstruction algorithms must be generalized to 4D. Several algorithms generalize to any number of dimensions, but they may be difficult to implement, the conditions for an accurate reconstruction may be unclear, and the time and space complexity may make the method infeasible. Here, we focus on one method that has recently been experimentally demonstrated, then mention a few more that could be explored in future work.

ART

Each measured projection on the screen produces the following set of equations:

$$\boldsymbol{\rho} = \mathbf{P}\boldsymbol{\psi}. \quad (4.11)$$

$\boldsymbol{\rho}$ is a vector of the pixel intensities on the screen and $\boldsymbol{\psi}$ is a vector of the phase space coordinates on the reconstruction grid. To form $\mathbf{P}$, we place a particle at the center of each bin in the reconstruction grid and track the particles to the screen using the transfer matrix. $\mathbf{P}_{i,j} = 1$ if particle j landed in bin i on the screen; otherwise, $\mathbf{P}_{i,j} = 0$. The equations produced by subsequent measurements are stacked, and the resulting system of equations is solved using a sparse least squares solver. This method has been used to reconstruct the phase space distribution in the Compact Linear Accelerator for Research and Applications (CLARA), a low-energy test facility [94].

For an $N \times N \times N \times N$ reconstruction grid, an $N \times N$ measurement grid, and n measurements, $\boldsymbol{\rho}$ has nN^2 elements, $\boldsymbol{\psi}$ has nN^4 elements, and $\mathbf{P}$ has $nN^2 \times N^4$ elements. In practice, these significant storage requirements limit the resolution of the reconstruction grid to $N \approx 50$ [94]. In Fig. 4.12, the method was applied to the same simulated distribution, but 8×8 projections were used instead of 15×15 . Although the main features of the distribution are present in the reconstruction, there are streaking artifacts outside the beam core that are not present in Fig. 4.11, although it is likely that the performance could improve if a larger number of projections were used. Unfortunately, the algorithm took hours to execute as opposed to minutes for the previous example, even with the reduced grid resolution.

Additional methods

Here are two additional methods to reconstruct the 4D phase space distribution that could be explored in future work.

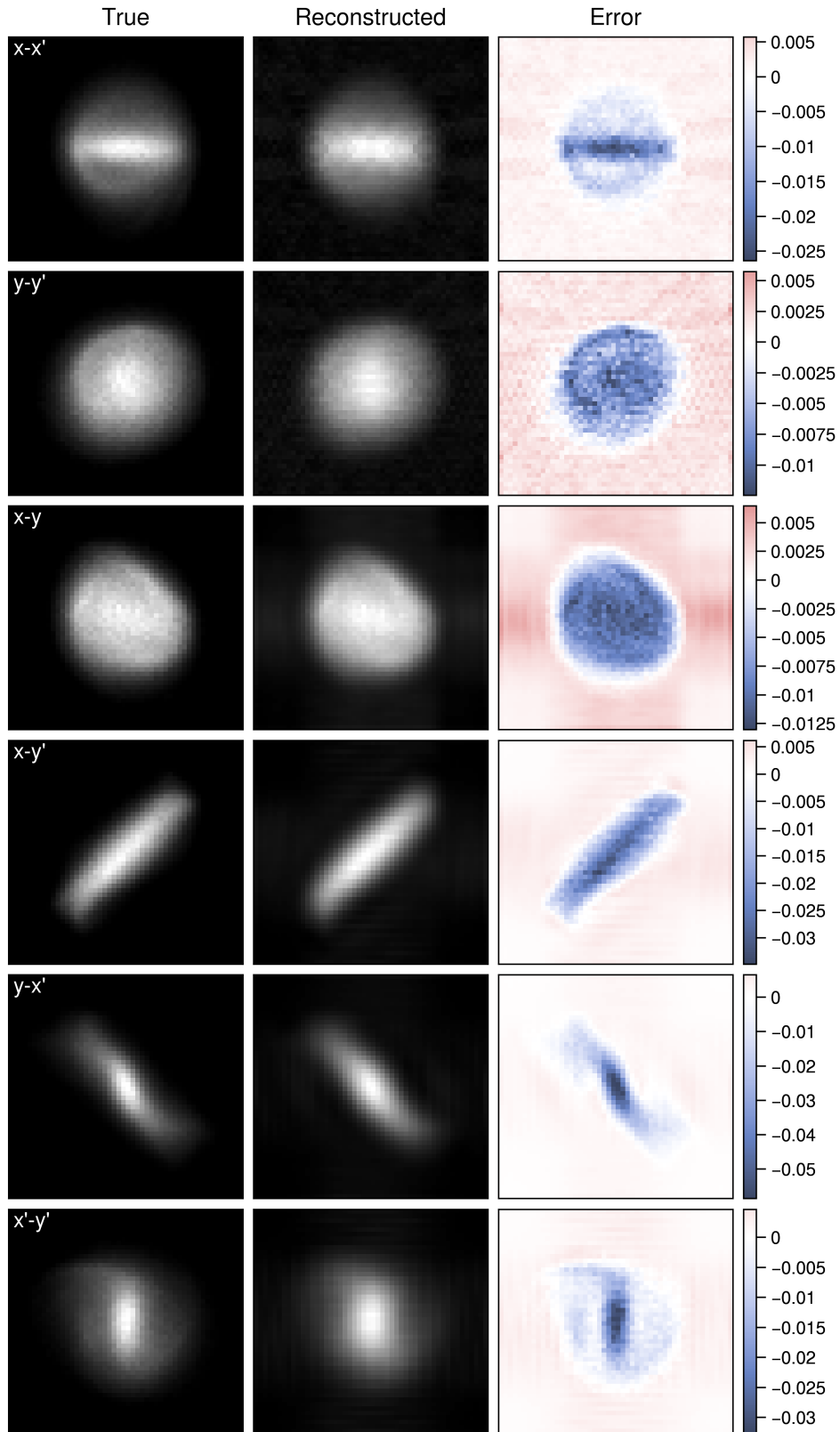


Figure 4.12: Simulated reconstruction using algebraic reconstruction (normalized phase space).

Among the distributions consistent with the measured projections, MENT selects the distribution with the maximum entropy. For example, without any measurements constraining the solution, MENT will produce a uniform distribution. It can perform well with few projections and has been used for 2D reconstruction in particle accelerators [97]. The downside is that the iterative numerical solution is difficult to implement and may struggle to converge when the number of projections is large. For 4D reconstruction from x - y projections, the MENT could be used to perform the 2D reconstructions in Hock’s method, which may result in improved performance over SART. Alternatively, just as ART was generalized to four dimensions in the previous section, MENT could be generalized to four dimensions.⁹ An analytic MENT solution has recently been derived and used for the 4D reconstruction of an SNS minipulse using x - x' and y - y' projections from a laser wire [100].

Another method is to generate a particle bunch, track the bunch to the screen, weight each particle by the measured signal at the bin where it fell on the screen, and generate new particles in the region of that particle according to its weight. The advantage of this method is that it does not assume linear transport and that it can perform well with few projections. It was experimentally demonstrated by Wang et. al. in the Xi’an Proton Application Facility (XiPAF) using six projections [93].

4.3.4 Implementation in the SNS

The idea to use SNS target images for tomographic reconstruction of the phase space distribution was proposed late in this research. Due to this fact, as well as time constraints and unexpected machine downtime, the methods described in the previous subsection were not able to be applied to real data; this is left for future work. Nonetheless, the following paragraphs describe how the reconstruction can be performed in the SNS.

Optics control

We desire independent control of the horizontal and vertical phase advances. The optics control developed for the wire-scanner measurement can be used here. The constraints are now that the β functions remain below 30 m/rad in the wire-scanner region, below 100 m/rad before the target, and stay within 15% of their nominal values at the target. Fig. 4.13 shows that both the horizontal and vertical phase advances can be independently scanned in a 180° range. Fig. 4.14 overlays the β functions and phase advances throughout the RTBT for every step in the scan, showing that the beam size constraints are not violated. The horizontal axis starts at the first varied quadrupole and ends at the target.

Computing each optics setting takes approximately sixteen seconds using an OpenXAL solver. It also takes time to change the magnet strengths in the machine, trigger the beam, and collect a batch of target images. The time available in most accelerator physics studies is eight to ten hours at a maximum, so we place an upper limit on the number of images collected during the scan at 15×15 , for which it takes around one hour to calculate the optics and one hour to collect the images.

⁹In principle, MENT can perform a 4D reconstruction using 1D projections [99]; however, it seems a priori unlikely for this to produce an accurate result: imagine reconstructing a 3D image from 1D projections.

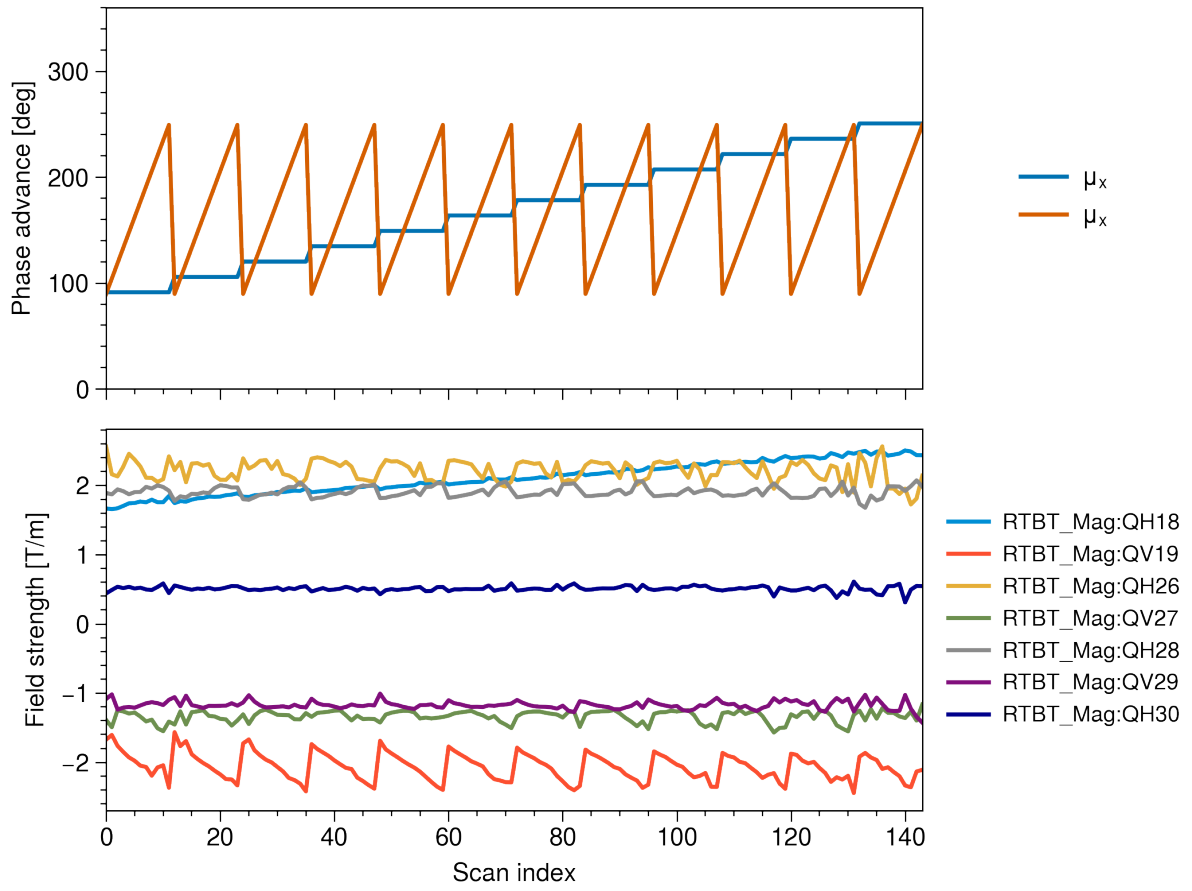


Figure 4.13: Scan of the phase advances at the target.

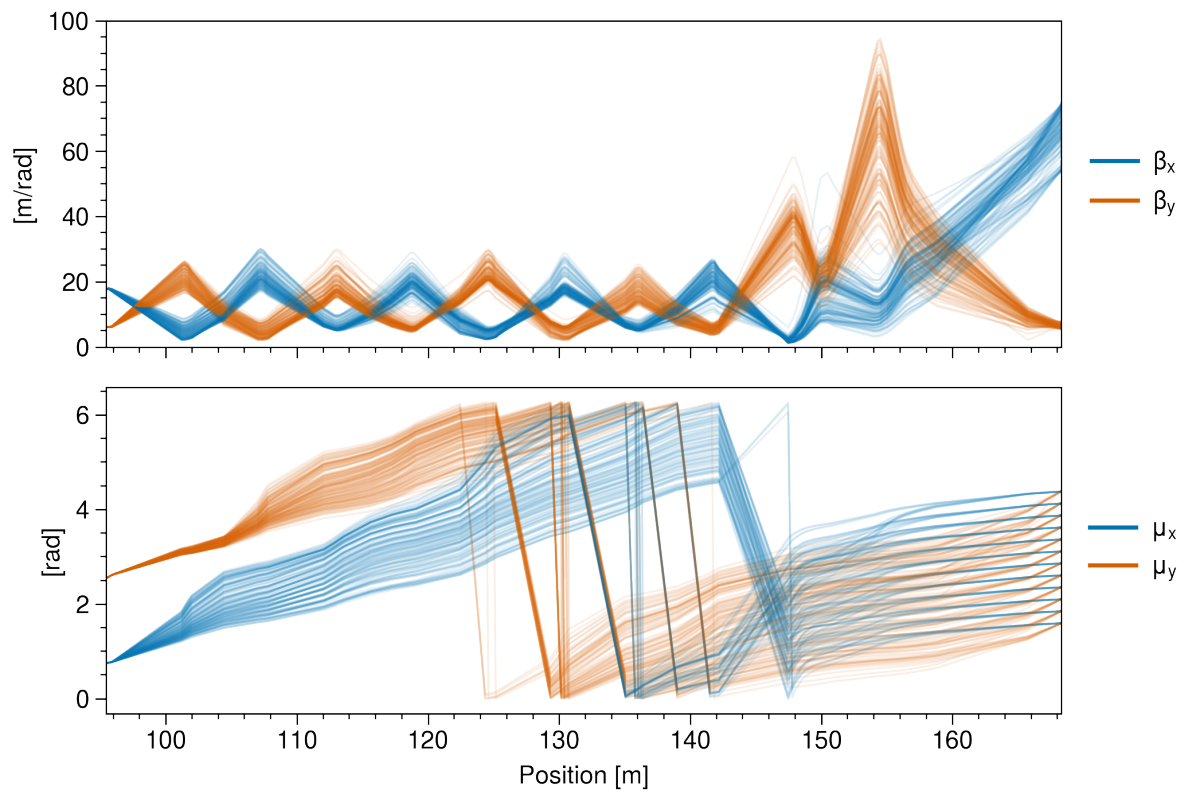


Figure 4.14: β functions and phase advances vs. position for the scan in Fig. 4.13.

Image acquisition and processing

Target image acquisition is handled entirely by the target imaging system software. Live target images are displayed in the SNS control room. It is straightforward to access the image from an OpenXAL script as an 80,000 element array. The script to perform the target scan repeatedly modifies the RTBT quadrupoles, triggers the beam, and saves the image array to a file.

The unprocessed target images are not ideal. First, to reduce pulse-to-pulse variation, the images can be averaged over a few pulses. Second, the beam passes through 2 meters of Helium at atmospheric pressure before the target; due to radiation damage, light from the gas appears as a streaking artifact on the lower-right of the image [80]. Although this has been corrected by delaying the shutter opening by a few microseconds, the issue has occasionally resurfaced when the beam energy is different than 1 GeV. If these images are collected, they can be identified later by placing a maximum value on the pixels far from the image center, particularly in the lower-right region. Third, there are visible grid lines from the fiber bundle. A Gaussian blur is therefore applied to the image as in Fig. 4.15. Finally, there are four dark spots on the image that serve as fiducial markers; they are visible when the beam is large. In this work, the dark spots are left in the image.

Other uses of 2D projections

There is information to be gained from 2D projections of the distribution in addition to the tomographic 4D reconstruction just described. The projections can be compared to a uniform density ellipse. Additionally, one can observe the variation in the x - y correlation coefficient as the difference between the horizontal and vertical phase advances is varied. This reveals any “hidden” cross-plane correlations, as in Fig. 4.10. Finally, by computing the RMS moments of the images, the covariance matrix can be reconstructed using the least squares method described in Section 4.2; the advantage would be that data collection is much faster than for the wire-scanners and that the $\langle xy \rangle$ moment is computed directly.

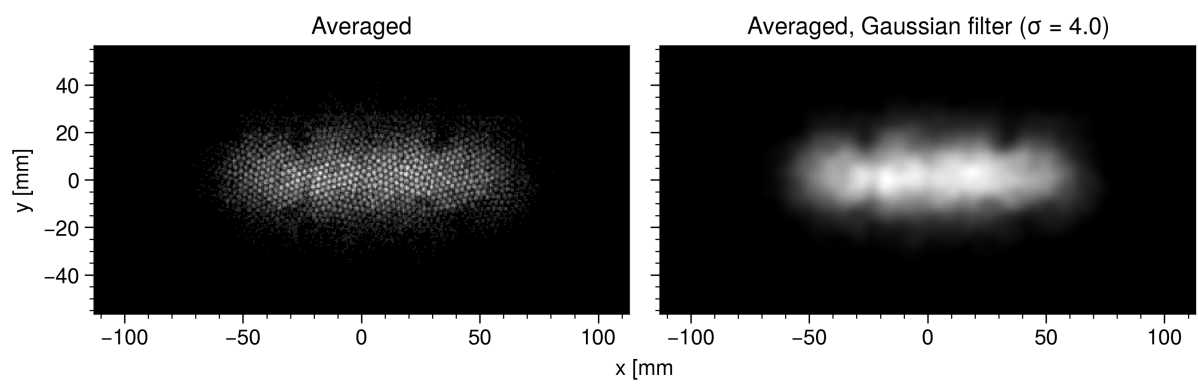


Figure 4.15: Image of the beam on the target.

Chapter 5

Experiments

This chapter presents the results of initial experiments to create an approximate Danilov distribution in the SNS. The computational studies in Chapter 2 and Chapter 3 were used to guide the experiments, and the diagnostics described in Chapter 4 were used to measure the painted distribution.

We repeat the following from Chapter 1: Elliptical painting requires the creation of elliptical modes in the ring. The SNS ring is uncoupled, but elliptical modes can be created by equating the horizontal and vertical tunes. Simulations predict that the addition of solenoids to the ring will stabilize the beam against nonlinearities which strongly influence the motion in this setup (see Chapter 3 and Appendix C). Solenoid magnets were planned to be installed in the SNS ring in 2021, but their installation was delayed until late 2022, outside the time frame of this work.

Therefore, in the following experiments, the quality of the painted beam was not expected to approach the “best-case scenario” simulated in Chapter 3. But it was hoped that the measured beam would be distinguishable from one produced by normal injection methods. The signatures we desire are a reduced 4D emittance and a uniform charge density.

A brief outline of this chapter: First, the experimental setup and data collection procedure are described. In Experiment 1, a production beam is measured for comparison and elliptical painting is attempted at a beam energy of 1 GeV. In Experiment 2, the beam energy is lowered to 0.8 GeV to allow proper scaling of the injection coordinates. In Experiment 3, several parameters are varied to study their effect on the measured 4D emittance. Finally, the implications of these experiments are discussed.

5.1 Procedure

Accelerator physics experiments are performed in the SNS control room using the OpenXAL framework, which provides a high-level interface to perform tasks such as changing magnet strengths, triggering the beam, etc. It can also perform single-particle or envelope tracking using an online model of the accelerator. OpenXAL scripts are written in Java or Jython and are executed from the command line. Many graphical user interface (GUI) OpenXAL applications have been developed over the history of the SNS and are available for use in the control room.

The following steps are taken during the experimental setup:

1. To increase the maximum injection angle, the beam energy is lowered from 1.0 GeV to 0.8 GeV by turning off several RF cavities at the end of the linac, then scaling every subsequent magnet in the machine. Lowering the energy can cause other accelerator components to trip or malfunction due to the modified timing of the beam pulses, and these issues must be corrected one-by-one. The first attempt to lower the energy to 0.8 GeV was successful and took approximately six hours. The task can now be performed by machine operators in less than half that time.¹
2. The horizontal and vertical tunes are set to the same value using the Ring Optics Control (ROC) application. ROC varies several quadrupoles until the model tunes are equal to the desired tunes. The tunes are measured using turn-by-turn BPM readings from a single minipulse in the ring. Generally, the measured and model tunes are not quite equal; we therefore shift the ROC input tunes until the measured tunes converge to the desired tunes.
3. Optional: The injection region is modified to increase the maximum injection angle.²
4. The eight injection kicker magnets are calibrated using the Ring Injection Control (RIC) application, as described in Chapter 1.
5. The kicker voltages required to obtain the desired injection coordinates at the start and end of injection are determined (as described in Chapter 1).
6. Square root waveforms connecting the initial/final voltages are applied to the kicker magnets. The duration of the waveforms is chosen to be consistent with the desired number of injected turns, i.e., beam intensity.
7. The number of injected turns before extraction is chosen. This allows measurement of the beam at different times during accumulation. It is also possible to store the beam in the ring after it reaches full intensity, but this is not attempted here.

The next task is to prepare for the measurements. For the wire-scanner measurement, the first step is to modify the RTBT optics using the application developed in Chapter 4. If the fixed-optics method is used, the optics are changed immediately. If the multi-optics method is used, the optics are pre-computed and stored for later use. The second possible measurement is the tomographic reconstruction from x - y projections on the target. Since the optics calculation is time-consuming, it can be run in the background while wire-scans are collected.

¹A lower beam energy is possible but requires significantly more effort, especially when the number of accumulated turns is large. Reduction of the energy requires the reduction of a master reference oscillator frequency, and the phase-locked loops of the various accelerator components become unstable if this frequency becomes too small. Circumvention of this issue requires changes to firmware that affect many other systems in the machine. An initial attempt to lower the energy to 0.6 GeV was successful but took over thirty-six hours.

²One option is to utilize orbit corrector dipoles to provide a closed bump in either plane, thus moving the ring orbit closer to the foil. Another option is to steer the injected beam; this is not ideal because it requires modification of the trajectory of the unstripped H^- ions after the foil, which must be guided to the beam dump. Finally, the Chicane dipole magnets can be modified, but again, this is complicated by the beam dump trajectory. The optimization of this system is an ongoing problem, and no modifications to the injection region are made in this work.

5.2 Experiment 1

At the time of our first experiment, setup of the injection region had not yet been completed. Although simulations indicated that a sizeable beam could not be painted at 1 GeV beam energy, this had not been tested. Furthermore, the SNS energy had not yet been decreased — a time-consuming task. Therefore, the goal of Experiment 1 was to push the injection coordinates x and y' to their limits at 1 GeV. We decided to measure the distribution not only at its final state but also at intermediate states during accumulation. The number of injected turns was reduced from 1000 to 500, halving the beam intensity, and the beam was measured every 50 turns using the fixed-optics method.

5.2.1 Experiment 1a: correlated painting

We first performed correlated painting for later comparison. The measured wire-scanner profiles are shown in Fig. 5.1. Each subplot shows the evolution of the projection onto a single wire; each row corresponds to a different wire-scanner and each column to a different projection axis — x , y , or u . Recall that in correlated painting, the injection angles are zero and the positions are increased from an initial offset. The initial offset is evident from the two peaks in the measured x and y profiles. The hollow center of the distribution eventually fills in due to nonlinear effects.

The reconstructed emittances and covariance ellipses at BPM17, just before QH18, are shown in Fig. 5.2. The size and location of the error bars were computed by repeating the reconstruction multiple times with 3% random noise added to the measured moments, then taking the mean and standard deviation over the trials. For our purposes, the most important feature of Fig. 5.2 is that the measured cross-plane correlation is small throughout injection, demonstrating that there is very little coupling from electromagnetic fields of the beam or ring in the standard SNS painting scheme.

5.2.2 Experiment 1b: attempted elliptical painting

We then attempted elliptical painting. First, the horizontal and vertical tunes were set to 6.18. The next step was to move the closed orbit to the foil, which was found to be possible in the vertical plane but impossible in the horizontal plane. The minimum horizontal distance from the foil was 10 mm, and the maximum vertical injection angle was 0.7 mrad; assuming $\alpha_y \approx 0$ and $\beta_y \approx 10$, the painted vertical emittance can be estimated as

$$\varepsilon_y \approx \frac{1}{4} \beta_y y'_{max}{}^2 = 1.22 \text{ mm mrad}, \quad (5.1)$$

which is only four times larger than the emittance from the linac. Although this is not ideal, we continued using initial coordinates $(x, x', y, y') \approx (10 \text{ mm}, 0 \text{ mrad}, 0 \text{ mm}, 0 \text{ mrad})$ and final coordinates $(x, x', y, y') \approx (21 \text{ mm}, 0 \text{ mrad}, 0 \text{ mm}, 0.7 \text{ mrad})$.

Let us pause to predict the beam evolution using these settings, assuming linear transport: initial particles would oscillate along a flat horizontal ellipse in the x - y plane, and as time progressed, the horizontal and vertical size of the ellipse would grow at different rates depending on the maximum injected x and y' coordinates. The inclusion of space charge complicates the analysis due to the non-uniformity introduced by the initial horizontal offset.

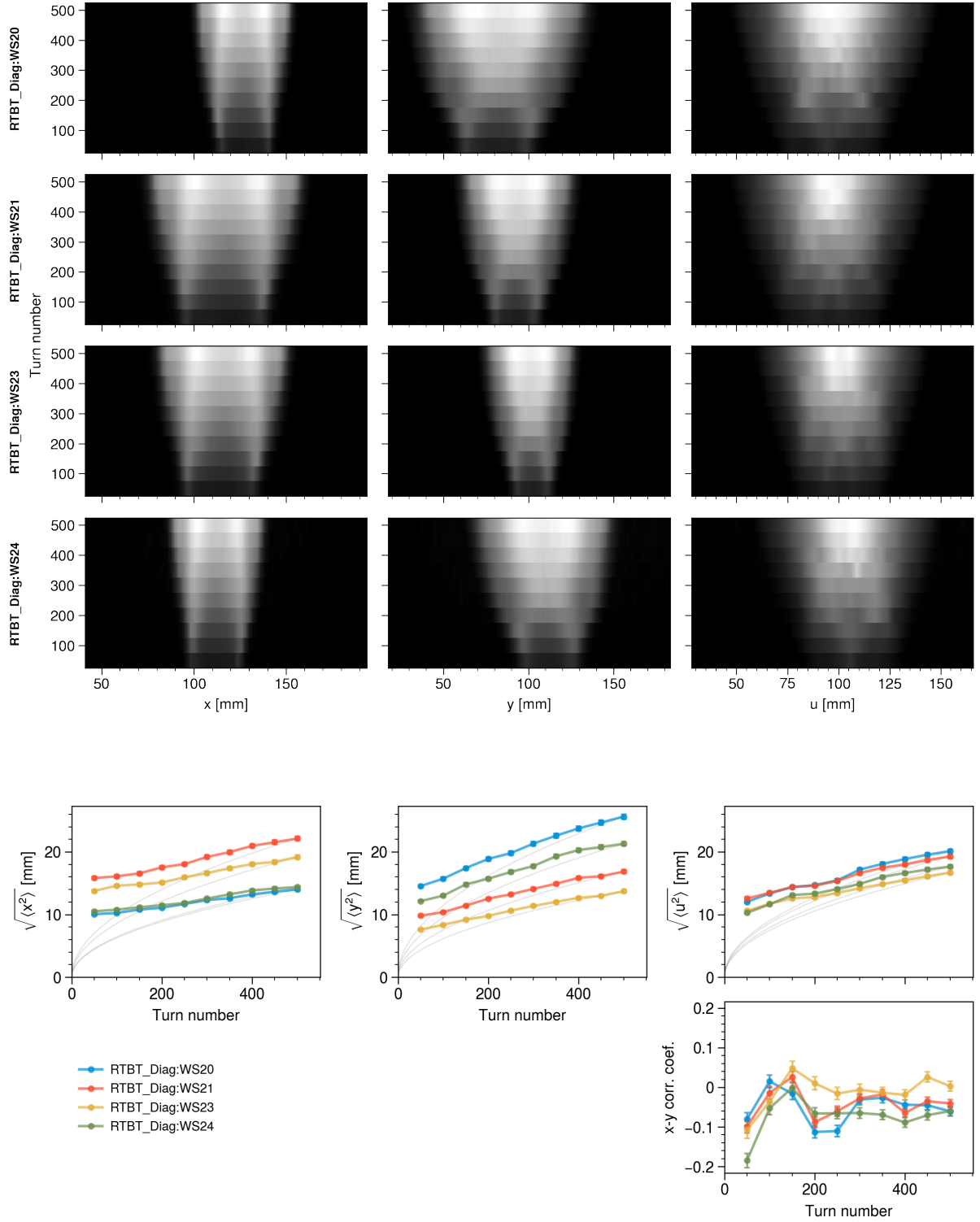


Figure 5.1: Measured wire-scanner profiles from Experiment 1a. The top figure shows the measured profiles on each wire as a function of time. The bottom plots show the moments extracted from the profiles.

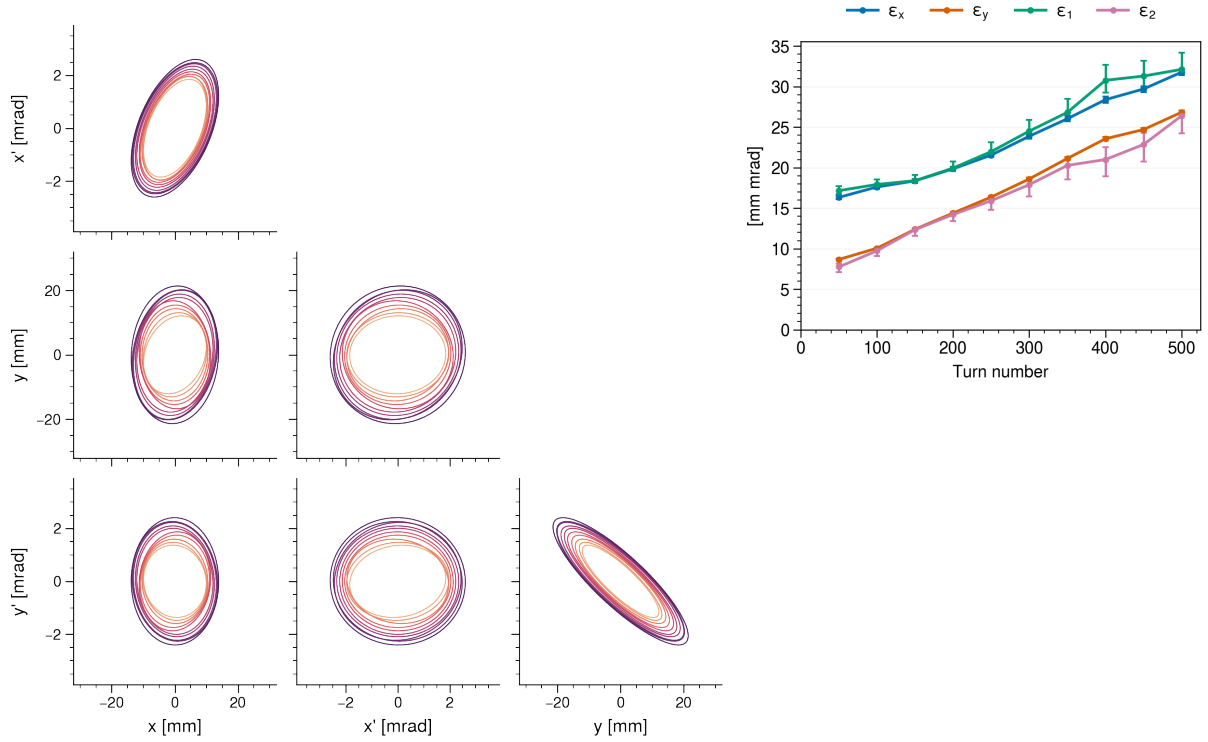


Figure 5.2: Reconstructed emittances and covariance ellipses from Experiment 1a. In this and subsequent figures, the reconstruction is performed at BPM17 and the light/dark ellipses correspond to the start/end of injection.

The measured wire-scanner profiles are shown in Fig. 5.3. The horizontal projection at 50 turns is hollow — evidence of the initial offset of the closed orbit — but quickly filaments. The most important feature of Fig. 5.3 is the vertical beam size, which starts at a small value and increases throughout injection. This is an indication that the injection kicker waveforms are correct. Notice that the beam is much smaller than in Experiment 1a.

We now turn to the reconstructed emittances and covariance ellipses in Fig. 5.4. There is a clear separation between the intrinsic and apparent emittances throughout accumulation. A possible explanation for the larger-than-expected increase in vertical emittance is that space charge coupled the motion during injection, causing emittance exchange as new particles were added to the bunch; another possible explanation is emittance blow-up due to many particles being injected into a small region of vertical phase space; the answer is left for future work. The most important feature of Fig. 5.4 is that the measured distribution was significantly different than in the previous experiment and was closer to the desired case; in other words, the ring modifications seem to have worked as intended. We close with a PyORBIT simulation of this experiment in Fig. 5.5. Keep in mind that the β functions of the ring at the injection point are not the same as in the experiment, so the exact values of the emittances are not expected to agree. Qualitative agreement with the measured emittance growth appears to be present.

5.3 Experiment 2

In Experiment 2, the beam energy was lowered to 0.8 GeV. At this energy, the closed orbit was able to reach the foil with zero slope, i.e., $x = x' = y = y' = 0$, as required to paint a uniform density beam. Again, only x and y' need to change during injection. There is no limit on x_{max} since increasing x involves decreasing the horizontal kickers, but the vertical kickers could only reach $y'_{max} \approx 1.1$ mrad. In the linear approximation, this angle is expected to produce a vertical emittance of 3-4 mm mrad. We estimate the ratio of painted emittances as

$$\frac{\varepsilon_y}{\varepsilon_x} \approx \beta_x \beta_y \left(\frac{y'_{max}}{x_{max}} \right)^2. \quad (5.2)$$

To paint equal emittances would require $x_{max} \approx 10$ mm — a small beam. We decided to use $x_{max} = 21$ mm, maintaining the same beam intensity as in Experiment 1.

The measured wire-scanner profiles are shown in Fig. 5.6. One important feature of Fig. 5.6 is that the beam must have some rotational symmetry in the x - y plane since the growth in beam size is similar on all wires. A second important feature is that the beam sizes start at a small value and increase at approximately the square root of time (light grey curves have been added showing the ideal square root time dependence given the final beam size). A third important feature is that the profiles appear to be more consistent with a Gaussian distribution than a uniform density distribution. This will be discussed more at the end of the chapter.

The reconstructed emittances and covariance ellipses are shown in Fig. 5.7. The apparent emittances grow linearly from a small value, as intended. The measured apparent emittances are equal throughout injection, which is not expected from Eq. 5.2. A possible explanation is that the beam experienced space-charge-driven emittance exchange as it

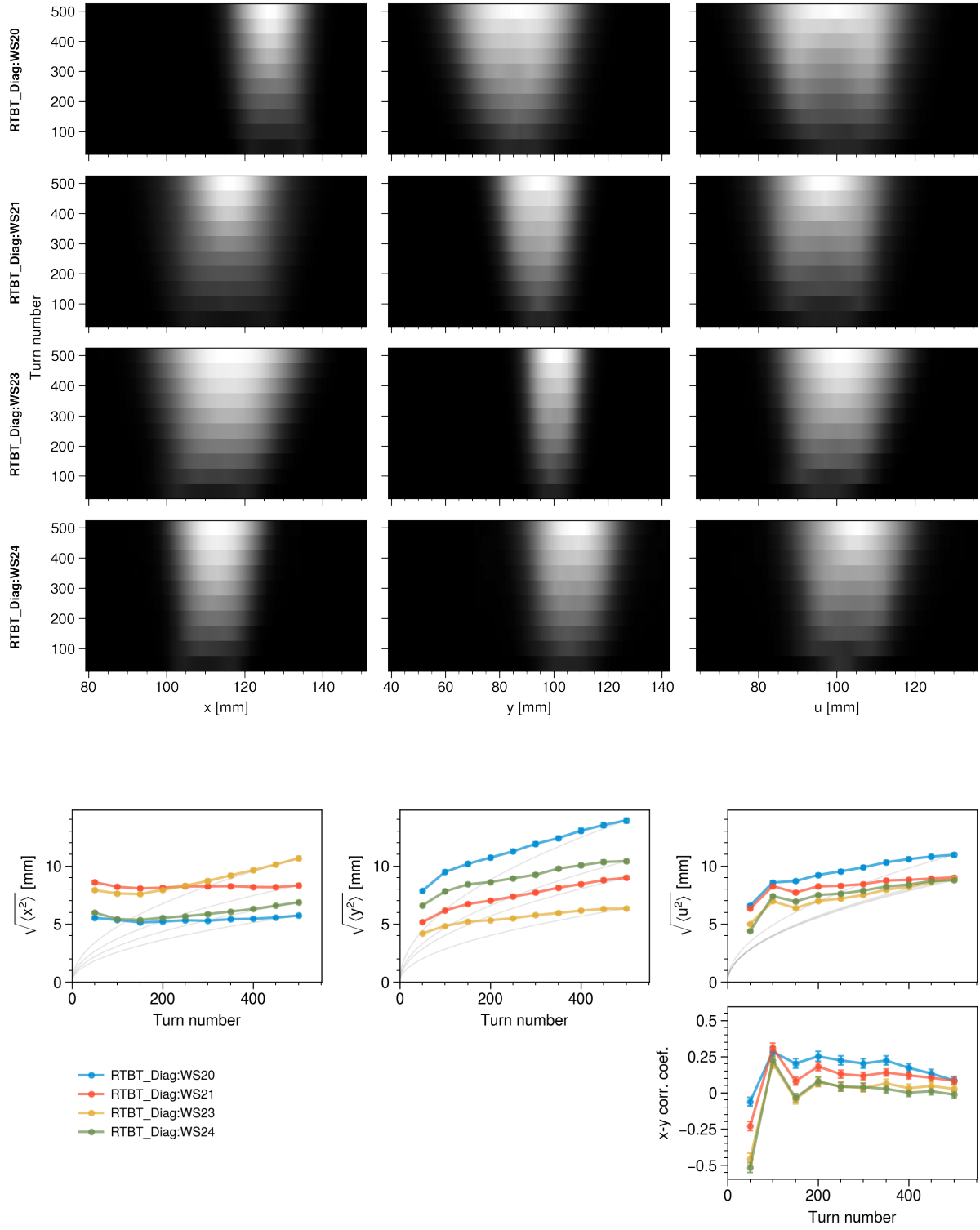


Figure 5.3: Measured wire-scanner profiles from Experiment 1b.

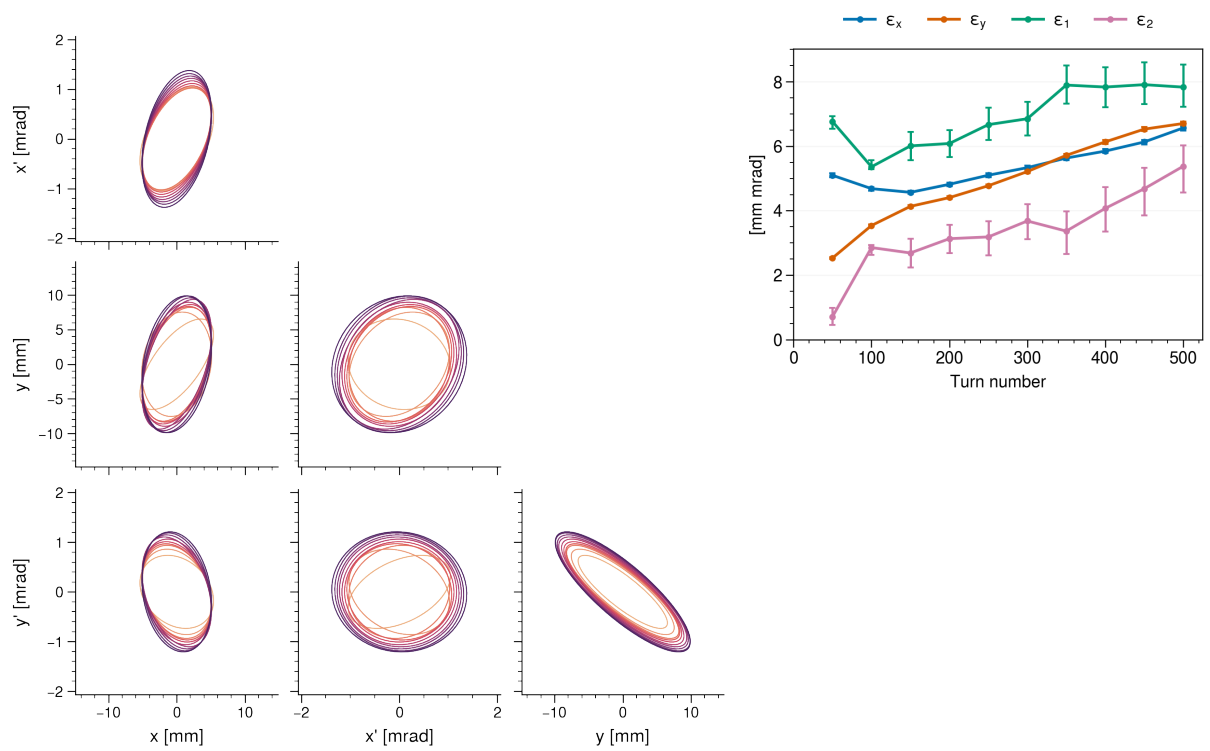


Figure 5.4: Reconstructed emittances and covariance ellipses from Experiment 1b.

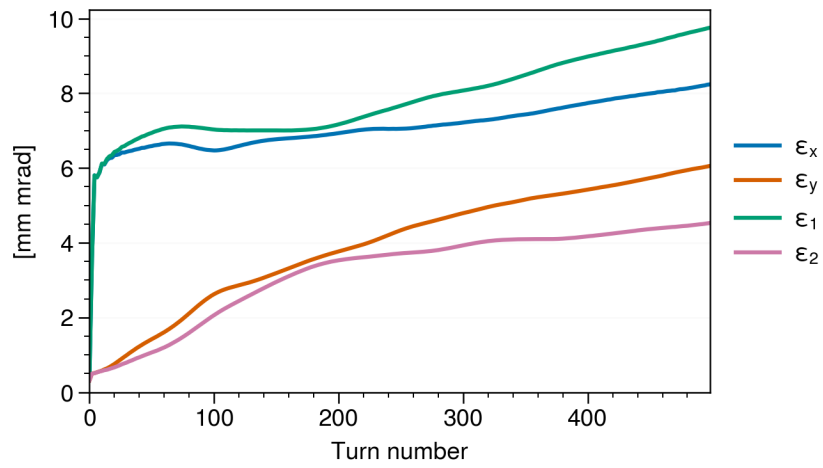
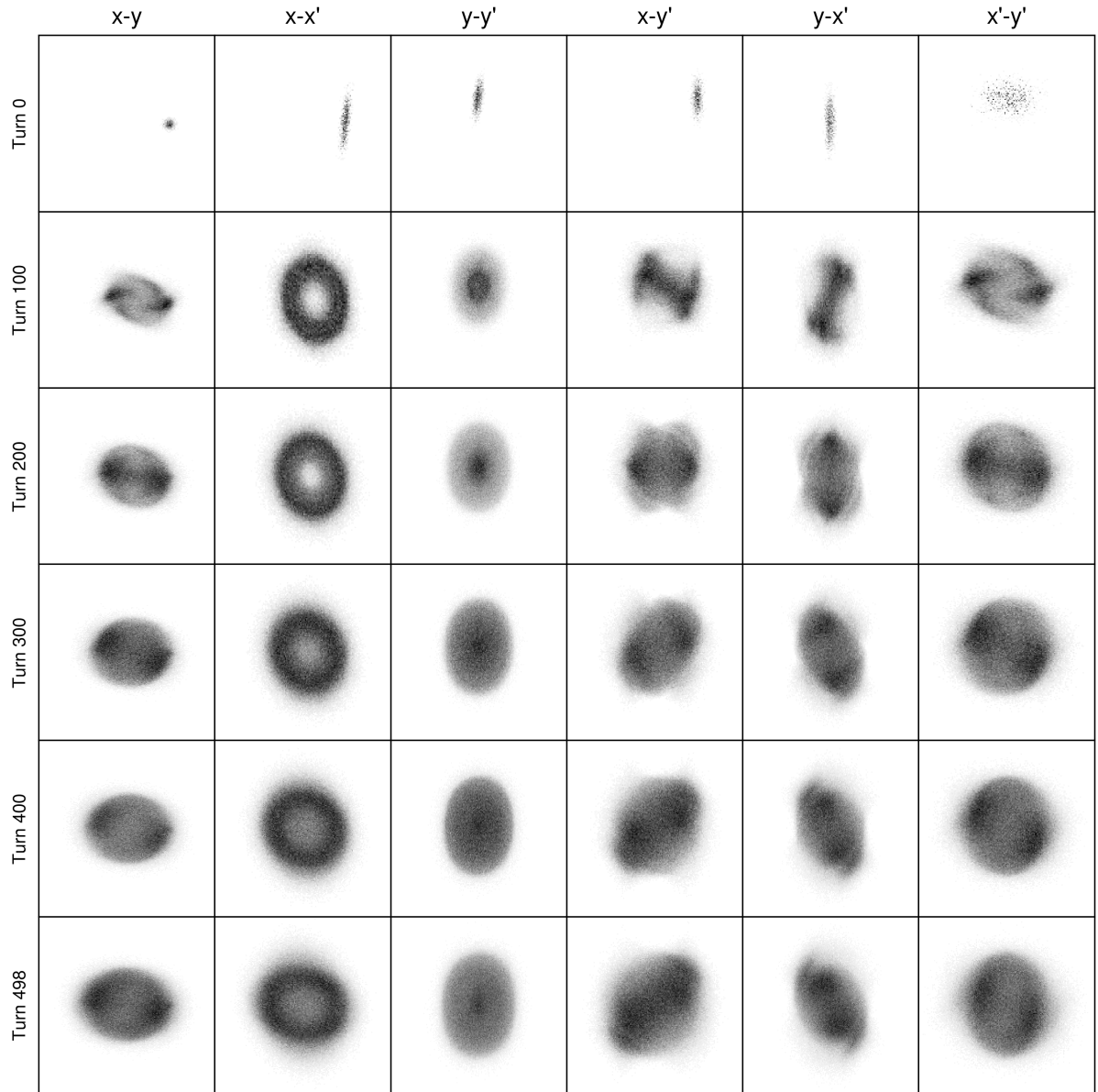


Figure 5.5: Simulation of Experiment 1b.

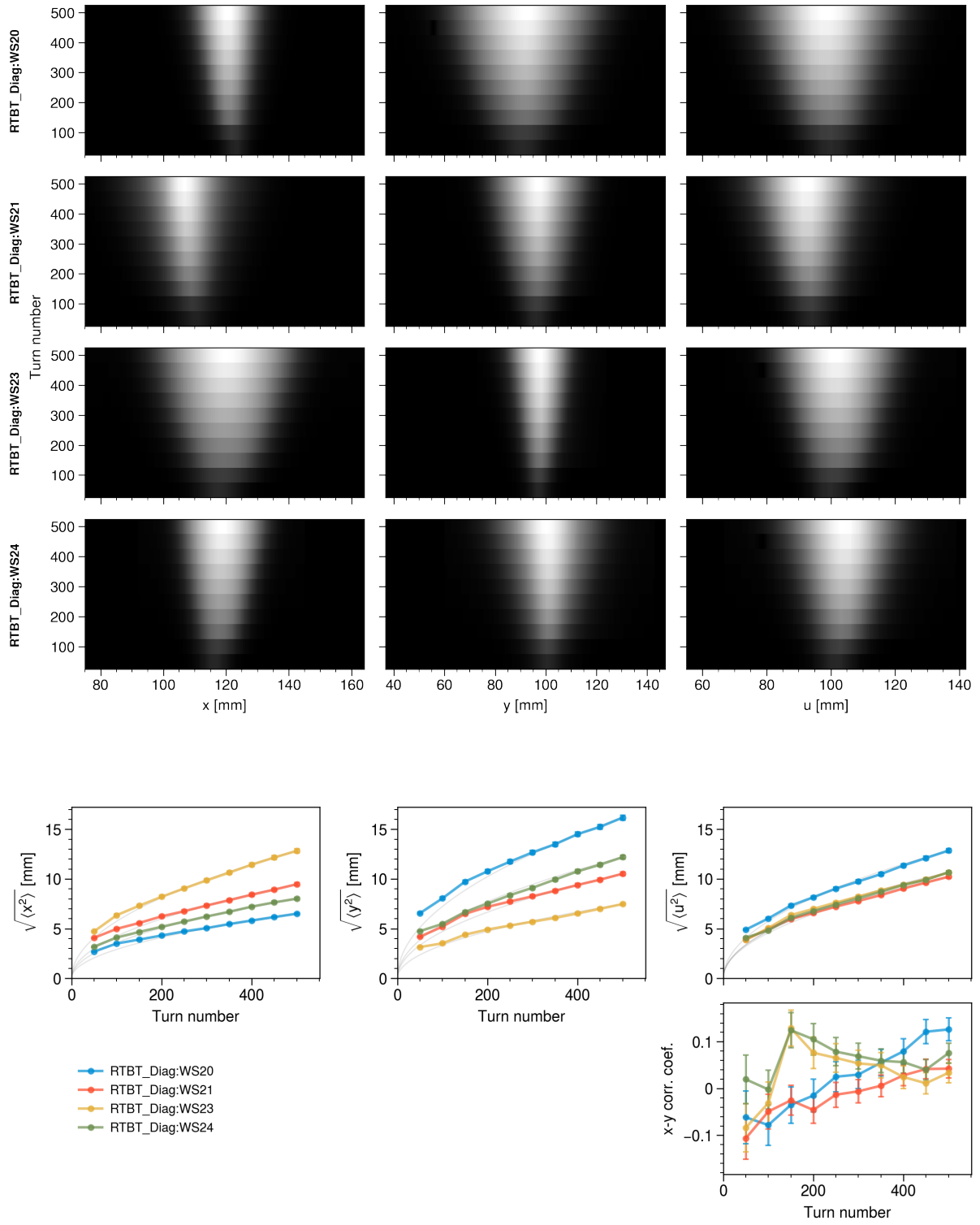


Figure 5.6: Measured wire-scanner profiles during injection from Experiment 2.

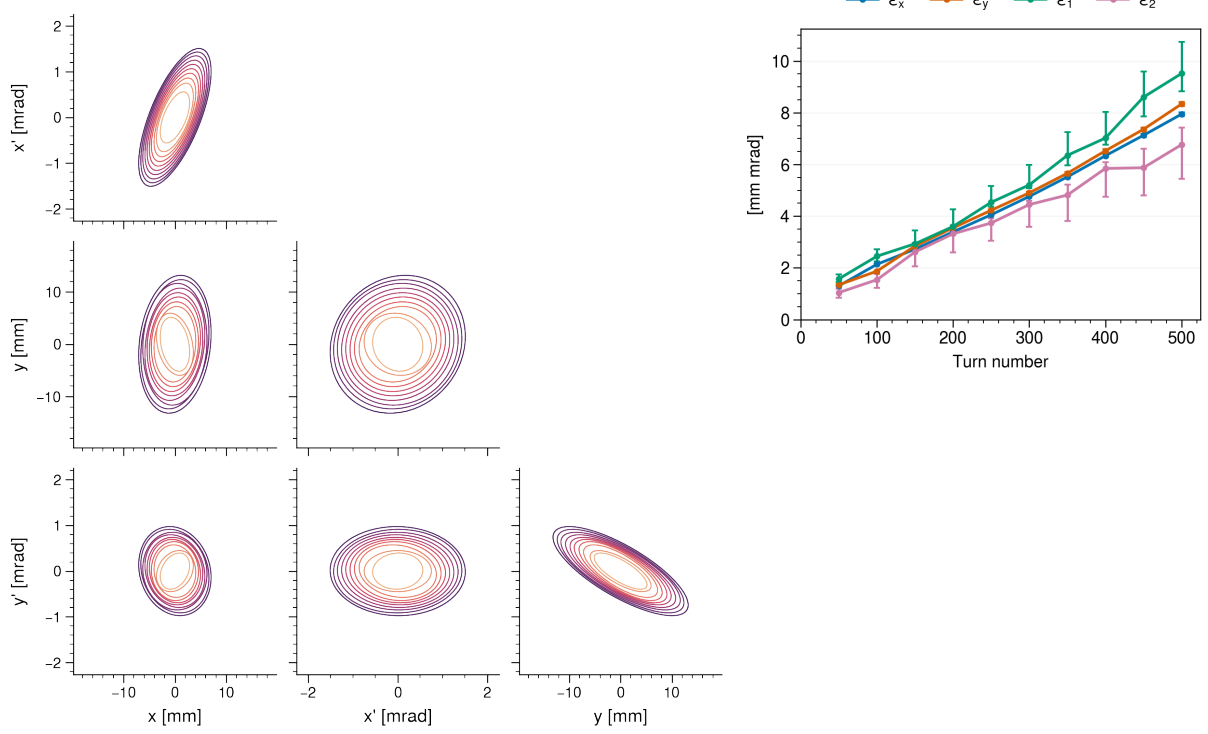


Figure 5.7: Reconstructed emittances and covariance ellipses from Experiment 2.

circulated, resulting in equal emittances as new particles were added to the distribution (see Chapter 2).

The reconstructed intrinsic emittances begin to diverge at the end of injection, but the measured cross-plane correlation is small. Additionally, the error bars are larger than in the previous experiment; this is most likely due to a larger mismatch of the beam Twiss parameters at the RTBT entrance, which is expected given the increase in beam perveance at 0.8 GeV energy and reduced beam size. Possible bias in the measurement, expected to be near the 10% level, must also be kept in mind. Given the small measured cross-plane correlation and larger error bars, it cannot be claimed that the 4D emittance is significantly reduced.

A simulation of this case is shown in Fig. 5.8. Notice that ε_2 begins to flatten after turn 100, but does not remain flat, and although the final x - y' projection has a higher density along the painting path, the linear correlation is significantly blurred. Space charge has a strong effect on the evolution at this intensity, energy, and beam size, providing some explanation for the measurements in Fig. 5.7.

5.4 Experiment 3

The goal of this final experiment was to vary the free parameters of the machine and record the intrinsic emittances in each case, with the plan to examine the most promising case in more detail. To begin, the setup from Experiment 2 was repeated. One difference was that the bunch length was increased from roughly 30/64 of the ring length to 40/64 of the ring length to better approximate a coasting beam. This was done by modifying the chopper settings before the linac and should have increased the total charge of the bunch without changing its charge density. Beam current monitor (BMC) measurements of the longitudinal distribution in the ring are shown in Fig. 5.9; it is clear that there are no strong peaks from RF bunching.

Since the vertical injection angle could not be increased, the remaining free parameters were the beam intensity and horizontal beam size: measured emittances for three different intensities and two different painting settings are shown in Fig. 5.10.³ Collective effects seem to affect the final distribution. It is somewhat surprising that the split in the intrinsic emittances increased with the beam intensity. The measurement process of the previous two experiments was repeated for the center cluster in the right subplot: which amounts to a 50% increase in horizontal beam size and 20% reduction in beam size.

The measured wire-scanner profiles are shown in Fig. 5.11. The beam size curves do not fit the square root dependence as well as in Experiment 2. The measured profiles look similar to those in Experiment 2, although the horizontal projection at WS21 has developed a sharp peak surrounded by a lower density cloud.

Fig. 5.12 shows the reconstructed emittances and covariance ellipses. From Eq. 5.2, the ratio $\varepsilon_y/\varepsilon_x$ is expected to be quite small — around 1/8 — but again, the apparent emittances remain nearly equal. A future study could split the tunes and perform correlated painting with zero offset from the foil, varying x_{max} , and y_{max} to examine whether this phenomenon is caused by coupled space charge forces.

³The intensities are not exact; they are obtained by multiplying the nominal minipulse intensity by the number of injected turns.

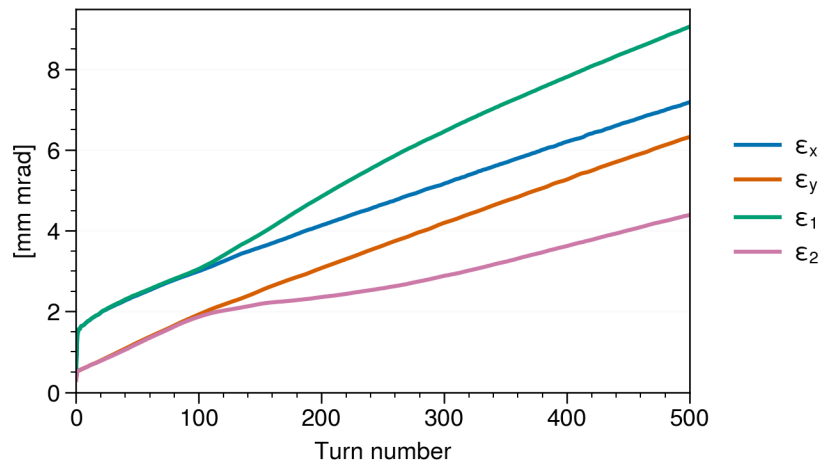
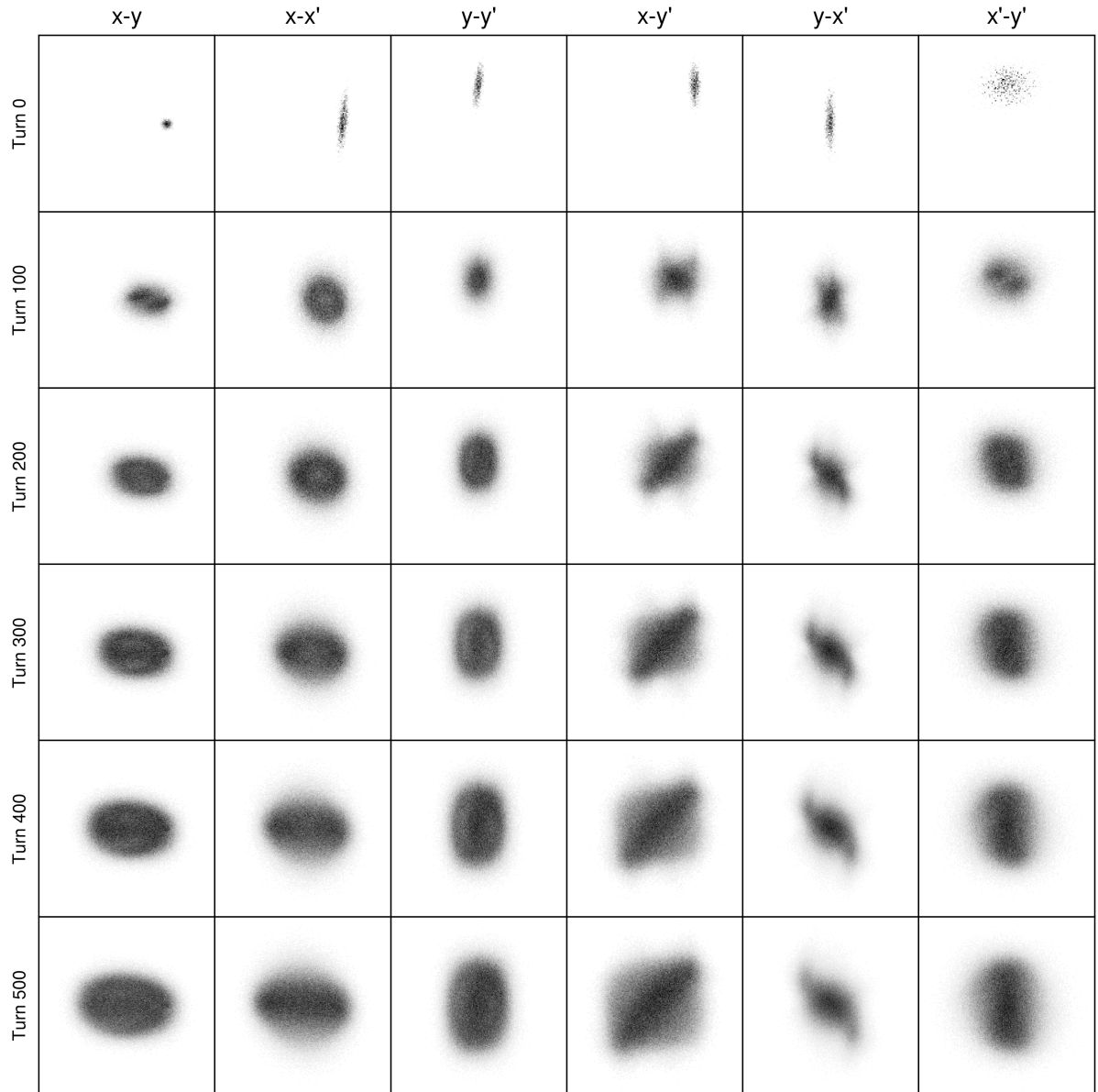


Figure 5.8: Simulation of Experiment 2.

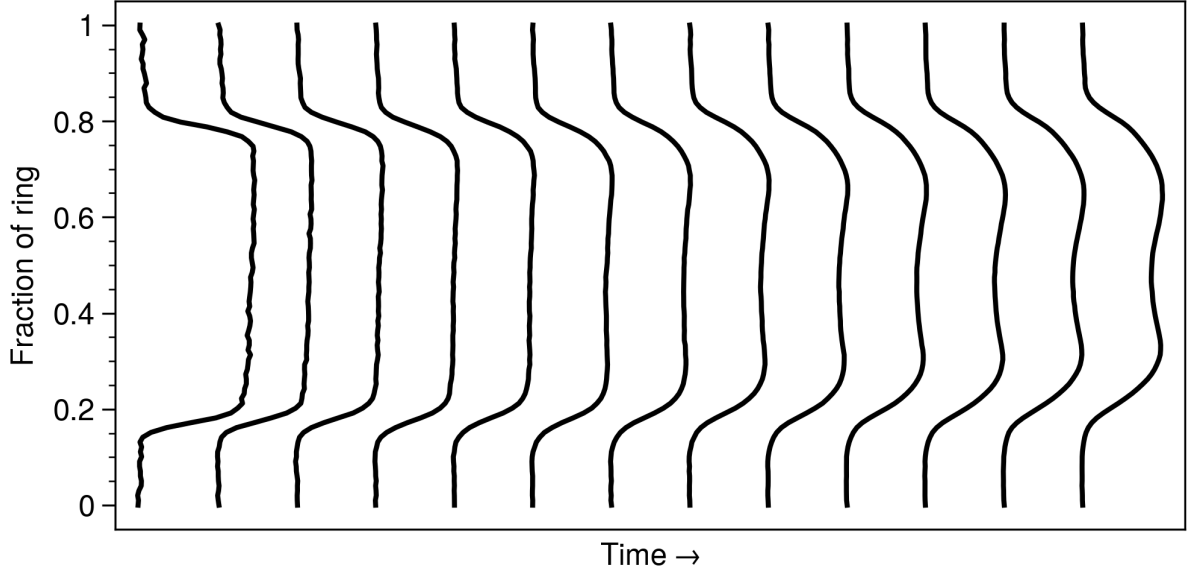


Figure 5.9: Evolution of the longitudinal distribution in the ring as measured by a beam current monitor (BCM).

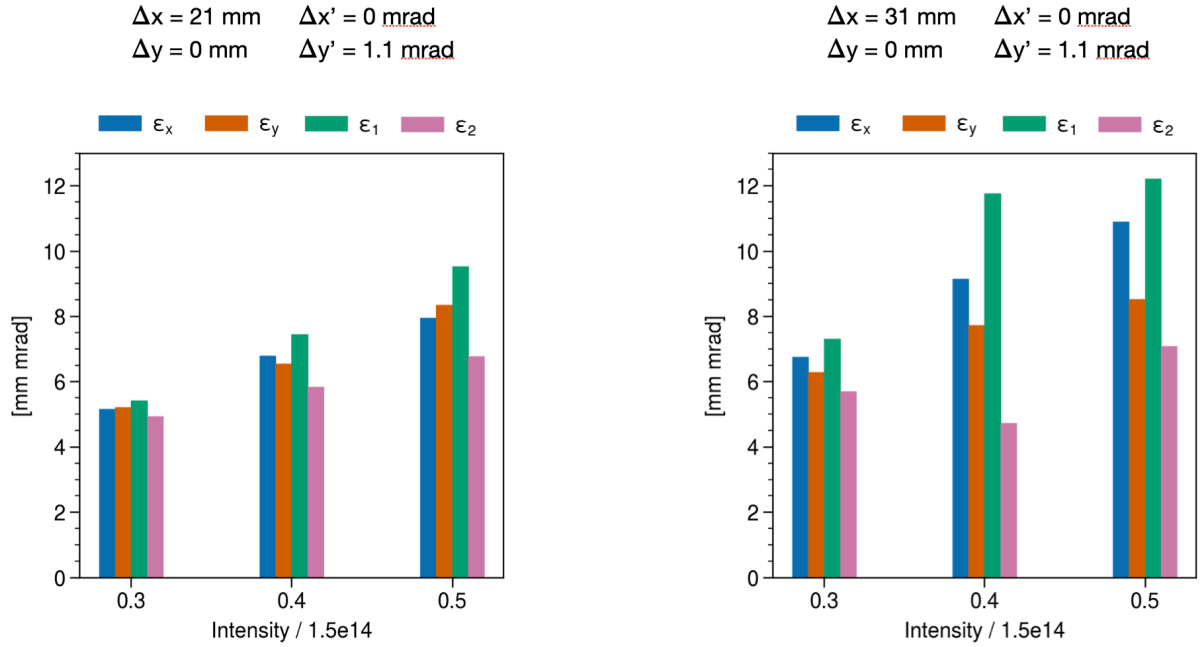


Figure 5.10: Measured emittances vs. beam intensity for two sets of injected coordinates. (Error bars not shown).

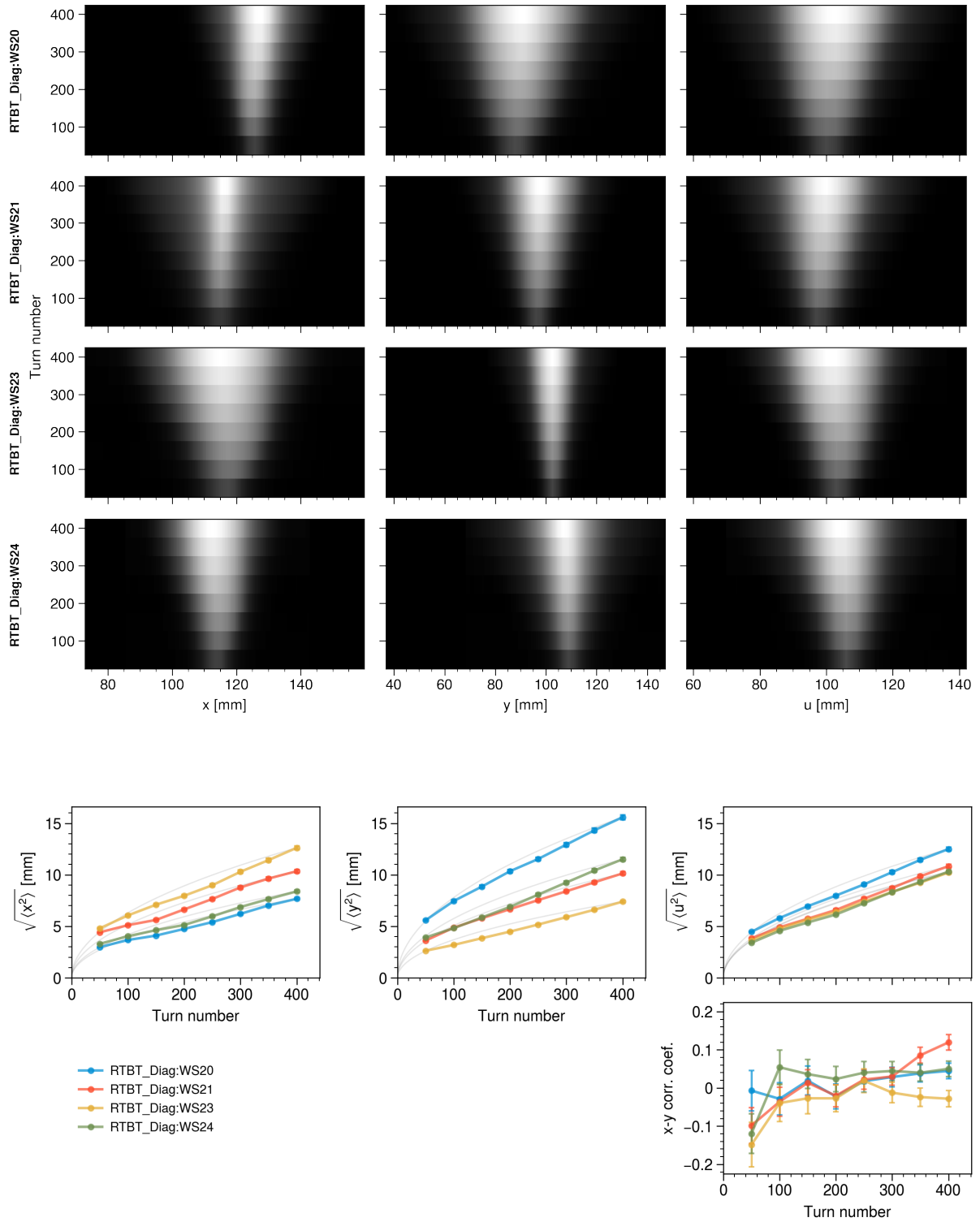


Figure 5.11: Measured wire-scanner profiles from Experiment 3.

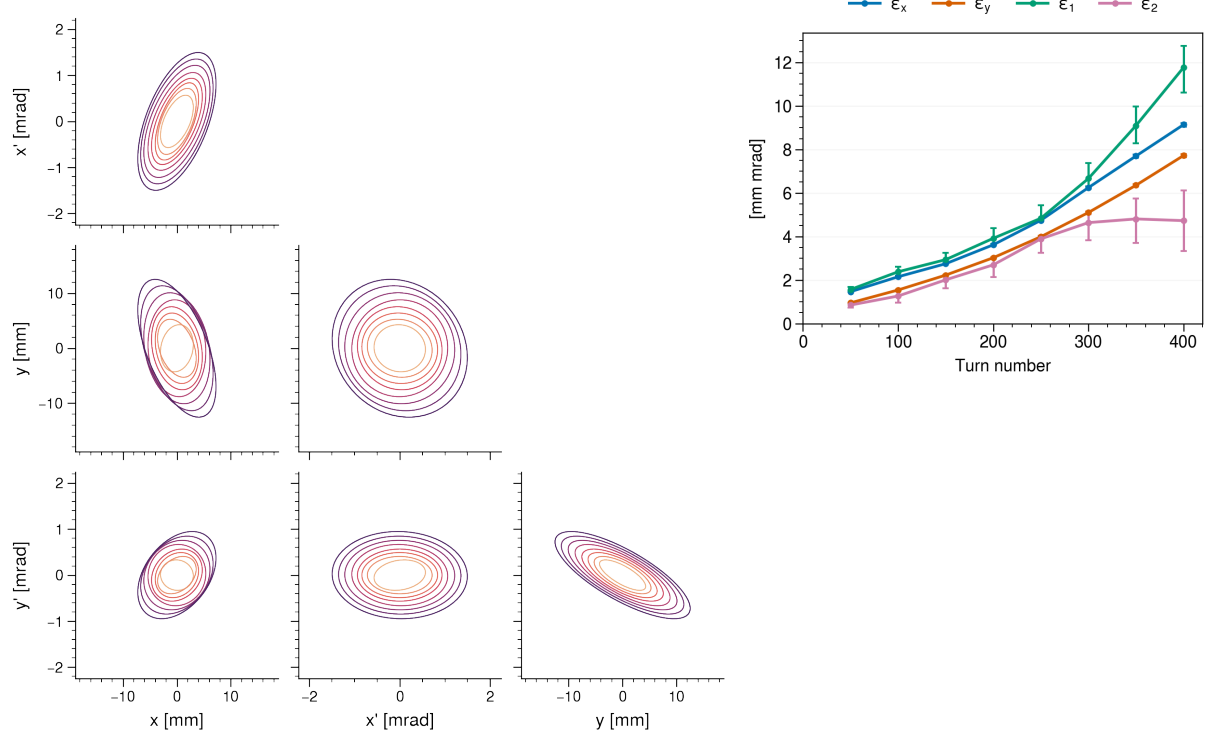


Figure 5.12: Reconstructed emittances and covariance ellipses from Experiment 3.

Although the error bars remain large, the reconstructed $\varepsilon_{1,2}$ significantly deviate from $\varepsilon_{x,y}$ after 300 turns, which manifests in the tilting of the reconstructed ellipses in the cross-plane projections. This was the largest correlation measured so far. Beam images on the target were also collected as the horizontal and vertical phase advances were scanned. The bunch length was inadvertently decreased by a factor of three beforehand, precluding direct comparison with the wire-scanner measurements. Two of the images are shown in Fig. 5.13. The x - y correlation coefficient, although small, clearly depends on the phase advances, demonstrating that there were cross-plane correlations in the beam.⁴

It is recommended that this setup be repeated in a future experiment. It should be examined whether additional slight changes to the RTBT optics can reduce the uncertainty in the fixed-optics measurement, and the multi-optics measurement should be performed on the final distribution for comparison. Additionally, images of the same beam should be collected as the phase advances are scanned at the target.

We conclude with a simulation of this experiment in Fig. 5.14. This looks closer to the best-case scenario from Chapter 3, even though solenoids are not present in the ring and the vertical injection angle is limited. Again, the predicted ratio $\varepsilon_1/\varepsilon_2$ is larger than what was measured. We repeated the simulation as the horizontal tune was varied in steps of 0.005 around its original value of 6.18. At $\nu_x = 6.2$, all cross-plane correlation in the beam was eliminated. Fig. 5.15 shows the case when $\nu_x = 6.195$. The time at which the intrinsic emittances diverge from the apparent emittances has been pushed towards the end of injection. Although it is difficult to make detailed comparisons between the measurements and the simulations due to possible differences in the ring Twiss parameters at the injection point (resulting in a different beam size for the same kicker settings), uncertainty in the total beam charge, some uncertainty in the measured injection position/angle, and relatively large error bars on the measured emittances, the similarities between the measured emittances in Fig. 5.12 and the simulated emittances in Fig. 5.15 are striking.⁵ These simulations indicate that the elliptical painting method (without solenoids) is very sensitive to the tune split in the ring, which may place a practical lower limit on the 4D emittance at this time; however, the qualitative agreement between measurement and simulation obtained here leaves open the possibility that tuning of the injection region and the addition of solenoid magnetic fields to the ring will bring the beam closer to a self-consistent state.

5.5 Summary and additional comparison between experiments

Let us make two additional comparisons between the final distributions in the three experiments. First, we reconstruct the covariance matrix at different locations in the RTBT. This will not change the emittances but will change the correlations between the phase space coordinates: the smaller the 4D emittance, the larger the variation in these correlations. See Fig. 5.16. The black lines represent the reconstructed values and

⁴The x - y correlation coefficient is calculated directly from the image.

⁵Although the tune split was measured to be ≈ 0.01 with small expected uncertainty [55], the uncertainty in the tune measurement should be re-examined and work should be done to synchronize the ring model in PyORBIT with the online model in OpenXAL. The measurement should also be repeated as the tune split is varied.

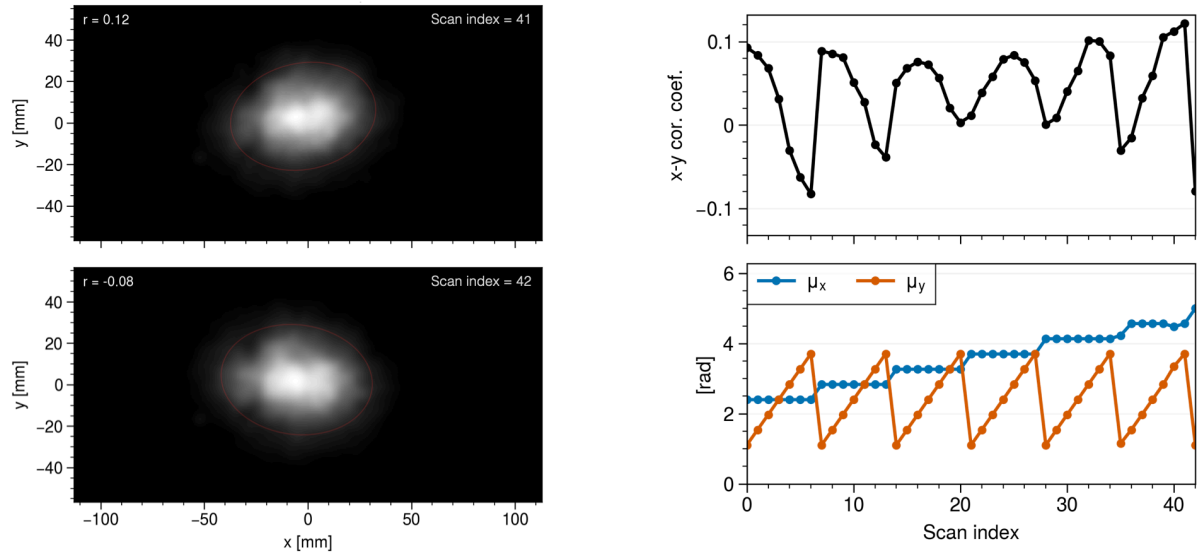


Figure 5.13: Scan of the phase advances at the target. Left: processed images on last two steps in the scan. Top right: x - y correlation coefficients computed from the images. Bottom right: Phase advances at the target.

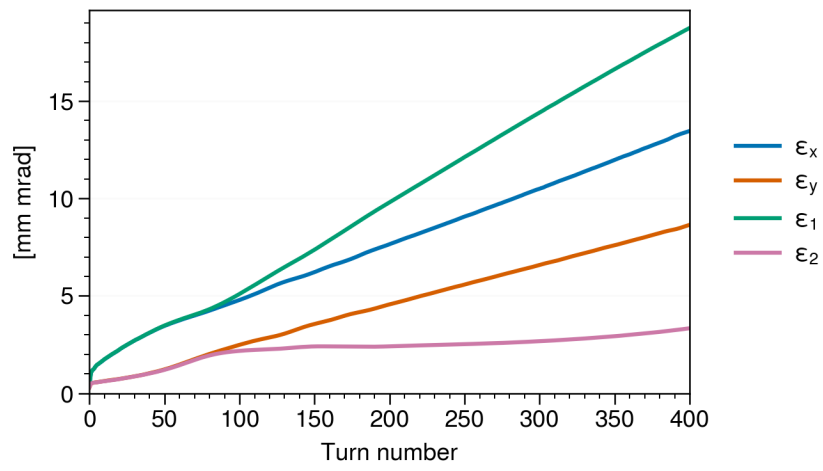
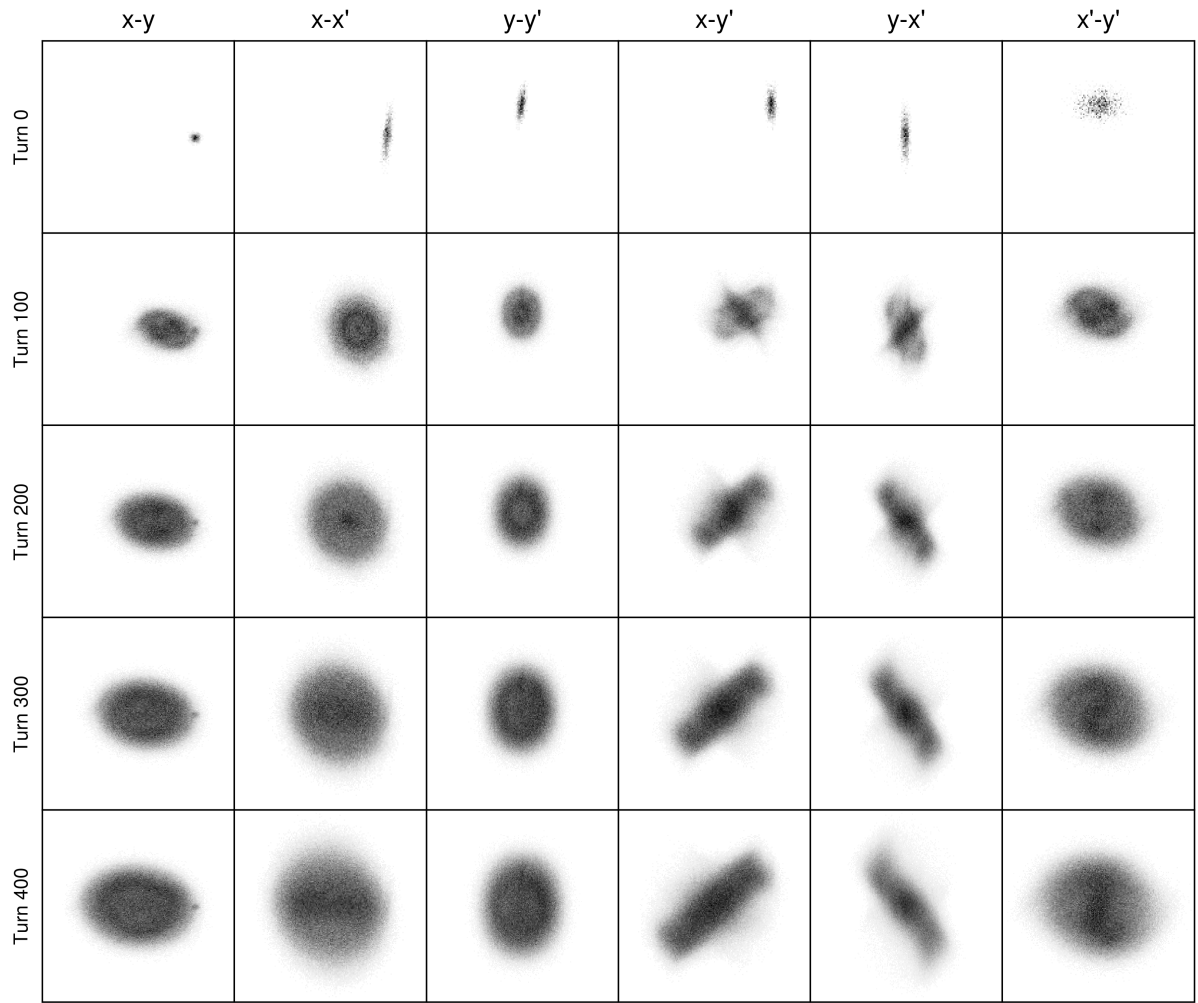


Figure 5.14: Simulation of Experiment 3.

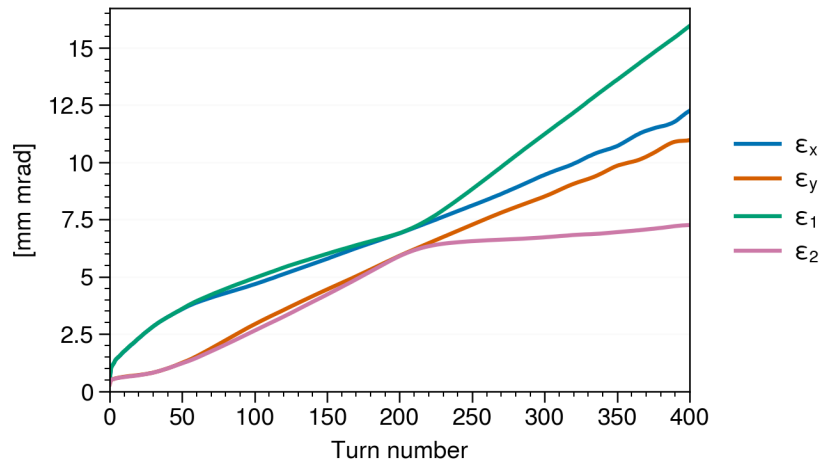
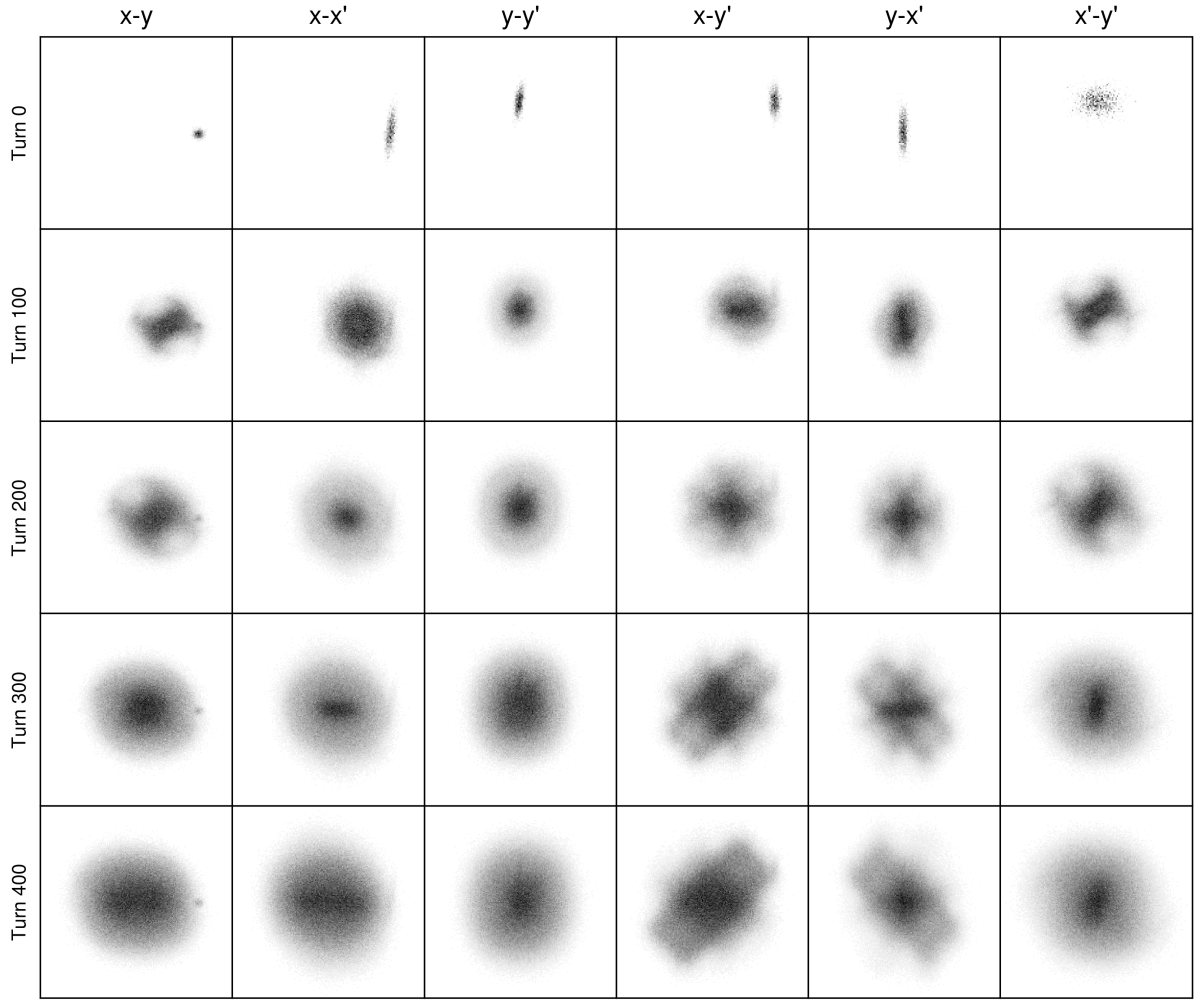


Figure 5.15: Simulation of Experiment 3 with $\nu_x = 6.195$, $\nu_y = 6.18$.

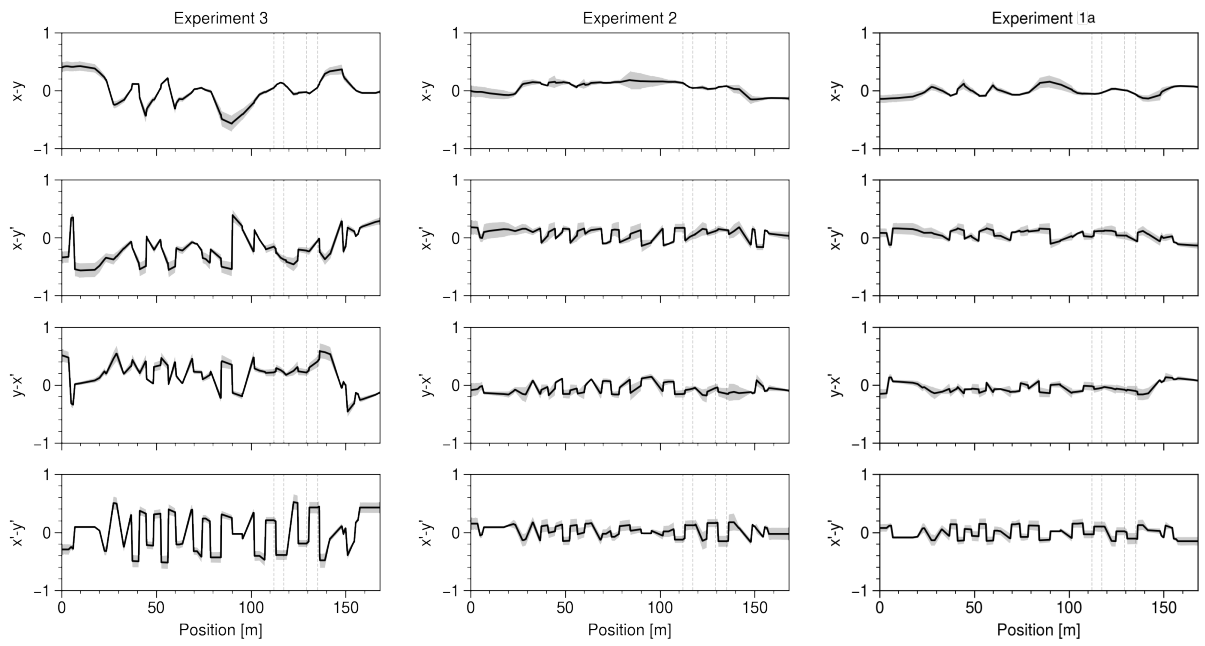


Figure 5.16: Reconstructed cross-plane correlation coefficients for Experiments 3, 2, and 1a.

the grey regions represent the standard deviation. This is simply an alternative way to visualize the measured reduction in 4D emittance in Experiment 3.

Second, although the color histograms in Fig. 5.1, Fig. 5.3, Fig. 5.6, and Fig. 5.11 are useful to show the measured beam evolution in one figure, the 1D profiles may be more difficult to interpret than a normal histogram plot. We therefore include Fig. 5.17, Fig. 5.18, Fig. 5.19, and Fig. 5.20, which show the final measured wire-scanner profiles from each experiment. Also plotted are the projections of a Gaussian distribution (red) and uniform density elliptical distribution (blue) with the same standard deviation as the RMS calculation from the measurement.

We now review the main results of the experiments in this chapter, commenting on these wire-scanner profiles along the way. Recall that our goal was to carry out the elliptical painting method in the SNS, measure the painted distribution, and compare the measurements to an ideal Danilov distribution. This has been accomplished.

In Experiment 1, the Ring Injection Control (RIC) application was tested at 1 GeV beam energy, and the beam emittance was efficiently measured throughout injection. First, correlated painting was used in Experiment 1a; the measured intrinsic emittances remained close to the apparent emittances, showing that there was very little cross-plane correlation in the beam. Second, in Experiment 1b, it was found that lowering the beam energy was necessary to inject particles onto the closed orbit, which is a necessary condition to perform the elliptical painting method. Nonetheless, setting equal tunes in the ring and varying the vertical injection angle resulted in a measured split in the intrinsic emittances. Furthermore, some of the measured wire-scanner profiles were more consistent with the projection of a uniform density elliptical distribution than with a Gaussian distribution. This gave us confidence that RIC was working as intended.

In Experiment 2, the beam energy was lowered to 0.8 GeV. BPM measurements verified that the initial kicker settings could be achieved so that particles were injected onto the closed orbit. The final kicker settings were chosen so that $(x, x', y, y') \approx (21 \text{ mm}, 0 \text{ mrad}, 0 \text{ mm}, 1.1 \text{ mrad})$. Wire-scanner measurements showed that the beam size grew with approximate square root time-dependence, as intended, but only a small split in the intrinsic emittances was measured at the end of injection. Additionally, the wire-scanner profiles were more consistent with a Gaussian distribution.

In Experiment 3, the beam size and intensity were varied. The most promising case — 20% reduction in beam intensity and 50% increase in horizontal beam size — was investigated in more detail. A larger split in the intrinsic emittances was measured during the last hundred turns of injection. Additionally, the tilt angle of the beam image on the target was shown to depend on the phase advances at the target (even though the bunch length was inadvertently reduced). Although most of the wire-scanner profiles still appeared to be consistent with a Gaussian distribution, one could argue that there were subtle differences from the previous experiment. For example, in Fig. 5.20, the horizontal projection at WS21 exhibits a sharp peak, resembling the y' projection in Fig. 3.3. (This is also present at WS20 to a lesser extent.) These features only appeared after 300 turns, when the intrinsic emittances began to split. Finally, simulations were performed to reproduce the measured emittance growth during injection. By splitting the tunes by 0.015 in the simulation, the qualitative behavior of the emittance growth was reproduced. This showed that the distribution is quite sensitive to the tune split in

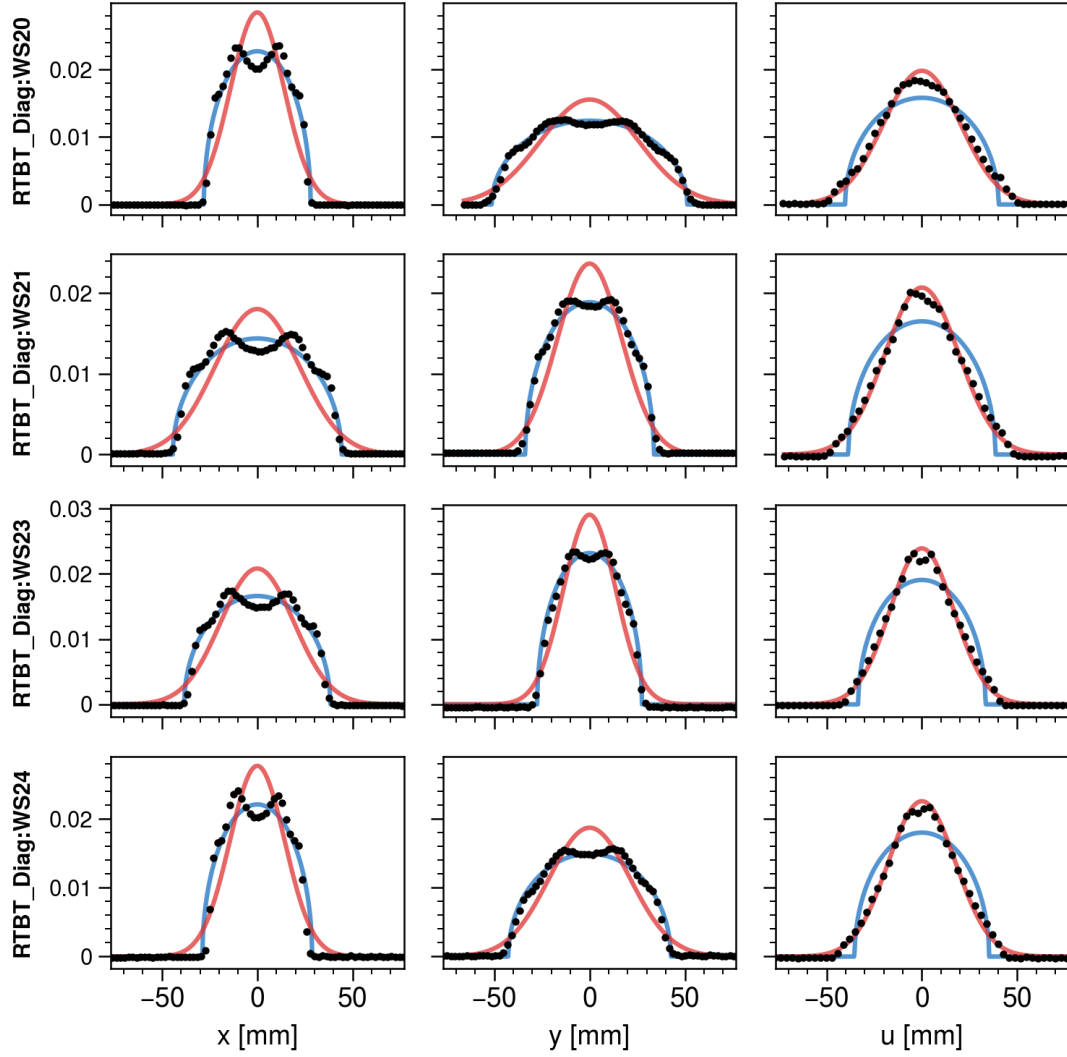


Figure 5.17: Measured wire-scanner profiles for the final distribution in Experiment 1a. Also plotted are the projections of a Gaussian distribution (red) and uniform density elliptical distribution (blue) with the same standard deviation as the rms calculation from the measurement.

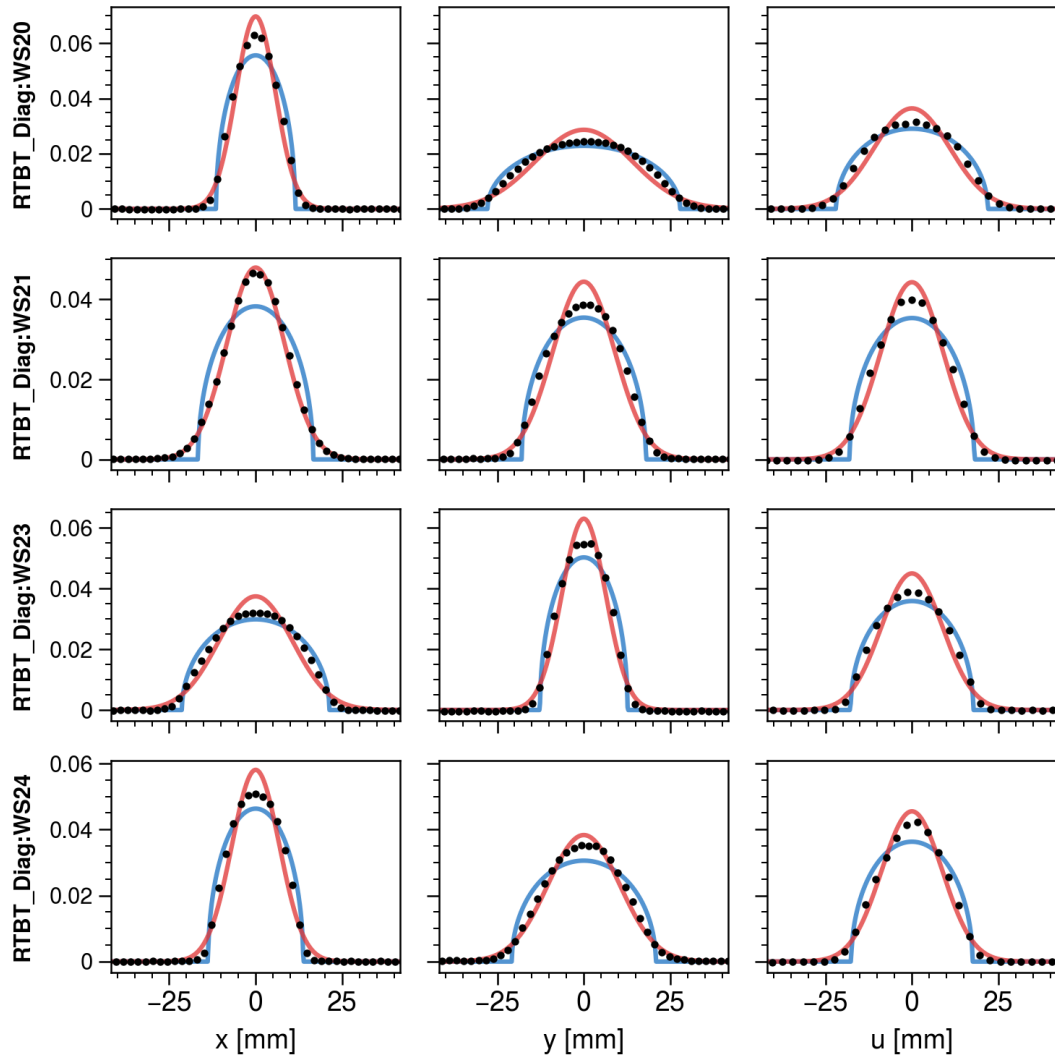


Figure 5.18: Measured wire-scanner profiles for the final distribution in Experiment 1b.

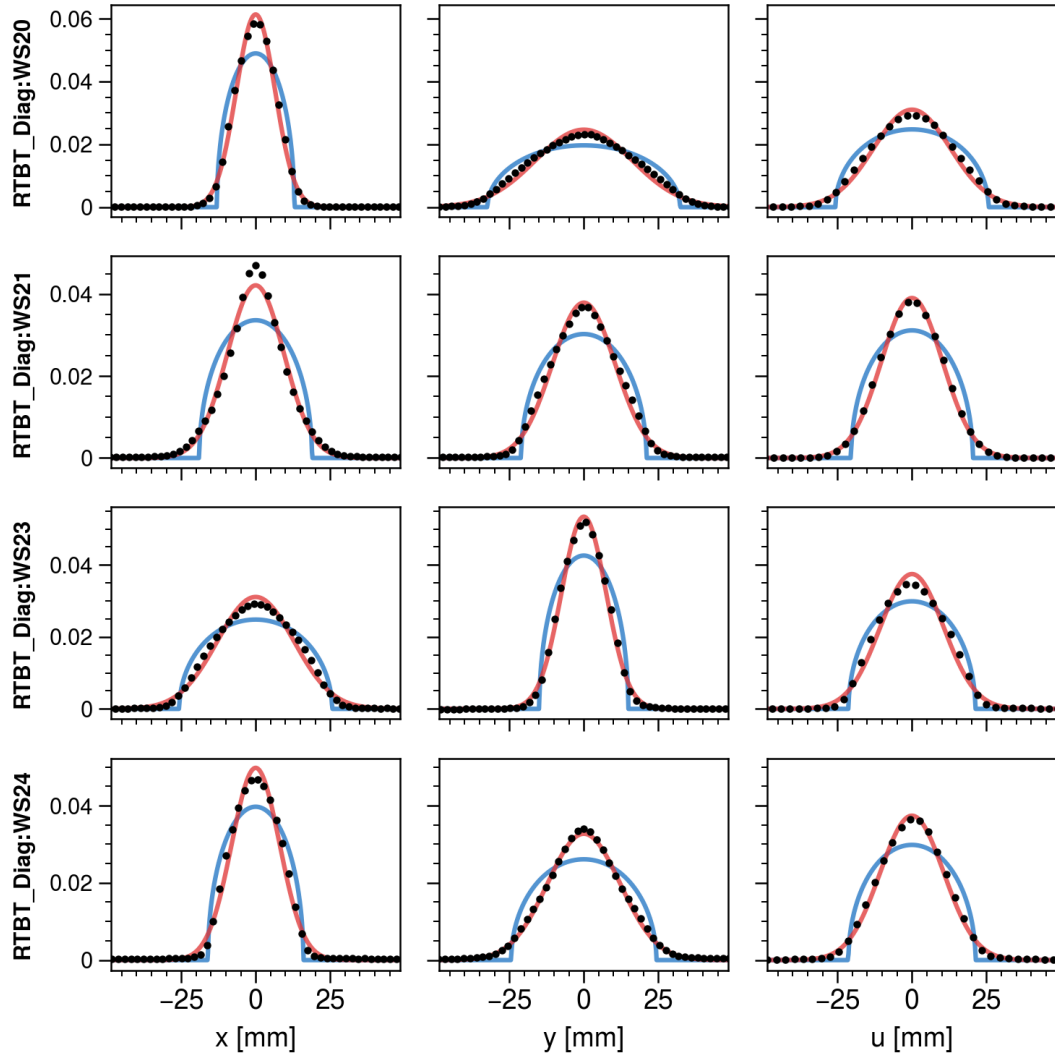


Figure 5.19: Measured wire-scanner profiles for the final distribution in Experiment 2.

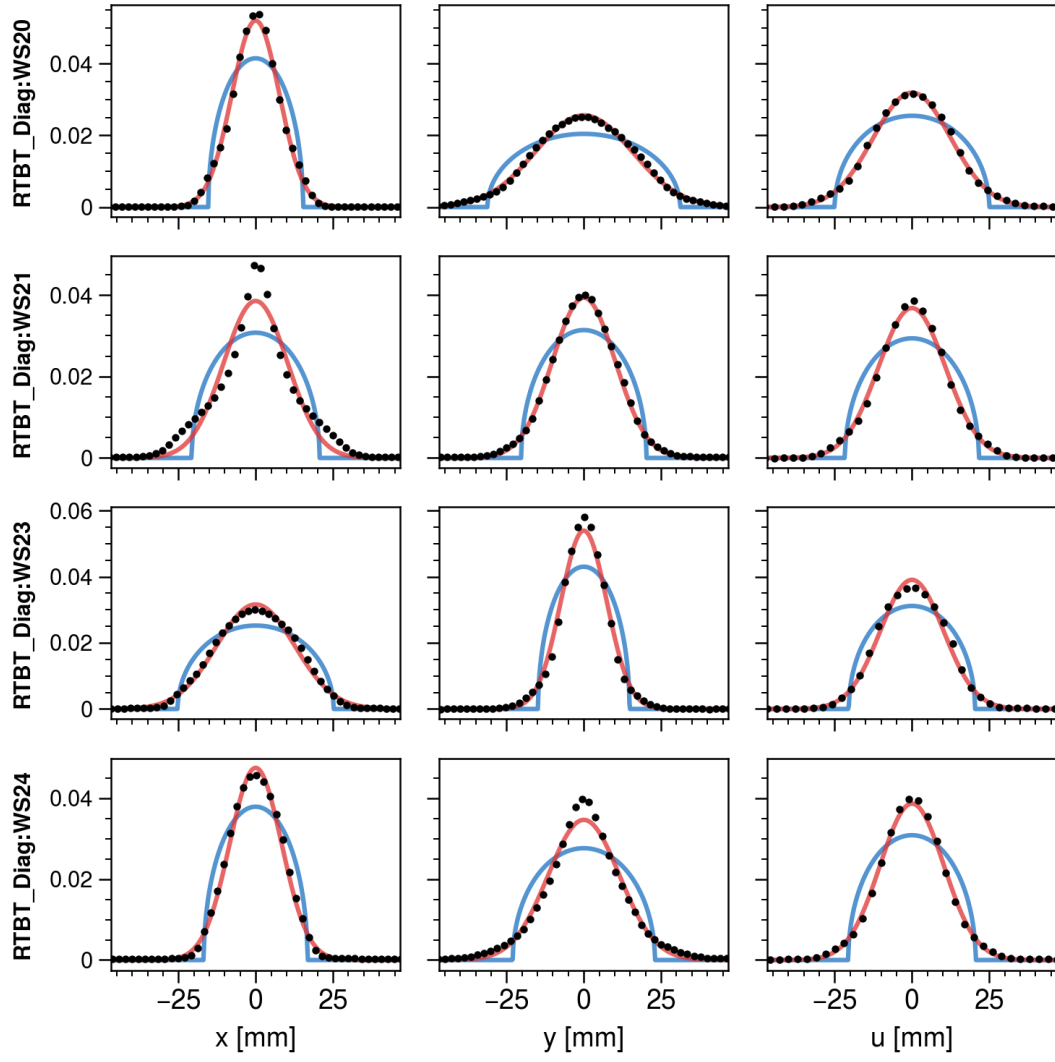


Figure 5.20: Measured wire-scanner profiles for the final distribution in Experiment 3.

the ring.

These results are promising: extensive troubleshooting has occurred during machine setup, the ring orbit has been measured and controlled, and modifications to the machine have been shown to have a positive effect on the painted distribution. Furthermore, the simulation-measurement agreement in Experiment 3 offers hope that future optimization of the SNS will result in the production of a Danilov-like distribution in the ring. One immediate need is to modify the injection region to increase the maximum vertical injection angle so that a larger, rounder beam can be painted.

Chapter 6

Conclusion

The following is a summary of this work.

- The attractive properties of the self-consistent Danilov distribution were described, as well as elliptical painting — a method to produce an approximate Danilov distribution in a circular accelerator.
- The dynamics of the Danilov distribution in linear focusing systems with space charge were investigated using envelope equations. An iterative algorithm was developed to find the matched beam in coupled or uncoupled focusing systems. The matched beam in the SNS was calculated, placing constraints on the elliptical painting method in the SNS.
- A previous simulation of elliptical painting in the SNS was revisited, more clearly defining the best expected case in the SNS in terms of the measurable beam parameters such as the intrinsic emittances.
- Methods to measure the 4D transverse phase space distribution were identified and implemented in the SNS using existing diagnostics. The first method was to use 1D projections from wire-scanners; the machine optics were adjusted to reduce the sensitivity to errors and minimize the measurement time. The second method was to use 2D projections on the target to reconstruct the phase space distribution.
- Elliptical painting was carried out for the first time in the SNS. The painted distribution was measured throughout accumulation and compared to simulations. In the final experiment, the measured four-dimensional emittance was reduced relative to a distribution produced by normal injection methods. Simulations indicated that the current setup is sensitive to the tune split in the ring, but that future modifications to the ring should bring the beam closer to a self-consistent state.

Work will continue at the SNS. It may be possible to increase the maximum injection angle by modifying the foil position, orbit corrector dipoles, Chicane dipoles, or HEBT trajectory, which would allow a larger, rounder beam to be painted and potentially allow elliptical painting at the nominal beam energy of 1 GeV. There may also be some use for the skew quadrupole correctors in the ring to reduce the required angular kicks. Additionally, solenoids, which will be installed in the SNS ring in late 2022, should stabilize the distribution against nonlinearities and reduce the sensitivity to the tune split in the ring. If the beam is measured and found to be close to a Danilov distribution,

it will be measured as it is stored in the ring after accumulation to determine its stability. Such experiments could address some of the questions in [41].

Diagnostics can be improved. The electron-scanner, which can measure turn-by-turn 1D projections of the distribution in real-time, will be recommissioned soon; although the phase space distribution cannot be reconstructed from these projections, they can be compared to the projections of a uniform density ellipse. For the beam images on the target, it would be ideal to use a camera instead of a fiber bundle and to eliminate the effect of the fiducial markers to produce a cleaner image. Additionally, the methods described in Chapter 4 can be used to reconstruct the 4D phase space distribution from the target images; unfortunately, a quality set of beam images was not collected in this work.

Related research problems may be pursued in the future. For example, it may be possible to design a lattice with so-called “circular mode optics” that keep the beam as round as possible throughout the ring, with a very small 4D emittance, using locally rotational-invariant optics [40]. This problem is being studied in a collaboration between the SNS, Argonne National Laboratory, and Fermi National Accelerator Laboratory [101]. Parallel design studies could be performed to optimize the injection region for elliptical painting. Realistic simulations of elliptical painting should be carried out with such an injection region, and losses should be compared to simulations of correlated and/or anti-correlated painting, especially as the beam intensity is scaled, to see if elliptical painting could outperform these methods.

There are also theoretical problems related to the Danilov distribution and/or self-consistent distributions in general. The stability properties of the envelope equations, studied in detail for the KV distribution in [8], have not yet been studied specifically for the Danilov distribution. There will be so-called odd modes, or tilting modes, due to the cross-plane correlation in the beam. A framework to study the stability of all second-order modes was presented in [102]. 3D envelope equations have also been studied using the KV model [103], but they are not closed since the KV distribution does not exist in three spatial dimensions; several self-consistent distributions were derived in three spatial dimensions in [9], and the resulting closed set of 3D envelope equations may be of interest. Finally, the high-order coherent instabilities present in the KV distribution can be investigated numerically for the Danilov distribution and/or other self-consistent distributions.

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Appendix A

Nonlinear resonances

The study of nonlinear resonances is important in many areas of physics [104]. A derivation of the nonlinear resonance condition in Eq. 1.15 (in one dimension) is included in this appendix. The derivation follows [105] closely.

We return to one-dimensional motion and write

$$x'' + k(s)x = \Delta B, \quad (\text{A.1})$$

where ΔB represents all the nonlinear terms in the magnetic field expansion (and also linear deviations from the design fields). The stable solution x_0 when $\Delta B = 0$ is given by Eq. (1.4). We now define

$$\phi(s) = \frac{1}{\nu} \oint \frac{ds}{\beta(s)}, \quad (\text{A.2})$$

where ν is the tune. Moving to the normalized coordinate $u = x/\sqrt{\beta}$, with $\dot{u} = du/d\phi$ we have

$$\ddot{u} + \nu^2 u = -\nu^2 \sum_{n=0}^{\infty} \left(\beta^{\frac{n+3}{2}} b_{n+1} \right) u^n. \quad (\text{A.3})$$

β (the oscillation amplitude of the unperturbed motion) and b_n (a multipole coefficient) are periodic in ϕ since they depend only on the position in the ring. Grouping these terms and Fourier expanding gives

$$\ddot{u} + \nu^2 u = -\nu^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} C_{n,k} u^n e^{ik\phi}. \quad (\text{A.4})$$

We then perturb around u_0 , the solution to the homogeneous equation, writing $u = u_0 + \delta u$, and keep only linear powers of δu .

$$\ddot{\delta u} + \nu^2 \delta u \approx -\nu^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} C_{n,k} u_0^n e^{ik\phi}. \quad (\text{A.5})$$

Noting that

$$u_0^n \propto \cos^n(\nu\phi) = \frac{1}{2^n} \sum_{m=0}^n \binom{n}{m} e^{i(n-2m)\nu\phi}, \quad (\text{A.6})$$

leads to

$$\ddot{\delta u} + \nu^2 \delta u \approx -\nu^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{m=0}^n \binom{n}{m} \frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu+k]\phi}. \quad (\text{A.7})$$

A resonance condition may occur when any of the frequency components of the driving terms are close to the tune ν ; i.e., when

$$(n - 2m)\nu + k = \pm\nu. \quad (\text{A.8})$$

Dipole terms correspond to integer tunes, quadrupole terms to 1/2 integer tunes, sextupole terms to 1/3 integer tunes, and so on. The same is true in the vertical dimension. The inclusion of coupling between x and y leads to the following resonance conditions:

$$M_x \nu_x + M_y \nu_y = N, \quad (\text{A.9})$$

where M_x , M_y , and N are integers and $|M_x| + |M_y|$ is the order of the resonance. These resonance lines are plotted in Fig. A.1.

It is helpful to visualize the particle trajectory when a resonance line is encountered; therefore, a numerical experiment from [14] is reproduced here. We consider a sextupole perturbation in an otherwise linear lattice, modeling the sextupole as a thin-lens kick. The turn-by-turn trajectories of particles with several different initial amplitudes are plotted in the top row of A.2 for different tunes ν_x . The third-order resonance leads to a well-known triangular region of stability as the tune approaches 2/3. The bottom plot reveals fourth and fifth-order resonances only obtained from second-order perturbation analysis.

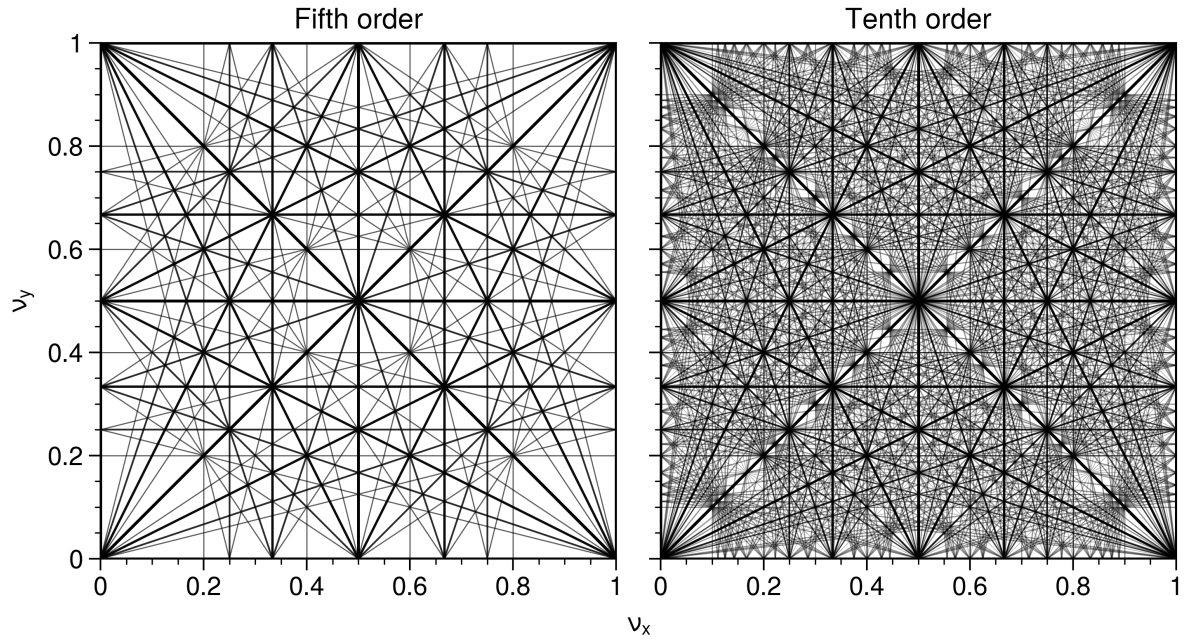


Figure A.1: Resonance lines in tune space defined by Eq. (A.9).

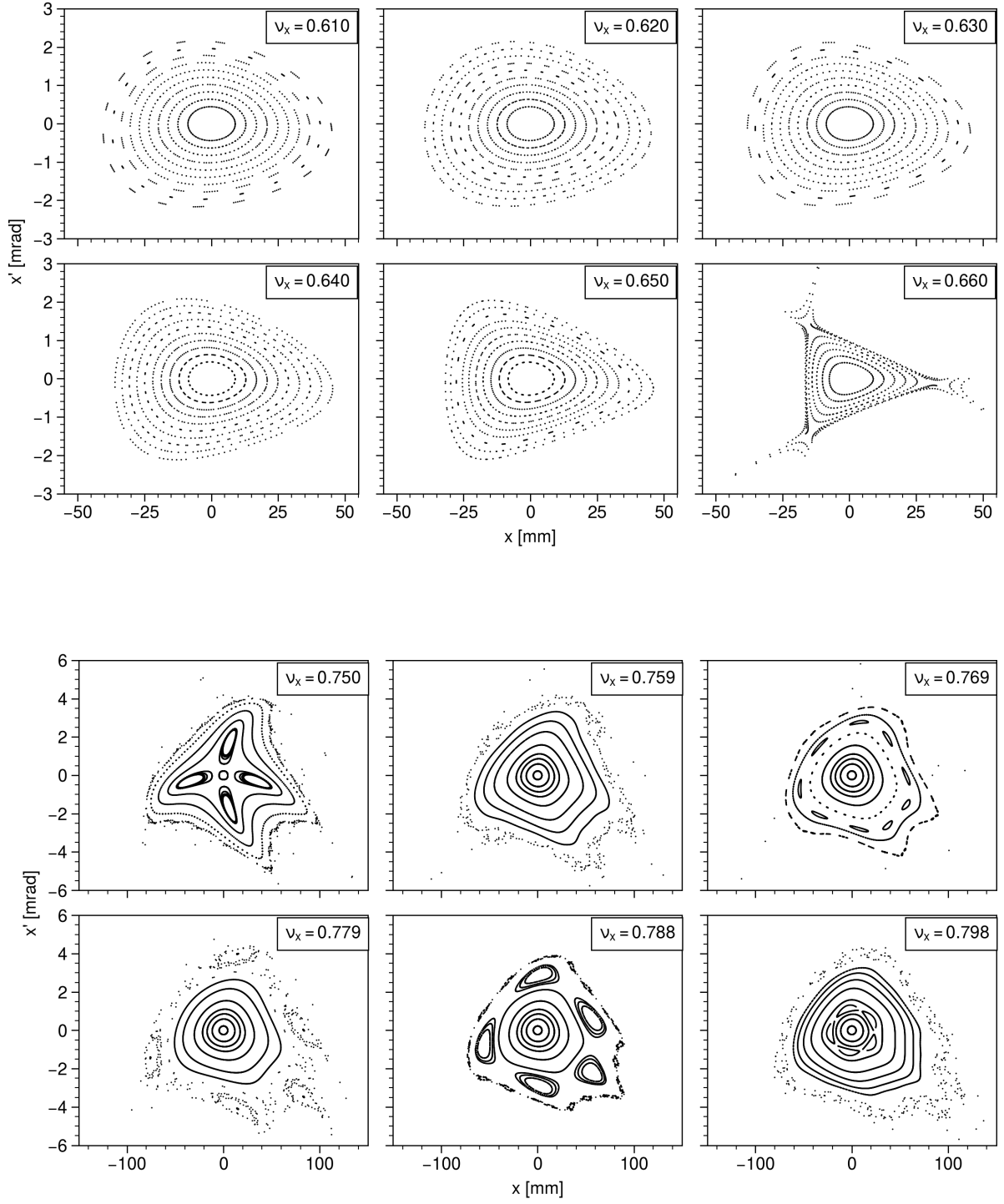


Figure A.2: Third-order (top) and fourth/fifth-order (bottom) resonances excited by a sextupole perturbation to a linear lattice. (Adapted from [14].)

Appendix B

Space charge resonances and instabilities

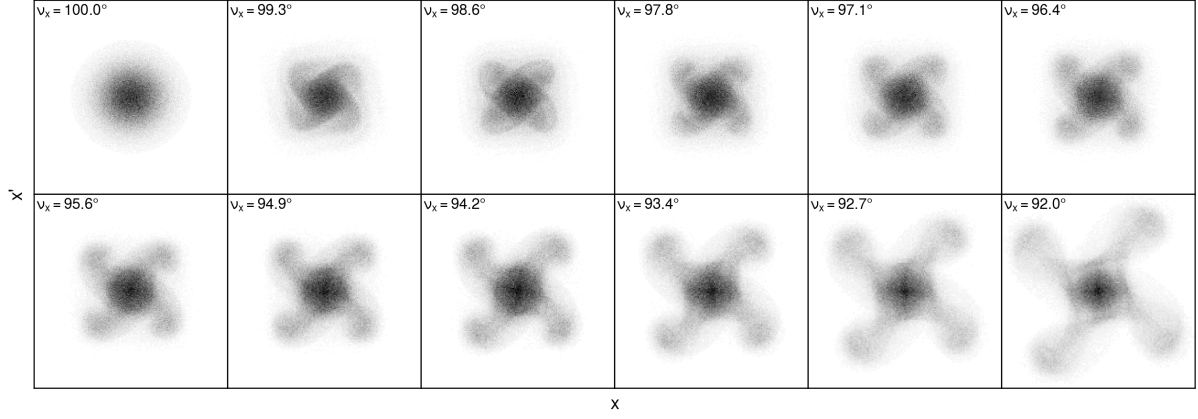
In Chapter 1, it was mentioned that space charge can be divided into two categories: incoherent effects involving the motion of single particles, and coherent effects involving the self-consistent motion of the entire beam. Here, these effects are explored by carrying out several of the numerical experiments in [4] using the PyORBIT code.

Incoherent space charge resonances

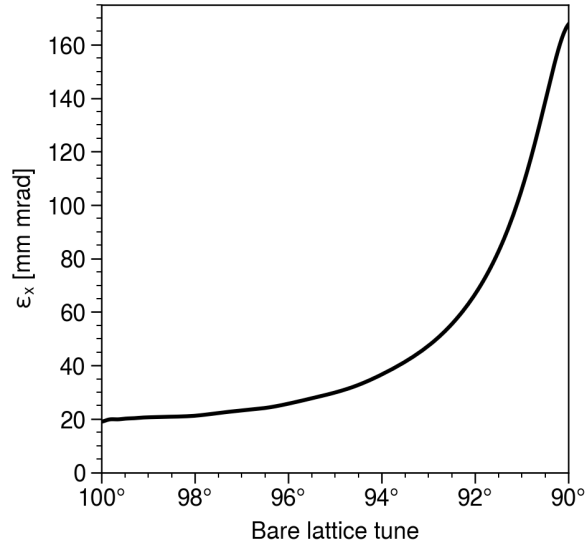
We first assume that the beam is matched — i.e., oscillates with the same periodicity as the external focusing — and track a particle in the field of the matched beam. In Chapter 1, we stated that the primary concern in circular accelerators is that the shifted single-particle tunes cross low-order machine resonance lines. But it is also possible for the beam’s electric field to drive single-particle resonances [30, 106–109]. For suppose the transverse electric field is expanded in powers of x and y : these so-called “psuedo-multipoles” can then be treated in a similar way to the magnetic multipoles in Appendix A. For illustration, we reproduce a numerical study from [4] using PyORBIT. Fig. B.1 shows a simulation of a truncated Gaussian distribution in a FODO lattice as the zero-current tune is decreased from 100° to 90° over 500 cells. The initial distribution has equal emittances in both planes and is matched to the lattice with a depressed tune of 92° . A fourth-order resonance is excited as the depressed tune approaches 90 degrees. The smooth emittance growth during most of the simulation shows that the core of the beam remains matched, justifying the use of “incoherent” describe the resonance. Higher-order resonances can also occur for different combinations of beam intensity and focusing strength.

Coherent instabilities

In [61], Hofmann et al. analytically studied perturbations of a round ($\varepsilon_x = \varepsilon_y$) KV distribution using the Vlasov equation in one of the simplest time-dependent cases: a FODO lattice with equal horizontal and vertical tunes. The result is shown in Fig. B.2, which plots the depressed tune as a function of beam intensity. Each thin line represents a different zero-current tune, and the thick lines represent regions of instability.



(a)



(b)

Figure B.1: Simulation of a truncated Gaussian distribution in a FODO lattice. The zero-current tune is decreased from 100° to 90° over 500 cells. (a) x - x' distribution. (b) RMS horizontal emittance. (Reproduced from [4].)

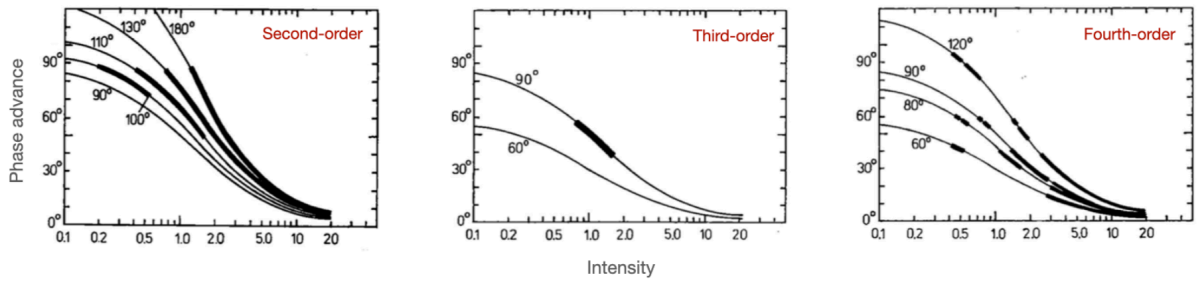


Figure B.2: Instability stopbands obtained from perturbations of a KV distribution with equal emittances in a FODO lattice. (From [61]).

The second-order instabilities involve linear forces only, so they should appear in the KV envelope equations. We use a FODO cell with a zero-current tune of 100° corresponding to the second-to-bottom line on the left-most plot in Fig. B.2. The initial distribution is first matched to the lattice, then tracked for 500 cells by integrating the KV envelope equations. Fig. B.3 shows the horizontal and vertical envelopes as the depressed KV tune is decreased from 90° to 71° , crossing the stopband. This is known as the envelope instability.

Observation of the higher-order stopbands requires PIC simulation. We choose a zero-current tune of 90° ; according to Fig. B.2, a third-order and fourth-order instability should occur at a depressed tune of 45° and 30° , respectively. Fig. B.4 shows the simulated evolution in PyORBIT for three different distributions: KV, Waterbag, and Gaussian. (Note that while the simulations in [4] used a bunched beam, coasting beams were used in this simulation.) The instabilities violently affect the KV distribution, but their effect is less pronounced in the other distributions. Thus, it is assumed that high-order coherent instabilities, while interesting, are not important in typical beams with large tune spreads.

The Danilov distribution and/or other self-consistent distributions may exhibit similar coherent instabilities. Lund, Kikuchi, and Davidson noted this in 2009 [22]:

Although the low-order properties of the KV distribution are appealing physically, the full four-dimensional structure corresponds to a singular, hyperellipsoidal shell in phase space. For strong space charge, this singular structure drives unphysical, higher-order instabilities which limit practical use of the KV distribution for initializing simulations. The KV distribution is the only exact Vlasov equilibrium known that is a function of linear-field Courant-Snyder invariants. Danilov et al. [9] investigate alternative classes of exact kinetic equilibrium distributions for linear forces. These distributions are highly singular, and based on elementary plasma physics considerations, can be expected to be unstable (similar to the KV distribution) in regimes of high space-charge intensity.

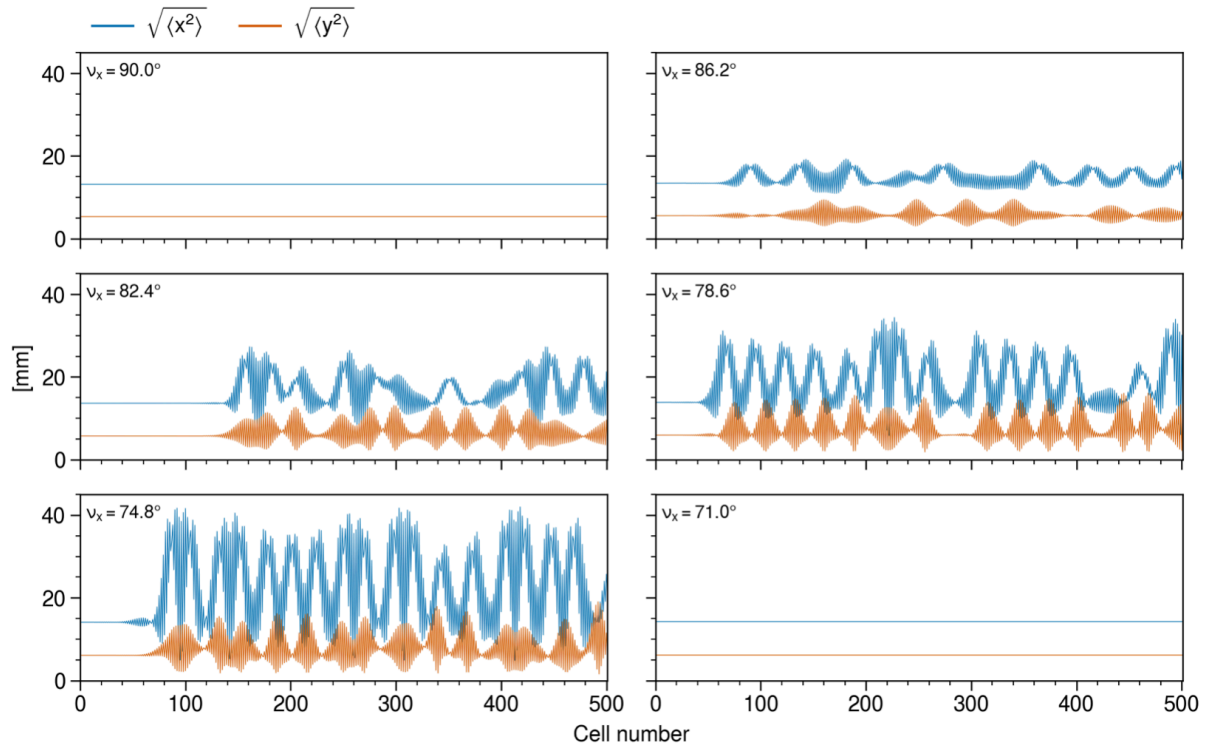


Figure B.3: Integrated KV envelope equations in a FODO lattice as the depressed KV tune ν_x is decreased. The zero-current tune is 100°.

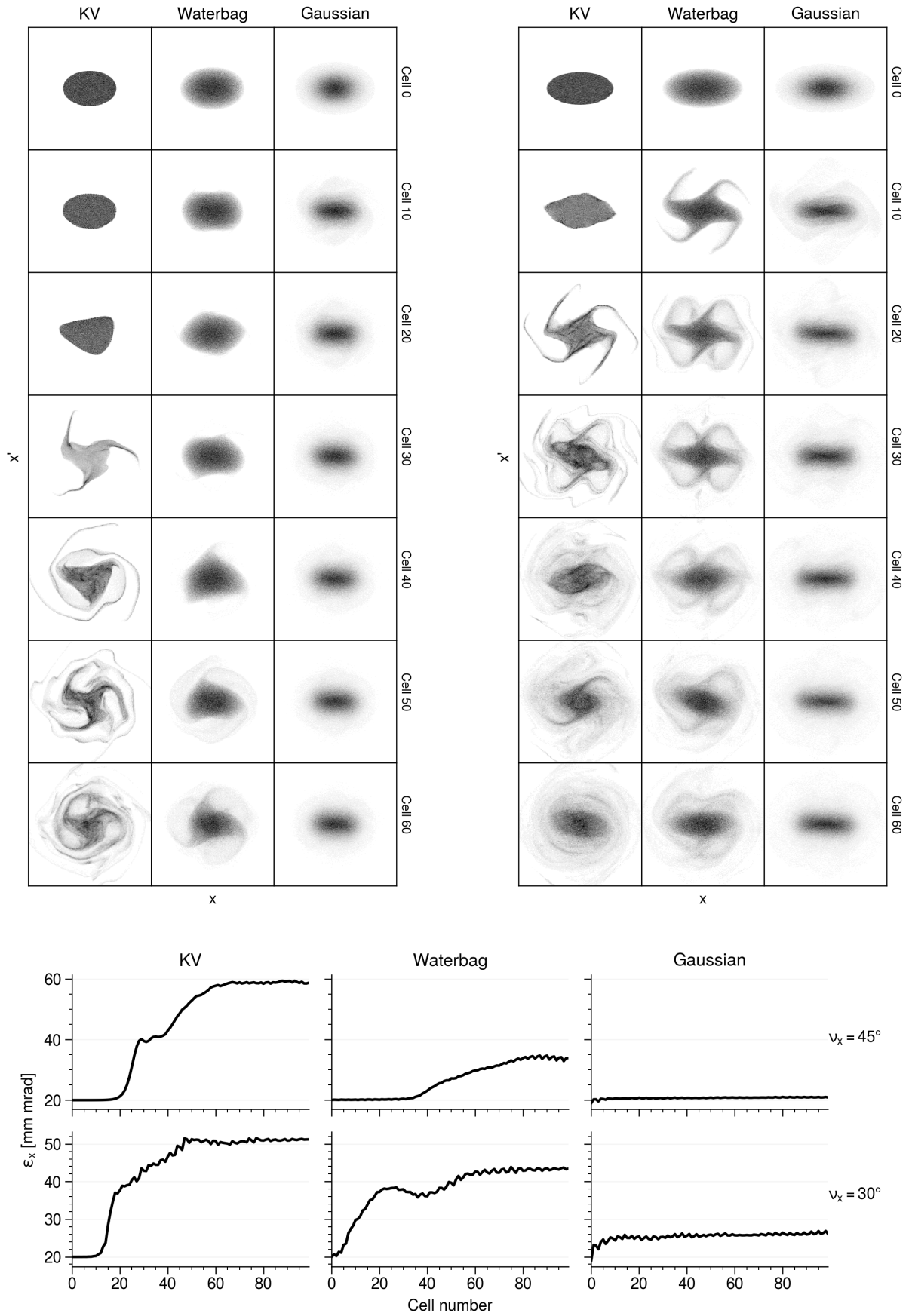


Figure B.4: Simulated Gaussian, Waterbag, and KV distributions in a FODO lattice with a zero-current tune of 90° and depressed KV tunes of $\nu_x = 45^\circ$ (top left) and 30° (top right).

Appendix C

Fringe field compensation using solenoids

In this appendix, we illuminate the finding from [10] that in an otherwise linear lattice with equal tunes, nonlinear fringe fields tend to eliminate any cross-plane correlations in the beam, but that this effect can be mitigated by solenoid magnetic fields and/or the beam's electric field.

To demonstrate this, a Danilov distribution matched to a linearized version of the SNS ring was generated. Fringe fields were turned on and the distribution was tracked without space charge. Fig. C.1 shows the turn-by-turn evolution.

There is nonlinear coupling between the horizontal and vertical motion, and the final distribution is a superposition of rotating and counter-rotating modes. We take this to be due to the difference resonance $\nu_x - \nu_y \approx 0$. In Fig. C.2, a solenoid magnet is added to the ring. The cross-plane correlations are now mostly maintained. One explanation for this phenomenon is that since the solenoid is a coupled element, it splits the tunes of the system — $\nu_{x,y} \rightarrow \nu_{1,2}$ with $\nu_1 \neq \nu_2$ — so the difference resonance is avoided.

In Fig. C.3, the simulation is repeated with the inclusion of space charge instead of the solenoid magnet. An intensity of 10^{14} is used and the bunch length is equal to the ring length. Since the beam's electric field also introduces linear coupled forces, space charge has a stabilizing effect like that of the solenoid magnetic field.

It is recommended, however, that solenoid magnets be added to the SNS ring to produce a Danilov-like distribution using the elliptical painting method. The difficulty is that fringe fields seem to dominate at the beginning of injection when the transverse displacement is maximum and before a round beam has been formed. Additionally, without a solenoid in the ring, the quality of the final distribution is very sensitive to the difference in horizontal and vertical tunes. With a solenoid in the ring, elliptical trajectories at the injection point are produced for a wide range of original horizontal and vertical tunes.

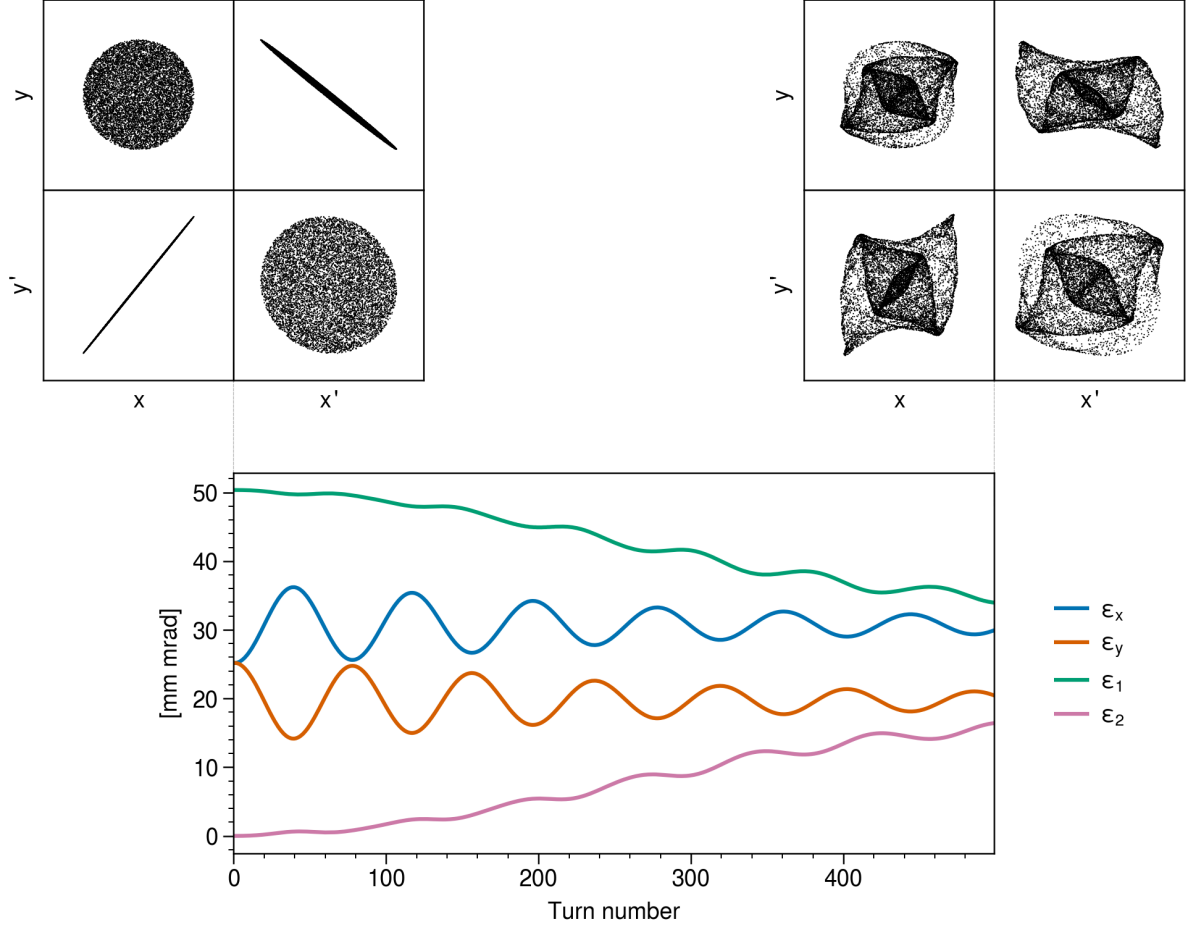


Figure C.1: Danilov distribution tracked in the SNS ring. Space charge *is not* included. Fringe fields are the only nonlinear external effect.

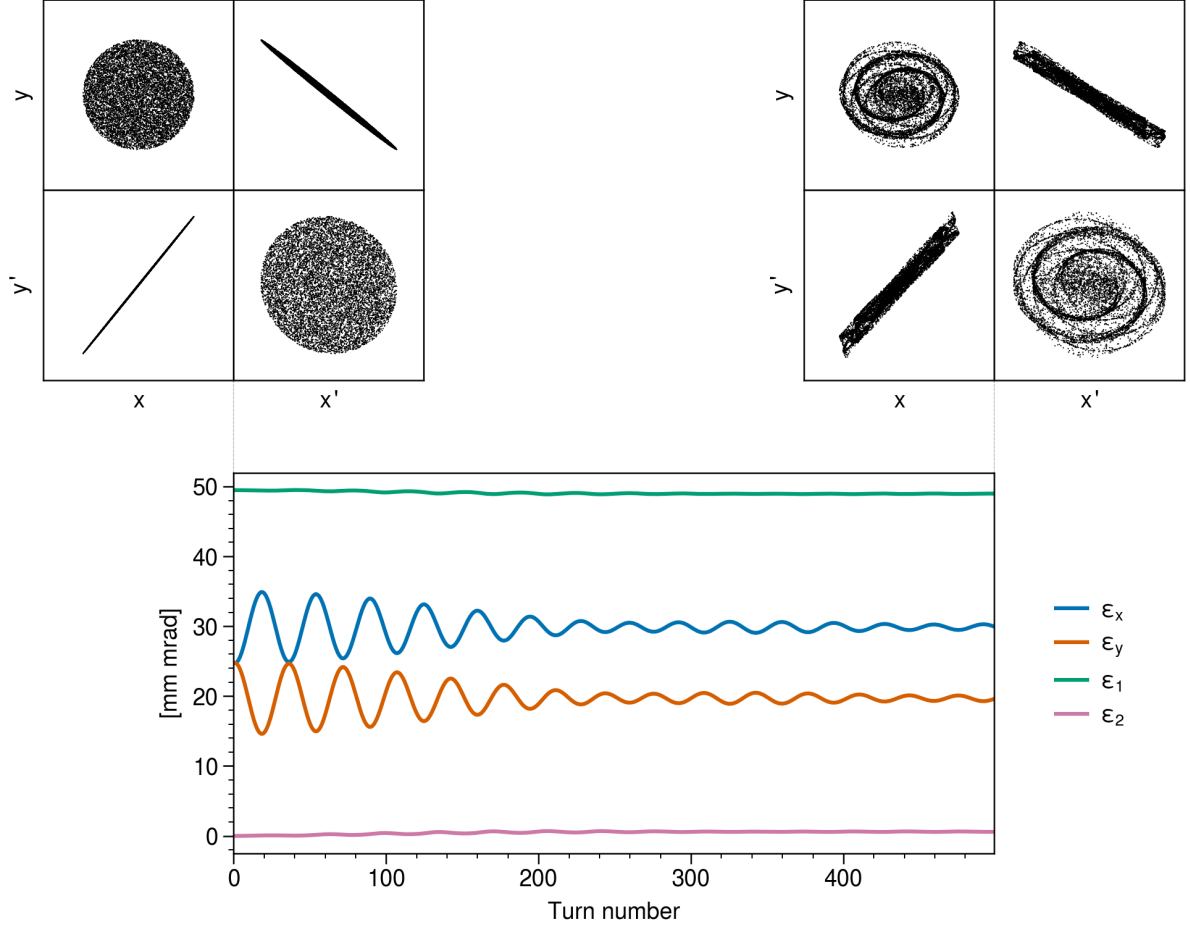


Figure C.2: Danilov distribution tracked in the SNS ring with a solenoid added to the ring. Space charge *is not* included. Fringe fields are the only nonlinear external effect.

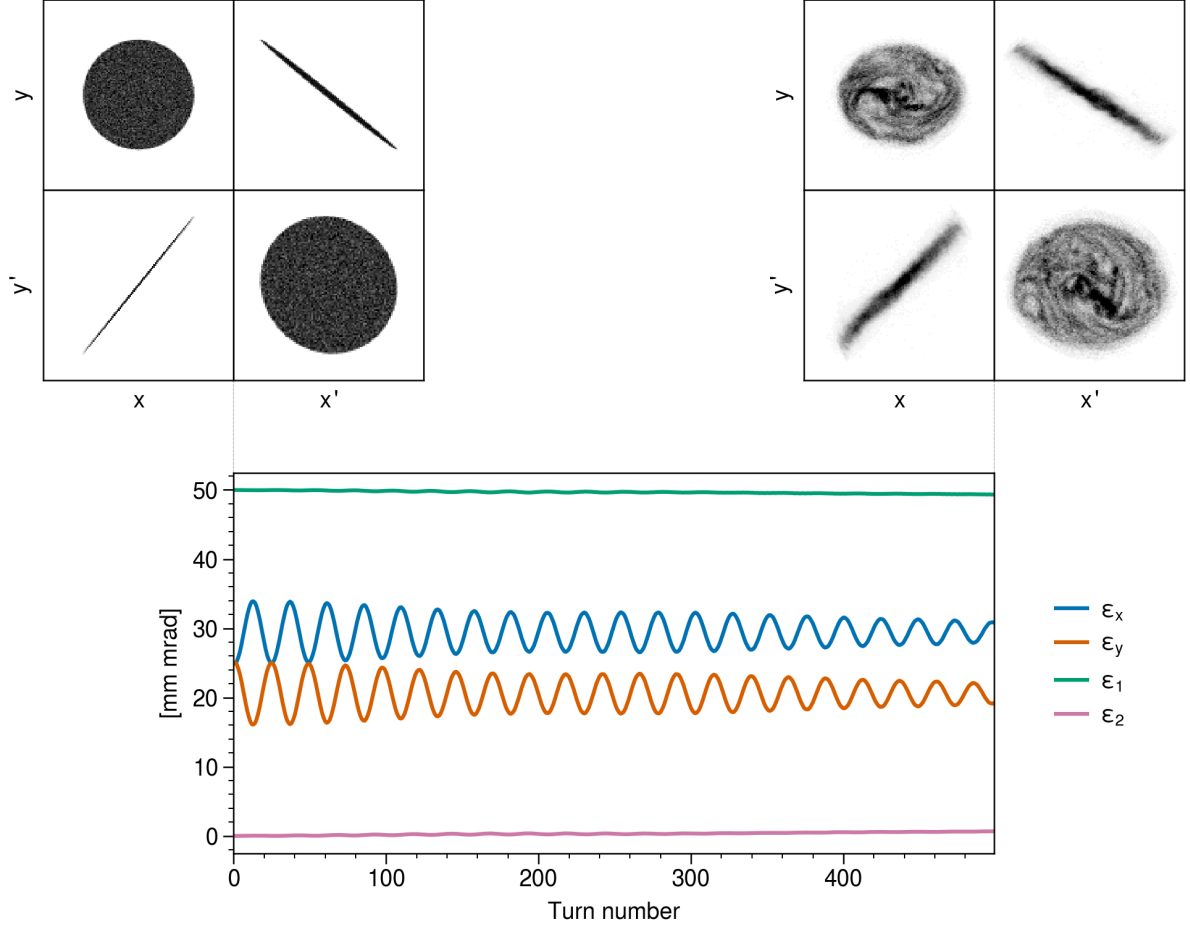


Figure C.3: Danilov distribution tracked in the SNS ring. Space charge *is* included. Fringe fields are the only nonlinear external effect.

Vita

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