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I am submitting herewith a thesis written by Slawomir Kolodynski entitled "Group Invariant Stable Processes". I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Dedication

This dissertation is dedicated to my wife Anna and my daughter Katarzyna

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Abstract

We study symmetric $\alpha-$stable (SαS) processes defined on a separable metric space T whose finite dimensional distributions are invariant under a group of transformations of T. This extends the classical notion of stationarity of stochastic processes.

Minimal integral representation on standard Borel space S of SαS process which is group stationary ($G-$stationary) in the above sense corresponds to a group of isometries of $L^\alpha(S,\mu)$. We show that for $1 < \alpha < 2$ this group of isometries is generated by a group action and a measurable cocycle on S. The pair of group action and cocycle induced this way is unique up to an isomorphism of group actions and cohomology equivalence relation for cocycles. We show that if a group action admits a Borel selector, then cocycles are cohomologous if and only if the induce the same homomorphisms on the isotropy groups. As an application we give characterization of isotropic SαS random fields on $\mathbb{R}^2$ in terms of their minimal representations.

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CHAPTER 1

Introduction

Traditional way of understanding the notion of stationarity in the theory of probability and stochastic processes is that a stochastic process $t \mapsto X(t)$ defined on $\mathbb{R}$ or $\mathbb{Z}$ is stationary if its finite dimensional distributions are the same as the finite dimensional distributions of the shifted process $t \mapsto X(t+s)$, for each $s \in \mathbb{R}$ or $\mathbb{Z}$. However, other types of invariance of stochastic processes are also of interest, such us exchangeability, isotropy, self-similarity etc. A notion that may unify these different kinds of invariance is stationarity with respect to a group action. Namely, let G be a group acting on a parameter set T. That is, there is a map $G \times T \ni (g,t) \mapsto g \cdot t \in T$, satisfying $(g_1 g_2) \cdot t = g_1 \cdot (g_2 \cdot t)$ and $e \cdot t = t$ for every $g_1, g_2 \in G$, $t \in T$ and the identity element e of G.

Definition 1.1. A stochastic process $X : T \to L^0(\Omega, P)$ is called *G-stationary* if for every $g \in G$ we have

$$(X(t),\, t \in T) \stackrel{d}{=} (X(g \cdot t),\, t \in T).$$

Here, and throughout this work, "$\stackrel{d}{=}$" means "equal in distribution" , i.e. all finite dimensional distributions are equal.

Definition 1.1 includes quite broad range of types of invariance considered in the theory of probability and stochastic processes. We give some examples to illustrate this point.

Example 1.2. Classical notion of stationarity of random fields on $\mathbb{R}^d$ corresponds to G-stationarity, where $G := \mathbb{R}^d$ acts on $T := \mathbb{R}^d$ as translations, $g \cdot t := g + t$.

Example 1.3. Instead of shifting the process by real numbers (or vectors in $\mathbb{R}^d$ for random fields) as in the classical notion of stationarity we may be interested in processes, whose distribution does not change when we shift the process by a multiplicity of an integer (or a point of $\mathbb{Z}^d$). This type of "periodic in distribution" processes correspond to $G-$stationary processes, where for fixed positive numbers $r_1, ..., r_2$ the group $G := \{(r_1 k_1, ..., r_d k_d) : (k_1, ..., k_d) \in Z^d\}$, acts on $T := \mathbb{R}^d$ as translations.

Example 1.4. Suppose X is a random field on $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ a fixed vector. We can consider the type of invariance in which the distribution of fields of the form $t \mapsto X(t + \lambda a)$, where $\lambda \in \mathbb{R}$, are the same as the distribution of $t \mapsto X(t)$. In this case we have the group $G := \mathbb{R}$ acting on the index set $T := \mathbb{R}^d$, $\lambda \cdot t := t + \lambda a$.

Example 1.5. Let G be the group of permutations of the set $T = \{1, .., n\}$. A random vector $(X_t,\ t = 1, ..., n)$ is called *symmetrically dependent* if $(X_{\sigma(t)},\ t = 1, ..., n) = (X_t,\ t = 1, ..., n)$ for every $\sigma \in G$. Here $\sigma \cdot t := \sigma(t)$.

Example 1.6. Let G be a group of bijections $\sigma : \mathbb{N} \to \mathbb{N}$, such that for every $\sigma \in G$ we have $\sigma(n) = n$ for all but a finite number of n. Then exchangeability of a sequence of random variables $X = (X_n, n \in \mathbb{N})$ means that the process X is G-stationary, where the action of G on $\mathbb{N}$ is $\sigma \cdot n := \sigma(n)$.

Example 1.7. Set $G := \mathbb{R}$ and $T := \mathbb{R} \times \{1, ..., d\}$. Then a random process $(X_t^i, t \in \mathbb{R}, i = 1, ..., d)$ can be thought of as a vector-valued process on $\mathbb{R}$. Stationarity of such process corresponds to $G-$stationarity, where we set the action η on T by $\eta(g, (t, i)) = (g + t, i)$, $g \in G = \mathbb{R}$, $t \in \mathbb{R}$, $i = 1, ..., d$.

Example 1.8. A class of random fields on $\mathbb{R}^d$ which often comes up in applications is the class of *isotropic* random fields. These are the fields $X : \mathbb{R}^d \to L^0(\Omega, P)$ such that

$$(X(g \cdot t), t \in \mathbb{R}^d) \stackrel{d}{=} (X(t), t \in \mathbb{R}^d)$$

for every $g \in \mathbf{SO}(n)$. We discuss isotropic SαS random fields in Chapter 7.

Example 1.9. Expanding Example 1.8 we can consider random fields on $\mathbb{R}^d$ whose distribution does not change when under the action of the group of rigid motions (generated by rotations and translations).

Example 1.10. Let $d \in \mathbb{N}$ and T be the Schwartz space of all infinitely differentiable rapidly decreasing functions on $\mathbb{R}^d$ and denote by T' the space real generalized functions (distributions) which is dual to T. Let Q be a probability measure on T'. Then Q defines a real

valued random process (field) $(X_\phi, \phi \in T)$ indexed by T with finite dimensional distributions given by

$$P((X_{\phi_1}, \ldots, X_{\phi_k}) \in B_k) = Q(\{F \in T' : (F(\phi_1), \ldots, F(\phi_k)) \in B_k\}),$$

where B_k is a Borel subset of $\mathbb{R}^k$. Dobrushin in [**6**] considered random processes of this type which are $G-$stationary with respect to group actions $\eta_i : G_i \times T \to T$, $i = 1, 2, 3$, where
$(\eta_1(a, \phi))(x) = \phi(x + a)$, $x \in \mathbb{R}^d$, $a \in G_1 := \mathbb{R}^d$, (additive), $\phi \in T$,
$(\eta_2(b, \phi))(x) = \phi(b(x))$, $x \in \mathbb{R}^d$, $b \in G_2 :=$ group of Euclidean rotations of $\mathbb{R}^d$, $\phi \in T$,
$(\eta_3(\lambda, \phi))(x) = \lambda^{-d}\phi(x/\lambda)$, $x \in \mathbb{R}^d$, $\lambda \in G_3 := (0, \infty)$ (multiplicative), $\phi \in T$.

The class of stable, non-Gaussian stochastic processes is especially convenient for studying the general notion of stationarity given in Definition 1.1. The tool of minimal integral representations of such processes introduced by Hardin in [**12**] and developed by Rosinski in [**18**], [**19**] and [**20**] makes studying $G-$invariant SαS processes equivalent to studying the groups of isometries of L^p spaces. That such groups induce a pair consisting of a group of transformations of the underlying measure space and a cocycle (see Chapter 4 for definitions) was first proven by Rosinski in [**18**, Th. 3.1] and allowed for important insight into the structure of stationary (in the traditional sense) SαS processes. The proof of that fact was originally given for the additive groups $\mathbb{R}$ and $\mathbb{C}$. We provide the general version of [**18**, Th. 3.1] in Theorem 4.5. The rest of Chapter 4 contains the general versions of

some other results from [18]. We hope that the general versions prove to be as useful in revealing the structure of $G-$stationary processes, whose examples are given above, as the original versions were in the case of stationary processes.

In Chapter 5 we propose a scheme for decomposition of $G-$invariant SαS processes in a special case when G is a compact Polish group. The situation when the G is compact is much easier than the general locally compact case. This allows to describe more closely the elements of the decomposition. In Chapter 6 we define and investigate basic properties of the analogs of the notions of moving average and mixed moving average in the context of $G-$stationarity. Chapter 7 is an application of the scheme from Chapter 5 to a special case of SαS isotropic random fields on $\mathbb{R}^2$ (Example 1.8). Finally in the Appendix we provide the survey of the results in the theory of $G-$spaces needed for studying $G-$stationary SαS processes or groups of isometries of L^p spaces. We also generalize the results by Varajaradan [26] on the structure of cohomology classes of cocycles in transitive $G-$spaces to the $G-$spaces which admit a Borel transversal.

Now we will set up some basic notational conventions used used in this work.

$\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural $(1, 2, ...)$, real and complex numbers, respectively. $\Re$, $\Im$ denotes the real and imaginary parts of a complex number. (Ω, P) is the underlying probability space with probability measure P. $L^0_{\mathbb{R}}(\Omega, P)$ and $L^0_{\mathbb{C}}(\Omega, P)$ are the spaces of real

and complex valued random variables. Many statements in the present work have two almost identical versions, one for real valued random variables or functions and one for complex valued random variables and functions. To avoid repetitions we will use the symbol $L^0(\Omega, P)$ to mean both $L^0_{\mathbb{R}}(\Omega, P)$ and $L^0_{\mathbb{C}}(\Omega, P)$, depending on the context.

A *measurable space* is a pair $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a $\sigma-$algebra of subsets of X. A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where $(X, \mathcal{A})$ is a measurable space and $\mu : \mathcal{A} \to [0, \infty)$ is a measure on $(X, \mathcal{A})$. We will often write a measure space as a pair (X, μ).

If $(X, \mathcal{A}, \mu)$ is a measure space, $\alpha > 0$, $\mathcal{L}^\alpha_{\mathbb{R}}(X, \mu)$ denotes the space of $\mathcal{A}-$measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X |f(x)|^\alpha \mu(dx) < \infty.$$

$L^\alpha_{\mathbb{R}}(X, \mu)$ is the space of equivalence classes in $\mathcal{L}^\alpha_{\mathbb{R}}(X, \mu)$ of the relation $f_1 \sim f_2 \Leftrightarrow \mu(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$. Similarly as with $L^0(\Omega, P)$ we introduce symbols $L^\alpha_{\mathbb{C}}(X, \mu)$, $\mathcal{L}^\alpha_{\mathbb{C}}(X, \mu)$, $L^\alpha(X, \mu)$ and $\mathcal{L}^\alpha(X, \mu)$. The distinction between $L^\alpha(X, \mu)$ and $\mathcal{L}^\alpha(X, \mu)$ is especially important in Chapter 3.

We denote by S^d the d-dimensional sphere

$$S^d = \left\{ (x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1} : \sum_{j=1}^{d+1} x_j^2 = 1 \right\}$$

in $\mathbb{R}^{d+1}$. The group of real $n \times n$ matrices R such that $RR^T = I$ and $\det R = 1$ is called $\mathbf{SO}(n)$. We can identify $\mathbf{SO}(2)$ with the unit circle in $\mathbb{C}$ with usual multiplication of complex numbers. Matrices in $\mathbf{SO}(3)$

represent rotations of the space $\mathbb{R}^3$ or S^2 about an axis passing through the origin.

CHAPTER 2

Stable random vectors, processes and measures

Definition 2.1. A random vector $\mathbb{X} = (X_1, ..., X_d)$ is called *stable* if for any positive numbers A, B there is a positive number C and vector $D \in \mathbb{R}^d$ such that

$$A\mathbb{X}^{(1)} + B\mathbb{X}^{(2)} \stackrel{d}{=} C\mathbb{X} + D, \tag{2.1}$$

where $\mathbb{X}^{(1)}$ and $\mathbb{X}^{(2)}$ are independent copies of $\mathbb{X}$. $\mathbb{X}$ is called *strictly stable* (2.1) holds with $D = 0$ for any $A > 0$, $B > 0$ and some $C > 0$. $\mathbb{X}$ is called symmetric stable if it is stable and satisfies $\mathbb{X} \stackrel{d}{=} -\mathbb{X}$.

Proposition 2.2. [**22**, Th. 2.1.2]*Let $\mathbb{X} = (X_1, ..., X_d)$ be a stable vector in $\mathbb{R}^d$. Then there is a constant $\alpha \in (0, 2]$ such that in (2.1)*

$$C = (A^\alpha + B^\alpha)^{1/\alpha}.$$

Moreover, $\mathbb{X}$ is strictly stable (resp. symmetric) if and only if any linear any linear combination $Y = \sum_{i=1}^n a_i X_i$ is strictly stable (resp. symmetric) in $\mathbb{R}^1$.

The constant α in Proposition (2.2) is called the *index of stability* of $\mathbb{X}$. Stable vectors with index of stability α are called *α-stable*.

The following theorems give the characterization of α-stable, strictly α-stable and symmetric α-stable (SαS) random vectors in terms of their characteristic function.

Theorem 2.3. [**22**, Th. 2.3.1]*Let* $0 < \alpha < 2$.

i) A random vector $\mathbb{X} = (X_1, X_2, ..., X_d)$ *is* α*-stable if and only if there exist a finite measure* Γ *on the unit sphere* S^{d-1} *of* $\mathbb{R}^d$ *and a vector* $\mu \in \mathbb{R}^d$ *such that for any* $a \in \mathbb{R}^d$

a) if $\alpha \neq 1$

$$\mathbb{E}(\exp i(a \cdot \mathbb{X})) =$$

(2.2)

$$\exp\left(-\int_{S^{d-1}} |a \cdot \theta|^\alpha \left(1 - i \operatorname{sign}(a \cdot \theta) \tan\frac{\pi\alpha}{2}\right) \Gamma(d\theta) + i(a \cdot \mu)\right)$$

b) if $\alpha = 1$

$$\mathbb{E}(\exp i(a \cdot \mathbb{X})) =$$

(2.3)

$$\exp\left(-\int_{S^{d-1}} |a \cdot \theta| \left(1 + i\frac{2}{\pi} \operatorname{sign}(a \cdot \theta) \ln|a \cdot \theta|\right) \Gamma(d\theta) + i(a \cdot \mu)\right).$$

The pair (Γ, μ) *is unique.*

Theorem 2.4. [**22**, Th. 2.4.1] *Let* $0 < \alpha < 2$, $\alpha \neq 1$. *An* α*-stable random vector* $\mathbb{X} = (X_1, X_2, ..., X_d)$ *is strictly* α*-stable if and only if* $\mu =$

0 in 2.2. A 1−stable random vector is strictly stable iff $\int_{S^{d-1}} \theta_k \Gamma(d\theta) = 0$ *for* $k = 1, 2, ..., d$ *in (2.3).*

Theorem 2.5. [22, Th. 2.4.3] $\mathbb{X}$ *is a symmetric* α*-stable random vector in* $\mathbb{R}^d$ *if and only if there exist a symmetric finite measure* Γ *on the unit sphere* S^{d-1} *such that for any* $a \in \mathbb{R}^d$

$$\mathbb{E}(\exp i(a \cdot \mathbb{X})) = \exp\left(-\int_{S^{d-1}} |a \cdot \theta|^\alpha \Gamma(d\theta)\right).$$

The following important corollary follows from Theorem 2.3:

Corollary 2.6. *Let* $\mathbb{X}$ *be an* $\alpha-$*stable random vector,* $0 < \alpha < 2$, $\alpha \neq 1$ *and* μ *as in Theorem 2.3. Then* $\mathbb{X} - \mu$ *is strictly* $\alpha-$*stable.*

Suppose $d = 1$ in Theorem 2.3. Then Γ is a measure on $S^0 = \{-1, 1\}$ and $\mu \in \mathbb{R}$. Writing

$$\sigma := (\Gamma(\{1\}) + \Gamma(\{-1\}))^{1/\alpha}, \quad \beta := \frac{\Gamma(\{1\}) - \Gamma(\{-1\}}{\Gamma(\{1\}) + \Gamma(\{-1\}}$$

we obtain a special case of Theorem 2.3 when $\mathbb{X}$ is a random variable.

Theorem 2.7. *Let* $0 < \alpha < 2$. *A random variable* X *is* α*-stable iff there are parameters* $\sigma \geq 0$, $-1 \leq \beta \leq 1$ *and* $\mu \in \mathbb{R}$ *such that its characteristic function has the following form:*
a) if $\alpha \neq 1$

$$\mathbb{E}(\exp iaX) = \exp\left(-\sigma^\alpha |a|^\alpha \left(1 - i\beta(\operatorname{sign} a) \tan\frac{\pi\alpha}{2}\right) + ia\mu\right),$$

b) if $\alpha = 1$

$$\mathbb{E}(\exp iaX) = \exp\left(-\sigma|a|\left(1 - i\beta\frac{2}{\pi}(\operatorname{sign} a)\ln|a|\right) + ia\mu\right).$$

The parameters σ, μ *are unique and* β *is unique if* $\alpha < 2$.

Notation 2.8. Since by Theorem 2.7 the distribution of an α-stable random variable X is characterized by parameters α, σ, β and μ we will denote stable distribution by $S_\alpha(\sigma, \beta, \mu)$ and write

$$X \sim S_\alpha(\sigma, \beta, \mu)$$

to indicate that X has the stable distribution $S_\alpha(\sigma, \beta, \mu)$.

One of the the reasons that many techniques valid for the Gaussian case do not apply when $\alpha < 2$ is that $\alpha-$stable random variables have infinite second moment. We have the following result concerning the existence of moments of α-stable random variables.

Proposition 2.9. [**22**, Prop. 1.2.17] *Let* $X \sim S_\alpha(\sigma\,\beta, 0)$ *with* $0 < \alpha < 2$ *and* $\beta = 0$ *in the case* $\alpha = 1$. *Then, for every* $0 < p < \alpha$, *there is a constant* $c_{\alpha,\beta}(p)$ *such that*

$$(\mathbb{E}|X|^p)^{1/p} = c_{\alpha,\beta}(p)\sigma.$$

Moreover, $c_{\alpha,\beta}(p) = (\mathbb{E}|X_0|^p)^{1/p}$, *where* $X_0 \sim S_\alpha(1, \beta, 0)$.

Definition 2.10. A complex random variable X is called *symmetric* α*-stable* (SαS) if the random vector $(\Re(X), \Im(X))$ is a symmetric α-stable random vector in $\mathbb{R}^2$. A complex symmetric α-stable random

variable is called *rotationally invariant* if

$$e^{i\theta}X \stackrel{d}{=} X$$

for all $\theta \in [0, 2\pi)$.

If $\alpha = 2$, then SαS complex random variable X is rotationally invariant if and only if $\Re(X)$, $\Im(X)$ are i.i.d. normal random variables. When $\alpha < 2$, this not true, in fact the real and imaginary parts of rotationally invariant X are always dependent, as we can see from the following proposition.

Proposition 2.11. [22, Cor. 2.6.4] *Let $\alpha < 2$. A complex SαS random variable $X = X_1 + iX_2$ is isotropic if and only if there are two i.i.d zero mean normal random variables Z_1, Z_2 and a random variable $Y \sim S_{\alpha/2}((\cos \pi\alpha/4)^{2/\alpha}, 1, 0)$, independent of Z_1, Z_2 such that $X = Y^{1/2}(Z_1 + iZ_2)$.*

Proposition 2.12. [22, Cor. 2.6.4]*Let $\alpha < 2$ and suppose that $X = X_1 + iX_2$ is rotationally invariant complex SαS random variable. Then*

$$\mathbb{E}\exp i(\theta_1 X_1 + \theta_2 X_2) = \exp\left(-c_0\Gamma(S^1)(\theta_1^2 + \theta_2^2)^{\alpha/2}\right), \tag{2.4}$$

where Γ is the spectral measure of (X_1, X_2) and $c_0 = \frac{1}{2\pi}\int_0^\pi |\cos\varphi| d\varphi$.

Definition 2.13. A real valued (resp., complex valued) *stochastic process (random field)* is a function $X : T \to L^0_{\mathbb{R}}(\Omega, P)$ (resp., $X : T \to$

$L^0_{\mathbb{C}}(\Omega, P)$). The *finite dimensional distributions* of X are the distributions of the vectors

$$(X(t_1), ..., X(t_d)), \quad t_1, t_2, ..., t_d \in T, \ d \geq 1.$$

A stochastic process (field) $X : T \to L^0(\Omega, P)$ is called α-stable (resp., strictly α-stable, symmetric α-stable) if all its finite dimensional distributions are α-stable (resp., strictly α-stable, symmetric α-stable). We say that random processes $X : T \to L^0(\Omega, P)$ and $Y : T \to L^0(\Omega, P)$ are *equal in distribution* and write

$$X \stackrel{d}{=} Y$$

or, more often,

$$(X(t), \, t \in T) \stackrel{d}{=} (Y(t), \, t \in T)$$

if for any finite collection of indices $t_1, ..., t_d \in T$ the vectors $(X(t_1), ..., X(t_d))$ and $(Y(t_1), ..., Y(t_d))$ have the same distributions.

Remark 2.14. We use the term random field and stochastic process interchangeably. We consider both real and complex valued stochastic processes - see Chapter 1 for notational conventions for $L^0(\Omega, P)$, $L^0_{\mathbb{R}}(\Omega, P)$ and $L^0_{\mathbb{C}}(\Omega, P)$.

Theorem 2.15. [**22**, Th. 3.1.2] *Let $X : T \to L^0_{\mathbb{R}}(\Omega)$ be a stochastic process.*

(a) X is strictly α-stable if and only if all linear combinations

$$\sum_{k=1}^{d} b_k X(t_k), \quad d \geq 1,\ t_1, t_2, ..., t_d \in T,\ \ b_1, b_2, ..., b_d \in \mathbb{R} \tag{2.5}$$

are strictly α-stable.

(b) X is symmetric α-stable if and only if all linear combinations (2.5) are symmetric α-stable.

(c) If $\alpha \geq 1$, then X is α-stable if and only if all linear combinations (2.5) are α-stable.

From Corollary 2.6 we easily obtain

Corollary 2.16. *If $X : T \to L^0_{\mathbb{R}}(\Omega)$ is an $\alpha-$stable process, $\alpha \neq 1$, then there exist a function $\mu : T \to \mathbb{R}$ such that $X - \mu$ is strictly $\alpha-$stable.*

We will also consider complex valued strictly α-stable, symmetric α-stable and rotationally invariant α-stable processes.

Definition 2.17. A stochastic process $X : T \to L^0_{\mathbb{C}}(\Omega, P)$ is called *complex valued strictly α-stable* (resp. *symmetric α-stable, rotationally invariant α-stable*) if for all vectors $a = (a_1, ..., a_d) \in \mathbb{C}^d$ and indices $t_1, ..., t_d \in T$ the linear combination $\sum_{k=1}^{d} a_k X(t_k)$ is a strictly α-stable, (resp., symmetric α-stable, rotationally invariant α-stable) complex random variable.

We will now turn to the definition of **stable random measures**. It is convenient to view random measures as stochastic processes indexed

by sets. We will adopt measure-theoretic terminology from [**11**]. A *Polish space* is a separable topological space, metrizable by a complete metric. The collection of *Borel sets* $\mathbb{B}(S)$ of a topological space S is σ-algebra generated by open sets in S.

A *standard Borel space* is a measurable space (i.e. a set S equipped with a σ-algebra $\mathcal{A}$ of subsets of S), such that there exist a topology on S which makes S a Polish space and $\mathcal{A}$ the σ-algebra of Borel sets in S. A *Borel measure* on S is a measure defined on $\mathcal{A} = \mathbb{B}(S)$.

Let S be a standard Borel space with a Borel measure μ defined on the σ-algebra $\mathbb{B}(S)$ of Borel sets of S; $\mathbb{B}_0(S) := \{B \in \mathbb{B}(S) : \mu(B) < \infty\}$, and let $\beta : S \to [-1, 1]$ be a measurable function.

Definition 2.18. A set function $M : \mathbb{B}_0(S) \to L^0_{\mathbb{R}}(\Omega, P)$ is called *independently scattered* if for any finite collection of disjoint sets $A_1, ..., A_n \in \mathbb{B}_0(S)$ the random variables $M(A_1), ..., M(A_n)$ are independent. M is called σ-*additive* if for any countable collection of disjoint sets $\{A_i\}_{i\in\mathbb{N}} \subset \mathbb{B}_0(S)$ such that $\bigcup_{j=1}^{\infty} A_i \in \mathbb{B}_0$ we have

$$M\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} M(A_j) \quad \text{a.s.}$$

An independently scattered σ-additive set function $M : \mathbb{B}_0(S) \to L^0_{\mathbb{R}}(\Omega, P)$ such that

$$M(A) \sim S_\alpha\left(\mu(A)^{1/\alpha}, \frac{\int_A \beta d\mu}{\mu(A)}, 0\right), \quad \text{for all } A \in \mathbb{B}_0(S)$$

is called an *α-stable random measure* on S with *control measure* μ and *skewness intensity* β. An α-stable random measure with skewness intensity $\beta = 0$ is called symmetric *α-stable ($S\alpha S$)*.

To show the existence of random measures we can treat M as a stochastic process defined on $\mathbb{B}_0(S)$, specify its finite dimensional distributions and then use Kolomogorov's existence theorem (see [**22**, Ch. 3.3] for details).

Since the control measure defines finite dimensional distributions of a SαS measure uniquely, we will often write M_μ to mean a SαS random measure with control measure μ. Similarly we will write $M_{\mu,\beta}$ to denote an α-stable random measure with control measure μ and skewness intensity β.

Definition of complex valued α-stable random measure is a little more complicated. Let S^1 be the unit circle in $\mathbb{R}^2$ and let k be a measure on $\mathbb{B}(S \times S^1)$, satisfying the following condition:

Condition 2.19. For every $B \in \mathbb{B}(S)$ such that $k(B \times S^1) < \infty$, $k(B \times \cdot)$ is a symmetric Borel measure on S^1.

Denote $\mathbb{B}_0 = \{B \in \mathbb{B}(S) : k(B \times S^1) < \infty\}$. The meaning of the notions of independently scattered and σ-additive set functions is analogous to the case of real random measures.

Definition 2.20. A *complex valued $S\alpha S$ random measure* on S with circular control measure k is an independently scattered σ-additive

complex valued set function

$$M : \mathbb{B}_0(S) \to L^0_{\mathbb{C}}(\Omega)$$

such that for every $B \in \mathbb{B}_0$ the vector

$$(\Re M(B), \Im M(B))$$

is SαS with spectral measure $k(B \times \cdot)$.

SαS complex valued random measure with circular control measure of the form

$$k = \mu \times m, \tag{2.6}$$

where μ is a Borel measure on S and m is the uniform measure on S^1, is called *rotationally invariant SαS random measure.*

The reason for calling a SαS complex valued random measure with control measure (2.6) rotationally invariant is that

$$e^{it}M(B) \stackrel{d}{=} M(B)$$

for every $B \in \mathbb{B}_0$ and $t \in \mathbb{R}$.

Similarly as for real valued SαS random measures the distribution of complex valued rotationally invariant SαS random measure is defined by the measure μ, thus we will use notation M^c_μ to mean the complex valued SαS random measure with control measure $k = \mu \times m$. In this case we will also call μ the control measure.

Having defined SαS random measure we will now turn to the construction of **integrals** with respect to these measures. Let M be an α-stable random measure with control measure μ and skewness intensity β. We will want to define

$$I_M(f) = \int_S f(s)\, M(ds)$$

for all Borel functions $f : S \to \mathbb{R}$ such that

$$\int_S |f(s)|^\alpha \mu(ds) < \infty \tag{2.7}$$

and, if $\alpha = 1$, also the condition

$$\int_S |f(s)|\beta(s) \ln |f(s)|\, \mu(ds) < \infty. \tag{2.8}$$

We will denote by F the set of functions satisfying (2.7) and (2.8) if $\alpha = 1$.

As usual, we start from defining the integral for simple functions of the form $f = \sum_{j=1}^n a_j 1_{A_j}$, where $A_j \in \mathbb{B}_0(S)$, $j = 1, ..., n$, A_j disjoint:

$$I(f) = \int_S f(s)\, M(ds) := \sum_{j=1}^n a_j M(A_j). \tag{2.9}$$

Consider now any $f \in F$. We choose a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n(x) \to f(x)$ μ-a.e. and $|f_n| \leq \theta$ for some $\theta \in F$. Such sequence always exists and it can be shown (see [**22**, Ch. 3.4]) that $I(f_n)$ as defined by (2.9) converges in probability to a random variable which does not depend on the choice of the sequence approximating f. We define this limit to be the integral of f with respect to M.

We have also

$$I(f) \sim S_\alpha(\sigma_f, \beta_f, 0), \tag{2.10}$$

where $\sigma_f = \int_S |f(s)|^\alpha \mu(ds)$ and $\beta_f = \int_S f^{\langle\alpha\rangle}(s)\beta(s)\mu(ds) / \int_S |f(s)|^\alpha \mu(ds)$. (2.10) is easy to check for simple functions f, and then it extends to general $f \in F$ because convergence in probability implies convergence in distribution. The property (2.10) and the linearity of integral imply

Proposition 2.21. *For any $f_1, f_2, ..., f_d \in F$, the characteristic function of the random vector $(I(f_1), ..., I(f_d))$ is given by*
(i) if $\alpha \neq 1$

$$\mathbb{E}\exp\left(i\sum_{j=1}^{d} a_j I(f_j)\right) = \exp\left(-\int_S |a \cdot f|^\alpha \left(1 - i\beta \operatorname{sign}(a \cdot f) \tan\frac{\pi\alpha}{2}\right) d\mu\right),$$

(ii) if $\alpha = 1$

$$\mathbb{E}\exp\left(i\sum_{j=1}^{d} a_j I(f_j)\right) = \exp\left(-\int_S |a \cdot f|^\alpha \left(1 + i\frac{2}{\pi}\beta \operatorname{sign}(a \cdot f) \ln|a \cdot f|\right) d\mu\right)$$

where $a = (a_1, ..., a_d) \in \mathbb{R}^d$ and $(a \cdot f)(x) = \sum_{j=1}^d a_j f_j(x)$.

The definition of integrals with respect to complex valued SαS random measures is completely analogous to the real case. Let M be a complex valued SαS random measure on a Borel space S with circular control measure k. As for the real random measures if $f = \sum_{j=1}^n c_j 1_{A_j}$, where $c_j \in \mathbb{C}$, $A_i \in \mathbb{B}_0$ are disjoint and satisfy condition 2.19 we define

$$I(f) = \int_S f\, dM := \sum_{j=1}^{n} c_j M(A_j)$$

and extend this definition to $L^\alpha(S,\mu)$, approximating $f \in L^\alpha(S,\mu)$ by a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$, such that $f_n(x) \to f(x)$ μ-a.e. and $|f| \leq \theta$ for some $\theta \in L^\alpha(S,\mu)$.

The next proposition summarizes the properties of the resulting complex valued integral..

Proposition 2.22. [22, Prop. 6.2.1]*i) If $f \in L^\alpha_{\mathbb{C}}(S,\mu)$, then $I(f)$ is a complex valued SαS random variable with characteristic function given by*

$$\mathbb{E}\exp i(a_1\Re I(f) + a_2\Im I(f)) =$$

$$\exp\left(-\int_S\int_{S^1}\left|a_1(\theta_1 f^1(s) - \theta_2 f^2(s) + a_2(\theta_1 f^2(s) + \theta_2 f^1(s)\right|^\alpha k(ds,d\theta)\right),$$

ii) for any $f_1, f_2 \in L^\alpha(S,\mu)$, $a_1, a_2 \in \mathbb{C}$

$$I(a_1 f_1 + a_2 f_2) = a_1 I(f_1) + a_2 I(f_2),$$

iii) for any $f_1, f_2, ..., f_d \in L^\alpha(S,\mu)$ and for any pairs (a_1^1, a_2^1), $(a_1^2, a_2^2), ..., (a_1^d, a_2^d) \in R^2$

$$-\ln\mathbb{E}\exp\left(\sum_{j=1}^{d}(a_j^1\Re I(f_j) + a_j^2\Im I(f_j))\right) =$$

$$\int_S \int_{S^1} \left| \sum_{j=1}^{d} (\theta_1 a_j^1 + \theta_2 a_j^2) f_j^1(s) + (\theta_1 a_j^2 - \theta_2 a_j^2) f_j^2(s) \right|^\alpha k(ds, d\theta),$$

where $f^1 = \Re f$, $f^2 = \Im f$.

An important special case is when the complex random measure is of the form

$$k = \mu \times m,$$

where m is the uniform probability measure on S^1.

Theorem 2.23. [**22**, Th. 6.3.1]*Let M be a complex rotationally invariant $S\alpha S$ random measure with control measure μ. Then for any complex numbers $z = a_1 + ia_2$*

$$\begin{aligned} \mathbb{E} \exp i(a_1 \Re I(f) + a_2 \Im I(f)) &= \mathbb{E} \exp (i \Re (I(\overline{z} f)) \\ &= \exp(-|z|^\alpha c_0 \textstyle\int_S |f|^\alpha) d\mu, \end{aligned} \tag{2.11}$$

where $c_0 = \frac{1}{2\pi} \int_0^\pi |\cos \varphi| d\varphi$.

Remark. If we normalize the uniform measure on S^1 by $m(S^1) = \frac{1}{c_0}$, then the constant c_0 does not appear in (2.11). We also note that comparing (2.11) with formula (2.4) in Corollary 2.12 we can conclude that if M is rotationally invariant, then $I(f)$ is rotationally invariant $S\alpha S$ r.v. as well.

Many properties of stable integrals in the complex case are similar to those in the real-valued case. The following result, for example, can be stated conveniently both for real an complex case.

Proposition 2.24. [22, Props. 3.5.1, 6.2.3] *Let* $1 < \alpha < 2$, $X_j = \int_S f_j(s)\, M(ds)$, $j = 1, 2, ...,$ *and* $X = \int_S f(s)\, M(ds)$, *where* M *is a* α*-stable random measure with control measure* μ*(complex valued random* $S\alpha S$ *measure with circular control measure* k *in the complex case). Suppose* f, f_j, $j = 1, 2, ...,$ *are in* $L^\alpha(\mu)$ *(*f, f_j, $j = 1, 2, ...,$ *are in* $L_c^\alpha(\mu)$, *where* $\mu = k(\cdot \times S^1)$ *in the complex case). Then the sequence* X_j *converges in probability to* X *if and only if*

$$\lim_{j\to\infty} \int_S |f_j(s) - f(s)|^\alpha \mu(ds) = 0.$$

Remark 2.25. An important consequence of Proposition 2.24 is that the integral $I(f)$ depends only on the equivalence class of the function f, that is that I is a linear $L^0(\Omega, P)$ valued operator defined on $L^\alpha(S, \mu)$. Proposition 2.24 also implies that this operator is one-to-one.

CHAPTER 3

Minimal integral representations of α-stable processes

It is very convenient to study the properties of α-stable processes via integral representations of the type

$$(X(t),\, t \in T) \stackrel{d}{=} \left(\int_S f_t(s)\, M(ds),\, t \in T \right).$$

A natural question is to determine the class of α-stable processes which can be represented this way. It turns out that this class is quite large, even if we limit ourselves to a random measure on the interval $(0,1)$ with Lebesgue measure as the control measure .

Definition 3.1. We say that $X : T \to L^0(\Omega, P)$ is *separable in probability* if there is a countable subset $T_0 \subset T$ such that for every $t \in T$, there is a sequence of linear combinations $\left\{ \sum_{j=1}^{k_n} \lambda_j^n X_{t_j^n} \right\}_{n \in N}$ which converges to $X(t)$ in probability.

Theorem 3.2. [**22**, Th. 13.2.1]*(i) Let $\alpha \neq 1$. If $X : T \to L^0_{\mathbb{R}}(\Omega, P)$ is separable in probability and strictly stable, then there exist functions $f_t \in L^\alpha_{\mathbb{R}}((0,1), m)$, $t \in T$, such that*

$$X \stackrel{d}{=} \left(\int_0^1 f_t(x) M(dx),\, t \in T \right), \tag{3.1}$$

where M is an α-stable random measure on $(0,1)$ with skewness intensity $\beta \equiv 1$ and Lebesgue control measure.

(ii) If $X : T \to L^0(\Omega, P)$ is separable in probability and $S\alpha S$, then there exist $f_t \in L^\alpha((0,1), m)$ such that (3.1) holds for M a $S\alpha S$ random measure on $(0,1)$ with Lebesgue control measure.

Studying the question of uniqueness of integral representation (3.1) Hardin defined the notion of minimal representation for a real valued $S\alpha S$ and complex valued rotationally invariant stable processes. Later, Rosinski [**20**] placed this notion in more general framework suitable for study of strictly stable and other infinitely divisible processes. We will use Rosinski's approach. The material of this section comes from [**20**], unless indicated otherwise. The reader is referred there for proofs of the results included below.

Definition 3.3. Let $\mathcal{X}$ be a subset of $L^0(\Omega, P)$. A transformation $V : \mathcal{X} \to L^0(\Omega, P)$ is called distribution preserving (d.p.) if for every $d \geq 1$ and $X_1, ..., X_d \in \mathcal{X}$

$$(X_1, ..., X_d) \stackrel{d}{=} (VX_1, ..., VX_d).$$

Let M be an α-stable random measure on a standard Borel space S with control measure μ and skewness intensity β and denote

$$L(M) = \left\{ \int_S f \, dM : \, f \in L^\alpha(S, \mu) \right\} \subset L^\alpha(\Omega, P).$$

For a stochastic process $X : T \to L^0(\Omega, P)$ by $X(T)$ denote the set $\{X(t) : t \in T\} \subset L^0(\Omega, P)$. Let $\mathcal{M}$ be one of the classes $\mathcal{M}_0^\alpha$, $\mathcal{M}_1^\alpha$ of all SαS and α-stable random measures, respectively.

Definition 3.4. We say that $V : X(T) \to L(M)$ is an *integral representation* of X in $L(M)$ if V is distribution preserving. An integral representation is minimal in $\mathcal{M}$ if for every integral representation $V' : X(T) \to L(M')$, $M' \in \mathcal{M}$ there exist a unique d.p. transformation $W : L(M) \to L(M')$ such that $V' = WV$.

One of the motivations for the notion of minimal representation is the following. Suppose $M : \mathbb{B}_0(S) \to L^p(\Omega, P)$ (as is the case for the classes $\mathcal{M}_0^\alpha$, $\mathcal{M}_1^\alpha$, see Proposition 2.9), and let $L^p(M)$ be the L^p closure of stochastic integrals of simple functions, $p \geq 0$. Let V be a representation of X in $L^p(M)$ and the L^p closure of span$(V(X(T)))$. Thus we have

$$L^p(X) \subset L^p(M). \tag{3.2}$$

One can easily find examples of SαS processes such that the inclusion in (3.2) is proper for any SαS random measure (see Examples 4.4, 4.5 in [20]). In such cases, intuitively speaking, minimal representation provides the "smallest gap" between $L^p(X)$ and $L^p(M)$.

Recall from Chapter 2 that the operator $I_M : L^\alpha(S, \mu) \to L(M)$, $I_M(g) = \int_S g\, dM$ is one-to-one. Thus, with some abuse of terminology we will call the mapping $f : T \to L^\alpha(S, \mu)$, $f_t = I_M^{-1}(V(X(t)))$

a representation or minimal representation of X if V is a representation or minimal representation, respectively. This way we associate the notions of stable processes and subsets of spaces $L^\alpha(S,\mu)$.

Definition 3.5. Let $p \in (0,\infty)$. A set $F \subset L^p(S,\mu)$ is said to be *rigid* if, for every space $L^p(S_1,\mu_1)$ and linear isometry $U_0 : F \to L^p(S_1,\mu_1)$ there exist a unique linear isometry $U : L^p(S,\mu) \to L^p(S_1,\mu_1)$ such that $U|_F = U_0$.

The following theorem, which combines Proposition 4.1 and Theorem 4.11 from **[20]**, explains the need for studying rigid subset in the context of minimal representations.

Theorem 3.6. *Let $V : X(T) \to L(M)$ be a representation of strictly α-stable process X, where $M \in \mathcal{M}$ is a random measure on S with control measure μ. Then V is minimal in $\mathcal{M}$ if and only if the set $(I_M^{-1} \circ V)(X(T))$ is rigid in $L^\alpha(S,\mu)$.*

Although minimal representations by their uniqueness property seem to be natural in studying integral representations of stable processes, the minimality of specific representations was difficult to establish because of the lack of workable tests for this property. This difficulty was overcome in **[20]**, where such tests were provided.

Recall that $\mathcal{L}^\alpha(S,\mu)$ is a collection of Borel functions $f : S \to R$ ($\mathbb{C}$ in the complex case) satisfying condition $\|f\|_\alpha^\alpha = \int_S f(s)d\mu < \infty$ (as opposite to $L^\alpha(S,\mu)$, which is a collection of $\mu-$equivalence classes of functions). $f \mapsto \|f\|_\alpha$ is a semi-norm on $\mathcal{L}^\alpha(S,\mu)$. By rigidity of a

collection of functions we will mean the rigidity of the corresponding collection of equivalence classes.

If $f, g : S \to \mathbb{R}$ ($\mathbb{C}$) are measurable, then we will write $f \overset{\mu}{=} g$ if $\mu(\{s \in S : f(s) \neq g(s)\}) = 0$. For $A, B \in \mathbb{B}(S)$ we will write $A \overset{\mu}{=} B$ to mean $1_A \overset{\mu}{=} 1_B$ (or, equivalently $\mu(A \triangle B) = 0$). For collections of sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{B}(S)$ we will write $\mathcal{A}_1 \overset{\mu}{=} \mathcal{A}_2$ if for every $A_1 \in \mathcal{A}_1$ there exist $A_2 \in \mathcal{A}_2$ such that $A_1 \overset{\mu}{=} A_2$ and conversely, for every $A_2 \in \mathcal{A}_2$ there exist $A_2 \in \mathcal{A}_2$ such that $A_2 \overset{\mu}{=} A_1$.

Let F be a collection of measurable functions on S.

Definition 3.7. For $f \in F$ define the support of f to be the measurable set $\{s \in S : f(s) \neq 0\}$, which we denote by $\text{supp}(f)$. We say that f has full support in F if $f \in F$ and $\mu(\text{supp}(g) \setminus \text{supp}(f)) = 0$ for any $g \in F$.

Functions with full support in F always exist if F is a closed linear subspace of $L^\alpha(S, \mu)$, where $0 < \alpha < \infty$ and S is a standard Borel space (see [**12**], Lemma 3.2).

Definition 3.8. Let $\psi : S \to S$ be a measurable map. ψ is non-singular if for all $B \in \mathbb{B}(S)$, $\mu(B) = 0$ implies $\mu(\psi^{-1}(B)) = 0$. We say that F is a ψ-quasi invariant if there exists a measurable function $k : S \to \mathbb{R} \setminus \{0\}$ ($\mathbb{C} \setminus \{0\}$ in the complex case) such that

$$f \circ \psi = k \cdot f \quad \mu - \text{a.e.},$$

for every $f \in F$.

We also adopt a convention that if $b = 0$, then $a/b = \partial$, the infinity point of one-point compactification of $\mathbb{R}$ (or $\mathbb{C}$ in the complex case).

Now we are ready to state the conditions for rigidity of a subset of $\mathcal{L}^\alpha(S,\mu)$, given in [**20**].

Theorem 3.9. *Let $F \subset {}^\alpha(S,\mu)$, where $\alpha \neq 2,4,6,...$, and suppose $\operatorname{supp}(f) \stackrel{\mu}{=} S$ for some f with full support in $\overline{\operatorname{lin}}_\alpha(F)$. Then the following are equivalent:*

(i) F is rigid.

(ii) If F is a $\psi-$quasi-invariant with respect to some null-preserving Borel isomorphism $\psi : S \to S$ with $\psi \circ \psi = \mathrm{id}_S$, then $\psi \stackrel{\mu}{=} \mathrm{id}_S$.

(iii) If F is a $\psi-$quasi-invariant with respect to some nonsingular $\psi : S \to S$ with $\psi \circ \psi = \mathrm{id}_S$, then $\psi \stackrel{\mu}{=} \mathrm{id}_S$.

(iv) $\sigma(f/g : f,g \in F) \stackrel{\mu}{=} \mathbb{B}(S)$.

(v) For any $f^* \in \overline{\operatorname{lin}}_\alpha(F)$ with $\operatorname{supp}(f^*) \stackrel{\mu}{=} S$ we have

$$\sigma(f/f^* : f \in F) \stackrel{\mu}{=} \mathbb{B}(S).$$

(vi) $\sigma(f/g : f,g \in \overline{\operatorname{lin}}_\alpha(F)) \stackrel{\mu}{=} \mathbb{B}(S)$.

(vii) There exist a sequence $\{f_n\}_{n\in\mathbb{N}} \subset F$ and $f^* \in \overline{\operatorname{lin}}_\alpha(F)$ with $\operatorname{supp}(f^*) \stackrel{\mu}{=} S$ such that the map

$$s \mapsto \left(\frac{f_1(s)}{f^*(s)}, \frac{f_2(s)}{f^*(s)}, \frac{f_2(s)}{f^*(s)}, ...,\right)$$

is one-to-one μ-a.e.

(viii) There exist a sequences $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}} \subset \overline{\operatorname{lin}}_\alpha(F)$ such that the

map

$$s \mapsto \left(\frac{f_1(s)}{g_1(s)}, \frac{f_2(s)}{g_2(s)}, \frac{f_2(s)}{g_3(s)}, ..., \right)$$

is one-to-one μ-a.e.

The following Lemma will be useful in Chapter 4.

Lemma 3.10. *Let $\{f_t : t \in T\} \subset L^\alpha(S, \mu)$ be a minimal representation of a $S\alpha S$ process. Then there exist a sequence $\{t_n\}_{n\in\mathbb{N}} \subset T$ such that* $\text{supp}\{f_{t_n} : n \in \mathbb{N}\} \stackrel{\mu}{=} S$.

Proof. Let $f^* \in \overline{\text{span}}_{L^\alpha(S,\mu)}\{f_t : t \in T\}$ be a function with full support in (S, μ) and let $a_i^k \in \mathbb{R}$, $k \in \mathbb{N}\ 1 \leq i \leq k$, $\{f_{t_k^i}\ k \in \mathbb{N}\ 1 \leq i \leq k\} \subset \{f_t\, t \in T\}$ be such that $\sum_{i=1}^k a_i^k f_{t_k^i}$ converges to f^* μ-almost everywhere as $k \to \infty$. Now $S \stackrel{\mu}{=} \text{supp}\, f^* \subset \text{supp}\{f_{t_k^i}, k \in \mathbb{N}\ 1 \leq i \leq k\}$. □

CHAPTER 4

Representations of G-stationary random fields

Let G be a group and X a set. An *action of G on X* is a map $\rho : G \times X \to X$, such that $\rho(e, x) = x$ and $\rho(g, \rho(h, x)) = \rho(g \cdot h, x)$ for all $g, h \in G$ and $x \in X$. The pair (X, ρ) is called a *$G-$space*. We will usually write $g \cdot x$ instead of $\rho(g, s)$. For each $g \in G$ we define the mapping $L_g : X \to X$, $L_g(x) := g \cdot x$.

In this chapter we focus on $G-$stationary SαS real and complex valued rotationally invariant random fields $X : T \to L^0(\Omega, P)$, defined on a G-space (T, η). We will also assume that X is separable in probability, that is there exist a countable set $T_1 \subset T$, such that for each $t \in T$ there is a sequence of linear combinations $\left\{\sum_{j=1}^{k_n} \lambda_j^n X_{t_j^n}\right\}_{n \in N}$ which converges to $X(t)$ in probability.

Hardin (1982) [12] showed that a minimal representation $f : T \to L^\alpha(S, \mu)$ of a $G-$stationary stable process (with $G = T$) induces a group of linear isometries of the space $L^\alpha(S, \mu)$. Using Lamperti's representation of isometries of L^p and Mackey's result on point realizations of groups of isomorphisms of Boolean algebras, Rosinski in [18] showed that, under some general measurability assumptions, such group of isometries generates an action of G on S and a cocycle $c : G \times S \to \{-1, 1\}$ ($\{z \in \mathbb{C} \ |z| = 1\}$ in the complex case). This fact, interesting by itself, allowed to show that stationary SαS field

on $\mathbb{R}^d$ can be uniquely decomposed into independent stationary fields with different properties. We write more about this decomposition in Chapter 5.

Since Rosinski's proof in [18] was given for the group $G = \mathbb{R}$ acting on $T = \mathbb{R}$ as translations, in Theorem 4.5 below we generalize Theorem 3.1 [18] to the case of locally compact group G acting on an arbitrary Polish space T.

Definition 4.1. Let T be a Polish space. A SαS process $X : T \to L^0(\Omega, P)$ is called measurable if the map $(t) \mapsto X(t) \in L^0(\Omega, P)$ is measurable. An integral representation $f : T \to L^\alpha(S, \mu)$ on a standard Borel space S with Borel measure μ is called measurable if the map $(t, s) \mapsto f_t(s)$ is measurable with respect to $\mathbb{B}(T) \times \mathbb{B}(S) = \mathbb{B}(T \times S)$.

It is known (see e.g. Rosinski and Woyczynski [21]) that if $f : T \to L^\alpha(S, \mu)$ is a representation of a measurable SαS process, then for each $t \in T$, f_t can be modified on a $\mu-$null set, so that the modified representation $\widetilde{f} : T \to L^\alpha(S, \mu)$ is measurable.

From now on all processes and their representations are assumed to be measurable and the group G is locally compact and Polish.

Definition 4.2. A *cocycle* for a group action $\rho : G \times S \to S$ on S is a Borel map $c : G \times C \to W$, where W is a locally compact Polish group such that

1) $c(e, s) = 1$ for μ-a.a. $s \in X$ and

2) for every $g_1, g_2 \in G$ and μ-a.a. $s \in S$ we have $c(g_2 g_1, s) = c(g_2, g_1 s) c(g_1, s)$.

If conditions 1) and 2) hold for *every* $s \in S$, then the cocycle c is called *strict*.

An action $\rho : G \times S \to S$ is called *nonsingular* if ρ is a Borel map and for every $g \in G$ and $B \in \mathbb{B}(S)$, $\mu(B) = 0$ implies $\mu(L_g(B)) = 0$.

The proof of Theorem 4.5, which is the main result of this chapter, uses Theorem 2.2 (b) from [**18**]. We include this theorem below for easy reference.

Theorem 4.3. [**18**, Th. 2.2 (b)] *Let $f^i : S_i \to L^\alpha(S_i, \mu_i)$, $i = 1, 2$ be minimal representations of $S\alpha S$ (rotationally invariant in the complex case) process $X : T \to L^0(\Omega, P)$. Then there exist functions $\Phi : S_2 \to S_1$ and $h : S_2 \to \mathbb{R} \setminus \{0\}$ ($\mathbb{C} \setminus \{0\}$ in the complex case) such that*
(i)Φ is one-to-one and onto,
(ii) for each $t \in T$, $f_t^2(s) = h(s) f_t^1(\Phi(s))$ for $\mu_2 -$ a.a $s \in S$,
(iii) $\frac{d(\mu_1 \circ \Phi)}{d\mu_2} = |h|^\alpha$, $\mu_2 -$ a.e.

Moreover, the functions Φ, h are unique modulo μ_2, i.e. if $\Phi_1 : S_2 \to S_1$ and $h_1 : S_2 \to \mathbb{R} \setminus \{0\}$ ($\mathbb{C} \setminus \{0\}$ in the complex case) satisfy (i)-(iii), then $\Phi(s) = \Phi_1(s)$ and $h(s) = h_1(s)$ for μ_2-a.a. $s \in S_2$.

Remark 4.4. It is easy to see from the form of characteristic function for a random vector $(I(f_1), ..., I(f_d))$ (Proposition 2.21 (i) with $\beta = 0$) that, under hypotheses of Theorem 4.3, there exist a linear isometry between the sets $\{f_t^i\}_{t \in T} \subset L^\alpha(S_i, \mu_i)$. If $f^i : S_i \to L^\alpha(S_i, \mu_i)$ are **minimal**, as in the theorem above, this isometry extends uniquely to a linear isometry between $L^\alpha(S_2, \mu_2)$ onto $L^\alpha(S_2, \mu_2)$ (see Theorem 3.6

and Hardin [12]). In [18], Rosinski used Lamperti's [14] characterization of such isometries in terms of set isomorphisms and Sikorski's Theorem 32.5 from [24] about point realisations of set isomorphisms to express this isometry in terms of Φ and h.

Theorem 4.5. *Let $X : T \to L^0(\Omega, P)$ a be $G-$stationary $S\alpha S$ (rotationally invariant in the complex case) process with a minimal representation $f : T \to L^\alpha(S, \mu)$ on a standard Borel space S. Then there exist a nonsingular action ρ of G on S and a strict cocycle $c : G \times S \to \{-1, 1\}$ ($\{z \in \mathbb{C} \ |z| = 1\}$ in the complex case) for ρ such that (S, ρ, μ) is a standard Borel $G-$space and for all $t \in T$, $g \in G$ we have*

$$f_{g^{-1}t} = c(g, \cdot) \left(\frac{d\mu \circ L_g}{d\mu} \right)^{\frac{1}{\alpha}} (f_t \circ L_g), \tag{4.1}$$

where $L_g : S \ni s \longmapsto \rho(g, s) \in S$.

Proof. The proof follows the lines of the proof of Theorem 3.1 in [18] in the group setting. Since $t \mapsto f_t$ is minimal, then, for each $g \in G$, $t \mapsto f_{g^{-1}t}$ is also a minimal representation of the same $S\alpha S$ process. By Theorem 4.3 for every $g \in G$ there exist a one-to-one and onto measurable functions $\Phi_g : S \to S$ and functions $h_g : S \to \mathbb{R} \setminus \{0\}$ such that, for each $t \in T$

$$f_{g^{-1}t} \stackrel{\mu}{=} (h_g)(f_t \circ \Phi_g) \tag{4.2}$$

and

$$\frac{d(\mu \circ \Phi_g)}{d\mu} \stackrel{\mu}{=} |h_g|^\alpha. \tag{4.3}$$

Thus, for any $g_1, g_2 \in G$, $t \in T$ we have

$$f_{(g_2g_1)^{-1}t} = f_{g_1^{-1}(g_2^{-1}t)} \stackrel{\mu}{=} h_{g_1}(f_{g_2^{-1}t} \circ \Phi_{g_1}) \stackrel{\mu}{=} h_{g_1}(h_{g_2} \circ \Phi_{g_1})(f_t \circ \Phi_{g_2} \circ \Phi_{g_1})$$

and

$$f_{(g_2g_1)^{-1}t} \stackrel{\mu}{=} h_{g_2g_1}(f_t \circ \Phi_{g_2g_1}).$$

By the uniqueness in Theorem 4.3 we get

$$h_{g_2g_1} \stackrel{\mu}{=} h_{g_1}(h_{g_2} \circ \Phi_{g_1}) \tag{4.4}$$

and

$$\Phi_{g_2g_1} \stackrel{\mu}{=} \Phi_{g_2} \circ \Phi_{g_1} \quad \mu - \text{a.e.} \tag{4.5}$$

Now we will use Theorem 1 from **[15]** to show that Φ_g can be modified on a set of μ measure 0 so that (4.5) would hold for all $s \in S$. To do this we need to verify that the $\sigma-$Boolean algebra $\mathbb{B}_\mu$ of the classes $[B] = \{A \in \mathbb{B}(S) : \mu(A \triangle B) = 0\}$ together with the map

$$\psi : G \times \mathbb{B}_\mu \ni (g, [B]) \longrightarrow \psi(g, [B]) = [\Phi_g^{-1}(B)] \in B_\mu$$

is a Boolean $G-$space. By (4.5) $(\mathbb{B}_\mu, \psi)$ is a $G-$space and by (4.3) the map $[B] \to \psi(g, [B])$ is a Boolean algebra isomorphism for every $g \in G$. Therefore we need only to check that for every finite measure

$\widetilde{\nu}$ on $\mathbb{B}_\mu$ and class $[B]$ the map

$$g \longmapsto \widetilde{\nu}\left([\Phi_g^{-1}(B)]\right)$$

is measurable. Let ν be the measure on $\mathbb{B}(S)$ defined by $\nu(B) := \widetilde{\nu}([B])$. Since we have $\nu \ll \mu$, put $k = d\nu/d\mu$. It is enough to show that for each Borel set $B \in \mathbb{B}(S)$ the map

$$g \longmapsto \int_S 1_B(\Phi_g(s))k(s)\mu(ds) \tag{4.6}$$

is measurable. Choose $f^* \in \overline{\text{span}}_{L^\alpha(S,\mu)}\{f_t : t \in T\}$ such that $f^*(s) \neq 0$ μ−a.e. on S. Let $f_n^* \in \text{span}\{f_t : t \in T\}$, $f_n^* = \sum_{i=1}^n c_{n,i} f_{t_{n,i}}$ be functions such that $f_n^*(s) \to f^*(s)$ μ−a.e. as $n \to \infty$. Then for every $g \in G$ we have

$$\sum_{i=1}^n c_{n,i} f_{g^{-1}t_{n,i}}(s) = h_g(s)(f_n^* \circ \Phi_g(s)) \ \mu \text{ a. e. on } S. \tag{4.7}$$

Since the mappings $(g,t) \longmapsto gt$ and $(t,s) \longmapsto f_t(s)$ are measurable, the map $(g,t,s) \longmapsto f_{gt}(s)$ is measurable and hence the map $(g,s) \longmapsto \sum_{i=1}^n c_{n,i} f_{gt_{n,i}}(s)$ is measurable for each $(t_{n,i},\ n \in \mathbb{N} i = 1, \ldots, n)$. Since the right hand side of (4.7) converges to $h_g(s)f^*(\Phi_g(s))$ μ−a.e. as $n \to \infty$, we conclude that there exist a measurable mapping $(g,s) \longmapsto f_g^*(s)$ such that

$$h_g(s)f^*(\Phi_g(s)) = f_g^*(s) \text{ for each } g \in G \text{ and } \mu \text{ a. a. } s. \tag{4.8}$$

Since $f : T \to L^\alpha(S,\mu)$ is minimal, by Theorem 3.9 (g)

$$\sigma(f/f^* : f \in F) \stackrel{\mu}{=} \mathbb{B}(S).$$

Since $\mathbb{B}(S)$ is countably generated, for every $B \in \mathbb{B}(S)$ there exist $(t_i,\, i \in \mathbb{N}) \subset T$ and $A \subset R^\infty$ such that

$$B \stackrel{\mu}{=} \{s \in S : (f_{t_1}(s)/f^*(s), f_{t_2}(s)/f^*(s), \dots) \in A\}$$

In view of (4.2) and (4.8),

$$1_B(\Phi_g(s)) = 1_A(f_{g^{-1}t_1}(s)/f^*(s), f_{g^{-1}t_2}(s)/f^*(s), \dots) \text{ for } \mu - \text{a.a. } s \in S.$$

Since the right side is measurable in (g, s) we infer that the map in (4.6) is measurable. Applying [**15**, Th. 1] we get that there exist a Borel $G-$space (S', β'), a finite Borel measure μ' on S', and a $\sigma-$Boolean algebra isomorphism $T : \mathbb{B}_{\mu'} \to \mathbb{B}_\mu$ such that

$$[\Phi_g^{-1}(T[B'])] = T([\phi_g'^{-1}(B')),$$

where $\phi_g'^{-1}(s') = \beta'(g, s')$. By Sikorski [**24**, Th. 32.5], there exist a Borel isomorphism $\Psi : S \to S'$ such that $T([B']) = [\Psi^{-1}(B')]$. Hence for $\phi_g := \Psi^{-1} \circ \phi_g' \circ \Psi$ we can replace Φ by ϕ in (4.5) to obtain a $G-$space S with desired properties.

It follows from (4.8) that $(g, s) \to h_g(s)$ has a jointly measurable modification. Define $c(g, \cdot) := h_g/|h_g|$. Then (4.4) shows that $(g, s) \longmapsto a_g(s)$ is a cocycle for the group action $(g, s) \to \phi_g(s)$. Now by Theorem B.9 in Appendix B of [**29**] this cocycle can be modified for each $g \in G$ on a set of μ measure 0 to so that the modified cocycle is strict. (4.3) completes the proof. □

If G is a topological group, S is a topological space and the action $\rho : G \times S \to S$ is continuous, then we call (S, ρ) a topological G-space. If, in addition, both G and S are Polish, we call (S, ρ) a *Polish G-space.*

Theorem 4.6. *(i) Let $X : T \to L^0(\Omega, P)$ be $G-$stationary $S\alpha S$ or complex valued rotationally invariant) process, $(X_t,\, t \in T) \stackrel{d}{=} (X_{g\cdot t},\, t \in T)$. Then there exist a Polish $G-$space (S, ρ, μ) (with ρ a* continuous *action), functions $\{f_t,\, t \in T\} \subset L^\alpha(S, \mu)$ and a strict cocycle c for ρ such that $t \mapsto f_t$, $t \in T$ is a minimal representation of X and*

$$f_{g^{-1}t} \stackrel{\mu}{=} c(g, \cdot) \left(\frac{d\mu \circ L_g}{d\mu} \right)^{\frac{1}{\alpha}} (f_t \circ L_g), \quad \text{for all } g \in G,\, t \in T.$$

(ii) If, in addition to hypotheses in (i), the group G is compact, then we can choose μ to be an invariant probability measure and obtain

$$f_{g^{-1}t} \stackrel{\mu}{=} c(g, \cdot)(f_t \circ L_g), \quad \text{for all } g \in G,\, t \in T.$$

Proof. (i) is an easy consequence of the fact that every $S\alpha S$ process admits a minimal representation on a standard Borel space, theorem 4.5 and Theorem 1.2 in the Appendix, which states that every standard Borel $G-$space is Borel isomorphic to a Polish $G-$space. This ends the proof of the first part.

Now if G is compact we can use a minimal representation from (i) to construct another minimal representation on (S, ρ, λ), where λ is an invariant Borel probability measure. By Lemma 3.10 we can find a sequence of indices $\{t_n,\, n \in \mathbb{N}\} \subset T$ such that $\text{supp}\{f_{t_n} : n \in \mathbb{N}\} \stackrel{\mu}{=} S$. There exist a sequence of positive numbers $\{a_n : n \in \mathbb{N}\}$ such that

$\sum_{n=1}^{\infty} a_n \|f_{t_n}\|^{\alpha}_{L^{\alpha}(S,\mu)} < \infty$. Given such sequence define

$$f^*(s) := \sum_{n\in\mathbb{N}} a_n \int_G |f_{g^{-1}t_n}(s)|^{\alpha} H(dg),$$

where H is the (probability) left Haar measure on G. Then we have

$$\int_S f^*(s)\mu(ds) = \sum_{n=1}^{\infty} a_n \int_G \int_S \frac{d\mu \circ L_g}{d\mu} |f_{t_n} \circ L_g|^{\alpha} \mu(ds) H(dg)$$

$$= \sum_{n=1}^{\infty} a_n \|f_{t_n}\|^{\alpha}_{L^{\alpha}(S,\mu)} < \infty,$$

which shows that $f^*(s) < \infty$ for $\mu-$a.a. s. Define

$$\lambda(ds) := f^*(s)\mu(ds).$$

Clearly $\lambda \ll \mu$. To see that λ is in fact equivalent to μ let $W := \{s \in S : f^*(s) = 0\}$. Then if $s \in W$ we have $f_{t_n}(s) = 0$ for all $n \in \mathbb{N}$, hence $s \notin \text{supp}\,\{f_{t_n} : n \in \mathbb{N}\} \stackrel{\mu}{=} S$ which implies that $\mu(W) = 0$. It is easy to see that λ is also invariant with respect to the group action ρ. Setting $h_t := (f^*)^{-\frac{1}{\alpha}} f_t$ we obtain a minimal representation $\{h_t\}_{t\in T} \subset L^{\alpha}(S,\lambda)$ of X satisfying

$$h_{g^{-1}t} = c(g,\cdot)(h_t \circ L_g), \quad \lambda - \text{a.e. for all } g \in G,\ t \in T.$$

□

Recall that for $t \in T$, $G_t := \{t \in T : g \cdot t = t\}$ is the isotropy group at t.

The following proposition gives a way of constructing $G-$stationary random processes.

Proposition 4.7. *Let (S, ρ, μ) be a standard Borel $G-$space, T_0 a transversal for a $G-$space (T, η), $\{f_{t_0}, t_0 \in T\} \subset L^\alpha(S, \mu)$, $0 < \alpha \leq 2$, c a strict cocycle for ρ such that for all $t \in T_0$, and $g \in G_t$ we have*

$$f_{t_0}(s) = c(g, s) \left(\frac{d\mu \circ L_g}{d\mu}(s) \right)^{\frac{1}{\alpha}} f_{t_0}(g \cdot s), \quad \mu - \text{a.e.} \tag{4.9}$$

where $L_h : S \ni s \longmapsto \rho(h, s) \in S$. For $t \in T$, $t = g \cdot t_0$ define

$$f_t := c(g, \cdot) \left(\frac{d\mu \circ L_g}{d\mu} \right)^{\frac{1}{\alpha}} (f_{t_0} \circ L_g), \tag{4.10}$$

This defines f_t for every $t \in T$. Let M be a $S\alpha S$ random measure on S with control measure μ. Then the process $(X_t, t \in T)$ defined by $X_t := \int_S f_t(s) M(ds)$ is $G-$stationary.

Proof. First we check that f_t, $t \in T$ is well defined (as an element of $L^\alpha(S, \mu)$) by (4.10). Suppose that $t = g_1^{-1} \cdot t_0 = g_2^{-1} \cdot t_0$ for some $g_1, g_2 \in G$. This implies that $g := g_1 g_2^{-1} \in G_{t_0}$ and $g_2 = g^{-1} g_1$. It is easy to check (or see the proof of Theorem 4.5), that $h : G \times S \to S$, $h(g, s) := c(g, s) \left(\frac{d\mu \circ L_g}{d\mu} \right)^{\frac{1}{\alpha}}$ is an $\mathbb{R} \setminus \{0\}$ ($\mathbb{C} \setminus \{0\}$ in the complex case) valued cocycle. Thus we have

$$h(g_2, s) f_{t_0}(g_2 \cdot s) \stackrel{\mu}{=} h(g^{-1} g_1, s) f_{t_0}(g^{-1} g_1 \cdot s) \stackrel{\mu}{=} h(g^{-1}, g_1 \cdot s) h(g_1, s) f_{t_0}(g^{-1} g_1 \cdot s).$$

By (4.9) written with $g_1 \cdot s$ in place of s this implies that

$$h(g_2, \cdot)(f_{t_0} \circ L_{g_2}) \stackrel{\mu}{=} h(g_1, \cdot)(f_{t_0} \circ L_{g_1}),$$

i.e. definition (4.10) does not depend on the choice of g to represent t as $t = g^{-1}t_0$.

Now we fill show that (4.10) defines a representation of a $G-$stationary process. Fix $g, g_1 \in G$ and $t_0 \in T$. Since the maps $g \mapsto g \cdot s = L_g(s)$, $g_1 \mapsto g_1 \cdot s = L_{g_1}(s)$, are nonsingular, we can check that

$$\frac{d(\mu \circ L_{g_1 g})}{d\mu}(s)\frac{d\mu}{d(\mu \circ L_{g_1})}(g \cdot s) = \frac{d(\mu \circ L_g)}{d\mu}(s) \quad \mu - \text{a.e.} \tag{4.11}$$

From (4.10) we obtain

$$f_{g^{-1}\cdot(g_1^{-1}\cdot t_0)} = f_{(g_1 g)^{-1}\cdot t_0} = c(g_1 g, \cdot)\left(\frac{d\mu \circ L_{g_1 g}}{d\mu}\right)^{\frac{1}{\alpha}} (f_{t_0} \circ L_{g_1 g}) \tag{4.12}$$

and

$$f_{g_1^{-1}t_0}(g \cdot s) = c(g_1, g \cdot s)\left(\frac{d\mu \circ L_{g_1}}{d\mu}(g \cdot s)\right)^{\frac{1}{\alpha}} f_{t_0}(g_1 g \cdot s). \tag{4.13}$$

(4.13) implies that

(4.14)

$$f_{t_0}(g_1 g \cdot s) = c(g, s)c(g_1 g, s)^{-1}\left(\frac{d\mu}{d(\mu \circ L_{g_1})}(g \cdot s)\right)^{\frac{1}{\alpha}} f_{g^{-1}t_0}(g \cdot s)$$

for $\mu-$a.a $s \in S$. Combining (4.12), (4.14) and (4.11) we obtain

$$f_{g^{-1}\,(g_1^{-1}\cdot t_0)}(s) = c(g, s)\left(\frac{d\mu \circ L_g}{d\mu}(s)\right)^{\frac{1}{\alpha}} (f_{g_1^{-1}\cdot t_0} \circ L_g).$$

Therefore (4.10) implies that (4.1) holds.

Now let $t_1, t_2, ..., t_n \in T$, $a_1, ..., a_n \in \mathbb{R}$ ($\mathbb{C}$ in the complex case). (4.1) gives

$$\| \sum_{i=1}^{n} a_i f_{g^{-1}t_i} \|_\alpha = \| \sum_{i=1}^{n} a_i c(g, \cdot) \left(\frac{d\mu \circ L_g}{d\mu} \right)^{\frac{1}{\alpha}} f_{t_i} \|_\alpha = \| \sum_{i=1}^{n} a_i f_{t_i} \|_\alpha.$$

This shows that $X : T \to L^0(\Omega, P)$, $X(t) := \int_S f_t(s) M_\mu(ds)$ is $G-$stationary.

□

Remark 4.8. Condition (4.9) assures that f_t is well defined by (4.10) when $t = g_1 t_0 = g_2 t_0$ for some $g_1 \neq g_2$.

The following theorem shows that a group action induced by a $G-$stationary process is determined up to certain equivalence relation between the triplets (S, ρ, c), where (S, ρ) is a standard Borel G-space and c a strict cocycle.

Definition 4.9. A cocycle (or strict cocycle) $c_1 : G \times S \to W$ is *cohomologous* to cocycle $c_2 : G \times S \to W$ if there exist a measurable function $b : S \to W$ such that for every $g \in G$ we have $c_1(g, s) = c_2(g, s)(b(L_g(s)))/b(s)$ $\mu-$ a.e. on S.

Theorem 4.10. *Let $f^i : T \to L^\alpha(S_i, \mu_i)$, $i = 1, 2$ be minimal representations of a $G-$stationary $S\alpha S$ (rotationally invariant in the complex case) process satisfying*

$$f^i_{g^{-1}t} = c_i(g, \cdot) \left(\frac{d\mu \circ L^{\rho_i}_g}{d\mu} \right)^{\frac{1}{\alpha}} (f^i_t \circ L^{\rho_i}_g),$$

for some group actions ρ_i and strict cocycles c_i on S_i, $i = 1, 2$. Then there exist invariant Borel subsets N_1 and N_2 of S_1 and S_2, respectively

and a Borel isomorphism $\Phi : S_2 \to S_1$ *such that*

(i) μ_1 *and* $\mu_2 \circ \Phi^{-1}$ *are mutually absolutely continuous,*

(ii) $\mu_i(N_i) = 0,\ i = 1, 2$ *and* $\Phi(N_1) = N_2$,

(iii) $\Phi(gs) = g\Phi(s)$ *for all* $s \in S_2 \setminus N_2$.

(iv) the map $(g, s) \mapsto c_1(g, \Phi(s))$ *restricted to* $G \times (S_2 \setminus N_2)$ *is a strict cocycle cohomologous to* $c_2|_{G\times(S_2\setminus N_2)}$.

Proof. By Theorem 2.2 in [18]

$$f^1_{g^{-1}t} = h(f_{g^{-1}t} \circ \Phi) \quad \mu_2 - \text{a.e.},\ \ t \in T,\ g \in G.$$

Denote

$$F^i_g = c_i(g, \cdot)\left(\frac{d\mu \circ L^{\rho_i}_g}{d\mu}\right)^{\frac{1}{\alpha}},\quad g \in G,$$

$i = 1, 2$. For every $t \in T$, $g \in G$ we have

$$f^2_{g^{-1}t} = h(f^1_{g^{-1}t} \circ \Phi) = h(F^1_g \circ \Phi)(f^1_t \circ L^{\rho_1}_g \circ \Phi)$$

$$= h(f^1_t \circ L^{\rho_1}_g \circ \Phi)[h \circ \Phi^{-1} \circ L^{\rho_1}_g \circ \Phi]^{-1}(f^2_t \circ \Phi^{-1} \circ L^{\rho_1}_g \Phi).$$

Since we also have $f^2_{g^{-1}t} = (F^2_g)(f^2_t \circ L^{\rho_2}_g)$ μ_2-a.e., for each $g \in G$ and $t \in T$, by the uniqueness result of Theorem 2.2 in [18] we obtain for each $g \in G$ μ_2-a.e.

$$L^{\rho_2}_g = \Phi^{-1} \circ L^{\rho_1}_g \circ \Phi$$

and

$$F^2_g = h(F^1_g \circ \Phi)[h \circ \Phi^{-1} \circ L^{\rho_1}_g]^{-1}.$$

Since $c_i(g,\cdot) = F_g^i/|F_g^i|$, cocycle $(g,s) \mapsto c_1(g,\Phi(s))$ is cohomologous to c_2.

Thus we have shown that there exist a Borel isomorphism $\Phi : S_2 \to S_1$ such that for every $g \in G$ we have $\Phi(gs) = g\Phi(s)$ for μ-a.a. $s \in S$ and (non-strict) cocycle $(g,s) \mapsto c_1(g,\Phi(s))$ is cohomologous to c_2. Now the existence of the invariant Borel sets N_i, $i = 1,2$ required in conditions (ii) and (iv) follows from Theorem 2 in [**15**]. □

Proposition 4.11. *Let $f : T \to L^\alpha(S,\mu)$ be a minimal representation of a G-stationary $S\alpha S$ process $X : T \to L^0(\Omega,P)$ and $\rho : G \times S \to S$, $c : G \times S \to \{-1,1\}$ ($\{z \in \mathbb{C} : |z| = 1\}$ in the complex case) a group action and strict cocycle, respectively, induced by f as in Theorem 4.5. Suppose $c_1 : G \times S \to \{-1,1\}$ ($\{z \in \mathbb{C} : |z| = 1\}$ in the complex case) is a strict cocycle cohomologous to c. Then there exist a minimal representation $f' : T \to L^\alpha(S,\mu)$ of X such that for every $g \in G$, $t \in T$*

$$f'_{g^{-1}t} = c_1(g,\cdot)\left(\frac{d\mu \circ L_g}{d\mu}\right)^{\frac{1}{\alpha}} (f'_t \circ L_g), \quad \mu - \text{a.e.} \tag{4.15}$$

Proof. Let $b : S \to \{-1,1\}$ ($\{z \in \mathbb{C} : |z| = 1\}$ in the complex case) be a function such that for all $g \in G$

$$c(g,s) = b(g \cdot s)c_1(g,s)b(s)^{-1} \quad \mu - \text{a.e.}$$

For each $t \in T$, $s \in S$ define

$$f'_t(s) := f_t(s)b(s). \tag{4.16}$$

The clearly $t \mapsto f'_t$ is a minimal representation of X and for every $g \in G$, $t \in T$ we have

$$f'_{g^{-1}\cdot t} = f_{g^{-1}\cdot t}(s)b(s) = c(g,s)\left(\frac{d\mu \circ L_g}{d\mu}(s)\right)^{\frac{1}{\alpha}} f_t(g\cdot s)b(s)^{-1} =$$

$$c_1(g,s)\left(\frac{d\mu \circ L_g}{d\mu}(s)\right)^{\frac{1}{\alpha}} f'_t(g\cdot s), \quad \mu - \text{a.e.}$$

□

Remark 4.12. If we drop the minimality requirement in the hypotheses of Proposition 4.11, then $f' : T \to L^\alpha(S,\mu)$ defined by (4.16) is also a representation of X (not necessarily minimal) and (4.15) holds.

CHAPTER 5

Structural analysis for G-stationary stable processes

In his paper [**18**] followed by [**19**] Rosinski showed that each stationary SαS process (field) $X : \mathbb{R}^d \to L^0_{\mathbb{C}}(\Omega, P)$ can be uniquely decomposed into three independent parts

$$X \stackrel{d}{=} X^{(1)} + X^{(2)} + X^{(3)}$$

where $X^{(1)}$ is a mixed moving average process, $X^{(2)}$ is harmonizable and $X^{(3)}$ is a SαS stationary process "of the third type" without mixed moving average or harmonizable components. Following [**4**] we will call the process "of the third type" *evanescent*. The structure of evanescent stable processes is not well understood at present. The goal of this chapter is to prove similar decomposition theorem for the case of G-stationary SαS process. We also characterize the parts of the decomposition in terms of types of group action and cocycle induced by a minimal representation.

Before we proceed let us recall the terminology. Let (S, μ) be a standard Borel space with a σ-finite Borel measure μ. A set $A \subset S$ is called a *$\mu-$null set* if $A \subset B$ for some $B \in \mathbb{B}(S)$ with $\mu(B) = 0$. The measure μ can be uniquely extended to a measure $\overline{\mu}$ defined on the σ-algebra $\overline{\mathbb{B}}_\mu$ generated by $\mathbb{B}(S)$ and μ-null sets. The members of $\overline{\mathbb{B}}_\mu$ are called

μ-measurable sets. We say that a property holds μ-almost everywhere ($\mu-$a.e.) if the set of $s \in S$ such that the property does not hold is μ-null.

Let G be a Polish group acting on a standard Borel space S. If the action $\rho : G \times S \to S$ is a Borel measurable map, then (S, ρ) is called a *standard Borel $G-$space*. If μ is a Borel measure on S such that $\mu(B) = 0$ implies $\mu(g^{-1} \cdot B) := \mu(\{x \in X : g \cdot x \in B\}) = 0$, for all $B \in \mathbb{B}(S)$, then we call the measure μ *quasi-invariant*. We will write a standard Borel space (S, ρ) as a triple (S, ρ, μ) to indicate that μ is a $\sigma-$finite quasi-invariant Borel measure on S.

Let $X : T \to L^0(\Omega, P)$ be a SαS G-stationary process, where G is a locally compact Polish group, and let $f : T \to L^\alpha(S, \mu)$ be a minimal representation of X on a standard Borel space S. As we know from Theorem 4.5 there exist a nonsingular action ρ of G on S and a strict cocycle c for ρ such that (S, ρ, μ) is a standard Borel $G-$space and for all $t \in T$, $g \in G$ we have

$$f_{g^{-1}t} = c(g, \cdot) \left(\frac{d\mu \circ L_g}{d\mu} \right)^{\frac{1}{\alpha}} (f_t \circ L_g), \tag{5.1}$$

where $L_g : S \ni s \longmapsto \rho(g, s) \in S$. We are seeking a way to decompose X into at most countable number of independent G-stationary SαS processes, whose structure is simpler than the structure of X. We can notice that every partition of S into a countable number of ρ-invariant Borel sets B_i, $i \in \mathbb{N}$ generates a decomposition of X into independent

G-stationary processes given by

$$X_i = \int_{B_i} f_t \, dM_\mu,$$

where M is SαS random measure with control measure μ. However, such decomposition typically depends on particular choice of the minimal representation, which is defined up do a null-preserving isomorphism and cohomology class of a cocycle. To avoid this difficulty we may partition S into invariant Borel subsets defined by an equivalence relation which is preserved under null-preserving isomorphisms and change of cocycle within the some cohomology class. To define such relation we need an extension of the notion of the orbit type (see, for example [**3**, Ch. I.4]) to G-space-cocycle pairs.

Recall that the isotropy group G_s at s is the group $G_s := \{g \in G : g \cdot s = s\}$. It is easy to check that $G_{g \cdot s} = g G_s g^{-1}$.

Definition 5.1. Let (S, ρ) be a standard Borel G-space, $c : G \times S \to S$ a strict cocycle, G_0 a closed subgroup of G and $m : G_0 \to W$ a measurable homomorphism with values in a locally compact Polish Abelian group W. We say that $s \in S$ is of orbit-cocycle type (G_0, m) if there exist $g \in G$ such that $G_{gs} = G_0$ and $c(h, gs) = m(h)$ for every $h \in G_0$. We say that s_1, $s_2 \in S$ have the same orbit-cocycle type and write $s_1 \overset{t}{\sim} s_2$ if both s_1 and s_2 are of orbit-cocycle type (G_0, m) for some closed subgroup $G_0 \subset G$ and measurable homomorphism $m : G_0 \to W$.

Proposition 5.2. *Under assumptions of Definition 5.1 the following are equivalent:*

(i) $s \in S$ is of orbit-cocycle type (G_0, m).

(ii) gs is of orbit-cocycle type (G_0, m) for every $g \in G$.

(ii) For every $g \in G$, s of orbit-cocycle type (gG_0g^{-1}, m_g), where $m_g : gG_0g^{-1} \to W$ is defined by $m_g(h) = m(g^{-1}hg)$, $h \in gG_0g^{-1}$.

Proof. Fix $g \in G$ and let $g_1 \in G$ be such that $G_{g_1 s} = G_0$ and $c(h, g_1 s) = m(h)$ for every $h \in G_0$.

(i)⇒(ii): Set $g_2 = g_1 g^{-1}$. Then $G_{g_2(gs)} = G_{g_1 s} = G_0$ and $c(h, g_2(gs)) = c(h, g_1 s) = m(h)$ for every $h \in G_0$. Thus gs is of orbit-cocycle type (G_0, m).

(i)⇒(iii): Set $g_2 = gg_1$. Then $G_{g_2 s} = G_{gg_1 s} = gG_{g_1 s}g^{-1} = gG_0g^{-1}$ and for every $h' \in gG_0g^{-1}$, $h' = ghg^{-1}$, $h \in G_0$ we have

$$c(h', g_2 s) = c(ghg^{-1},\ gg_1 s) = c(gh,\ g_1 s)c(g, g_1 s)^{-1} =$$

$$c(g,\ hg_1 s)c(h,\ g_1 s)c(g,\ g_1 s)^{-1} = c(g,\ g_1 s)c(g,\ g_1 s)^{-1}c(h,\ g_1 s) = m(h) = m_g(h'),$$

therefore s if of orbit-cocycle type (gG_0g^{-1}, m_g).

The implications (ii)⇒(i) and (iii)⇒(i) are obvious. □

Remark 5.3. The equivalence of (i) and (ii) in Proposition 5.2 shows that the orbit-cocycle type characterizes the orbit of $s \in S$ under the group action, rather than s itself. The equivalence of (i) and (iii) shows that there is a 1:1 correspondence between possible orbit-cocycle types and pairs consisting of a conjugacy class of a closed subgroup of G and

a homomorphism defined on a representant of this class. We note also that every $s \in S$ is of orbit-cocycle type $(G_s, h \mapsto c(h,s))$.

The following proposition follows immediately from the definition of the relation $\overset{t}{\sim}$, Proposition 5.2 and the fact that $G_{gs} = gG_sg^{-1}$ for all $g \in G$ and $s \in S$.

Proposition 5.4. *Under assumptions of Definition 5.1* $s_1 \overset{t}{\sim} s_2$ *if and only if there exist a* $g \in G$ *such that*

$$G_{s_1} = G_{gs_2} \text{ and } c(h,s_1) = c_2(ghg^{-1}, s_2) \text{ for all } h \in G_{s_1}. \tag{5.2}$$

The following proposition shows that each orbit-cocycle type corresponds to distinct class of G-stationary SαS processes.

Proposition 5.5. *Let* G *be a locally compact group and* $f_i : T \to L^\alpha(S_i, \mu_i)$, $i = 1,2$ *be minimal representations of a* G*-stationary* SαS *(rotationally invariant in the complex case) process satisfying*

$$f^i_{g^{-1}t} = c_i(g,\cdot)\left(\frac{d\mu \circ L^{\rho_i}_g}{d\mu}\right)^{\frac{1}{\alpha}} (f_t \circ L_g),$$

for some group actions ρ_i *and strict cocycles* c_i *on* S_i, $i = 1,2$. *Suppose further that* G_0 *is a closed subgroup of* G *and* $m : G_0 \to W$ *is a measurable homomorphism and*

$$s \text{ is of } \text{orbit} - \text{cocycle type } (G_0, m) \text{ for } \mu_1 - \text{a.a } s \in S_1.$$

Then

$$s \text{ is of } \text{orbit} - \text{cocycle type } (G_0, m) \text{ for } \mu_2 - \text{a.a } s \in S_2.$$

Proof. By Theorem 4.10 there exist invariant Borel subsets N_1 and N_2 of S_1 and S_2, respectively and a Borel isomorphism $\Phi : S_2 \to S_1$ such that $\Phi(N_2) = N_1$, μ_1 and $\mu_2 \circ \Phi^{-1}$ are mutually absolutely continuous,$\mu_i(N_i) = 0$, $i = 1, 2$, $\Phi(gs) = g\Phi(s)$ for all $s \in S_2 \setminus N_2$ and cocycle $(g, s) \mapsto c_1(g, \Phi(s))$ is cohomologous to c_2 on $S_2 \setminus N_2$. This implies that $G_{\Phi(s)} = G_s$ for all $s \in S_2 \setminus N_2$ and by Theorem 1.12 in the Appendix the cocycles $(g, s) \mapsto c_1(g, \Phi(s))$ and c_2 generate the same homomorphisms on the isotropy groups G_s for μ-a.a $s \in S_2 \setminus N_2$. Therefore s is of the same orbit-cocycle type as $\Phi(s)$ for μ-a.a. $s \in S_2$. □

Proposition 5.5 demonstrates that the following definition does not depend on the choice of minimal representation of X.

Definition 5.6. If a process X satisfies the hypotheses of Proposition 5.5, then we say that X is of orbit-cocycle type (G_0, m).

To show that the equivalence relation $\overset{t}{\sim}$ decomposes S into Borel sets (up to μ- null sets) we will a use a result about measurability of the map $s \mapsto G_s$ for standard Borel G-spaces, which can be found in [1]. To state this result we will need some additional background material on Polish spaces.

Definition 5.7. Let X be a Polish space. The *Effros Borel space* is the collection $\mathcal{F}(X)$ of closed subsets of X with the Borel structure generated by the sets (collections of sets in $\mathcal{F}(X)$) of the form

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

for some open $U \subset X$. Effros Borel space is standard (see [**13**, Th. II.12.6]).

Lemma 5.8. *Let G a locally compact Polish group and (S, ρ) a standard Borel G-space. Then the map $\Phi : S \to \mathcal{F}(G)$, $\Phi(s) = G_s$ is Borel.*

Proof. It is shown in [**1**, Th. 7.1.2] that the if group G is Polish and the equivalence relation induced by the group action is a Borel subset of $S \times S$, then the map $s \mapsto G_s$ is Borel. By Proposition 1.4 in the Appendix if the group is locally compact, this equivalence relation is a Borel subset of $S \times S$ and the statement follows. □

Definition 5.9. Let X be a Polish space. The set $A \subset X$ is called *analytic* if there is a Polish space Y and a continuous mapping $f : Y \to X$ with $f(Y) = A$. If S is a standard Borel space and $A \subset S$, then A is called analytic if there is a Polish space Y and Borel isomorphism $f : S \to Y$, such that $f(A)$ is analytic in Y.

All Borel sets are analytic, but the opposite inclusion is proper (for uncountable standard Borel spaces). The following proposition combines Proposition 14.4 and Theorems 14.11, 29.7 in [**13**] and lists the properties of analytic sets we will need for the statement and proof of Proposition 5.11.

Proposition 5.10. *(i) If X is a Polish space and $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of analytic subsets of X, then $\bigcup_n A_n$ and $\bigcap_n A_n$ is analytic.*
(ii) If $A \subset X$ is analytic and A^c is analytic, then A is a Borel set.

(ii) If $A \subset X$ is analytic and μ is a Borel measure on X, then A is $\mu-$measurable.

Proposition 5.11. *Relation "$s_1 \overset{t}{\sim} s_2$" is an equivalence relation on S. Moreover, if (S, ρ, μ) is a standard Borel G-space with a σ-finite quasi-invariant Borel measure μ and G is locally compact Polish group, then for every $s \in S$ the equivalence class $[s]_{\overset{t}{\sim}}$ is analytic (hence μ-measurable).*

Proof. Clearly $s \overset{t}{\sim} s$ for every $s \in S$. Suppose that $s_1 \overset{t}{\sim} s_2$ and let $g \in G$ be such that $G_{s_1} = gG_{s_2}g^{-1}$. Then for $g' := g^{-1}$ we get $G_{s_2} = g^{-1}G_{s_1}g = G_{g's_2}$. Now let $h \in G_{s_2}$. Then $h' = ghg^{-1} \in G_{s_1}$, thus

$$c(h, s_2) = c(g^{-1}h'g, s_2) = c(h', s_1) = c(ghg^{-1}, s_1) = c(g'^{-1}hg', s_1)$$

and we can conclude that $s_2 \overset{t}{\sim} s_1$.

Now suppose $s_1 \overset{t}{\sim} s_2$ and $s_2 \overset{t}{\sim} s_3$ and let $g_1, g_2 \in G$ be such that $G_{s_1} = g_1G_{s_2}g_1^{-1}$ and $c(h_1, s_1) = c(g_1^{-1}h_1g_1, s_2)$ for all $h_1 \in G_{s_1}$ and $G_{s_2} = g_2G_{s_3}g_2^{-1}$ and $c(h_2, s_2) = c(g_2^{-1}h_2g_2, s_3)$ for all $h_2 \in G_{s_2}$. Set $g_3 = g_1g_2$. Then we have

$$G_{s_1} = g_1G_{s_2}g_1^{-1} = g_1g_2G_{s_3}g_2^{-1}g_1^{-1} = g_3G_{s_3}g_3^{-1}.$$

For any $h \in G_{s_1}$ we also have

$$c(h, s_1) = c(g_1^{-1}hg_1, s_2) = c(g_2^{-1}g_1^{-1}hg_1g_2, s_3) = c(g_3^{-1}hg_3, s_3),$$

hence $s_1 \overset{t}{\sim} s_3$. This concludes the proof that $\overset{t}{\sim}$ is an equivalence relation on S.

To see the second part of the claim fix $s_1 \in S$ and let $\{h_n\}_{n\in\mathbb{N}}$ be a dense set in G_{s_1}. Since for every $s \in S$ the mapping $h \mapsto c(h,s)$ is a measurable homomorphism of G_s into M, it must be continuous by a theorem due to Banach, see for example [5, Ch. 5]. Thus the condition $c(h,s_1) = c_2(ghg^{-1}, s_2)$ for all $h \in G_{s_1}$ in (5.2) is equivalent to $c(h_n, s_1) = c_2(gh_ng^{-1}, s_2)$ for all $n \in \mathbb{N}$.

Now consider the functions $\Phi : G \times S \to \mathcal{F}(G)$, $\Phi(g,s) = G_{gs}$ and $\Psi_n : G \times S \to M$, $\Psi_n(g,s) = c(g^{-1}h_ng, s)$, $n \in \mathbb{N}$. Φ is Borel by Lemma 5.8 and Ψ_n is Borel for every n by the definition of the cocycle. Thus the sets

$$A := \{(g,s) : \Phi(g,s) = G_{s_1}\}, \quad B_n := \{(g,s) : \Psi(g,s) = c(h_n,s)\}, \quad n \in \mathbb{N}$$

are Borel. Now by Proposition 5.4 $s_2 \in [s_1]_{\overset{t}{\sim}}$ iff there exist $g \in G$ such that

$$(g, s_2) \in \left(A \cap \bigcap_{n\in\mathbb{N}} B_n \right),$$

thus $[s_1]_{\overset{t}{\sim}} = \pi(A \cap \bigcap_{n\in\mathbb{N}} B_n)$, where $\pi : G \times S \to S$ is the projection on S. This shows that $[s_1]_{\overset{t}{\sim}}$ is a projection image of a Borel set, therefore an analytic set in S. □

For further analysis we assume that W is one of the (multiplicative) groups $W_1 = \{z \in \mathbb{C} : |z| = 1\}$, $W_0 = \{-1, 1\}$. For a topological group G consider the following relation R_G defined on the set of pairs of the

form (H, m), where H is a closed subgroup of G and $m : H \to W$ is a measurable (hence continuous) homomorphism

$(H_1, m_1) R (H_2, m_2)$ iff there exist $g \in G$ such that $H_1 = gH_2g^{-1}$ and $m_1(h) = m_2(g^{-1}hg)$, , $h \in H_1$.

Denote by I_G the collection of equivalence classes for R_G. It is easy to see that I_G indexes all possible orbit-cocycle types of standard Borel G-spaces for this specific G. For the equivalence relation $\overset{t}{\sim}$ to be useful in decompositions of G-stationary stable processes we need I_G to be at most countable. We currently do not know the characterization of topological groups for which I_G is countable. Thus to characterize all G-stationary SαS processes for a specific group G we need to check if I_G is countable for this group. In particular for the groups $\mathbf{SO}(2)$, $\mathbf{SO}(3)$ treated in Chapter 7 I_G is countable and for finite groups I_G is finite. An example of group for which I_G is **not** countable is the additive group $\mathbb{R}$ of real numbers. Indeed, closed (proper) subgroups of $\mathbb{R}$ are of the form $H_r := \{rk, k \in \mathbb{Z}\}$, $r \in \mathbb{R}$, $r \geq 0$ and each subgroup H_r forms its own conjugacy class. As for the group $\mathbb{Z}$ the situation is different depending on whether we consider real or complex valued processes. $\mathbb{Z}$ has countably many subgroups of the form $H_k := \{jk, j \in \mathbb{Z}\}$, $k = 0, 1, 2....$ If $W = \{z \in \mathbb{C} : |z| = 1\}$, then for each $\theta \in [0, 2\pi)$ we can define a homomorphism $m_\theta : H_k \to W$,

$$m(jk) := e^{jk\theta}.$$

However, if $W = \{-1, 1\}$, then for each group H_k we have only two homomorphisms: the trivial one $m_0 \equiv 1$ and the one defined by

$m_1(jk) := (-1)^j$, thus I_G is countable for $G := \mathbb{Z}$ in the real valued context and not countable in the complex context.

Now we are ready to state the main theorem on decomposition of stable G-stationary processes.

Theorem 5.12. *Let $X : T \to L^0(\Omega, P)$ be a G-stationary $S\alpha S$ (rotationally invariant in the complex case) process, where G is a locally compact group for which I_G is at most countable. Then there exist a collection $\{X_i\}_{i \in I_G}$ of independent G-stationary processes $X_i : T \to L^0(\Omega, P)$ such that*

$$X \stackrel{d}{=} \sum_{i \in I_G} X_i \tag{5.3}$$

and each X_i is of orbit-cocycle type (G_0^i, m_i), where (G_0^i, m_i) is a representant of the class $i \in I_G$.

Proof. Let $f : T \to L^\alpha(S, \mu)$ be a minimal representation of X satisfying 5.1. For every $i \in I_G$ define

$$B_i := \{s \in S : s \text{ is of } \text{orbit} - \text{cocycle type } (G_0^i, m_i)\},$$

where $i \in I_G$ is a representant of class $i \in I_G$. By Proposition 5.2 (ii) the set B_i is analytic and invariant and by (iii) it does not depend on the choice of group and homomorphism to represent the class $i \in I_G$. By Proposition 5.10 (i)

$$\bigcup_{j \in I_G, j \neq i} B_j = B_i^c$$

is also analytic, therefore by Proposition 5.10 (ii) B_i is a Borel set, $i \in I_G$.

Setting

$$X_i(t) := \int_{B_i} f_t \, dM_\mu$$

we obtain decomposition (5.3). □

CHAPTER 6

Moving average processes

The best known examples of stationary random fields on $\mathbb{R}^d$ are moving average random fields. A moving average SαS random field is given by

$$X \stackrel{d}{=} \left(\int_{\mathbb{R}^d} f(t+s) M_H(ds),\, t \in \mathbb{R}^d \right),$$

where H is the Lebesgue measure on $\mathbb{R}^d$, $f \in L^\alpha(\mathbb{R}^d, H)$ and M_H is the SαS (rotationally invariant in the complex case) random measure with control measure H. A mixed moving average field is represented by

$$X \stackrel{d}{=} \left(\int_{\mathbb{R}^d \times W} f(t+s,w) M_{H\times\nu}(ds,dw),\, t \in \mathbb{R}^d \right).$$

See [25] for more information on stable mixed moving averages.

In this chapter we consider analogous types of random fields defined on a $G-$space (T,η), where T is a Polish space and η is a measurable action. We will assume that (T,η) admits a Borel selector T_0.

Definition 6.1. Let G be a locally compact group with a left Haar measure H and (T,η) a $G-$space. A SαS process $X : T \to L^0(\Omega, P)$ is called a *$G-$moving average process* if it has a minimal representation of the form

$$(X_t, t \in T) \stackrel{d}{=} \left(\int_G F(g^{-1}\cdot t) M(dg),\, t \in T \right)$$

for some $F : T \to \mathbb{R}$ ($\mathbb{C}$ in the complex case) such that $(g \mapsto F(g \cdot t)) \in L^\alpha(G, m)$ for every $t \in T$, where M is a SαS random measure on G with control measure H. A *mixed $G-$moving average* has a minimal representation

$$(X_t, t \in T) \stackrel{d}{=} \left(\int_{G \times W} F(g^{-1} \cdot t, w) M(dg, dw),\ t \in T \right) \tag{6.1}$$

where W is a standard Borel space with $\sigma-$finite Borel measure λ, $((g, w) \longmapsto F(g^{-1}t, w)) \in L^\alpha(G \times W, m \times \lambda)$ for each $t \in T$ and the control measure of M is $H \times \lambda$.

The following two propositions demonstrate the importance of the minimality requirement in Definition 6.1.

Proposition 6.2. *If G is compact, then every $G-$stationary SαS process $(X_t,\ t \in T)$ has a (not necessary minimal) representation of the form (6.1).*

Proof. Let $\{f_t,\}_{t \in T} \subset L^\alpha(S, \mu)$ be a representation of $(X_t,\ t \in T)$. Then the collection of functions $F_t : G \times S \to \mathbb{R}$, given by $F_t(g, s) = f_{g^{-1}t}(s)$ is also a representation of $(X_t,\ t \in T)$. Indeed, for $a_1, ..., a_n \in \mathbb{R}$ and $t_1, ..., t_n \in T$ we have

$$\int_{G \times S} \left| \sum_{i=1}^n a_i F_{t_i}(g, s) \right|^\alpha (H \times \mu)(dg, ds) = \int_G \int_S \left| \sum_{i=1}^n a_i f_{g^{-1}t_i}(s) \right|^\alpha \mu(ds) H(ds)$$

$$= \int_S \left| \sum_{i=1}^n a_i f_{t_i}(s) \right|^\alpha \mu(ds).$$

□

If we require minimality in Definition 6.1, then the class of mixed moving averages becomes a proper subclass of $G-$stationary processes.

Proposition 6.3. *Let (S, ρ, μ) be a standard Borel $G-$space, G a compact Polish group, $f : T \to L^\alpha(S, \mu)$ a minimal representation of a $S\alpha S$ process X such that for all $t \in T$, $g \in G$*

$$f_{g^{-1}t} = c(g, \cdot) \left(\frac{d\mu \circ L_g^\rho}{d\mu} \right)^{\frac{1}{\alpha}} (f_t \circ L_g) \quad \mu - \text{a.e.} \tag{6.2}$$

for some strict cocycle c on S. Then X is a mixed moving average iff the group action ρ is free $\mu-$a.e. (that is $G_s = \{e\}$ for μ-a.a. $s \in S$).

Proof. Using representation f we will construct a (minimal) mixed moving average representation required in Definition 6.1.

By Lemma 5.8 the set $S_1 := \{s \in S : G_s = \{e\}\}$ is Borel. Clearly S_1 is invariant and $\mu(S \setminus S_1) = 0$. Thus, without loss of generality we may assume that $G_s = \{e\}$ for all $s \in S$.

By Theorem 4.6 (ii) we can find a minimal representation $f' : T \to L^\alpha(S, \lambda)$ of X such that λ is an invariant Borel probability measure. We then have for each $g \in G$, $t \in T$

$$f'_{g^{-1}t} = c(g, \cdot)(f'_t \circ L_g) \quad \mu - \text{a.e.}$$

Obviously the only homomorphism defined on the trivial group is the trivial one. By Theorem 1.12 in the Appendix this implies that that the cocycle c in (6.2) is cohomologous to the cocycle $c_1 \equiv 1$. Modifying the representation f' in a way described in Proposition 4.11 we can find

a representation $f'' : T \to L^\alpha(S, \mu)$ such that for every $g \in G$, $t \in T$

$$f''_{g^{-1}t} = f''_t \circ L_g \quad \mu - \text{a.e.}$$

Let $\beta : S \to S$ be a Borel selector for the group action ρ and $S_0 := \{s \in S : \beta(s) = s\}$ be a Borel transversal (both Borel selector and Borel transversal exist if G is compact Polish, see Corollary 1.7 in the Appendix). S_0 together with the relative Borel structure $\mathbb{B}(S_0) := \mathbb{B}(S) \cap S_0$ is a standard Borel space. Define $\Phi : G \times S_0 \to S$ by

$$\Phi(g, s) := g \cdot s.$$

Φ is a 1:1 Borel map, thus a Borel isomorphism. Let λ' be a Borel measure on $G \times S_0$ defined by

$$\lambda'(B) := \lambda(\Phi(B)), \ \ B \in \mathbb{B}(G \times S_0).$$

Note that for any $g \in G$ and $B \in \mathbb{B}(G \times S_0)$ we have

$$\lambda'(\{(gg_1, s_0) \in G \times S_0 : (g, s_0) \in B\}) = \lambda(g \cdot \Phi(B)) = \lambda(\Phi(B)) = \lambda'(B).$$

This means that the measure λ' is invariant with respect to the natural action ζ of G on $G \times S_0$ defined by $\zeta(g, (g_1, s)) := (gg_1, s)$, $g, g_1 \in G$, $s \in S_0$. To see that $\lambda' = H \times \nu$ for some measure ν on S_0 we employ an argument analogous to the argument in Step 5 of the proof of Theorem 2.1 in [**19**]. For fixed $A \in \mathbb{B}(S_0)$ consider the measure on G given by $\lambda'(\cdot \times A)$. Then for any $B \in \mathbb{B}(G)$ and $g \in G$

$$\lambda'((gB) \times A) = \lambda'(g(B \times A)) = \mu(\Phi(g(B \times A)) =$$

$$\mu(g \cdot \Phi(B \times A)) = \mu(\Phi(B \times A)) = \lambda'(B \times A).$$

Thus the measure $B \mapsto \lambda'(B \times A)$ is proportional to the Haar measure. Writing $\nu(A)$ to denote the proportionality constant we see that $A \mapsto \nu(A)$ is a measure on S_0 and $\lambda'(B \times A) = H(B)\nu(A)$ for all $B \in \mathbb{B}(G)$ and $A \in \mathbb{B}(S_0)$. Therefore $\lambda' = H \times \nu$.

Now let us define $F : T \times S_0 \to \mathbb{R}$ ($\mathbb{C}$ in the complex case) by

$$F(g \cdot t_0) := f''(g^{-1} \cdot s), \;\; g \in G, \, t_0 \in T_0, \, s \in S_0,$$

where T_0 is a Borel selector for the $G-$space (T, η). Then for any $t \in T$, $t = g_1^{-1} \cdot t$ we have

$$F(g^{-1} \cdot t_0, s) = F((g_1 g)^{-1} \cdot t_0) = f''_{t_0}(g_1 g \cdot s) = f''_{g_1^{-1} \cdot t_0}(g \cdot s) = (f''_t \circ \Phi)(g, s)$$

$H \times \nu$ a.e. on $G \times S_0$. This implies

$$\left(\int_{G \times S_0} F(g^{-1} \cdot t, s) \, M_{H \times \nu}(dg, ds), \, t \in T \right) \stackrel{d}{=} \left(\int_S f''_t(s) M_\lambda(ds), \, t \in T \right) \stackrel{d}{=} X,$$

hence F defines a minimal representation required in the Definition 6.1.

Now suppose that X is a mixed moving average, i.e. admits a minimal representation of the form (6.1). We can see that the action ζ of G on $G \times W$, $\zeta(g, (g_1, w)) = (gg_1, w)$, is free. Then by Theorem 4.10 the action induced by any other minimal representation is free μ-a.e. $\square$

Remark 6.4. Using terminology introduced in Chapter 5, Proposition 6.3 says that for compact Polish groups mixed $G-$moving averages can be identified with the processes of orbit-cocycle type $(\{e\}, 1)$.

CHAPTER 7

Isotropic SαS random fields on $\mathbb{R}^2$

An interesting class of random fields on $\mathbb{R}^n$ that often arises in applications is the class of *isotropic* random fields. These are characterized by the property that the distribution of the field is invariant under rotations of $\mathbb{R}^n$, so that

$$X(g\cdot) \stackrel{d}{=} X(\cdot), \tag{7.1}$$

where $g \in \mathbf{SO}(n)$.

The characterization of Gaussian (2-stable) isotropic fields due to Yadrenko [28] is known. Different forms of this characterization are written usually as series and integral expansions of the covariance function of the field in terms of special functions. Since there is no covariance function for SαS fields if $1 < \alpha < 2$, the most convenient for comparisons is the characterization of the Gaussian isotropic fields in terms of spherical harmonics, which does not use the notion of covariance function. We quote the result from [28] in Theorem 7.7 for easy reference.

The characterization of SαS isotropic random fields for $1 < \alpha < 2$ is unknown. In Theorems 7.1, 7.3, 7.5, 7.6 we present partial results which characterize real valued and complex valued SαS non-Gaussian isotropic random fields on the unit circle in $\mathbb{R}^2$ and on $\mathbb{R}^2$, respectively. The main tool in the proofs is the decomposition Theorem 5.12 applied

to the group $G = \mathbf{SO}(2)$ acting on the index set $T = \mathbb{R}^2$. Relatively simple structure of $\mathbf{SO}(2)$ makes it a rather straightforward application of Theorem 5.12. The proof of Theorem 7.1 can be written in a much more compact way if we use more general results from Appendix concerning the connection between cohomology classes of strict cocycles and homomorphisms on isotropy groups. However, we choose not to use those results to give the reader the opportunity to follow the details of the proof in a special case.

To simplify the notation we will treat $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and identify $\mathbf{SO}(2)$ with the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, which we will denote G, for consistency with notation used in Chapter 5. The group action of G on $\mathbb{C}$ is the usual multiplication of complex numbers. The Haar measure on G is the uniform measure with total mass 1 and is denoted H. Measure m on $[0, 2\pi)$ is the Lebesgue measure normalized so that $m([0, 2\pi)) = 1$.

Theorem 7.1. *Let $X : G \to L^0(\Omega, P)$ be a SαS (real valued) random field, $1 < \alpha < 2$, such that*

$$X(g\cdot) \stackrel{d}{=} X(\cdot) \text{ for all } g \in G.$$

Then there exist independent random fields $Y_k^i : G \to L^0(\Omega, P)$, $k = 1, 2, \ldots$,$i = 1, 2$ and a SαS random variable Y_0 such that
(i) the field defined by

$$(7.2)\quad Y(e^{i\theta}) := Y_0 + \sum_{k=1}^{\infty} \left(Y_k^1(e^{ik\theta}) + (-1)^{\left[\frac{k\theta}{\pi}\right]} Y_k^2(e^{2ki\theta}) \right), \quad \theta \in [0, 2\pi)$$

is equal in distribution to X.

(ii) the fields Y_k^1, $k = 1, 2, \ldots$ *admit a minimal representation of the form*

(7.3)

$$\left(Y_k^1(e^{i\theta}),\ \theta \in [0, 2\pi)\right) \stackrel{d}{=} \left(\int_{[0,2\pi)\times W_k^1} F_k^1(e^{i(\theta-\xi)}, w) M_{m\times\nu_k^1}(d\xi, dw),\ \theta \in [0, 2\pi)\right),$$

for some functions $F_k^1 \in L^\alpha([0, 2\pi) \times W_k^1, m \times \nu)$, *where* W_k^1 *is Polish space with Borel measure* ν_k^1, $k = 1, 2, \ldots$ *(that is,* Y_k^1 *is a mixed* $G-$*moving average, see Chapter 6).*

(iii) the fields Y_k^2, $k = 1, 2, \ldots$ *admit a minimal representation of the form*

$$\left(Y_k^2(e^{i\theta}),\ \theta \in [0, 2\pi)\right) \stackrel{d}{=}$$

$$\left(\int_{[0,2\pi)\times W_k^2} \operatorname{sign}(\theta - \xi) F_k^2(e^{i(\theta-\xi)}, w) M_{m\times\nu_k^2}(d\xi, dw),\ \theta \in [0, 2\pi)\right),$$

for some functions $F_k^2 \in L^\alpha([0, 2\pi) \times W_k^2, m \times \nu_k^2)$.

Remark 7.2. Note that the terms in (7.2) have the property, that rotating the field by an angle $\theta = 2\pi/k$ either does not change the value of the field (for the terms corresponding to Y_k^1), or multiplies it by -1 (for the tems corresponding to Y_k^2). In this sense decomposition (7.2) can be interpreted as spectral decomposition of X into random fields which are periodic or "antiperiodic", with different periods (frequencies).

Proof. We start by listing all possible orbit-cocycle types for the group G. Since G is Abelian in this case, it amounts to listing all closed subgroups of G with $\{-1,1\}$ valued measurable homomorphisms.

Let H_n, $n = 1,2,...$ be the group of n'th complex roots of 1. If $n = 2k-1$, $k = 1,2,...$, then the only homomorphism $m : H_n \to \{-1,1\}$ is the trivial one, $m \equiv 1$. This is true also for G. If $n = 2k$, $k = 1,2,...$, then there are two homomorphisms $m : H_k \to \{-1,1\}$, the trivial one and the one defined by

$$m_n\left(e^{ij\frac{2\pi}{n}}\right) := (-1)^j, \quad j = 0,1,2,...,n-1. \tag{7.4}$$

Thus by Theorem 5.12 we have the following decomposition:

$$X \stackrel{d}{=} X_\infty + \sum_{n=1}^{\infty} X_n^1 + \sum_{k=1} X_k^2, \tag{7.5}$$

where X_∞ is a G-stationary SαS process of orbit-cocycle type $(G,1)$, X_n^1, $n = 1,2,...$ is of orbit-cocycle type $(H_n,1)$ and X_k^2, $k = 1,2,...$ is of orbit-cocycle type (H_{2k}, m_{2k}), where m_{2k} is defined by (7.4). Now we need to identify the terms of (7.5) as those described in the statement of the theorem.

Being of orbit-cocycle type $(G,1)$ means that the induced action is trivial and for all $g \in G$ we have

$$f_{g^{-1}}^{\infty}(s) = f_1(g \cdot s) = f_1(s) \quad \mu - \text{a.e.}$$

for some minimal representation of X_∞. Here we use "$\cdot$" to indicate the group action of G on S. This implies that that $X_\infty(g) = X_\infty(1)$ for all $g \in G$ and X_∞ is a random variable.

Let us consider random fields of orbit-cocycle type $(H_n, 1)$, $n = 1, 2, ...$ Suppose that $Z : G \to L^0(\Omega, P)$ is such field for a fixed n and $f : G \to L^\alpha(S, \mu)$ is a minimal representation on a Polish $G-$space (S, ρ, μ), satisfying for each $g \in G$,

$$f_{g^{-1}}(s) = c(g, s) f_1(g \cdot s) \ \ \mu - \text{a.e} \tag{7.6}$$

for some continuous group action ρ, invariant finite measure μ and strict cocycle ρ. The existence of such representation is provided by Theorem 4.6.

If $h \in H_n$ we have from (7.6)

$$f_{h^{-1}g^{-1}}(s) \stackrel{\mu}{=} c(gh, \cdot) f_1(gh \cdot s) = c(g, s) c(h, s) f_1(gs) = c(g, s) f_1(s) \stackrel{\mu}{=} f_{g^{-1}}(s),$$

which implies

$$Z(hg) = Z(g) \ \text{ for all } \ g \in G, \ h \in H_n. \tag{7.7}$$

Let $\beta : S \to S$ be a Borel selector for this group action, that is a Borel map $\beta : S \to S$ such that $\beta(s) \in G \cdot s$ and $s \in G \cdot s_1$ implies $\beta(s) = \beta(s_1)$ for all $s, s_1 \in S$. Such Borel selector always exists if G is a compact group - see Chapter A. Every $s \in S$ can be written as

$$s = g\beta(s) \tag{7.8}$$

for some $g \in G$. g is not unique in (7.8), however, for every s there is exactly one $g = e^{i\theta}$ such that (7.8) holds and $\theta \in [0, 2\pi/n)$.

Since Z is of orbit-cocycle type $(H_n, 1)$, we have

$$G_s = H_n \text{ and } \mathrm{c}(h, \beta(s)) = 1 \text{ for all } h \in H_n,\ s \in S.$$

For $s = g\beta(s)$ define

$$b(s) := c(g, \beta(s)). \tag{7.9}$$

Since

$$c(gh, \beta(s)) = c(g, h\beta(s))c(h, \beta(s)) = c(g, \beta(s))$$

for all $g \in G,\ s \in S$, the definition (1.5) does not depend on the choice of g to represent s in (7.8). Now for $s = g\beta(s)$ we have

$$c(g, s)b(s) = c(g_1, g_2\beta(s))c(g_2, \beta(s)) =$$

$$c(g_1 g, \beta(s))c(g, \beta(s))^{-1}c(g, \beta(s)) = c(g_1 g, \beta(s)) = b(g_1 s).$$

This shows that c is cohomologous to 1. Thus modifying $\{f_g\}_{g\in G}$ as described in Proposition 4.11 we can find $f' : G \to L^\alpha(S, \mu)$ such that for all $g \in G$ we have

$$f'_{g^{-1}}(s) = f'_1(gs) \ \ \mu - \text{a.e.}$$

Now we will move the representation to the space $G \times (S/G)$. Let $\Pi : S \to S/G$ be the canonical projection and $S_0 = \beta(S)$. S_0 is a

Borel set which intersects each orbit at exactly one point. Then $\Pi|_{S_0}$ is a Borel bijection between standard Borel spaces, hence it is a Borel isomorphism and so is $\Pi_0 : S/G \to S_0$, $\Pi_0 := (\Pi|_{S_0})^{-1}$. This implies that the map

$$\Phi : G \times (S/G) \to S, \quad \Phi(g, w) = g^{\frac{1}{n}} \Pi_0(w), \tag{7.10}$$

where $g^{1/n} = e^{i\theta/n}$ if $g = e^{i\theta}$, $\theta \in [0, 2\pi)$ is also a Borel isomorphism. Note that in this notation in general $(gg_1)^{\frac{1}{n}} \neq g^{\frac{1}{n}} g_1^{\frac{1}{n}}$, however we do have

$$(g^{-1}g_1)^{\frac{1}{n}} \cdot s = (g^{\frac{1}{n}})^{-1} \cdot (g_1^{\frac{1}{n}} \cdot s) \tag{7.11}$$

for all $g, g_1 \in G$ and $s \in S$.

$G \times (S/G)$ is a Polish $G-$space with a natural action $\eta(g, (g_1, w)) = (gg_1, w)$. By (7.10) and (7.11)

$$\Phi(gg_1, w) = g^{\frac{1}{n}} \Phi(g_1, w).$$

Define a measure λ on $G \times (S/G)$ by

$$\lambda(B) := \mu(\Phi(B)), \quad B \in \mathbb{B}(G \times (S/G)).$$

To see that $\lambda = H \times \nu$ for some measure ν on S/G we employ an argument analogous to the argument in Step 5 of the proof of Theorem 2.1 in [**19**]. For fixed $A \in \mathbb{B}(S/G)$ consider the measure on G given by $\lambda(\cdot \times A)$. Then for any $B \in \mathbb{B}(G)$ and $g \in G$

$$\lambda((gB) \times A) = \lambda(g(B \times A)) = \mu(\Phi(g(B \times A)) =$$

$$\mu(g^{\frac{1}{n}} \cdot \Phi(B \times A)) = \mu(\Phi(B \times A)) = \lambda(B \times A).$$

Thus the measure $B \mapsto \lambda(B \times A)$ is proportional to the Haar measure. Writing $\nu(A)$ to denote the proportionality constant we see that $A \mapsto \nu(A)$ is a measure on S/G and $\lambda(B \times A) = H(B)\nu(A)$ for all $B \in \mathbb{B}(G)$ and $A \in \mathbb{B}(S/G)$. Therefore $\lambda = H \times \nu$.

Now set

$$F(g, w) := f_1'\left((g^{-1})^{\frac{1}{n}} \cdot \Pi_0(w)\right), \quad g \in G,\, w \in S/G.$$

Then, for every $g \in G$

$$F(g_1^{-1}g, w) = f_1'\left((g^{-1}g_1)^{\frac{1}{n}} \cdot \Pi_0(w)\right) = f_1'\left((g^{\frac{1}{n}})^{-1} g_1^{\frac{1}{n}} \cdot \Pi_0(w)\right) = f'_{g^{\frac{1}{n}}}(\Phi(g_1, w))$$

for $H \times \nu$ a.a. (g_1, w). Thus if we define a random field

$$Y(g) := \int_{G \times (S/G)} F(g_1^{-1}g, w)\, M_{H \times \nu}(dg, dw)$$

we obtain

$$\left(Y(e^{in\theta}),\, \theta \in [0, 2\pi/n)\right) \stackrel{d}{=} \left(Z(e^{i\theta}),\, \theta \in [0, 2\pi/n)\right). \tag{7.12}$$

Representation (7.3) follows by extending (7.12) to $\theta \in [0, 2\pi)$ using (7.7).

Let us turn to the fields of orbit-cocycle type (H_n, m_n), where $n = 2k$, $k = 1, 2, \ldots$ and m_n is defined by (7.4). The reasoning in this case is

very similar to that for $(H_n, 1)$, although complicated by the presence of nontrivial cocycle. Suppose Z is of orbit-cocycle type (H_n, m_n) and $f : T \to L^\alpha(S, \mu)$ is a minimal representation of Z provided by Theorem 4.6 (ii). Similarly as in (7.7) we obtain that for $h = e^{2\pi ij/n} \in H_n$, $j = 0, 1, ..., n-1$

$$Z(hg) = (-1)^j Z(g). \tag{7.13}$$

Let $\beta : S \to S$ be a Borel selector for the group action and $c_1 : G \times S \to \{-1, 1\}$ be defined

$$c_1(e^{i\theta}, e^{i\xi} \cdot \beta(s)) := (-1)^{\left[\frac{k(\theta+\xi)}{\pi}\right]+\left[\frac{k\xi}{\pi}\right]}. \tag{7.14}$$

Since if $h \in H_n$, $h = e^{im\frac{\pi}{k}}$ we have

$$c_1(e^{i\theta}, e^{i\xi}h \cdot \beta(s)) = (-1)^{\left[\frac{k(\theta+\xi)}{\pi}\right]+m+\left[\frac{k\xi}{\pi}\right]+m} = c_1(e^{i\theta}, e^{i\xi}\beta(s)),$$

the definition of c_1 does not depend on the choice of ξ to represent $s = e^{i\xi} \cdot \beta(s)$. We can also check easily that c_1 is a strict cocycle and that

$$c_1(e^{im\frac{\pi}{k}}, s) = m_{2k}(e^{im\frac{\pi}{k}}),$$

that is c_1 and c generate the same homomorphisms on the isotropy group H_n:

$$c_1(h, s) = c_2(h, s) \text{ for all } h \in H_n, \ s \in S. \tag{7.15}$$

Now we will show that c_1 is cohomologous to c. For $s = g\beta(s)$ define

$$b(s) := c(g, \beta(s))c_1(g, \beta(s))^{-1}. \tag{7.16}$$

We need to check that b is well defined, that is (7.16) does not depend on the choice of g to represent s as $s = g\beta(s)$. Indeed, if we write $s = g\beta(s) = g_1\beta(s)$, then $h := g^{-1}g_1 \in H_n$ and

$$c(g_1, \beta(s))c_1(g_1, \beta(s))^{-1} = c(gh, \beta(s))c_1(gh, \beta(s))^{-1} =$$

$$c(g, h \cdot \beta(s))c(h, \beta(s))^{-1} \left(c_1(g, h \cdot \beta(s))c_1(h, \beta(s))^{-1}\right)^{-1} =$$

$$c(g, \beta(s))c_1(g, \beta(s))^{-1}c(h, \beta(s)) \left(c_1(h, \beta(s))\right)^{-1} = c(g, \beta(s))c_1(g, \beta(s))^{-1},$$

the last equality due to (7.15).

Now a simple calculations hows that c is cohomologous to c_1:

$$c(g_1, s)b(s) = c(g_1 g, \beta(s))c_1(g, \beta(s))^{-1} = b(g_1 \cdot s)c_1(g_1, s).$$

Therefore we can modify the representation of Z so that for every $g \in G$

$$f'_{g^{-1}} \stackrel{\mu}{=} c_1(g, \cdot)f'_1,$$

where c_1 is the strict cocycle defined by (7.14).

We will use the same Φ defined in (7.10) to transfer the representation f' to the Polish G-space $(G \times (S/G), \eta, H \times \nu)$.

Namely, set $F_{\cdot}: G \times (S/G) \to R$,

$$F(g,w) := -f_1'(\Phi(g^{-1},w)) = -f_1'\left((g^{-1})^{\frac{1}{n}}\Pi_0(w)\right).$$

For fixed $g = e^{i\theta}$, $\theta \in [0,2\pi)$ consider the function $(g_1,w) \mapsto F(g_1^{-1}g,w)c_1(g_1^{\frac{1}{n}},\Pi_0(w))$. Writing $g_1 = e^{i\xi}$, $\xi \in [0,2\pi)$ we get

$$\operatorname{sign}(\theta-\xi)F(g_1^{-1}g,w)c_1(g_1^{\frac{1}{n}},\Pi_0(w)) =$$

$$\operatorname{sign}(\theta-\xi)f_1'\left(\left(g^{\frac{1}{n}}\right)^{-1}g_1^{\frac{1}{n}}\cdot\Pi_0(w)\right)c_1(g_1^{\frac{1}{n}},\Pi_0(w)) \stackrel{H\times\nu}{=}$$

$$\operatorname{sign}(\theta-\xi)c_1\left(\left(g^{\frac{1}{n}}\right)^{-1},g_1^{\frac{1}{n}}\cdot\Pi_0(w)\right)^{-1}f'_{g^{\frac{1}{n}}}\left(g_1^{\frac{1}{n}}\cdot\Pi_0(w)\right) =$$

(7.17)

$$-\operatorname{sign}(\theta-\xi)c_1\left(\left(g^{\frac{1}{n}}\right)^{-1}g_1^{\frac{1}{n}},\Pi_0(w)\right)f'_{g^{\frac{1}{n}}}(\Phi(g_1,w)) \stackrel{H\times\nu}{=} f'_{g^{\frac{1}{n}}}(\Phi(g_1,w)),$$

the last equality following from an easy to check identity

$$\operatorname{sign}(\theta-\xi)c_1\left(e^{\frac{2\pi-\theta+\xi}{n}},\Pi_0(w)\right) = -1, \quad \theta,\xi \in [0,2\pi), \quad \theta \neq \xi, \quad w \in S/G.$$

Thus if we define

$$Y(e^{i\theta}) := \int_{[0,2\pi)\times(S/G)} \operatorname{sign}(\theta-\xi)F\left(e^{i(\theta-\xi)},w\right)M_{m\times\nu}(d\xi,dw), \quad \theta \in [0,2\pi)$$

we obtain

$$\left(Y(e^{in\theta},\ \theta \in [0,2\pi/n)\right) \stackrel{d}{=} \left(Z(e^{i\theta}),\ \theta \in [0,2\pi/n)\right). \tag{7.18}$$

Using (7.13) to extend (7.18) to $\theta \in [0, 2\pi)$ we get

$$\left((-1)^{\left[\frac{n\theta}{2\pi}\right]} Y(e^{in\theta},\, \theta \in [0, 2\pi\right) \stackrel{d}{=} \left(Z(e^{i\theta}),\, \theta \in [0, 2)\right).$$

This concludes the proof of Theorem 7.1. □

Theorem 7.3. *Let $X : G \to L_c^0(\Omega, P)$ be a rotationally invariant complex $\alpha-$stable random field such that for all $g_1 \in G$*

$$(X(g_1 g),\, g \in G) \stackrel{d}{=} (X(g),\, g \in G).$$

Then there exist independent random fields $Y_{n,k}^1 : G \to L_{\mathbb{C}}^0(\Omega, P)$, $n = 1, 2, ...,$ $k = -n+1, ..., n-1$ and rotationally invariant stable random variables $Y_j^0 \in L_{\mathbb{C}}^0(\Omega, P)$, $j \in Z$, such that
(i) random Y field defined by

$$Y(e^{i\theta}) := \sum_{j \in Z} e^{-ij\theta} Y_j^0 + \sum_{n=1}^{\infty} \sum_{k=-n+1}^{k=n-1} e^{-ik\frac{2\pi}{n}\left[\frac{n\theta}{2\pi}\right]} Y_{n,k}^1(e^{in\theta}), \tag{7.19}$$

is equal in distribution to X,
(ii) $Y_{n,k}^1$ admits a minimal representation of the form

$$\left(Y_{n,k}^1(e^{i\theta}),\, \theta \in [0, 2\pi)\right) \stackrel{d}{=}$$

$$\left(\int_{[0,2\pi)\times W_{n,k}} \exp\left(\frac{2k\pi}{n} 1_{\theta > \xi}\right) F_{n,k}(e^{i(\theta-\xi)}, w) M_{m \times \nu_{n,k}}(d\xi, dw),\, \theta \in [0, 2\pi)\right),$$

for some Borel spaces $W_{n,k}$ with Borel measures $\nu_{n,k}$ and functions $F_{n,k} \in L_{\mathbb{C}}^{\alpha}(G \times W_{n,k}, H \times \nu_{n,k})$, $n = 1, 2, ...,$ $k = -n+1, ..., n-1$.

Remark 7.4. Similarly as in Theorem 7.1, different terms in decomposition (7.19) correspond to random fields which are "periodic with

phase shift". Namely shifting the angle parameter in the term corresponding to the field $Y^1_{n,k}$ by $\theta = 2\pi/n$ multiples the value of the term by $e^{-ik\frac{2\pi}{n}}$.

Proof. The proof repeats the ideas of the proof of Theorem 7.1. The main difference now is that the cocycle takes values in G, rather than $\{-1, 1\}$. Let $H_n = \{e^{ij\frac{2\pi}{n}} : j = 0, 1, ..., n-1\}$ be the group of n'th complex roots of one. Each group H_n admits $2n-1$ G-valued homomorphisms given by

$$m_{n,k}\left(e^{ij\frac{2\pi}{n}}\right) = e^{ijk\frac{2\pi}{n}}, \quad n = 1, 2, ..., \; k = -n+1, ..., n-1, \; j = 0, 1, ...n-1.$$

The G-valued homomorphisms on G are given by

$$m_{\infty,k}(e^{i\theta}) = e^{ik\theta}, \quad k \in \mathbb{Z}.$$

Thus the collection $\{(H_n, m_{n,k}),\ n = 1, 2, ...,\ k = -n+1, ..., n-1\} \cup \{(G, m_{\infty,k}),\ k \in \mathbb{Z}\}$ exhausts the list of possible orbit-cocycle types for G in the complex case.

Fix $k \in \mathbb{Z}$ and suppose that Z is a random field of orbit-cocycle type $(G, m_{\infty,k})$. We get

$$f_{e^{-i\theta}}(s) = e^{ik\theta} f_1(e^{i\theta} \cdot s) = e^{ik\theta} f_1(s) \;\; \mu - \text{a.e.}$$

for a minimal representation of Z on a Polish G-space with invariant measure μ. This implies that

$$Z(e^{i\theta}) = e^{-ik\theta} Z(1).$$

Setting $Y_j^0 := Z$ we obtain the single sum term in (7.19).

Now fix $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $-n < k < n$ and suppose Z is of orbit-cocycle type $(H_n, m_{n,k})$. Let $f : G \to L^\alpha(S, \mu)$ be a minimal representation of Z on a Polish $G-$space (S, ρ, μ) with an invariant measure μ provided by Theorem 4.6. Similarly as in (7.13) we obtain

$$Z(hg) = e^{-ijk\frac{2\pi}{n}} Z(g), \tag{7.20}$$

for all $h = e^{2\pi ij/n} \in H_n$, $j = 0, 1, ..., n-1$ and $g \in G$.

Let $\beta : S \to S$ be a Borel selector for the $G-$space (S, ρ) and define the map $c_1 : G \times S \to G$ by

$$c_1(e^{i\theta}, s) = e^{i\frac{2\pi k}{n}\left(\left[\frac{n(\theta+\xi)}{2\pi}\right]-\left[\frac{n\xi}{2\pi}\right]\right)}, \ \theta \in \mathbb{R} \tag{7.21}$$

where $s = e^{i\xi} \cdot \beta(s)$, $\xi \in \mathbb{R}$. We can check easily that the definition (7.21) does not depend on the choice of ξ to represent s as $s = e^{i\xi} \cdot \beta(s)$, that c_1 is a strict cocycle and

$$c_1(h, s) = m_{n,k}(h) = c(h, s) \ \text{ for all } \ h \in H_n, \ s \in S.$$

Using the reasoning as in the proof of Theorem 7.1, starting from (7.15) we can show that c_1 is cohomologous to c.

This way we obtain a minimal representation $f' : G \to L^\alpha_{\mathbb{C}}(S, \mu)$ of Z such that for each $g \in G$

$$f'_{g^{-1}}(s) = c_1(g, s) f'_1(gs) \ \ \mu - \text{a.e.},$$

where c_1 is defined by (7.21).

Now we will use Π_0 and Φ defined in (7.10) to transfer this representation to the G–space $(G \times (S/G), \eta, H \times \nu)$.

Define $F : G \times (S/G) \to \mathbb{C}$ by

$$F(g,w) := f_1'(\Phi(g^{-1},w)) = f_1'\left((g^{-1})^{\frac{1}{n}}\Pi_0(w)\right).$$

Let $g = e^{i\theta}$, $g_1 = e^{i\xi}$, $\theta, \xi \in [0, 2\pi)$. Using an argument analogous to the one leading to (7.17) and the identity

$$c_1\left(\left(g^{\frac{1}{n}}\right)^{-1} g_1^{\frac{1}{n}}, \Pi_0(w)\right) = \exp\left(-i\frac{2\pi k}{n}1_{\theta>\xi}\right)$$

we can show that for all $g \in G$

$$e^{i\frac{2\pi k}{n}1_{\theta>\xi}} F(g_1^{-1}, g, w) c_1(g_1^{\frac{1}{n}}, \Pi_0(w)) = f'_{g^{\frac{1}{n}}}(\Phi(g,w))$$

for $H \times \nu$ a.a. (g_1, w). Since $c_1(g_1^{\frac{1}{n}}, \Pi_0(w))$ does not depend on g, this implies that if we define

$$Y(e^{i\theta}) := \int_{[0,2\pi)\times(S/G)} \exp\left(\frac{2k\pi}{n}1_{\theta>\xi}\right) F_{n,k}(e^{i(\theta-\xi)}, w) M(d\xi, dw),\ \theta \in [0, 2\pi),$$

then

$$(7.22) \qquad \left(Y(e^{in\theta}),\ \theta \in [0, 2\pi/n)\right) \stackrel{d}{=} \left(Z(e^{i\theta}),\ \theta \in [0, 2\pi/n)\right).$$

Using (7.20) to extend (7.22) to $\theta \in [0, 2\pi)$ we obtain

$$\left(e^{-ik\frac{2\pi}{n}\left[\frac{n\theta}{2\pi}\right]} Y(e^{in\theta}),\ \theta \in [0, 2\pi\right) \stackrel{d}{=} \left(Z(e^{i\theta}),\ \theta \in [0, 2\pi)\right).$$

This concludes the characterization of a field of orbit-cocycle type $(H_n, m_{n.k})$ for fixed n, k. Taking the sum over $n = 1, 2, \ldots$ and $k =$

$-n+1, ..., n-1$ we obtain the double sum term of (7.3), which finishes the proof. □

Now we will demonstrate similar characterizations of SαS and rotationally invariant complex valued $\alpha-$stable random fields on $\mathbb{C}$.

Theorem 7.5. *Let $X : \mathbb{C} \to L^0(\Omega, P)$ be a SαS (real valued) random field, $1 < \alpha < 2$, such that*

$$X(g\cdot) \stackrel{d}{=} X(\cdot) \text{ for all } g \in G.$$

Then there exist independent random fields $Y_k^i : \mathbb{C} \to L^0(\Omega, P)$, $k = 1, 2, ...$,$i = 1, 2$ and a SαS process $Y_0 : [0, \infty) \to L^0_{\mathbb{R}}(\Omega, P)$ such that
(i) the field defined by

(7.23)

$$Y(re^{i\theta}) := Y_0(r) + \sum_{k=1}^{\infty} \left(Y_k^1(re^{ik\theta}) + (-1)^{[\frac{k\theta}{\pi}]} Y_k^2(re^{2ki\theta}) \right), \quad \theta \in [0, 2\pi)$$

is equal in distribution to X,
(ii) the fields Y_k^1, $k = 1, 2, ...$ admit a minimal representation of the form

$$\left(Y_k^1(z),\, z \in \mathbb{C}\right) \stackrel{d}{=} \left(\int_{[0,2\pi)\times W_k^1} F_k^1(e^{-i\xi} z, w) M_{m\times\nu_k^1}(d\xi, dw),\, z \in \mathbb{C} \right),$$

for some functions $F_k^1 : \mathbb{C} \to \mathbb{R}$, such that for all $r \in [0, \infty)$ $((\theta, w) \mapsto F_k^1(re^{i\theta})) \in L^\alpha([0, 2\pi) \times W_k^1, m \times \nu)$, where W_k^1 is Polish space with Borel measure ν_k^1, $k = 1, 2, ...$.

(iii) the fields Y_k^2, $k = 1, 2, ...$, $Y_k^2(0) = 0$, *admit a minimal representation of the form*

$$\left(Y_k^2(re^{i\theta}),\, r \in (0,\infty), \theta \in [0, 2\pi)\right) \stackrel{d}{=}$$

$$\left(\int_{[0,2\pi)\times W_k^2} \operatorname{sign}(\theta - \xi) F_k^2(re^{i(\theta-\xi)}, w) M_{m\times\nu_k^2}(d\xi, dw), r \in (0,\infty), \theta \in [0, 2\pi)\right),$$

for some functions $F_k^2 : \mathbb{C} \to W_k^1$, *such that for all* $r \in (0,\infty)$ $((\theta, w) \mapsto F_k^2(re^{i\theta}, w) \in L^\alpha([0, 2\pi) \times W_k^2, m \times \nu_k^2)$, *where* W_k^2 *is Polish space with Borel measure* ν_k^2, $k = 1, 2, ...$

Proof. Let H_n, $m_n, n = 1, 2, ...$ be as in the proof of Theorem 7.1. We need to describe the fields of orbit-cocycle type $(H_k, 1)$, (H_{2k}, m_{2k}), $k = 1, 2, ...$ and $(G, 1)$.

Suppose first that Z is of orbit-cocycle type $(G, 1)$ and $f : \mathbb{C} \to L^\alpha(S, \mu)is$ a minimal representation of Z on a Polish $G-$space with invariant measure μ. Then, for $r \geq 0$ and $g \in G$ we have

$$f_{g^{-1}r}(s) = f_r(g \cdot s) = f_r(s), \;\; \mu - \text{a.e.}$$

This implies that

$$Z(re^{i\theta}) = Z(r)$$

for all $r \geq 0$, $\theta \in \mathbb{R}$. This way we obtain the $Y_0(r)$ term in the sum (7.23).

Now suppose that Z is of orbit-cocycle type $(H_n, 1)$ for a fixed n and again let $f : \mathbb{C} \to L^\alpha(S, \mu)$ be a minimal representation of Z on a

Polish $G-$space satisfying for each $r \geq 0$ and $g \in G$

$$f_{g^{-1}r}(s) = c(g,s) f_r(g \cdot s) \quad \mu - \text{a.e.} \tag{7.24}$$

If $h \in H_n$, $g \in G$ and $r \geq 0$ we have from (7.24)

$$f_{h^{-1}g^{-1}r}(s) \stackrel{\mu}{=} f_{g^{-1}r}(s)$$

which implies

$$Z(hg) = Z(g) \tag{7.25}$$

for all $h \in H_n$, $g \in G$.

Modifying the representation f in a way described in the proof of Theorem 7.1 we obtain a minimal representation $f' : \mathbb{C} \to L^{\alpha}_{\mathbb{C}}(S,\mu)$ such that for all $g \in G$, $r \geq 0$ we have

$$f'_{g^{-1}r}(s) = f'_r(g \cdot s) \quad \mu - \text{a.e.}$$

Let us define $F : \mathbb{C} \times (S/G) \to \mathbb{C}$, by

$$F(re^{i\theta}) = f'_r(e^{-i\frac{\theta}{n}} \cdot \Pi_0(w)),$$

where Π_0 is defined as in the proof of Theorem 7.1. Then, for every $\theta \in [0, 2\pi)$, $r \geq 0$ we obtain

$$F(re^{i(\theta-\xi)}, w) = f'_{re^{i\theta/n}}\left(\Phi(re^{i(\theta-\xi)}, w)\right)$$

for $m \times \nu$−a.a. $(\xi, w) \in [0, 2\pi) \times (S/G)$. Next we define a random field

$$Y(re^{i\theta}) := \int_{G\times(S/G)} F(re^{i(\theta-\xi)}, w) M_{m\times\nu}(d\xi, dw)$$

to get

$$\left(Y(re^{in\theta}),\, \theta \in [0, 2\pi/n)\right) \stackrel{d}{=} \left(Z(re^{i\theta}),\, \theta \in [0, 2\pi/n)\right). \tag{7.26}$$

We extend the equality in (7.26) to $\theta \in [0, 2\pi)$ using (7.25) to get

$$\left(Y(re^{in\theta}),\, \theta \in [0, 2\pi) \stackrel{d}{=} \left(Z(re^{i\theta}),\, \theta \in [0, 2\pi)\right).$$

This concludes the description of the fields of orbit-cocycle type $(H_n, 1)$, $n = 1, 2, ...$ and results in condition (ii) of the Theorem.

To obtain a similar description for of the fields of orbit-cocycle type (H_{2k}, m_{2k}), $k = 1, 2, ...$ we repeat the appropriate part of the proof of Theorem 7.1, with obvious modifications presented above. The only detail worth mentioning is that the condition

$$Z(hgr) = (-1)^j Z(gr), \;\; h = e^{i\frac{2\pi}{n}} \in H_n, \;\; g \in G, \;\; r \geq 0,$$

which is the counterpart of (7.13), forces $Z(0) = 0$. □

Theorem 7.6. *Let $X : \mathbb{C} \to L^0_{\mathbb{C}}(\Omega, P)$ be a rotationally invariant complex α−stable random field such that for all $g_1 \in G$*

$$(X(g_1 g),\, g \in G) \stackrel{d}{=} (X(g),\, g \in G).$$

Then there exist independent random fields $Y^1_{n,k} : \mathbb{C} \to L^0_c(\Omega, P)$, $n = 1, 2, ...,$ $k = -n+1, ..., n-1$ and rotationally invariant stable processes

$Y_j^0 : [0,\infty) \to L_c^0(\Omega, P)$ *such that*

(i) random Y *field defined by*

$$(7.27)\quad Y(re^{i\theta}) := \sum_{j\in Z} e^{-ij\theta} Y_j^0(r) + \sum_{n=1}^{\infty} \sum_{k=-n+1}^{k=n-1} e^{-ik\frac{2\pi}{n}\left[\frac{n\theta}{2\pi}\right]} Y_{n,k}^1(re^{in\theta}),$$

is equal in distribution to X,

(ii) $Y_{n,k}^1$ *admits a minimal representation of the form*

$$\left(Y_{n,k}^1(re^{i\theta}),\, \theta \in [0,2\pi)\right) \stackrel{d}{=}$$

$$\left(\int_{[0,2\pi)\times W_{n,k}} \exp\left(\frac{2k\pi}{n} 1_{\theta>\xi}\right) F_{n,k}(re^{i(\theta-\xi)}, w) M_{m\times\nu_{n,k}}(d\xi, dw),\, \theta \in [0,2\pi) \right),$$

for some functions $F_{n,k} : \mathbb{C} \to \mathbb{C}$, *such that for all* $r \in [0,\infty)$ $((\theta, w) \mapsto F_{n,k}(re^{i\theta})) \in L^\alpha([0,2\pi) \times W_{n,k}, m \times \nu_{n,k})$, *where* $W_{n,k}$ *is Polish space with Borel measure* $\nu_{n,k}$, $n = 1, 2, ...$, $k = -n+1, ..., n-1$.

Proof. The proof is essentially the same as the proof of Theorem 7.3, modified in a way presented in the proof of Theorem 7.5. □

To present classical results ([**28**], Sec. 5 and 6) on characterization of of Gaussian (2-stable) complex valued isotropic random fields on S^{d-1} and $\mathbb{R}^d$, we introduce a set of functions $h_{m,l}^d : S^{d-1} \to \mathbb{C}$, $m = 0, 1, ...$, $l = 1, 2, ..., d_m$, where $d_m := (2m+d-2)\frac{(n+d-3)!}{(n-2)!m!}$, $m > 0$, $d_0 := 1$, known as the set of spherical harmonics on S^{d-1} (for details on these functions, see [**9**]). Spherical harmonics form an orthonormal basis in $L^2(S^{d-1}, m)$, where m is the uniform probability measure on S^{d-1}. In particular, if $d = 2$, $d_m = 2$ for any $m > 0$, and the corresponding

spherical harmonics have the form

$$h^d_{m,1}(e^{i\theta}) = e^{im\theta} \text{ and } h^d_{m,-1}(e^{i\theta}) = e^{-im\theta},\ m = 0, 1, 2, ...$$

Recall that as mentioned after Definition 2.10 in Chapter 2, if $\alpha = 2$, then a complex SαS random variable is rotationally invariant if and only if its real and imaginary parts are independent. We will use interchangeably the terms "complex symmetric Gaussian" and a "complex 2-stable rotationally invariant" random variable or field.

Theorem 7.7. *Let $X : S^{d-1} \to L^0_{\mathbb{C}}(\Omega, P)$ be a complex 2−stable rotationally invariant random field. Then the following are equivalent:*
(i) X is isotropic (that is satisfies (7.1),
(ii) There exist a sequence of independent complex symmetric Gaussian random variables $\{Y^l_m,\ m = 0, 1, 2, ..., l = 1, 2, ..., d_m\}$, such that $\mathbb{E}|Y^l_m|^2 = \mathbb{E}|Y^1_m|^2$, $\sum_{m=0}^{\infty} d_m \mathbb{E}|Y^1_m|^2 < \infty$ and

$$\left(X(t),\ t \in S^{d-1}\right) \stackrel{d}{=} \left(\sum_{m=0}^{\infty}\sum_{l=1}^{d_m} h^d_{m,l}(t) Y_{m,l},\ t \in S^{d-1}\right).$$

The description of isotropic random fields on $\mathbb{R}^d$ is similar.

Theorem 7.8. *Let $X : \mathbb{R}^d \to L^0_{\mathbb{C}}(\Omega, P)$ be a complex 2−stable rotationally invariant random field. Then the following are equivalent:*
(i) X is isotropic,
(ii) There exist a sequence of independent complex symmetric Gaussian random processes $Y^l_m : [0, \infty) \to L^0_{\mathbb{C}}(\Omega, P)$, such that $Y^l_m \stackrel{d}{=} Y^1_m$, $\sum_{m=0}^{\infty} d_m \mathbb{E}|Y^1_m(r)|^2 < \infty$ for all $m = 0, 1, 2, ..., l = 1, 2, ..., d_m$, $r \in$

$[0,\infty)$ *and*

$$\left(X(rt),\, t\in S^{d-1},\, r\in[0,\infty)\right) \stackrel{d}{=} \left(\sum_{m=0}^{\infty}\sum_{l=1}^{d_m} h_{m,l}^{d}(t)Y_{m,l}(r), t\in S^{d-1},\, r\in[0,\infty)\right).$$

We can see from the proofs of Theorems 7.3 and 7.6 that the description of 2-stable rotationally invariant isotropic random fields on S^1 and $\mathbb{R}^2$ are identical as the descriptions of their $\alpha-$stable counterparts generated by a trivial group action (the single sum terms in (7.19) and (7.27)). Analogous statement is true for stationary random fields in $\mathbb{R}^d$. In our terminology these are $G-$stationary random fields, where $G := \mathbb{R}^d$ acts on $T := \mathbb{R}^d$ as translations. Namely, every stationary and continuous in probability complex Gaussian random field X on $\mathbb{R}^d$ can be written as

$$X \stackrel{d}{=} \left(\int_{\mathbb{R}^d} e^{it\cdot s} M(s),\, t\in\mathbb{R}^d\right), \tag{7.28}$$

where M is independently scattered Gaussian random measure on $\mathbb{R}^d$. Also every rotationally invariant α-stable, $1<\alpha<2$, random field generated by a trivial group action (that is, a harmonizable one, see **[19]** for details), can be written in the form (7.28) for some α-stable random measure on $\mathbb{R}^d$.

However, these analogies do not hold for isotropic random fields on $\mathbb{R}^3$ or S^2. Indeed, since all 1-dimensional (complex) representations of $\mathbf{SO}(3)$ are constant, a cocycle for a trivial action is constant and we have $X(t) = X(t_0)$ for all $t\in S^2$ and a fixed $t_0\in S^2$, if X is generated

by a trivial action. On the other hand, we can see from Theorem 7.6 that Gaussian isotropic random fields on $\mathbb{R}^d$ do not have to be trivial.

We will finish this chapter with some remarks on the structure of $\mathbf{SO}(3)$ and an example of a nontrivial isotropic SαS random field on S^2.

Each element $R \in \mathbf{SO}(3)$ can be written as

$$R = \Phi(\varphi, \theta, \psi) = R^z_\varphi R^y_\theta R^z_\psi, \tag{7.29}$$

where

$$R^z_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a rotation about the Z axis by angle φ and

$$R^y_\varphi = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

is a rotation about the Y axis by angle θ. φ, θ, ψ are called the Euler angles of a rotation R.

The mapping $\Phi : [0, 2\pi) \times [0, \pi] \times [0, 2\pi) \to \mathbf{SO}(3)$ is not 1:1, but $\Phi|_{(0,2\pi)\times(0,\pi)\times(0,2\pi)}$ is 1:1 and $H((0, 2\pi) \times (0, \pi) \times (0, 2\pi)) = 1$. The density of the Haar measure in parameterization of $\mathbf{SO}(3)$ by (φ, θ, ψ) is

$$H(d\Phi(\varphi, \theta, \psi)) = \frac{1}{8\pi^2} \sin\theta d\varphi d\theta d\psi. \tag{7.30}$$

See, for example, [17] for details.

Proposition 7.9. *Let $X : S^2 \to L^0_{\mathbb{R}}(\Omega, P)$ be a random field defined by*

$$X(t) := \int_{(0,2\pi)\times\mathbb{R}} f_t(\varphi, s) M(d\varphi, s),$$

where for $t = (x, y, z) \in S^2 \subset \mathbb{R}^3$ we set $f_t : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}$

$$f_t(\varphi, s) := \begin{cases} 1 \text{ if } x\cos\varphi + y\sin\varphi + zs > 0, \\ 0 \text{ otherwise,} \end{cases}$$

M is a SαS random measure on $(0, 2\pi) \times \mathbb{R}$ with control measure $\nu(d\varphi, ds) = (1 + s^2)^{-3/2} d\varphi ds$.

Then X is an isotropic SαS random field on S^2.

Proof. Consider the random field on S^2 given by

$$Y(t) := \int_{\mathbf{SO}(3)} F(R^{-1}t) M_H(dR), \tag{7.31}$$

where we set $F(t) = 1_{z>0}$ for $t = (x, y, z) \in S^2$ and M_H is a SαS random measure on $\mathbf{SO}(3)$ with control measure H. It is easy to see that a process defined as in (7.31) is isotropic if only the map $R \mapsto F(R^{-1}t)$ is in $L^\alpha(\mathbf{SO}(3), H)$. From (7.29) we obtain that for $t = (x, y, z) \in S^2$

(7.32)

$$F(\Phi(\varphi, \theta, \psi)^{-1}t) = \begin{cases} 1 \text{ if } x\sin\theta\cos\varphi + y\sin\theta\sin\varphi + z\cos\theta > 0, \\ 0 \text{ otherwise,} \end{cases}$$

Note that for all $t \in S^2$ the function $(\varphi, \theta, \psi) \mapsto F(\Phi(\varphi, \theta, \psi)^{-1}t)$ does not depend on ψ, thus representation (7.31) is not minimal. □

Using (7.30) and (7.32) we obtain a representation of X on $(0, 2\pi) \times (0, \pi)$:

$$(Y(x, y, z),\ (x, y, z) \in S^2) \stackrel{d}{=}$$

$$\left(C \int_{(0,2\pi)\times(0,\pi)} 1_{x \sin\theta\cos\varphi + y \sin\theta \sin\varphi + z\cos\theta > 1} M_1(d\varphi, d\theta),\ (x, y, z) \in S^2\right),$$

where M_1 is a SαS random measure on $(0, 2\pi) \times (0, \pi)$ with control measure $\mu(d\varphi, d\theta) = \sin\theta d\varphi d\theta$ and $C = \left(2^{3-1/\alpha}\pi^{2-1/\alpha}\right)^{-1}$. Substituting $s := \cot\theta$ we get a on $(0, 2\pi) \times \mathbb{R}$

$$(Y(x, y, z),\ (x, y, z) \in S^2) \stackrel{d}{=}$$

$$\left(C \int_{(0,2\pi)\times R} 1_{x\cos\varphi + y\sin\varphi + zs > 0} M_2(d\varphi, ds),\ (x, y, z) \in S^2\right),$$

where M_2 is a SαS random measure on $(0, 2\pi) \times \mathbb{R}$ with control measure $\nu(d\varphi, ds) = (1 + s^2)^{-3/2} d\varphi ds$. This implies

$$\left(X(t),\ t \in S^2\right) \stackrel{d}{=} \left(C^{-1} Y(t),\ t \in S^2\right),$$

therefore X is $G-$stationary.

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Appendix

APPENDIX A

G–spaces and cohomology classes of cocycles

In this chapter we set up terminology and survey some results on the actions of topological groups on topological and measurable spaces. Most of the terminology, notation and results come from books by Kechris [**13**] and Becker and Kechris [**1**]. We also generalize some results on strict cocycles and their cohomology classes from [**26**] from the case of transitive G–spaces to the case of G–spaces which admit a Borel selector.

A *Polish space* is a separable topological space, metrizable by a complete metric. The collection of *Borel sets* $\mathbb{B}(X)$ of a topological space X is σ-algebra generated by open sets in X.

A *standard Borel space* is a measurable space (i.e a set X equipped with a σ-algebra $\mathcal{A}$ of subsets of X), such that there exist a topology on X which makes X a Polish space and A the σ-algebra of Borel sets in X. A *Borel measure* on X is a measure defined on $\mathcal{A} = \mathbb{B}(X)$.

A *topological group* is a group G together with a topology in G, such that the map $(g, h) \mapsto g \cdot h^{-1}$ is continuous (from G^2 into G). A topological group which is a Polish space is called a *Polish group*. The neutral element of a group G will be denoted e or 1.

Let G be a group and X a set. An *action of G on X* is a map $\rho : G \times X \to X$, such that $\rho(e,x) = x$ and $\rho(g,\rho(h,x)) = \rho(g \cdot h, x)$ for all $g, h \in G$ and $x \in X$. The pair (X,ρ) is called a *G–space*. We will often write $g \cdot x$ or gx instead of $\rho(g,x)$. A set $A \subset X$ is called *invariant* if $g \cdot A := \{g \cdot x : x \in A\} = A$ for all $g \in G$.

For each G–space (X,ρ) we denote E_ρ the associated equivalence relation defined by

$$xE_\rho y \;\Leftrightarrow\; y = g \cdot x \;\text{ for some }\; g \in G.$$

The equivalence classes of E_ρ are called orbits and denoted

$$G \cdot x = \{g \cdot x : g \in G\}.$$

We also denote by X/G the set of orbits. The map

$$\pi : X \to X/G, \;\; \pi(x) = G \cdot x$$

is called the canonical projection. A *selector* for ρ is a map $\beta : X \to X$ such that $\beta(x) \in G \cdot x$ and $x \in G \cdot y$ implies $\beta(x) = \beta(y)$ for all $x, y \in X$. A *transversal* for ρ is a set $X_0 \subset X$ meeting each orbit at exactly one point. An action of G on X is called transitive if there is only one orbit.

An *isomorphism* of G-spaces (X,ρ) and (Y,η) is a map $\Phi : X \to Y$ which is 1:1 and onto and satisfies

$$\Phi(g \cdot x) = g \cdot \Phi(x)$$

for all $g \in G$, $x \in X$.

An *isotropy group* of a point $x \in X$ is a subgroup $G_x \subset G$ defined by

$$G_x := \{g \in G : g \cdot x = x\}.$$

Note that if $y = g \cdot x$, then $G_y = gG_xg^{-1}$. An action of G on X is called *free* if $G_x = \{e\}$ for all $x \in X$. An action is trivial if $G_x = G$ for all $x \in X$ (that is $g \cdot x = x$ for all $g \in G$, $x \in X$).

If G is a topological group, X is a topological space and the action $\rho : G \times X \to X$ is continuous, then we call (X, ρ) a topological G-space. If, in addition, both G and X are Polish, we call (X, ρ) a *Polish G-space*.

If (X, ρ) is a topological G-space, we can use the canonical projection to define a topology on X/G, called the *quotient topology*. Namely $U \subset X/G$ is open in the quotient topology if $\pi^{-1}(U)$ is open in X. This makes π a continuous and open map. In general, this topology does not have to be T_0 (a topological space is T_0 iff $x \neq y$ implies $\overline{\{x\}} \neq \overline{\{y\}}$) even if both G and X are Hausdorff (T_2). Similarly we can define a $\sigma-$algebra of subsets of S/G

$$\mathcal{A} := \{A \subset S/G : \pi^{-1}(A) \in \mathbb{B}(X)\},$$

called the *quotient Borel σ-algebra*. In general, the quotient Borel σ-algebra is not the same as the σ-algebra $\mathbb{B}(S/G)$ of Borel sets generated by the quotient topology. However, we have the following sufficient condition for X/G to be a Polish space:

Proposition 1.1. [1, Prop. 2.2.6] *Let (X, ρ) be a Polish space. If for any closed set $A \subset X$, the saturation $G \cdot A := \{g \cdot x : g \in G, x \in X\}$ is closed, then X/G is Polish.*

Let G be a Polish group acting on a standard Borel space. If the action $\rho : G \times X \to X$ is a Borel measurable map (that is, $\rho^{-1}(B) \in \mathbb{B}(G \times X)$ for every $B \in \mathbb{B}(X)$), then (X, ρ) is called a *standard Borel G–space.*

If μ is a Borel measure on X such that $\mu(B) = 0$ implies $\mu(g^{-1} \cdot B) := \mu(\{x \in X : g \cdot x \in B\}) = 0$, for all $B \in \mathbb{B}(X)$, then we call the measure μ *quasi-invariant*. If $\mu(B) = \mu(g \cdot B)$ for all $B \in \mathbb{B}(X)$, then we say that the measure μ is *invariant*. We often write a standard Borel G–space as a triple (X, ρ, μ) to indicate that μ is a σ–finite quasi-invariant Borel measure on X.

Let (X, ρ), (Y, η) be standard Borel G–spaces. If there exist an isomorphism of G–spaces (X, ρ) and (Y, η), which is a Borel map, then we say that (X, ρ) and (Y, η) are Borel isomorphic. We have the following important theorem due to Becker and Kechris [1] :

Theorem 1.2. *Let (X, ρ) be a standard Borel G–space. Then there exist a Polish G–space (Y, η) such that (X, ρ) is Borel isomorphic to (Y, η).*

A convenient reformulation of Theorem 1.2 is the following: If (X, ρ) is a standard Borel G–space, then there exist a Polish topology on X, giving the same Borel structure, for which the action becomes continuous. This allows to generalize many results proven for topological G-spaces to the situation when the group action is measurable.

Let (X, ρ) be a standard Borel G–space and μ a Borel probability measure on X. We say that μ is *ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ for any invariant set $B \in \mathbb{B}(X)$. The measure μ is called *nonatomic* if $\mu(G \cdot x) = 0$ for all $x \in X$.

A very important question is whether a particular standard Borel G–space admits a selector $\beta : X \to X$, which is a Borel map, or a transversal $X_0 \subset X$, such that $X_0 \in \mathbb{B}(X)$. In this respect, we have the following result, proved by Effros [7], following up the work of Glimm [10], who dealt with the case when G and X are locally compact.

Theorem 1.3. *Let (X, ρ) be a Polish G–space with associated equivalence relation $E_\rho \subset X \times X$. Assuming that E_ρ is an F_σ subset of $X \times X$, the following are equivalent:*

(i) X/G is T_0,

(ii) there exist a Borel selector for ρ,

(iii) there exist a Borel transversal for ρ,

(iv) the quotient topology on S/G generates the quotient Borel structure,

(v) every orbit is a G_δ subset of X,

(vi) every orbit is a difference of two closed sets in X,

(vii) E_ρ is G_δ in X^2,

(viii) for each $x \in X$ the mapping $gG_x \mapsto g \cdot x$ is a homeomorphism of G/G_x onto $G \cdot x$,

(viii) (X, ρ) does not admit an ergodic nonatomic probability measure,

(ix) (X, ρ) admits a countable separating family, i.e. a family $\{B_n\}_{n \in \mathbb{N}} \subset$

$\mathbb{B}(X)$, *such that*

$$x \in G \cdot y \;\Leftrightarrow\; (x \in B_n \;\Leftrightarrow\; y \in B_n) \text{ for all } n \in \mathbb{N}.$$

The next proposition shows that if G is locally compact, then the assumption in the previous theorem that E_ρ is F_σ in X^2 is superfluous.

Proposition 1.4. *(i) If (X, ρ) is a Polish G–space and G is locally compact, then E_ρ is F_σ in X^2.*
(ii) If (X, ρ) is a standard Borel G–space and G is locally compact, then E_ρ is a Borel subset of X^2.

Proof. Since G is locally compact and separable, there exist a sequence of compact sets $K_n \subset G$, $n = 1, 2, ...$ such that $G = \bigcup_{n=1}^{\infty} K_n$. For fixed n consider the set

$$A_n := \{(x, g \cdot x) \in X^2 : x \in X,\, g \in K\}.$$

Clearly $E_\rho = \bigcup_{n=1}^{\infty} A_n$. To show that A_n is closed, $n = 1, 2, ...$ let $\{(x_k, g_k \cdot x_k)\}_{k \in \mathbb{N}} \subset A_n$ be a sequence convergent (in a Polish metric consistent with topology on X^2) to $(x_0, y) \in X^2$. Selecting a subsequence $\{g_{k_l}\}_{l \in \mathbb{N}}$ such that $g_{k_l} \to g_0 \in K_n$ as $l \to \infty$, by continuity of the action we obtain $y = g_0 x_0$, hence $(x_0, y) \in A_n$. This shows (i). Now (ii) is a consequence of (i) and Theorem 1.2. □

Remark 1.5. Proposition 1.4 (i) is given as Exercise (15.13) in [**13**]. In [8] one can find a proof under additional assumption that X is also locally compact, but that proof does not apply when X is merely Polish.

The following theorem can be found in [**3**, Th. I.3.1].

Theorem 1.6. *Let (X, ρ) be a topological $G-$space, where G is a compact group and X is Hausdorff. Then*
(i) X/G is Hausdorff,
(ii) the canonical projection $\pi : X \to X/G$ is closed,
(iii) if $K \subset X/G$ is compact, then $\pi^{-1}(K)$ is compact,
(iv) X is compact iff (X/G) is compact,
(v) X is locally compact iff X/G is compact.

As a corollary we obtain that many nice properties of Polish $G-$spaces enumerated in Theorem 1.3 hold automatically when G is compact.

Corollary 1.7. *Let (X, ρ) be a Polish G-space, where G is a compact group. Then all properties (i)-(ix) of Theorem 1.3 hold. Moreover, X/G is a Polish space.*

Proof. Since metrizable spaces are T_2, by Theorem 1.6 X/G is T_2, hence T_0 and we get the property (i) in Theorem 1.3. Consequently, we get all properties (ii)-(ix). Since π is continuous and by Theorem 1.6 π is closed, the saturation $G \cdot A = \pi^{-1}(\pi(A))$ is closed for any closed set A. By Proposition 1.1 this implies that X/G is Polish. □

Consider the set of strict cocycles with values in an Abelian locally compact Polish group W for a given standard Borel $G-$space (S, ρ, μ) with equivalence relation

$$c_1 \sim c_2 \Leftrightarrow \quad c_1 \text{ is cohomologous to } c_2.$$

By the definition, if $c : G \times S \to W$ is a strict cocycle, then for all $s \in S$ the function $G_s \ni h \mapsto m_{c,s}(h,s) := c(h,s) \in W$ is a measurable homomorphism of G_s into W. In fact, the mapping $h \mapsto c(h,s)$ must be continuous by a theorem due to Banach, see for example [**5**, Ch. 5]. It is shown by Varadarajan in his book [**26**] that if the group action is transitive, then $c_1 \sim c_2$ iff these strict cocycles induce the same homomorphisms of the isotropy group G_s, where s is a fixed (any) point of S. This way the equivalence classes for relation $\sim$ can be identified with Borel homomorphisms of G_s into W. In this Appendix we generalize this result to certain class of not necessarily transitive G–spaces.

We will need the following simple lemma for the proof of Proposition 1.9.

Lemma 1.8. *Let (S, ρ, μ) be a standard Borel G–space, where G is a locally compact group with a Haar measure H and suppose $B \in \mathbb{B}(S)$, $\mu(B) = 0$. Then*

$$(H \times \mu)(\{(g,s) : g \cdot s \in B\}) = 0. \tag{1.1}$$

Proof. Since μ is quasi-invariant, for every $g \in G$ we have

$$\mu(\{s \in S : g \cdot s \in A\}) = \mu(g^{-1} \cdot A) = 0.$$

Now (1.1) follows from Fubini's theorem. □

Proposition 1.9. *Let (S, ρ, μ) be a standard Borel G–space. with strict cocycles $c_i : G \times S \to W$. where W is locally compact Abelian*

Polish group. Then $c_1 \sim c_2$ *implies that there exist a set* $N \subset S$, $\mu(N) = 0$ *such that*

$$c_1(h, s_1) = c_2(h, s) \text{ for all } s \in S \setminus N \text{ and } h \in G_s.$$

Proof. Let $b : S \to W$ a measurable map be such that for every $g \in G$

$$c_2(g, s) = c_1(g, s)b(g \cdot s)(b(s))^{-1} \text{ for } \mu - \text{a.a. } s \in S.$$

By Lemma 1.8 and Fubini's theorem this implies that for all $s \in S \setminus N$, $\mu(N) = 0$

$$c_2(g, g_1 \cdot s) = b(gg_1 \cdot s)c_1(gg_1, s)(b(g_1 \cdot s))^{-1}$$

$(H \times H)(dg, dg_1)$-a.e. on $G \times G$. By simple calculation we get for all $s \in S \setminus N$

$$c(gg_1, s)^{-1}b(gg_1 \cdot s)c_2(gg_1, s) = c(g_1, s)^{-1}b(g_1 \cdot s)c_2(g_1, s) \tag{1.2}$$

for $(H \times H)$-a.a.(g, g_1).

Now fix $s \in S \setminus N$ and $h \in G_s$. Define $f_s : G \to W$,

$$f_s(g) := c_1(g, s)^{-1}b(g \cdot s)c_2(g, s) \ , g \in G.$$

Then (1.2) means that

$$f_s(gg_1) = f_s(g_1) \text{ for } H \times H - \text{a.a. } (g, g_1) \in G \times G.$$

By Corollary 8.21 in [**26**] f_s must be then constant H-a.e., say

$$c_1(g,s)^{-1}b(g\cdot s)c_2(g,s) = k_s \text{ for } H-\text{a.a } g\in G.$$

Writing gh instead of g in the above identity we obtain

$$k_s = c_1(gh,s)^{-1}b(gh\cdot s)c_2(gh,s) =$$

$$c_1(h,s)^{-1}c_1(g,s)b(g\cdot s)c_2(g,s)c_2(h,s)k_s$$

for H-a.a. $g\in G$. This implies

$$c_1(h,s)^{-1}c_2(h,s) = 1.$$

□

Remark 1.10. The assumption that the W is Abelian in the above proposition can be dropped. However, in that case the homomorphisms induced by cohomologous strict cocycles are conjugate ($c_2(h,s)k_s = k_s c_1(h,s)$ for some function $s\mapsto k_s$), rather than identical.

Proposition 1.9 suggests that the connection between cohomology classes of strict cocycles and measurable homomorphisms holds also for non-transitive G-spaces. However, to invert the implication in the statement of Proposition 1.9 we need to make additional assumptions on the structure of the G-space which allows us to construct a measurable function $b : S\to W$ needed for strict cocycles to be cohomologous. Namely, we will consider standard Borel G-spaces (S,ρ,μ) which admit a Borel selector, that is a Borel map $\beta : S\to S$ such that $\beta(s)\in G\cdot s$ and $s\in G\cdot s_1$ implies $\beta(s) = \beta(s_1)$ for all $s, s_1\in S$. By Corollary 1.7

in this Appendix Borel selector always exist if G is a compact Polish group.

We will need the following lemma in the proof of Theorem 1.12.

Lemma 1.11. *Let (S, ρ, μ) be a standard Borel G-space, with G locally compact, such that $\beta : S \to S$ is a Borel selector for the action ρ. Then there exist a Borel map $\gamma : S \to G$ such that*

$$s = \gamma(s) \cdot \beta(s) \text{ for } \mu - \text{a.a } s \in S.$$

Proof. Since $\beta(s) \in G \cdot s$, for every $s \in S$ we can find a $\gamma \in G$ such that $s = \gamma \cdot \beta(s)$. Our objective is to choose γ for every $s \in S$ such that the map $s \mapsto \gamma(s)$ is measurable. To do this we will use a theorem from [27], which a good survey of the results on measurable selections.

Denote $G_{s,s_1} := \{g \in G : g \cdot s = s_1\}$. G_{s,s_1} is a closed set in G for every $s, s_1 \in S$. Since for locally compact Polish group G the equivalence relation induced on S by the group action is Borel (Proposition 1.4), we can use Theorem 7.1.2 from [1] to conclude the map $(s, s_1) \mapsto G_{s,s_1}$ from S^2 into the Effros Borel space $\mathbb{F}(G)$ is Borel (see Chapter 5 Definition 5.7 for information on the Effros Borel space).

Consider the set-valued function defined on S by

$$F(s) := G_{\beta(s),s} = \{g \in G : g \cdot \beta(s) = s\}.$$

For every $s \in S$ $F(s)$ is a closed set and the map $s \mapsto F(s)$ from S into $\mathbb{F}(G)$ is Borel. By Definition 5.7 of the Effros Borel space this implies

that for every open set $U \subset G$ the set

$$F^{-}(U) := \{s \in S : F(s) \cap U \neq \emptyset\}$$

is a Borel set in S. In the terminology used in [27] this means that F is closed valued and weakly measurable. Then by part (g) of Theorem 4.2 in [27] there exist a function $\gamma' : S \to G$ such that $\gamma'(s) \in F(s)$ for every $s \in S$ and γ' is measurable with respect to the completion $\mathcal{M}$ of σ–algebra $\mathbb{B}(S)$. Using Lusin's theorem we can find $\gamma : S \to G$ such that $\gamma(s) = \gamma'(s)$ μ–a.e. and γ is Borel measurable. □

Theorem 1.12. *Let (S, ρ, μ) be a standard Borel G–space and suppose (S, ρ, μ) admits a Borel selector. Let $c_i : G \times S \to W$, $i = 1, 2$ be strict cocycles which μ–a.e. generate the same homomorphisms on the isotropy groups, that is for μ–a.a. $s \in S$*

$$c_1(h, s) = c_2(h, s) \text{ for all } h \in G_s.$$

Then c_1 is cohomologous to c_2.

Proof. We need to exhibit a Borel function $b : S \to W$ such that for every $g \in G$

$$c_2(g, s)b(s) = b(g \cdot s)c_1(g, s) \text{ for } \mu - \text{a.a. } s \in S. \tag{1.3}$$

Let $\beta : S \to S$ be a Borel selector for the action ρ and $\gamma : S \to G$ a Borel map such that $s = \gamma(s) \cdot \beta(s)$ for μ–a.a. $s \in S$. The existence of γ follows from Lemma 1.11. Let N_1, N_2 be Borel sets such that

$\mu(N_i) = 0$, $i = 1, 2$, $s \in S \setminus N_1$ implies

$$c_1(h, s) = c_2(h, s) \text{ for all } h \in G_s \tag{1.4}$$

and $s \in S \setminus N_2$ implies $s = \gamma(s) \cdot \beta(s)$. Define function $b : S \to W$ by

$$b(s) := c_2(\gamma(s), \beta(s))c_1(\gamma(s), \beta(s))^{-1}. \tag{1.5}$$

We will show that b satisfies (1.3) for all $g \in G$ and $s \in S \setminus N_g$, where $N_g := N_1 \cup N_2 \cup g^{-1} \cdot N_2$ is a Borel set, $\mu(N_g) = 0$.

Fix $g \in G$ and $s \in S \setminus N_g$. Since we have

$$g \cdot s = \gamma(g \cdot s) \cdot \beta(g \cdot s) = \gamma(g \cdot s) \cdot \beta(s) = \gamma(g \cdot s)\gamma(s)^{-1} \cdot s$$

we see that $h := g^{-1}\gamma(g \cdot s)\gamma(s)^{-1} \in G_s$ and

$$\gamma(g \cdot s) = gh \cdot \gamma(s).$$

Writing $g \cdot s$ in place of s in (1.5) we obtain

$$b(g \cdot s) := c_2(\gamma(g \cdot s), \beta(s))c_1(\gamma(g \cdot s), \beta(s))^{-1}.$$

By the properties of cocycles and identity $s = \gamma(s) \cdot \beta(s)$ we obtain

$$b(g \cdot s) = c_2(g, s)c_2(h\gamma(s), \beta(s))c_1(h\gamma(s), \beta(s))^{-1}c_1(g, s)^{-1} =$$

$$c_2(g, s)c_2(h, s)c_2(\gamma(s), \beta(s))c_1(\gamma(s), \beta(s))^{-1}c_1(h, s)c_1(g, s)^{-1}$$

Because W is Abelian we can rearrange the factors and use (1.4) to get

$$b(g \cdot s)c_1(g, s) = c_2(g, s)b(s).$$

Therefore c_1 is cohomologous to c_1. □

Although the previous theorem identifies cohomology classes of strict cocycles with homomorphisms of the isotropy groups, in practice it is useful to be able to construct specific examples of cocycles from a cohomology class corresponding to a given homomorphism. The next theorem shows how we can it with the aid of a Borel selector for the right action of G_0 on G. We choose to formulate the next theorem for the case of transitive G-spaces, but the proof can be easily extended as in the proof of Theorem 1.12 to the case of G-spaces which admit a Borel selector and the isotropy groups are conjugate to a fixed closed subgroup of G.

Theorem 1.13. *Let G be a locally compact Polish group and G_0 be a closed subgroup of G, $S = G/G_0 = \{gG_0 : g \in G\}$ the space of left cosets of G_0. Suppose $\beta : G \to G$ is a Borel selector for the right action $g \mapsto gh$, $g \in G$, $h \in G_0$ of G_0 on G such that $\beta(h) = e$ for $h \in G_0$. Let $m : G_0 \to M$ be a measurable homomorphism and $c : G \times S \to M$ be defined*

$$c(g_1, g_2G_0) := m(\beta(g_1g_2)^{-1}g_1\beta(g_2)). \tag{1.6}$$

Then c is a well defined strict cocycle c belongs to the cohomology class corresponding to the homomorphism m.

Proof. Since for any $h \in G_0$ we have $\beta(gh) = \beta(g)$ the definition in (1.6) does not depend on the choice of g_2 to represent the coset g_2G_0.

To show that c is a strict cocycle we need to check that for all $g_1, g_2, g_3 \in G$ we have

$$c(g_1, g_2g_3G_0) = c(g_1g_2, g_3G_0)c(g_2, g_3G_0)^{-1}. \tag{1.7}$$

Indeed, using definition (1.6) we write the RHS of (1.7) as

$$c(g_1g_2, g_3G_0)c(g_2, g_3G_0)^{-1} = m(\beta(g_1g_3g_2)^{-1}g_1g_3\beta(g_2))\ \left(m(\beta(g_3g_2)^{-1}g_3\beta(g_2))\right)^{-1} =$$

$$m(\beta(g_1g_3g_2)^{-1}g_1g_3\beta(g_2)(\beta(g_3g_2)^{-1}g_3\beta(g_2))^{-1}) =$$

$$m(\beta(g_1g_2g_3)^{-1}g_1\beta(g_2g_3)) = c(g_2, g_2g_3G_0).$$

Thus c is a strict cocycle and clearly, for $h \in G_0$

$$m(h) = c(h, G_0) = m(\beta(h)^{-1}h) = m(h),$$

therefore c is in the cohomology class corresponding to the homomorphism m. □

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Vita

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